ON THE EXISTENCE OF A MINIMAL SUFFICIENT SUBFIELD

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ABSTRACT

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Let (X, S) be a measurable space and M a collection of probability measures on S. Pitcher (Pacific J. Math. (1965), p. 597-611) introduced a condition called compactness on the statistical structure (X, S, M), more general than domination by a fixed σ-finite measure. Under this condition he gave a construction of a minimal sufficient subfield. In this paper a counterexample invalidating this construction is presented. We give a non-constructive proof of the existence of a minimal sufficient subfield under a condition slightly weaker than compactness. The proof proceeds by considering the intersection of an uncountable collection of sufficient subfields, and relies on a martingale convergence theorem with directed index set, due to Krickeberg.

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1. Introduction

Let S be a σ-field of subsets of a space X and let M be a collection of probability measures on (X, S). A sub-σ-field (abbreviated hereafter as "subfield") $T \subseteq S$ is said to be sufficient for M if for each A in S, there exists a T-measurable function $f(x)$ such that $f = P[A|T]$ for each P in M (f depends on A but not on P and may be chosen so that $0 \leq f \leq 1$). All subfields considered in this paper are subfields of S.

A set A in S is called M-null if $P(A) = 0$ for each P in M. Let N denote the subfield consisting of all M-null sets and their complements. For two subfields T_1, T_2 of S, write $T_1 \subseteq T_2[M]$ if for each A_1 in T_1 there exists A_2 in T_2 such that $(A_1 - A_2) \cup (A_2 - A_1)$ is M-null. Note that $T_1 \subseteq T_2[M]$ if and only if $T_1 \subseteq T_2 \vee N$, where $T_2 \vee N$ denote the smallest subfield containing T_2 and N. Following Bahadur (1954) we say that a subfield T_0 is minimal sufficient for M if T_0 is sufficient and if $T_0 \subseteq T\ [M]$ for any other sufficient subfield T.

If there exists a σ-finite measure λ such that every P in M is absolutely continuous with respect to λ we speak of the dominated case, which has been studied by Halmos and Savage (1949) and Bahadur (1954). If every P in M is a discrete measure, if S is the collection of all subsets of X, and if $N = \{\emptyset, X\}$ we speak of the discrete case, which has been studied by Basu and Ghosh (1969) and Morimoto (1972). In both the dominated and discrete cases, sufficiency can be characterized by a Factorization Criterion, and by means of this criterion a minimal sufficient subfield can be constructed (see Bahadur (1954) and Basu and Ghosh (1969)).

Pitcher (1965) introduced a more general condition on (X, S, M), which he called compactness. The compact case includes both the dominated case (Pitcher (1965), Theorem 1.2) and the discrete case (Kusama and Yamada (1972), Theorem 1.1) and is not exhausted by these two cases (Pitcher (1965), Example 1). In his Theorem 2.5 Pitcher (1965) states that under the assumption of compactness on (X, S, M) there exists a minimal sufficient subfield and in the proof claims to construct such a subfield. This proof is in error, however, as we show in section 5 of this paper.

The main purpose of this paper is to provide a veracious proof of a slight generalization of Pitcher's Theorem 2.5, demonstrating the existence of a minimal sufficient subfield under a restriction on (X, S, M) slightly weaker than compactness. Our proof differs from that attempted by Pitcher in that it is non-constructive and relies on a convergence theorem of Krickeberg (1960) for a martingale with a directed index set. Based on our discussion in section 5, we feel it is unlikely that a constructive proof can be found.

In section 2 we outline our approach and state the main result, Theorem 2.3. Section 3 contains some preliminary results needed for the proof of the main theorem, which proof is given in section 4. In section 5 we present an example to show that Pitcher's construction of a minimal sufficient subfield is incorrect.

2. The existence of a minimal sufficient subfield.

In the dominated and discrete cases a minimal sufficient subfield can be constructed using the Factorization Criterion for sufficiency. In more general cases where one does not have such a criterion available, it is natural to investigate the existence of a minimal sufficient subfield by asking whether the intersection of a collection of sufficient subfields is sufficient. Our starting point is a result of Burkholder (1961, Theorem 4 and Corollary 2) which states that without any restrictions on (X, S, M), the intersection of a finite or countable collection of sufficient subfields is sufficient, provided that each contains N (Burkholder also gives an example to show that the result is not true without the proviso concerning N).

Theorem 2.1. (Burkholder). Let $\{T_i \mid i = 1, 2, \ldots\}$ be a finite or countable collection of sufficient subfields, for M such that $N \subset T_i$ for each T_i. Then $\cap T_i$ is also sufficient for M.

Furthermore, in the dominated case Theorem 2.1 is true for an arbitrary (not necessarily countable) collection:

Theorem 2.2. Suppose M is dominated. Let $\{T_\alpha\}$ be an arbitrary collection of sufficient subfields for M such that $N \subset T_\alpha$ for each T_α. Then $\cap T_\alpha$ is also sufficient for M.

Proof. Let T^* be a minimal sufficient subfield for M. Then $T^* \subseteq T_\alpha[M]$ for each α, i.e., $T^* \subseteq T_\alpha \vee N = T_\alpha$. Hence $T^* \subseteq \cap T_\alpha$. Since T^* is sufficient and M is dominated, this implies that $\cap T_\alpha$ is sufficient (Bahadur (1954), Theorem 6.4).

A similar argument shows that Theorem 2.2 is true in the discrete case. Dropping the assumptions of dominance or discreteness, suppose that, conversely, the conclusion of Theorem 2.2 is satisfied for (X, S, M). Then a minimal sufficient subfield T^* exists. To see this, simply let $\{T_\alpha\}$ be the collection of all sufficient subfields which contain N (S is one such subfield), and let $T^* = \cap T_\alpha$. By our supposition, T^* is sufficient. If T is any other sufficient subfield then $T \vee N$ is also sufficient and contains N, hence belongs to $\{T_\alpha\}$. Thus $T^* = \cap T_\alpha \subseteq T \vee N$, i.e. $T^* \subseteq T[M]$, so T^* is minimal sufficient.

The conclusion of Theorem 2.2 does not hold without some restriction on (X, S, M), for Pitcher (1957) has shown that a minimal sufficient subfield does not always exist. Our main result, Theorem 2.3, states that if (X, S, M) is <u>coherent</u> (Definition 2.3) then the conclusion of Theorem 2.2 holds and a minimal sufficient subfield exists. This result includes Theorem 2.5 of Pitcher (1965).

Let B be the collection of all S-measurable functions f such that $0 \leq f(x) \leq 1$ for all x in X. For each P in M let $B(P)$ be a copy of B, and let $\Pi B(P)$ denote the Cartesian product of the $B(P)$ as P ranges over M. An element of $\Pi B(P)$ is denoted by (f_P).

<u>Definition 2.1</u>. An element (f_P) of $\Pi B(P)$ is said to be <u>finitely coherent</u> (<u>countably coherent</u>) if for each finite (countable) subset $\varphi = \{P_i\}$ of M, there exists a function f_φ in B such that $f_\varphi = f_{P_i}$ a.e. $[P_i]$ for each P_i in φ.

<u>Definition 2.2</u>. An element (f_P) of $\Pi B(P)$ is said to be <u>coherent</u> if there exists a function f in B such that $f = f_P$ a.e. $[P]$ for each P in M.

<u>Remark</u>. The notions of finite coherence and countable coherence coincide.

For , if (f_P) is finitely coherent and if $\varphi = \{P_i\}$ is a countable subset of M, we can define f_φ by

$$f_\varphi(x) = \begin{cases} \lim f_{\varphi_n} & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

where $\varphi_n = \{P_1, \ldots, P_n\}$. A countably coherent element (f_P) need not be coherent - see the Example at the end of this section.

<u>Definition 2.3</u>. (X, S, M) is said to be <u>coherent</u> if every countably coherent element (f_P) of $\Pi B(P)$ is coherent.

The following is our main result. Its proof is given in section 4.

<u>Theorem 2.3</u>. Suppose that (X, S, M) is coherent.
Let $\{T_\alpha\}$ be an arbitrary collection of sufficient subfields for M such that $N \subset T_\alpha$ for each T_α. Then $\cap T_\alpha$ is sufficient for M. Thus, if (X, S, M) is coherent there exists a minimal sufficient subfield.

If (X, S, M) is compact according to the definition of Pitcher (1965, p.598) then it is coherent. This follows easily from the statement at the top of p. 599 and Lemma 1.2, both in Pitcher (1965). It is easy to verify directly that (X, S, M) is coherent in the discrete and dominated cases. If (X, S, M) is dominated, for example, then M contains a countable subset $\varphi^* = \{P_i\}$ which is equivalent to M (Halmos and Savage (1949), Lemma 7). Suppose that (f_P) is countably coherent. We claim that $f_{\varphi^*} = f_P$ a.e. [P] for each P in M, so (f_P) is coherent. To see this, let $\varphi = \{P\} \cup \varphi^*$. Then $f_\varphi = f_{P_i} = f_{\varphi^*}$ a.e. $[P_i]$ for each P_i in φ^*, so $f_\varphi = f_{\varphi^*}$ a.e. [M]. Hence $f_{\varphi^*} = f_\varphi = f_P$ a.e. [P], as required.

Finally, consider the following example of Pitcher (1965, Example 2). Let X be $[0, 1]$, S the Borel σ-field on X, and M the collection of all measures on S which are either degenerate at a single point of X or else are absolutely continuous with respect to Lebesgue measure. For each P in M and x in X define $f_P(x) = P(\{x\})$. Then (f_P) is a countably coherent element of $\Pi B(P)$ but is not coherent. Thus (X, S, M) is not coherent. However, S itself is minimal sufficient: for any sufficient subfield T and any A in S, let $g = P[A|T]$ for all P in M. If P_x is the probability measure degenerate at x, then

$$g(x) = \int_X g dP_x = P_x(A) = I_A(x)$$

for all x, where I_A denotes the indicator function. Since g is T-measurable, A must belong to T, hence $T = S$, so S is minimal sufficient. This example shows that coherence is a sufficient but not necessary condition for the existence of a minimal sufficient subfield. Nonetheless, we think it unlikely that a significantly more general condition for the existence of a minimal sufficient subfield can be found.

3. Preliminary results.

In the course of the proof of Theorem 2.3 we shall need the fact (Theorem 3.2) that in the dominated case, Theorem 2.2 remains true if the proviso that $N \subset T_\alpha$ for each T_α is replaced by the assumption that $\{T_\alpha\}$ is directed downward by inclusion (defined below). Without the assumption that $N \subset T_\alpha$, the proof of this assertion is substantially more difficult than the simple proof of Theorem 2.2. (In Theorem 3.2 the collection of sufficient subfields is denoted by $\{U_t\}$ rather than $\{T_\alpha\}$.)

We shall rely on a reverse martingale convergence theorem of Krickeberg (1960), for which the following terminology is needed. Let (X, S, μ) be a probability space. Let Θ be an arbitrary index set and let $\{U_\tau \mid \tau \text{ in } \Theta\}$ be a non-empty collection of subfields of S <u>directed downward by inclusion</u>, i.e., for each pair σ, τ in Θ there exists ρ in Θ such that $U_\rho \subseteq U_\sigma \cap U_\tau$. Then Θ becomes a directed set under the partial ordering $<<$ defined by $\rho << \tau \Leftrightarrow U_\tau \subseteq U_\rho$. For each τ in Θ let h_τ be a real-valued, U_τ-measurable, μ-integrable function. The collection $\{(h_\tau, U_\tau) \mid \tau \text{ in } \Theta\}$ is said to be a <u>martingale</u> (relative to μ) if

$$h_\tau = E_\mu[h_\rho \mid U_\tau] \quad \text{a.e. } [\mu]$$

whenever $\rho << \tau$. The collection $\{h_\tau\}$ is said to be μ-<u>uniformly integrable</u> if

$$\int_{\{|h_\tau| > \gamma\}} |h_\tau| \, d\mu \to 0 \qquad \text{as} \quad \gamma \to \infty$$

uniformly in τ. Let $\| h \|_\mu$ denote the $L_1(\mu)$-norm of h, and set $U_{-\infty} = \bigcap U_\tau$. As a special case of Theorem 2.2 of Krickeberg (1960), we state the following result.

Theorem 3.1. (Krickeberg) Let $\{(h_\tau, U_\tau) \mid \tau \text{ in } \Theta\}$ be a μ-uniformly integrable martingale. Then there exists a $U_{-\infty}$-measurable, μ-integrable function $h_{-\infty}$ such that

$$\lim_\tau \| h_\tau - h_{-\infty} \|_\mu = 0.$$

Theorem 3.2. Suppose M is dominated. Let $\{U_\tau\}$ be an arbitrary collection of sufficient subfields for M such that $\{U_\tau\}$ is directed downward by inclusion. Then $U_{-\infty} \equiv \bigcap U_\tau$ is also sufficient for M.

Proof. Since M is dominated, there exists a countable subset $\{P_i\} \subset M$ such that every P in M is absolutely continuous with respect to $\lambda \equiv \sum 2^{-i} P_i$. A well-known factorization criterion for sufficiency in the dominated case states that a subfield T is sufficient for M if and only if for each P in M there exists a T-measurable version of the Radon-Nikodym derivative $dP/d\lambda$ on S (Halmos and Savage (1949), Bahadur (1954)).

Fix P in M. Since each U_τ is sufficient, there exist nonnegative U_τ-measurable versions f_τ of $dP/d\lambda$ on S. For any ρ, τ we have

$$(4.1) \qquad \int_A f_\tau \, d\lambda = P(A) = \int_A f_\rho \, d\lambda$$

for all A in S, so $\{(f_\tau, U_\tau)\}$ is a martingale relative to λ. Furthermore, (4.1) implies that

$$(4.2) \qquad f_\tau = f_\rho \quad \text{a.e. } [\lambda].$$

Since $\int f_\tau d\lambda = 1 < \infty$ we have that

$$\int_{\{f_\tau > \gamma\}} f_\tau d\lambda \to 0 \qquad \text{as } \gamma \to \infty$$

and by (4.2) this convergence is uniform in τ, so $\{f_\tau\}$ is λ-uniformly integrable. Thus by Theorem 3.1 there exists a $U_{-\infty}$-measurable function $f_{-\infty}$ such that

$$\lim_\tau \| f_\tau - f_{-\infty} \|_\lambda = 0.$$

In particular, for each A in S we have

$$\int_A f_{-\infty}\, d\lambda = \lim_\tau \int_A f_\tau\, d\lambda = P(A),$$

so $f_{-\infty}$ is a $U_{-\infty}$-measurable version of $dP/d\lambda$ on S. Since this holds for each P in M, $U_{-\infty} = \cap U_\tau$ is sufficient, which completes the proof.

We shall also need the following lemma, which is similar to Theorem 2.3 of Pitcher (1965). For each P in M let N_P denote the subfield of all P-null sets and their complements. For each subfield T, define

$$\tilde{T} = \bigcap_{P \text{ in } M} (T \vee N_P);$$

$\tilde{T}$ is also a subfield of S. It is easy to see that a function $\tilde{f}$ is $\tilde{T}$-measurable iff for each P in M there exists a T-measurable function f_P such that $\tilde{f} = f_P$ a.e. $[P]$. Clearly, $T \subseteq T \vee N \subseteq \tilde{T}$ for each subfield T.

<u>Lemma 3.3</u>. If T is a sufficient subfield of S, then $T \vee N = \tilde{T}$.

Proof. Suppose $\tilde{f}$ is $\tilde{T}$-measurable and bounded. Since T is sufficient there exists a bounded T-measurable function f such that $f = E_P[\tilde{f} \mid T]$ for each P in M. Since $\tilde{f} - f$ is $\tilde{T}$-measurable, for each P there exists a T-measurable function f_P such that $\tilde{f} - f = f_P$ a.e. [P]. Therefore, for each P,

$$\begin{aligned}\int (\tilde{f} - f)^2 \, dP &= \int (\tilde{f} - f) f_P dP \\ &= \int E_P[(\tilde{f} - f) f_P \mid T] \, dP \\ &= \int f_P E_P [\tilde{f} - f \mid T] \, dP \\ &= 0.\end{aligned}$$

Hence $\tilde{f} = f$ a.e. [P] for each P. Since f is T-measurable, this implies that $\tilde{f}$ is $T \vee N$-measurable, completing the proof.

Remark. The subfields $\tilde{T}$ in our paper replace the $\hat{T}$ considered by Pitcher (1965, p.601). $\tilde{T}$ and $\hat{T}$ are defined similarly, except that $\tilde{T} \subseteq S$ whereas Pitcher, by considering outer measures, does not restrict $\hat{T}$ to be contained in S. There does not appear to be any essential difference in the two treatments.

4. Proof of Theorem 2.3.

Let $\{U_\tau\}$ be the collection of all finite intersections formed from the given collection $\{T_\alpha\}$. By Theorem 2.1 each subfield U_τ is sufficient for M and contains N. Furthermore, $\{U_\tau\}$ is closed under finite intersections, hence is directed downward by inclusion. We are to show that $U_{-\infty} \equiv \cap U_\tau = \cap T_\alpha$ is sufficient for M. For this, we fix A in S and will find a $U_{-\infty}$-measurable function f such that $f = P[A \mid U_{-\infty}]$ for every P in M.

For each P in M let $0 \le f_P \le 1$ be a version of $P[A \mid U_{-\infty}]$. We claim that (f_P) is countably coherent. To see this, let $\varphi = \{P_i\}$ be a countable subset of M. Each U_τ is sufficient for φ, and φ is a dominated family, so by Theorem 3.2, $U_{-\infty}$ is sufficient for φ. Hence there exists a $U_{-\infty}$-measurable function f_φ such that $f_\varphi = P_i\,[A \mid U_{-\infty}]$ for each P_i in φ. Thus $f_\varphi = f_{P_i}$ a.e.$[P_i]$; hence (f_P) is finitely coherent. Thus, by the hypothesis that (X, S, M) is coherent, there exists an S-measurable function $0 \le f \le 1$ such that $f = f_P$ a.e.$[P]$ for each P in M.

Since each f_P is U_τ-measurable for every τ, the preceding implies that f is $\widetilde{U}_\tau$-measurable. However, by Lemma 3.3, $\widetilde{U}_\tau = U_\tau \vee N = U_\tau$. Thus f is U_τ-measurable for every τ, hence f is $U_{-\infty}$-measurable. Since $f = f_P$ a.e. $[P]$ for every P in M, this implies that $f = P[A \mid U_{-\infty}]$ for every P, as required.

5. A counterexample to Pitcher's construction of a minimal sufficient subfield.

Pitcher (1965) denotes probability measures in M by μ, ν, etc. In the proof of his Theorem 2.5 he begins the construction of a minimal sufficient subfield as follows: "Let T_0 be the subfield generated by all the functions $[d\mu / d(\mu + \nu)]$ for μ and ν in M". Then, using his assumption that (X, S, M) is compact (coherence would suffice) he correctly shows that the subfield $\hat{T}_0$ (or, $\tilde{T}_0$ in our treatment) is sufficient. Lastly, he claims that $\hat{T}_0$ is minimal sufficient - however, this need not be true.

The difficulty is that the Radon-Nikodym derivatives $d\mu / d(\mu + \nu)$ are well-defined only up to sets of $(\mu + \nu)$-measure 0, but Pitcher does not specify which versions are to be chosen. The following example shows that if the choice of versions is unfortunate, $\hat{T}_0$ (or $\tilde{T}_0$) need not be minimal sufficient. Let X be the interval $[0, 3]$, S the Borel σ-field on X, and $M = \{\mu_1, \mu_2, \mu_3\}$, where μ_i is the uniform probability measure on the interval $[i-1, i]$. One possible choice of version of $d\mu_i / d(\mu_i + \mu_j)$ is the indicator function $I_{[i-1, i]}$. With this choice T_0 is simply the field generated by the three intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, and $\hat{T}_0$ (or $\tilde{T}_0$) is easily seen to be minimal sufficient. Suppose, however, that we choose the versions of $d\mu_i / d(\mu_i + \mu_j)$ to assume the value 1 on $[i-1, i]$, 0 on $[j-1, j]$, and to be linear with positive slope on $[k-1, k]$, where $k \neq i, j$. Then T_0 is the entire Borel σ-field S and $\hat{T}_0$ (or $\tilde{T}_0$) is sufficient, but not minimal sufficient.

In this example the proper choice of version for $d\mu / d(\mu + \nu)$ is clear. However, in the generality assumed by Pitcher and in the present paper, with no topological structue on (X, S), we do not see that a proper version can be specified. It was for this reason that we have sought to prove the existence of a minimal sufficient subfield by an indirect approach, examining the sufficiency of the intersection of an uncountable collection of sufficient subfields.

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