ON THE NONEXISTENCE OF RELEVANT SUBSETS IN LOCATION MODELS

by

Robert J. Buehler*

Technical Report No. 295

July, 1977

Abstract

Let X denote a vector of observations from a location parameter model whose first moment exists, and let A denote an invariant set of (X,θ) values such that $S_x = \{\theta | (x,\theta) \in A\}$ are level γ confidence regions and for each x, S_x has fiducial probability γ. It is shown that there is no subset C of X values such that $\inf_\theta P_\theta(A|C) > \gamma$ or such that $\sup_\theta P_\theta(A|C) < \gamma$.

*Supported by National Science Foundation Grant MCS76-06284.

1. Introduction.

A set A of (x,θ) values can be called a level γ confidence procedure if $P_\theta\{(X,\theta) \in A\} = \gamma$ for all θ. A set C of x values has been called a relevant subset (Buehler 1959) if for some $\varepsilon > 0$

$$(1) \quad P_\theta(A|C) \geq \gamma + \varepsilon \quad \text{or} \quad P_\theta(A|C) \leq \gamma - \varepsilon$$

for all θ.

If $f(x,\theta) = f(x-\theta)$, and the first moment of X exists, and $A = \{x,\theta | c_1 < x-\theta < c_2\}$ where c_1 and c_2 are finite, then it is known that there are no relevant subsets (Buehler 1959). In the present paper this result is generalized in three ways: (i) Finiteness of c_1, c_2 is not required, nor need the confidence region be an interval. (ii) The subset C is replaced by a "selection" $\varphi(x)$, $0 \leq \varphi(x) \leq 1$. (iii) Arbitrary sample size n is allowed, with the confidence procedure based on the Fisher-Pitman fiducial distribution (Fisher (1934), Pitman (1939)).

For this more general case we find that existence of the first moment still guarantees nonexistence of relevant subsets. It is not known whether the moment condition is actually necessary.

For estimation of the mean in joint location-scale models an example involving Student's distribution shows that relevant subsets can exist (Buehler and Feddersen (1963), Brown (1967)). In the Student example and also for Behrens-Fisher confidence intervals, Robinson (1976) has shown that only a positive bias is possible so that in this sense the intervals are conservative. For other examples of relevant subsets see Robinson (1975).

A modified criterion involving $P_\theta(C)$ as well as $P_\theta(A|C)$ has been suggested by Stone (1972) and studied for group-invariant models by Bondar (1977).

For the case of a multivariate translation family with density $f(x_1-\theta_1,\ldots,x_k-\theta_k)$ Stein (1961) showed nonexistence of relevant subsets when sample size $n=1$. For $k=2$ we have attempted to extend this result to general n using the methods below. Using notation analogous to that below, let

$$\mu(R) = \int_{t_1=-R}^{R} \int_{t_2=-R}^{R} \int_{u_1} \int_{u_2} \varphi(t_1,u_1,t_2,u_2)d\lambda(u_1,u_2)dt_1dt_2 .$$

The proof seems to extend either if $\mu(\infty) < \infty$ or if $R/\mu(R) \to 0$ as $R \to \infty$, but unfortunately not in general.

2. The Main Result.

Let $X = (X_1,\ldots,X_n)$ have density

$$(2) \qquad \prod_{i=1}^{n} f(x_i-\theta)$$

and define T , U by

$$(3) \qquad T = \frac{1}{n}\sum_{i=1}^{n} X_i \quad , \quad U = (X_1-X_2, X_2-X_3, \ldots, X_{n-1}-X_n) \; .$$

Let the joint density of (T,U) be denoted by

$$(4) \qquad g(t-\theta|u)\, dt\, d\lambda(u) \; .$$

For fixed $0 < \gamma < 1$ suppose we determine for each u a set A_u such that

$$(5) \qquad \int_{A_u} g(t|u)\, dt = \gamma \; .$$

With $x = (t,u)$ we then define

$$(6) \qquad \begin{aligned} A &= \{x,\theta \,|\, t-\theta \in A_u\} \\ A_x &= \{\theta \,|\, (x,\theta) \in A\} \; . \end{aligned}$$

The sets A_x are then level γ invariant confidence regions for θ , and each A_x has fiducial probability γ . We will call these Pitman confidence regions. In practice A_x is usually a finite or infinite interval, but there is no need here for any such restriction. The conditioning set C is replaced by the more general "selection function" (Tukey (1958), Wallace (1959), Stein (1961)) $\varphi(x)$, with $0 \le \varphi(x) \le 1$, interpreting

$\varphi(x)$ as the probability that observation x goes into the selected subset. Then $P_\theta(A|C) = P_\theta(AC)/P_\theta(C)$ is replaced by $E_\theta(1_A\varphi)/E_\theta\varphi$, the two being equal when $\varphi = 1_C$.

<u>Theorem</u> 1. Let X have density (2) where EX_i exists. Let A define level γ Pitman confidence regions. Then there does not exist a selection $\varphi(x)$ $(0 \le \varphi(x) \le 1)$ such that $E_\theta\varphi > 0$ for all θ and for some $\varepsilon > 0$

$$\frac{E_\theta 1_A\varphi}{E_\theta\varphi} \ge \gamma + \varepsilon \quad \text{or} \quad \frac{E_\theta 1_A\varphi}{E_\theta\varphi} \le \gamma - \varepsilon \quad \text{for all } \theta .$$

<u>Proof</u>. We have

$$E_\theta\varphi = \int_u \int_t \varphi(t,u)g(t-\theta|u)dt\, d\lambda(u)$$

$$E_\theta 1_A\varphi = \int_u \int_{t\in A_u+\theta} \varphi(t,u)g(t-\theta|u)dt\, d\lambda(u) .$$

(Where limits are not given, integrals are over the full range of the variable.) Define

$$\mu(R) = \int_{t=-R}^{R} \int_u \varphi(t,u)\, d\lambda(u)\, dt$$

$$\alpha(R) = \int_{\theta=-R}^{R} (E_\theta\varphi)\, d\theta$$

$$\beta(R) = \int_{\theta=-R}^{R} (E_\theta 1_A\varphi)\, d\theta$$

A contradiction will be established by showing that $\alpha(R)/\beta(R)$ tends to γ as R tends to ∞.

Case 1: $\mu(\infty) = M < \infty$.

$$\beta(\infty) = \int_\theta \int_u \int_{t' \in A_u} \varphi(t'+\theta,u)g(t'|u)dt' \, d\lambda(u) \, d\theta$$

$$= \int_u \int_{t' \in A_u} \int_\theta \varphi(t'+\theta,u)g(t'|u)d\theta \, dt' \, d\lambda(u)$$

$$= \int_u \int_{t' \in A_u} \int_{\theta'} \varphi(\theta',u)g(t'|u)d\theta' \, dt' \, d\lambda(u)$$

$$= \gamma \, \mu(\infty) = \gamma \, M .$$

The key to variable changes and integral reversals is: $t \to t' = t-\theta$, $(\theta,u,t') \to (u,t',\theta)$, $\theta \to \theta' = \theta + t'$, $(u,t',\theta') \to (u,\theta',t')$, and integrate over t' .

The special case $\gamma = 1$, $A_u = (-\infty,\infty)$ gives $\alpha(\infty) = M$, establishing the desired contradiction for Case 1.

Case 2. $\mu(R) \to \infty$ as $R \to \infty$. In the Case 1 calculation replace $\beta(\infty)$ by $\beta(R)$ and the integral $-\infty < \theta < \infty$ by $-R < \theta < R$. The same steps then give (dropping primes on θ and t)

$$\beta(R) = \int_u \int_{t \in A_u} \int_{\theta=t-R}^{t+R} \varphi(\theta,u)g(t|u)d\theta dt \, d\lambda(u) .$$

We will compare $\beta(R)$ with $\beta'(R)$ defined by

$$\beta'(R) = \int_u \int_{t \in A_u} \int_{\theta=-R}^{R} \varphi(\theta,u)g(t|u)d\theta dt \, d\lambda(u) .$$

In the last expression we can integrate first over t and then reverse the two remaining integrals to get

$$\beta'(R) = \gamma\ \mu(R)\ .$$

The difference $\beta(R)-\beta'(R)$ is the sum of four integrals, two positive and two negative, over regions

$$S_1 = \{u,\ \theta < -R,\ t\in A_u,\ \theta-R < t \leq \theta+R\}$$

$$S_2 = \{u,\ -R < \theta < R,\ t\in A_u,\ t < \theta-R\}$$

$$S_3 = \{u,\ -R < \theta < R,\ t\in A_u,\ t > \theta+R\}$$

$$S_4 = \{u,\ \theta > R,\ t\in A_u,\ \theta-R < t < \theta+R\}\ .$$

Let $I_1,\ldots,I_4$ denote the corresponding integrals. We will show that all four are bounded as $R\to\infty$. Replacing $\varphi(t,u)$ by 1 and A_u by $(-\infty,\infty)$ we have

$$I_1 \leq \int_u \int_{\theta=-\infty}^{-R} \int_{t=\theta-R}^{\theta+R} g(t|u)dtd\theta\ d\lambda(u)$$

$$= \int_{\theta=-\infty}^{-R} \int_{t=\theta-R}^{\theta+R} g(t)dtd\theta\ ,$$

where $g(t)$ is the marginal density of t . Denoting the corresponding c.d.f. by $G(t)$ and putting $t = \theta+R$ we have

$$I_1 \leq \int_{\theta=-\infty}^{-R} [G(\theta+R)-G(\theta-R)]d\theta$$

$$= \int_{t=-\infty}^{0} [G(t)-G(t-2R)]dt\ .$$

Since EX_i exists, we know that ET exists, and this implies $\int_{-\infty}^{0} G(t)dt = K < \infty$. Thus $I_1 \leq K$ for all $0 < R < \infty$.

For I_2 we have

$$I_2 \le \int_u \int_{\theta=-R}^{R} \int_{t=-\infty}^{\theta-R} g(t|u)\,dt\,d\theta\; d\lambda(u)$$

$$= \int_{\theta=-R}^{R} \int_{t=-\infty}^{\theta-R} g(t)\,dt\,d\theta$$

$$= \int_{\theta=-R}^{R} G(\theta-R)\,d\theta$$

$$= \int_{t=-2R}^{0} G(t)\,dt \le K \; .$$

The integrals I_3 and I_4 are of course similar so that $\beta(R)-\beta'(R) = O(R)$. The special case $A_u = (-\infty,\infty)$ gives $\alpha(R)-\alpha'(R) = O(R)$ where $\alpha'(R) = \mu(R)$. Thus $\beta(R)/\alpha(R) \to \gamma$, giving the desired contradiction.

References

1. Bondar, James V. (1977). On a conditional confidence principle. Ann. Statist. 5 (in press).

2. Brown, L. (1967). The conditional level of Student's test. Ann. Math. Statist. 38 1068-1071.

3. Buehler, Robert J. (1959). Some validity criteria for statistical inferences. Ann. Math. Statist. 30 845-863.

4. Buehler, Robert J. and Feddersen, A. P. (1963). Note on a conditional property of Student's t. Ann. Math. Statist. 34 1098-1100.

5. Fisher, R. A. (1934). Two new properties of maximum likelihood. Proc. Roy. Soc. Ser. A. 14 285-307.

6. Pitman, E. J. G. (1939). The estimation of location and scale parameters of a continuous population of any given form. Biometrika 30 391-421.

7. Robinson, G. K. (1975). Some counterexamples to the theory of confidence intervals. Biometrika 62 155-161.

8. Robinson, G. K. (1976). Properties of Student's t and of the Behrens-Fisher solution to the two means problem. Ann. Statist. 4 963-971.

9. Stein, Charles (1961). Estimation of many parameters. Inst. Math. Statist. Wald Lectures. Unpublished.

10. Stone, M. (1972). Review of "Foundations of Statistical Inference" (V. P. Godambe and D. A. Sprott, editors). Biometrika 59 236-237.

11. Tukey, John W. (1958). Fiducial inference. Inst. Math. Statist. Wald Lectures. Unpublished.

12. Wallace, David L. (1959). Conditional confidence level properties. Ann. Math. Statist. 30 864-876.