August 1963

A SURVEILLANCE MODEL: TWO MACHINE CASE

Vidya Sagar Taneja [1/]

Technical Report No. 30

University of Minnesota

Minneapolis, Minnesota

[1/] Work done in part under contract Nonr-710(31), NR 042 003 of the Office of Naval Research.

ERRATA SHEET

for

A SURVEILLANCE MODEL: TWO MACHINE CASE

Department of Statistics, University of Minnesota, Technical Report No. 30

by Vidya Sagar Taneja

NOTATION: An underscored line number (e.g., page 2, line <u>4</u>, means the correction is to be made in the 4th line from the bottom of page 2), etc.

Page 1, line 2 "consist" should read "consists".

Page 2, line <u>6</u> "The machines (components) change states independently of each other." delete the entire sentence.

Page 3, line 2 "follow" should read "follows".

Page 3, line <u>3</u> "ubounded" should read "unbounded".

Page 4, line 4 "$0 \le x < b;\ 0 \le y < d$" should read "$0 \le x \le b;\ 0 \le y \le d$".

Page 4, line 12 "In the more general case where at the time..." should read "In a more general case, at the time...".

Page 5, Lemma 1 At the beginning of the statement, insert "If $E(T) < \infty$, then"

Page 5, Proof At the beginning of the proof, insert "If $\sum_{(x, y) \in B} F_G(x, y) < 1$, then with positive probability the cycle may not terminate, and in this case $E(T) = \infty$.".

Page 5, line <u>6</u> "Since" should read "Hence".

Page 6, line 4 "exist" should read "exists".

Page 11, lines <u>2</u> and <u>1</u> "and U are any constants. Then if..." should read "and U is a constant. If...".

Page 15, line 5 "is greater than the" should read "is greater than or equal to the"

Page 15, line <u>4</u> "because" should read "if".

Page 15, line <u>2</u> "Therefore" should read "Then".

Page 17, line 9 "z(t) process" should read "production process".

Page 17, line 12 "z(t) process" should read "production process".

Page 19, line 3 "provide" should read "with".

Page 22, line <u>3</u> "1.e." should read "i.e.".

A SURVEILLANCE MODEL: TWO MACHINE CASE

Vidya Sagar Taneja

0. Summary.

This paper deals with an economic model for the surveillance of a production process. The process consits of a single machine with two components or of two machines. The output of the process is a single stream of goods. At each instant the distribution of the quality of the output depends on the current state of the two components. These components are assumed to be statistically independent. The production process has the tendency to wear after each adjustment. This paper deals with "continuous surveillance" where it is possible to observe the production process without cost at all times of production. In this case optimal strategies are found; a strategy which tells the producer when to make adjustments.

In Section 1, the model is described in detail. It is assumed that the production process is a two dimensional random walk with state space a lattice of points in the plane. This walk has the property that in a transition the system moves one unit either to the right with probability p or upwards with probability $q=1-p$. It is proved that if $P^*(x, y)$ denotes the steady state probabilities of being at (x, y), then

$$P^*(x, y) = P^*(0, 0) P(x, y)$$

where $P(x, y)$ denotes the probability that a path starting at $(0, 0)$ goes through (x, y) before the process stops for adjustments (Lemma 2).

The set of points on which the random walk occurs is called the continuation set and is denoted by C. When $x(t)=x$, $y(t)=y$; $i(x, y)$ denotes the income per unit time. C is optimal if and only if it maximizes the long run income per

unit time. Let K denote the cost of repair per unit time. It is proved in Section 2, that a sufficient conditon that the optimal set C be finite and non empty is that $\limsup i(x, y) < -K$ and that $i(0, 0) > -K$.

Section 2 also deals with the properties of optimal C. A continuation set C is full if $(x, y) \in C$ implies that $(x', y') \in C$ for $0 \leq x' \leq x$, $0 \leq y' \leq y$. When $i(x, y)$ is non increasing in each coordinate, it is proved that the optimal C is full (Theorem 1) and is of the form C_λ, where

$$C_\lambda = \{(x, y) | i(x, y) \geq \lambda\}$$

and λ is real (Theorem 3).

Finally Section 3 considers in detail an application of the above model. It is assumed that the components of the production process $(x(t), y(t))$ are independent Poisson processes. It is indicated how the discrete model is applicable to the continuous case. Section 3 also contains numerical examples.

1. Introduction.

This paper deals with a production process which tends to wear unless repairs are made. Under the assumption that the process is kept under surveillance, optimal strategies are found. An optimal strategy is a rule which tells the producer when to stop production and make repairs in order to maximize the long run average income per unit time.

We consider the case where the production process is vector valued. To be specific, mathematical procedures for the case where the production process consists of two machines (two components of the same machine) are developed. The machines (components) change states independently of each other. Production is continued when neither of the two machines (components) are in repair.

The following assumptions are made concerning the inspection procedure:

(1.1a) It is assumed that the results of the inspection are available immediately.

(1.1b) It is assumed that the decision to continue production or stop and repair follow immediately after inspection.

The strategy consists in specifying a set of points C (with non negative integer coordinates) in the plane. As long as $(x(t), y(t)) \in C$, production is continued. As soon as $(x(t), y(t)) \notin C$, the production process is stopped and repairs begin.

It is assumed that the production process is a two dimensional random walk with $x(0)=y(0)=0$. This random walk consists of moving to the right one unit with probability p or moving upwards with probability $q=1-p$, for all points of C. A realization of this random walk is a 'path' in the plane. When the random walk (or more exactly the path of the random walk) leaves C, it returns to (0, 0). The production process is a discrete parameter Markov chain with transition probabilities:

$$(1.2)\quad \begin{cases} P\{(j, k), (j+1, k)\} = p \; ; \quad P\{(j, k), (j, k+1)\} = 1-p = q \; , \quad j,k=0,1,2,\ldots \\ \text{and all other transition probabilities are zero except that the process} \\ \text{returns to } (0, 0) \text{ with probability one as soon as it leaves } C. \end{cases}$$

The only form of surveillance considered in this paper is "continuous surveillance", where it is possible to observe the production process without cost at all steps of production. The basic problem is to form a strategy (or equivalently a set C described above) as to when to stop production and send the machines (components) to repair. Associated with each point (x, y) in the plane is a number i(x, y) called the "income". For most of the results it is assumed that for $x,y \geq 0$, i(x, y) is non increasing in each coordinate and may be undounded below in each coordinate. Hence the most desirable state is at (0, 0). Some interesting forms of i(x, y) are:

$$(1.3a)\quad i(z) = i(x, y) = \begin{cases} 0 & x \text{ or } y < 0 \\ A-Bx-Dy & x \geq 0; \quad y \geq 0; \quad A \geq 0; \quad B > 0; \quad D > 0 \end{cases}$$

$$(1.3b) \quad i(z) = i(x, y) = \begin{cases} 0 & x \text{ or } y < 0 \\ A-Bxy & x \geq 0; \; y \geq 0; \; A \geq 0; \; B > 0 \end{cases}$$

$$(1.3c) \quad i(z) = i(x, y) = \begin{cases} 0 & x \text{ or } y < 0 \\ A & 0 \leq x < b; \; 0 \leq y < d; \; b > 0; \; d > 0; \; A \geq 0 \\ 0 & x > b \text{ or } y > d \end{cases}$$

If the production is stopped when $x(t)=x$, $y(t)=y$; then it is assumed that m "time" units will be required to bring the process to $x(0)=y(0)=0$ through repair and the cost per unit time of repair is K. For the purpose of Section 1 and 2, "time" is measured in terms of steps during the random walk and m is expressed in the same units. The case where m and K depend on x, y, is not considered. The objective is to maximize the long run average income per unit time.

In the more general case where at the time when the production is stopped, we make one of the following decisions: 1) repair both the machines (components), i.e., bring the process back to (0, 0), 2) repair one of the two machines (components), i.e., project either on the x-axis or y-axis (in cases where repair is very costly) and start production. This general case is not considered in this paper.

Definition 1: The set of points C on which the random walk occurs is called the continuation set. Every non empty continuation set includes (0, 0). A set in C consisting of points which cannot be reached by paths from the origin through points of C is said to be a null set in C.

Definition 2: The set of points characterized by the following two properties and denoted by B is the boundary of the set C:

(a) The process reaches a point of B as soon as it leaves C.

(b) The process stops for repair as soon as it reaches any point of B.

Let T denote the length of the path, i.e., the number of steps required to reach B, and I be the income associated with a path, i.e., the sum of values of $i(x, y)$ over all points in C which the path passes through. A fundamental economic quantity is,

$$\frac{E(I)-mK}{E(T)+m} . \tag{1.4}$$

This corresponds to the long run income per unit of "time" (see Johns and Miller [1963]). The value of the fraction (1.4) depends on the choice of C. An optimal strategy consists of finding a C which maximizes the above quantity. In this paper we give several properties of such optimal C's including algorithms helpful in finding the best choice of C as well as the maximum income per unit time.

Definition 3: A cycle is the "time" from beginning production, i.e., starting at (0, 0), through repair until the recurrence of that event, i.e., the beginning of production.

Let $P_C(x, y)$ denote the probability that a path starting from (0, 0) goes through (x, y) before reaching B. Then

Lemma 1:

$$E(T) = \sum_{(x, y) \in C} P_C(x, y) = \sum_{(x, y) \in B} (x+y) P_C(x, y) . \tag{1.5}$$

Proof:

Since $\sum_{(x, y) \in B} P_C(x, y) = 1$, i.e., the process reaches B with probability one and the number of steps required to reach any point $(\bar{x}, \bar{y}) \in B$ is $(\overline{x+y})$. it is clear that

$$E(T) = \sum_{(x, y) \in B} (x+y) P_C(x, y) .$$

For each point the path goes through before leaving the continuation set, the number of steps to reach the boundary of C is increased by one. Hence we have

$$E(T) = \sum_{(x, y) \in C} P_C(x, y) .$$

This proves the result.

Let $P^*_C(x, y)$ denote the steady state probability of being at (x, y) in C. To see that $P^*_C(x, y)$ exist we note that the Markov chain is irreducible, with all states ergodic and therefore possesses a unique stationary distribution (see Theorem on page 356 of Feller [1957]).

<u>Lemma 2:</u>

$$P^*_C(x, y) = P^*_C(0, 0)\, P_C(x, y) .$$

<u>Proof:</u>

Suppose we observe a very large number N of paths. For n=1,2,3,...,N; let t_n be the number of steps to complete the n^{th} path. Observe that t_n does not include the repair "time". Let

$$L_{x,y,n} = \begin{cases} 1 & \text{if } n^{th} \text{ path goes through } (x, y) \\ 0 & \text{otherwise.} \end{cases}$$

note that $L_{0,0,n} \equiv 1$.

If $P'_C(x, y)$ denotes the observed proportion of steps at (x, y) in C, then

(1.6) $$P'_C(x, y) = \frac{\sum_{n=1}^{N} L_{x,y,n}}{\sum_{n=1}^{N} t_n} = \frac{\sum_{n=1}^{N} L_{x,y,n} \Big/ N}{\sum_{n=1}^{N} t_n \Big/ N} .$$

Note that

(1.7) $$P'_C(0, 0) = \frac{1}{\sum_{n=1}^{N} t_n \Big/ N} .$$

But the long run value of $P'(x, y)$ is $P^*(x, y)$ (Chung [1960]) and by the strong law of large numbers (Feller [1957] p. 374),

$$\lim_{N \to \infty} \left[\frac{1}{N} \sum_{n=1}^{N} t_n \right] = E(T) \tag{1.8a}$$

and

$$\lim_{N \to \infty} \left[\frac{1}{N} \sum_{n=1}^{N} L_{x,y,n} \right] = P_C(x, y) \, . \tag{1.8b}$$

Therefore from (1.6), (1.7), (1.8a) and (1.8b),

$$P_C^*(0, 0) = \frac{1}{E(T)} , \qquad P_C^*(x, y) = \frac{P_C(x, y)}{E(T)}$$

or

$$P_C^*(x, y) = P_C^*(0, 0) \, P_C(x, y) \, .$$

In general $P_C(x, y) = np^x q^y$, where n=(number of paths from (0, 0) to (x, y)) and if all paths are possible, i.e., if the set C is full, then

$$P_C(x, y) = \binom{x+y}{x} p^x q^y \, . \tag{1.9}$$

If the region C is not full, the computation of $P_C(x, y)$ can be tedious.

Let I_n and T_n denote respectively the "observed" income and "observed" length in the $n^{\underline{th}}$ cycle. Define

$$I_n(C) = \frac{\sum_{n=1}^{N} I_n}{\sum_{n=1}^{N} T_n} , \qquad \text{for } n=1,2,3,\ldots \, .$$

Then Johns and Miller [1963] have shown that with probability one the limit of $I_n(C)$ exists. Call this limit $I(C)$. They have shown that

$$I(C) = \frac{E(I)-mK}{E(T)+m}$$

with probability one. Hence from (1.5)

$$\text{(1.10)} \qquad I(C) = \frac{\sum_{(x,\,y)\,\epsilon\,C} P_C(x,\,y)\; i(x,\,y) - mK}{\sum_{(x,\,y)\,\epsilon\,C} P_C(x,\,y) + m}$$

with probability one.

2. Properties of optimal C.

Lemma 3:

There exists a continuation set C such that

$$I(C) \geqq -K\ .$$

Proof:

Consider the empty continuation set, C_0. In this case the process starts at (0, 0) and goes to repair immediately; comes back to (0, 0) after repair and goes back to repair without any production. In fact the process always remains in repair. From (1.10),

$$I(C_0) = \frac{0-mK}{0+m} = -K\ .$$

This proves Lemma 3.

Definition 4: A continuation set C is said to be optimal if and only if it maximizes the long run income per unit time.

Lemma 4:

If

$$\lim\sup\ i(x,\,y) < -K\ ,$$

then there exists a finite optimal set.

Proof:

Since $\lim\sup\ i(x,\,y) < -K$, there exists R^2 such that $i(x,\,y) < -K$, if $x^2+y^2 > R^2$. Let $\overline{C}_R = \{(x, y) | x^2+y^2 > R^2\}$ and C_R be its complement. Let C^* be the optimal set. Define

$$C_1 = C^*C_R \; ; \qquad C_2 = C^*\overline{C}_R \, .$$

From Lemma 3, we have

$$\frac{\sum_{(x, y) \in C_1} P_{C*}(x, y)\; i(x, y) + \sum_{(x, y) \in C_2} P_{C*}(x, y)\; i(x, y) - mK}{\sum_{(x, y) \in C_1} P_{C*}(x, y) + \sum_{(x, y) \in C_2} P_{C*}(x, y) + m} \geq -K$$

or

$$\sum_{(x, y) \in C_1} P_{C*}(x, y)[i(x, y)+K] + \sum_{(x, y) \in C_2} P_{C*}(x, y)[i(x, y)+K] \geq 0.$$

But for each $(x, y) \in C_2$, $i(x, y) < -K$, therefore

(2.1) $$\sum_{(x, y) \in C_1} P_{C*}(x, y)]i(x, y)+K] \geq 0.$$

Now

$$I(C_1) \geq I(C^*)$$

or

$$\frac{\sum_{(x, y) \in C_1} P_{C*}(x, y)\; i(x, y) - mK}{\sum_{(x, y) \in C_1} P_{C*}(x, y) + m}$$

$$\geq \frac{\sum_{(x, y) \in C_1} P_{C*}(x, y)\; i(x, y) + \sum_{(x, y) \in C_2} P_{C*}(x, y)\; i(x, y) - mK}{\sum_{(x, y) \in C_1} P_{C*}(x, y) + \sum_{(x, y) \in C_2} P_{C*}(x, y) + m}$$

if

$$\sum_{(x, y) \in C_2} P_{C*}(x, y)[\sum_{(x, y) \in C_1} P_{C*}(x, y)\; i(x, y) - mK]$$

$$\geq \sum_{(x, y) \in C_2} P_{C*}(x, y)\; i(x, y)[\sum_{(x, y) \in C_1} P_{C*}(x, y) + m]$$

or, if

$$(2.2) \qquad \sum_{(x,\, y)\, \epsilon\, C_1} P_{C*}(x, y)\; i(x, y) \;\geq\; -K \sum_{(x,\, y)\, \epsilon\, C_1} P_{C*}(x, y) \;.$$

(2.2) is true because of (2.1). Hence

$$I(C_1) \;\geq\; I(C*) \;.$$

But C* is optimal, therefore

$$I(C_1) \;=\; I(C*) \;.$$

Note that C_1 is finite. This proves Lemma 4.

Lemma 5:

A sufficient condition for the optimal set to be non empty is that

$$i(0, 0) \;>\; -K \;.$$

Proof:

Consider the continuation set C = {(0, 0)}. From (1.10),

$$I(C) \;=\; \frac{i(0,\, 0)-mK}{1+m} \;>\; -K \qquad (\text{since } i(0, 0) > -K).$$

Hence the empty set cannot be the optimal set (Lemma 3).

From Lemmas 4 and 5, we conclude that a sufficient condition for the optimal set to be finite and non empty is that

$$(2.3) \qquad \begin{cases} \limsup\, i(x, y) < -K, & \text{and that} \\ i(0, 0) > -K \;. & \end{cases}$$

Definition 5: A set C is said to be full if (x, y) ϵ C implies (x', y') ϵ C for $0 \leq x' \leq x$, $0 \leq y' \leq y$.

Theorem 1:

If i(x, y) is non increasing in each coordinate and lim sup i(x, y) < -K, then an optimal C is full with the exception of null sets.

Proof:

Since exclusion of null sets does not change the value of I(C), they can be disregarded.

Let I* be the maximum of I(C), i.e., the value of I(C) for optimal C.

At this point optimal C and I* are unknown, but it is assumed that they exist (Lemma 4). Then for any other continuation set C

$$I^* - \frac{\sum_{(x, y) \in C} P_C(x, y)\, i(x, y) - mK}{\sum_{(x, y) \in C} P_C(x, y) + m} \geq 0 .$$

Or

$$(2.4) \qquad \sum_{(x, y) \in C} P_C(x, y)[i(x, y) - I^*] - m(K+I^*) \leq 0.$$

Equality holds when and only when C is optimal. From (2.4) we notice that

(a) for C optimal, $i(x, y) - I^* \geq 0$, $(x, y) \in C$. Otherwise if for some $(x^*, y^*) \in C$, $i(x^*, y^*) - I^* < 0$, then since $i(x, y)$ is non increasing in each coordinate, we can increase the value of the left hand side of (2.4) by taking out of C, the point (x^*, y^*) and all other points which can be reached by paths through (x^*, y^*).

(b) again the value of the left hand side of (2.4) would be increased by adding to C any point such that $i(x, y) - I^* > 0$. This follows since $i(x, y)$ is non increasing in each coordinate.

These two steps assure that optimal set C is full.

Let the optimal C be denoted by C* and I(C*) = I*.

Corollary 1:

Suppose there are n points for which $i(x, y) = U$, where $n \geq 0$ and U are any constants. Then if one or more of these n points are included in C*, then all the n points are included in C*, i.e., C* is determined by the contours of the income function $i(x, y)$.

Proof:

From (2.4), for any C,

$$(2.4') \qquad \sum_{(x,\, y) \notin C} P_C(x, y)[i(x, y)-I^*] - m(K+I^*) \leq 0$$

with equality for optimal C.

From part (a) of the proof of Theorem 1,

$$i(x, y) \geq I^*, \quad \text{for} \quad (x, y) \in C^* .$$

Now suppose that a part of a particular contour of $i(x, y)$ is included in C*. Then for each point of this contour, $i(x, y) \geq I^*$. Hence the left hand side of (2.4') can only increase by adding to C* the points of this particular contour which are not included in C*. This proves the result.

Theorem 2:

If $i(x, y)$ is non increasing in each coordinate, then for each choice of λ, the set C_λ, where

$$C_\lambda = \{(x, y) | i(x, y) \geq \lambda\}$$

is full.

Proof:

Let $(x^*, y^*) \in C_\lambda$. Then since $i(x, y)$ is non increasing in each coordinate, it follows from the definition of the set C_λ that $(x', y') \in C_\lambda$ where

$$0 \leq x' \leq x^* \quad \text{and} \quad 0 \leq y' \leq y^*$$

which assures that C_λ is full.

Theorem 3:

If $i(x, y)$ is non increasing in each coordinate, then the optimal choice of C is a C_λ.

Proof:

Let C* denote the optimal set. From Theorems 1 and 2, C* and all C_λ are full. From Corollary 1, we know that the optimal set is determined by the contours of the income function. Hence it follows that optimal C is of the form

(2.5) $$C_\lambda = \{(x, y) | i(x, y) \geq \lambda\}$$

which proves the theorem.

Lemma 6:

If

$$\frac{N-ab}{D-a} \geq \frac{N}{D},$$

then

$$\frac{N}{D} \geq \frac{N+a'b'}{D+a'}$$

where $D > 0$, $a > 0$, $a' > 0$ and $b \geq b'$.

Proof:

From the given condition

$$-abD \geq -aN ,$$

therefore, since $a > 0$,

(2.6) $$N \geq bD.$$

Now

$$\frac{N}{D} \geq \frac{N+a'b'}{D+a'}$$

only if

$$a'N \geqq a'b'D.$$

But $a' > 0$, therefore the desired condition holds only if

(2.7) $$N \geqq b'D.$$

Since $b \gtreqless b'$, (2.7) is satisfied because of (2.6), which completes the proof of the lemma.

Theorem 4:

Let $\lambda_1 > \lambda_2 > \lambda_3$. If $I(C_{\lambda_1}) \geqq I(C_{\lambda_2})$, then $I(C_{\lambda_2}) \geqq I(C_{\lambda_3})$.

Proof:

$\lambda_1 > \lambda_2 > \lambda_3$ implies

$$C_{\lambda_1} \subset C_{\lambda_2} \subset C_{\lambda_3} \quad \text{since} \quad C_\lambda = \{(x, y) | i(x, y) \geqq \lambda\} .$$

Let I_1, I_2, I_3 ; T_1, T_2, T_3 be the total expected income and expected duration of a cycle when the continuation set is C_{λ_1}, C_{λ_2}, C_{λ_3} respectively.

Since $C_{\lambda_1} \subset C_{\lambda_2}$,

$$T_1 \leqq T_2 .$$

Similarly

$$T_2 \leqq T_3 .$$

Let $T_2 > 0$, $T_1 = T_2 - a$, $T_3 = T_2 + a'$ where $a > 0$ and $a' > 0$. Denote

$$I_2 = N, \quad I_1 = N - ab, \quad I_3 = N + a'b' .$$

The given condition $I(C_{\lambda_1}) \geqq I(C_{\lambda_2})$, can be written as

(2.8) $$\frac{N-ab}{T_2-a} \geqq \frac{N}{T_2} .$$

From (2.8) and Lemma 6, we have

$$I(C_{\lambda_2}) = \frac{N}{T_2} \geq \frac{N+a'b'}{T_2+a'} = I(C_{\lambda_3})$$

provided $b \gtrless b'$.

Since $i(x, y)$ is non increasing in each coordinate, the income per unit "time" in the region $C_{\lambda_2}-C_{\lambda_1}$ is greater than the income per unit "time" in the region $C_{\lambda_3}-C_{\lambda_2}$, therefore $b \gtrless b'$. This completes the proof of Theorem 4.

3. Applications.

As an application of the above model consider the case when the production process is a vector valued stochastic process $\{(x(t), y(t)), t \geq 0\}$ with state space a lattice of points in the non negative part of the plane. When the process moves, it moves one step either to the right with probability p or upwards with probability $q=1-p$. Let the waiting times between moves be independent and identically distributed random variables with finite expectation H. Consider the maximization of the quantity

$$(1.4') \qquad \frac{E(I')-mK}{E(T')+m},$$

where T' and I' are the time and income associated with a path and m and K are defined above. The quantity in (1.4') corresponds to the long run income per unit time (Johns and Miller [1963]).

In this case the inspection process corresponds to observing the process $\{(x(t), y(t)), t \geq 0\}$ continuously. The problem is simplified because it is necessary to observe the process only at the times $\{t_n\}$ corresponding to the moments when the successive moves occur. Therefore T' and I' can be expressed respectively as H times the number of steps in a path and H times the sum of the values of $i(x, y)$ over all points in C which the path passes through, i.e.,

$$T' = HT , \qquad I' = HI .$$

Thus the discrete model discussed in Section 1 can be used in this continuous case and the quantity in (1.4') can be maximized. It follows from Johns and Miller [1963] that $I(C)$, the limit of $I_n(C)$, the observed incomes per unit time in the n^{th} cycles, exists with probability one and that $I(C) = \frac{E(I')-mK}{E(T')+m}$ with probability one. Therefore, from (1.10),

$$(1.10') \qquad I(C) = \frac{E(I')-mK}{E(T')+m} = \frac{HE(I)-mK}{HE(T)+m} = \frac{H \sum_{(x,\, y) \in C} P_C(x, y)\, i(x, y) - mK}{H \sum_{(x,\, y) \in C} P_C(x, y) + m} .$$

3.1. Poisson Process.

As a special case let the two components of the production process $(x(t), y(t))$ be independent Poisson processes with parameters Δ_1 and Δ_2 respectively. This case is a generalization of the problem considered by Savage [1962].

The following are some of the basic properties of $(x(t), y(t))$ process (disregarding repair state):

a. $P(x(t')-x(t^*) = x) \; = \; e^{-\Delta_1 T}(\Delta_1 T)^x/x!$

$P(y(t')-y(t^*) = y) \; = \; e^{-\Delta_2 T}(\Delta_2 T)^y/y!$

where $x \geqq 0$, $y \geqq 0$, $t'-t^* = T \geqq 0$.

b. As functions of t, both $x(t)$, $y(t)$ are non decreasing and with probability one whenever $x(t)$ or $y(t)$ increases; the size of increase is one.

c. The changes in the values of $x(t)$ are independent of the changes in $y(t)$, i.e., each machine change states independently of the other.

d. For $t_4 \geq t_3 \geq t_2 \geq t_1$ and $x_1, x_2 \geq 0$, $P(x(t_4)-x(t_3) = x_1$ and $x(t_2)-x(t_1) = x_2) = P(x(t_4)-x(t_3) = x_1) P(x(t_2)-x(t_1) = x_2)$ and a similar expression for $y(t)$.

e. The waiting time between the points of increase of $x(t)$ and $y(t)$ are exponential with parameters Δ_1 and Δ_2, i.e., the exprected waiting times respectively are Δ_1^{-1} and Δ_2^{-1}. The respective variances are Δ_1^{-2} and Δ_2^{-2}.

f. Each component of the $z(t)$ process is Markovian, i.e., to compute the probability distribution for the future of either of the two components, only the most recent history of that component is required.

g. Each component of the $z(t)$ process has stationary increments, i.e., the distribution of $x(t')-x(t^*)$ or $y(t')-y(t^*)$ depends only on $T = t'-t^* \geq 0$.

Properties (a. through c.) imply (d. through g.).

In this case the process starts at (0, 0) and when it moves, it moves a step either to the right with probability p or a step upwards with probability q. It can be shown that

$$p = \frac{\Delta_1}{\Delta_1+\Delta_2} \quad \text{and} \quad q = 1-p = \frac{\Delta_2}{\Delta_1+\Delta_2} .$$

Note further that $H = \frac{1}{\Delta_1+\Delta_2}$, i.e., the expected time spent by the process at any point during production is $\frac{1}{\Delta_1+\Delta_2}$.

From (1.10'),

$$\text{(1.10")} \qquad I(C) = \frac{\sum_{(x,\, y) \in C} P_C(x, y)\, i(x, y) - mK(\Delta_1+\Delta_2)}{\sum_{(x,\, y) \in C} P_C(x, y) + m(\Delta_1+\Delta_2)} .$$

Now we consider various examples in the Poisson case to illustrate the results obtained in Section 2 and to compare our results with Savage [1962].

3.2. Examples (Poisson case):

Example 1.

Consider the symmetric case of the income function (1.3a) when B = D, i.e.,

(3.1) $$i(x, y) = A-B(x+y) .$$

It is required to find λ so that

$$C_\lambda = \{(x, y) | i(x, y) = A-B(x+y) \geqq \lambda\}$$

is the optimal set.

C_λ can be written as

$$C_\lambda = \{(x, y) | x+y \leq h\} \qquad \text{where } h = \frac{A-\lambda}{B} .$$

Without loss of generality we can assume h to be an integer. Otherwise we define $h = [\frac{A-\lambda}{B}]$ where [x] stands for the largest integer less than or equal to x. Now

(3.2) $$\sum_{C_\lambda} P_{C_\lambda}(x, y) = \sum_{j=0}^{h} \sum_{x+y=j} P_{C_\lambda}(x, y) = \sum_{j=0}^{h} [\sum_{x=0}^{j} \binom{j}{x} p^x q^{j-x}] = h+1 = \frac{A+B-\lambda}{B}$$

(3.3) $$\sum_{C_\lambda} P_{C_\lambda}(x, y)\, i(x, y) = \sum_{j=0}^{h} \sum_{x+y=j} \binom{x+y}{x} p^x q^y (A-Bx-By)$$

$$= \sum_{j=0}^{h} [A-Bj] = A(h+1) - B\frac{h(h+1)}{2} = \frac{(A+\lambda)(A+B-\lambda)}{2B} .$$

Hence (1.10") reduces to

(3.4) $$I(C_\lambda) = \frac{2A(h+1)-Bh(h+1)-2(\Delta_1+\Delta_2)mK}{2(h+1)+2(\Delta_1+\Delta_2)m} , \qquad \text{or}$$

(3.5) $$I(C_\lambda) = \frac{(A+B-\lambda)(A+\lambda)-2BmK(\Delta_1+\Delta_2)}{2(A+B-\lambda)+2Bm(\Delta_1+\Delta_2)} .$$

Note that equations (3.2), (3.3) and (3.4) correspond exactly to Savage [1962] equation (18), equation below (21') and equation (22), provided $\Delta = \Delta_1+\Delta_2$. In working with critical equation (3.5), λ is treated as if it is a continuous variable and it follows from equation (23') of Savage [1962] that

(3.6) $$h^* = \frac{A-\lambda^*}{B} = \Delta m([1+(2A+B+2K)/\Delta Bm]^{\frac{1}{2}}-1) - 1 ,$$

or

(3.7) $$\lambda^* = A+B+Bm(\Delta_1+\Delta_2)-Bm(\Delta_1+\Delta_2)[1 + \frac{2A+B+2K}{Bm(\Delta_1+\Delta_2)}]^{\frac{1}{2}}$$

To find $I^* = I(C_{\lambda^*})$, we can use (3.5) or from equations (24), (25) and (26) of Savage [1962], we have

(3.8) $$I^* = I(C_{\lambda^*}) = \frac{2\lambda^*-B}{2} .$$

Note that equations (3.4) and (3.7) are of correct dimensionality when it is realized that:

h is a pure number,

m is dimensionality (time),

$\Delta = \Delta_1+\Delta_2$ is dimensionality $(\text{time})^{-1}$, and

A, B, K are dimensionality (money/time).

<u>Example 2.</u>

Consider again the income function (1.3a) and let

$B = 2,\quad D = 3,\quad m = 1,\quad k = 5$

$\Delta_1 = \frac{1}{2},\quad \Delta_2 = \frac{1}{2}$ and therefore

$$p = \frac{\Delta_1}{\Delta_1+\Delta_2} = \frac{1}{2}, \qquad q = 1-p = \frac{1}{2} .$$

We consider three cases corresponding to three values of A, i.e., $A = 10, 11, 12$. Note that C_j denotes the set of points $\{(x, y) | i(x, y) \geq j\}$. Therefore for $A = 11$,

C_4 contains (0, 0), (1, 0), (0, 1), (2, 0), (0, 2), (1, 1), (3, 0), (2, 1)

and

C_7 contains (0, 0), (1, 0), (0, 1), (2, 0) etc.

<u>For A = 10</u> [values of I(C) are approximated to 2-decimal places.]

$$\sum_{(x, y) \in C_4} P_C(x, y) = \sum_{C_4} \binom{x+y}{x} p^x q^y = \frac{25}{8}$$

$$\sum_{C_5} P_C(x, y) = \frac{11}{4}, \qquad \sum_{C_6} P_C(x, y) = \frac{9}{4}$$

$$\sum_{(x, y) \in C_4} P_C(x, y)\, i(x, y) = \sum_{C_4} \binom{x+y}{x} p^x q^y [10-2x-3y] = 23$$

$$\sum_{C_5} P_C(x, y)\, i(x, y) = 21.5, \qquad \sum_{C_6} P_C(x, y)\, i(x, y) = 19$$

and

$$I(C_4) = \frac{23-5}{25/8 + 1} = 4.36$$

$$I(C_5) = 4.40, \qquad I(C_6) = 4.31 .$$

We find that C_5 is optimal and λ^* (optimal λ) satisfies; $4 < \lambda^* \leq 5$ and $I^* = 4.40$.

<u>For A = 11</u> [values of I(C) are approximated to 3-decimal places.]

$$\sum_{C_4} P_C(x, y) = \frac{7}{2}, \qquad \sum_{C_5} P_C(x, y) = \frac{25}{8}$$

$$\sum_{C_6} P_C(x, y) = \frac{11}{4}, \qquad \sum_{C_7} P_C(x, y) = \frac{9}{4}$$

$$\sum_{C_4} P_C(x, y)\ i(x, y) = \frac{221}{8}, \qquad \sum_{C_5} P_C(x, y)\ i(x, y) = \frac{209}{8}$$

$$\sum_{C_6} P_C(x, y)\ i(x, y) = \frac{97}{4}, \qquad \sum_{C_7} P_C(x, y)\ i(x, y) = \frac{85}{4}$$

and

$$I(C_4) = 5.028, \qquad I(C_5) = 5.121$$

$$I(C_6) = 5.133, \qquad I(C_7) = 5.000 .$$

Therefore C_6 is optimal and λ^* satifies; $5 < \lambda^* \leq 6$ and $I^* = 5.133$.

For $A = 12$

$$\sum_{C_5} P_C(x, y) = \frac{7}{2}, \qquad \sum_{C_6} P_C(x, y) = \frac{25}{8}$$

$$\sum_{C_7} P_C(x, y) = \frac{11}{4}, \qquad \sum_{C_8} P_C(x, y) = \frac{9}{4}$$

$$\sum_{C_5} P_C(x, y)\ i(x, y) = \frac{249}{8}, \qquad \sum_{C_6} P_C(x, y)\ i(x, y) = \frac{117}{4}$$

$$\sum_{C_7} P_C(x, y)\ i(x, y) = 27, \qquad \sum_{C_8} P_C(x, y)\ i(x, y) = \frac{47}{2}$$

and

$$I(C_5) = 5.806, \qquad I(C_6) = 5.879$$

$$I(C_7) = 5.867, \qquad I(C_8) = 5.692 .$$

Therefore λ^* satisfies; $5 < \lambda^* \leq 6$ and $I^* = 5.879$.

Example 3:

Consider again the income function (1.3a) and let

$B = 4, \quad D = 3, \quad m = 1, \quad K = 4$

$\Delta_1 = 1/3, \quad \Delta_2 = 2/3$ and therefore

$p = 1/3, \quad q = 1-p = 2/3$.

Again we consider three cases corresponding to three values of A, i.e., A = 14, 15, 16.

For A = 14

$$I(C_7) = 7.029 , \quad I(C_8) = 7.031 , \quad I(C_9) = 6.889 .$$

Therefore $7 < \lambda^* \leq 8$ and $I^* = 7.031$.

For A = 15

$$I(C_6) = 7.63 , \quad I(C_7) = 7.75$$

$$I(C_8) = 7.77 , \quad I(C_9) = 7.74 .$$

Therefore $7 < \lambda^* \leq 8$ and $I^* = 7.77$.

For A = 16

$$I(C_7) = 8.39 , \quad I(C_8) = 8.50$$

$$I(C_9) = 8.514 , \quad I(C_{10}) = 8.45 .$$

Therefore $8 < \lambda^* \leq 9$ and $I^* = 8.514$.

Example 4:

In example 3, we specialize the case when A = 15 and D = 4 instead of 3, i.3., the income function is of the form (3.1) of example 1 with A = 15, B = 4. In this case;

$$I(C_4) = I(C_5) = I(C_6) = I(C_7) = 7.250 ,$$

$$I(C_8) = I(C_9) = I(C_{10}) = I(C_{11}) = 7.333 ,$$

$$I(C_{12}) = 5.50 .$$

Note that $I(C_{12})$ is a one point set.

(3.9) $\therefore \quad 7 < \lambda^* \leq 11$

and

(3.10) $I^* = 7.333 .$

Now from (3.7),

(3.11) $\lambda^* = 19+4-4[1 + \frac{42}{4}]^{\frac{1}{2}} \cong 9.4$

and from (3.8),

(3.12) $I^* = \frac{2\lambda^*-B}{2} \cong \frac{14.8}{2} = 7.4 .$

Note that the value of I* given by (3.10) is sufficiently close to the approximate value of I* given by (3.12); hence example 4 illustrates numerically the results obtained in (3.7) and (3.8).

Example 5:

Now consider the income function (1.3b). In this case the contours of $i(x, y)$ are rectangular hyperbolas; let

$$B = 2, \quad m = 1, \quad K = 3$$

$$\Delta_1 = \tfrac{1}{2}, \quad \Delta_2 = \tfrac{1}{2} \quad ; \quad p = \tfrac{1}{2}, \quad q = \tfrac{1}{2}$$

and A takes the values 9, 10 and 11.

For A = 9

$$I(C_2) = I(C_3) = 5.69, \quad I(C_4) = I(C_5) = 5.86$$

$$I(C_6) = I(C_7) = 6.11, \quad I(C_8) = 6.00 .$$

Therefore $5 < \lambda^* \leqq 7$ and $I^* = 6.11$.

For A = 10

$$I(C_3) = I(C_4) = 6.52, \quad I(C_5) = I(C_6) = 6.71$$

$$I(C_7) = I(C_8) = 6.89, \quad I(C_9) = 6.75 .$$

Therefore, in this case, $6 < \lambda^* \leqq 8$ and $I^* = 6.89$.

For A = 11

$$I(C_4) = I(C_5) = 7.35, \quad I(C_6) = I(C_7) = 7.57$$

$$I(C_8) = I(C_9) = 7.67, \quad I(C_{10}) = 7.25 .$$

Therefore $7 < \lambda^* \leqq 9$ and $I^* = 7.67$.

Examples 2, 3 and 5 can be summarized by the table on page 25.

Note: From examples 2, 3 and 5, we note the following:

(a) The optimal C is full.

(b) As A (sure rate of income) increases, the continuation set does not decrease.

(c) I* increases with A.

(d) Consider the set C_λ:

$$C_\lambda = \{(x, y) | i(x, y) \geqq \lambda\} .$$

Note that $i(x, y)$ is defined only for integer values of x and y. Therefore for any choice of λ, there exists a set of λ's which give the same continuation set C_λ and we have the following conjecture.

Conjecture: Let λ^*, λ^{**} denote the minimum and maximum of the set of λ's which correspond to the optimal continuation set and if the income function is of the form (1.3a) or (1.3b), then

$$\lambda^* \leqq I^* \leqq \lambda^{**} . \tag{3.13}$$

Summary of Examples 2, 3 and 5.

Form of Income Function	A	B	D	m	p	q	K	Δ_1	Δ_2	C*	I*
1. Linear: $i(x, y) = A-Bx-Dy$	10	2	3	1	$\frac{1}{2}$	$\frac{1}{2}$	5	$\frac{1}{2}$	$\frac{1}{2}$	(0, 0),(1, 0),(0, 1),(2, 0),(1, 1)	4.40
	11	2	3	1	$\frac{1}{2}$	$\frac{1}{2}$	5	$\frac{1}{2}$	$\frac{1}{2}$	(0, 0),(1, 0),(0, 1),(2, 0),(1, 1)	5.133
	12	2	3	1	$\frac{1}{2}$	$\frac{1}{2}$	5	$\frac{1}{2}$	$\frac{1}{2}$	(0, 0),(1, 0),(0, 1),(2, 0), (0, 2),(1, 1),(3, 0)	5.879
	14	4	3	1	$\frac{1}{3}$	$\frac{2}{3}$	4	$\frac{1}{3}$	$\frac{2}{3}$	(0, 0),(1, 0),(0, 1),(0, 2)	7.031
	15	4	3	1	$\frac{1}{3}$	$\frac{2}{3}$	4	$\frac{1}{3}$	$\frac{2}{3}$	(0, 0),(1, 0),(0, 1),(0, 2),(1, 1)	7.77
	16	4	3	1	$\frac{1}{3}$	$\frac{2}{3}$	4	$\frac{1}{3}$	$\frac{2}{3}$	(0, 0),(1, 0),(0, 1),(0, 2),(1, 1)	8.514
2. Quadratic: $i(x, y) = A-Bxy$	9	2	-	1	$\frac{1}{2}$	$\frac{1}{2}$	3	$\frac{1}{2}$	$\frac{1}{2}$	(0, 0),(1, 1);(0, j) and (j, 0) for j=1,2,3,...	6.11
	10	2	-	1	$\frac{1}{2}$	$\frac{1}{2}$	3	$\frac{1}{2}$	$\frac{1}{2}$	(0, 0),(1, 1);(0, j) and (j, 0) for j=1,2,3,...	6.89
	11	2	-	1	$\frac{1}{2}$	$\frac{1}{2}$	3	$\frac{1}{2}$	$\frac{1}{2}$	(0, 0),(1, 1);(0, j) and (j, 0) for j=1,2,3,...	7.67

Acknowledgment:

I wish to express my indebtedness to Professor I. R. Savage for suggesting this problem to me, for his inspiring guidance, constant encouragement and valuable suggestions. It has been both a privilege and a pleasure to work under his direction.

References

[1] Chung, Kai Lai, "Markov Chains with Stationary Transition Probabilities," Springer-Verlag, Berlin (1960).

[2] Feller, W, "An Introduction to Probability Theory and its Applications," John Wiley and Sons, Inc. New York.

[3] Girshick, M. A. and Rubin, H. "A Bayes Approach to a Quality Control Model," Ann. Math. Statist. 23 (1952) pp. 114-125.

[4] Johns, M. V. Jr. and Miller, R. G. Jr. "Average Renewal Loss Rates," Ann. Math. Statist. 34, No. 2 (1963) pp. 396-401.

[5] Savage, I. R., "Surveillance Problems" Naval Research Logistics Quarterly, Vol. 9, Nos. 3 and 4, Sep-Dec (1962) pp. 187-209.