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In [1] the author and van Eeden considered, as prior distributions for the cumulative, F, of the bio-assay problem, processes whose sample functions are, with probability one, distribution functions. The example we considered there had the undesirable property that its mean, E(F), was singular with respect to Lebesgue measure. In fact, Dubin and Freedman [2] have shown that a class of such processes, which includes the example we considered, has sample functions F which are, with probability one, singular.

In this note, a class of such processes is given, which, with probability one, have sample functions that are absolutely continuous with respect to Lebesgue measure.

Let $\{Z(\frac{k}{2^n});\ k \in (1,3,5,\ldots,2^n-1),\ n=1,2,\ldots\}$ be a completely independent set of random variables defined on (Ω, A, P) and such that

a) $0 \leq Z(\frac{k}{2^n}) \leq 1$

b) $EZ(\frac{k}{2^n}) = \frac{1}{2}$.

Let $\{F(x),\ 0 \leq x \leq 1\}$ be the process defined by $F(x) = \lim_n F_n(x)$, where $F_1(0) = 0$, $F_1(\frac{1}{2}) = Z(\frac{1}{2})$, $F_1(1) = 1$,

$$F_n(\frac{k}{2^n}) = F_{n-1}(\frac{k-1}{2^n})[1-Z(\frac{k}{2^n})] + F_{n-1}(\frac{k+1}{2^n})[Z(\frac{k+1}{2^n})]$$

and $F_n(x) = F_n(\frac{k}{2^n}) + (x-\frac{k}{2^n})\cdot \dfrac{F_n(\frac{k+1}{2^n}) - F_n(\frac{k}{2^n})}{2^{-n}}$ for $\frac{k}{2^n} < x < \frac{k+1}{2^n}$.

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F_n has, except at $\frac{k}{2^n}$, a derivative, f_n, whose value is

$$f_n(x) = [F_n(\frac{k+1}{2^n}) - F_n(\frac{k}{2^n})]\cdot 2^n \quad \text{for} \quad \frac{k}{2^n} < x < \frac{k+1}{2^n} .$$

If x is written as $x = \sum_{i=1}^{\infty} \frac{\epsilon_i(x)}{2^i}$ and $k_n(x)$ is defined by $\frac{k_n(x)}{2^n} < x < \frac{k_n(x)+1}{2^n}$, (in what follows, $x \neq \frac{k}{2^n}$, and, $\epsilon_i(x)$, $k_n(x)$, will be written as ϵ_i, k_n.) then this derivative can be written as

$$f_n(x) = \prod_{i=1}^{n} 2[Z(\frac{k_i+1}{2^i})]^{1-\epsilon_i} [1-Z(\frac{k_i}{2^i})]^{\epsilon_i} .$$

With these definitions the following theorem, giving sufficient conditions that F be absolutely continuous, can be stated.

<u>Theorem.</u> If there exists $K < \infty$, such that for every n,

$$\int_0^1 E[f_n^2(x)]dx < K$$

then, with probability one (P),

i) $F_n' = f_n$ converges, for almost all x in [0, 1], to a finite limit, f.

ii) $F(t) = \int_0^t f(x)dx, \qquad 0 \leqq t \leqq 1.$

<u>Corollary.</u> If $\sup_{0 < k < 2^n} \sigma^2[Z(\frac{k}{2^n})] \leqq b_n$, and $\Sigma\, b_n < \infty$ then i) and ii) hold.

<u>Proof of i).</u> Let $Q = P\times L$ on $\Omega\times I$ (L = Lebesgue measure on I=[0, 1]). Straight forward calculation establishes that, on $\Omega\times[0, 1]$, f_n is a martingale (with respect to Q) such that $E_Q f_n = 1$. Therefore, by the

martingale convergence theorem, Doob [3], $f = \lim_n f_n$ is defined, finite, with probability one (Q). By Fubini's theorem i) follows.

The proof of ii) will be given in several steps.

A) $\{f_n\}$ is uniformly integrable (Q).

Proof.

$$\int_{[f_n > M]} f_n dQ = \int_0^1 dx \int_\Omega f_n I(f_n > M) dP$$

$$\leq \int_0^1 [\tfrac{1}{M} E_P(f_n^2)]^{\frac{1}{2}} dx$$

$$\leq \frac{1}{\sqrt{M}} \sqrt{\int E_P(f_n^2) dx}$$

$$\leq \frac{\sqrt{K}}{\sqrt{M}} .$$

B) $\int_0^1 f(x)dx \leq 1$ with probability one (P).

Proof. Let $T = [\omega: \int_0^1 f(x)dx > 1]$. It follows from A) and the martingale convergence theorem that $\int |f_n - f| dQ \to 0$, so that $\int_{A\times I} f_n dQ \to \int_{A\times I} fdQ$.

However,

$$\int_{A\times I} f_n dQ = \int_A dP \int_0^1 f_n dx = P(A)$$

and $$\int_{A\times I} fdQ = \int_A dP \int_0^1 fdx > P(A) \quad \text{if} \quad P(A) > 0.$$

Therefore, $P(A) = 0$.

C) $\int_0^1 fdx = 1$ with probability one (P).

Proof. Since $\int |f_n - f| dQ \to 0$ and $\int f_n dQ = 1$ it follows that $\int fdQ = 1$. However, $\int fdQ = \int dP \int_0^1 fdx$ and since, by B) $\int_0^1 fdx \leq 1$ with probability one (P), it must be that $\int_0^1 f dx = 1$ with probability one(P).

D) For any t, $0 \leq t \leq 1$, $F(t) = \int_0^t f(x)dx$ with probability one (P).

Proof. f is a density since $\int_0^1 f dx = 1$. By Scheffe's theorem $\int |f_n - f| dx \to 0$ which implies that $\int_0^t f_n dx \to \int_0^t f dx$. But $\int_0^t f_n dx = F_n(t) \to F(t)$.

This completes the proof of the theorem.

The corollary follows from the considerations,

$$E_P[f_n^2(x)] = \prod_{i=1}^{n} 4\{E[Z^2(\frac{k_i + 1}{2^i})]\}^{1-\epsilon_i} \{E[1-Z(\frac{k_i}{2^i})]^2\}^{\epsilon_i}$$

which converges if

$$\Sigma\{1-4[\{E[Z^2(\frac{k_i + 1}{2^i})]\}^{1-\epsilon_i} \{E[1-Z(\frac{k_i}{2^i})]^2\}^{\epsilon_i}]\}$$

converges. Since $E[Z(\frac{k}{2^n})] = \frac{1}{2}$ this last sum is

$$4\ \Sigma\ \sigma^2[\{Z(\frac{k_i + 1}{2^i})\}^{1-\epsilon_i} \{1-Z(\frac{k_i}{2^i})\}^{\epsilon_i}],$$

and, by hypothesis, this series converges uniformly in x. Therefore, $E_P[f_n^2(x)]$ converges uniformly in x, that is $E_P[f_n^2(x)] < K$ and so is $\int_0^1 E_P[f_n^2(x)]dx$.

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[1] Kraft, Charles H. and Eeden, Constance van., "Bayesian Bio-Assay." Note to appear in Ann. Math. Stat. June, 1964.

[2] Dubins, L. E., and Friedman, David A., "Random Distribution Functions." Bull. of Amer. Math. Soc. 69 (1963) 548-551.

[3] Doob, J. L. "Stochastic Processes" John Wiley & Sons, 1953.