Coverage of Generalized Chess Boards
by Randomly Placed Rooks

by

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Introduction.

At a recent colloquium on combinatorial structures, Kamps and van Lint [2] presented a paper on the minimal number of rooks $\sigma(n, k)$ required to 'cover' a generalized chessboard; the latter is represented by R_k^n, the set of n-vectors (or cells) with components in the ring of integers mod k. To explain the notion of 'cover' we first define the Hamming distance $d_H(\underset{\sim}{x}, \underset{\sim}{y})$ between two vectors ('squares' of the chessboard) as the number of components in which they differ; under the metric d_H, the board R_k^n is a metric space. R_8^2 is the familiar chessboard. Then the rook domain or region covered by a rook at $\underset{\sim}{x}$ is the unit sphere $B(x, 1) = \{y \in R_k^n | d_H(\underset{\sim}{x}, \underset{\sim}{y}) \le 1\}$.

Kamps and van Lint gave the following table of $\sigma(n, k)$ which represents almost all the known results to date for the above deterministic problem.

Table 1

Known Values of $\sigma(n, k)$

k \ n	3	4	5	6	7	8	...	13
2	2	4	7	12	16	2^5		
3	5	9	3^3					3^{10}
4	8	24	4^3					
5	13			5^4				
6	18	72						
7	25					7^6		

The only general results known (see their references) are

(1) $\sigma(2, k) = k$,

(2) $\sigma(3, k) = [(k^2+1)/2]$, where $[x]$ = integer part of x, and

(3) $\sigma(n, k) = \frac{k^n}{1+n(k-1)}$, provided

(a) the right side of (3) is an integer (which implies $n > 3$) and

(b) the integer k is the power of a prime.

For example, from (3) $\sigma(4, 3) = 9$ and from (2), we have $\sigma(3, 3) = 5$. Many values of $\sigma(n, k)$ were computed by Stanton [4], Stanton and Kalbfleisch [5], [6] and others.

We consider two stochastic versions of the rook coverage problem. Rooks are placed in cells (vectors) sequentially and independently with uniform probabilities. We consider the distribution (in particular, the expectation) of the number of rooks Y required to cover R_k^n for the first time. In the multinomial case (Case M) the cells have constant probability k^{-n} and repetition of occupancy is permitted. In the hypergeometric case (Case H) each successive occupancy is permitted only in the currently unoccupied cells, with uniform probability over these cells.

By introducing the stochastic version of the problem we feel that the problem has been broadened in an interesting and non-trivial manner. Indeed, although the deterministic problem is trivial for $n = 2$, the corresponding stochastic problem is by no means trivial. Moreover it is hoped that the more general approach used in the stochastic version would lead to further extensions in the deterministic version, especially in the case of higher dimensions.

Exact Solution for the Multinomial Case with $n = 2$.

Consider a 2-dimensional $k \times k$ chessboard. For Case M let Y_M denote the random number of rooks required to cover the $k \times k$ board and let y denote values of Y_M. The event 'covering a row (column)' is equivalent to 'occupying a row (column)'.

Coverage of the board R_k^2 is characterized by occupancy either of all the rows or or all the columns. We also use the fact that for any given number of rooks, N, the number of rows occupied is independent of the number of columns occupied. Finally, occupancy of rows (similarly for columns) is a direct consequence of the classical Maxwell-Boltzmann statistics (see, for example, p. 59 of Feller [1]). In particular, the probability that all k rows are occupied by x randomly placed rooks is given exactly by

$$(4)\quad F_k(x) = \sum_{\alpha=0}^{k} (-1)^{\alpha}\binom{k}{\alpha}\left(1 - \frac{\alpha}{k}\right)^{x}$$

and the same result holds for columns. By virtue of the independence of row and column occupancy, the cdf $G_k(y)$ of Y_M is given by

$$(5)\quad G_k(y) = 1 - [1 - F_k(y)]^2.$$

The corresponding probability law $g_k(y)$ of Y_M is obtained by taking differences in equation (5). Expectations are then obtained from $g_k(y)$ or by summing the complement of $G_k(y)$ over $y \geq 0$; this yields the two equivalent exact expressions

$$(6a)\quad E\{Y_m\} = k + \sum_{\beta=k}^{\infty}\left[\sum_{\alpha=1}^{k-1}(-1)^{\alpha}\binom{k}{\alpha}\left(1 - \frac{\alpha}{k}\right)^{\beta}\right]^2$$

$$(6b)\quad = k + \frac{1}{k^{2(k-1)}}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(-1)^{i+j}\frac{\binom{k}{i}\binom{k}{j}(ij)^k}{k^2 - ij},$$

both of which are useful for computing (cf. Table 2).

Exact Solution of the Hypergeometric Case for $n = 2$.

Here rooks are placed one-at-a-time independently and with uniform probability in the unoccupied cells. This case requires extensive modification of the solution strategy, mainly due to the loss of independence between row occupancies and column occupancies. We employ the method of inclusion-exclusion and Fréchet sums [1, p. 99] but the basic events have to be defined carefully.

First, we note that the k^2-vector space (chessboard) is <u>not</u> covered by y rooks if and only if at least one cell is not covered and this, in turn, holds if and only if at least one row is not occupied <u>and</u> at least one column is not occupied. The event that a single cell is not covered, in positive terms, requires that all y rooks currently placed are in some $(k-1) \times (k-1)$ product subspace defined by the offending cell. Intersections of these subspaces are again product subspaces, which may be indexed by the deleted rows and columns. Thus, we define our basic events $E_{ij}^{(y)}$ as the event {row i and column j are not covered when y rooks are randomly placed}. We now proceed to apply the Fréchet sum technique as follows.

In this hypergeometric set-up, y rooks can be placed without repetition in $\binom{k^2}{y}$ ways. They can fall in a subspace avoiding r specified rows and c specified columns in $\binom{(k-r)(k-c)}{y}$ ways and the probability of this event (not necessarily basic) is given by

$$(7) \quad \binom{(k-r)(k-c)}{y} / \binom{k^2}{y} \quad (y = 0,1,2,\ldots).$$

Since the r rows and c columns can be specified in $\binom{k}{r}\binom{k}{c}$ ways, the Fréchet sums, for a fixed total $t = r + c$ $(r \geq 1,\ c \geq 1)$ of rows and columns not covered, are given by

$$(8)\quad s_t(y) = \sum_{r=1}^{t-1} \binom{k}{r}\binom{k}{t-r}\binom{(k-r)(k-t+r)}{y}/\binom{k^2}{y}.$$

According to the discussion above, if a cell is not covered then the sum t of the rows and columns not covered is at least 2 and clearly $t \leq 2k - y - 1$. Hence the probability of realization of at least one of the basic events is

$$(9)\quad 1 - H_k(y) = \sum_{t=2}^{2k-y-1} (-1)^{t+1} s_t(y)$$

where $H_k(y)$ is the cdf of the number of rooks required Y_H for Case H.

The expected value of Y_H is obtained by summing (9) over $y \geq 0$. However the first k terms are all equal to 1. Since $Y_H \leq 1 + (k-1)^2$, it follows that $1 - H_k(y) = 0$ for $y \geq 1 + (k-1)^2$ and hence

$$(10)\quad E\{Y_H\} = k + \sum_{y=k}^{(k-1)^2} (1 - H_k(y)).$$

This completes the exact solution for $E\{Y_H\}$ in Case H (cf. Table 2).

Asymptotic Evaluations.

In Case M we have from (4) asymptotically $(k \to \infty)$

$$(11)\quad F_k(x) = \sum_{\alpha=0}^{k} (-1)^{\alpha}\binom{k}{\alpha}\left(1 - \frac{\alpha}{k}\right)^x \sim (1 - e^{-x/k})^k \sim e^{-ke^{-x/k}}.$$

Using the normalizing transformation [1, p. 106]

$$(12)\quad X = k \ln k + kZ$$

we obtain for large k the limiting cdf of Z (which takes on values z)

$$(13)\quad V_k(z) = e^{-e^{-z}} \qquad -\infty < z < \infty,$$

the (standardized) extreme-value distribution.

In our application Y_M is the smaller of two independent chance variables each having the same cdf $F_k(x)$ and it follows from (12) that for $k \to \infty$

(14) $E\{Y_M\} \sim k \ln k + k\, E\{Z_{1:2}\} = k(C + \ln k)$,

where $Z_{1:2}$ is the smaller of 2 independent chance variables with cdf $V_k(z)$ in (13) and its expectation $C = -.1159315$ by the table of Lieblein and Salzer [3].

In Case H we no longer have independence of row and column coverage and have to resort to an 'ad hoc method' to obtain a useful approximation which is as good as the approximation already obtained for Case M. Indeed one reason for considering the two cases together in the same paper is that we suspected that asymptotically the expectations for Case M and Case H would be the same to the first order approximation.

We make use of the fact that if we delete repetitions in placing rooks at random by the multinomial scheme, then the remaining observations Y_H are formally indistinguishable from a hypergeometric sample sequence. The difference $D = Y_M - Y_H$ is the repetition in the multinomial sampling and our evaluation of $E\{Y_H\}$ arises by using

(15) $E\{Y_H\} = E\{Y_M\} - E\{D\}$.

To evalute $E\{D\}$ we first write $D = \sum_{i=1}^{k} \sum_{j=1}^{k} D_{ij}$, where D_{ij} is the number of repetitions of extra rooks placed in the (i,j) cell. The total number of rooks placed in the (i, j) cell under multinomial sampling has a binomial distribution with parameters Y_M and $1/k^2$. Our 'ad hoc method' is to replace Y_M by $E\{Y_M\}$ in evaluating $E\{D_{ij}\}$;

we justify this by noting that the error introduced in the last expressions of (16) and (17) below is of the order of magnitude $O(\frac{E\{Y_M\}}{k^2}) = O(\frac{C + \ln k}{k}) \to 0$ as $k \to \infty$. We now obtain

$$(16) \quad E\{D_{ij}\} \sim \sum_{\alpha=2}^{E\{Y_M\}} (\alpha-1)\binom{E\{Y_M\}}{\alpha}\left(\frac{1}{k^2}\right)^{\alpha}\left(1 - \frac{1}{k^2}\right)^{E\{Y_M\}-\alpha}$$

$$= \frac{1}{k^2} E\{Y_M\} - 1 + \left(1 - \frac{1}{k^2}\right)^{k(C+\ln k)}.$$

Using (14) for $E\{Y_M\}$ and expanding the last term in (16) gives

$$(17) \quad E\{D_{ij}\} \sim \frac{1}{2}\left(\frac{C + \ln k}{k}\right)^2 + O\left(\frac{\ln^3 k}{k^3}\right)$$

and the error term in (17) can also be disregarded. Thus, for the total set of k^2 cells we have from (17)

$$(18) \quad E\{D\} \sim \frac{1}{2}(C + \ln k)^2 + O(1),$$

and hence by (15)

$$(19) \quad E\{Y_H\} \sim k(C + \ln k) - \frac{1}{2}(C + \ln k)^2$$

where the error, which tends to zero as $k \to \infty$, is now omitted.

Table 2 gives exact values of $E\{Y_M\}$ for $k = 2(1)30$ using (6) and approximate values based on (14). It also gives exact values of $E\{Y_H\}$ for $k = 2(1)12$ using (10) and approximate values based on (19). Roundoff errors in this table are estimated to be at most one in the last digit shown.

Coverage of k^n-board for $n > 2$.

Define a skeletal axis centered at cell C as the n mutually perpendicular lines of cells parallel to the sides of the hypercube and having the cell C in common; for $n = 3$ denote the cell by $C_{\alpha,\beta,\gamma}$ $(\alpha, \beta, \gamma = 1, 2, \ldots, k)$ and the corresponding skeletal axis by $C^{\alpha,\beta,\gamma}$. For any n, a cell $C_{\alpha,\beta,\gamma}$ is not covered if and only if the skeletal axis $C^{\alpha,\beta,\gamma}$ has no occupancies. Hence we can use as our basic sets for an inclusion-exclusion argument the sets $C^{\alpha,\beta,\gamma}$ $(\alpha, \beta, \gamma = 1, 2, \ldots, k)$. However the intersections of these skeletal axes are not simple and the corresponding analysis is complicated even for $n = 3$. A complete discussion of this analysis will not be considered here. Thus the stochastic problem becomes more difficult as n increases as it does in the deterministic case of Kamps and van Lint [2]. Mr. Theodore Levy, a student of one of the authors at Michigan State University, is working on a class of such problems; the results are not yet very encouraging.

Use of Independence in Higher Dimensions.

It is of some interest to find a way to generalize the independence of row occupancy and column occupancy that was used above for $n = 2$. For this purpose we define a piece that starts at a cell C in n dimensions and moves (anywhere) inside any Hamming sphere centered at C and of radius $n - 1$. For $n = 2$ this reduces to the usual rook move. For $n = 3$ and starting at cell C the piece moves inside the horizontal plane (H-plane) through C or inside the north-south plane (NS-plane) through C or inside the east-west plane (EW-plane) through C. Hence one such piece covers all the cells in 3 mutually perpendicular slabs that contain the starting cell.

The cube R_k^3 will be covered as soon as either all L slabs or all NS slabs or all EW slabs are occupied. Hence the same argument as for $n = 2$ (Case M) gives for general n (Case M) the exact solution for the cdf of Y_M

$$(20) \quad G_k(y) = 1 - [1 - F_k(y)]^n$$

where $F_k(y)$ is given by (4). For $n = 3$ the expectation becomes

$$(21) \quad E\{Y_M\} = k + \sum_{\beta=k}^{\infty} [\sum_{\alpha=1}^{k-1} (-1)^{\alpha} \binom{k}{\alpha} (1 - \frac{\alpha}{k})^{\beta}]^3$$

$$= k + \frac{1}{k^{3(k-1)}} \sum_{\alpha=1}^{k-1} \sum_{\beta=1}^{k-1} \sum_{\gamma=1}^{k-1} (-1)^{k-\alpha-\beta-\gamma} \frac{\binom{k}{\alpha}\binom{k}{\beta}\binom{k}{\gamma}(\alpha\beta\gamma)^k}{k^3 - \alpha\beta\gamma} ,$$

both of which can be used for computing.

In the corresponding asymptotic $(k \to \infty)$ evaluation for $n = 3$ we need the expectation of the smallest of 3 independent observations on the cdf (13); this is given in [3] as -.4036136. This analysis is easily generalized to any number of dimensions n. This type of solution became possible only after we defined a 'super piece' that moved in more than one dimension. No similar analysis was found for the original definition of a rook move in the Hamming sphere of radius 1.

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Table 2

EXPECTED VALUES OF NUMBERS OF

RANDOM ROOKS REQUIRED TO COVER THE k^2-CHESSBOARD

k	$E\{Y_M\}$	Approximation to $E\{Y_M\}$	$E\{Y_H\}$	Approximation to $E(Y_H)$
2	2.3333333333	1.5544	2.0000000	1.3878
3	4.1821428571	2.9480	3.5000000	2.4652
4	6.3655677654	5.0815	5.3522478	4.2746
5	8.7938685820	7.4675	7.4723892	6.3522
6	11.4171670989	10.0550	9.8091916	8.6508
7	14.2030879491	12.8099	12.3278253	11.1355
8	17.1286506847	15.7081	15.0029299	13.7804
9	20.1766249904	18.7316	17.8152024	16.5657
10	23.3335906237	21.8665	20.7494692	19.4758
11	26.5887915430	25.1016	23.7935002	22.4979
12	29.9334107812	28.4277	26.9372363	25.6217
13	33.3600877782	31.8372		28.8384
14	36.8625841610	35.3238		32.1407
15	40.4355447768	38.8818		35.5223
16	44.074322209	42.5065		38.9776
17	47.77484495	46.1938		42.5020
18	51.5335164	49.9399		46.0911
19	55.3471359	53.7416		49.7414
20	59.212836	57.5960		53.4494
21	63.12803	61.5004		57.2121
22	67.09038	65.4524		61.0268
23	71.09771	69.4499		64.8910
24	75.1481	73.4909		68.8026
25	79.2396	77.5736		72.7595
26	83.3704	81.6963		76.7597
27	87.539	85.8574		80.8015
28	91.743	90.0556		84.8834
29	95.981	94.2896		89.0039
30	100.250	98.5580		93.1615

References

[1] Feller, W. (1950). An Introduction to Probability Theory and its Applications, Volume 1. John Wiley and Sons, New York. Page references are to the third edition (1968).

[2] Kamps, H. J. L. and van Lint, J. H. (1970). A covering problem, to appear in a volume for the Colloquium on Combinatorial Structures held at Balatonfured, Hungary, in August, 1969.

[3] Lieblein, J. and Salzer, H. E. (1957). Table of the first moment of ranked extremes. J. Res. Nat. Bur. Stand. 59 203-206.

[4] Stanton, R. G. (1969). Covering theorems in groups (or: How to win at football pools). A paper included in Recent Progress in Combinatorics. Academic Press, New York.

[5] Stanton, R. G. and Kalbfleisch, J. G. (1968). Covering problems for dichotomized matchings. Aequat. Math. 1 94-103.

[6] Stanton, R. G. and Kalbfleisch, J. G. (1969). Intersection inequalities for the covering problem. Siam J. Appl. Math. 17 1311-1316.