TWO-WAY ORDER STATISTICS

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Summary. If a complete two-way table with one observation per cell is written so that the rows and columns are ordered, least to greatest for each, then the entry in the ij-th cell of the reordered table is called the ij-th two-way order statistic. Under the random effects model, the moments of the two-way order statistics can be simply expressed in terms of the moments of the usual (one-way) order statistics. Using these relationships, linear unbiased estimates of row and column effects are obtained. Tests for normality among row effects and among column effects similar to the W test proposed by Shapiro and Wilk (1965) are also derived.

1. Two-way order statistics. Consider an rxc table with one observation per cell. The usual random effects model is as follows:

$$\Omega_{II} \left\{ \begin{array}{l} X_{ij} = \mu + a_i + b_j + e_{ij} \qquad i = 1,2,\ldots,r;\ j = 1,2\ldots,c \\ \{a_i\}, \{b_j\}, \{e_{ij}\}, \text{ are independent, normally} \\ \text{distributed with zero means, and variances} \\ \text{given by } \sigma_A^2, \sigma_B^2 \text{ and } \sigma_e^2 \text{ respectively} \end{array} \right. \tag{1}$$

We shall assume this model throughout.

A subscript replaced by a "." denotes an average over that subscript. Thus, for example, $X_{i.} = \sum_j X_{ij}/c$ is the i-th row mean. Let $X_{(i\cdot)}$ be the i-th smallest row mean, $i = 1,2,\ldots,r$, and $X_{(\cdot j)}$ be the j-th smallest column mean, $j = 1,2,\ldots, c$. We shall call the observation in the row with the i-th smallest mean and in the column with the j-th smallest mean the ij-th two-way order statistics. Thus, when we reorder the rows and columns of the table according to means, least to greatest for each, then $X_{(ij)}$ is the observation in the ij-th cell of the reordered table. For example, Table 1a gives a realization of a 5x5 table of unordered observations from Ω_{II} with $\sigma_A^2 = \sigma_B^2 = \mu = 0$ and $\sigma_e^2 = 1$ (this is the null case of no row or column effects). Table 1b gives the reordered table of the $X_{(ij)}$. Note the general orderliness of the table, with large values near the southeast corner and small values near the northwest corner. In this table, $X_{(55)}$ is not the largest observation in the table.

Table 1 about here

Let $M = X_{..}$ be the grand mean of the observations, $R_i = (X_{i.} - X_{..})$ be the i-th (unordered) row effect, $C_j = (X_{.j} - X_{..})$ be the j-th (unordered) column effect, and $D_{ij} = (X_{ij} - X_{i.} - X_{.j} + X_{..})$ be the ij-th residual. As in analysis of variance, we have the identity

$$X_{ij} = M + R_i + C_j + D_{ij}. \tag{2}$$

The four terms M, R_i, C_j, and D_{ij} lie in orthogonal subspaces and are independent (by normality). Let g be any function of $R' = (R_1,\ldots,R_r)$ and h be any function of $C' = (C_1,\ldots C_c)$. Then M, g(R), h(C), and D_{ij} are independent, and the distributions of M and D_{ij} are unchanged. Specifically, let g be the function that orders the R_i's: $g(R) = (R_{(1)},\ldots,R_{(r)})$ where $R_{(i)} = (X_{(i\cdot)} - X_{\cdot\cdot})$, and let h order the C_j's: $h(C) = (C_{(1)},\ldots,C_{(c)})$ where $C_{(j)} = (X_{(\cdot j)} - X_{\cdot\cdot})$. The D_{ij} are independent and identically distributed and hence are uneffected by permutation of subscripts. Thus the ij-th two-way order statistic can be written

$$X_{(ij)} = M + R_{(i)} + C_{(j)} + D_{(ij)} \tag{3}$$

where the four terms on the R.H.S. of (3) are independent, and $D_{(ij)} = X_{(ij)} - X_{(i\cdot)} - X_{(\cdot j)} + X_{\cdot\cdot}$ is distributed as any of the unordered D_{ij}.

2. Moments of two-way order statistics. We can now prove the following theorem:

Theorem. Let $S_A^2 = \sigma_A^2 + \sigma_e^2/c$, $S_B^2 = \sigma_B^2 + \sigma_e^2/r$ and $S^2 = \sigma_A^2 + \sigma_B^2 + \sigma_e^2$. Then, writing $u_{(i|n)}$ for the i-th order statistics from an independent normal sample of size n, we have

$$E(X_{(ij)}) = \mu + S_A E(u_{(i|r)}) + S_B E(u_{(j|c)}) \tag{4}$$

$$\mathrm{Var}(X_{(ij)}) = S^2 + S_A^2[1 - \mathrm{Var}(u_{(i|r)})] + S_B^2[1 - \mathrm{Var}(u_{(j|c)})] \tag{5}$$

$$\mathrm{Cov}(X_{(ij)},X_{(i\ell)}) = \sigma_A^2 - S_A^2[1 - \mathrm{Var}(u_{(i|r)})] + S_B^2\mathrm{Cov}(u_{(j|c)},u_{(\ell|c)}) \ , \ (j\neq\ell) \tag{6}$$

$$\mathrm{Cov}(X_{(ij)},X_{(kj)}) = \sigma_B^2 + S_B^2\mathrm{Cov}(u_{(i|r)},u_{(k|r)}) - S_B^2[1 - \mathrm{Var}(u_{(j|c)})] \ , \ (i\neq k) \tag{7}$$

$$\mathrm{Cov}(X_{(ij)},X_{(k\ell)}) = S_A^2\mathrm{Cov}(u_{(i|r)},u_{(k|r)}) + S_B^2\mathrm{Cov}(u_{(j|c)},u_{(\ell|c)}) \ , \ (i\neq k, j\neq\ell) \tag{8}$$

Proof. Taking the expected value on both sides of (3) gives

$$E(X_{(ij)}) = E(X_{(i\cdot)}) + E(X_{(\cdot j)}) - \mu. \tag{9}$$

The unordered row means are identically distributed with mean μ, variance $\sigma_A^2 + \sigma_B^2/c + \sigma_e^2/c$ and common correlation $\rho_A = (\sigma_B^2/c)/(\sigma_A^2 + \sigma_B^2/c + \sigma_e^2/c)$.

To find the expectations of the order statistics $X_{(i\cdot)}$ we consider the variables $y_0, y_1, \ldots, y_r$, each independent standard normal. We can then write, following Dunnett and Sobel (1955),

$$X_{i\cdot} = [\sigma_A^2 + \sigma_B^2/c + \sigma_e^2/c]^{\frac{1}{2}}[(1 - \rho_A)^{\frac{1}{2}} y_i + \rho_A^{\frac{1}{2}} y_0] + \mu \ , \ i = 1,2,\ldots,r. \tag{10}$$

Now ordering the y_i induces the same ordering on the $X_{i\cdot}$; hence we can write

$$X_{(i\cdot)} = [\sigma_A^2 + \sigma_B^2/c + \sigma_e^2/c]^{\frac{1}{2}}[(1 - \rho_A)^{\frac{1}{2}} y_{(i)} + \rho_A^{\frac{1}{2}} y_0] + \mu. \tag{11}$$

Taking expected values on both sides of (11) and writing $S_A^2 = \sigma_A^2 + \sigma_e^2/c$ gives

$$E(X_{(i\cdot)}) = \mu + S_A E(u_{(i|r)}). \tag{12}$$

A similar argument leads to

$$E(X_{(\cdot j)}) = \mu + S_B E(u_{(j|c)}) \tag{13}$$

where $S_B^2 = \sigma_B^2 + \sigma_e^2/r$. Substituting (12) and (13) into (9) proves equation (4).

For variances and covariances, noting that the four terms on the R.H.S. of (3) are independent, we have , for some i', j', k', and ℓ',

$$\begin{aligned} \mathrm{Cov}(X_{(ij)}, X_{(k\ell)}) = &\ \mathrm{Cov}(X_{i'j'}, X_{k'\ell'}) \\ &+ [\mathrm{Cov}(X_{(i\cdot)}, X_{(k\cdot)}) - \mathrm{Cov}(X_{i'\cdot}, X_{k'\cdot})] \\ &+ [\mathrm{Cov}(X_{(\cdot j)}, X_{(\cdot \ell)}) - \mathrm{Cov}(X_{\cdot j'}, X_{\cdot \ell'})]. \end{aligned} \tag{14}$$

since all other terms add to zero for all i, j, k, ℓ. With the assistance of (10) and (11), and similar expressions for columns, (14) simplifies to become (5) through (8) in special cases, and the theorem is proved.

For the special case where $r = c = 10$, $\sigma_A^2 = \sigma_B^2 = \mu = 0$, and $\sigma_e^2 = 1$ (the null case), the expected values of the two-way order statistics and their standard deviations are given in Table 2.

Table 2 about here

Properties of moments. For simplicity we shall discuss only the null case with $r = c = n$, say, although the results will be qualitatively applicable to the more general case. For the null case, we find that $E(X_{(nn)})$

reaches its maximum value of 1.05 at $n = 5$ and then slowly returns to zero at rate $\sqrt{(\log n/n)}$ (recall that the expected value of the largest (one-way) order statistic grows large at rate $\sqrt{\log n}$), Gumbel (1958)), implying that, for very large tables, the ordering has only minor effect (on means), as can be seen in Figure 1.

Figure 1

Again setting $r = c = n$, if we expand $\text{S.D.}(X_{(ij)}) = \sqrt{\text{Var}(X_{(ij)})}$ in Taylor series, we get the approximation

$$\text{S.D.}(X_{(ij)}) = 1 - \frac{1}{n+1} + o(n^{-1}) \tag{15}$$

In Figure 2 the largest and smallest standard deviation in an nxn table for $n \leq 20$ is graphed as well as the approximation (15). The maximum fractional error due to using the approximation is approximately $\frac{1}{n+1}$ or about 9% for a 10x10 table and 2% for a 50x50 table.

Figure 2 about here

Unlike the usual (one-way) order statistics, we see from (7), (8), and (9) that some of the covariances between the two-way order statistics will be negative, although all the covariances are small.

For further discussion of the moments, see Weisberg (1973).

3. Linear estimation of S_A and S_B. Let $\theta' = (\mu, S_A, S_B)$ be a parameter vector;

$$X' = (X_{11}, \ldots, X_{1c}, X_{21}, \ldots, X_{2c}, \ldots, X_{rc})$$

be the observation vector and

$$A' = \begin{pmatrix} 1 & ,\ldots, & 1 & , & 1 & ,\ldots, & 1 & ,\ldots, & 1 \\ E(u_{(1|r)}) & ,\ldots, & E(u_{(1|r)}) & , & E(u_{(2|r)}) & ,\ldots, & E(u_{(2|r)}) & ,\ldots, & E(u_{(r|r)}) \\ E(u_{(1|c)}) & ,\ldots, & E(u_{(c|c)}) & , & E(u_{(1|c)}) & ,\ldots, & E(u_{(c|c)}) & ,\ldots, & E(u_{(c|c)}) \end{pmatrix}$$

the design matrix. Then equation (4) can be rewritten

$$E(X) = A\theta. \tag{16}$$

If the variance-covariance matrix of the $X_{(ij)}$ is known (up to a constant $S^2 = \sigma_A^2 + \sigma_B^2 + \sigma_e^2$), then the generalized Gauss-Markov theorem would give the best linear unbiased estimate (BLUE, Lloyd, 1952) of θ. This is, writing $S^2\Sigma$ for the covariance matrix,

$$\hat{\theta} = (A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1}X. \tag{17}$$

In random effects models, assuming Σ known is equivalent to assuming that σ_A^2 and σ_B^2 are known.

To find linear estimates in the one-dimensional case, Gupta (1952) considered the simplification of setting $\Sigma = I$. For the normal distribution the resulting estimates of μ and σ are asymptotically maximum likelihood estimates (Ali and Chan (1964)) and hence are efficient. Even in small samples, the efficiency of Gupta's estimate relative to the BLUE is high (for $n \leq 10$, the efficiency is $\geq 99.2\%$, Chernoff and Lieberman (1954)).

Analogously for the two dimensional case, we set $\Sigma = I$ in (17). The resulting estimates are thus

$$\tilde{\theta} = (A'A)^{-1}A'X. \tag{18}$$

We call $\tilde{\theta}' = (\tilde{\mu}, \tilde{S}_A, \tilde{S}_B)$ the simplified linear unbiased estimate (SLUE). Since $\sum E(u_{(i|r)}) = 0$, $\tilde{\theta}$ simplifies to

$$\tilde{\theta} = \begin{pmatrix} \tilde{\mu} \\ \tilde{S}_A \\ \tilde{S}_B \end{pmatrix} = \begin{pmatrix} X_{\cdot\cdot} \\ \sum \lambda_{i|r} X_{(i\cdot)} \\ \sum \lambda_{j|c} X_{(\cdot j)} \end{pmatrix} \tag{19}$$

where

$$\lambda_{j|c} = E(u_{(j|c)}) / \sum [E(u_{(j|c)})]^2.$$

The coefficients $\lambda_{j|c}$ are given for $2 \leq r, c \leq 20$ in Table 3. Missing values can be filled in by the relations $\lambda_{i|r} = -\lambda_{i+1-r|r}$ and, if $r = 2m+1$, $\lambda_{m+1|r} = 0$. The last column of Table 3 gives $\sum[E(u_{(j|c)})]^2$, which will be needed in Section 4.

Table 3 about here

4. Relation to analysis of variance. Since $\tilde{S}_A$ is a contrast among the (ordered) row means, it follows that

$$\widetilde{SS}_A = \frac{c\tilde{S}_A^2}{\sum \lambda_{i|r}^2} = c\ (\sum [E(u_{(i|r)})]^2)\ S_A^2 \tag{20}$$

is a sum of squares in the row space. It follows from Ali and Chan (1964) that S_A is asymptotically the maximum likelihood estimate of S_A (given normality), so that $\widetilde{SS}_A$ has the same asymptotic distribution as the usual maximum likelihood estimator, $SS_A = c\sum X_{(i\cdot)}^2 - rcX_{\cdot\cdot}^2$. Hence, asymptotically, $\widetilde{SS}_A \sim (r-1)S_A^2 X_{r-1}^2$. Since $\widetilde{SS}_A$ is independent of MS_e, writing $\widetilde{MS}_A = \widetilde{SS}_A/(r-1)$,

$$\tilde{F} = \frac{\widetilde{MS}_A}{MS_e} \tag{21}$$

is asymptotically distributed as $F(r-1, (r-1)(c-1))$ and provides a test of $\sigma_A^2 = 0$. Consideration of Scheffé's (1959) S-method shows that $\tilde{F} \leq MS_A/MS_e$ and hence the test $\tilde{F}$ is conservative. Here, MS_e is the usual within-mean square estimate of σ^2, and MS_A is the rows mean square.

If the row effects are not normally distributed, then expectation of $\widetilde{SS}_A$ will be reduced. For testing the normality assumption in one dimension, Shapiro and Wilk (1965) considered the statistic

$$W = \frac{\widetilde{SS}_A}{SS_A} \tag{22}$$

except that their estimate of S_A was the BLUE rather than the SLUE used here. They have tabulated percentage points of the null distribution of W for $3 \leq n \leq 50$ for $p = .01, .02, .05, .10, .50, .90, .95, .98, .99$. The hypothesis of normality can be rejected if the observed value of W is <u>less than</u> the tabled critical value. Sampling experiments (Weisberg, 1973) indicate that the percentage points of the statistic given at (22) are well approximated by the values given by Shapiro and Wilk, often providing a slightly conservative test.

The statistic given at (22) was discussed by Shapiro and Francia (1972) for use in large samples (they call it W'). They give percentage points for large samples.

<u>Table 4 about here</u>

After similar arguments are applied in the column space, the results can be summarized in the Analysis of Variance Table given as Table 5.

Table 5. Analysis of Variance

Source	d.f.	SS	MS	F	W
Rows	r-1	SS_A	MS_A	MS_A/MS_e	
$\tilde{S}_A$		$\widetilde{SS}_A$	$\widetilde{MS}_A$	$\widetilde{MS}_A/MS_e$	
Lack of Fit		$SS_A-\widetilde{SS}_A$			$\widetilde{SS}_A/SS_A$
Columns	c-1	SS_B	MS_B	MS_B/MS_e	
$\tilde{S}_B$		$\widetilde{SS}_B$	$\widetilde{MS}_A$	$\widetilde{MS}_B/MS_e$	
Lack of Fit		$SS_B-\widetilde{SS}_B$			$\widetilde{SS}_B/SS_B$
Residual	(r-1)(c-1)	SS_e	MS_e		

5. Example. As an example, a 10x10 table was generated such that the row effects were independent and identically drawn from a normal distribution, mean 0, $\sigma_A^2 = 0.1$ (the actual numbers drawn had mean .061 and sample variance .115). The column effects were all set to zero, except in one column the effect was 0.953, chosen so that the sample variance of the column effects would equal 0.1. Table entries were then generated by (row effect) + (column effect) + standard normal (N(0,1)) deviate. The resulting array is to be given in Table 6, in which the ordering according to row and column means has already been accomplished. Thus the cell entries are the two-way order statistics. The ordered table appears to be relatively well behaved with larger values near the southeast corner and smaller values near the northwest corner of the table.

Tables 6 and 7 about here

The Analysis of Variance for this data is given in Table 7. Either F test for rows (H: $\sigma_A^2 = 0$) is significant ($.025 < p < .05$), while the F test for columns is not significant. Furthermore, W for rows is not significant, while W for columns is significant ($.025 < p < .05$). The significant value of W leads to rejection of the random effects model for columns.

The conclusions to be drawn from the analysis are that σ_A^2 is non-zero (it can be estimated in the usual way--c.f. Scheffé (1959)--or, as a more conservative approach, by using $\widetilde{MS}_A$ in place of MS_A) and, while we have no reason to suspect that σ_B^2 is non-zero, we reject the hypothesis of normality among the column means, so that there is some indication that not all the columns are the same.

Residual analysis. The usual residuals from the model are computed as

$$d_{(ij)} = X_{(ij)} - X_{(i\cdot)} - X_{(\cdot j)} + X_{\cdot\cdot} \; . \tag{23}$$

As an added benefit of computing $\widetilde{S}_A$ and $\widetilde{S}_B$, another set of residuals can be computed:

$$\widetilde{d}_{(ij)} = X_{(ij)} - [\, \widetilde{\mu} + \widetilde{S}_A E(u_{(i|r)}) + \widetilde{S}_B E(u_{(j|c)}) \,] . \tag{24}$$

The usual residuals add to zero in each row or column: $\sum_i d_{(ij)} = \sum_j d_{(ij)} = 0$, while the new residuals only add to zero over the whole table: $\sum_{ij} \widetilde{d}_{(ij)} = 0$. Of course, $\sum \widetilde{d}^2_{(ij)} \geq \sum d^2_{ij}$; however, since the error is spread throughout the table in a different fashion, the new set of residuals may be useful in finding failures of the model.

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Table 1.

ØRIGINAL DATA

						RØW MEANS
	.79	-.26	.36	-.44	-.09	.07
	-1.07	.54	2.11	.47	1.58	.72
	.75	2.12	.48	-.12	-.02	.64
	-1.54	1.01	.11	1.13	-1.27	-.11
	.37	-.02	-.98	1.28	-1.17	-.10
CØLUMN MEANS	-.14	.68	.42	.46	-.19	

ØRDERED DATA

						RØW MEANS
	-1.27	-1.54	.11	1.13	1.01	-.11
	-1.17	.37	-.98	1.28	-.02	-.10
	-.09	.79	.36	-.44	-.26	.07
	-.02	.75	.48	-.12	2.12	.64
	1.58	-1.07	2.11	.47	.54	.72
CØLUMN MEANS	-.19	-.14	.42	.46	.68	

Table 2. Expected values and standard deviations of two-way order statistics for a 10x10 null table.

										Row means
	-.9732	-.8033	-.6941	-.6054	-.5254	-.4478	-.3678	-.2791	-.1699	0	-1.5388
	.9321	.9251	.9230	.9221	.9217	.9217	.9221	.9230	.9251	.9321	.5868
	-.8033	-.6333	-.5241	-.4354	-.3554	-.2779	-.1979	-.1092	0	.1699	-1.0014
	.9251	.9181	.9159	.9150	.9146	.9146	.9150	.9159	.9181	.9251	.4632
	-.6941	-.5241	-.4149	-.3263	-.2463	-.1687	-.0887	0	.1092	.2791	-.6561
	.9230	.9159	.9138	.9128	.9125	.9125	.9128	.9138	.9159	.9230	.4183
	-.6054	-.4354	-.3263	-.2376	-.1576	-.0800	0	.0887	.1979	.3678	-.3756
	.9221	.9150	.9128	.9119	.9115	.9115	.9119	.9128	.9150	.9221	.3974
	-.5254	-.3554	-.2463	-.1576	-.0776	0	.0800	.1687	.2779	.4478	-.1227
	.9217	.9146	.9125	.9115	.9112	.9112	.9115	.9125	.9146	.9217	.3887
	-.4478	-.2779	-.1687	-.0800	0	.0776	.1576	.2463	.3554	.5254	.1227
	.9217	.9146	.9125	.9115	.9112	.9112	.9115	.9125	.9146	.9217	.3887
	-.3678	-.1979	-.0887	0	.0800	.1576	.2376	.3263	.4354	.6054	.3756
	.9221	.9150	.9128	.9119	.9115	.9115	.9119	.9128	.9150	.9221	.3974
	-.2791	-.1092	0	.0887	.1687	.2463	.3263	.4149	.5241	.6941	.6561
	.9230	.9159	.9138	.9128	.9125	.9125	.9128	.9138	.9159	.9230	.4183
	-.1699	0	.1092	.1979	.2779	.3554	.4354	.5241	.6333	.8033	1.0014
	.9251	.9181	.9159	.9150	.9146	.9146	.9150	.9159	.9181	.9251	.4632
	0	.1699	.2791	.3678	.4478	.5254	.6054	.6941	.8033	.9732	1.5388
	.9321	.9251	.9230	.9221	.9217	.9217	.9221	.9230	.9251	.9321	.5868
Column	-1.5388	-1.0014	-.6561	-.3756	-.1227	.1227	.3756	.6561	1.0014	1.5388	
means	.5868	.4632	.4183	.3974	.3887	.3887	.3974	.4183	.4632	.5868	

Table 3.

COEFFICIENTS FOR SIMPLIFIED LINEAR UNBIASED ESTIMATES OF S_A AND S_B FOR r AND c LESS THAN OR EQUAL TO 20

r,c	1	2	3	4	5	6	7	8	9	10	$\sum E(u_{(i\|r)})^2$
2	.8862										.6366
3	.5908										1.4324
4	.4484	.1294									2.2956
5	.3640	.1549									3.1950
6	.3078	.1559	.0490								4.1166
7	.2676	.1499	.0698								5.0528
8	.2373	.1420	.0788	.0254							5.9995
9	.2136	.1341	.0823	.0395							6.9539
10	.1944	.1265	.0829	.0475	.0155						7.9143
11	.1787	.1196	.0821	.0520	.0253						8.8793
12	.1654	.1133	.0805	.0545	.0317	.0104					9.8481
13	.1542	.1076	.0785	.0557	.0359	.0176					10.8200
14	.1444	.1024	.0764	.0561	.0386	.0227	.0075				11.7945
15	.1359	.0977	.0742	.0560	.0404	.0263	.0129				12.7712
16	.1284	.0934	.0720	.0555	.0415	.0288	.0170	.0056			13.7497
17	.1218	.0895	.0699	.0548	.0421	.0306	.0200	.0099			14.7299
18	.1158	.0860	.0678	.0540	.0423	.0319	.0223	.0132	.0044		15.7114
19	.1105	.0827	.0659	.0531	.0423	.0328	.0241	.0158	.0078		16.6942
20	.1056	.0796	.0640	.0521	.0422	.0334	.0254	.0178	.0106	.0035	17.6782

NOTE: ALL ENTRIES HAVE A SUPPRESSED MINUS SIGN

Table 4. Critical values for Shapiro and Wilk's W.*

r,c	Level 0.01	0.02	0.05	0.10
3	0.753	0.756	0.767	0.789
4	.687	.707	.748	.792
5	.686	.715	.762	.806
6	.713	.743	.788	.826
7	.730	.760	.803	.838
8	.749	.778	.818	.851
9	.764	.791	.829	.859
10	.781	.806	.842	.869
11	.792	.817	.850	.876
12	.805	.828	.859	.883
13	.814	.837	.866	.889
14	.825	.846	.874	.895
15	.835	.855	.881	.901
16	.844	.863	.887	.906
17	.851	.869	.892	.910
18	.858	.874	.897	.914
19	.863	.879	.901	.917
20	.868	.884	.905	.920

*Abstracted from a larger table in Shapiro and Wilk(1965).

Table 6. Artificial data set arranged to give the two-way order statistics.

											Column means
	-1.66	-1.40	.03	-.38	-1.06	-.70	.86	-2.95	-1.83	1.54	-.75
	.17	.12	-2.19	-1.01	.30	-.38	-2.74	-.21	-1.28	.50	-.67
	-.24	.11	-.57	-1.05	-1.24	-.73	-.02	-.52	1.32	.70	-.22
	-.13	-.86	-2.12	-.46	-1.45	-.34	.52	1.89	-1.11	2.20	-.19
	.20	-1.16	1.68	-.49	-.57	.55	1.09	.49	-.35	-.93	.05
	.01	.17	-.60	-.05	-.24	.75	.28	.48	1.12	.23	.21
	-2.32	1.05	.73	-1.98	2.08	1.12	-.48	1.00	.85	.93	.30
	.21	-.15	1.38	1.70	-.65	-.81	.68	-.53	.53	2.61	.50
	-.38	-.35	-.92	1.92	2.04	.53	.98	1.37	1.00	-.54	.57
	.77	.29	.56	.60	.36	.16	-.71	.09	3.51	2.31	.79
Row means	-.34	-.22	-.20	-.12	-.04	.01	.05	.11	.38	.96	

Table 7. Analysis of Variance for the data in Table 6.

Analysis of Variance

Source	d.f.	SS	MS	F	W
Rows	9	24.076	2.675	2.16	
S_A		23.342	2.594	2.10	
Lack of Fit		.734			.970
Columns	9	12.545	1.394	1.13	
S_B		10.336	1.148	.93	
Lack of Fit		2.209			.824
Residual	81	100.165	1.237		

SLUE Effects

Mean = .0580

$\tilde{S}_A$ = .5431

$\tilde{S}_B$ = .3614

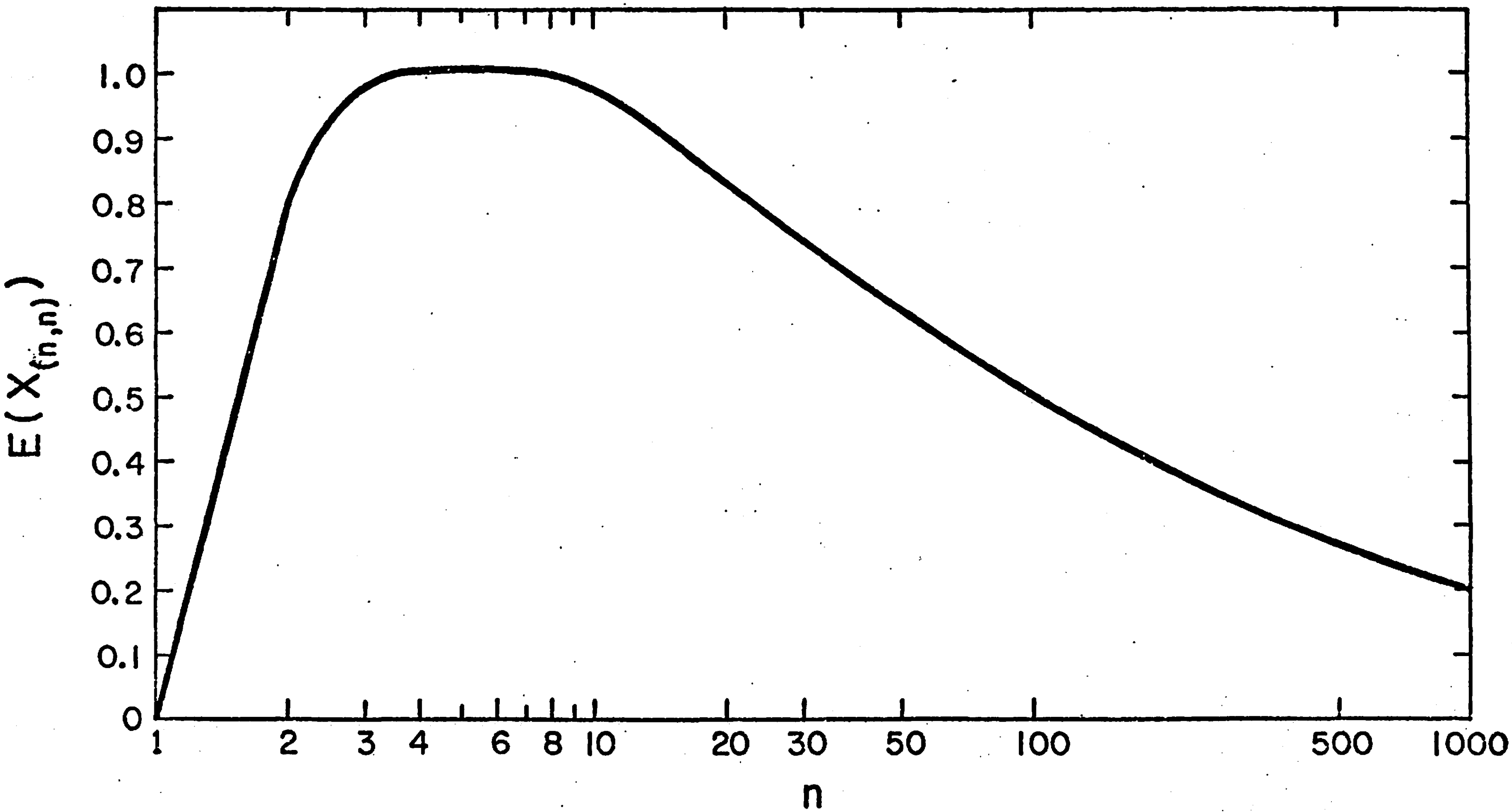

Figure 1. Largest expected value from an n × n standard normal table.

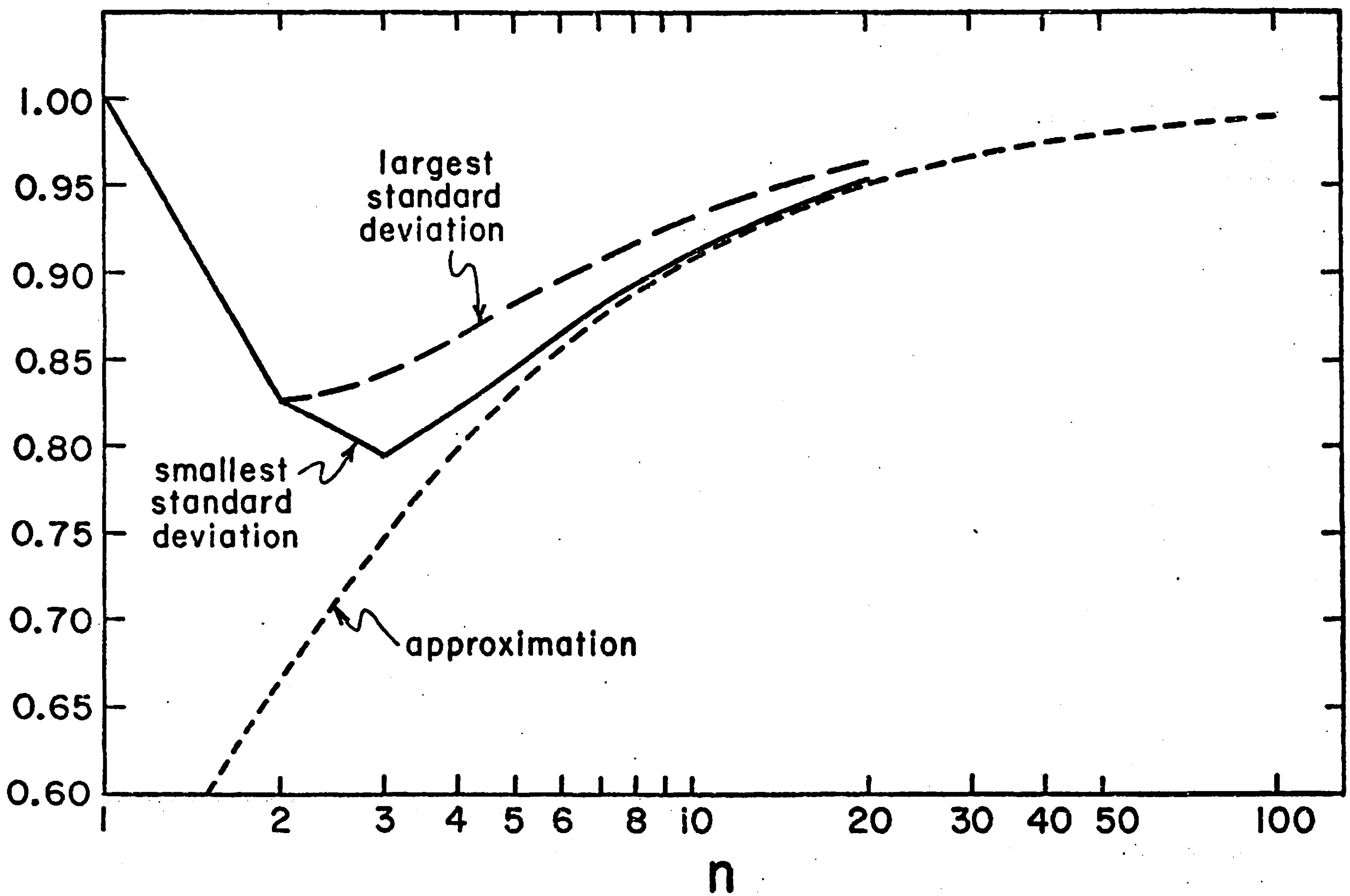

Figure 2. Largest and smallest standard deviation from an n × n normal table along with the approximation $1 - \frac{1}{n+1}$.