ON TUKEY'S TEST OF ADDITIVITY

by

Basilio A. Rojas

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Basilio A. Rojas

School of Statistics, Department of Applied Statistics
University of Minnesota

SUMMARY

It is shown that Tukey's test of additivity is equivalent to the test of a regression coefficient in a model with a single covariate. The derivation given is general for the standard experimental designs, like complete and incomplete randomized block and Latin squares. Expressions for the covariate or concommitant variable are given. An example of a Latin square is shown.

INTRODUCTION

Tukey [1949] singled out one degree of freedom from the error sum of squares in a two-way classification to detect non-additivity between rows and columns (treatments and blocks). He later, Tukey [1955], generalized the method to other classifications. Several books on statistical methods and linear hypothesis have been written and all follow the approach Tukey used to present his additivity test, some with emphasis on calculation procedures and others on the distribution properties. However, the author feels that there has been a degree of mystery on the source of not very simple expressions for non-additivity tests.

This paper shows that Tukey's tests of additivity can be obtained from a usual analysis of covariance. This knowledge may be useful to applied statisticians who can investigate non-additivity in several experimental situations in a straightforward manner.

ANALYSIS OF COVARIANCE FOR ADDITIVITY TESTS

A three-way classification with n_{ijk} observations per cell, where $n_{ijk} = 0$ or 1, will be used for procedures described here. Application to randomized complete and incomplete blocks, complete and incomplete Latin squares and other experimental designs will follow directly.

Analysis of Covariance.

Let us assume the following three-way classification model with dependent variable y_{ijk}, covariate x_{ijk}, and additive effects α_i, β_j and γ_k,

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \theta x_{ijk} + \epsilon_{ijk} \tag{1}$$

where θ is the regression coefficient. As stated above, the number of observations in cell ijk is one or zero. The error ϵ_{ijk} is assumed normally and independently distributed with mean zero and variance σ^2.

We are here interested in inferences about θ for which, as it is well known, the analysis of covariance offers a systematic procedure. This technique basically makes the following transitory assumption about y_{ijk} and x_{ijk}:

$$y_{ijk} = \mu^o + \alpha_i^o + \beta_j^o + \gamma_k^o + \epsilon_{ijk}^o \tag{2}$$

$$x_{ijk} = \mu' + \alpha_i' + \beta_j' + \gamma_k' + \epsilon_{ijk}' \tag{3}$$

This means that both the dependent variable and the covariate are made equal to linear models with respective parameters and errors as shown in (2) and (3). Errors ϵ_{ijk}^o and ϵ_{ijk}' are assumed uncorrelated random variables with means zero and constant variances. If least-squares method is applied to models (2) and (3) in turn, we obtain the respective b.l.u.e.'s, (best linear unbiased estimator), and we can write

$$\hat{y}_{ijk} = \hat{\mu}^o + \hat{\alpha}_i^o + \hat{\beta}_j^o + \hat{\gamma}_k^o$$

$$\hat{x}_{ijk} = \hat{\mu}' + \hat{\alpha}_i' + \hat{\beta}_j' + \hat{\gamma}_k'$$

where $\hat{\alpha}_i^o$, $\hat{\alpha}_i'$ are, for example, the b.l.u.e.'s of α_i^o, α_i', respectively.

The sums of squares for error in the analysis of covariance are equivalent to $E_{yy} = \Sigma(y_{ijk} - \hat{y}_{ijk})^2$ and $E_{xx} = \Sigma(x_{ijk} - \hat{x}_{ijk})^2$. The sum of cross products for error is $E_{xy} = \Sigma(y_{ijk} - \hat{y}_{ijk})(x_{ijk} - \hat{x}_{ijk})$. The b.l.u.e. for θ in model (1) is then $\hat{\theta} = E_{xy}/E_{xx}$. It is also known that the sums of squares for reduction in error due to the regression, denoted by R, is

$$R = \frac{(E_{xy})^2}{E_{xx}} = \frac{[\Sigma(y_{ijk} - \hat{y}_{ijk})(x_{ijk} - \hat{x}_{ijk})]^2}{\Sigma(x_{ijk} - \hat{x}_{ijk})^2} . \qquad (4)$$

It is also known that R/σ^2 and $(E_{yy} - R)/\sigma^2$ are independently distributed and follow χ^2 distributions with 1 and $(f-1)$ degrees of freedom, respectively, where f is the number of degrees of freedom for error E_{yy}. Under the hypothesis that $\theta = 0$ both distributions are central χ^2 and therefore we use the variance ratio

$$F = R(f-1)/(E_{yy} - R) \qquad (5)$$

as the criterium to test $\theta = 0$.

The procedure described before is valid for model (1) if the errors follow the assumptions stated above and also the values of covariate x_{ijk} are known without error. They are still valid when x_{ijk} is subject to an error normally distributed, which may be correlated with ϵ_{ijk}, error in (1) for the same subscripts, but all other correlations must be zero. See, for example, Scheffé [1959].

Covariate in Additivity Test.

Effects α, β, and γ are additive in a three-way classification when

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \varepsilon^o_{ijk} \ . \tag{7}$$

There are cases, however, in which the existence of non-additive effects is suspected. Model (7) may be extended as follows

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\beta)_{ik} + (\beta\gamma)_{jk} + \varepsilon_{ijk} \tag{8}$$

where the first-order interactions have been inserted. However, when the number of observations per cell is at most one the complete set of parameters involved in (8) can be nonestimable. There are many ways to circumvent this difficulty and one could think of more appropriate solutions suited to particular circumstances. Perhaps the simplest approach is taken by assuming that the non-additivity part is explained as follows

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \theta(\alpha_i\beta_j + \alpha_i\gamma_k + \beta_j\gamma_k) + \varepsilon_{ijk} \tag{9}$$

where θ is an unknown constant. However, model (9) has the great disadvantage of being non-linear with all inherent complications. To avoid this difficulty we modify model (9) by introducing a known or estimable covariate x_{ijk} instead of $\mathcal{E}_{ijk} = \alpha_i\beta_j + \alpha_i\gamma_k + \beta_j\gamma_k$. Covariate x_{ijk} must be highly correlated with $\mathcal{E}_{ijk}$ in order that the following model

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \theta x_{ijk} + \varepsilon_{ijk} \tag{10}$$

fulfills the objective of explaining non-additivity. In model (10) the importance of the non-additive part will be given by the magnitude of θ and the test of the hypothesis $\theta = 0$ will correspond to a test of additivity.

An intuitively suggested expression for x_{ijk} is the following

$$x_{ijk} = \hat{\alpha}_i\hat{\beta}_j + \hat{\alpha}_i\hat{\gamma}_k + \hat{\beta}_j\hat{\gamma}_k \tag{11}$$

where $\hat{\alpha}_i$, $\hat{\beta}_j$ and $\hat{\gamma}_k$'s are the b.l.u.e.'s of α_i, β_j and γ_k's, respectively in the additive model (7).

If x_{ijk} values are assumed known without error and ε_{ijk}'s in model (10) are normally and independently distributed, then the corresponding analysis of covariance provides a valid test for $\theta = 0$ conditional on the given set of x_{ijk} values. However, covariate x_{ijk} enjoys interesting properties under the hypothesis of complete additivity: i) the true value of x_{ijk}, $\mathcal{E}_{ijk}$, is zero whatever values α_i, β_j and γ_k might take, ii) $E(x_{ijk}) = \mathcal{E}_{ijk} = 0$ for orthogonal designs, like complete block and Latin squares; iii) $E(x_{ijk}) = \mathcal{E}_{ijk} + Cov(\hat{\alpha}_i, \hat{\beta}_j) + Cov(\hat{\alpha}_i, \hat{\gamma}_k) + Cov(\hat{\beta}_j, \hat{\gamma}_k)$ = zero plus a constant bias not depending on α_i, β_j and γ_k; and iv) x_{ijk}, as a function of $\hat{\alpha}_i$, $\hat{\beta}_j$ and $\hat{\gamma}_k$'s, contains errors which are independently distributed of the errors $\hat{\varepsilon}^o_{ijk}$ in model (7), which are equal to errors $\hat{\varepsilon}_{ijk}$ in model (10), under additivity. Furthermore, the sum of squares for regression R, given in (4), is invariant for any effects $\hat{\alpha}_i$, $\hat{\beta}_j$, and $\hat{\gamma}_k$'s added to covariate x_{ijk}. This means that we could use a new covariate $x'_{ijk} = x_{ijk} + A_i + B_j + C_k + D$ and the corresponding R would be the same as for x_{ijk}. This result follows from the fact that the estimated residuals $\hat{\varepsilon}'_{ijk}$ in model (3) remain the same for x_{ijk} and x'_{ijk}. It is for all previous reasons that we can say that the variance ratio F given in (5) is an unconditional test of additivity, for the F ratio and the distributional properties of R and $E_{yy} - R$ are unaffected under the hypothesis $\mathcal{E}_{ijk} = 0$. The rationalization developed here is in fact an extension of Scheffě's [1959], pages 132-133, proof of the distribution properties of Tukey's test for the two-way classification.

It was indicated in the previous paragraph that covariate x_{ijk} can be changed into expressions which contain any additive effects A_i, B_j and C_k. Therefore, the covariate can take the following forms, without affecting the value R,

$$z_{ijk} = a(\hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_k)^2;\ w_{ijk} = (\hat{y}_{ijk})^2;\ u_{ijk} = a(\hat{y}_{ijk} - b)^2 \quad (12)$$

where $\hat{y}_{ijk} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_k$, a and b are arbitrary constants, and $\hat{\mu}$, $\hat{\alpha}_i$, $\hat{\beta}_j$ and $\hat{\gamma}_k$'s are the b.l.u.e.'s of the respective parameters in the additive model (7). Formulas (11) and (12) of covariate lead to the same F value (5) to test additivity. The particular form to be chosen depends on circumstances. For a p-way classification form x_{ijk} is the most elaborate for it requires the calculation of $p(p-1)/2$ products for each x_{ijk}. There are computer programs which provide the $\hat{y}_{ijk}$ values and forms w_{ijk} of u_{ijk} can be preferred. Form u_{ijk} can be suggested in order to avoid the large numbers that w_{ijk} can have in some cases.

TUKEY'S TESTS OF ADDITIVITY

It will be shown now that Tukey's tests of additivity for the randomized complete blocks and Latin square designs are equivalent to the use of an analysis of covariance with a covariate in one of forms (11) or (12). First we will obtain a simpler form to the sum of squares R given in (4) for a three-way table such that the groups $\{\alpha_i\}$, $\{\beta_j\}$ and $\{\gamma_k\}$ are orthogonal. If the general form of the covariate is denoted by v_{ijk}, where v_{ijk} can take any of expressions (11) or (12), it is well known that we can write

$$\left.\begin{aligned} \hat{v}_{ijk} &= v_{i..} + v_{.j.} + v_{..k} - 2v_{...}\ ;\ \hat{\varepsilon}'_{ijk} = v_{ijk} - \hat{v}_{ijk} \\ \hat{y}_{ijk} &= y_{i..} + y_{.j.} + y_{..k} - 2y_{...}\ ;\ \hat{\varepsilon}^{o}_{ijk} = y_{ijk} - \hat{y}_{ijk} \end{aligned}\right\} \quad (13)$$

It is clear that

$$\sum_i \hat{\varepsilon}^o_{ijk} = \sum_j \hat{\varepsilon}^o_{ijk} = \sum_k \hat{\varepsilon}^o_{ijk} = 0$$

with equivalent expressions for $\hat{\varepsilon}'_{ijk}$. From which we can write

$$\Sigma(y_{ijk} - \hat{y}_{ijk})(v_{ijk} - \hat{v}_{ijk}) = \Sigma(y_{ijk} - \hat{y}_{ijk})v_{ijk}$$

and expression (4) can be written as follows

$$R = [\Sigma(y_{ijk} - y_{ijk})v_{ijk}]^2/\Sigma(v_{ijk} - \hat{v}_{ijk})^2 \tag{14}$$

where R can be called the sum of squares for non-additivity. The F test for additivity is given by (5).

Randomized Complete Blocks.

The model for that design is $y_{ij} = \mu + T_i + \beta_j + \varepsilon_{ij}$. If we use form (11) for the covariate we have $x_{ij} = \hat{T}_i\hat{\beta}_j$ and it is known that $\hat{T}_i = y_{i\cdot} - y_{\cdot\cdot}$ and $\hat{\beta}_j = y_{\cdot j} - y_{\cdot\cdot}$. Therefore $x_{i\cdot} = x_{\cdot j} = x_{\cdot\cdot} = 0$ and using expression (13) we obtain $\hat{x}_{ijk} = 0$ and also $\Sigma(y_{ij} - \hat{y}_{ij})(x_{ij}) = \Sigma y_{ij}x_{ij}$. Therefore R becomes

$$R = \Sigma y_{ij}x_{ij}/\Sigma x^2_{ij} \; . \tag{15}$$

Due to the fact that with covariate x_{ij} the values $x_{i\cdot} = x_{\cdot j} = 0$, the sums of squares in x and the sums of cross product xy for treatments and blocks will be zero. The analysis of covariance procedure is then simplified to the direct use of formula (15).

Snedecor and Cochran [1969], page 333, describe Tukey's test as follows: i) Obtain $\hat{T}_i = y_{i\cdot} - y_{\cdot\cdot}$ and $\hat{\beta}_j = y_{\cdot j} - y_{\cdot\cdot}$; ii) calculate $w_i = \sum_j y_{ij}\hat{\beta}_j$; iii) calculate $N = \Sigma w_i\hat{T}_i$; iv) obtain $D = (\sum_i \hat{T}^2_i)(\sum_j \hat{\beta}^2_j)$; and finally iv) $R = N^2/D$. We can see that

$$N = \sum_i (\sum_j y_{ij}\hat{\beta}_j)\hat{T}_i = \sum_{i,j} y_{ij}\hat{T}_i\hat{\beta}_j = \sum_{i,j} y_{ij}x_{ij} \; ; \text{ and } D = \sum_{i,j} \hat{T}^2_i\hat{\beta}^2_i = \Sigma x^2_{ij} \; .$$

so that N^2/D is the value R given by (15). All this means that Snedecor and Cochran's procedure could be reduced as follows: i) Calculate covariate $x_{ij} = (y_{i.} - y_{..})(y_{.j} - y_{..})$, ii) obtain $N = \Sigma y_{ij} x_{ij}$; iii) obtain $D = \Sigma x_{ij}^2$; and iv) obtain $R = N^2/D$.

Latin Square.

Covariate x_{ijk} in the Latin square does not lead to the simple case found in the complete blocks design and an analysis of covariance is the easiest procedure. However, Snedecor and Cochran [1969], pages 335-336, following Tukey's [1955], arrive at the test of additivity as follows: i) Obtain $\hat{y}_{ijk}$, ii) calculate deviates $d_{ijk} = y_{ijk} - \hat{y}_{ijk}$, and adjust them so that $\sum_i d_{ijk} = \sum_j d_{ijk} = \sum_k d_{ijk} = 0$; iii) obtain $U_{ijk} = a(\hat{y}_{ijk} - b)$, where a and b are appropriate constants, iv) work out an analysis of variance for a Latin square using U_{ijk} as data and obtain E_{uu}, sum of squares for error; v) calculate $N = \Sigma U_{ijk}(\hat{y}_{ijk} - \hat{y}_{ijk})$; and finally, vi) determine the sum of squares for non-additivity equals N^2/E_{uu}.

We can see that if U_{ijk} is assumed a covariate, of form u_{ijk} in (12), the expression N^2/E_{uu} corresponds to (14). This implies that the procedure for a test of additivity can be simplified as follows: i) calculate covariate u_{ijk}; ii) work out an analysis of covariance; iii) test the regression coefficient as in (5).

RANKING PROCEDURES

Model (9) was given as a simplified conception to explain non-additivity. It was stated that the term $\varepsilon_{ijk} = \alpha_i\beta_j + \alpha_i\gamma_k + \beta_j\gamma_k$ could be substituted by a known covariate highly correlated. Covariate x_{ijk} given by (11) was suggested. Approximate results can then be obtained by using ranks instead of the estimated means $\hat{\alpha}_i, \hat{\beta}_j$, and $\hat{\gamma}_k$'s. The covariate would be

$$x_{ijk} = r_{\alpha(i)}r_{\beta(j)} + r_{\alpha(i)}r_{\gamma(k)} + r_{\beta(j)}r_{\gamma(k)} \qquad (16)$$

where $r_{\alpha(i)}$ is the rank corresponding to $\hat{\alpha}_i$ in the group of $\hat{\alpha}_1, \hat{\alpha}_2, \ldots$. We can also use any of the forms (12) expressed in ranks.

Professor O. Kempthorne, after reading a preliminary version of this paper, referred the author to Giesbrecht [1967] who used the ranking procedure to test non-additivity in the two-way classification. He used linear, second, third, etc., degree orthogonal polynomials to obtain different measures of non-additivity. Tests of additivity will be in general less sensitive with the use of ranks and the calculations required are basically the same. The only case of simplification arises in the randomized block design when using as ranks the corresponding orthogonal polynomial terms; the covariate is given directly by $x_{ij} = r_{\alpha(i)}r_{\beta(j)}$ and we apply (15) to test additivity.

EXAMPLE

It was shown that for a randomized complete blocks design the use of covariate $x_{ij} = \hat{T}_i\hat{\beta}_j$ simplifies the analysis of covariance considerably and the sum of squares for non-additivity becomes $R = \Sigma\, y_{ij}x_{ij}/\Sigma\, x_{ij}^2$, which can be calculated directly. Such simple procedure is not maintained for the Latin square with covariate x_{ijk} in (11) or the equivalent forms (12).

The example chosen has been taken from Snedecor and Cochran [1969] and our objective is to show that their lengthy procedure to test additivity in a Latin square can be simplified and reduced to the wellknown analysis of covariance technique with the same results. Table 1 shows the Latin square data with 5 treatments A, B, C, D and E. The upper numbers correspond to the dependent variable y_{ijk} and the lower ones to values of covariate u_{ijk}, as given by the authors referred to. They define $u_{ijk} = 1000(\hat{y}_{ijk} - y_{...})^2$ which is equivalent to the form u_{ijk} in (12). Their procedure required the calculation of all $\hat{y}_{ijk}$'s for they needed the values $d_{ijk} = \hat{y}_{ijk} - y_{ijk}$. Actually, we can see that

$$u_{ijk} = 1000(\hat{\rho}_i + \hat{\gamma}_j + \hat{T}_k)^2 = 1000(y_{i..} + y_{.j.} + y_{..k} - 3y_{...})^2 \qquad (17)$$

where $\hat{\rho}_i$, $\hat{\gamma}_j$ and $\hat{T}_k$ are the estimates of the true effects of row ρ_i, column γ_j and treatment T_j, respectively. The last expression for u_{ijk} in (17) makes the calculation of covariate values easier.

Table 2 shows the analysis of covariance for y_{ijk} and u_{ijk}'s. It also contains the value for the sums of squares obtained by Snedecor and Cochran [1969]. The very small difference found is due entirely to accuracy in calculations.

TABLE 1

Latin Square 5 x 5. y_{ijk} = log of numbers of responses by pairs of monkeys under five stimuli.

Pair	Week 1	2	3	4	5	$y_{i..}$
1	B 1.99 37	D 2.25 3	C 2.18 0	A 2.18 17	E 2.51 92	2.222
2	D 2.00 70	B 1.85 80	A 1.79 132	E 2.14 4	C 2.31 0	2.018
3	C 2.17 7	A 2.10 18	E 2.34 18	B 2.20 1	D 2.40 66	2.242
4	E 2.41 58	C 2.47 61	B 2.44 23	D 2.53 97	A 2.44 71	2.458
5	A 1.85 125	E 2.32 1	D 2.21 2	C 2.05 3	B 2.25 0	2.136
$y_{.j.}$	2.084	2.198	2.192	2.220	2.382	
Treat $y_{..k}$	A 2.072	B 2.146	C 2.236	D 2.278	E 2.344	$y_{...}$ = 2.215

TABLE 2

Analysis of Covariance and Test of Additivity

Source	D.F.	Σy^2	Σuy	Σu^2
Total	24	1.0576	-37.92	42740
Rows	4	.5243	12.45	6984
Columns	4	.2294	- 7.05	3658
Treatments	4	.2313	-23.04	9765
Error	12	.0726	-20.28	22333
Non-additivity	1	.0184 = $(-20.28)^2/22333$ = R		
Remainder	11	.0542		

F = .0184x11/.0542 = 3.73

In Snedecor and Cochran [1969], page 336, R = .0186, F = 3.76

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