On the Adequacy of Stationary Plans

for Gambling Problems

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Abstract

Whether stationary families of strategies are uniformly adequate for a leavable, analytically measurable, nonnegative gambling problem whose optimal return function is everywhere finite is a question which remains open but is given an affirmative answer if, for example, the fortune space is Euclidean and all nontrivial, available gambles are absolutely continuous with respect to Lebesgue measure.

1. Introduction. Consonant with [6], a gambling problem is an ordered couple (Γ,u), where Γ is a gambling house and u is a real-valued function, both defined on the same set F. In [6], u was assumed to be bounded; here u may be unbounded, but it is assumed that u is nonnegative and that the optimal return function V is everywhere finite.

If, for each $f \in F$, $\bar{\sigma}(f)$ is a strategy, $\bar{\sigma}$ is a plan. If $\wp$ is a set of plans available in Γ and if, for each $\epsilon > 0$ and $f \in F$, $\exists\, \bar{\sigma} \in \wp$ such that

$$(1.1) \qquad u(\bar{\sigma}(f)) \geq (1-\epsilon)\, V(f) ,$$

then $\wp$ is an adequate set of plans for (Γ,u). If, for each $\epsilon > 0$, $\exists\, \bar{\sigma} \in \wp$ such that (1.1) holds for all f, then $\wp$ is uniformly adequate for (Γ,u).

A Markov kernel is a gamble-valued function γ defined on F. If $\gamma(f) \in \Gamma(f)$ for all f, then γ is a Γ-selector or a Γ-kernel. The plan γ^{∞} that prescribes $\gamma(f)$ whenever the current fortune is f is stationary. Let $\mathcal{S}$ be the set of all stationary plans available in Γ. The question raised in [6] as to whether stationary plans are uniformly adequate for leavable problems was settled in the negative by a surprising example of Ornstein [13]. (A later example [9] shows that stationary plans need not even be adequate.) It is natural, therefore, to ask whether stationary plans are uniformly adequate if the problem (Γ,u) is Borel measurable, but we have succeeded in answering this query only if Γ is Borel absolutely continuous.

A Borel, or even analytically, measurable house Γ is <u>Borel absolutely continuous</u> if, for some probability measure α countably additive on the Borel subsets of F, every nontrivial gamble γ available in Γ, assigns probability zero to every Borel subset of F to which α assigns probability zero. (If $\gamma \in \Gamma(f)$ and $\gamma \neq \delta(f)$, then γ is <u>nontrivial</u>.) The principal purpose of this paper is to show that if it is also supposed that Γ is Borel absolutely continuous, then stationary plans are indeed uniformly adequate.

Two preliminary facts require no assumptions of countable additivity or of measurability. The first, Proposition 4.1 and its Corollary, is a characterization of optimal stationary plans for leavable, stop-or-go houses, that is, those houses in which, at each f, at most one nontrivial gamble is available; it can be viewed as another version of the fundamental theorem of optimal stopping (cf. [5] and [8]). The second, Proposition 5.1, states that, for each $\epsilon > 0$ and $n \geq 1$, there is available a stationary plan γ^∞ such that

$$(1.2) \qquad u(\gamma^\infty(f)) \geq (1-\epsilon)\, U_n(f) ,$$

where U_n is the optimal return if gambling must terminate by time n.

2. <u>A few conventions and some notation</u>. Every gamble γ is assumed to be defined on the set of all nonnegative, extended-real-valued functions with domain F and to satisfy the usual conditions:

(a) $\gamma(u_1 + u_2) = \gamma u_1 + \gamma u_2$,

(b) $\gamma(tu) = t\gamma u$ for $t \geq 0$,

(c) $u_1 \leq u_2 \Rightarrow \gamma u_1 \leq \gamma u_2$,

(d) $\gamma c = c$ for all constants c .

Using the same argument as for [6, Theorem 2.8.1], one can verify that every nonnegative, extended-real-valued, finitary function g can be integrated in a unique way by every strategy σ so as to satisfy $\sigma c = c$ and

$$(2.1)\qquad \sigma g = \int \sigma[f](gf)\, d\sigma_0(f)\ .$$

Notation, such as $\sigma[f]$, used often in [6], will ordinarily not be explained here. From (2.1) follows the more general

$$(2.2)\qquad \sigma g = \int \sigma[p_t](gp_t)\, d\sigma,$$

which holds for every stop rule t, as in [6, Equation 3.7.1]. Here $p_t(h)$ is the partial history $p = (f_1,\ldots,f_n)$ where $h = (f_1,f_2,\ldots)$ and $t(h) = n$. It is natural to regard $\sigma[p_t](gp_t)$ as the conditional σ-expectation of g given the past up to time t. The <u>time remainin after</u> p, say $t[p]$, is defined by

$$t[p](h) = t(ph)-n\ ,$$

where $h \in H$, ph is the history which consists of p followed by h, and n is the number of coordinates of p. Notice that $t[p]$ is a stop rule if $t(ph) > n$; it is a nonpositive constant otherwise. As in [6, Section 2.5], $\sigma[p]$ denotes the conditional strategy given p. If π is the policy (σ,t), then the <u>conditional policy given</u> p is $\pi[p] = (\sigma[p],t[p])$. The utility $u(\pi[p]) = \int u(f_{t[p]})\, d\sigma[p]$ is well-defined if $t[p]$ is a stop rule. If, on the other hand, $t[p]$ is nonpositive, set $u(\pi[p]) = u(f_k)$ where $p = (f_1,\ldots,f_n)$ and $t(f_1,\ldots,f_n,\ldots) = k \le n$. Let s be a stop rule. Then the formula

$$(2.3)\qquad u(\pi) = \int u(\pi[p_s])\, d\sigma$$

is a special case of (2.2) as well as an extension to stop rules of [6, Formula 2.10.2].

Many of the definitions and results in [6], which were established there for bounded u's, extend without difficulty to nonnegative u's, and will be used here without comment. Recall that, as defined in [6], the two optimal return functions for a gambling problem, V and U, satisfy $V \leq U$. In this paper, whenever it is considerably simpler to do so, the problem will be assumed to be a leavable one, in which event $V = U$. Nevertheless, when greater generality is to be hinted at or when the logic of an argument is clarified by doing so, "V" will often be used in lieu of, and in addition to, "U" even when $V = U$.

3. Preliminary lemmas. For $\beta \in \Gamma(f)$, let V_β be the supremum of $u(\sigma)$ over all σ available in Γ at f for which the initial gamble σ_0 is β.

Lemma 3.1. $V_\beta = \beta V$.

Proof: Apply [6, Corollary 3.3.4].

Lemma 3.2. Let γ be a Γ-selector. Then γ^∞ is optimal if and only if, for all f, both of these conditions hold:

(a) $\gamma(f)V = V(f)$,

(b) $u(f) < V(f) \Rightarrow V(f) \leq \sup u(\gamma^\infty(f), t)$,

where the supremum is taken over all stop rules t.

<u>Proof</u>: Suppose γ^∞ is optimal. Plainly, Lemma 3.1 implies (a), and

$$(3.1)\qquad V(f) = u(\gamma^\infty(f)) = \limsup_t u(\gamma^\infty(f),t) \le \sup_t u(\gamma^\infty(f),t) \ ,$$

so (b) holds. For the converse, suppose (a) and (b) hold for all f. Fix $\epsilon > 0$, $f \in F$, and a stop rule s. It suffices to find a stop rule $t \ge s$ such that $u(\gamma^\infty(f),t) \ge V(f)-\epsilon$. If $u(f') < V(f')$, then by (b), there is a stop rule $\bar{t}(f')$ such that

$$(3.2)\qquad u(\gamma^\infty(f'),\bar{t}(f')) \ge V(f')-\epsilon \ .$$

If $u(f') \ge V(f')$, let $\bar{t}(f') = 0$. Define t to be the composition of s with the family $\bar{t}$; that is,

$$(3.3)\qquad t(h) = s(h) + \bar{t}(f_{s(h)})(f_{s(h)+1}, f_{s(h)+2}, \ldots)$$

for all h. Then

$$\begin{aligned} u(\gamma^\infty(f),t) &= \int u(\gamma^\infty(f_s),\bar{t}(f_s))\, d\gamma^\infty(f) \\ &\ge \int V(f_s)\, d\gamma^\infty(f)-\epsilon \\ &= V(f)-\epsilon \ . \end{aligned}$$

The first equality is an instance of (2.3); the inequality is by definition of $\bar{t}$; and the final equation holds for every stop rule s as can be seen using (a) and an induction on the structure of f_s. □

<u>Lemma</u> 3.3. Suppose γ^∞ is optimal. Then

(c) $\gamma(f) = \delta(f) \Rightarrow u(f) = V(f)$.

<u>Lemma</u> 3.4. Let Γ' be a subhouse of Γ. Suppose that, for every f at which $u(f) < U(f)$, $\Gamma'(f)$ includes every $\gamma \in \Gamma(f)$ except possibly $\delta(f)$. Then $U' = U$.

Proof: Obviously, $U' \leq U$. The reverse inequality will follow from [6, Theorem 2.12.1] once it is verified that U' is excessive for Γ. For the verification, fix f and $\gamma \in \Gamma(f)$. If $u(f) < U(f)$, then either $\gamma = \delta(f)$ or $\gamma \in \Gamma'(f)$. The desired inequality, $\gamma U' \leq U'(f)$, is obvious in the first case and a consequence of [6, Theorem 2.14.1] in the second. If $u(f) = U(f)$, then

$$\gamma U' \leq \gamma U \leq U(f) = u(f) \leq U'(f) ,$$

where the first inequality holds because $U' \leq U$; the second because of [6, Theorem 2.14.1]; the equality by hypothesis; and the final inequality by definition of U'. □

Lemma 3.5. At any f at which $u(f) < U(f)$, there is a $\gamma \in \Gamma(f)$ distinct from $\delta(f)$.

4. Stop-or-Go Houses. Throughout this section, Γ is a leavable, stop-or-go house which means that, for some gamble-valued function α defined on F, $\Gamma(f)$ is $\{\alpha(f), \delta(f)\}$. A stationary plan γ^{∞} is promising if, for all f,

(a) $\gamma(f)V = V(f)$,

and

(b) $\gamma(f) = \delta(f) \Rightarrow u(f) = V(f)$.

Proposition 4.1. For a stationary plan to be everywhere optimal it is necessary and sufficient that it be promising.

(For predecessors of, and for results closely related to, Proposition 4.1, consult [5], [6], [8], and [16].)

Proof: The necessity is evident even without the help of Lemmas 3.2 and 3.3. Suppose therefore that γ^∞ is promising, in which event condition (a) of Lemma 3.2 certainly holds. To see that condition (b) also holds, let $\Gamma'(f)$ be the one-gamble house $\{\gamma(f)\}$. If $u(f) < U(f)$, then $u(f) < V(f)$, for $U = V$ for leavable Γ. For such f, $\gamma(f) \neq \delta(f)$ because γ^∞ is promising. Of course, $\gamma(f) \neq \delta(f)$ implies that $\gamma(f) = \alpha(f)$. In sum if $u(f) < U(f)$, then $\Gamma'(f)$ contains $\alpha(f)$ so that the hypothesis of Lemma 3.4 is satisfied. So $U' = U$. If $u(f) < V(f)$, as is now plain, $u(f) < U'(f)$, and there must be, for each $\epsilon > 0$, a policy π available in Γ' at f for which $u(\pi) > U'(f)-\epsilon$. Equivalently, if $u(f) < V(f)$, $\sup u(\gamma^\infty(f),t) = U'(f)$, where the sup is taken over all stop rules t. Since $U' = U = V$, condition (b) of Lemma 3.2 holds. That lemma now yields the conclusion that γ^∞ is optimal for Γ. □

The problem of showing that optimal stationary plans exist has been reduced to showing that promising stationary plans exist. For showing that the latter exist, this simple lemma is useful.

Lemma 4.1. At any f at which $u(f) < V(f)$, $\alpha(f)$ is distinct from $\delta(f)$ and $\alpha(f)V = V(f)$. At any f at which $\alpha(f)V < V(f)$, $\delta(f) \in \Gamma(f)$ and $u(f) = V(f)$.

Proof: Suppose $u(f) < V(f)$. Then Lemma 3.5 applies to show that $\alpha(f)$ is distinct from $\delta(f)$. Moreover, since $u(f) < V(f)$, there must be for each $\epsilon > 0$ an ϵ-optimal strategy available at f whose initial gamble is $\alpha(f)$. As Lemma 3.1 now implies, $\alpha(f)V = V(f)$.

Suppose $\alpha(f)V < V(f)$. Then by Lemma 3.1, there is available at f some γ other than $\alpha(f)$ which γ can be nothing but $\delta(f)$. That $u(f) = V(f)$ is the main content of the first sentence of Lemma 4.1. □

Corollary 4.1. There exist everywhere-optimal stationary plans. In fact, there exist Γ-selectors γ with this property: at each f at which $u(f) < V(f)$, $\gamma(f) = \alpha(f)$; and at each f at which $\alpha(f)V < V(f)$, $\gamma(f) = \delta(f)$; for each such γ, γ^∞ is everywhere optimal. Moreover, there are no other everywhere-optimal stationary plans. If $\gamma(f)$ is $\alpha(f)$ or $\delta(f)$ according as $u(f) < V(f)$ or not, then γ^∞ is the optimal stationary plan for which the time until stagnation is a minimum for every history.

Proof: That there exist Γ-selectors γ with the stated property and that, for such γ, γ^∞ is promising is immediate from Lemma 4.1. That each such γ^∞ is everywhere-optimal is implied by Proposition 4.1 as is the assertion that there are no other everywhere-optimal stationary plans. The final assertion is now evident. $\square$

5. There is a stationary family which yields at least $(1-\epsilon)\, U_n$. Let $U_0 = u$ and, for $k \geq 1$ and $f \in F$, let $U_k(f)$ be the most a gambler with initial fortune f can attain if play is allowed to continue up to time k but not beyond. By [6, Theorem 2.15.2], for $k \geq 0$ and $f \in F$,

$$(5.1) \qquad U_{k+1}(f) = \sup\{\gamma U_k : \gamma \in \Gamma(f)\},$$

which obviously implies that, for $0 < \beta < 1$, there exists a (Γ,β)-sequence, that is, $\gamma_1, \gamma_2, \ldots$ such that, for all $k \geq 1$ and all f,

$$(5.2) \qquad \gamma_k(f)\, U_{k-1} \geq \sqrt{\beta}\, U_k(f).$$

For each (Γ,β)-sequence, $\gamma_1, \gamma_2, \ldots$, each $n \geq 1$, and each f, if $k = k(f) = k(f,n)$ is a nonnegative integer at most n and satisfies

$$(5.3) \qquad \beta^k U_k(f) = \max_{0 \leq j \leq n} \beta^j U_j(f),$$

and if

$$(5.4) \qquad \gamma(f) = \begin{cases} \gamma_{k(f)}(f) & \text{if } k(f) \geq 1 , \\ \delta(f) & \text{if } k(f) = 0 , \end{cases}$$

then the Γ-selector γ is called a (Γ,β,n)-selector. Here is a generalization of [15, Theorem 1.2].

Proposition 5.1. For each $\epsilon > 0$ and $n \geq 1$, there is a Γ-selector γ such that

$$(5.5) \qquad u(\gamma^\infty(f)) \geq (1-\epsilon)\, U_n(f) \quad \text{for all } f .$$

Indeed, for each β, n and each (Γ,β,n)-selector γ,

$$(5.6) \qquad u(\gamma^\infty(f)) \geq \beta^n U_n(f) .$$

Proof: Fix β and n, let $k = k(f)$ satisfy (5.3), define γ as in (5.4), let $W(f)$ be the right-hand side of (5.3), let $\alpha = 1/\sqrt{\beta}$, and, for any f for which $k(f) \geq 1$, calculate thus.

$$\begin{aligned} (5.7) \qquad \gamma(f)W &\geq \gamma(f)\{\beta^{k-1} U_{k-1}\} \\ &= \beta^{k-1}\gamma_k(f)\, U_{k-1} \\ &\geq \alpha\beta^k U_k(f) \\ &= \alpha W(f) . \end{aligned}$$

For any f at which $k(f) = 0$, $\gamma(f)W = \delta(f)W = W(f)$.

Fix f and let $\sigma = \gamma^\infty(f)$. The process $W(f), W(f_1), \ldots$ is, by the previous paragraph, expectation increasing under σ and, hence, $W(\sigma) \geq W(\sigma,t) \geq W(\sigma,s) \geq W(f)$ for all stop rules s,t with $t \geq s$. Since $W \geq \beta^n U_n$, for (5.6), it suffices to show $u(\sigma) \geq W(\sigma)$. This is obviously true if $u(f) = W(f)$. So assume $u(f) < W(f)$, or in particular, that $\gamma(f) \neq \delta(f)$.

Let $h = (f_1, f_2, \ldots)$ and let $t_0(h)$ be the first k (if any) such that $u(f_k) = W(f_k)$, where it is understood that $t_0(h)$ is $+\infty$ if there is no such k .

For each stop rule t , write $W(\sigma,t) = a_t + b_t$ where $a_t = \int_{t\geq t_0} W(f_t)\, d\sigma$ and $b_t = \int_{t<t_0} W(f_t)\, d\sigma$. Then

$$a_t = \int_{t\geq t_0} W(f_{t_0})\, d\sigma = \int_{t\geq t_0} u(f_{t_0})\, d\sigma = \int_{t\geq t_0} u(f_t)\, d\sigma \leq u(\sigma,t) .$$

The first and third equalities hold because σ stagnates at time t_0 [6, Theorem 3.4.3], the second equality holds by the definition of t_0 , and the inequality holds because $u \geq 0$.

It now suffices to show $b_t \to 0$ since, in that case,

$$u(\sigma) = \limsup_t u(\sigma,t) \geq \limsup_t a_t = \limsup_t W(\sigma,t) = W(\sigma) . \tag{5.8}$$

To each stop rule t , associate another stop rule $\hat{t}$ given by

$$\hat{t}(h) = \begin{cases} t(h) & \text{if } t(h) \geq t_0(h) , \\ t(h)+1 & \text{if } t(h) < t_0(h) . \end{cases}$$

Then

$$\begin{aligned} W(\sigma,\hat{t}) &= \int W(\sigma[p_t],\hat{t}[p_t])\, d\sigma = \int_{t\geq t_0} W(f_t)\, d\sigma + \int_{t<t_0} \gamma(f_t)W\, d\sigma \\ &\geq a_t + \int_{t<t_0} \alpha W(f_t)\, d\sigma \\ &= a_t + \alpha b_t . \end{aligned} \tag{5.9}$$

The first equality is an instance of (2.3) and the inequality is by (5.7).

Now let $\epsilon > 0$ and choose a stop rule s such that $W(\sigma,s) > W(\sigma)-\epsilon$. (The choice is possible because $W(\sigma) \leq U(\sigma) \leq U(f) = V(f) < +\infty$. The first inequality holds because $U \leq W$, the second by [4, Corollary 3.3.4] and the standing hypothesis that V is finite.) Then, for every stop rule $t \geq s$,

$$(5.10) \qquad W(\sigma,t) \geq W(\sigma,s) > W(\sigma)-\epsilon \geq W(\sigma\hat{t})-\epsilon$$

because the process $\{W(f_n)\}$ is expectation increasing. By (5.9) and (5.10),

$$a_t + b_t = W(\sigma,t) \geq a_t + \alpha b_t - \epsilon ,$$

so $b_t \to 0$. □

Here is an example which shows that there may be no stationary family which yields as much as $U_n-\epsilon$ even when $n = 2$.

<u>Example</u> 5.1. (A modification of an example of Blackwell in [3]). Let F be the set of integers; let $u(n) = 0$ for $n \geq 0$, $u(n) = 2^{-n}-1$ for $n < 0$; $\Gamma(n) = \{\delta(n)\}$ for $n \leq 0$, $\Gamma(n) = \{\delta(n), \frac{1}{2}(\delta(n+1)+\delta(0)), \delta(-n)\}$ for $n > 0$. Then $U_2(n) = 2^n-1/2$ for $n > 0$. But, if γ is a Γ-selector, then either $\gamma(n) = \frac{1}{2}(\delta(n+1)+\delta(0))$ and $u(\gamma^\infty(n)) = 0$ for all $n > 0$ or there is a positive n with $\gamma(n) = \delta(n)$ or $\delta(-n)$ in which case $u(\gamma^\infty(n)) \leq 2^n-1 = U_2(n)-1/2$.

6. <u>Analytically measurable gambling problems</u>.

Consonant with Blackwell, Freedman, and Orkin's paper [4], a gambling problem (Γ,u) is called <u>analytic</u> if Γ is analytic and u is semi-analytic. Analytic problems include the measurable problems defined and studied by Strauch [14], and, a fortiori, the continuous gambling problems studied in [6].

This section shows that, for leavable, analytic problems, measurable stationary plans are adequate.

Recall that a separable metric space X is <u>analytic</u> if there is a continuous function from the set of irrationals in the unit interval onto X (Kuratowski [11]). Let $\mathcal{B}(X)$ and $\mathcal{G}(X)$ denote the sigma-field of Borel subsets of X and the sigma-field generated by the analytic subsets of X respectively. An extended real-valued function g defined on X is called <u>semi-analytic</u> if it is nonnegative and, for all real numbers a, the set of x for which $g(x) > a$ is analytic. For a discussion of these concepts see [8, Section 39, XI] or [2].

Denote by $\mathcal{P}(X)$ the set of countably additive probability measures defined on $\mathcal{B}(X)$. Equip $\mathcal{P}(X)$ with the weak-star topology. Then $\mathcal{P}(X)$ is again an analytic set [2, Lemma 25].

A gambling house Γ is <u>analytic</u> if F is analytic and the set $\{(f,\gamma): \gamma \in \Gamma(f)\}$ is an analytic subset of $F \times \mathcal{P}(F)$. (Here each gamble γ is identified with its restriction to $\mathcal{B}(F)$.)

<u>Lemma</u> 6.1. (Lemma 1(3) of Meyer and Traki [9]). Let u be semi-analytic on F. Then the mapping $\gamma \to \gamma u$ from $\mathcal{P}(F)$ to the extended real numbers is semi-analytic.

Define the operator Γ^* by

$$(6.1) \qquad (\Gamma^* u)(f) = \sup \{\gamma u : \gamma \in \Gamma(f)\}, \quad f \in F .$$

If (Γ,u) is a leavable gambling problem, then, as was noted in [6, Theorem 2.15.2], for all $n \geq 0$, $\Gamma^* U_n = U_{n+1}$ where U_0 is u.

Throughout the remainder of this paper assume that (Γ,u) is a leavable, analytic gambling problem. The assumption usually in force that V is everywhere finite is needed in this section only for Proposition 6.1.

Lemma 6.2. $\Gamma^* u$ and, consequently, each U_n is semi-analytic. Furthermore $U_n \uparrow U$ as $n \uparrow \infty$, so U, too, is semi-analytic.

Proof: For any real number a, the set of f such that $(\Gamma^* u)(f) > a$ is the projection on F of $\{(f,\gamma): \gamma u > a, \gamma \in \Gamma(f)\}$ which set is analytic by Lemma 6.1 and the hypothesis that Γ is an analytic house. Hence, its projection is also analytic. Consequently, $\Gamma^* u = U_1$ is semi-analytic.

Use induction and the comment following (6.1) to see that each U_n is semi-analytic. By [6, Theorem 2.15.5g], $U_n \uparrow U$. So U too is semi-analytic. $\square$

The construction of measurable strategies requires the measurable choice of gambles for which purpose the following selection lemma is useful.

Lemma 6.3. Let π be the projection of the product $X \times Y$ of the two analytic sets X and Y onto X; $\ldots \supset A_{-1} \supset A_0 \supset A_1 \ldots$ be a doubly infinite sequence of analytic subsets of $X \times Y$; $B_i = \pi(A_i)$ for all i; and let $B_\infty = \cap B_i$. Then, for each integer k, there is an analytically measurable mapping, s, of the union of the B_i into Y which satisfies these two conditions:

(i) $x \in B_i - B_{i+1}$ implies $(x,s(x)) \in A_i$ for all i,

(ii) $x \in B_\infty$ implies $(x,s(x)) \in A_k$.

(That s is analytically measurable means $s^{-1}(D) \in \mathcal{A}(X)$ for each $D \in \mathcal{B}(Y)$.)

Proof: According to a selection theorem of Mackey and von Neumann [2, Proposition 15], for each i including $i = \infty$, there exists an $\mathcal{Q}(X)$-measurable mapping $s_i : B_i \to Y$ such that, for all $x \in B_i$, $(x, s_i(x))$ is an element of B_i or B_k according as i is finite or not. If $x \in B_\infty$, let $s(x) = s_\infty(x)$. If $x \in \cup B_i - B_\infty$, let $s(x) = s_i(x)$ where i is the unique integer such that $x \in B_i$ and $x \in B_{i+1}$. That s satisfies (i) and (ii) is easily verified. □

The above lemma and its proof were abstracted out of Blackwell, Freedman, and Orkin [2].

A selector γ for Γ is Borel (analytic) if it is a Borel measurable (analytically-measurable) function from F to $\mathcal{P}(F)$. In each case, $\mathcal{P}(F)$ is equipped with the sigma-field of its Borel subsets. It can happen that there is no Borel selector for a nonleavable, Borel measurable house [14]. There are, however, analytic selectors for such houses. Indeed, they exist for all analytic houses, including those which are not leavable, as is immediate from the Mackey-von Neumann selection theorem, which is the important special case of Lemma 6.3 in which the A_i do not vary with i.

Lemma 6.4. For each $\varepsilon > 0$, there is an analytic Γ-selector γ such that

$$(6.2) \qquad \gamma(f)u \geq \begin{cases} (1-\varepsilon)U_1(f) & \text{if } U_1(f) < \infty , \\ 1/\varepsilon & \text{if } U_1(f) = \infty . \end{cases}$$

(The assumption, otherwise in force, that Γ is leavable, is not needed for this lemma.)

Proof: Choose δ and k so that

$$(6.3) \qquad (1+\delta)^{-1} > 1-\varepsilon , \text{ and } (1+\delta)^k > \frac{1}{\varepsilon} .$$

For each n, let A_n be the set of all (f,γ) such that γ is available at f and γu exceeds $(1+\delta)^n$. By Lemma 6.1, each A_n is analytic. Hence, with X and Y replaced by F and $\mathcal{P}(F)$, the hypothesis of Lemma 6.3 is satisfied. Let s be the map which Lemma 6.3 delivers and let γ be s on the domain of s which is the set $[U_1 > 0]$ or UB_n in the notation of Lemma 6.3. That (6.2) holds for f in this set is easily verified because $f \in B_n$ if and only if $U_1(f)$ is $+\infty$. Define γ on the set $[U_1 = 0]$ to agree with any analytic -selector. □

Of course, if U_1 is everywhere finite, then the statement of Lemma 6.4 and its proof becomes slightly simpler. F · predecessors of Lemma 6.4, see [4, No. 43] and [6, Section 16].

<u>Corollary</u> 6.1. For each $\epsilon > 0$ and each $\alpha \in \mathcal{P}(\cdot)$ there is a Borel Γ-selector γ such that (6.2) holds except for a set of f's having α-probability zero.

<u>Proof</u>: By Lemma 6.4, there is an analytic selector γ' which makes (6.2) true when γ is replaced by γ'. Choose a Borel Markov kernel β such that the set of f for which $\gamma(f)$ is different from $\beta(f)$, call it A_0, has α-probability zero. Then choose a Borel subset A of F such that $A_0 \subset A$ and $\alpha(A) = 0$. Define $\gamma(f) = \beta(f)$ if $f \notin A$ and $\gamma(f) = \delta(f)$ if $f \in A$. □

A stationary family γ^∞ is called <u>Borel</u> (<u>analytic</u>) if the selector γ is Borel (analytic).

<u>Proposition</u> 6.1. For each positive integer n and $\epsilon > 0$, there is an analytic Γ-selector γ such that

$$(6.4) \qquad u(\gamma^\infty(f)) \geq (1-\epsilon)\, U_n(f)$$

for all $f \in F$.

Proof: Choose β such that $0 < \beta < 1$ and $\beta^n > 1-\epsilon$. By Lemma 6.4, there is, for $k = 1,2,\ldots$, an analytic Γ-selector γ_k such that $\gamma_k(f)U_{k-1} > \sqrt{\beta}U_k(f)$ for all f. Proposition 5.1 now applies. □

Since $U_n \to U$, it follows from Proposition 6.1 that analytic, stationary plans are adequate. In fact, Borel stationary plans are adequate as Theorem 7.1 below implies. No assertion about the measurability of the left-hand side of (6.4) is made here. Indeed, we do not know whether $u(\gamma^\infty(\cdot))$ is analytically measurable even if γ is Borel measurable, unless u is bounded [17, Theorem 2].

7. Borel stationary plans are almost uniformly adequate. There is a notion of the adequacy of a set $\wp$ of plans which is intermediate in strength between ordinary adequacy and uniform adequacy. Namely, a set $\wp$ of plans available in Γ is almost uniformly adequate if, for each $\epsilon > 0$ and each measure α countably additive on the Borel subsets of F, there is a $\bar{\sigma} \in \wp$ such that the set of F for which (1.1) fails to hold has measure zero under α.

Proposition 7.1. Borel stationary plans are almost uniformly adequate for Γ.

This proposition has predecessors in [1], [2], [13], and [15]. Because the present proposition treats unbounded, albeit nonnegative, utilities and because it covers analytic problems rather than Borel problems only, the proof given here differs from that of its predecessors. But the reader will easily discern the underlying similarity of the arguments and, in particular, our debt to Ornstein [13], who was the first to settle the problem of stationarity for a large class of countably additive houses Γ based on a denumerable fortune space F. The result which corresponds to Proposition 7.1 in the case of positive dynamic programming was stated by Frid [10, Theorem 1], but his proof has an error. (The sets G and H defined in Lemma 3 of [10] need not be Borel.)

A leavable house Γ' defined on the analytic set F is (Borel) countably parametrized if it is the union of (the graphs of) a countable number of Borel measurable Markov kernels. A house Γ' is a subhouse of Γ, written $\Gamma' \subseteq \Gamma$, if, for each f, $\Gamma'(f) \subseteq \Gamma(f)$. A house Γ' is a union of houses Γ_n if, for each f, $\Gamma'(f)$ is the set-theoretic union of $\Gamma_n(f)$.

For each $\alpha \in \mathcal{P}(F)$ and each Borel Markov kernel γ, let $\alpha\gamma$ be that element of $\mathcal{P}(F)$ defined by

$$(7.1) \qquad (\alpha\gamma)(A) = \int \gamma(f)(A)\, d\alpha(f)$$

for $A \in \mathcal{B}(F)$. The trivial fact that a subset A of F which has probability one under the completion of $\alpha\gamma$ also has probability one under the completion of $\gamma(f)$ for α-almost all f will be used twice in the proof of Lemma 7.1.

Lemma 7.1. For each $\alpha \in \mathcal{P}(F)$, there is a countably parametrized subhouse Γ' of Γ and a Borel measurable, nonnegative function u' on F such that

(i) $u' \le u$ everywhere, and

(ii) $U' \ge U$ α-almost everywhere,

where U' is the optimal return function for (Γ', u').

Proof: First the lemma obtained by replacing (ii) with the weaker condition

$$U' \geq (1-\varepsilon)\, U_n \qquad \alpha\text{ - a.s.} \tag{7.2}$$

will be proved. Here $\varepsilon > 0$ and n is a positive integer. To this end, choose λ in $(0,1)$ such that $\lambda^n > (1-\varepsilon)$. By (5.1) and Corollary 6.1, there is a Borel Γ-selector γ_1 such that

$$\gamma_1(f)\, U_{n-1} \geq \lambda\, U_n(f) \qquad \alpha\text{ - a.s.} \tag{7.3}$$

Use the notation of (7.1), set $\alpha_1 = \alpha\gamma_1$, and again call on Corollary 6.1 to obtain another Borel Γ-selector γ_2 with

$$\gamma_2(f)\, U_{n-2} \geq \lambda\, U_{n-1}(f) \qquad \alpha_1\text{ - a.s.}$$

Continue thus to define inductively Borel Γ-selectors $\gamma_1,\ldots,\gamma_n$ and measures $\alpha_0 = \alpha$, $\alpha_1,\ldots,\alpha_{n-1}$ so that, for $1 \leq k \leq n$, $\alpha_k = \alpha_{k-1}\gamma_k$ and

$$\gamma_k(f)\, U_{n-k} \geq \lambda\, U_{n-k+1}(f) \qquad \alpha_{k-1}\text{ a.s.} \tag{7.4}$$

Let $\Gamma'(f) = \{\gamma_1(f),\ldots,\gamma_n(f),\ \delta(f)\}$. Since u is measurable with respect to the completion of α_n, there is a Borel $u' \geq 0$ which satisfies (i) and which agrees with u on a set of α_n-probability one. So $\gamma_n(f)[u'=u] = 1$ α_{n-1}- a.s. and, hence,

$$\gamma_n(f)u' \geq \gamma_n(f)u \qquad \alpha_{n-1}\text{- a.s.} \tag{7.5}$$

Let U_j' be the optimal j-day return for the problem (Γ',u').

As will now be shown, for $1 \leq j \leq n$, (7.6) $\quad U_j' \geq \lambda^j U_j \qquad \alpha_{n-j}$ a.s.

To verify (7.6) for $j=1$, calculate thus.

$$\begin{aligned} U_1'(f) &\geq \gamma_n(f)u' \\ &\geq \gamma_n(f)u \qquad \alpha_{n-1} \text{ a.s.} \\ &\geq \lambda\, U_1(f) \qquad \alpha_{n-1} \text{ a.s.} \end{aligned}$$

The first inequality is by definition of U_1', the second is (7.5), and the third is the instance of (7.4) in which $\;$ = n . Suppose (7.6) holds for $j = n-1$. That is, the set of f_1 such that

$$U_{n-1}'(f_1) \geq \lambda^{n-1} U_{n-1}(f_1)$$

has $(\alpha\gamma_1)$-probability one. So, for α-almost all f, the same set has $\gamma_1(f)$-probability one. Now calculate:

$$\begin{aligned} U_n'(f) &\geq \int U_{n-1}'(f_1)\ \gamma_1(df_1|f) \\ &\geq \int \lambda^{n-1} U_{n-1}(f_1)\ \gamma_1(df_1|f) \qquad \alpha\text{-a.s.} \\ &\geq \lambda^n U_n(f) \qquad \alpha\text{ - a.s.} \end{aligned}$$

The first inequality is by (5.1) and the third is by (7.3). The proof of (7.2) is complete.

Thus, for $n = 1,2,\ldots$, there is a countably parametrized house $\Gamma_n \subseteq \Gamma$ and a nonnegative Borel utility function $u_n \leq u$ such that, if R_n is the return function for Γ_n, then $R_n \geq (1-1/n)\, U_n$ α-almost surely. Let Γ' be the union of the Γ_n, let $u' = \sup u_n$. Obviously, (i) holds and, since $U = \sup U_n$, (ii) is easily verified. □

Lemma 7.2. Suppose Γ is countably parametrized, u is nonnegative Borel, and ϵ is a positive number. Then

(a) the functions $U_1, U_2, \ldots$ and U are Borel;

(b) for each positive integer n, there is a Borel Γ-selector γ such that

$$u(\gamma^{\infty}(f)) \geq (1-\epsilon/2)\, U_n(f) \quad \text{for all } f , \tag{7.7}$$

$u(\gamma^{\infty}(f)$ is a Borel measurable function of f and, for each $\alpha \in \mathcal{P}(F)$ and all n sufficiently large,

$$u(\gamma^{\infty}(f)) \geq (1-\epsilon)\, U(f) \quad \text{with } \alpha\text{-probability at least } 1-\epsilon . \tag{7.8}$$

Proof: For a proof of (a), use Formula 5.1 and Lemma 6.2 or [15, Theorem 4.1].

For (b), choose β in $(0,1)$ such that $\beta^n > 1-\epsilon/2$. Let $\gamma_1', \gamma_2', \ldots$ be the Borel Markov kernels comprising Γ. For each k and f, let $\gamma_k(f)$ be the first element of the sequence $\gamma_1'(f), \gamma_2'(f), \ldots$ satisfying (5.2). Then $\gamma_1, \gamma_2, \ldots$ are Borel measurable and constitute a (Γ, β) sequence as defined in Section 5. That the corresponding Γ-selector γ satisfies (7.7) is evident in view of Proposition 5.1. Since $U_n \uparrow U$ (Lemma 6.2), (7.8) holds for all sufficiently large n.

The proof would be complete if $u(\gamma^{\infty}(f))$ could be shown to be a Borel measurable function of f. Whether it is or not, we do not know and, for present purposes, need not know, for as will now be shown, there is a Borel Γ-selector λ such that $u(\lambda^{\infty}(f)) \geq u(\gamma^{\infty}(f))$ for all f and $f \to u(\lambda^{\infty}(f))$ is Borel. Consider the Borel gambling problem (Γ', u) where $\Gamma'(f) = \{\delta(f), \gamma(f)\}$ for all f. By (a), U' is Borel measurable. So, if $\lambda = \gamma$ or δ according as $u < U'$ or $u = U'$, then λ too is Borel measurable.

Of course, $U'(f) \geq u(\gamma^\infty(f))$ and by Corollary 4.1, $u(\lambda^\infty(f))$ is $U'(f)$ for all f. So $u(\lambda^\infty(f))$ is Borel measurable and is no less than $u(\gamma^\infty(f))$. □

Incomplete stop rules, as defined in [6], are here called stopping times. A stop rule is simply a stopping time which has only finite values. As in [6], the partial history $(f_1,\ldots,f_n)$ of the history $h = (f_1,\ldots,f_n, \ldots)$ is denoted by $p_n(h)$. Strategies σ and σ' agree prior to a time τ if $\sigma_0 = \sigma_0'$ and, for every h and $0 \leq n < \tau(h)$, $\sigma(p_n(h)) = \sigma'(p_n(h))$. If $\tau(h) < +\infty$, abbreviate $p_{\tau(h)}(h)$ to $p_\tau(h)$.

The next two lemmas are proved in [9], and do not require the measurability and leavability assumptions of Theorem 7.1.

Lemma 7.3. If σ is available in Γ, σ and σ' agree prior to time τ, $\varepsilon \geq 0$, and $u(\sigma'[p_\tau(h)]) \geq (1-\varepsilon)\, V(f_\tau(h))$ whenever $\tau(h) < +\infty$, then $u(\sigma') \geq (1-\varepsilon)\, u(\sigma)$.

Proof: This lemma is one of the implications proved in [9, Lemma 3]. □

Given σ and $\varepsilon > 0$, introduce $\tau = r(\sigma,\varepsilon)$ as the first time (if any) when σ is not conditionally ε-optimal; that is,

$$(7.9) \qquad \begin{aligned} \tau(h) &= r(\sigma,\varepsilon)(h) \\ &= \inf \{n : u(\sigma[p_n(h)]) < (1-\varepsilon)\, V(f_n)\}. \end{aligned}$$

The infimum of an empty set of n's is, by convention, $+\infty$.

The next lemma states that a strategy which agrees with a very good strategy until that strategy is conditionally less than good is itself a good strategy.

Lemma 7.4. Let σ be a strategy available at f and let $\epsilon > 0$. If σ' is any strategy which agrees with σ prior to time $\tau = r(\sigma,\epsilon)$, then $u(\sigma') \geq u(\sigma) - \epsilon^{-1}[V(f) - u(\sigma)]$. Therefore, if $u(\sigma) \geq (1-\epsilon^2/2)V(f)$, then $u(\sigma') \geq (1-\epsilon)V(f)$.

Proof: This lemma is part of [9, Lemma 4]. □

For the rest of this section, assume that Γ is countably parametrized and that u is Borel.

To each stop-or-go subhouse Σ of Γ, associate the house $\Gamma \circ \Sigma$ which is defined by

$$(\Gamma \circ \Sigma)(f) = \begin{cases} \Sigma(f), & \text{if } \Sigma(f) \text{ contains two elements,} \\ \Gamma(f), & \text{otherwise.} \end{cases}$$

Plainly, $\Sigma \subseteq \Gamma \circ \Sigma \subseteq \Gamma$ and $\Sigma \subseteq \Sigma' \Rightarrow \Gamma \circ \Sigma' \subseteq \Gamma \circ \Sigma$.

The leavable, stop-or-go house Σ such that $\Sigma(f) = \{\lambda(f), \delta(f)\}$ for all f is Borel if the mapping λ is Borel measurable. Such a house Σ is plainly countably parametrized, and, because Γ is countably parametrized, $\Gamma \circ \Sigma$ is also. So, by Lemma 7.2, the optimal return functions W and R for Σ and $\Gamma \circ \Sigma$, respectively, are Borel measurable.

Lemma 7.5. Suppose Σ is a leavable, Borel stop-or-go subhouse of Γ, $\alpha \in \mathcal{P}(F)$, and $\epsilon > 0$. Then there is a leavable, Borel stop-or-go house Σ' such that

(i) $\Sigma \subseteq \Sigma' \subseteq \Gamma$,

(ii) $\alpha[W' \geq (1-\epsilon)R] \geq 1-\epsilon$,

(iii) $R' \geq (1-\epsilon)R$.

(Here, R, R', and W' are the optimal return functions for $\Gamma \circ \Sigma$, $\Gamma \circ \Sigma'$, and Σ', respectively.)

<u>Proof</u>: By Lemma 7.2, there is a Borel $\Gamma \circ \Sigma$-selector γ such that

$$(7.10) \qquad \alpha(S) \geq 1-\epsilon ,$$

where $S = \{f: u(\gamma^{\infty}(f)) \geq (1-\epsilon^2/2)R(f)\}$. Let $T = \{f: u(\gamma^{\infty}(f)) \geq (1-\epsilon)R(f)\}$, and let Σ' be the smallest leavable, stop-or-go house which is at least as large as Σ and in which $\gamma(f)$ is available at each f in T. That is,

$$\Sigma'(f) = \begin{cases} \{\gamma(f), \delta(f)\} , & \text{if } f \in T , \\ \Sigma(f) , & \text{if } \Sigma(f) \text{ contains two elements,} \\ \{\delta(f)\} , & \text{otherwise.} \end{cases}$$

Obviously, Σ' satisfies (i). To check (ii), let λ be the Σ'-selector which equals γ on T and is δ on T^c. Then, for each $f \in S$, $\lambda^{\infty}(f)$ agrees with $\gamma^{\infty}(f)$ prior to the time of the first exit from T. So, by Lemma 7.4,

$$W'(f) \geq u(\lambda^{\infty}(f)) \geq (1-\epsilon)R(f)$$

for $f \in S$. Condition (ii) now follows from (7.10).

It remains to verify (iii). Since γ is a Σ'-selector, the inequality of (iii) certainly holds for $f \in T$. For $f \notin T$, let σ be any strategy available at f in $\Gamma \circ \Sigma$ and define σ' to be that strategy which agrees with σ prior to the time τ of first entrance into T and such that the conditional strategy $\sigma'[p_{\tau}(h)]$ is $\gamma^{\infty}(f_{\tau}(h))$ whenever $\tau(h) < \infty$. Then $R'(f) \geq u(\sigma') \geq (1-\epsilon)u(\sigma)$, where the first inequality holds because σ' is available in Γ' at f and the second is by Lemma 7.3. $\square$

__Lemma__ 7.6. Let $\alpha \in \mathcal{P}(F)$ and $\epsilon > 0$. There is a sequence $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots$ of leavable, Borel stop-or-go subhouses of Γ whose optimal return functions $W_0, W_1, \ldots$ satisfy

$$(7.11) \qquad \alpha[W_n \geq (1-\epsilon)U] \nearrow 1 .$$

__Proof__: Let Σ_0 be the trivial house in which only $\delta(f)$ is available at f for every f. Then $\Gamma \circ \Sigma_0 = \Gamma$ and R_0, the return function for $\Gamma \circ \Sigma_0$, is U.

Suppose that Σ_n has been defined and that $0 < \epsilon_n < 1$. Let R_n be the return function for $\Gamma \circ \Sigma_n$. Then, by the previous lemma, there exists Σ_{n+1} such that

$$(7.12) \qquad \alpha[W_{n+1} \geq (1-\epsilon_n)R_n] \geq 1-\epsilon_n$$

and

$$(7.13) \qquad R_{n+1} \geq (1-\epsilon_n)R_n .$$

For $n \geq 1$,

$$R_n \geq \left(\prod_{i=1}^{n-1} (1-\epsilon_i)\right)R_0 ,$$

as follows from (7.13). Thus the event

$$[W_{n+1} \geq \left(\prod_{i=1}^{n} (1-\epsilon_i)\right)U]$$

contains the event occurring in the left-hand side of (7.12) and, therefore, has α-probability at least $1-\epsilon_n$. The proof is complete once the ϵ_n are chosen so that

$$\prod_{n=1}^{\infty} (1-\epsilon_n) > 1-\epsilon . \quad \square$$

To complete the proof of Proposition 7.1, let $\{\Sigma_n\}$ be the sequence of houses given by Lemma 7.6 and Σ the union of the Σ_n. Then Σ is a leavable, Borel, stop-or-go subhouse of Γ whose return function W is at least W_n for every n. By (7.11), $W \geq (1-\epsilon)U$ α-almost surely. Suppose $\Sigma(f) = \{\lambda(f), \delta(f)\}$ for every f and γ is that Σ-selector which equals δ on the set $[u = W]$ and equals λ on the complementary set. Then γ is Borel measurable and, by Corollary 4.1, $u(\gamma^\infty(f)) = W(f)$ for all f. The proof of Proposition 7.1 is now complete.

8. Absolutely continuous houses. As defined in the introduction, an analytic house Γ is Borel absolutely continuous with respect to $\alpha \in \mathcal{P}(F)$ if, for every $f \in F$ and $\gamma \in \Gamma(f)$, $\gamma \neq \delta(f)$ implies γ is Borel absolutely continuous with respect to α.

Theorem 8.1. If Γ is leavable and Borel absolutely continuous with respect to $\alpha \in \mathcal{P}(F)$, U is everywhere finite, and $0 < \epsilon < 1$, then there is an analytic Γ-selector γ such that

$$u(\gamma^\infty(f)) \geq (1-\epsilon)\, U(f)$$

for all $f \in F$.

Proof: Choose ϵ_1 so that $0 < \epsilon_1 < 1$ and $(1-\epsilon_1)^2 > 1-\epsilon$. By Proposition 7.1, there is a Borel selector γ_1 and a Borel subset S of $\{f: u(\gamma_1^\infty(f)) \geq (1-\epsilon_1)\, U(f)\}$ for which $\gamma_0(S) = 1$. Since Γ is absolutely continuous with respect to α, S has probability one under every gamble available at a fortune $f \in S$. Thus, for each $f \in S$, $f_n \in S$ for all n with $\gamma_1^\infty(f)$-probability one. So, for any selector γ which agrees with γ_1 on S and for $f \in S$, $u(\gamma^\infty(f)) = u(\gamma_1^\infty(f)) \geq (1-\epsilon_1)\, U(f) > (1-\epsilon)\, U(f)$.

There remains to define γ on S^c and, for this, a lemma is helpful.

Lemma 8.1. There is an analytic Γ-selector γ_2 such that

(i) $\gamma_2(f)U \geq (1-\epsilon_1)\, U(f)$ for all f,

(ii) $[\gamma_2 = \delta] \subseteq [u = U]$.

Proof: Consider the analytic gambling problem (Γ',u') where $u' = U$ and and

$$\Gamma'(f) = \begin{cases} \Gamma(f) & \text{if } u(f) = U(f)\,, \\ \Gamma(f)-\{\delta(f)\} & \text{if } u(f) < U(f)\,. \end{cases}$$

The desired selector γ_2 can be obtained by an application of Lemma 6.4 once it is verified that U_1' equals U. The inequality $U_1' \leq U$ holds because U is excessive for Γ [6, Theorem 2.14.1]. Because Γ is leavable, the reverse inequality $U_1' \geq U$ holds on the set $[u = U]$. Suppose now that $u(f) < U(f)$. Then there must be, for each positive ϵ', a strategy which is ϵ'-optimal at f for the original problem (Γ,u) and whose initial gamble γ is not $\delta(f)$. Then

$$U_1'(f) \geq \gamma U \geq U(f)-\epsilon'$$

where the first inequality is by definition of U_1' and the second by Lemma 3.1. $\square$

Returning to the proof of the theorem, set $\gamma = \gamma_2$ on S^c where γ_2 is the analytic selector given by the lemma. If $f \in S^c$ and $\gamma(f) = \delta(f)$, then $u(\gamma^\infty(f)) = u(f) = U(f)$. If $f \in S^c$ and $\gamma(f) \neq \delta(f)$, then $\gamma(f)(S) = 1$ and, therefore,

$$\begin{aligned}
u(\gamma^{\infty}(f)) &= \int u(\gamma^{\infty}(f_1))\ d\gamma(f_1|f) \\
&= \int u(\gamma_1^{\infty}(f_1))\ d\gamma(f_1|f) \\
&\geq (1-\varepsilon_1) \int U(f_1)\ d\gamma(f_1|f) \\
&= (1-\varepsilon_1)\ \gamma(f)U \\
&= (1-\varepsilon_1)\ \gamma_2(f)U \\
&\geq (1-\varepsilon_1)^2\ U(f) \\
&\geq (1-\varepsilon)\ U(f)\ .
\end{aligned}$$

The proof is complete.

If Γ is Borel countably parametrized and u is Borel measurable, then, as is not difficult to verify, the selector γ_2 of Lemma 8.1 and, hence, the selector γ of Theorem 8.1, can be chosen to be Borel.

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