January, 1964

SURVEILLANCE PROBLEMS: WIENER PROCESSES*

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Technical Report No. 34

University of Minnesota

Minneapolis, Minnesota

*Work done in part under contract Nonr-710(31), NR 0420-003 of the Office of Naval Research.

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INTRODUCTION

In [6], problems of determining optimal strategies for the surveillance of production processes which behave as Poisson processes are studied. Some of these problems are solved explicitly, and, for others, qualitative properties of optimal strategies are derived. Analogous results are given here for the case of a production process which behaves as a Wiener process. Some of the results proved in [6] are directly applicable to the Wiener case as well as the Poisson case and are merely stated here; others are easily extended to the present case. This paper is largely self-contained, but for motivation, and some proofs, the reader is referred to [6]. The problems with which [6] is concerned are as follows:

> A production process produces output in a continuous stream when it is not in the repair state. While producing, income (or expected income) per unit of time depends on the state of the production process, which is assumed to be a Poisson process $x(t)$ where t denotes the elapsed production time since the process last emerged from the repair state. The time intervals defined by successive emergences from the repair state are called cycles. Income per unit of time when $x(t) = x$ is denoted by $i(x)$, and $i(x)$ is assumed to be nonincreasing. If production is stopped with $x(t) = x$ $(x \geq 0)$, repairs take m units of time and cost K per unit time; repairs enable the next cycle to start at $x(0) = 0$.
>
> Two kinds of surveillance are studied: <u>continuous surveillance</u>, in which errorless observations of $x(t)$ are immediately available at no cost at all times of production (a constant cost can be incorporated into $i(x)$), and <u>costly surveillance</u>, in which errorless and immediately available observations of $x(t)$ can be made at any times at a positive cost of L each. In either case, the objective is the determination of a strategy which maximizes average income per unit of time. For continuous surveillance, a strategy must specify for every possible process history whether or not the production process should be placed in the repair state. For costly surveillance, a strategy must specify,

as a function of previous and current observations and observation times, whether or not the production process should be placed in the repair state and if not, when the next observation should be taken. The only strategies which are considered, however, are those which depend only on the last observation. These could be described as the stationary Markov strategies. They appear to be the natural ones in view of the fact that the process being considered is Markovian and the income function is additive. Thus, for continuous surveillance, the only strategies considered are: never placing the production process in the repair state, and placing the process in the repair state as soon as $x(t) = w+1$ $(w \geq -1)$. For costly surveillance, the only strategies considered are those which are specified by a continuation set W, a function $T(x)$ defined on W, and the rule: if $x \notin W$, place the production process in the repair state and if $x \in W$, take the next observation in $T(x)$ time units. (It is assumed that an observation must be made immediately prior to starting repairs unless $w = -1$, in which case the production process is always in the repair state.)

This paper considers exactly the same problems under the assumptions that $x(t)$ is a Wiener process with $x(0) = 0$ and variance parameter Δ and $i(x)$ is symmetric about $x = 0$, finite for $x = 0$, and nonincreasing in $|x|$.[1]

WIENER PROCESSES

The following properties of the Wiener process with variance parameter Δ will be used in this paper:[2]

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[1]The case of $i(x)$ nonincreasing in $|x|$ but not symmetric involves the additional problem of determining the optimal point from which to start the process. For the symmetric case it follows from PROPOSITION 8 that the optimal starting point is 0.

[2]Properties (1)-(4) are basic (see, e.g., [5]). Properties (5)-(7) are proved in the Appendix.

(1) For each finite set of values $t_1, \ldots, t_r$, the random variables $x(t_1), \ldots, x(t_r)$ have a multivariate normal distribution with mean vector $(0,0, \ldots, 0)$ and covariance matrix $\Sigma = [\sigma_{jk}]$ where $\sigma_{jk} = \Delta \min(t_j, t_k)$ for $j, k=1, 2, \ldots, r$.

(2) If $x(t_1) = a$ then $E(x(t_2)) = a$ for any $t_2 > t_1$.

(3) If $t_1 < t_2 \leq t_3 < t_4$ then $x(t_2) - x(t_1)$ and $x(t_4) - x(t_3)$ are independent and normally distributed with mean values 0 and variances $\Delta(t_2-t_1)$ and $\Delta(t_4-t_3)$ respectively.

(4) With probability one, $x(\cdot)$ is a continuous function.

(5) If $x(t) = b$ and $a < b < c$ then the expected amount of time that $x(\cdot)$ will be in $(y, y+dy)$, where $a \leq y < y + dy \leq c$, before $x(\cdot)$ equals a or c is given by

$$2\Delta^{-1}[\min(b-a, y-a)-(b-a)(y-a)/(c-a)]dy.$$

(6) If $x(t) = b$ and $a < b < c$ then the expected waiting time for $x(\cdot)$ to reach either a or c is $(c-b)(b-a)/\Delta$.

(7) If $x(t) = b$ and $a < b < c$, the probability that $x(\cdot)$ will reach a before c is $(c-b)/(c-a)$, and the probability that $x(\cdot)$ will reach c before a is $(b-a)/(c-a)$.

CONTINUOUS SURVEILLANCE

The theory for continuous surveillance of a Wiener process follows easily from that given in [6] for Poisson processes. For the objective of maximizing average income per unit of time among the stationary Markov strategies being considered, an optimal decision depends only on the current observation. Hence, specification of an optimal strategy is equivalent to the specification of an optimal continuation set, i.e., a set of x values from which production is to be continued.

From the symmetry of the Wiener process and the assumed symmetry of the income function, it is clear that we need only consider continuation sets symmetric about 0 (including the empty set). Furthermore, since the sample paths are continuous with probability one, consideration can be restricted to the empty set and continuation sets of the form $W = (-w, w)$ where $0 < w \leq \infty$.[3]

Let R_w denote the strategy of using $W = (-w, w)$, let t_w denote the smallest value of t for which $x(t) \notin W$, and let $I(w)$ be the long run income per unit of time if the strategy R_w is used. For $w < \infty$, it is noted in [6] and proved in [3] (and [4]) that with probability one

$$I(w) = [EI(0, t_w)-mK]/[Et_w+m] \tag{8}$$

where $I(x, T)$ is defined to be the conditional expected income from production in the interval $(t, t+T)$ given that $x(t) = x$ and that production is allowed to continue to time $t+T$, i.e.,

$$I(x, T) = E[\int_t^{t+T} i(x(s))ds \,|\, x(t) = x]. \tag{9}$$

From (6) and (9), (8) can be written as

$$I(w) = \frac{E[\int_0^{t_w} i(x(t))dt \,|\, x(0) = 0]-mk}{w^2\Delta^{-1}+m} \tag{10}$$

where the expectation operates on both t_w and $x(t)$. To compute $E[\int_0^{t_w} i(x(t))dt \,|\, x(0) = 0]$, define the new stochastic process $t(y)$ where $t(y)dy$ is the amount of time that $x(t)$ satisfies $y < x(t) < y + dy$, given that $x(0) = 0$, before $|x(\cdot)| = w$. Then from (5), with $b = 0$, $a = -w$, and $c = w$,

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[3]Clearly, we can consider either open or closed intervals; for later convenience, let them be open, and let the empty set be denoted by $(-0, 0)$.

$$E\int_0^{t_w} i(x(t))dt = E\int_{-w}^{w} i(y)t(y)dy = \int_{-w}^{w} i(y)Et(y)dy \tag{11}$$

$$= 2\Delta^{-1}\int_{-w}^{w} i(y)[\min(w,\ y+w)-(y+w)/2]dy$$

$$= 2\Delta^{-1}\int_{0}^{w} i(y)(w-y)dy.$$

Substitution of (11) into (10) results in

$$I(w) = \frac{2\Delta^{-1}\int_0^w i(y)(w-y)dy-mk}{w^2\Delta^{-1}+m}.\tag{12}$$ [4]

The following analysis of (12) shows that except for three particular cases, there exists a unique finite value w^* of w which maximizes $I(w)$. First, since $i(y)$ is nonincreasing for $y \geq 0$, it can be shown that $I'(w) = dI(w)/dw$ exists for all $w > 0$ and is given by $I'(w) = N(w)(w^2\Delta^{-1}+m)^{-2}$ where

$$N(w) = (w^2\Delta^{-1} +m)2\Delta^{-1}\int_0^w i(y)dy - 2w\ \Delta^{-1}(2\Delta^{-1}\int_0^w i(y)(w-y)dy-mK)\tag{13}$$

$$= 2\Delta^{-1}(w^2\Delta^{-1}+m)(\int_0^w i(y)dy - wI(w)).$$

Routine computations yield

$$N'(w) = 2\Delta^{-1}[(w^2\Delta^{-1}+m)i(w) - (2\Delta^{-1}\int_0^w i(y)(w-y)dy - mK)]\tag{14}$$

$$= 2\Delta^{-1}\ (w^2\Delta^{-1}+m)(i(w) - I(w)),$$

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[4] Dimensionally, m is time, K and $i(y)$ are money/time, and, if the dimension of y and w is d, Δ is d^2/time. Thus, $I(w)$ has the dimension money/time, as it should.

(15) $$N''(w) = 2\Delta^{-1}[(w^2\Delta^{-1}+m)\, i'(w) + 2\Delta^{-1}\int_0^w (i(w) - i(y))dy].$$ [5]

Notice that $N''(w) \leq 0$ for all $w > 0$ and is 0 only if $i(y)$ is constant for $0 < y < w$. Notice also that $I(0) = -K$, $N(0) = 0$, $N'(0) = 2\Delta^{-1} m(i(0)+K)$, and $N''(0) = 2\Delta^{-1}mi'(0)$.

The first of the three particular cases obtains if $i(y) < -K$ for all $y > 0$. Then $I'(w) < 0$ and $I(w) < I(0)$ for all $w > 0$, and the optimal continuation set W^* is empty. The second particular case obtains if there exists a $w_0 > 0$ (possibly ∞) such that $i(y) = -K$ for $0 < y < w_0$ while $i(y) < -K$ for $y > w_0$. In this case, it is easily seen that W^* can be empty or any set of the form $(-w^*, w^*)$ where $0 < w^* \leq w_0$. The third particular case obtains if $i(y) = c(> -K)$ for all $y > 0$. In this case, it can be seen from (13) that $N(w)$, and hence $I'(w)$, is positive for all $w > 0$. Thus $W^* = (-\infty, \infty)$.

The alternative to the above three particular cases is that $i(y) > -K$ for some $y > 0$ and $i(y)$ is not constant. For this situation, $N(0) = 0$, $N'(0) > 0$, $N''(w) \leq 0$ for all $w > 0$, and $N''(w)$ is bounded away from 0 for all w sufficiently large. It follows that $N(w)$, and hence $I'(w)$, has a unique 0 at some w^* such that $0 < w^* < \infty$. From (13), it can be seen that w^* must be such that

(16) $$w^*I(w^*) = \int_0^{w^*} i(y)dy.$$

EXAMPLE 1:

Let $i(y) = -Ay^2$. From (12)

(17) $$I(w) = -[2\Delta^{-1}A\int_0^w y^2(w-y)dy+mK]/[w^2\Delta^{-1}+m] = -[\Delta^{-1}Aw^4/6 + mK]/[\Delta^{-1}w^2+m].$$

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[5] Since $i(y)$ is nonincreasing for $y \geq 0$, $i'(w)$ exists almost everywhere. If $i'(w)$ does not exist, (14) and (15) are still true if $i(w)$, $i'(w)$, $N'(w)$, and $N''(w)$ are interpreted as left limits and derivatives or right limits and derivatives.

The maximizing value w^* of w is given by

(18) $$w^* = \{\Delta m[(1+6K/\Delta mA)^{1/2}-1]\}^{1/2}$$

and from (16), $I^* = -Aw^{*2}/3$. Note that I^* is a decreasing function of m, K, A, and Δ, i.e., the optimal income per unit of time decreases with the time and costs for adjustment, with A, and with the variability of the process.

EXAMPLE 2:

Let $i(y) = A$ for $-1 < y < 1$ and 0 otherwise. Then, from (12)

(19) $$I(w) = \begin{cases} [A\Delta^{-1}w^2 - mK]/[\Delta^{-1}w^2+m] & 0 \le w \le 1 \\ [2A\Delta^{-1}(w-1/2)-mK]/[\Delta^{-1}w^2+m] & 1 < w < \infty. \end{cases}$$

The critical equation is

(20) $$\begin{aligned} &2Awm + 2wmK = 0 && 0 \le w \le 1 \\ &A\Delta^{-1}w^2 - (A\Delta^{-1}+mK)\,w - Am = 0 && 1 < w < \infty. \end{aligned}$$

The first part of (20) has the root $w = 0$, which corresponds to a minimum. The roots of the second part of (20) are

$$\{A\Delta^{-1} + mK \pm [(A\Delta^{-1}+mK)^2 + 4A^2\Delta^{-1}m]^{1/2}\}/2A\Delta^{-1}.$$

The positive root is w^*, and it can be written as

(21) $$w^* = (1/2)\{1 + \frac{mK\Delta}{A}\}\{1+[1 + \frac{4A^2\Delta m}{(A+mK\Delta)^2}]^{1/2}\}.$$

From (16), $I^* = A/w^*$. Notice that repairs are not made as soon as $|x(t)| > 1$; for $1 < x(t) < w^*$ there remains a favorable possibility that $x(t)$ will wander back inside the interval $(-1, 1)$. Note also that I^* is an increasing function of A and a decreasing function of m, K, and Δ.

The following table gives values of w^* (upper entries) and I^* (lower entries) for selected values of A, Δ, m, and K.

	K=1						K=3					
Δ	m=1			m=5			m=1			m=5		
A	1	2	5	1	2	5	1	2	5	1	2	5
1	2.41	3.56	6.74	6.74	11.84	26.93	4.24	7.28	16.31	16.31	31.32	76.33
	.41	.28	.15	.15	.08	.04	.24	.14	.06	.06	.03	.01
5	1.77	2.28	3.45	3.45	5.00	7.74	2.08	2.89	5.00	5.00	8.22	16.51
	2.83	2.19	1.45	1.45	1.00	.56	2.40	1.73	1.00	1.00	.58	.29
10	1.69	2.14	3.11	3.11	4.32	7.05	1.84	2.42	3.81	3.81	5.77	10.81
	5.91	4.68	3.22	3.22	2.32	1.42	5.43	4.12	2.62	2.62	1.74	.93
20	1.65	2.07	2.95	2.95	4.00	6.25	1.73	2.21	3.28	3.28	4.65	7.91
	12.09	9.67	6.79	6.79	5.00	3.20	11.57	9.06	6.10	6.10	4.30	2.53

COSTLY SURVEILLANCE-THEORY

The formulation presented in [6] of the problem of determining an optimal (stationary Markov) strategy for costly surveillance of a Poisson process extends easily to the Wiener process case. Hence, only a brief summary will be given here.

Assume that an optimal strategy does exist and that the resulting long run average income per unit of time is I^*. Consider a new problem for which income per unit of time is $i^*(x) = i(x) - I^*$ (symmetric about 0 since $i(x)$ is) and the repair cost per unit of time is $K^* = K + I^*$. It is shown in [6] that the original and new problems have the same optimal strategies, and that since the maximum expected income per cycle for the new problem is zero, the problem of finding an optimal strategy is equivalent to finding the largest value for I^* for which the maximum expected income per cycle for the new problem is zero. This latter formulation introduces another unknown (I^*) but allows the consideration of a maximum expected value rather than the more difficult maximum of a ratio of expected values.

It has been noted in the INTRODUCTION that for costly surveillance of a Poisson process the natural strategies to consider are those which specify for each x whether or not the process should be placed in the repair state and, if not, when the next observation should be taken. This is also true for the Wiener case since the Wiener process is Markovian with stationary increments. Thus, consideration will be restricted to strategies which specify a continuation set W and function $T(x)$, defined on W, giving the time to the next inspection if the process is observed at state x. As in [6], it will be assumed that an inspection must be made immediately prior to starting repairs unless the continuation set is empty, in which case no inspections are necessary.

For the new problem, let $F(x)$ denote the maximum expected income remaining in the cycle if the process is in state x. Notice that $F(0) = 0$. A functional equation for $F(x)$ developed in [6] for $x(t)$ a Poisson process extends easily to any Markov process with stationary increments. For $x(t)$ a Wiener process with $x(0) = 0$ and variance parameter Δ, it is

$$(22) \qquad F(x) = \max_{T(x),\, W} \begin{cases} -m[I^*+K] & , \quad x \notin W \\ I(x, T(x)) - I^*T(x) - L + \int_{-\infty}^{\infty} F(y) f_N(y|x, \Delta T(x))\,dy, & x \in W \end{cases}$$

where $f_N(y|x, \Delta T(x))$ denotes the normal density with mean x and variance $\Delta T(x)$. The problem of determining an optimal strategy can now be stated as that of finding a W, $T(x)$, and I^* satisfying (22) for all x, and the boundary condition $F(0) = 0$. Let W^* and $T^*(x)$ correspond to an optimal strategy.

No way of solving this problem analytically is known to the authors. Hence the following series of qualitative properties of an optimal strategy and related propositions are of interest.

PROPOSITION 1:

If $i(x)$ is unbounded below, the strategy of never repairing need not be considered.

PROOF:

This is an immediate consequence of the fact that

$$\lim_{t \to \infty} P(x(t) > |x|) = 1 \text{ for all } x.$$

Let I_k^* be the maximum income per unit of time for those strategies which have at most k inspections per cycle. Let ${}_kI^*$ be the maximum income per unit of time from those strategies (unfeasible) that involve at most k inspections, and are such that when a k^{th} inspection is used, it is made precisely at the time that $x(t)$ leaves a continuation set which is optimal for such strategies. The next six PROPOSITIONS are merely stated here. For proofs of PROPOSITIONS 2 and 7, see [6]; the proofs of PROPOSITIONS 3-6 are obvious.

PROPOSITION 2:

For $k \geq 1$

$$I_1^* \leq I_2^* \leq \cdots \leq I_k^* \leq I^* \leq {}_kI^* \leq \cdots \leq {}_2I^* \leq {}_1I^*.$$

PROPOSITION 3:

$$I_1^* = \max_{t > 0} [I(0, t)-L-mK]/[t+m].$$

PROPOSITION 4:

$${}_1I^* = \max_W [EI(0, t_w)-L-mK]/[Et_w+m]$$

where t_w is the first time that $x(t) \notin W$.

Expressions for the other bounds in PROPOSITION 2 are more complicated and will not be given here. For the Poisson case, expressions for I_2^* and ${}_2I^*$ are given in [6]; they can easily be extended to the Wiener case.

PROPOSITION 5:

If, for some k, I_k^* is greater than the long run average income per unit of time for the strategy of never repairing, then an optimal strategy will involve repairs.

PROPOSITION 6:

If $i(x)$ is constant then W^* is either empty or the entire real line.

If, for some $x > 0$ and all y such that $y > x$, $i(y) = i(x)$, then either $x \notin W^*$ or W^* is the entire real line.

PROPOSITION 7:

For all x,

$F(x) \geq -m(I^*+K)$ and $I^* \geq -K$.

PROPOSITION 8:

W^* is an interval symmetric about 0 and $F(x)$ is nonincreasing in $|x|$. If W^* is finite, $F(x)$ is strictly decreasing in $|x|$ for $x \in W^*$.

PROOF:

Assume W^* and $T^*(x)$ defining an optimal strategy are known. There is nothing to prove if W^* is empty. Otherwise, there exists an $x \neq 0$ such that $x \in W^*$, and, since $i(x) = i(-x)$. W^* can be taken to be symmetric about 0, and x can be assumed to be positive. Let x' be such that $0 \leq x' < x$ and let $\varepsilon = (x-x')/2$. Let $p(t)$ denote an arbitrary path on $[t_0, t_1]$ starting at x at time t_0, where t_1 denotes the first time at which $p(t)$ is observed, using an optimal strategy, to be out of W^*. Such a path may or may not have assumed the value ε for one or more values of $t \in [t_0, t_1]$. If it did, let t_ε denote the smallest value of t such that $p(t) = \varepsilon$.

Note that the optimal strategy provides observations of $x(t)$ only at isolated t values, and hence, if $p(t_1) > \varepsilon$, we might not know whether or not t_ε even exists and in no case would its value be known. Nevertheless, we can consider the one-to-one measure preserving transformation of the set of paths $p(t)$ onto a set of paths $p'(t)$ starting at x' at time t_0 defined by

$$\text{if } t_\varepsilon \text{ does not exist, let } p'(t) = p(t) - 2\varepsilon \qquad t_0 \leq t \leq t_1$$

$$(23) \qquad \text{if } t_\varepsilon \text{ exists, let } p'(t) = \begin{cases} p(t) - 2\varepsilon & t < t_\varepsilon \\ \text{reflection of } p(t) \text{ about } 0 & t_\varepsilon \leq t \leq t_1 . \end{cases}$$

Then for any path $p(t)$ and the corresponding value of t_1, the income associated with $p(t)$ is less than or equal to the income associated with the path $p'(t)$ during the interval $[t_0, t_1]$.

Consider the following nonfeasible, but suboptimal, strategy for $x(t)$ having just been observed at x' at time t_0: act as if $x(t_0) = x$, i.e., continue production (since $x \in W^*$) and take the next observation at time $t_0+T^*(x)$. Suppose that at time $t_0+T^*(x)$, this observation (a point on a path $p'(t)$ from x') is x''. Now assume that the entire history $p'(t)$ of the process between t_0 and $t_0+T^*(x)$ becomes available. Because the process if Markovian, the optimal strategy for x'' is independent of this history. Let $p(t_0+T^*(x))$ denote the value uniquely determined by x'' and the inverse of the transformation (23). Let the strategic decision for the observation x'' be that specified by an optimal strategy for $p(t_0+T^*(x))$. Continue this procedure until production is stopped. This suboptimal strategy for $x(t_0) = x'$ results in at least as large an expected income to the end of the cycle as does an optimal strategy for $x(t_0) = x$. Hence, $x' \in W^*$ and $F(x') \geq F(x)$. If W^* is finite and not empty, it is easily shown from PROPOSITION 6 and the above that $F(x') > F(x)$.

Thus, W^* can be denoted by $(-w^*, w^*)$.

In the following PROPOSITIONS it is assumed that $0 < w^* < \infty$.

PROPOSITION 9:

$$F(x) < 0 \quad \text{for} \quad x \neq 0.$$

For PROPOSITION 10 it will be assumed that $i'(x) = di(x)/dx$ exists for $x \in W^*$ and that $F''(x)$ exists throughout W^*, except possibly at 0.[6]

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[6]Whether or not any regularity conditions need to be imposed on $i(x)$ to insure the existence of $F''(x)$ for $x \neq 0$ has not as yet been determined. The same comment applies to the assumptions preceding PROPOSITION 11.

PROPOSITION 10:

$$i^*(x) + F''(x)/2 \geq 0 \quad \text{for} \quad x(\neq 0) \in W^*.$$

PROOF:

Let x and $\varepsilon > 0$ be such that $x \pm \varepsilon \in W^*$. Consider the superoptimal strategy for $x(t_0) = x$ of observing the process continuously at no cost until $x(t) = x \pm \varepsilon$ and then proceeding with the optimal strategy. Let t_0+t_ε denote the time at which $x(t)$ first assumes either of the values $x \pm \varepsilon$. Denote by $F_s(x, \varepsilon)$ the expected income (for the new problem) remaining in the cycle associated with this superoptimal strategy. Then

$$F_s(x, \varepsilon) = E[\int_{t_0}^{t_0+t_\varepsilon} i^*(x(s))ds \,|\, x(t_0) = x] + (1/2)[F(x-\varepsilon)+F(x+\varepsilon)].$$

If $t(y)\ dy$ is defined to be the time that $x(t) \in (y, y+dy)$ (where $x-\varepsilon \leq y < y + dy \leq x + \varepsilon$), given that $x(t_0) = x$, before $|x(t)-x| = \varepsilon$, then

$$E[\int_{t_0}^{t_0+t_\varepsilon} i^*(x(s))ds \,|\, x(t_0) = x] = E \int_{x-\varepsilon}^{x+\varepsilon} i^*(y)\ t(y)dy$$

From (6), $E \int_{x-\varepsilon}^{x+\varepsilon} t(y)dy = \varepsilon^2\Delta^{-1}$. Hence, for some x_1, $x_2 \in [x-\varepsilon, x+\varepsilon]$,

$$E \int_{x-\varepsilon}^{x+\varepsilon} i^*(y)t(y)dy = i^*(x_1)\ \varepsilon^2\Delta^{-1}$$

$$= [i^*(x)+(x_1-x)i^{*\prime}(x_2)]\,\varepsilon^2\Delta^{-1}$$

$$= i^*(x)\ \varepsilon^2\Delta^{-1} + O(\varepsilon^3).$$

Thus

$$0 < F_s(x, \varepsilon)-F(x) = \Delta^{-1} \varepsilon^2 i*(x) + 0(\varepsilon^3) + 2^{-1}[F(x+\varepsilon) - 2F(x) + F(x-\varepsilon)].$$

Letting $\varepsilon \longrightarrow 0$, we obtain

$$\Delta^{-1} i*(x) + 2^{-1} F''(x) \geq 0.$$

If $i*(x)$ has a discontinuity at x, the above proof can be extended to show that $(2\Delta)^{-1}[i*(x^+) + i*(x^-) + F''(x)] \geq 0$, where $i*(x^+)$ and $i*(x^-)$ are the right and left hand limits of $i*(y)$ as y approaches x.

The final two PROPOSITIONS involve $g(x, t)$, which is defined as the maximum expected income (for the new problem) remaining in the cycle, given that the production process is at x at time t_0 and that exactly t (>0) time units must elapse before the next inspection, i.e.,

$$g(x, t) = I*(x, t)-L + \int_{-\infty}^{\infty} F(y) f_N(y|x, \Delta t)dy \tag{24}$$

where, recalling that $i^*(x) = i(x) - I^*$,

$$I^*(x,\ t) = I(x,\ t) - I^*t = E[\int_{t_0}^{t_0+t} i^*(x(s))ds\,|\,x(t_0) = x]$$

$$= \int_0^t \int_{-\infty}^{\infty} i^*(y) f_N(y|x, \Delta t')dydt'.$$

Also, define

(25) $$g(x,\ 0) = \lim_{t \longrightarrow 0} g(x,\ t).$$

Note that $g(x,\ T^*(x)) = F(x)$ and $g(x,\ 0) = F(x) - L$.

It will be assumed that the second partial derivatives of $g(x,\ t)$ are continuous and that $g_{tt}(x, T^*_m(x))$, where $T^*_m(x)$ is defined in the proof of PROPOSITION 12, is negative for all $x \in W^*$. Also, for simplicity, let $\Delta=1$.

PROPOSITION 11:

$$g_t(x,\ t) = \partial g(x,\ t)/\partial t = \int_{-\infty}^{\infty} [i^*(y) + 2^{-1}F''(y)] f_N(y|x,\ t)dy$$

PROOF:

From (24), it is easily seen that

$$g_t(x,\ t) = \int_{-\infty}^{\infty} [i^*(y) + 2^{-1}F(y)t^{-1}((y-x)^2 t^{-1} - 1)]\ f_N(y|x,\ t)dy.$$

Now, letting $f_{N^*}(z) = f_N(z|0,\ 1)$

$$\int_{-\infty}^{\infty} F(y)t^{-1}((y-x)^2 t^{-1} - 1) f_N(y|x,\ t)dy = t^{-1} \int_{-\infty}^{\infty} F(zt^{1/2} + x)(z^2 - 1) f_{N^*}(z)dz$$

$$= t^{-1} \int_{-\infty}^{\infty} F(zt^{1/2} + x) f''_{N^*}(z)dz$$

and integrating this expression by parts twice, we get

$$\int_{-\infty}^{\infty} F''(zt^{1/2} + x) f_{N^*}(z)dz = \int_{-\infty}^{\infty} F''(y) f_N(y|x,\ t)dy.$$

PROPOSITION 12:

$T^*(x)$ is unique and nonincreasing in $|x|$ for $x \in W^*$.

PROOF:

The proof will be divided into several lemmas. First, let

$$h(y) = i^*(y) + F''(y)/2 \tag{26}$$

so that, from PROPOSITION 11,

$$g_t(x, t) = \int_{-\infty}^{\infty} h(y) f_N(y|x, t)dy. \tag{27}$$

Notice that $h(y) = h(-y)$.

LEMMA 1:

For $x \in W^*$, there exists a smallest optimal time $T_m^*(x)$ to the next inspection, and $T_m^*(x) > 0$.

PROOF:

Since $x \in W^*$, there exists at least one optimal time $T^*(x)$ to the next inspection. The assumed continuity of $g_t(x, t)$ insures the existence of a smallest optimal time, $T_m^*(x)$, and, since $g(x, T^*(x)) = F(x)$ and $g(x, 0) = F(x) - L$, $T_m^*(x) > 0$.

LEMMA 2:

As $y > 0$ increases from 0 to ∞, $h(y)$ changes sign exactly once, from positive to negative.

PROOF:

PROPOSITION 10 is that $h(y) \geq 0$ for $y \in W^*$. For $y \geq w^*$, $h(y) = i^*(y)$, which is nonincreasing and, by the assumption that $w^* < \infty$, ultimately negative. It remains only to show that for some $y \in W^*$ $(y > 0)$, $h(y) > 0$, and this is easily proved by contradiction from (27) and the meaning of W^*.

LEMMA 3:

$T^{*\prime}_m(x) = dT^*_m(x)/dx$ is continuous and nonincreasing in $|x|$ for $x \in W^*$.

PROOF:

Let $0 < x_0 < w^*$. From (27)

$$g_t(x_0, t) = \int_{-\infty}^{\infty} h(y)f_N(y|x_0, t)dy = \int_{-\infty}^{\infty} h(zt^{1/2})f_N(z|x_0t^{-1/2}, 1)dz. \tag{28}$$

By assumption, $g_t(x_0, T^*_m(x_0)) = 0$ and $g_{tt}(x_0, T^*_m(x_0)) < 0$. From LEMMA 2, (28), and Pólya theory

$$g_t(x, T^*_m(x_0)) \begin{cases} \geq 0 & \text{for } 0 < x < x_0 \\ \leq 0 & \text{for } x_0 < x. \end{cases} \tag{29}$$

Hence $g_{xt}(x_0, T^*_m(x_0)) \leq 0$. Applying the implicit function theorem (see, e.g., [1, p. 114]) to $g_t(x, T^*_m(x)) = 0$, we can conclude that $T^{*\prime}_m(x)$ is continuous and, for $0 < x < w^*$

$$T^{*\prime}_m(x) = -g_{xt}(x, T^*_m(x))/g_{tt}(x, T^*_m(x)) \leq 0.$$

For $x = 0$, it is easily seen that $g_{xt}(0, T^*_m(0)) = 0$, and hence that $T^{*\prime}_m(0) = 0$. Since $T^*_m(-x) = T^*_m(x)$ for $0 < x < w^*$, $T^{*\prime}_m(-x) = -T^{*\prime}_m(x)$.

To complete the proof of PROPOSITION 12, note, from (28) and symmetry, that

$$g_t(0, t) = 2\int_0^{\infty} h(y)f_N(y|0, t)dy.$$

Since $2f_N(y|0, t)$ has a monotone likelihood ratio in t, $g_t(0, t)$ changes sign at most once. Since $h(y) > 0$ for some $y > 0$ and $g_{tt}(0, T^*_m(0)) < 0$, $g_t(0, t)$ has a unique zero at $t = T^*_m(0)$. This shows that $T^*(0) = T^*_m(0)$ is unique. Now consider an arbitrary positive $x_0 \in W^*$. By LEMMA 3, $T^*_m(x_0) \leq T^*_m(x)$ for all $x \in [0, x_0)$. Suppose there exists a $T^*(x_0) > T^*_m(x_0)$. Then there exists an $\varepsilon > 0$ and an $x_1 \in [0, x_0)$ such that $g_t(x_1, T^*(x_0)) < -\varepsilon$. But then $g_t(x, T^*(x_0)) + \varepsilon = \int_{-\infty}^{\infty} (h(y) + \varepsilon)f_N(y|x, T^*(x_0))dy$, considered as a function of x for $x \geq 0$, changes sign at least twice, which contradicts Pólya theory. Hence, $T^*(x_0)$ is unique and equal to $T^*_m(x_0)$.

COSTLY SURVEILLANCE-EXAMPLES

The two examples of this section illustrate only one of several possible first steps towards the solution of the costly surveillance problem for Wiener processes.

EXAMPLE 3:

Let $x(t)$ be a Wiener process with $x(0) = 0$ and variance parameter Δ. Let $i(x) = A(>0)$ for $|x| \leq 1$ and 0 otherwise. Define $Q(x) = \int_{-x}^{x} f_{N*}(y)dy$.

Consider first I^*_1. From PROPOSITION 3 and (9)

$$I^*_1 = \max_{T>0}\left\{[I(0, T)-L-mK]/[T+m]\right\}$$

$$(30) \qquad = \max_{T>0}\left\{[A\int_0^T \int_{-1}^{1} f_N(x|0, \Delta t)dxdt-L-mK]/[T+m]\right\}$$

$$= \max_{T>0}\left\{[A\int_0^T Q((\Delta t)^{-1/2})dt-L-mK]/[T+m]\right\}.$$

The critical equation for the maximizing value of T is

$$(T+m)AQ((\Delta T)^{-1/2})-A\int_0^T Q((\Delta t)^{-1/2})dt+L+mK = 0.$$

Now, an integration by parts and a change of variable yields

$$\int_0^T Q((\Delta t)^{-1/2})dt = TQ((\Delta T)^{-1/2})+2^{-1}\pi^{-1/2}\int_{(2\Delta T)^{-1}}^{\infty} u^{-3/2}e^{-u}du.$$

Hence the critical equation can be written in the form

$$(31) \qquad mAQ((\Delta T)^{-1/2})-2^{-1}\pi^{-1/2}A\int_{(2\Delta T)^{-1}}^{\infty} u^{-3/2}e^{-u}du+L+mK = 0.$$

I^*_1 could be determined from (30) and (31).

The problem of finding ${}_1I^*$ is the same as the problem of EXAMPLE 2 with mK replaced by $mK+L$. Hence

$$(32)\qquad {}_1I^* = \max_{w>0} \begin{cases} (A\Delta^{-1}w^2 - mK-L)/(\Delta^{-1}w^2+m) & \text{for } w \le 1 \\ (2A\Delta^{-1}(w-1/2)-mK-L)/(\Delta^{-1}w^2+m) & \text{for } w > 1, \end{cases}$$

$$(33)\qquad w^* = \left\{1 + \frac{\Delta(mK+L)}{A}\right\}\left\{1 + \left[1 + \frac{4A^2\Delta m}{(A+\Delta(mK+L))^2}\right]^{1/2}\right\},$$

and

$$(34)\qquad {}_1I^* = A/w^*.$$

Letting $F_N(x|0, \Delta t) = \int_{-\infty}^{x} f_N(y|0, \Delta t)dy$, the functional equation for determining an optimal strategy has the form

$$(35)\qquad F(x) = \max_{w,\ T(x)>0} \left\{I(x, T)-I^*T-L + \int_{-w}^{w} F(y)f_N(y|x, \Delta T)dy - m(K+I^*)\int_{y \notin W} f_N(y|x, \Delta T)dy\right\}$$

where

$$(36)\qquad I(x, T) = A\int_0^T \int_{-1}^{1} f_N(y|x, \Delta t)dydt = A\int_0^T F_N(1|x,\Delta t)-F_N(-1|x, \Delta t)]dt$$

and $F(0) = 0$.

EXAMPLE 4:

Let $x(t)$ be a Wiener process with $x(0) = 0$ and variance parameter Δ. Let $i(x) = -x^2$.

From PROPOSITION 3 and (9) it is easily shown that

$$(37)\qquad I_1^* = \max_{T>0} \left\{[-\Delta T^2/2-L-mK]/[T+m]\right\}.$$

The maximizing value of T, call it T^*, is given by

(38) $$T^* = m[(1+2(mK+L)/m^2\Delta)^{-1/2}-1]$$

and

(39) $$I_1^* = -\Delta T^*.$$

The problem of finding ${}_1I^*$ is the same as the problem of EXAMPLE 1 with $A = 1$ and mK replaced by $mK+L$, i.e.,

(40) $${}_1I^* = \max_{w > 0} \left\{[(-w^4/6)-\Delta(mK+L)]/[w^2+m\Delta]\right\} .$$

The desired root w^* of the critical equation is given by

(41) $$w^* = \left\{m\Delta[(1+6(mK+L)/(\Delta m^2)^{1/2}-1]\right\}^{1/2}$$

and

(42) $${}_1I^* = -w^{*2}/3.$$

The functional equation for determining an optimal strategy is the same as in EXAMPLE 3 except that a new expression is required for $I(x, T)$. It is

(43) $$I(x, T) = -\int_0^T \int_{-\infty}^{\infty} (x+y)^2 f_N(y|x, \Delta t)dydt = -\int_0^T (2x^2+\Delta t)dt = -(2x^2T+\Delta T^2/2).$$

If, in this example, $m=K=L=2$, (38) becomes

$$T^* = 2[(1+3/\Delta)^{1/2}-1]$$

and (41) becomes

$$w^* = \left\{2\Delta[(1+9/\Delta)^{1/2}-1]\right\}^{1/2}.$$

For $\Delta = .2$, the specific results are

$$T^* = 6 \quad \text{and} \quad I_1^* = -1.2$$

$$w^* = 1.52 \quad \text{and} \quad {}_1I^* = -.77.$$

APPENDIX

Proofs of properties (5)-(7):

To prove (5), consider a symmetric random walk over the integers starting at b with absorbing barriers at a and c where $a \leq b \leq c$. Let p_{by} be the probability of reaching y $(a \leq y \leq c)$ before being absorbed. It is well known (see, e.g., [2]) that

$$p_{by} = \begin{cases} \frac{c-b}{c-y} & a \leq y \leq b \\ \frac{b-a}{y-a} & b \leq y \leq c. \end{cases}$$

Let P_{yy} be the probability of a return to y before absorption if the random walk starts at y. Then

$$1-P_{yy} = \begin{cases} \frac{c-a}{2(c-y)(y-a)} & a+1 < y < c-1 \\ \frac{c-a}{2(c-a-1)} & y = a+1 \text{ or } y = c-1 \\ 1 & y = a \text{ or } y = c. \end{cases}$$

This can be shown by noting that a path starting at y must move to $y-1$ or $y+1$, each with probability $1/2$, and then using the formula for p_{by} with $b = y-1$ or $b = y+1$, as is appropriate.

Now, let the values of a, b, c, and y be held constant and the number of steps in an interval increase by letting the size of each step be Δy $(\Delta y < 1)$. Let the time for each step be Δt. Then T_y, the expected time spent at y before absorption, equals Δt times the expected number of visits to y. For this new random walk

$$T_y = \frac{\Delta t}{\Delta y} \sum_{k=1}^{\infty} k p_{by} P_{yy}^{k-1}(1-P_{yy}) = \frac{\Delta t}{\Delta y} p_{by} \Big/ (1-P_{yy}).$$

It is easily verified that

$$T_y = 2\frac{\Delta t}{(\Delta y)^2}\left[\min(b-a,\ y-a) - \frac{(b-a)(y-a)}{(c-a)}\right]\Delta y.$$

If Δy and Δt approach zero in such a way that $(\Delta y)^2/\Delta t$ approaches Δ, the distribution of the displacement at time t approaches that for the Wiener process, and

$$T_y \sim 2\Delta^{-1}[\min(b-a,\ y-a) - (b-a)(y-a)/(c-a)]\,\Delta y.$$

This result was also derived directly by Professor Frank Spitzer (personal communication).

Property (6) follows from integrating the expression in property (5). Property (7) follows from the formula above for p_{by}.

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