# Sequential Estimation of the Mean with Prespecified Precision: A Stopping Rule Based on the Pairwise Range

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# ABSTRACT

A sequential estimation procedure for estimating the mean of a population is proposed. The technique uses as a measure of variability the sum of ranges observed in successive pairs of observations. By using this easy-to-compute statistic, the procedure can be conveniently used in the field; whereas other techniques requiring recomputation of the sums-of-squares at each step are difficult to apply. The properties of the procedure are examined under the usual normality assumption and also under a variety of non-normal conditions.

## 1. Introduction

The best-known sampling procedure for estimating the mean with prescribed precision in the absence of variance information is that of Stein [6]. Although his method guarantees (under the assumption of an underlying normal distribution) a confidence level not less than the specified probability, prudent choice of the first stage sample size is necessary to minimize the chance of obtaining an excessively large total sample size. In addition, whenever the nature of the sampling activity precludes the use of a calculator, there is the major inconvenience of having to manually compute the first stage sample variance before sampling can resume.

Alternatives which have been proposed in the literature include a modification of Stein's method (Wormleighton [8]), an analogous two-stage procedure based upon the range (Knight [2]), and various fully sequential methods requiring the repeated calculation of the sample variance until a stopping boundary is first crossed (Stein [7], Anscombe [1], Ray [3], Robbins [4], and Starr [5]). Each of these schemes shares with Stein's two-stage procedure at least one of the two aforementioned drawbacks.

The sequential rule presented in this paper is matched, in its ease of execution, only by Knight's range-based method; yet the proposed method yields excessive sample sizes with considerably less frequency. Unlike Knight's procedure the one we propose is relatively insensitive to moderate departures from normality. Finally, the new method has the advantage over both two-stage procedures in not requiring a prior estimate of the population variance in order to ensure a reasonable over-all sample size.

As with other fully sequential rules, the one we propose yields a coverage probability which differs slightly from the desired confidence coefficient.

However, if the underlying distribution is normal, the coverage probability converges asymptotically to the nominal value. For non-normal distributions, the coverage probability converges asymptotically to a value (usually close to the nominal level) that depends upon the degree of non-normality.

## 2. The Sequential Procedure

Ideally, what is sought is a sequential estimate, $\bar{X}_n$, of the population mean, $\mu$, satisfying the condition

$$P(|\bar{X}_n - \mu| \leq d) \geq 1 - \alpha \tag{2.1}$$

when the variance, $\sigma^2$, is unknown, and both d and $\alpha$ have been specified in advance of sampling. Stein's two-stage sampling procedure attains this goal, but has (in addition to the disadvantages mentioned above) the built-in inefficiency of estimating $\sigma^2$ using only data from the first stage sample.

Stein [7] also proposed a sequential estimation procedure that advocates sampling until the first time the usual fixed sample size confidence interval (calculated after each observation as if N were the prespecified sample size) has half-width less than d. The procedure we present is analogous to Stein's second proposal, except that the successive confidence intervals are based upon the mean range of successive observation-pairs rather than upon the standard deviation (a saving of computation very important in field work).

Suppose $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables sampled from a normal distribution. Stein's fully sequential (as opposed to two-stage) procedure samples until

$$S_N^2 \leq N\, d^2/t^2_{\alpha,N-1} \tag{2.2}$$

is first satisfied ($S_N^2$ is the sample variance and $t_{\alpha,N-1}$ is the two-sided critical value for the t-distribution). Suppose, now, that observations are taken two at a time, and that as sampling progresses, the ranges of successive pairs of observations are accumulated to form the sequence $\{T_2, T_4, T_6, T_8, \ldots\}$ defined by

$$T_n = |X_1 - X_2| + |X_3 - X_4| + \ldots + |X_{n-1} - X_n| \text{ , n even.} \tag{2.3}$$

The stopping rule we propose is to sample until

$$T_n \leq \frac{(n-1)^{3/2} d}{2\, \theta_{\alpha,n-1}} \tag{2.4}$$

where $\theta_{\alpha,n-1}$ is defined by

$$\theta_{\alpha,n-1} = \frac{Z_\alpha (n-1)^{\frac{1}{2}}}{[(4/\pi)(n-1) - 2Z_\alpha{}^2(2 - 4/\pi)]^{\frac{1}{2}}} \tag{2.5}$$

Our proposed $1-\alpha$ confidence interval for the mean is then

$$\bar{X}_N \pm d \tag{2.6}$$

where N is the first n satisfying (2.4). In the remainder of this section, we will develop some of the rationale for our choice of stopping rule.

Let $\bar{w}$ be the mean ($\bar{w} = 2\,T_n/n$) of the progressive range increments, and let $\theta_{\alpha,n}$ be defined such that the variable

$$Z_n = \sqrt{n}\,(\bar{X}_n - \mu)/\bar{w} \tag{2.7}$$

has probability $\alpha$ of falling outside the interval $(-\theta_{\alpha,n}, +\theta_{\alpha,n})$. Then, provided that the value of n at which sampling terminates is independent of the observations, it follows that

$$P(\sqrt{n}\,(\bar{X}_n - \mu)/\bar{w} \leq -\theta_{\alpha,n}) = \alpha/2 \tag{2.8}$$

or, equivalently, upon multiplying by $\bar{w}/\sigma$ and rearranging, that

$$P(\sqrt{n}\,(\bar{X}_n - \mu)/\sigma + \bar{w}\cdot\theta_{\alpha,n}/\sigma < 0) = \alpha/2 \text{ .} \tag{2.9}$$

Now if n is large enough so that the distribution of $\bar{w}$ can be assumed to be essentially normal, then, for any fixed n, the expression to the left of the inequality in (2.9) will also be normal, with mean $E(\bar{w}/\sigma)\cdot\theta_{\alpha,n}$ and variance $\sigma^2\{V(\bar{w}/\sigma)\cdot\theta_{\alpha,n}^2 + 1\}$. The value zero on the scale of this distribution may therefore be mapped directly onto the lower $\alpha/2$ percentage point, $-Z_\alpha$, of the standard normal distribution by the usual transformation

$$-Z_\alpha = \frac{-E(\bar{w}/\sigma)\theta_{\alpha,n}}{[V(\bar{w}/\sigma)\cdot\theta_{\alpha,n}^2 + 1]^{\frac{1}{2}}} \tag{2.10}$$

to yield, upon rearrangement,

$$\theta_{\alpha,n} = \frac{Z_\alpha}{[E^2(\bar{w}/\sigma) - Z_\alpha^2 V(\bar{w}/\sigma)]^{\frac{1}{2}}} \quad . \tag{2.11}$$

Since the mean and the variance of the standard range for pairs of independent normal variables are known to be $2/\sqrt{\pi} \simeq 1.12838$ and $(2 - 4/\pi) \simeq 0.762676$, respectively, and since $\bar{w}/\sigma$ is just the average of n/2 such ranges, the expression for $\theta_{\alpha,n}$ can be further simplified to that previously given in (2.5).

Having thus found a means of deriving approximate values for $\theta_{\alpha,n}$, we can proceed to compute the $1-\alpha$ confidence limits for the mean, given by

$$P\{|\bar{X}_n - \mu| \leq \bar{w}\,\theta_{\alpha,n}/\sqrt{n}\} = 1 - \alpha \ , \tag{2.12}$$

again provided n is not dependent on the observations.

Upon comparison of this result with (2.1), it is clear that if n were such as to satisfy the inequality

$$\bar{w}\,\theta_{\alpha,n}/\sqrt{n} \leq d \tag{2.13}$$

or its equivalent

$$T_n \leq \frac{n^{3/2} d}{2\,\theta_{\alpha,n}} \ , \tag{2.14}$$

then precision specifications would be fully met. We are thus led to consider what would happen if n were chosen to be the smallest sample size, N, fulfilling this requirement.

As it turns out, the true confidence coefficient under this stopping rule is unacceptably low unless the width of the confidence interval is made infinitesimally small. If, however, the stopping boundary is shifted ahead by a unit amount, producing the requirement that sampling should stop whenever

$$T_N \leq B_N = \frac{(N-1)^{3/2}\, d}{2\,\theta_{\alpha,N-1}} , \qquad (2.15)$$

then the discrepancy between the true and the nominal confidence levels, as is shown in what follows, is minor. A table of values of $B_N$ for d = 1 is given in Table 5. Section 6 contains a numerical example illustrating the procedure and the use of the table. The next three sections deal with properties of the procedure.

## 3. Properties of the New Procedure

In this section some properties of the stopping rule and the resulting estimation procedure are reported. In particular, results of exact computations for the normal distribution are presented.

The probability that the procedure stops after n observations is given by

$$P(N = n) = P(T_n \leq B_n\, ,\, T_{n-2} > B_{n-2}\, ,\, \ldots\, ,\, T_2 > B_2) \qquad (3.1)$$

where $T_n$ is the cumulative sum of pairwise ranges and the $B_n$'s are the critical values given by (2.15). Since the distribution of N is extremely difficult to handle analytically, exact computations were carried out on the computer for certain cases with the normal distribution. (In the robustness study reported in Section 5, Monte Carlo methods were used.) With $\sigma^2$ known, the optimal sample size satisfying the desired criteria is given by

$$n_0 = \frac{\sigma^2 Z_\alpha^{\,2}}{d^2} . \qquad (3.2)$$

Since finding the exact distribution of N involves a great deal of machine time, only cases for $n_0 = 5,10,15,\ldots,70$ and $\alpha = .05$ were computed.

<u>Figure A goes about here</u>

For $n_0 = 25$, the exact distribution of the stopping time N is given in Figure A. In this case, the mean stopping time is 31.5 with the median and mode of the distribution at 32. More than 50% of the distribution is concentrated in the points 26-38. Also, note that the righthand tail of the distribution is not too long. This implies that excessive sample sizes occur infrequently.

For notational purposes, let us follow Starr and define $D(n_0)$ and $C(n_0)$ as the expected sample size and coverage probability, respectively, for optimal sample size $n_0$. These are given by

$$D(n_0) = E(N|n_0) = \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} n\, P(N = n|n_0) \qquad (3.3)$$

and

$$C(n_0) = P(|\bar{X}_N - \mu| \leq d|n_0) = \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} P(|\bar{X}_n - \mu| \leq d|N=n) P(N=n|n_0) . \qquad (3.4)$$

<u>Table 1 goes about here</u>

Table 1 gives the values of $D(n_0)$ and $C(n_0)$ for the computed values of $n_0$. As can be seen from the table, the pairwise range procedure takes, on the average, about six more observations than the optimal sample size for the known variance case. Also, the coverage probability is not bounded below by .95, but goes slightly below and starts to come back to .95 by $n_0 = 70$. The shape of the coverage probability function is very similar to that found by Starr for the fully sequential procedures based on the sample variance (see the next section). It can be easily shown (via the central limit theorem and the law of large numbers) that as $n_0$ increases, the coverage probability approaches $1-\alpha$. Thus, for an average cost of a few observations and a minute increase in the error rate, the sequential pairwise range procedure provides a fixed-width confidence interval with $\sigma^2$ unknown.

## 4. Comparisons with Other Procedures

### 4.1 Stein's Two-Stage Sampling Scheme

Briefly, this method involves taking the sample in two stages, the first of which supplies the variance estimate $s_1^2$ from which the size of the second sample is derived. The total sample size N is then given by the rule:

$$N = \max \{ n_1 , 1 + [s_1^2 t^2_{\alpha,n_1-1}/d^2] \} \qquad (4.1)$$

where $t_{\alpha,n_1-1}$ is the two-sided critical value of a t-distribution with $n_1-1$ degrees of freedom, and [x] is the greatest integer less than x. When the underlying distribution is normal, then $1-\alpha$ is a guaranteed lower bound for the true confidence level or coverage probability. Despite this remarkable property, the Stein procedure still has the two unattractive features mentioned in the introduction.

Because the performance characteristics of the method are more readily obtained for an underlying normal distribution, we have confined our comparisons to this case only. And since the sample size distribution corresponding to Stein's method depends upon the initial sample size $n_1$, we have in all instances chosen the $n_1$ for which E(N) is a minimum. (Note that this biases the comparisons in favor of the Stein procedure.)

If we bypass the restriction that N be an integer, then the relationship between E(N) and $n_1$ is found to be (with error less than unity)

$$E(N) = n_1\{ F_{n_1-1}(\chi_0^2) + \frac{(n_1-1)}{\chi_0^2} (1 - F_{n_1+1}(\chi_0^2)) \} \qquad (4.2)$$

where $F_\nu(\chi_0^2)$ is the chi-square c.d.f. with $\nu$ degrees of freedom evaluated at

$$\chi_0^2 = \frac{n_1 (n_1 - 1) Z_\alpha^2}{n_0 t^2_{\alpha,n-1}} \qquad (4.3)$$

and is represented graphically in Figure B for selected values of $n_0$ $(= \sigma^2 Z_\alpha^2/d^2)$ and for the case $1-\alpha = .95$. Optimum $n_1$ values can be read directly from this graph.

Figure B goes about here

The performance of any sequential procedure is expressible not only in terms of expected sample size, but also by the magnitude of the upper 95-th percentage point of the sample size distribution, for there is little merit in having a small expected N if inordinately large values have a high probability of occurrence. In the case of Stein's procedure, the 95-th percentile of N is given by

$$n_{.95} = n_0 \left( \frac{t^2_{\alpha,n_1-1}}{Z_\alpha^2} \right) \left( \frac{\chi^2_{.05,n_1-1}}{n_1-1} \right) . \qquad (4.4)$$

Table 2 goes about here

Both the expected values and 95-th percentiles of the sample size distributions generated by the use of optimum $n_1$ are listed in Table 2. Compared to the other sequential methods, Stein's is best by the criterion of expected sample size and is close to the sequential variance procedure (see Section 4.3 for the sequential variance method) on the 95-th percentile. However, to quote Starr [5], "...it is clear that if we had precise knowledge of $\sigma$...we would not rely on <u>any</u> sequential procedure, but simply preassign n the smallest integer value satisfying $n > \sigma^2 Z_\alpha^2/d^2$. Therefore, the only case of interest to sequential analysis is when $\sigma$ is unknown. In this event an inappropriate (unlucky) choice of $n_1$ can have costly results."

Just how costly (in terms of additional observations) these results can be is illustrated by the consequences of choosing $n_1 = 10$ when $\sigma/d$ is such that $n_0 = 80$. Calculation reveals the 95-th percentile of N for Stein's procedure in this case to be 205, whereas it could have been as small as 110 had optimal $n_1$ been used instead. By comparison, $n_{.95}$ under the proposed pairwise range method is found to be only 118.

From these facts, even if allowance is made for the computational inconvenience of the Stein procedure, the fully sequential range-based method would appear to be the preferable choice for normally distributed populations with unknown variance.

4.2 Knight's Two-Stage Method

Knight's procedure is identical to Stein's two-stage method except that the scale parameter estimate is based upon the range in place of the standard deviation. Since the range is extremely sensitive to non-normality (this the major reason we use the pairwise range), the Knight procedure performs very poorly in cases with small departures from normality. Also, even for normal populations, the expected sample sizes and 95-th percentile points are unacceptably large. For example, in the example given above for Stein's procedure ($n_0 = 80$, $n_1 = 10$), the 95-th percentile for Knight's is 223. This should be compared with 205 for Stein and 118 for the pairwise range procedure.

From these considerations, it appears that practical use of the Knight procedure is precluded from all cases except those in which sampling costs are nil and populations can be assumed normal.

4.3 Sequential Stopping Rules

We refer here to fully sequential schemes in which observations are taken one by one (or two by two) until n first satisfies a criterion such as

$$s_N^2 \leq N d^2/t^2_{\alpha,N-1} . \tag{4.5}$$

Reference to this type of stopping rule seems first to have been made by Stein [7], following which Anscombe [1] derived some asymptotic results applicable both to this procedure and to some minor modifications thereof. Subsequently, Ray [3], Robbins [4], and Starr [5] published numerical values of the coverage probabilities and expected sample sizes corresponding to selected values of $\sigma/d$, for an underlying normal distribution.

Because all five authors cited have contributed to the development of this approach, it is properly termed the Stein-Anscombe-Ray-Robbins-Starr sequential rule. However, in the interest of brevity, we will refer to it here simply as the sequential variance procedure.

To simplify calculations, Ray, Robbins and Starr modified the stopping rule to the extent that sampling had to terminate on an odd integer. Other variations introduced by Starr included the setting of a minimum size for n and the addition of various preplanned numbers of observations to the sample after first crossing the stopping boundary.

In terms of coverage probabilities, his best results occurred under his rule "E," in which minimum N was set at 3 and the number of extra observations added to the sample was 4. Table 3 and Figure C compare the coverage probabilities for this rule to those for our range-based procedure. Although the sequential variance rule yields slightly better values than ours when $n_0$ is greater than 47, the magnitude of the difference is minor.

Table 3 and Figures C and D go about here

The sample size distribution for the sequential variance procedure with $n_0 = 25$ is given in Figure D. Expected values and 95-th percentiles for the sequential variance method as well as those for the Stein and pairwise range procedures are listed in Table 2.

Figure E goes about here

Figure E presents a comparison of the interquartile intervals of the sequential variance and pairwise range methods. Note that the variability of the sample size is greater with the pairwise range. This is caused by the inefficiency of using the range in place of the standard deviation for our scale estimate. The greater efficiency which is evident in the sequential variance procedure is attained, of course, at the cost of requiring recalculation of the sample variance with the addition of each new pair of observations, an inconvenience which our range-based method avoids.

## 5. The Effects of Non-Normality

Adopting the notation used by Starr [5], the consistency and efficiency of a sequential procedure are defined, respectively, by

$$\tau_{n_0} = C(n_0)/(1-\alpha) \tag{5.1}$$

and

$$\eta_{n_0} = D(n_0)/n_0 \tag{5.2}$$

where $C(n_0)$ and $D(n_0)$ are given by (3.3) and (3.4). In this section we examine the performance of the proposed method, as expressed by these measures, under various departures from normality.

Sequential procedures based upon the sample variance are known to be asymptotically consistent and efficient for any distribution with a finite variance [5]. However, our method is based upon the mean range of paired observations, not upon the sample variance. Since the relationship between the expected range and the standard deviation varies with the distribution, it is not surprising that both the asymptotic consistency and efficiency of the method vary as well.

Indeed, it can be shown that

$$\lim_{n_0 \to \infty} \eta_{n_0} = \{ E(R_2/\sigma)/(2/\sqrt{\pi}) \}^2 \qquad (5.3)$$

and

$$\lim_{n_0 \to \infty} \tau_{n_0} = \{ 1 - 2\Phi(-Z_\alpha \sqrt{\eta_\infty}) \}/(1-\alpha) \qquad (5.4)$$

where $R_2$ is the range of two independent observations from any given distribution and $\Phi(Z)$ is the standard normal distribution function.

Table 4 and Figure F go about here

Using Monte Carlo simulation, we estimated the small sample coverage probabilities $C(n_0)$ for various types of departure from normality. These simulation results are shown in Table 4 and Figure F. Six non-normal distributions were selected for study. The contaminated normal, double exponential, and t with 6 degrees of freedom all have coverage probabilities similar to the normal. For the uniform distribution, $C(n_0)$ is greater than the nominal .95 value, while for the exponential, the coverage probability is far below .95. Also, included in the study was a real finite population of tenth-acre forest plot volumes. For this population, the coverage probability was similar to that for the normal.

It is evident from our results that although the proposed procedure is affected by extreme departures from normality, moderate departures have only a minor influence on the coverage probabilities.

## 6. Numerical Example

A test of the practical utility of the pairwise range procedure was performed on a series of forest inventory tree counts obtained from the University of Minnesota's Cloquet Forestry Center. The sample path generated by the data is shown in Table 6 and graphically displayed in Figure G along with the stopping boundary corresponding to a confidence level of 95% and a confidence interval half-width of $d = 4$. As demonstrated in the figure, the stopping criterion was met by the first 32 sampling units.

Tables 5 and 6 and Figure G go about here

By way of comparison, the Starr sequential variance scheme (procedure E) was applied to the same set of data with results as shown in Figure G. Although this stopping boundary was crossed at $n = 31$, the procedure required the addition of four more sampling units.

Thus, in this example, the pairwise range procedure led to approximately the same sample size as the sequential variance scheme, although it required but a fraction of the computational effort.

References

[1] Anscombe, F.J. (1953) Sequential estimation. Journal of the Royal Statistical Society (B) 15, 1-20.

[2] Knight, W. (1963) The use of the range in place of the standard deviation in Stein's test. Annals of Mathematical Statistics 34, 346-347.

[3] Ray, W.D. (1957) Sequential confidence intervals for the mean of a normal population with unknown variance. Journal of the Royal Statistical Society (B) 19, 133-143.

[4] Robbins, H. (1959) Sequential estimation of the mean of a normal population. In Probability and Statistics, The Harold Cramer Volume. Uppsala, Sweden: Almquist and Wiksell, p. 235.

[5] Starr, N. (1966) The performance of a sequential procedure for the fixed-width interval estimation of the mean. Annals of Mathematical Statistics 37, 36-50.

[6] Stein, C. (1945) A two-sample test for a linear hypothesis whose power is independent of the variance. Annals of Mathematical Statistics 16, 243-258.

[7] Stein, C. (1949) Some problems in sequential estimation (abstract). Econometrica 17, 640-649.

[8] Wormleighton, R. (1960) A useful generalization of the Stein two-sample procedure. Annals of Mathematical Statistics 31, 217-221.

Table 1

Pairwise Range Procedure: Expected Sample Size and Coverage Probability for an Underlying Normal Distribution

| Ideal N | E(N) | $C(N_0)$ | Ideal N | E(N) | $C(N_0)$ |
|---|---|---|---|---|---|
| 5 | 12.3 | .9953 | 55 | 60.8 | .9481 |
| 10 | 17.2 | .9812 | 60 | 65.7 | .9480 |
| 15 | 22.0 | .9698 | 65 | 70.7 | .94790 |
| 20 | 26.7 | .9621 | 70 | 75.7 | .94792 |
| 25 | 31.5 | .9570 | 75 | 80.8[a] | .9485[a] |
| 30 | 36.3 | .9536 | 80 | 85.5[a] | .9479[a] |
| 35 | 41.2 | .9514 | 85 | 90.7[a] | .9451[a] |
| 40 | 46.0 | .9500 | 90 | 95.5[a] | .9468[a] |
| 45 | 50.9 | .9490 | 95 | 100.6[a] | .9485[a] |
| 50 | 55.9 | .9485 | 100 | 105.6[a] | .9491[a] |

[a]Estimates based upon Monte Carlo simulation; number of trials per estimate = 50,000.

Table 2

Expected Values and 0.95 Quantiles of
Sample Size Distributions Under Three Sequential Procedures

| Ideal | Pairwise Range | | Stein's Two-Stage | | Sequential Variance (Starr) | |
|---|---|---|---|---|---|---|
| N | E(N) | .95% | E(N) | .95% | E(N) | .95% |
| 5 | 12.3 | 18 | 9.1 | 17 | 11.6 | 17 |
| 10 | 17.2 | 26 | 14.2 | 24 | 15.7 | 23 |
| 15 | 22.0 | 34 | 19.3 | 32 | 20.4 | 29 |
| 20 | 26.7 | 40 | 24.2 | 39 | 25.1 | 37 |
| 25 | 31.5 | 48 | 29.1 | 45 | 29.9 | 43 |
| 30 | 36.3 | 54 | 34.1 | 51 | 34.8 | 49 |
| 35 | 41.2 | 60 | 39.0 | 58 | 39.8 | 55 |
| 40 | 46.0 | 68 | 43.9 | 64 | 44.8 | 61 |
| 45 | 50.9 | 74 | 48.9 | 70 | 49.8 | 67 |
| 50 | 55.9 | 80 | 53.9 | 75 | 54.9 | 73 |
| 55 | 60.8 | 86 | 58.9 | 81 | 59.9 | 77 |
| 60 | 65.7 | 92 | 63.8 | 87 | 64.9 | 83 |
| 65 | 70.7 | 98 | 68.7 | 93 | 70.0 | 89 |
| 70 | 75.7 | 104 | 73.7 | 99 | 75.0 | 95 |

Table 3

Interquartile Intervals and Coverage Probabilities for the Pairwise Range and Sequential Variance Procedures

| | Pairwise Range | | Sequential Variance (Starr) | |
|---|---|---|---|---|
| $n_0$ | Interquartile Interval | C(N) | Interquartile Interval | C(N) |
| 5 | (10,14) | .9953 | ( 9,13) | .9944 |
| 10 | (14,20) | .9812 | (13,19) | .9771 |
| 15 | (16,26) | .9698 | (17,25) | .9641 |
| 20 | (20,32) | .9621 | (21,31) | .9561 |
| 25 | (24,38) | .9570 | (25,35) | .9526 |
| 30 | (28,44) | .9536 | (29,41) | .9504 |
| 35 | (34,50) | .9514 | (35,47) | .9493 |
| 40 | (38,56) | .9500 | (39,53) | .9489 |
| 45 | (42,60) | .9490 | (43,57) | .9487 |
| 50 | (46,66) | .9485 | (49,63) | .9487 |
| 55 | (50,72) | .9481 | (53,69) | .9488 |
| 60 | (56,78) | .9480 | (59,73) | .9489 |
| 65 | (60,82) | .94790 | (63,79) | .9490 |
| 70 | (64,88) | .94792 | (67,85) | .9491 |

Table 4

Coverage Probabilities under the Pairwise Range Procedure for Various Underlying Distributions[a]

| $n_0$ | Contaminated Normal[c] | Double Exponential | Exponential | t (6 d.f.) | Uniform | Real[b] |
|---|---|---|---|---|---|---|
| 5 | .9969 | .9967 | .9986 | .9966 | .9906 | .998 |
| 10 | .9853 | .9870 | .9794 | .9850 | .9726 | .982 |
| 15 | .9746 | .9771 | .9516 | .9752 | .9602 | .970 |
| 20 | .9653 | .9678 | .9311 | .9641 | .9540 | .962 |
| 25 | .9586 | .9604 | .9122 | .9586 | .9504 | .954 |
| 30 | .9513 | .9555 | .9022 | .9533 | .9484 | .951 |
| 35 | .9526 | .9515 | .8943 | .9503 | .9462 | .946 |
| 40 | .9495 | .9476 | .8887 | .9498 | .9468 | .951 |
| 45 | .9470 | .9465 | .8879 | .9466 | .9462 | .953 |
| 50 | .9438 | .9453 | .8811 | .9456 | .9472 | .951 |
| 55 | .9430 | .9420 | .8814 | .9455 | .9470 | .948 |
| 60 | .9418 | .9401 | .8830 | .9439 | .9474 | .952 |
| 65 | .9413 | .9395 | .8772 | .9422 | .9460 | .945 |
| 70 | .9413 | .9400 | .8785 | .9430 | .9481 | .953 |
| 75 | .9404 | .9380 | .8808 | .9445 | .9476 | .949 |
| 80 | .9436 | .9383 | .8820 | .9431 | .9466 | .939 |
| 85 | .9412 | .9375 | .8845 | .9420 | .9494 | .947 |
| 90 | .9407 | .9385 | .8825 | .9420 | .9488 | .945 |
| 95 | .9393 | .9403 | .8818 | .9434 | .9495 | .949 |
| 100 | .9393 | .9372 | .8808 | .9435 | .9487 | .965 |

[a]Estimates based upon Monte Carlo simulation; number of trials per estimate = 50,000.

[b]Real finite population of tenth-acre forest plot volumes (hundreds of cubic feet per acre) sampled with replacement.

[c]Mixture of N(0,1) and N(0,9) with mixing probabilities .90 and .10.

Table 5

Standardized Stopping Boundaries (d = 1) for the Cumulative Pairwise Range Statistic

| N | Stopping Boundary 95% | Stopping Boundary 99% | N | Stopping Boundary 95% | Stopping Boundary 99% |
|---|---|---|---|---|---|
| 2 | - | - | 62 | 132.1 | 97.70 |
| 4 | - | - | 64 | 138.8 | 102.8 |
| 6 | 1.128 | - | 66 | 145.7 | 107.9 |
| 8 | 3.258 | - | 68 | 152.6 | 113.2 |
| 10 | 5.565 | 2.359 | 70 | 159.7 | 118.5 |
| 12 | 8.143 | 4.464 | 72 | 166.8 | 123.9 |
| 14 | 10.98 | 6.638 | 74 | 174.1 | 129.4 |
| 16 | 14.07 | 8.959 | 76 | 181.4 | 134.9 |
| 18 | 17.38 | 11.44 | 78 | 188.9 | 140.6 |
| 20 | 20.91 | 14.07 | 80 | 196.4 | 146.3 |
| 22 | 24.64 | 16.86 | 82 | 204.1 | 152.1 |
| 24 | 28.56 | 19.80 | 84 | 211.8 | 157.9 |
| 26 | 32.67 | 22.87 | 86 | 219.7 | 163.9 |
| 28 | 36.96 | 26.08 | 88 | 227.6 | 169.9 |
| 30 | 41.42 | 29.41 | 90 | 235.7 | 176.0 |
| 32 | 46.04 | 32.88 | 92 | 243.8 | 182.1 |
| 34 | 50.81 | 36.46 | 94 | 252.0 | 188.3 |
| 36 | 55.74 | 40.16 | 96 | 260.3 | 194.6 |
| 38 | 60.82 | 43.98 | 98 | 268.7 | 201.0 |
| 40 | 66.05 | 47.91 | 100 | 277.2 | 207.4 |
| 42 | 71.41 | 51.94 | 102 | 285.8 | 213.9 |
| 44 | 76.92 | 56.08 | 104 | 294.4 | 220.5 |
| 46 | 82.55 | 60.32 | 106 | 303.2 | 227.1 |
| 48 | 88.32 | 64.66 | 108 | 312.0 | 233.8 |
| 50 | 94.21 | 69.10 | 110 | 320.9 | 240.5 |
| 52 | 100.2 | 73.64 | 112 | 329.9 | 247.3 |
| 54 | 106.4 | 78.27 | 114 | 339.0 | 254.2 |
| 56 | 112.6 | 82.99 | 116 | 348.2 | 261.2 |
| 58 | 119.0 | 87.81 | 118 | 357.4 | 268.2 |
| 60 | 125.5 | 92.71 | 120 | 366.7 | 275.2 |

Note: For confidence interval of width d, multiply table value by d.

Table 6

Example of Pairwise Range Sampling

| N | $(X_{N-1}, X_N)$ | $\Delta X$ | $T_N$ | $B_N$ |
|---|---|---|---|---|
| 2 | ( 6, 7) | 1 | 1 | - |
| 4 | (29,16) | 13 | 14 | - |
| 6 | (32,25) | 7 | 21 | 4.51 |
| 8 | ( 3,24) | 21 | 42 | 13.26 |
| 10 | ( 1,18) | 17 | 59 | 22.26 |
| 12 | (18,27) | 9 | 68 | 32.57 |
| 14 | (23,18) | 5 | 73 | 43.93 |
| 16 | (11, 4) | 7 | 80 | 56.27 |
| 18 | ( 8,13) | 5 | 85 | 69.52 |
| 20 | (15,33) | 18 | 103 | 83.63 |
| 22 | (46,17) | 29 | 132 | 98.56 |
| 24 | ( 6, 4) | 2 | 134 | 114.26 |
| 26 | (27, 8) | 19 | 153 | 130.68 |
| 28 | (16,21) | 5 | 158 | 147.84 |
| 30 | (12,25) | 13 | 171 | 165.64 |
| 32 | (27,26) | 1 | 172 | 184.14 |
| 34 | ( 4,19) | 15 | 187 | 203.25 |
| 36 | ( 8, 2) | 6 | 193 | 222.96 |
| 38 | (14,18) | 4 | 197 | 243.30 |
| 40 | (21,32) | 11 | 208 | 264.19 |

Note: The data are tree counts from forest inventory plots established in the University of Minnesota's Cloquet Forestry Center. The stopping boundary corresponds to a 95% confidence interval of half-width d = 4.

Figures



[a]Estimates based upon Monte Carlo simulation; number of trials per estimate = 50,00

[b]Real finite population of tenth-acre forest plot volumes (hundreds of cubic feet per acre) samples with replacement.

[c]Tree counts from forest inventory plots in the University of Minnesota's Cloquet Forestry Center. The sequential variance procedure is Starr's method E. In both methods the 95% confidence interval half-width is specified at d = 4.

Figure A

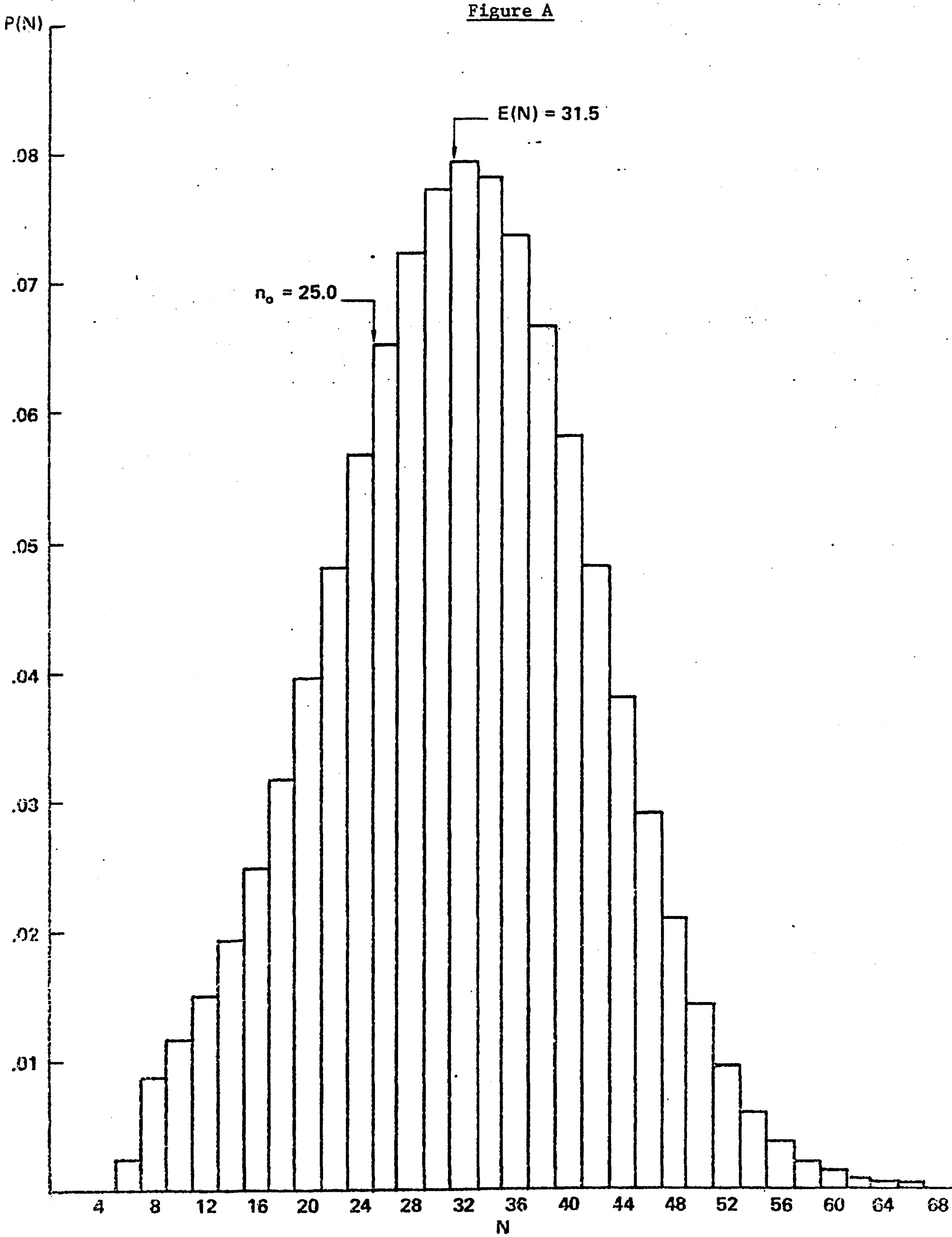
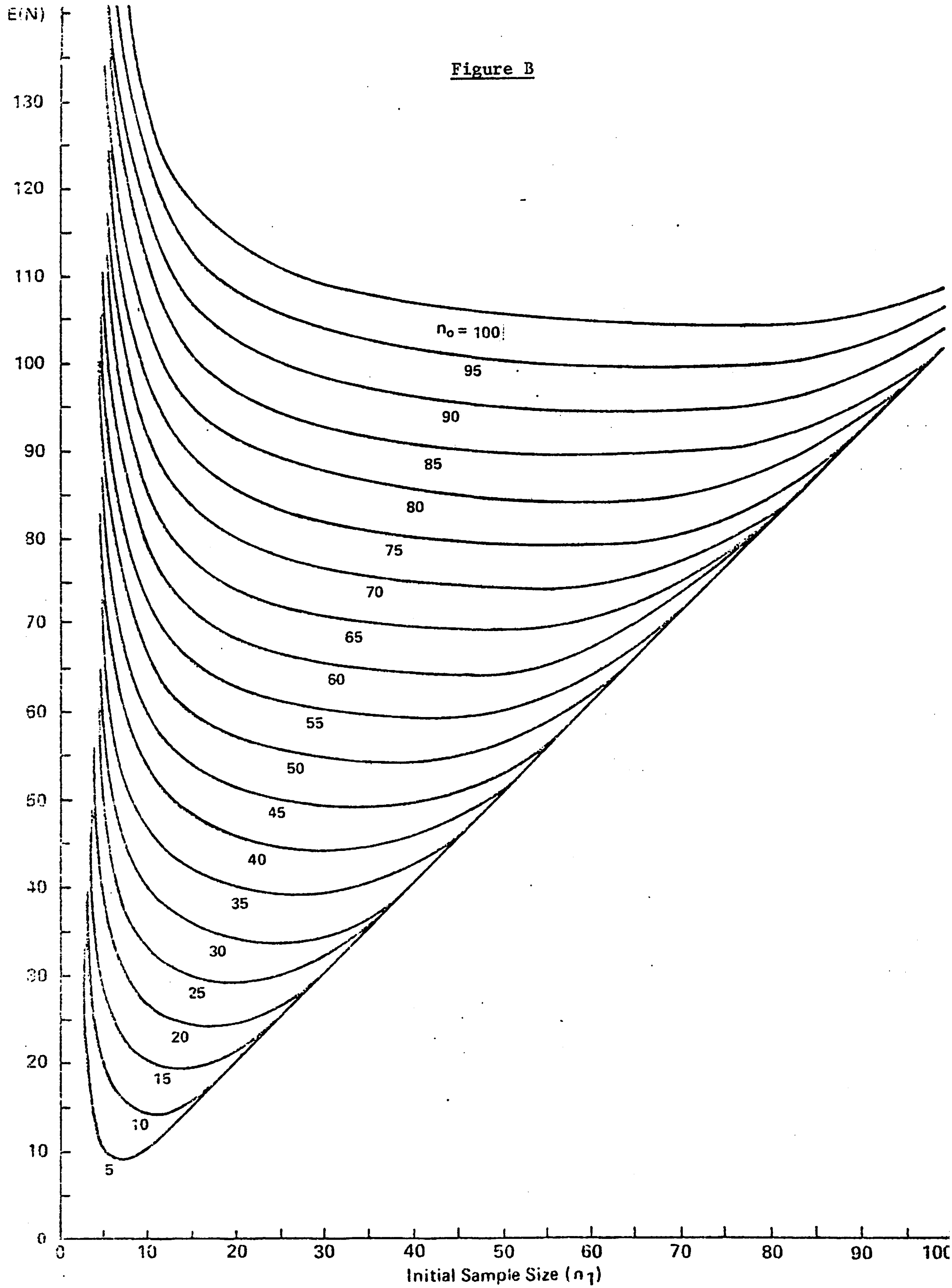
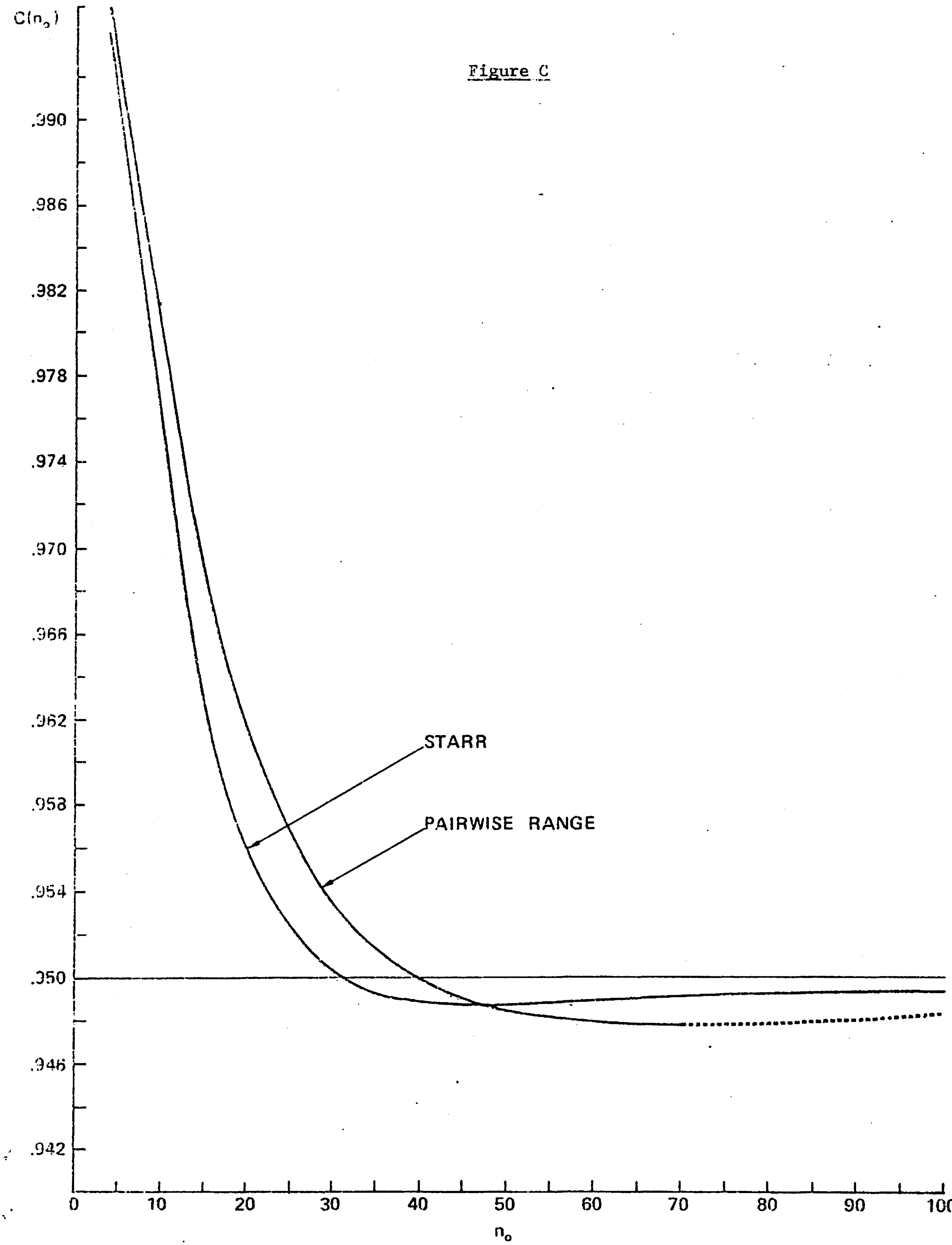

P(N)
E(N) = 31.5
$n_o$ = 25.0
.08
.07
.06
.05
.04
.03
.02
.01
4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68
N

**Figure B**



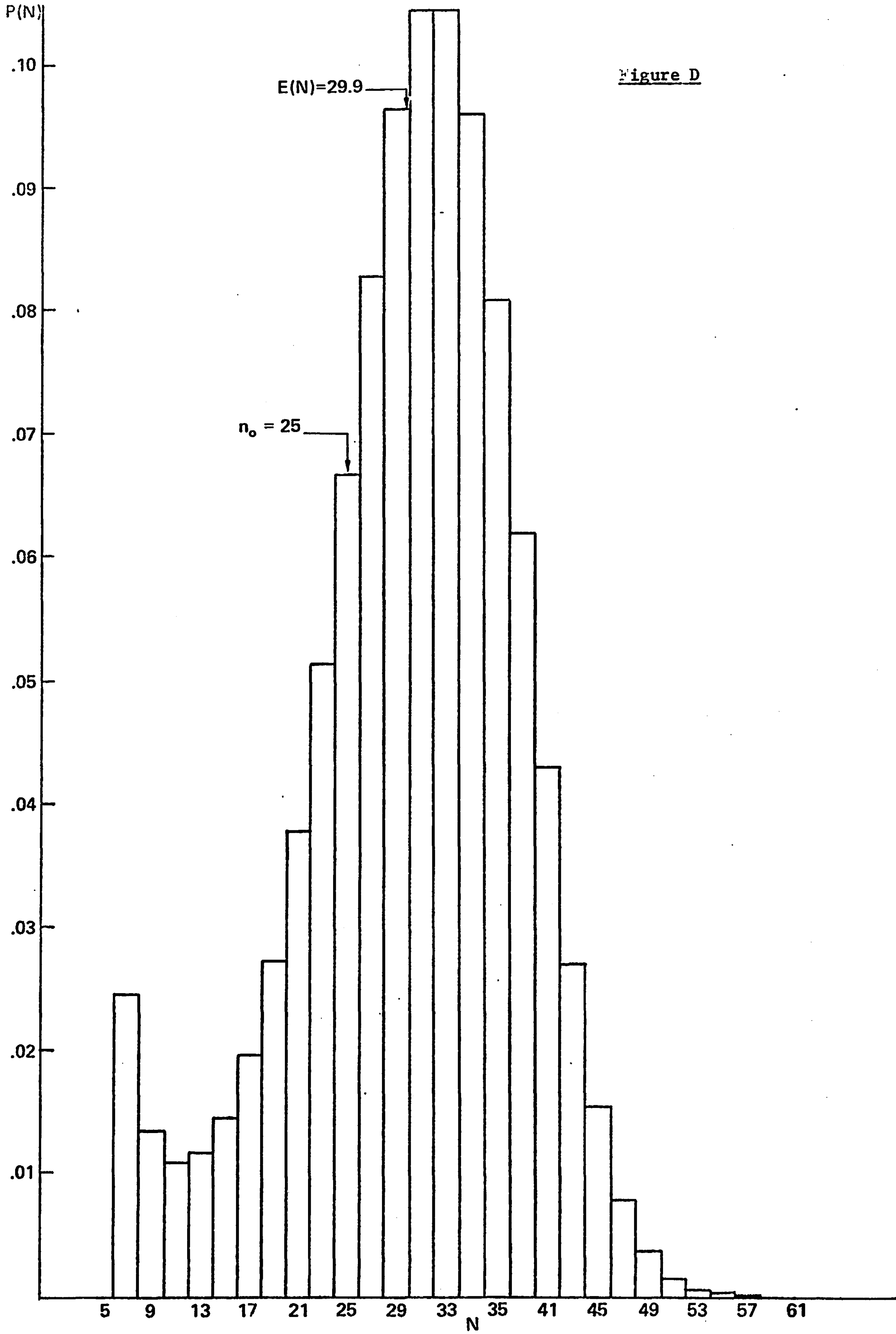
E(N)
130
120
110
100
90
80
70
60
50
40
30
20
10
0
n_o = 100
95
90
85
80
75
70
65
60
55
50
45
40
35
30
25
20
15
10
5
0
10
20
30
40
50
60
70
80
90
100
Initial Sample Size (n1)

**Figure C**



C(n₀)
.990
.986
.982
.978
.974
.970
.966
.962
.958
.954
.950
.946
.942
STARR
PAIRWISE RANGE
0
10
20
30
40
50
60
70
80
90
100
n₀

Figure D



P(N)
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
E(N)=29.9
n_o = 25
5
9
13
17
21
25
29
33
35
41
45
49
53
57
61
N

Figure E

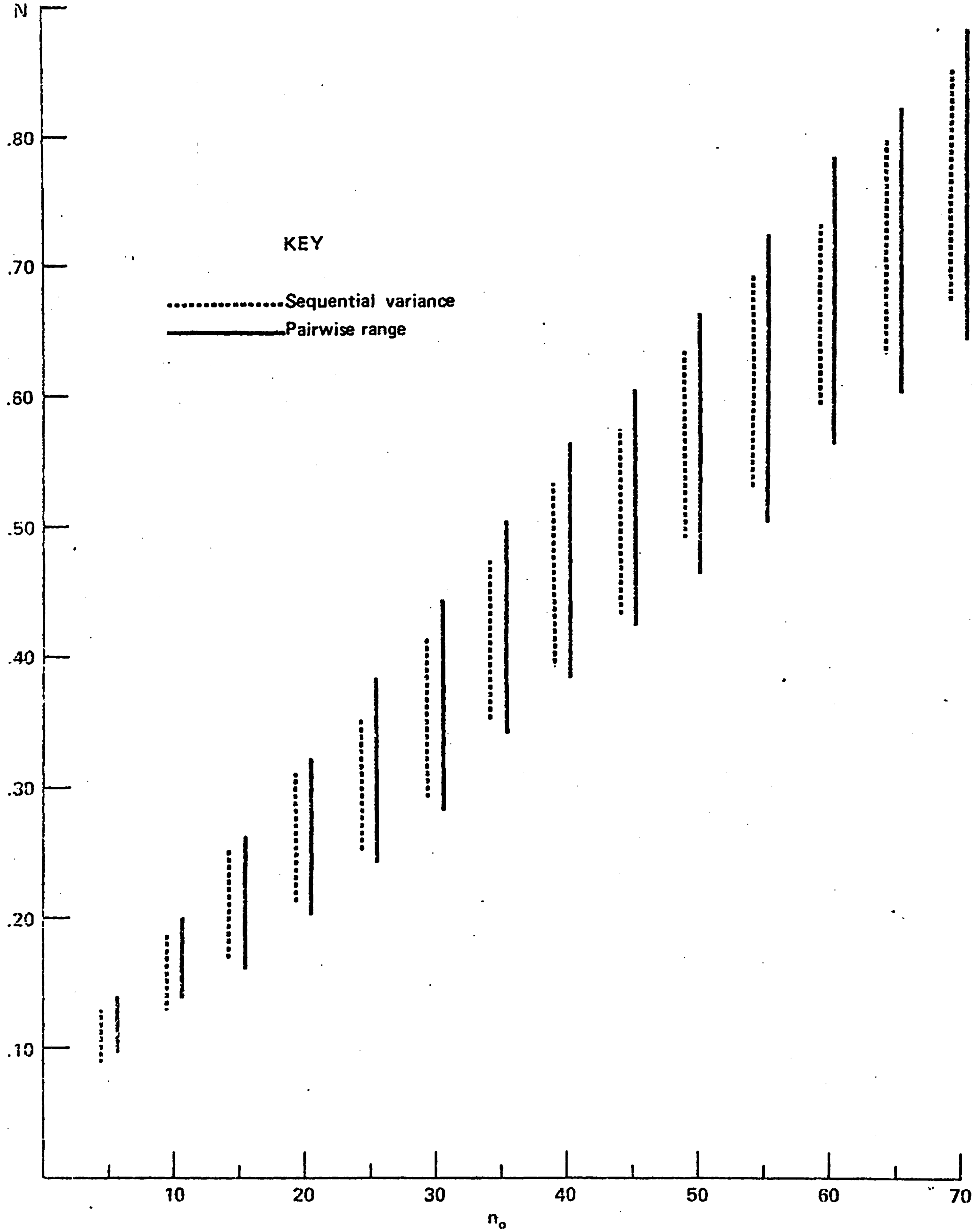
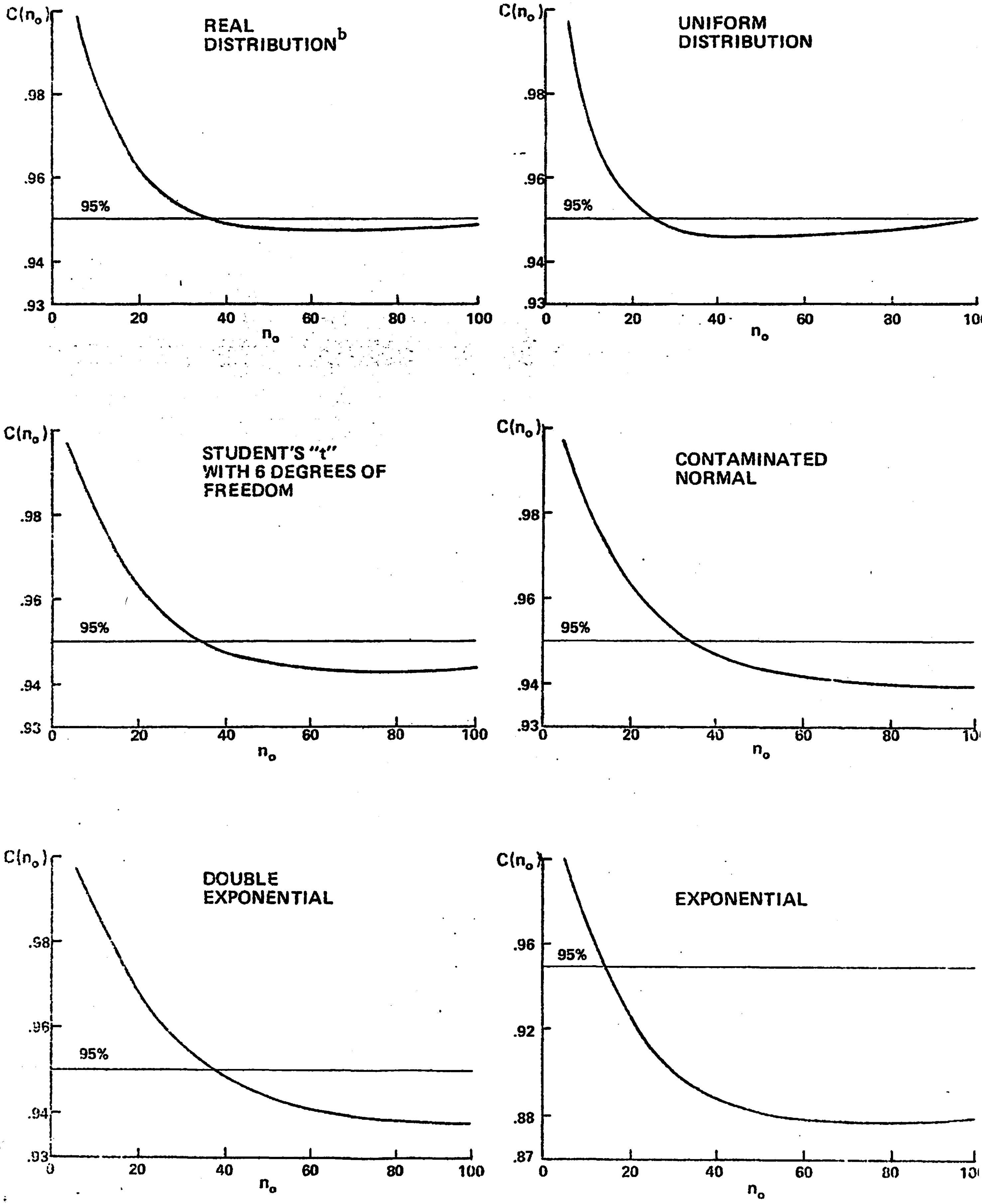
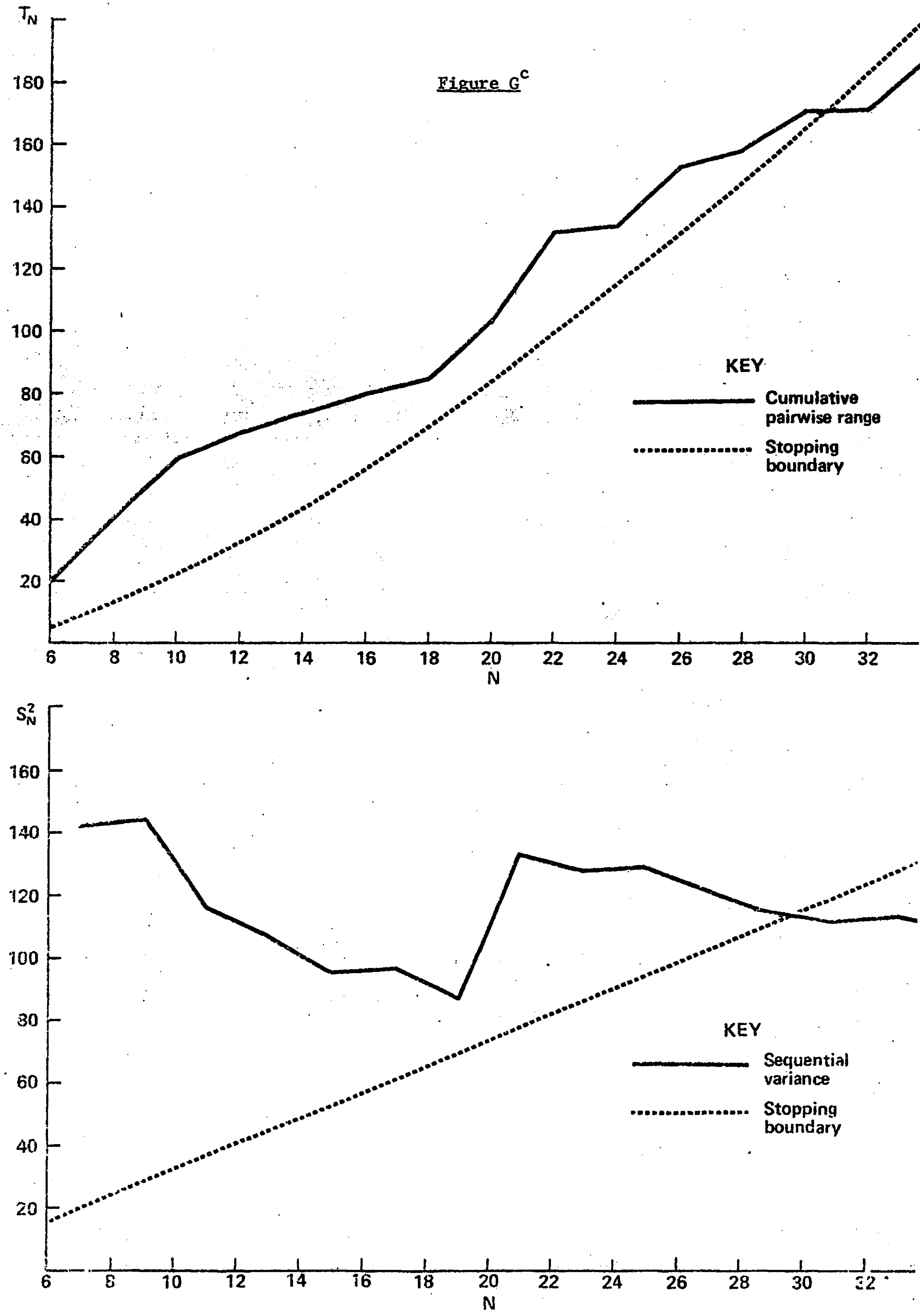

N
.80
.70
.60
.50
.40
.30
.20
.10
KEY
Sequential variance
Pairwise range
10
20
30
40
50
60
70
$n_o$

Figure F[a]



REAL DISTRIBUTION[b]
UNIFORM DISTRIBUTION
STUDENT'S "t" WITH 6 DEGREES OF FREEDOM
CONTAMINATED NORMAL
DOUBLE EXPONENTIAL
EXPONENTIAL
C(n_o)
n_o
95%
.98
.96
.94
.93
.92
.88
.87
0
20
40
60
80
100

Figure $G^c$



$T_N$
180
160
140
120
100
80
60
40
20
6
8
10
12
14
16
18
20
22
24
26
28
30
32
N
KEY
Cumulative pairwise range
Stopping boundary
$S_N^2$
160
140
120
100
80
60
40
20
6
8
10
12
14
16
18
20
22
24
26
28
30
32
N
KEY
Sequential variance
Stopping boundary