# ORDERINGS INDUCED ON $R^n$ BY COMPACT GROUPS OF LINEAR TRANSFORMATIONS WITH APPLICATIONS TO PROBABILITY INEQUALITIES

(Preliminary Report)

by

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## § 1. Motivation and History

In recent years, a result due to T. W. Anderson (1955) and its extentions by Mudholkar (1966) have received considerable attention. The main application of these results has been in the area of probability inequalities and their application to statistics. A few recent references are Das Gupta, Eaton, et al (1972) (and the references there), Mudholkar (1966), Šidák (1971), Fefferman, Jodiet and Perlman (1972) and Das Gupta (1974).

The setting of Mudholkar's extension of Anderson's Theorem is the following. Let $G$ be a group of linear transformations on $R^n$ to $R^n$ which preserve Lebesgue measure.

Definition 1.1: A measurable function $f$ on $R^n$ to $[0, \infty)$ is called unimodal if $\{x | f(x) \geq u\} \equiv K_u$ is convex for every $u > 0$.

Definition 1.2: Let $y \in R^n$ and let $C(y)$ (sometimes $C_G(y)$) denote the convex hull of the set $\{gy | g \in G\}$. $C(y)$ is the convex hull of the G-orbit of $y$.

Theorem 1.3 (Mudholkar): Suppose $f$ is a G-invariant unimodal function and $E \subseteq R^n$ is a convex $G$ invariant set. Then

$$(1.1) \qquad \int I_E(x)f(x+z)dx \geq \int I_E(x)f(x+y)dx$$

for $z \in C(y)$.

Proof: We include the proof of this theorem to indicate where the various assumptions are used. First note that

(1.2) $$f(w) = \int_0^\infty I_{K_u}(w)du$$

so it suffices to establish (1.1) when $f$ is the indicator of a convex G-invariant set - say $f = I_A$. It is easy to show that

(1.3) $$\int I_E(x)\, I_A(x+w)dx = \mu_n((E + w) \cap A)$$

where $\mu_n$ denotes n-dimensional Lebesgue measure. Define $\phi$ by

(1.4) $$\phi(w) = \mu_n^{1/n}((E + w) \cap A)$$

and let

(1.5) $$Z = \{z | \phi(z) \geq \phi(y)\} .$$

Since $E$ and $A$ are G-invariant, $\phi$ is $G$ invariant so $Z$ is $G$ invariant, and $gy \in Z$ for all $g \in G$. To complete the proof, it suffices to show that $Z$ is a convex set.

Consider $z_1$ and $z_2$ in $Z$ and $\alpha \in (0,1)$. Now, it is easy to show that

(1.6) $$(E + \alpha z_1 + (1-\alpha)z_2) \cap A$$

$$\supseteq \alpha[(E + z_1) \cap A] + (1-\alpha)[(E + z_2) \cap A]$$

where the second set is the Minkowski sum and $\alpha B \equiv \{\alpha b | b \in B\}$. The establishment of (1.6) uses only the convexity of $E$ and $A$. By the Brünn - Minkowski inequality

$$(1.7) \qquad \mu_n^{1/n}(\alpha[(E + z_1) \cap A] + (1-\alpha)[(E + z_2) \cap A])$$

$$\geq \alpha\, \mu_n^{1/n}((E + z_1) \cap A) + (1-\alpha)\mu_n^{1/n}((E + z_2) \cap A)$$

$$= \alpha\, \phi(z_1) + (1-\alpha)\phi(z_2) \geq \phi(y) .$$

From (1.6) and (1.7), $\phi$ is a concave function, so $Z$ is convex. Hence $Z \supseteq C(y)$ and the proof is complete.

Recall that if $h_1 \geq 0$ and $h_2 \geq 0$ are functions on $R^n$, the convolution of $h_1$ and $h_2$ is defined by

$$(1.8) \qquad (h_1 * h_2)(y) \equiv \int_{R^n} h_1(x)h_2(y-x)dx .$$

Theorem 1.3 may be restated as follows.

<u>Theorem 1.4:</u> If $f_1 \geq 0$ and $f_2 \geq 0$ are both G-invariant and unimodal, then

$$(1.9) \qquad (f_1 * f_2)(z) \geq (f_1 * f_2)(y)$$

for $z \in C(y)$.

<u>Proof:</u> Without loss of generality $f_1$ and $f_2$ are both indicators of G-invariant convex sets. Replacing $E$ by $-E$ in Theorem 1.3 shows Theorem 1.4 holds. The converse is clear.

The primary motivation for the current study is the following question: Can the assumption of unimodality in Theorem 1.4 be weakened - if so, what is the "right" condition on $f_1$ and $f_2$. Since $f_1 * f_2$ is linear in both $f_1$ and $f_2$, (1.9) clearly holds for all non-negative functions which are limits (in some sense) of positive linear combinations of indicators of convex G-invariant sets. This is the essence of a paper by Sherman (1955).

Recently, Marshall and Olkin (1974) have shown that when $G$ is the group of permutation matrices, unimodality may be weakened to Schur-concavity and (1.9) still holds. The definitions and discussion in the next section are basically generalizations of the notion of Schur concavity to general compact groups of matrices.

## § 2. Notation and Definitions

For the remainder of this paper $G$ will be a closed subgroup of $\mathcal{O}(n)$ - the group (compact) of $n \times n$ orthogonal matrices. This is not a restriction on compact $G$'s since every compact group of linear transformations acting on $R^n$ is isomorphic to such a group $G$ and the problems we will study are not affected by isomorphisms. As in Section 1, $C(y)$ will denote the convex hull of the G-orbit of $y$.

Definition 2.1: For $x,y \in R^n$, we write $x \leq y$ if $x \in C(y)$.

Definition 2.2: A set $B \subseteq R^n$ is called G-monotone if $x \in B$ implies $C(x) \subseteq B$.

Definition 2.3: A function $f:R^n \to [0, \infty)$ is called decreasing if

$$(2.1) \qquad f(x) \geq f(y)$$

whenever $x \leq y$.

Proposition 2.1: If $f$ is decreasing, then $f$ is G-invariant.

Proof: Since $x \leq gx$ and $gx \leq x$ for all $g$ and all $x$, when $f$ is decreasing, (2.1) implies that $f(x) = f(gx)$ for all $x$ and for all $g \in G$. Hence $f$ is G-invariant.

Proposition 2.2: The following are equivalent:

(a) $f$ is decreasing

(b) For all $u > 0$, $K_u \equiv \{x | f(x) \geq u\}$ is a G-monotone set.

Proof: Suppose $f$ is decreasing and let $z \in K_u$. If $w \in C(z)$, then $w \leq z$ so $f(w) \geq f(z) \geq u$. Hence $w \in K_u$ so $C(z) \subseteq K_u$. Thus $K_u$ is G-monotone.

Now, suppose $K_u$ is G-monotone for all $u > 0$ and let $x$ be given. If $f(x) = 0$, then $f(y) \geq f(x)$ for all $y$. If $f(x) > 0$, let $u = f(x)$. Then $x \in K_u$ so $C(x) \subseteq K_u$. Hence, if $y \in C(x)$, $y \in K_u$ so $f(y) \geq u = f(x)$. This completes the proof.

Let $M$ be the class of Borel measurable sets which are G-monotone. It follows immediately from the definition of G-monotonicity that $M$ is closed under countable unions and countable intersections - that is, $M$ is a $\sigma$-lattice. Much work has been done recently on $\sigma$-lattices. For example - see Brunk (1965) and the references there.

Let $F$ be defined as the set of all measurable non-negative decreasing functions. The next propositions describes the structure of $F$.

Proposition 2.3: The set $F$ is a convex cone which is closed under the algebraic operations of multiplication, minimum and maximum. Further, $F$ contains all the G-invariant non-negative unimodal functions.

Proof: It is clear that $F$ is a convex cone and that $F$ is closed under multiplication. That $F$ is closed under minimum and maximum follows immediately from Prop. 2.2 and the fact that $M$ is a $\sigma$-lattice.

If $f \geq 0$ is G-invariant and unimodal, then $\{x | f(x) \geq u\}$ is convex and $G$ invariant for all $u > 0$. But, any convex G-invariant set is clearly G-monotone. By Prop. 2.2, $f \in F$. This completes the proof.

The next result can be viewed as a general counterpart (for arbitrary G) to a Theorem of Hardy-Littlewood and Polya given in Berge (1963 - p. 184). Let $k$ be a concave function on $R^1$ and define $\hat{k}:R^n \to R^1$ by

$$\hat{k}(x) \equiv \int_G k(t'gx)\ \nu(dg) \tag{2.2}$$

where $t \in R^n$ is a fixed vector and $\nu$ is invariant probability measure on $G$. Then $\hat{k}$ is concave so if $y \le x$, $\hat{k}(y) \ge \hat{k}(x)$.

Proposition 2.4: If $\hat{k}(y) \ge \hat{k}(x)$ for all $k$ and all $t \in R^n$, then $y \le x$.

Proof: We will establish the contrapositive of the assertion. That is, if $y \notin C(x)$, we will construct a function $\hat{k}$ such that $\hat{k}(y) < \hat{k}(x)$. Since $C(x)$ is a compact convex set and $y \notin C(x)$, there exists a $t_0 \in R^n$ such that $t_0' u \le 1$ for $u \in C(x)$ and $t_0' y > 1$. Let $d = t_0' y > 1$. Let $k(r) = 1$ if $r \le 1$ and let $k$ be linear, continuous, and decreasing for $r > 1$ with $k(\frac{1+d}{2}) = \frac{1}{2}$. Then

$$\hat{k}(x) = \int_G k(t_0' gx)\ \nu(dg) = 1 . \tag{2.3}$$

But,

$$\begin{aligned} \hat{k}(y) &= \int_G k(t_0' gy)\ \nu(dg) \\ &= \int_{G\cap B} k(t_0' gy)\nu(dg) + \int_{G\cap B^c} k(t_0' gy)\nu(dg) \\ &\le \int_{G\cap B} k(t_0' gy)\nu(dg) + \nu(B^c) \end{aligned} \tag{2.4}$$

where $B = \{g | t_0' gy > \frac{1+d}{2}\}$ . Since $B$ is open and non-empty, we see that the last member of (2.4) is dominated by $k(\frac{1+d}{2})\nu(B) + \nu(B^c) = \frac{1}{2}\nu(B) + \nu(B^c) < 1$ since $\nu(B) > 0$. This completes the proof.

## § 3. A Necessary Condition that a Function be Decreasing and a Conjecture.

Let $F_1$ be the set of $f \in F$ which possess a differential on $R^n$.

**Theorem 3.1:** If $f \in F_1$, then

$$(3.1) \qquad (gx - x)' \nabla f(x) \geq 0$$

for all $x \in R^n$ and for all $g \in G$.

**Proof:** Let $\nabla f$ denote the gradient column vector for $f \in F_1$. Fix $x \in R^n$ and set $h(\alpha) = f(1-\alpha)x + \alpha gx)$ for $\alpha \in [0,1]$ and $g \in G$. Using Taylor's Theorem on $h$, we have

$$(3.2) \qquad h(\alpha) = h(o) + h'(o)\alpha + o(\alpha) .$$

But $h'(o) = (gx - x)' \nabla f(x)$. Since $(1-\alpha)x + \alpha gx \in C(x)$, we have from (3.2),

$$(3.3) \qquad f(x) \leq h(\alpha) = h(o) + h'(o)\alpha + o(\alpha)$$

$$= f(x) + (gx-x)'(\nabla f(x))\alpha + o(\alpha) .$$

Thus,

$$(3.4) \qquad (gx-x)' \nabla f(x)\alpha + o(\alpha) \geq 0 .$$

Dividing by $\alpha > 0$ and letting $\alpha$ converge to zero, the proof is complete.

By taking convex combinations of (3.1) for different $g \in G$, it is clear that (3.1) is equivalent to

$$(3.5) \qquad (y-x) \nabla f(x) \geq 0 \qquad \text{for all } y \in C(x) .$$

Another way to say 3.5 is to say that $-\nabla f(x)$ is in the dual cone of the convex set $C(x) - x$. We now make the following

Conjecture: If $f : R^n \to [0, \infty)$ has a differential and satisfies (3.1) for all $g \in G$, then $f \in F_1$.

The validity of this conjecture can be established directly in the case $G = \mathfrak{G}(n)$ or $G = \{\pm I\}$, as well as in other simple cases. Further, the conjecture is known to be correct when $G$ is the group of permutation matrices (see Berge (1963)).

## § 4: A Discussion of a Conjectured Convolution Theorem.

The question to be considered in this section is: under what conditions on the group $G$ is $F$ closed under convolution. It is not hard to show that $F$ is not closed under convolution when $G = \{\pm I\}$, but it is known (Marshall and Olkin (1973)) that $F$ is closed under convolution when $G$ is the group of $n \times n$ permutation matrices acting on $R^n$.

Let $X_r = \{x \mid \|x\| \le r\}$ and note that $X_r$ is G-invariant and all the concepts and results proved for $G$ acting on $R^n$ carry over to $G$ acting on $X_r$ with essentially no change. If the convolution theorem could be established for $X_r$ (for any $r$), then the result would hold for $R^n$ by approximation. $F_{(r)}$ will denote the set of non-negative decreasing functions defined on $X_r$. For $f \in F_{(r)}$ defined only on $X_{(r)}$, we will automatically extend the definition of $f$ to $R^n$ by setting $f(x) = 0$ for $x \notin X_{(r)}$. Thus, for $f_1, f_2 \in F_{(r)}$, $f_1 * f_2$ is well defined. Since $f_1 * f_2$ is linear in $f_1$ and $f_2$, it is enough to study the convolution when both $f_1$ and $f_2$ are indicators of G-monotone sets.

Let $B_0$ be a fixed compact G-monotone set with a non-empty interior, let $x_0 \in R^n$. Let $f_0$ be the indicator of $-B_0$ and let $f = I_A$ where $A$ is a compact $G$ monotone set. If $F$ is closed under convolution, we must have

$$(4.1) \qquad (f_0 * I_A)(y) \ge (f_0 * I_A)(x_0), \quad y \in C(x_0), \text{ all } A .$$

A bit of calculation shows that (4.1) is equivalent to

$$(4.2) \qquad \ell_n(B_0 + y \cap A) \ge \ell_n(B_0 + x_0 \cap A), \quad y \in C(x_0), \text{ all } A ,$$

where $\ell_n$ denotes Lebesgue measure on $R^n$. It is clear that (4.2) is equivalent to

$$(4.3) \qquad \int f(u)I_{B_0+y}(u)du \geq \int f(u)I_{B_0+x_0}(u)du \quad \text{for } y \in C(x_0) \text{ and } f \in F .$$

Proposition 4.4: Let $D = \bigcup_{w \in B_0+x_0} C(w)$. A necessary condition that (4.3) hold is

$$(4.5) \qquad D \supseteq B_0 + y \quad \text{for } y \in C(x_0) .$$

Proof: It is clear that $D$ is a compact G-monotone set. Suppose there exists a $y^* \in C(x_0)$ such that $D \not\supseteq B_0 + y^*$. Thus $D \not\supseteq (B_0)^0 + y^*$ so $D^c \cap \{(B_0)^0 + y^*\} \neq \phi$. Since $D^c$ and $(B_0)^0 + y^*$ are open, $\ell_n(D^c \cap B_0 + y^*) > 0$. Let $f = I_D$. Since $B_0 + x_0 \subseteq D$, it is easy to see (4.3) does not hold. Thus (4.5) must hold. This completes the proof.

In order that a convolution theorem hold, (4.5) must hold for all compact $B_0$ with non-empty interior and for all $x_0$ and $y \in C(x_0)$. With this in mind, let

$$(4.6) \qquad \mathcal{B} = \left\{ B \;\middle|\; \begin{array}{l} B \text{ is G-monotone,} \\ \bigcup_{w \in B+x_0} C(w) \supseteq B + y, \quad \forall x_0, \quad \forall y \in C(x_0) \end{array} \right\} .$$

We now want a sufficient condition that $\mathcal{B}$ contain all the G-monotone sets. Since $\mathcal{B}$ is closed under unions, it suffices to show that $C(u) \in \mathcal{B}$ for all $u$ to show $\mathcal{B}$ contains all G-monotone sets. Thus, we want to find conditions under which

$$(4.7) \qquad \bigcup_{v \in C(u)+x_0} C(v) \supseteq C(u) + y, \quad \forall y \in C(x_0) .$$

**Proposition 4.8:** (4.7) holds iff

$$(4.9) \qquad C(u) + C(x_0) \subseteq \bigcup_{v \in C(u)+x_0} C(v) \qquad \text{for all } u \text{ and } x_0 .$$

**Proof:** Clearly, (4.9) implies (4.7). If (4.7) holds, then

$$\bigcup_{y \in C(x_0)} (C(u) + y) \subseteq \bigcup_{v \in C(u)+x_0} C(v) .$$

But $C(u) + C(x_0) = \bigcup_{y \in C(x_0)} (C(u) + y)$ . This completes the proof.

The following is obvious.

**Lemma 4.9:** For any compact set $\mathfrak{F}$, $\bigcup_{y \in \mathfrak{F}} C(y)$ is the smallest compact G-monotone set containing $\mathfrak{F}$.

**Proposition 4.10:** $C(u) + C(x_0)$ is a compact convex G-monotone set. Further, $C(u) + C(x_0)$ is the smallest compact convex G-monotone set which contains $C(u) + x_0$.

**Proof:** Since $C(u)$ and $C(x_0)$ are both compact and convex, $C(u) + C(x_0)$ is compact and convex being the Minkowski sum. If $w \in C(u) + C(x_0)$, then $w = w_1 + w_2$ where $w_1 \in C(u)$ and $w_2 \in C(x_0)$. To show $C(u) + C(x_0)$ is G-monotone, we must show that $\int gw\, \nu(dg) \in C(u) + C(x_0)$ for all probability measures $\nu$ on G. But $\int gw\, \nu(dg) = \int gw_1 \nu(dg) + \int gw_2 \nu(dg)$ $\in C(u) + C(x_0)$ since $C(u)$ and $C(x_0)$ are G-monotone.

To show $C(u) + C(x_0)$ is the smallest compact convex G-monotone set which contains $C(u) + x_0$, let D be a compact convex set containing $C(u) + x_0$. Noting that $\bigcup_{y \in C(u)+x_0} C(y) = \bigcup_{y \in C(u)+gx_0} C(y)$ for all $g \in G$, since $\bigcup_{y \in C(u)+x_0} C(y) \subseteq D$ we see that $g_1 u + g_2 x_0 \in D$ for all $g_1, g_2 \in G$. But, since D is convex, $C(u) + C(x_0) \subseteq D$. This completes the proof.

**Proposition 4.11:** $\mathcal{B}$ is all G-monotone sets iff

(4.12) $$\bigcup_{y\in C(u)+x_0} C(y) = C(u) + C(x_0)$$

iff

(4.13) $$\bigcup_{y\in C(u)+x_0} C(y) \text{ is convex .}$$

**Proof:** Since $C(u) + x_0 \subseteq C(u) + C(x_0)$ and $C(u) + C(x_0)$ is G-monotone, the first iff follows from Prop. 4.8. Obviously (4.12) implies (4.13). If (4.13) holds, then from Lemma 9, $\bigcup_{y\in C(u)+x_0} C(y)$ is the smallest compact convex G-monotone set containing $C(u) + x_0$. (4.12) now follows from Prop. 4.10. The proof is complete.

Now, let

(4.14) $$E = \{e = g_1u + g_2x_0 | g_1, g_2 \in G\}$$

and note that $E$ contains the set of extreme points of $C(u) + C(x_0)$, since the convex hull of $E$ is clearly $C(u) + C(x_0)$.

**Proposition 4.15:** If $\bigcup_{e\in E} C(e)$ is convex, then (4.13) holds.

**Proof:** If $\bigcup_{e\in E} C(e)$ is convex then $\bigcup_{e\in E} C(e) \supseteq$ Convex Hull $(E) = C(u) + C(x_0)$ $\supseteq \bigcup_{y\in C(u)+x_0} C(y) = \bigcup_{g\in G} \bigcup_{y\in C(u)+gx_0} C(y) \supseteq \bigcup_{e\in E} C(e)$. The proof is complete.

**Proposition 4.16:** If for each $w \in C(u) + C(x_0)$, there exists an $e \in E$ such that $w \leq e$, then $\bigcup_{e\in E} C(e)$ is convex, and conversely.

**Proof:** Clearly $\bigcup_{e\in E} C(e) \subseteq C(u) + C(x_0)$. If $w \in C(u) + C(x_0)$ and $w \leq e$

then $w \in C(e)$ so if this holds for all $w \in C(u) + C(x_0)$, then $C(u) + C(x_0) \subseteq \bigcup_{e \in E} C(e)$. Thus $\bigcup_{e \in E} C(e)$ is convex.

To establish the converse, note that

$$(4.17) \qquad \bigcup_{e \in E} C(e) = \bigcup_{g \in G} C(gu + x_0) .$$

Thus $gu + x_0 \in \bigcup_{e \in E} C(e)$ for all $g \in G$ so convex combinations (over $g$'s) of $gu + x_0$ is in $\bigcup_{e \in E} C(e)$ when $\bigcup_{e \in E} C(e)$ is convex. Hence $C(u) + x_0 \subseteq \bigcup_{e \in E} C(e)$ and applying Proposition 4.10 completes the proof.

Proposition 4.18: A sufficient condition that $\bigcup_{g \in E} C(e)$ be convex is that there exists a $g_0 \in G$ such that

$$(4.19) \qquad g_1 u + g_2 x_0 \leq u + g_0 x_0 \quad \text{for all } g_1, g_2 \in G .$$

Proof: If $g_1 u + g_2 x_0 \leq u + g_0 x_0$ for all $g_1, g_2 \in G$, then $\bigcup_{e \in E} C(e) \subseteq C(u + g_0 x_0)$ which is convex. Since the reverse inclusion is obvious, the proof is complete.

Definition 4.20: $G$ is called full if for each $(w, v) \in R^n$, there exists a $g_0 \in G$ such that

$$(4.21) \qquad g_1 u + g_2 v \leq u + g_0 v \quad \text{for all } g_1, g_2 \in G .$$

Note that the permutation group is full.

(4.22) Conjecture: If $G$ is full, then $F_{(r)}$ is closed under convolution for all $r > 0$.

The conjecture is known to hold when $G$ is the permutation group (Marshall and Olkin (1972)) or $G$ is the full orthogonal group.

## § 5. Orderings on Measures

The purpose of this section is to discuss the relationship between the conjectured convolution theorem and orderings on measures defined by convex cones of functions as described in Meyer (1965) (Section 3 of Chapter XI). Let $K$ be a fixed compact convex set in $R^n$ (this can be generalized to more general linear spaces, but that will not be needed here), and let $F$ be a given convex cone of functions on $K$ such that $F$ is closed under the operation of minimum and $1 \in F$.
Further, let $N^+$ denote the set of non-negative finite measures on $K$.

Definition 5.1: If $\mu$ and $\lambda$ are in $N^+$, write $\lambda < \mu$ to mean $\int f d\lambda \geq \int f d\mu$ for all $f \in F$.

Recall that a kernel $T$ is a function $T: \mathcal{B} \times K \to [0,1]$ ($\mathcal{B}$ is the $\sigma$-algebra of Borel sets of $K$) such that

(5.2)
$T(B, \cdot)$ is $\mathcal{B}$-measurable for each $B \in \mathcal{B}$
$T(\cdot, x)$ is a probability measure on $\mathcal{B}$ for each $x \in K$.

Definition 5.3: $T$ is called an F-dilation if for each $x$ and $f \in F$, $f(x) \geq \int f(y)\, T(dy,x)$.

The main result which relates F-dilations to the ordering induced by $F$ is the following (Meyer (1966) - Theorem 53, Chapter XI).

Theorem 5.4: The following are equivalent:

(i) $\lambda < \mu$

(ii) $\mu(B) = \int T(B,x)\lambda(dx)$.

To apply the above to the conjectured convolution theorem, let $B_0$ be a fixed compact G-monotone set and let $K = S_r$ (in the notation of

Section 4). Also, fix $x_0 \in K$ and let $y \in C(x_0)$. Then, define $\lambda$ and $\mu$ by

$$
(5.5) \qquad \begin{aligned} \lambda(B) &= \ell_n((B_0 + y) \cap B) \\ \mu(B) &= \ell_n((B_0 + x_0) \cap B) . \end{aligned}
$$

Now, (4.3) holds (for $y$ fixed) iff $\lambda \prec \mu$ where $\prec$ is defined by $F$ - the class of non-negative G-monotone functions. Note that $F$ is a convex cone closed under minimum and contains the constants. Thus, Theorem 5.4 is directly applicable. However, this Theorem seems to shed little light (at least so far) on the conjectured convolution theorem.

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