COUNTABLY ADDITIVE GAMBLING

and

OPTIMAL STOPPING

by

Lester E. Dubins* and William D. Sudderth*
University of California, Berkeley and University of Minnesota

Technical Report No. 272

*Research supported by National Science Foundation Grants NSF GP 43085 (for Dubins) and MPS 75-06173 and MCS76-03940 (for Sudderth).

ABSTRACT

A theory of discrete-time optimal stopping is presented within the more general framework of countably additive gambling theory. In particular, offered here is a substitute for a fundamental theorem of optimal stopping due to Chow, Robbins, and Siegmund.

Key words: gambling, optimal strategies, optimal stopping, probability, dynamic programming, decision theory.

A.M.S. classification numbers: 6000,60G99, 62C05

Introduction: This paper offers a substitute for what is perhaps the most fundamental theorem of *The Theory of Optimal Stopping* by Chow, Robbins, and Siegmund, namely [3, Theorem 4.10]. From the point of view of this paper, optimal stopping problems are special gambling problems, namely those associated with *stop-or-go* gambling houses. In such a house there is, at each fortune f, at most one gamble $\alpha(f)$ other than the trivial one-point, Dirac delta measure, $\delta(f)$, which may or may not be available. A basic question is to determine those strategies, if any, which are optimal for stop-or-go problems.

Suppose that $u(f)$ is the utility of the fortune f and that $W(f)$ is the supremum over all strategies σ available at f of the utility of σ. If σ stagnates at f, that is, if σ uses $\delta(f)$ forever, then the utility of σ is taken to be $u(f)$. Plainly, then, if f is *inadequate*, that is, if $u(f) < W(f)$, it is not optimal to stagnate at f. Nor can a strategy σ be persistently optimal, that is, be conditionally optimal given every partial history $p = (f_1, \ldots, f_n)$, if, after any p for which f_n is inadequate, σ then stagnates at f_n. Other strategies, too, are easily seen not to be optimal. In particular, if Γ is a gambling house, $\gamma \in \Gamma(f)$ and $\gamma W < W(f)$, then no strategy which employs γ initially can be optimal at f. Nor can a strategy be persistently optimal if after any $p = (f_1, \ldots, f_n)$ for which $f_n = f$ it then employs γ.

A strategy which avoids the two non-optimal modes of behavior described above will be called *promising*. A fundamental problem for conventional stop-or-go problems is to determine whether every promising strategy is persistently optimal and, if not, what additional condition is necessary.

A Markov kernel γ for which $\gamma(f) \in \Gamma(f)$ for every f is a Γ-*selector*. Let $\gamma^{\infty}(f)$ be that strategy whose initial gamble is $\gamma(f)$ and which, after the partial history $(f_1, \ldots, f_n)$ employs $\gamma(f_n)$, and call γ^∞ a <u>stationary</u> family of strategies.

As is implied by Theorem 3 below, all stationary families of promising strategies are indeed optimal for conventional stop-or-go problems if u is bounded.

For unbounded u's additional difficulties are encountered. First, two natural methods--henceforth to be called the "gambling" and the "conventional" methods--of evaluating the worth of a strategy, which are equivalent for bounded u's [14, Theorem 2], can yield different values for unbounded u's. Second, at least for the conventional method, it is no longer enough for a stationary family of strategies to be promising for it to be optimal. A growth condition must also be imposed. For the conventional method, Siegmund [10, Theorem 4] found such a condition (domination in the sense of Lebesgue) that is sufficient, and Chow, Robbins, and Siegmund [3, Theorem 4.10] a less restrictive one (uniform integrability) which is necessary and sufficient, for a promising, stationary strategy to be optimal.

One problem encountered in studying stopping time problems, and more general gambling problems, in a countably additive framework, is that suprema of uncountably many measurable functions are encountered, and these suprema are not automatically measurable. In the theory as presented in [11] and [3], this problem is met by considering, when necessary, essential suprema rather than suprema. The present paper concerns itself with a class of problems including, but not confined to, stop-or-go problems, in which W is demonstrably measurable. Hence, the simpler and more satisfactory notion of supremum is adequate for this paper, and in particular, for Theorem 3 below.

The study of stop-or-go problems in which the so-called gambling evaluation of strategies is employed is to be included in a forthcoming paper concerned with the general topic of stationary strategies.

§1. Summary. Throughout this paper, (Γ, g) is a Borel measurable, invariant, gambling problem, and W is the most that is achievable if only analytically measurable σ available in Γ are employed. (For precise definitions, see §2.)

The main purpose of this paper is to report three results: a characterization of W (Theorem 1), a characterization of optimal strategies (Theorem 2), and a determination, as in Theorem 3, of all stationary families of optimal strategies for stop-or-go problems with conventional utilities, a species of gambling problem defined in §7.

These results, especially Theorems 1 and 3, were strongly influenced by similar results already in the literature. For predecessors of Theorem 1, see [11, Theorem 3.6], [5, Theorems 2.12.1, 2.14.1, and 3.3.1] and [17, Theorems 3.1 and 3.2]. Among the predecessors of Theorem 3 are [11, Theorem 3.7], [5, Theorem 3.9.5], [16, Theorem 2], and [3, Theorem 4.10].

Of more fundamental interest than W is W', the most that is achievable by all strategies, including of course those which are not measurable. Left open by the present paper, however, is the basic question whether W' can exceed W. Under certain assumptions on (Γ, g), W' is demonstrably no larger than W [12, Theorem 1], [14, Theorem 6.4]. The question whether W equals W' leads to the preliminary question whether W' is absolutely measurable. These problems have been open at least since 1960 (when the second mimeographed edition of [5] appeared), so this paper confines its attention to W only.

§2. The formulation of Theorem 1, a characterization of W. Throughout this paper, $(F, \mathcal{B})$ is a standard space ($\mathcal{B}$ is the set of Borel subsets of a Borel subset F of a Polish space); $\mathcal{P}$ is the set of all countably additive probability measures defined on $\mathcal{B}$, and, as in [12], the Borel gambling house Γ is a Borel subset of $F \times \mathcal{P}$ such that, for each $f \in F$, $\Gamma(f)$, the f-section of Γ, is nonempty; the utility of a history $h = (f_1, f_2, \ldots)$ is an extended real number $g(h)$. It is assumed that g is Borel measurable and invariant. That g is invariant means $g(fh) = g(h)$ for all f and all h where, as in [5], $fh = (f, f_1, f_2, \ldots)$. The product of a finite or a denumerable number of standard spaces is a standard space, as is the set of countably additive probability measures defined on the Borel subsets of a standard space ([9], Chapter II). So, for example, F^n, $\mathcal{P}$, and $F \times \mathcal{P}$ are standard spaces, as is $H = F \times F \times \ldots$.

A mapping from one standard space to another is analytically measurable or $\mathcal{G}$-measurable, for short, if the inverse image of every Borel set is in the sigma-field generated by the analytic sets. In conformance with a notion given in [1], a strategy σ is analytically measurable or $\mathcal{G}$-measurable if $\sigma_0 \in \mathcal{P}$ and every σ_n is analytically measurable from F^n to $\mathcal{P}$. (Terms such as "strategy" and symbols such as "σ_n" which are frequently used in [5] will not be defined here.) Each $\mathcal{G}$-measurable σ determines, as usual, by the theorem of Ionescu Tulcea (Proposition V-1-1,[8]), a countably additive probability measure, say $m(\sigma)$, defined on the Borel subsets of H. Let $\Gamma^{\infty}(f)$ be the set of all $m(\sigma)$ as σ ranges over all $\mathcal{G}$-measurable strategies available in Γ at f. Often, $m(\sigma)$ will simply be designated by σ. Thus 'σ' refers sometimes to a strategy and sometimes to its distribution, a harmless ambiguity.

In some previous papers, [12], [13], and [14], attention was restricted to a subfamily of the $\mathcal{A}$-measurable strategies, namely, the Borel, or $\mathcal{B}$-measurable, strategies. This restriction is somewhat unsatisfactory for, as mentioned in [13], there exist Borel houses in which no $\mathcal{B}$-measurable strategies are available. Furthermore, as can easily be seen, if there are $\mathcal{B}$-measurable strategies available, then the set $\Gamma^\infty(f)$ of their distributions is the same whether σ ranges over $\mathcal{B}$-measurable or $\mathcal{A}$-measurable strategies available at f. This explains why here, as in [1], it is the full class of all $\mathcal{A}$-measurable σ which is of primary interest.

Assume henceforth that, for all $\mathcal{A}$-measurable σ available in Γ, $\int g\, d\sigma$, or σg for short, is well-defined, possibly $-\infty$, but strictly less than $+\infty$.

Introduce the optimal return function, thus.

$$(2.1) \qquad W(f) = \sup\{\sigma g\colon \sigma \in \Gamma^\infty(f)\},$$

and assume throughout this paper that W is everywhere finite. Of course, W is the most the gambler can achieve if he employs nothing but analytically measurable strategies.

Associated with any extended real-valued function Q--in particular W--defined on F, are the functions $Q_1, Q_2, \ldots, Q^*$, and Q_* defined for all $h = (f_1, f_2, \ldots) \in H$, thus.

$$(2.2) \qquad Q_n(h) = Q(f_n) \quad \text{for} \quad n = 1, 2, \ldots,$$

$$Q^* = \lim_n \sup Q_n$$

$$Q_* = \lim_n \inf Q_n$$

If, for all $\mathcal{G}$-measurable σ available in Γ for which σg is finite, the integral σQ^* exists and

(2.3) $\sigma Q^* \geq \sigma g,$

then Q Γ-dominates, or, more briefly, dominates, g.

For each σ, let $T(\sigma)$ be the collection of all σ-stopping times, that is, Borel mappings t from H to the positive integers with $+\infty$ adjoined such that the event $[t < +\infty]$ has σ-probability one and, for every n, the event $[t \leq n]$ is measurable with respect to the first n coordinates of H. For $t \epsilon T(\sigma)$ and $h = (f_1, f_2, \ldots)$, set

$$Q_t(h) = Q(f_{t(h)})$$

when $t(h) < \infty$. Say that Q is excessive (for Γ) if, for all $f \epsilon F$, all $\sigma \epsilon \Gamma^{\infty}(f)$, and all $t \epsilon T(\sigma)$, $\sigma(Q_t)$ exists and

(2.4) $\sigma Q_t \leq Q(f).$

The characterization of W can now be given.

Theorem 1. W is the smallest $\mathcal{G}$-measurable function which is excessive and dominates g.

The proof is given in §4.

§3. Regular supermartingales. Let Q be an extended real-valued, $\mathcal{G}$-measurable function defined on F and let σ be an $\mathcal{G}$-measurable strategy. If the expectation σQ_t exists for each $t \in T(\sigma)$, then the collection $\{\sigma Q_t, t \in T(\sigma)\}$ becomes a net when $T(\sigma)$ is given its natural partial ordering, so

$$\lim_t \sup \sigma Q_t = \inf_s \sup_{t \geq s} \sigma Q_t$$

is well-defined as is the $\lim_t \sup$

Lemma 3.1. Each of the following inequalities holds whenever the expectations occurring in it exist:

$$(3.1) \qquad \lim_t \inf \sigma Q_t \leq \sigma Q_* \leq \sigma Q^* \leq \lim_t \sup \sigma Q_t .$$

Proof: For each n, there is a Borel measurable function $\tilde{Q}_n(h) = \tilde{Q}_n(f_n)$ which equals $Q_n(h)$ σ-almost surely, for analytic sets are universally measurable [7, III.24]. The final inequality holds for the $\tilde{Q}_n$ [15, Theorem 1], and, hence, for the Q_n as well. The first inequality is equivalent to the final one, and the middle one is obvious. □

If φ is a function with domain H and $p = (f_1, \ldots, f_n)$, then, as in [5], φp is that function on H whose value at $h' = (f_1', f_2', \ldots)$ is $\varphi(ph') = \varphi(f_1, \ldots, f_n, f_1', f_2', \ldots)$. If, in addition, φ is extended real-valued and $\sigma\varphi$ exists, then $\sigma[p](\varphi p)$, or $\sigma(\varphi|p)$ for short, is a version of the conditional expectation of φ given p under σ. Consequently, for each σ-stopping time t, (3.2) $\int\varphi d\sigma = \int\sigma(\varphi | p_t(h))d\sigma(h)$

where $p_t(h) = p_n(h) = (f_1, \ldots, f_n)$ when $t(h) = n$ [8, Prop. III.2.1].

The sequence $\{Q_n\}$ is a regular supermartingale under σ if, for s, $t \epsilon T(\sigma)$ and $s \leq t$, σQ_t exists and the inequality

(3.3) $\sigma(Q_t | p_s(h)) \leq Q_s(h)$

holds σ-almost surely. The next lemma includes a criterion for almost sure convergence which is similar to that of Dvoretzky [6].

Lemma 3.2. Each of the following conditions implies its successor.

(a) $\{Q_n\}$ is a regular supermartingale under σ.

(b) $\sigma Q_s \geq \sigma Q_t$ whenever $s \leq t \in T(\sigma)$.

(c) The net $\{\sigma Q_t, t \in T(\sigma)\}$ converges.

(d) The equality $\sigma Q^* = \sigma Q_*$ holds if both expectations occurring in it are well-defined.

(e) $\{Q_n\}$ converges σ-almost surely if σQ^* and σQ_* are finite.

Proof: To see that (a) implies (b), integrate with respect to σ in (3.3) and use (3.2). Obviously, (b) implies (c). Lemma 3.1 applies to show (c) implies (d). The final implication is trivial. □

Here is a lemma which indicates why the study of gambling problems leads to an interest in regular supermartingales.

Lemma 3.3. If Q is excessive for Γ, then $\{Q_n\}$ is a regular supermartingale under every σ available in Γ.

Proof: Let $s \leq t \in T(\sigma)$. If $s(h) = t(h)$, then (3.3) obviously holds with equality. So suppose $t(h) > s(h) = n$. Set $p = p_s(h) = (f_1, \ldots, f_n)$ and define $t[p](h') = t(ph') - n$ for all $h' \in H$. Then for σ-almost every p,

$$\sigma(Q_t|p) = \sigma[p](Q_t p) = \sigma[p](Q_{t[p]}) \leq Q(f_n) = Q_s(h).$$

The first equality is by definition of $\sigma(Q_t|p)$; the second holds because $Q_t p = Q_{t[p]}$; the inequality is by the excessiveness of Q together with the facts that $\sigma[p] \in \Gamma^\infty(f_n)$ and $t[p] \in T(\sigma[p])$ σ-almost surely. □

§4. Proof of Theorem 1. What must first be shown is that W is $\mathcal{G}$-measurable. To this end, let $\mathcal{P}(H)$ be the Borel set of countably additive probability measures defined on the Borel subsets of H, and let Γ^∞ be the subset of $F \times \mathcal{P}(H)$ such that, for each f, the f-section of Γ^∞ is $\Gamma^\infty(f)$.

Lemma 4.1. Γ^∞ is a Borel subset of $F \times \mathcal{P}(H)$.

Proof: By a theorem of Mackey and von Neumann [1, Prop. 15], there is an $\mathcal{G}$-measurable mapping γ from F to $\mathcal{P} = \mathcal{P}(F)$ such that $\gamma(f) \in \Gamma(f)$ for all f. The argument given for [13, Theorem 2.1] now applies and completes the proof. □

Lemma 4.2. The set of $\sigma \in \mathcal{P}(H)$ such that $\int g \, d\sigma$ exists is a Borel set, and, when restricted to this set, the map $\sigma \to \int g \, d\sigma$ is Borel measurable.

Proof: For non-negative g, the conclusion holds in view of the lemma in [14]. Since any g is the difference of two non-negative g's, the conclusion for general g easily follows. □

Preparations have now been made to establish the first part of Theorem 1:

Lemma 4.3. W is $\mathcal{G}$-measurable. Indeed, for each real number r, the event $(W > r)$ is an analytic subset of F.

Proof: The set $(W > r)$ is the projection onto F of the set, S, of all (f, σ) such that $\sigma \in \Gamma^\infty(f)$ and $\sigma g > r$. Of course, S is a Borel set by Lemmas 4.1 and 4.2. So $(W > r)$ is analytic. □

As was shown by Strauch [12], W need not be Borel measurable.

The purpose of the next three lemmas is to prove that W is excessive.

Let $\epsilon \geq 0$. A strategy $\sigma \in \Gamma^{\infty}(f)$ is ϵ-optimal at f if $\sigma g \geq W(f) - \epsilon$. Call $\bar{\sigma}$ an $\mathcal{G}$-measurable family of ϵ-optimal strategies if $\bar{\sigma}$ is an $\mathcal{G}$-measurable mapping from F to $\mathcal{P}(H)$ such that for all f, $\bar{\sigma}(f)$ is ϵ-optimal.

Lemma 4.4. For every $\epsilon > 0$, there is an $\mathcal{G}$-measurable family $\bar{\sigma}$ of ϵ-optimal strategies.

Proof: For every integer n, let

$$A_n = \{(f, \sigma): \sigma \in \Gamma^{\infty}(f) \text{ and } \sigma g > n\epsilon\}.$$

That A_n is Borel is clear from Lemmas 4.1 and 4.2. Let π be the projection mapping, $\pi(f, \sigma) = f$ and notice that $\pi(A_n) = \{f: W(f) > n\epsilon\}$. By [1, Prop. 15], there is an $\mathcal{G}$-measurable map $\bar{\sigma}_n: \pi(A_n) \to \mathcal{P}(H)$ such that $(f, \bar{\sigma}_n(f)) \in A_n$ for all $f \in \pi(A_n)$. Define $\bar{\sigma}(f)$ to equal $\bar{\sigma}_n(f)$ when $(n + 1)\epsilon \geq W(f) > n\epsilon$. As is easily verified, $\bar{\sigma}$ is an $\mathcal{G}$-measurable family of ϵ-optimal strategies. □

Let $I(g)$ be the collection of $\mathcal{G}$-measurable σ for which σg exists.

Lemma 4.5. Each of the following conditions on g implies its successor.

(a) g is invariant.

(b) $gp = g$ for all p.

(c) $\sigma(gp) = \sigma g$ for all p and all $\sigma \in I(gp)$.

(d) $\sigma[p]gp = \sigma[p]g$ for all p and all σ with $\sigma[p] \in I(gp)$.

(e) $\sigma g = \int \sigma[p_t(h)]g \, d\sigma(h)$ for all $\sigma \in I(g)$ and all σ-stopping times t.

(f) $\sigma g = \int \sigma[p_n(h)]g \, d\sigma(h)$ for all $\sigma \in I(g)$ and all integers n.

(g) $\sigma[f_1, \ldots, f_n]g \to g(f_1, f_2, \ldots)$ with σ-probability 1 for all σ for which g has finite expectation.

If g assumes only finite values, then the conditions are equivalent.

Proof: Assume (a). An induction on the length n of $p = (f_1, \ldots, f_n)$ establishes (b). Obviously, (b) implies (c). If (c) holds, then (c) holds with σ replaced by $\sigma[p]$, which yields (d). That (d) implies (e) is evident in the light of (3.2). Plainly, (e) $\to$ (f). If (f) holds and σg is finite, then the left-hand side of (e), being the integrand in the right-hand side of (f), is a version of the conditional expectation of g under σ given $f_1, \ldots, f_n$. Now Paul Levy's martingale convergence theorem applies and yields (g). Lastly, assume g has only finite values and suppose (g) holds. Fix $h = (f_1, f_2, \ldots)$, let σ_0 be $\delta(f_1)$, and let $\sigma_n(f_1, \ldots, f_n)$ be $\delta(f_{n+1})$, so σ is the one-point measure $\delta(h)$. The left-hand side of (g) is then equal to $g(f_{n+1}, f_{n+2}, \ldots)$ and, by (g), converges to $g(f_1, f_2, \ldots)$. Plainly, it then necessarily converges to $g(f_2, f_3, \ldots)$ also. So $g(f_1, f_2, \ldots)$ equals $g(f_2, f_3, \ldots)$, and (a) is established. □

The strategies σ and σ' agree prior to a stopping time t if $\sigma_0 = \sigma_0'$ and $\sigma_n(p_n(h)) = \sigma_n'(p_n(h))$ for all $h \in H$ such that $t(h) > n$.

Lemma 4.6. For $f \in F$, $\sigma \in \Gamma^\infty(f)$, and $t \in T(\sigma)$, the integral $\sigma W(f_t)$ exists and

$$(4.1) \qquad \sigma W(f_t) = \sup\{\sigma' g\colon \sigma' \in A(\sigma, t)\},$$

where $A(\sigma, t)$ is the set of σ' in $\Gamma^\infty(f)$ which agree with σ prior to t.

Proof: Let $\varepsilon > 0$, and choose $\bar{\sigma}$ as in Lemma 4.4. Let σ' be that element of $A(\sigma, t)$ such that $\sigma'[p_t(h)] = \bar{\sigma}(f_t(h))$ for $t(h) < \infty$. Calculate thus.

$$\begin{aligned}
+\infty &> W(f) \\
&\geq \sigma' g \\
&= \int \sigma'[p_t(h)] g \, d\sigma(h) \\
&= \int \bar{\sigma}(f_t(h)) g \, d\sigma(h) \\
&\geq \int W(f_t(h)) \, d\sigma(h) - \varepsilon.
\end{aligned}$$

This calculation demonstrates the existence of $\sigma W(f_t)$ and proves one of the inequalities needed to establish (4.1).

For the reverse inequality, let $\sigma' \varepsilon A(\sigma, t)$. Then

$$\begin{aligned}
\sigma' g &= \int \sigma'[p_t] g \, d\sigma(h) \\
&\leq \int W(f_t) \, d\sigma.
\end{aligned}$$ □

Corollary 4.1. W is excessive.

Proof: Use (2.1) and (4.1). □

That W dominates g is established next.

Lemma 4.7. For every $\mathcal{G}$-measurable σ available in Γ for which σg is finite finite,

(4.2) $\quad g \leq W_* \quad \sigma$-almost certainly.

Consequently, W dominates g.

Proof: Since $\sigma[f_1, \ldots, f_n] \in \Gamma^\infty(f_n)$, $\sigma[f_1, \ldots, f_n]g \le W(f_n)$, so (4.2) follows, as Lemma 4.5(g) makes evident. □

Lemma 4.8. If Q is excessive, and dominates g, then $Q \ge W$.

Proof: For $\sigma \in \Gamma^\infty(f)$,

$$\begin{aligned} Q(f) &\ge \sup_{t \in T(\sigma)} \sigma Q(f_t) \\ &\ge \sigma Q^* \\ &\ge \sigma g. \end{aligned}$$

The three inequalities hold because Q is excessive, by Lemma 3.1, and because Q dominates g, respectively. Since $W(f)$ is the supremum of σg over all such σ, $Q(f) \ge W(f)$. □

In view of Lemmas 4.3, 4.7, and 4.8, and Corollary 4.1, the proof of Theorem 1 is complete.

Remark. Lemma 4.8 does not require the assumption otherwise in force that g is Borel measurable; it suffices that σg exist for all $\mathcal{G}$-measurable σ available in Γ, for then W is well-defined.

§5. <u>A Characterization of Optimal Strategies</u>. The W which was characterized in the preceding section exists for each (Γ, g). In contrast, optimal strategies need not exist. But when they do exist, they can be characterized as in Theorem 2. Assume, until the proof of Theorem 2 is completed, that $f \in F$, $\sigma \in \Gamma^{\infty}(f)$, and σg is finite.

<u>Theorem 2</u>. For σ to be optimal at f it is necessary and sufficient that $\sigma g = \sigma W^*$ and any (all) of the following three conditions be satisfied.

(a) $\sigma W^* = W(f)$.

(b) $W(f), W(f_1), \ldots$ is a uniformly integrable martingale under σ.

(c) $W(f), W(f_1), \ldots$ is an L_1-bounded martingale under σ which satisfies:

$$\sigma(W(f_t)) \geq W(f) \quad \text{for all} \quad t \in T(\sigma). \tag{5.1}$$

To say that $W(f), W(f_1), \ldots$ is L_1-<u>bounded</u> under σ means, of course, that for some constant $K < \infty$, $\int |W(f_n)| d\sigma \leq K$ for all n.

The next three lemmas comprise part of the proof of Theorem 2.

<u>Lemma 5.1</u>. (i) $\sigma W^* \leq W(f)$ and, for all n and σ-almost every h, (ii) $\sigma[p_n(h)]W^* \leq W(f_n)$.

<u>Proof</u>: The expectation σW^* exists because σg is finite and $W^* \geq g$ σ-almost surely by Lemma 4.7. Inequality (i) now follows from Lemma 3.1 and Corollary 4.1. By Lemma 4.5(f), $\sigma[p_n(h)]g$ is finite with σ-probability one and, hence, (ii) follows from (i). □

As an immediate corollary to (4.2) and Lemma 5.1(i), one obtains a part of Theorem 2.

Corollary 5.1. For σ to be optimal at f it is necessary and sufficient that

(5.2) $\sigma g = \sigma W^{*}$ and $\sigma W^{*} = W(f)$.

Lemma 5.2. If $\sigma W^{*} = W(f)$, then, for all n and σ-almost all h,

(5.3) $\sigma[p_n(h)]W^{*} = W(f_n)$.

Proof: Calculate thus.

$$\begin{aligned} \sigma W^{*} &= \int \sigma[p_n(h)]W^{*}\, d\sigma(h) \\ &\leq \int W(f_n)\, d\sigma(h) \\ &\leq W(f) \\ &= \sigma W^{*}, \end{aligned}$$

where the first equality holds by Lemma 4.5(f); the first inequality is by Lemma 5.1(ii); the second inequality holds because W is excessive, and the final equality is by hypothesis. (5.3) now follows with the aid of Lemma Lemma 5.1(ii). □

Lemma 5.3. Conditions (a), (b) and (c) of Theorem 2 are equivalent.

Proof: Assume (a). Since σW^{*} is finite, $\sigma(W^{*}|f_1, \ldots, f_n)$ is a uniformly integrable martingale, as is well-known [7, V, T18). In view of Lemmas 4.5 and 5.2, this martingale is almost certainly the same as $W(f_n)$. So (b) holds. That (b) implies (c) is part of standard measure theory. Suppose now that (c) holds. Since W is excessive, (5.1) holds with equality and consequently, (b) too holds, as the Corollary in [4] asserts. Now assume that (b) holds. Then $W(f)$, $W(f_1)$, ... converges σ-almost surely to W^*, and $\sigma W^* = \lim \sigma(W(f_n))$. But $\sigma(W(f_n))$ is independent of n and equals $W(f)$. So (a) holds. □

Theorem 2 now follows from Corollary 5.1 and Lemma 5.3.

Corollary 5.2. If σ is available in Γ and σg is finite, then $W^* = W_* \ \sigma$-almost surely.

Proof: By Lemmas 5.1 and 4.7, the integrals σW^* and σW_* exist as finite numbers. Now use Corollary 4.1, and Lemmas 3.3 and 3.2. □

To be applied in §6 is this easy consequence of (4.2) and Lemma 5.1.

Lemma 5.4. For each $f \in F$ and all ϵ, $\delta > 0$, $\exists\, \sigma \in \Gamma^\infty(f)$ such that

$$(5.4) \qquad \sigma(g > W^* - \epsilon) \geq 1 - \delta.$$

In fact, (5.4) holds for all σ which are $\epsilon\delta$-optimal at f.

This section concludes with a result on ϵ-optimal stationary families.

Proposition 1. Let $\epsilon \geq 0$ and let γ be an $\mathcal{G}$-measurable Γ-selector. If γ^∞ is an ϵ-optimal family, then, for every f, (i) $\gamma^\infty(f)g \geq \gamma^\infty(f)W^* - \epsilon$ and (ii) the process $\{W(f_n)\}$ is uniformly integrable under $\gamma^\infty(f)$. Conversely, if, for all f, (i) and (ii) hold and, in addition, (iii) $\gamma(f)W = W(f)$, then γ^∞ is ϵ-optimal.

Proof: First assume (i), (ii) and (iii) for all f. Then, $W(f)$, $W(f_1)$, ... is a uniformly integrable martingale under $\gamma^\infty(f)$ and, hence, $\gamma^\infty(f)W^* = W(f)$. Use (i) to conclude that $\gamma^\infty(f)$ is ϵ-optimal.

For the other implication, assume γ^∞ is everywhere ϵ-optimal and compute as follows:

$$(5.5) \qquad \gamma^\infty(f)W^* \geq \gamma^\infty(f)g \geq W(f) - \epsilon \geq \gamma^\infty(f)W^* - \epsilon.$$

The first inequality is by Lemma 4.7, the second is by the assumed ϵ-optimality, and the third by Lemma 5.1. Condition (i) is now clear, and (ii) is a consequence of (5.5) and the next lemma, which applies to all Markov kernels γ.

Lemma 5.5. Let $Q:F \to R$ and $\gamma:F \to \mathcal{P}$ be analytically measurable. If Q^* has a finite integral under every $\gamma^\infty(f)$ and

$$\sup_f |\gamma^\infty(f)Q^* - Q(f)| < \infty,$$

then $Q(f_1)$, $Q(f_2)$, ... is uniformly integrable under $\gamma^\infty(f)$ for all f.

Proof: Plainly, $\gamma^\infty(f_n)Q^*$ is a version of the conditional expectation $\gamma^\infty(f)(Q^*|f_1, \ldots, f_n)$, which sequence is uniformly integrable because Q^* has a finite $\gamma^\infty(f)$-integral [7, V, T 18]. Since $Q(f_n)$ differs from $\gamma^\infty(f_n)Q^*$ by at most a fixed constant, $Q(f_n)$ too is uniformly integrable under $\gamma^\infty(f)$. □

Except for the special class of problems studied in §7, we do not know necessary and sufficient conditions for the existence of ϵ-optimal stationary families.

§6. Conventional utilities. Throughout this section and the next, u is a fixed, real-valued, Borel measurable function defined on F and g is specialized to be the conventional utility u^* where $u^*(h) = \limsup u(f_n)$. The purpose of this section is to record several facts for Borel houses Γ with conventional utilities u^*.

Suppose, for the next two lemmas, that σ is $\mathcal{G}$-measurable and σu^* is finite.

Lemma 6.1. If σ is available in Γ, then

$$(6.1) \qquad \sigma(\liminf (W(f_n) - u(f_n)) \leq 0) = \sigma(W^* = u^*).$$

Proof: By Lemmas 4.7 and Corollary 5.2,

$$(6.2) \qquad u^* \leq W_* = W^*$$

almost surely under σ. A routine calculation now suffices to deduce (6.1). □

Lemma 6.2. For σ in $\Gamma^\infty(f)$ to be optimal at f it is necessary and sufficient that $W(f), W(f_1), \ldots$ be a uniformly integrable martingale under σ and

$$(6.3) \qquad \sigma(\liminf (W(f_n) - u(f_n)) \leq 0) = 1.$$

Proof: Use (4.2), (6.1), and Theorem 2. □

For $\epsilon > 0$, define

$$A_\epsilon = \{f: u(f) \geq W(f) - \epsilon\}, \text{ and } B_\epsilon = \{h: \exists k \ni f_k \in A_\epsilon\}.$$

In view of Lemma 4.3, A_ϵ is analytically measurable, from which it easily follows that B_ϵ too is analytically measurable.

Lemma 6.3. For each $f \in F$ and $\epsilon, \delta > 0$, there is a $\sigma \in \Gamma^{\infty}(f)$ such that $\sigma(B_\epsilon) \geq 1 - \delta$.

Proof: Plainly, $B_\epsilon \supseteq (u^* > W^* - \epsilon)$. Now use Lemma 5.3. □

An examination of the proof of Lemma 6.3 reveals that it is applicable to various other g's, for instance, to $\bar{u}$ and u_* where:

$$\bar{u}(h) = \limsup \frac{u(f_1) + \ldots + u(f_n)}{n}; \; u_*(h) = \liminf u(f_n).$$

However, this is in contrast to Lemma 6.2 which would be false were g equal to either $\bar{u}$ or u_*, as Example 7.3 shows.

Suppose γ is an $\mathcal{G}$-measurable Γ-selector, and define

$$(6.4) \qquad \begin{aligned} Q_\epsilon(f) &= 1 \quad \text{for } f \in A_\epsilon, \\ &= \gamma^{\infty}(f)(B_\epsilon) \quad \text{for } f \in F - A_\epsilon. \end{aligned}$$

Lemma 6.4. Each Q_ϵ is universally measurable, and

$$(6.5) \qquad \gamma(f)Q_\epsilon = Q_\epsilon(f) \quad \text{for } f \in F - A_\epsilon.$$

Proof: Let B_ϵ^n be the event that, for some $k \leq n$, $f_k \in A_\epsilon$, and let $Q_\epsilon^n(f) = \gamma^{\infty}(f)(B_\epsilon^n)$. As [1, Cor. 41] implies, each Q_ϵ^n is universally measurable. Therefore, so is $\lim_n Q_\epsilon^n$. Since this limit agrees with Q_ϵ on $F-A_\epsilon$, Q_ϵ is also universally measurable. It is now simple to verify (6.5). □

Lemma 6.5. For any $\epsilon > 0$ for which Q_ϵ is excessive for Γ, $Q_\epsilon \equiv 1$ and

(6.6) $\gamma^\infty(f)(\underline{\lim}\,(W(f_n) - u(f_n)) \leq \epsilon) = 1,$

for all f. Consequently, for any such ϵ, $\gamma^\infty(f)(u^* \geq W_* - \epsilon) = 1$.

Proof: Consider a new gambling problem (Γ^L, v^*) where Γ^L is the leavable closure of Γ, that is, $\Gamma^L(f) = \Gamma(f) \cup \{\delta(f)\}$ for all f, and where v is the indicator function of A_ϵ. Because Q_ϵ is excessive for Γ, it is also excessive for Γ^L. Obviously, $Q_\epsilon \geq v$. So $Q_\epsilon^* \geq v^*$; that is, Q_ϵ dominates v^*. By Lemma 4.8 and the remark which follows it, W^L, the optimal return function for (Γ^L, v^*), is well-defined and $Q_\epsilon \geq W^L$. Thus, to show $Q_\epsilon \equiv 1$, it suffices to show $W^L \equiv 1$. Let $f \in F$, $\delta > 0$ and, by Lemma 6.3, choose $\sigma \in \Gamma^\infty(f)$ such that $\sigma(B_\epsilon) \geq 1 - \delta$. Define σ' to be that strategy which agrees with σ prior to the time of first entrance into A and which then stagnates. Calculate thus: $W^L(f) \geq \sigma' v^* = \sigma'(B_\epsilon) \geq 1 - \delta$. This completes the proof that $Q_\epsilon \equiv 1$. As is now easily seen, for example, with the help of [2, Exercise 9, Chapter 5], for every f, the $\gamma^\infty(f)$-probability that $f_k \in A_\epsilon$ for infinitely many k is 1. This completes the proof of (6.6). □

§7. Stop-or-go houses with a conventional utility. In this final section, u and the utility $g = u^*$ are as in §6, but Γ is now specialized to be a Borel stop-or-go house, that is,

$$\Gamma(f) = \{\alpha(f), \delta(f)\} \quad \text{for } f \in D,$$
$$= \{\alpha(f)\} \quad \text{for } f \in F-D,$$

where D is a Borel subset of F, possibly equal to F, and α is a Borel mapping from F to $\mathcal{P}$. Plainly, the mapping $f \to \delta(f)$ is a continuous and, hence, Borel mapping from F to $\mathcal{P}$. The graphs of the Borel mappings α and δ are Borel subsets of $F \times \mathcal{P}$ [9, Theorem I.3.3]; so their union, namely Γ, is also.

Let γ be an $\mathcal{G}$-measurable Γ-selector. The associated stationary family γ^∞ is called promising if (i) $\gamma(f) = \delta(f)$ implies $u(f) = W(f)$ and, (ii) for all f, $\gamma(f)W = W(f)$.

Lemma 7.1. If γ agrees with α on $F - A_\epsilon$, in particular, if γ^∞ is promising, then Q_ϵ is excessive for Γ, where Q_ϵ is defined in (6.4).

Proof: By (6.5), $\alpha(f)Q_\epsilon = Q_\epsilon(f)$ for $f \epsilon F-A_\epsilon$. For all other (γ, f) with $\gamma \epsilon \Gamma(f)$, it is trivial that $\gamma Q_\epsilon \leq Q_\epsilon(f)$. Thus, under every available σ, the process $Q_\epsilon(f), Q_\epsilon(f_1), \ldots$ is a bounded supermartingale and therefore, $\sigma Q_\epsilon(f_t) \leq Q_\epsilon(f)$ for every $t \in T(\sigma)$ [7, V, T 28]. □

Theorem 3. For every $\mathcal{G}$-measurable Γ-selector γ, these two conditions are equivalent.

(a) γ^∞ is everywhere optimal.

(b) γ^∞ is promising and $W(f_1), W(f_2), \ldots$ is uniformly integrable under every strategy $\gamma^\infty(f)$.

Proof: Assume (a). By Theorem 2, $W(f)$, $W(f_1)$, ... is a uniformly integrable martingale under $\gamma^\infty(f)$. Plainly then $\gamma(f)W = W(f)$. To complete the proof that γ^∞ is promising, let $\gamma(f) = \delta(f)$, and verify that $u(f) = \gamma^\infty(f)u^* = W(f)$. Therefore, (b) holds.

Now assume (b). By Lemmas 7.1 and 6.5, Formula (6.3) plainly holds. Furthermore, $W(f)$, $W(f_1)$, ... is a martingale under $\gamma^\infty(f)$ because γ^∞ is promising. So, by Lemma 6.2, $\gamma^\infty(f)$ is optimal at f. □

There exist (Γ, u^*) for which no stationary family is optimal as Example 7.1 below illustrates. Consequently, there is interest in the possible existence of ϵ-optimal stationary families. Define

$$(7.1) \qquad \gamma_\epsilon(f) = \delta(f) \quad \text{if} \quad u(f) \geq W(f) - \epsilon \quad \text{and} \quad \delta(f)\epsilon\Gamma(f),$$
$$= \alpha(f) \quad \text{otherwise.}$$

Proposition 2. For each $\epsilon \geq 0$, these three conditions are equivalent:

(a) There is available an $\mathcal{G}$-measurable, ϵ-optimal stationary family.

(b) The family γ_ϵ^∞ is ϵ-optimal.

(c) For every f, $\{W(f_n)\}$ is uniformly integrable under $\gamma_\epsilon^\infty(f)$.

Proof: First assume (a) and let γ^∞ be an $\mathcal{G}$-measurable, ϵ-optimal stationary family. By Proposition 1 of §5, $\{W(f_n)\}$ is uniformly integrable under each $\gamma^\infty(f)$. Since, in addition, $\gamma_\epsilon(f) = \gamma(f)$ whenever $\gamma_\epsilon(f) \neq \delta(f)$, (c) holds.

Observe that $\gamma_\epsilon(f)W = W(f)$ for all f and, therefore, the implication

(c) ⇒ (b) will follow from Proposition 1 once it is verified that

(7.2) $\qquad \gamma_\epsilon^\infty(f)[u^* \geq W^* - \epsilon] = 1$

for all f. To check (7.2), use Lemma 7.1 to see that Q_ϵ is excessive and then use Lemma 6.5 together with the fact that $W^* = W_*$ $\gamma_\epsilon^\infty(f)$-almost surely as follows from the martingale convergence theorem [7, V, T 17].

Since (b) obviously implies (a), the proof is complete. □

Here are two examples in which F is the set of nonnegative integers and u is the identity function. In the first, there is no optimal stationary family and yet, for every $\epsilon > 0$, there is an ϵ-optimal stationary family. In the second, for some $\epsilon > 0$, there is not even an ϵ-optimal stationary family.

Example 7.1. Let $\Gamma(0) = \{\delta(0)\}$ and, for $n > 0$, let $\Gamma(n) = \{\delta(n), \alpha(n)\}$, where $\alpha(n) = (\frac{1}{2} + 1/(n+1)^2)\delta(2n) + (\frac{1}{2} - 1/(n+1)^2)\delta(0)$.

Example 7.2. Let $\Gamma(0) = \{\delta(0)\}$ and, for $n > 0$, let $\Gamma(n) = \{\delta(n), \alpha(n)\}$, where $\alpha(n) = \frac{1}{2}\delta(2n+1) + \frac{1}{2}\delta(0)$.

The following example, noticed during a conversation with David Gilat, shows that Theorem 3 would not hold if the conventional payoff u^* were replaced, say, by $\underline{u}$ or by u_*.

Example 7.3. Let $F = \{1, 2, \ldots\}$; $u(f) = 1 - 1/f$ for $f \in F$; $\Gamma(f) = \{\alpha(f), \delta(f)\}$ for every f where $\alpha(f) = \delta(1)$ for $f > 1$ and $\alpha(1)$ gives positive measure to every set of the form $\{f, f+1, \ldots\}$. If $g = \overline{u}$ or if $g = u_*$, then $W \equiv 1$ and the family α^∞ is promising, but not optimal. In fact, there are no optimal strategies.

Some comment on the relationship of the work in this paper to the formulation of optimal stopping problems in [3] is in section 4 of [16]. Except for measurability technicalities, the stopping problem of [3] corresponds to a leavable, stop-or-go problem with a conventional utility.

References

[1] Blackwell, D., Freedman, D., and Orkin, M. (1974). The optimal reward operator in dynamic programming. Ann. Prob. 2, 926-941.

[2] Breiman, Leo (1968). Probability. Addison-Wesley Publishing Co., Reading, Mass.

[3] Chow, Y. S., Robbins, Herbert, and Siegmund, David (1971). Great Expectations: The Theory of Optimal Stopping. Houghton-Mifflin, Boston.

[4] Dubins, Lester E., and Freedman, David A. (1966). On the expected value of a stopped martingale. Ann. Math. Statist. 37, 1505-1509.

[5] Dubins, Lester E., and Savage, Leonard J. (1976). Inequalities for Stochastic Processes (How to Gamble if You Must). Dover, New York.

[6] Dvoretzky, Aryeh (1976). On stopping time directed convergence. Bull. Amer. Math. Soc. 82, 347-349.

[7] Meyer, Paul A. (1966). Probability and Potentials. Blaisdell, Waltham, Mass.

[8] Neveu, Jacques (1965). Mathematical Foundations of the Calculus of Probability. Holden-Day, San Francisco.

[9] Parthasarathy, K. R. (1967). Probability Measures on Matric Spaces. Academic Press, New York.

[10] Siegmund, David Oliver. (1967). Some problems in the theory of optimal stopping rules. Ann. Math. Statist. 38, 1627-1640.

[11] Snell, J. L. (1952). Applications of martingale system theorems. Trans. Amer. Math. Soc. 73, 293-312.

[12] Strauch, Ralph E. (1967). Measurable gambling houses. Trans. Amer. Math. Soc. 126, 64-72.

[13] Sudderth, W. D. (1969). On the existence of good stationary strategies. Trans. Amer. Math. Soc. 135, 399-414.

[14] ______________ (1971). On measurable gambling problems. Ann. Math. Statist. 42, 260-269.

[15] ______________ (1971). A "Fatou Equation" for randomly stopped variables. Ann. Math. Statist. 42, 2143-2146.

[16] ______________ (1971). A gambling theorem and optimal stopping theory. Ann. Math. Statist. 42, 1697-1705.

[17] ______________ (1972). On the Dubins and Savage characterization of optimal strategies. Ann. Math. Statist. 43, 498-507.