RED-AND-BLACK WITH UNKNOWN WIN PROBABILTY

by

Donald A. Berry[1], David C. Heath[2] and William D. Sudderth[3]

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ABSTRACT

A gambler seeks to maximize his probability of reaching a goal in a game where he is allowed at each stage to stake any amount of his current fortune. He wins each bet with a certain fixed probability w. Lester E. Dubins and Leonard J. Savage found optimal strategies for a gambler who knows w. Here strategies are found which are nearly optimal for all w and, therefore, also for a gambler with an unknown w.

1. Introduction.

In continuous red-and-black gambling problems the gambler can stake any amount s of his current fortune f, $0 \leq s \leq f$. If he stakes s his fortune becomes $f + s$ with probability w and $f - s$ with probability $\bar{w} = 1 - w$ where $0 \leq w \leq 1$. The gambler is allowed to gamble repeatedly. We consider a gambler whose objective is to reach fortune 1, so the utility of his strategy is the probability that he attains fortune 1 using that strategy. The problem, which has content only for $0 < f < 1$, is to find a strategy which makes this probability as large as possible.

In the more precise notation and terminology of Dubins and Savage (1965), red-and-black is, for fixed w, a gambling problem in which the set of fortunes, utility of a fortune, and available gambles are as follows: $F = [0, +\infty)$; $u(f) = 1$ or 0 according as $f \geq 1$ or $0 \leq f < 1$; $\Gamma_w(f) = \{w\delta(f+s) + \bar{w}\delta(f-s) : 0 \leq s \leq f\}$ for all $f \in F$. The symbol $\delta(f)$ denotes the measure which assigns mass 1 to $\{f\}$.
A strategy σ available at f in Γ_w is a sequence $\sigma_0, \sigma_1, \ldots$ where $\sigma_0 \in \Gamma_w(f)$ and, for each positive n and each finite sequence $(f_1, \ldots, f_n)$ of elements of F, $\sigma_n(f_1, \ldots, f_n) \in \Gamma_w(f_n)$. The utility of a strategy σ is denoted by $u(\sigma)$ and $U_w(f) = \sup u(\sigma)$ where the supremum is taken over all σ available at f in Γ_w. A strategy σ is ϵ-optimal at f in Γ_w if $u(\sigma) \geq U_w(f) - \epsilon$ for $\epsilon \geq 0$ and is optimal if it is 0-optimal.
Specifying a gamble in $\Gamma_w(f)$ is equivalent to specifying a stake, so that a strategy can as well be defined in terms of stakes. This definition of strategy has the advantage that the same stakes are available at a fortune f for every value of w.

Consider first the case in which w is fixed with $0 \leq w \leq 1/2$. Dubins and Savage (1965) show that in these subfair games it is optimal to play boldly by always staking $\min(f, \bar{f})$ for $0 \leq f \leq 1$ where $\bar{f} = 1 - f$. Denote the utility of the bold strategy starting from f by $B_w(f)$ for $0 \leq w \leq 1$. Some characteristics of $B_w(f)$ and a description of other optimal strategies will be given in Section 3.

Now consider the case in which $1/2 < w \leq 1$. It follows from the general theory of proportional strategies in Chapter 10 of Dubins and Savage (1965), or from a simple direct argument based on the strong law of large numbers, that the strategy which always stakes αf at fortune f for $\alpha > 0$ is optimal for α sufficiently small depending on w, and the utility of such strategies is 1. However, none of these proportional strategies is simultaneously optimal for all w in $(\frac{1}{2}, 1]$.

Here we show that a strategy which always stakes f^2 at f is optimal for $1/2 < w \leq 1$. This result serves as an aid in finding a strategy which is nearly optimal for values of w less than $1/2$ and greater than $1/2$ simultaneously. Roughly, one such strategy is to stake f^2 for f near zero and to make nearly bold stakes elsewhere. Such a strategy is also nearly optimal when the gambler does not know the value of w but has a probability distribution on w. Thus, a gambler can ignore prior information about w without significant loss. A gambler cannot always ignore such information without loss in discrete red-and-black, a problem we briefly consider in which the goal is a positive integer and the fortunes and permitted stakes are also integers.

2. Timid Play is Optimal for $\frac{1}{2} < w \leq 1$.

The timid stake at f is defined by

$$t(f) = f^2 \quad \text{if} \quad 0 \leq f < 1$$
$$\quad = 0 \quad \text{if} \quad f \geq 1.$$

The <u>timid strategy</u> at f in Γ_w is that strategy, available at f in Γ_w, which always stakes $t(f')$ whenever f' is the current fortune.

<u>Theorem 1.</u>

If $1/2 < w \leq 1$ and $f > 0$, then the timid strategy at f has utility 1 and is, therefore, optimal.

<u>Proof:</u>

Let $1/2 < w \leq 1$. Suppose $f > 0$, $0 < \alpha < 1$. A gambler who has fortune f and stakes αf will, at the next stage, have a fortune f_1 which is $(1+\alpha)f$ with probability w and $(1-\alpha)f$ with probability $\overline{w}$. Thus, the expected value of the logarithm of his resultant fortune is

$$E \log f_1 = g(\alpha) + \log f,$$

where $g(\alpha) = w \log(1+\alpha) + \overline{w} \log(1-\alpha)$. Since $g(0) = 0$ and $g'(0) > 0$ there is an $\alpha(w)$ in $(0, 1)$ such that $g(\alpha) > 0$ for $0 < \alpha \leq \alpha(w)$. Therefore, if a gambler has fortune f with $0 < f \leq \alpha(w)$ and makes the timid stake at f, then $E \log f_1 > \log f$.

Consider now the sequence of fortunes $f, f_1, f_2, \ldots$ of a gambler who starts with f and plays timidly. Suppose $0 < f < \alpha(w)$ and set $X = \log f$, and, for $n = 1,2,\ldots,$

$$\begin{aligned} X_n &= \log f_n \quad \text{if} \quad f_i < \alpha(w),\ i = 1,\ldots, n, \\ &= \log f_k \quad \text{if} \quad f_i < \alpha(w),\ i = 1,\ldots, k-1 \quad \text{and} \quad f_k \geq \alpha(w) \text{ for some } k \leq n. \end{aligned}$$

The argument above shows that the process $X, X_1, X_2, \ldots$ is a submartingale. Since the process is also bounded above by $\log(\alpha(w) + \alpha(w)^2)$, it converges almost surely to a random variable Y such that $EY \geq X$ (Theorem VII.4.1s, p. 324 in Doob (1953).). But convergence of the X_n to a finite number less than $\log \alpha(w)$ is clearly impossible and convergence to $-\infty$ can only occur with probability zero since otherwise $EY = -\infty$. Hence, $Y \geq \log \alpha(w)$

almost surely. In other words, a timid gambler who reaches a fortune less than $\alpha(w)$ will, with probability one, attain a fortune larger than $\alpha(w)$ at some future stage. Whenever his current fortune is larger than $\alpha(w)$, his fortune will become 1 before becoming less than $\alpha(w)$ after k or more plays with probability at least w^k, where k is finite and chosen so that, starting from $\alpha(w)$, k consecutive wins enable a timid gambler to reach 1. Since his fortune will be greater than $\alpha(w)$ an unlimited number of times with probability 1 it follows that a timid gambler with positive initial fortune will eventually reach fortune 1 with probability 1. $\square$

There are, of course, many other strategies which are optimal for all w in $(1/2, 1]$. For example, consider a strategy which stakes $f^{1+\delta}$ with $\delta > 0$ whenever the current fortune is f. A trivial modification of the proof of Theorem 1 shows that such a strategy is optimal in superfair red-and-black. Roughly speaking, any strategy will do which makes stakes near zero small enough to ensure that the $\log f_n$ process is expectation increasing for f_n near zero and stakes away from zero large enough (but never the entire fortune) to guarantee that the f_n process does not remain forever in $(0, 1)$ with positive probability. We have not been able to characterize those strategies which are optimal for all w in $(1/2, 1]$, at least not in any satisfactory way.

3. δ-bold strategies are ε-optimal for all w.

Let $0 < \delta < 1/2$ and consider the family of strategies which always stake $b_\delta(f)$ at f where

$$b_\delta(f) = \begin{cases} f^2 & \text{if } 0 < f \leq \delta, \\ f-\delta & \text{if } \delta < f < (1+\delta)/2, \\ \bar{f} & \text{if } (1+\delta)/2 \leq f < 1, \\ 0 & \text{if } f \geq 1. \end{cases}$$

The δ-bold strategy at f in Γ_w is that strategy available at f in Γ_w which stakes $b_\delta(f')$ whenever f' is the current fortune. Playing δ-boldly means playing timidly for $f \in (0, \delta]$ and boldly for $f \in (\delta, 1)$, but as though the fortune is $f - \delta$ instead of f.

Theorem 2.

For every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that for all $\delta \in (0, \delta(\varepsilon))$, $w \in [0, 1]$, and $f \in F$, the δ-bold strategy is ε-optimal at f in Γ_w.

Proof:

If $w \in \{0\} \cup (1/2, 1]$, then δ-bold play is optimal at every $f \in F$ for all $\delta \in (0, 1/2)$. This is clear if w is 0 or 1, so suppose $1/2 < w < 1$. By Theorem 1 a δ-bold gambler whose initial fortune is in $(0, \delta)$ will attain a fortune at least as large as δ with probability 1. A δ-bold gambler whose initial fortune is in $[\delta, 1)$ will attain fortune 1 after at most $k + 1$ plays with probability at least $\bar{w}w^k$, where k is a positive integer such that a δ-bold gambler starting from δ reaches 1 after k consecutive wins. Therefore the fortune of a δ-bold gambler reaches 1 with probability 1 for $1/2 < w < 1$.

Now fix $\varepsilon > 0$ and consider $w \in (0, 1/2)$. For all f, $U_w(f) = B_w(f)$ which is continuous and increasing on $[0, 1]$. Also, $B_w(f) < B_{w'}(f)$ for $w < w'$ and $f \in (0, 1)$. It follows from formula 5.2.1 of Dubins and Savage (1965) that $B_w(3/4) = w + \bar{w}w$, which tends to 0 as $w \to 0$. Choose $w_0 < 1/2$ such that, for $0 \leq w \leq w_0$, $B_w(3/4) < \varepsilon$. For $w \leq w_0$ and $f \leq 3/4$, $B_w(f) < \varepsilon$ and every strategy is ε-optimal at f. For $w \leq w_0$ and $f > 3/4$ the δ-bold strategy at f with $0 < \delta < 1/2$ makes the same stakes as an optimal strategy (namely, the bold strategy) at fortunes greater than $3/4$ and continues with an ε-optimal strategy if the gambler's fortune ever becomes less than $3/4$. It follows that, for $0 < \delta < 1/2$ and $0 < w \leq w_0$, the δ-bold strategies are ε-optimal for all f.

The case $w \in (w_0, 1/2]$ remains to be considered. For $f \in (\delta, 1)$, the probability that the fortune of a δ-bold gambler starting from f reaches 1 before δ is $B_w(\frac{f-\delta}{1-\delta})$ and so the utility of the δ-bold strategy at f is larger than $B_w(f-\delta)$. Suppose k and n are nonnegative integers with $k < 2^n$ and

$$(k+1)2^{-n} \geq f > f-\delta \geq k2^{-n} .$$

Use first the fact that $B_w(f)$ is increasing in f for each w and then formula 5.2.2 of Dubins and Savage (1965) to get

$$B_w(f) - B_w(f-\delta) \leq B_w((k+1)2^{-n}) - B_w(k2^{-n}) = \bar{w}^a w^{n-a},$$

where a is the number of 1's in the binary expansion of $k2^{-n}$. It follows that

$$B_w(f) - B_w(f-\delta) \leq [\max(1-w_0, 1/2)]^n.$$

Let n be large enough to guarantee that the right side is less than ε and let $\delta(\varepsilon) = \min(\varepsilon, 2^{-n})$. It follows that, for $\delta < \delta(\varepsilon)$, $w \in (w_0, 1/2]$, and $f \in (\delta, 1)$, δ-bold play is ε-optimal.

The conclusion holds trivially for $f \in [0, \delta]$. For such an f, if $\delta < \delta(\varepsilon)$ and $w \leq 1/2$, then $B_w(f) \leq B_w(\delta) \leq B_{\frac{1}{2}}(\delta) = \delta < \varepsilon$. □

As was the case with the timid strategy in the previous section, the δ-bold strategies are not uniquely ε-optimal for all f and w. The argument used to prove Theorem 2 can be modified to show that any family of strategies which makes timid or similar stakes near zero, bold stakes near 1, and sufficiently close to bold stakes (but never the entire fortune) elsewhere has the desired property. For example, a strategy which stakes $\min(f^{1+\delta}, \bar{f})$ for $\delta > 0$ whenever the current fortune is f is ε-optimal for all w if δ is sufficiently small.

Another sort of nonuniqueness arises from the fact that bold play is not uniquely optimal in subfair red-and-black. Suppose n is a positive integer and k is a nonnegative integer with $k2^{-n} < f < (k+1)2^{-n}$. If $w < 1/2$, then according to Dubins and Savage (1965, Section 5.4) an optimal strategy at f is to play boldly to scale on the interval $(k2^{-n}, (k+1)2^{-n})$ followed by some optimal continuation after reaching either endpoint. Likewise, an ε-optimal strategy for all w is to play $\delta(\varepsilon/2)$-boldly to scale on such an interval followed by some continuation which is $\varepsilon/2$-optimal for all w.

4. Red-and-black with a probability distribution on w.

Suppose w is not known precisely, but is a random variable whose distribution is a probability measure π on $[0, 1]$. The outcome of each gamble may now affect future stakes for two reasons: the gambler's fortune f changes and his information regarding w changes. The object is to reach 1 as before.

Consider the following gambling problem. The set of fortunes is $G = \{(f, \pi): f \in F, \pi \in \mathcal{P}\}$, where $\mathcal{P}$ can be taken to be the collection of countably additive probabilities defined on the Borel subsets of $[0, 1]$ or the collection of finitely additive probabilities defined on all subsets of $[0, 1]$. The gambler's utility for $(f, \pi) \in G$ is $u(f, \pi) = u(f)$ which is again 0 or 1 according as $f < 1$ or $f \geq 1$. For $\pi \in \mathcal{P}$, let $E\pi = \int w d\pi(w)$ and $\overline{E\pi} = \int \bar{w} d\pi(w)$ in a slight departure from standard notation, and define $\pi_W, \pi_L \in \mathcal{P}$ by

$$\pi_W(B) = \frac{1}{E\pi} \int_B w d\pi(w) \quad \text{if } E\pi \neq 0,$$

$$= \pi(B) \quad \text{if } E\pi = 0,$$

$$\pi_L(B) = \frac{1}{\overline{E\pi}} \int_B \bar{w} d\pi(w) \quad \text{if } \overline{E\pi} \neq 0,$$

$$= \pi(B) \quad \text{if } \overline{E\pi} = 0,$$

for appropriate sets B. π_W and π_L reflect information present about

w after observing a win and loss, respectively. The convention that $\pi_W = \pi$ when $E\pi = 0$ and $\pi_L = \pi$ when $E\pi = 1$ is somewhat arbitrary, but of little importance since win and loss, respectively, are events of probability zero. To stake s at (f, π) is to use the gamble $g(s, f, \pi) = (E\pi)\delta(f+s, \pi_W) + (\overline{E\pi})\delta(f-s, \pi_L)$ and the available gambles are defined by $\Gamma(f, \pi) = \{g(s, f, \pi): 0 \leq s \leq f\}$. Let U be the utility of the house Γ. That is, $U(f, \pi)$ is the supremum over all strategies σ available at (f, π) in Γ of the σ-probability that 1 is reached. The <u>δ-bold</u> <u>strategy</u> at (f, π) in Γ is that strategy available at (f, π) in Γ which stakes $b_\delta(f')$ whenever the current fortune is (f', π').

The next result is a natural consequence of Theorem 2.

<u>Theorem 3</u>.

For $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that, for all $\delta \in (0, \delta(\varepsilon))$ and all $(f, \pi) \in G$, the δ-bold strategy at (f, π) is ε-optimal. Furthermore, for all $(f, \pi) \in G$, $U(f, \pi) = \int U_w(f)d\pi(w) = \pi(w > 1/2) + \int_{w\leq\frac{1}{2}} B_w(f)d\pi(w)$.

<u>Proof:</u>

For $(f, \pi) \in G$, let $Q(f, \pi) = \int U_w(f)d\pi(w)$. Then $Q \geq u$ since $U_w \geq u$ for all w. Also, for $0 \leq s \leq f$,

$$g(s, f, \pi)Q = (E\pi)Q(f+s, \pi_W) + (\overline{E\pi})Q(f-s, \pi_L)$$

$$= \int [wU_w(f+s) + \bar{w}\, U_w(f-s)]d\pi(w) \leq \int U_w(f)d\pi(w),$$

since U_w is excessive for Γ_w. Therefore, $g(s, f, \pi)Q \leq Q(f, \pi)$.

Thus Q is excessive for Γ and, by Theorem 2.12.1 of Dubins and Savage (1965), $Q \geq U$. (A different proof of this is based on the obvious fact that the gambler can do at least as well knowing the value of w as he can without such knowledge.)

Let $\varepsilon > 0$ and $\delta(\varepsilon)$ be as in Theorem 2. Suppose $0 < \delta < \delta(\varepsilon)$ and let σ_w, σ be the δ-bold strategies available at f, (f, π) in Γ_w, Γ, respectively. Then $U(f, \pi) \geq u(\sigma) = \int u(\sigma_w)d\pi(w) \geq \int (U_w(f)-\varepsilon)d\pi(w) = Q(f, \pi) - \varepsilon$. □

It is interesting that the δ-bold gambler does essentially as well as he could expect to do while ignoring his distribution π and the results of the gambles. Another ε-optimal strategy (at least in the countably additive case) is to stake 0 until the distribution of w assigns probability at least $1 - \varepsilon$ to the set $[0, 1/2)$ or $[1/2, 1]$ and play timidly or boldly thereafter. This strategy is less appealing since it seems less likely to be helpful in an analysis of a problem in which there is a minimum positive stake, as is the case in the problem considered next.

5. Discrete red-and-black.

Let n be a positive integer, $F = \{0,1,\ldots, n\}$, and $0 \leq w \leq 1$. If f is 0 or n, define $\Gamma_w(f) = \{\delta(f)\}$; if $f = 1,2,\ldots, n-1$, define $\Gamma_w(f) = \{w\delta(f+s) + \bar{w}\delta(f-s)\colon s = 1,\ldots, n-f\}$. Set $u(n) = 1$ and $u(f) = 0$ for $f = 0,\ldots, n-1$. This defines a discrete red-and-black gambling problem analogous to the continuous one considered in previous sections; now the goal is $f = n$. For a fixed value of w, optimal strategies are known. If $0 \leq w \leq 1/2$, then one optimal strategy is to play boldly by always staking the largest available amount; this result is an easy consequence of the corresponding one for continuous red-and-black. If $1/2 \leq w \leq 1$, then the timid strategy which always makes the least possible stake is optimal. This is easily proved using Theorem 2.12.1 of Dubins and Savage (1965) and the formula for the utility of timid play given in Section XIV.2 of Feller (1957). If $1/2 < w < 1$, timid play is uniquely optimal.

Unlike the continuous case, there need not exist strategies which are ε-optimal for all w. For example, if $n = 2m > 2$, then the uniquely optimal stake at $f = m$ is m for $w \in (0, 1/2)$ and is 1 for $w \in (1/2, 1)$. (Here an optimal (ε-optimal) stake at f is one which is the first stake in some optimal (ε-optimal) strategy at f.) Since there are only a finite number of available stakes, none of them is ε-optimal for all w when ε is sufficiently small.

Another feature of the problem which differs from the continuous case is that the gambler cannot safely ignore information about w. A striking illustration of this is that if $n = 8$ (chosen as the smallest interesting power of 2), it is optimal for any fixed value of w to stake 1 at the fortune $f = 5$ (and uniquely optimal for $w \in (1/2, 1)$), but, as computer calculations indicate, 1 is not an optimal stake when w has the prior distribution which gives probability 1/2 to each of the values 1/4 and 3/4 (3 is the optimal stake).

The discrete problem with a probability distribution on w can, of course, be formulated as a gambling problem by analogy with the continuous problem in Section 4. It is possible to get information about this discrete problem from the continuous one when n is large, but interesting, exact results about optimal strategies seem difficult to obtain.

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