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SINGLE SAMPLE RANKING PROBLEMS WITH POISSON POPULATIONS*

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1. Introduction

In this paper a standard ranking problem, that of selecting 'the best' population, is applied to the special case of Poisson distributions; the application to Poisson processes is essentially the same problem and is automatically also considered. More specifically, there are given k Poisson populations π_i with unknown parameters $\lambda_i > 0$ (i=1,2,...,k), respectively, and on the basis of a common number of observations from each population we wish to decide which population has the largest parameter value.

A similar ranking formulation has been applied to the problem of ranking Normal means by Bechhofer [1] and to the problem of ranking Normal variances by Bechhofer and Sobel [2]. The problem treated here differs from these and other ranking problems heretofore treated in that a simplified solution based only on the parameter differences or only on the parameter ratios does not exist. However, the exact solution for a fairly simple formulation based on the simultaneous consideration of differences and ratios is given and proved below.

Before experimentation starts, the parameter space Ω is partitioned into a zone of indifference Ω^0 and k symmetrical regions of preference (whose union will be called a zone of preference, Ω^+) and a constant P* with $1/k < P^* < 1$ is specified. This is done in accordance with the experimenter's requirement that if the vector of true parameter values $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ lies in the zone of preference then he wants a procedure which has a probability of at least P* of making a correct selection, i.e., of selecting the population

with the largest parameter value. The terminal decision rule will be an obvious one, and since it will be assumed that observations are taken k at a time (one from each population), the only problem remaining is to determine the common number n of observations to be taken from each population.

The formulation differs from the usual test of homogeneity in that we take the more realistic position that the k populations will hardly ever be equally good in any practical situation. Also it often happens that a test of homogeneity (expecially after rejection) is followed by an analysis that terminates in ranking the given populations. Hence it might be reasonable to make some aspect of ranking our primary goal and, instead of controlling the type I error, we might control the probability of a correct ranking. Furthermore, there is an analogue of power in this formulation; it is the probability of a correct selection, PCS, viewed as a function of the true parameter values.

2. Formulation of the Problem

Let π_i denote a Poisson population with parameter λ_i (i=1,2,...,k) and let the ordered values of these λ_i be denoted by

$$\lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[k]}. \tag{2.1}$$

This model allows equality of the parameter values in (2.1) in the sense that when such equalities hold the choice of any population with a parameter value equal to $\lambda_{[k]}$ is regarded as a correct selection; however, the experimenter can select only one of the populations as best. Indeed, a correct selection is here defined as the selection of any population with parameter value $\lambda_{[k]}$. Our goal is to make a correct selection using a common number n (to be determined) of observations from each of the k populations. The population π_i has a sufficient statistic

(2.2) $$S_i = \sum_{j=1}^{n} X_{ij} \qquad (i=1,2,\ldots,k)$$

and our decision rule will depend only on these S_i. The terminal decision rule used is simply to select the population with the largest S_i, if there are no ties for first place. If there are such ties, then the experimenter is free to

either i) select one of the contenders for first place by randomization, i.e., by an independent experiment giving equal probability to each of the contenders for first place,

or ii) select any one of the contenders for first place in an arbitrary way or on non-statistical grounds, provided the method or criterion is not positively correlated with wrong selections.

[As far as the basic requirement is concerned the experimenter may also choose to continue taking observations from the populations tied for first place (with or without subsequent elimination) until the tie is broken, but in this case the resulting procedure is no longer a single sample procedure and we are only including single sample procedures in the present discussion.]

Let $\vec{\lambda} = (\lambda_{[1]},\lambda_{[2]},\ldots,\lambda_{[k]})$ denote a point in the parameter space Ω where $0 < \lambda_{[i]} < \infty$ $(i=1,2,\ldots,k)$. The space Ω is partitioned into a preference zone $\Omega^{+}=\Omega^{+}(r^*,d^*)$, defined as the set of $\vec{\lambda}$ such that all the $2(k-1)$ inequalities

(2.3) $$(\lambda_{[k]}/\lambda_{[i]}) \geq r^* \quad \text{and} \quad \lambda_{[k]}-\lambda_{[i]} \geq d^* \qquad (i=1,2,\ldots,k-1)$$

hold, and its complement, the indifference zone Ω^{o}, where at least one of the inequalities in (2.3) is violated. The quantities $r^* > 1$ and $d^* > 0$ are preassigned by the experimenter. More explicitly, it is assumed that the experimenter is indifferent between the best population with parameter value

$\lambda_{[k]}$ and any other population with parameter value $\lambda_{[i]}$ if either one of the two inequalities associated with $\lambda_{[i]}$ in (2.3) is violated. It will be convenient to say that the population associated with any $\lambda_{[i]}$ that violates either inequality in (2.3) is in the indifference zone. Of course, in the preference zone the unique population with parameter value $\lambda_{[k]}$ is preferred.

In addition to specifying r* and d*, the experimenter specifies a constant P* with $1/k < P^* < 1$ and states that he requires a procedure δ for which the probability of a correct selection PCS satisfies the inequality

$$\underline{(2.4)} \qquad P\{CS;\delta,\vec{\lambda}\} \geq P^* \qquad \text{for all } \vec{\lambda}\epsilon\Omega^+(d^*,r^*).$$

Here the procedure δ is the obvious one based on the k sufficient statistics using randomization in the case of ties. The only remaining problem is to determine the smallest integer equal to or greater than the decimal solution $n_0=n_0(k,P^*,r^*,d^*)$ of (2.4); this is carried out in Section 4 below.

If we define a correct decision (CD) to be the selection of any population in the indifference zone Ω^0 then it is easily seen that for any integer $n \geq n_0$ and the above procedure δ, the probability of a correct decision PCD satisfies the inequality

$$\underline{(2.5)} \qquad P\{CD;\delta,\vec{\lambda}\} \geq P^* \qquad \text{for all } \vec{\lambda}\epsilon\Omega_0$$

Furthermore, if we introduce the associated simple loss function, which defines the loss to be zero for a correct decision and to be the same constant $C > 0$ for any incorrect decision, then $R(\delta,\vec{\lambda}) = C[1-P\{CD;\delta,\vec{\lambda}\}]$ and we can assert that the risk function $R(\delta,\lambda)$ for the above procedure δ with $n \geq n_0$ satisfies the inequality

$$\underline{(2.6)} \qquad R(\delta,\vec{\lambda}) \leq R^* \qquad \text{for all } \vec{\lambda}\epsilon\Omega$$

where $R^* = C(1-P^*)$.

Using the fact that observations from different populations are independent and that each population has a sufficient statistic whose distribution has a monotone likelihood ratio, it can be shown that the procedure δ is optimal in the sense that it yields the highest PCD [and also the highest PCS and the lowest $R(\delta,\vec{\lambda})$] for every n and every $\vec{\lambda}\epsilon\Omega$. Since the computation of n_0 for the procedure δ is carried out by setting the maximum of the risk function equal to R* (or the minimum of the PCD equal to P*) it follows from this optimal property that for any fixed value of n the procedure δ is minimax with respect to the above mentioned loss function.

3. Minimum of the PCS: Step I

Let $S_i=S_{i,n}$ denote the sum of the observations from population π_i and let $S_{(i)}=S_{(i),n}$ denote the sum from the population with parameter $\lambda_{[i]}$ $(i=1,2,\ldots,k)$. Let $\psi_{[i]}=n\lambda_{[i]}$ $(i=1,2,\ldots,k)$. Let $S^j_{(i_1)},S^j_{(i_2)},\ldots,S^j_{(i_\nu)}$ denote a particular set of S's indexed by j which does not include the best population; let $T_{j,\nu}$ denote the corresponding set of subscripts $(i_1,i_2,\ldots,i_\nu)$ so that for fixed ν the range of j is $j=1,2,\ldots,\binom{k-1}{\nu}=J$, (say). Let $T^c_{j,\nu}$ denote the complement of $T_{j,\nu}$ with respect to the set $(1,2,\ldots,k-1)$. Let

$$f(\alpha;\psi)=e^{-\psi}\psi^{\alpha}/\alpha! \qquad (\alpha=0,1,\ldots;\psi>0). \tag{3.1}$$

For any (true) parameter point $\vec{\lambda}$ in Ω the $P\{CS;\delta,\vec{\lambda}\}$ (or more simply the PCS(δ)) for the procedure δ described in Section 2 is given by

$$PCS(\delta)=\sum_{\nu=0}^{k-1}\frac{1}{\nu+1}\sum_{j=1}^{J}P\{S_{(k)}=S^j_{(i_1)}=\ldots=S^j_{(i_\nu)}>S_{(\beta)}\ \text{for all}\ \beta\epsilon T^c_{j,\nu}\} \tag{3.2}$$

$$=\sum_{\nu=0}^{k-1}\frac{1}{\nu+1}\sum_{j=1}^{J}\sum_{\alpha=0}^{\infty}f(\alpha;\psi_{[k]})\left[\prod_{i\epsilon T_{j,\nu}}f(\alpha;\psi_{[i]})\right]\left[\prod_{i\epsilon T^c_{j,\nu}}\sum_{\beta=0}^{\alpha-1}f(\beta;\psi_{[i]})\right]$$

Here the sum up to α-1 is understood to be zero for α=0 unless ν=k-1, in which case $T^{c}_{\nu,j}$ is empty and the associated product then becomes unity. This expression is useful for an exact computation but for proving monotonicity properties it is clumsy. To obtain another expression for PCS(δ) we define a 'continuous Poisson' density. Let {y} be the largest integer less than or equal to y; the continuous Poisson density is defined for all real y by

$$g(y;\psi) = f(\{y\};\psi) \qquad \text{for } 0 < y < \infty \tag{3.3}$$

and $g(y;\psi)$=0 otherwise. Let $G(y;\psi)$ denote the corresponding cumulative distribution function (c.d.f.). Let Y_i denote a chance variable with the density (3.3) with $\psi=\psi_i$ (i=1,2,...,k) where the ψ_i are the same as in the original problem. Consider the problem of selecting the chance variable associated with the largest parameter value, $\psi_{[k]}$, based on a single observation of each Y_i (i=1,2,...,k). Let δ' denote the obvious procedure of selecting the one with the largest observation and let PCS(δ') denote the probability of a correct selection using δ'. We now compare PCS(δ') with the PCS(δ) of the original problem.

On the one hand PCS(δ) = PCS(δ'). To prove this consider the disjoint, exhaustive set of half-open intervals $A_\alpha = [\alpha,\alpha+1]$ for α=0,1,2,... . Let $Y_{(i)}$ denote the Y associated with $\psi_{[i]}$ (i=1,2,...,k). For a correct selection we must have $Y_{[k]}$ and ν other Y's in A_α for some fixed $\alpha \geq 0$ and all the remaining Y's (if any) must be in intervals to the left of A_α; let this event be denoted by $W_{\nu,\alpha}$. Given $W_{\nu,\alpha}$ for any fixed α and any fixed subset $i_1,i_2,\ldots,i_\nu$ of size ν, the conditional probability that $Y_{(k)}$ yields the largest observation is easily seen to be $1/(\nu+1)$ which is exactly the coefficient in (3.2). The remaining factors summed over j from 1 to J, represent

the probability of $W_{\nu,\alpha}$ for each pair (ν,α). Thus the continuous problem with one observation per population and parameters ψ_i $(i=1,2,\ldots,k)$ has its PCS(δ') equal to the PCS(δ) of the original discrete problem with n observations per population; let the common value be denoted by PCS.

On the other hand the continuous problem allows us to write the PCS in the form

$$\underline{(3.4)} \qquad PCS = P\{Y_{(k)} > Y_{(i)} \ (i=1,2,\ldots,k-1)\}$$

$$= \int_0^\infty P\{Y_{(i)} < y \ (i=1,2,\ldots,k-1)\} \ g(y; \psi_{[k]})dy$$

$$= \int_0^\infty \left[\prod_{i=1}^{k-1} G(y; \psi_{[i]})\right] \ g(y; \psi_{[k]})dy$$

To find the minimum of the PCS (3.4) in that part of the parameter space Ω where (2.3) holds, we first point out that the statistic Y with density (3.3) has a monotone likelihood ratio (m.l.r.). For $y_1 < y_2$ in different intervals I_1, I_2 the result follows from the fact that the discrete Poisson has the same property and for $y_1 < y_2$ in the same interval the result is trivial since the likelihood ratio is constant. Hence for $y_1 < y_2$ and $\psi_1 < \psi_2$

$$\underline{(3.5)} \qquad g(y_1; \psi_1) \ g(y_2; \psi_2) \geq g(y_2; \psi_1) \ g(y_1; \psi_2).$$

Consider any y such that $y_1 < y < y_2$. If we integrate both sides of (3.5) over the range $0 < y_1 < y$, $y < y_2 < \infty$ then we find that for $\psi_1 < \psi_2$

$$\underline{(3.6)} \qquad G(y; \psi_1) \geq G(y; \psi_2) \qquad 0 < y < \infty,$$

i.e., the c.d.f. $G(y; \psi)$ is a monotone function of ψ for every fixed y. It

follows that the PCS in (3.4) can be made smaller by increasing $\psi_{[k-2]}$ to $\psi_{[k-1]}$, then increasing $\psi_{[k-3]}$ to $\psi_{[k-1]}$,..., and finally increasing $\psi_{[1]}$ to $\psi_{[k-1]}$. Thus we restrict our attention to $\vec{\lambda}\in\Omega$ such that

$$\underline{(3.7)} \qquad \lambda_{[1]} = \lambda_{[2]} = \ldots = \lambda_{[k-1]} = \lambda \text{ (say)}$$

and minimize the simpler expression

$$\underline{(3.8)} \qquad \text{PCS} = \int_0^\infty G^{k-1}(y;\psi)\, g(y;\psi_{[k]})dy$$

where $\psi=n\lambda$ and $\psi_{[k]}=n\lambda_{[k]}$, over the pairs $(\lambda,\lambda_{[k]})$ such that we have both

$$\underline{(3.9)} \qquad \lambda_{[k]}/\lambda \geq r^* \quad \underline{\text{and}} \quad \lambda_{[k]}-\lambda \geq d^*.$$

Formula (3.2) also takes on a simpler form now. In terms of $\psi_{[k]} = \psi'$(say) and the common ψ we can write (3.2) as

$$\underline{(3.10)} \qquad \text{PCS} = \sum_{\nu=0}^{k-1} \frac{\binom{k-1}{\nu}}{\nu+1} \sum_{\alpha=0}^{\infty} f(\alpha;\psi')\, f^{\nu}(\alpha;\psi) \left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^{k-1-\nu}.$$

Interchanging the order of summations in (3.10) and using the binomial theorem gives

$$\underline{(3.11)} \qquad \text{PCS} = \frac{1}{k} \sum_{\alpha=0}^{\infty} \frac{f(\alpha;\psi')}{f(\alpha;\psi)} \left(\left[\sum_{\beta=0}^{\alpha} f(\beta;\psi)\right]^k - \left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^k\right)$$

An alternative expression for (3.11) is

$$\underline{(3.12)} \qquad \text{PCS} = \sum_{\alpha=0}^{\infty} f(\alpha;\psi') \frac{1}{k} \sum_{j=0}^{k-1} \left[\sum_{\beta=0}^{\alpha} f(\beta;\psi)\right]^{k-1-j} \left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^{j},$$

where, as in (3.2), (3.10) and (3.11), the sum up to $\alpha-1$ is zero for $\alpha=0$ unless the power of the sum is zero, in which case it is set equal to unity. Changing

α to α-1 and vice versa in (3.12) yields the bounds

$$\underline{(3.13)} \quad \sum_{\alpha=1}^{\infty} f(\alpha;\psi') \left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^{k-1} \leq PCS \leq \sum_{\alpha=0}^{\infty} f(\alpha;\psi') \left[\sum_{\beta=0}^{\alpha} f(\beta;\psi)\right]^{k-1}.$$

Replacing the arithmetic mean in (3.12) by the geometric mean yields a lower bound closer than that in (3.13), namely,

$$\underline{(3.14)} \quad PCS \geq \sum_{\alpha=0}^{\infty} f(\alpha;\psi') \left(\left[\sum_{\beta=0}^{\alpha} f(\beta;\psi)\right]^{k-1} \left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^{k-1}\right)^{\frac{1}{2}}$$

If we replace $\frac{1}{(\nu+1)}$ by $\frac{1}{2}$ for $\nu \geq 2$ in (3.10) then we obtain an upper bound closer than that in (3.13), namely,

$$\underline{(3.15)} \quad PCS \leq \frac{1}{2} \sum_{\alpha=0}^{\infty} f(\alpha;\psi') \left(\left[\sum_{\beta=0}^{\alpha} f(\beta;\psi)\right]^{k-1} + \left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^{k-1}\right).$$

If we replace $\frac{1}{(\nu+1)}$ by 0 for $\nu \geq 2$ in (3.10) then we obtain another lower bound, namely,

$$\underline{(3.16)} \quad PCS \geq \sum_{\alpha=0}^{\infty} f(\alpha;\psi') \left(\left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^{k-1} + \frac{(k-1)}{2} f(\alpha;\psi) \left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right]^{k-2}\right).$$

The inequalities (3.14) and (3.15) suggest as an approximation for the PCS the geometric mean of the extreme members of (3.13); it is easily shown that the latter lies between the right hand members of (3.14) and (3.15) and it is easier to compute than the right hand side of (3.14). [Of course all sums of $f(\beta;\psi)$ in the above discussion can be replaced by integrals using the well known identity

$$\underline{(3.17)} \quad \sum_{\beta=0}^{\alpha} f(\beta;\psi) = \int_{\psi}^{\infty} \frac{e^{-x} x^{\alpha}}{\alpha!} dx \quad].$$

The difference between the upper bound (UB) and lower bound (LB) in (3.13) is bounded above by the probability that $X_{(k)}$ is equal to at least one of the remaining $X_{(i)}$. This, in turn, can be bounded above by Boole's inequality giving

$$\text{(3.18)} \qquad \text{UB-LB} \leq (k-1) \sum_{\alpha=0}^{\infty} f(\alpha;\psi')\, f(\alpha;\psi)$$

where the summation is the probability of equality of two Poisson chance variables with parameters $\psi=n\lambda$ and $\psi'=n\lambda_{[k]}$. For $\lambda_{[k]}=r\lambda$ (with $r>1$, $\lambda>0$ and both fixed) it is easily shown that this sum tends to zero as $n \to \infty$ and, in fact, it is strictly decreasing towards zero. It follows that the difference between the PCS and either bound in (3.13) (and hence also for the bounds in (3.14) and (3.15)) tends to zero as $n \to \infty$. Similarly, if we let UB_1 LB_1 denote the right members of (3.15) and (3.16), respectively, then it follows from the above discussion that

$$\text{(3.19)} \qquad UB_1\text{-}LB_1 \leq \binom{k-1}{2} \sum_{\alpha=0}^{\infty} f(\alpha;\psi')\, f^2(\alpha;\psi)$$

which tends to zero faster than the right hand member of (3.18) since the summation in (3.19) is the probability of a triple equality.

For the special case k=2 the right hand members of (3.15) and (3.16) are both equal and exact and we obtain the arithmetic mean of the two extremes in (3.13), i.e., for k=2

$$\text{(3.20)} \qquad \text{PCS} = \left[\sum_{\alpha=1}^{\infty} f(\alpha;\psi')\right]\left[\sum_{\beta=0}^{\alpha-1} f(\beta;\psi)\right] + \frac{1}{2}\sum_{\alpha=0}^{\infty} f(\alpha;\psi')\, f(\alpha;\psi).$$

4. Minimum of the PCS: Step II

To find the minimum of the PCS over pairs $\lambda,\lambda_{[k]}$ satisfying (3.9) we prove

a theorem after some preliminary results are derived. The c.d.f. $G(x;\psi)$ of the density $g(x;\psi)$ in (3.3) is given by

$$\underline{(4.1)} \qquad G(x;\psi) = \sum_{j=0}^{\{x\}-1} f(j;\psi) + (x-\{x\})\, f(\{x\};\psi).$$

The derivatives of $g(x;\psi)$ and $G(x;\psi)$ with respect to ψ are

$$\underline{(4.2)} \qquad \frac{d}{d\psi} g(x;\psi) = g(x-1;\psi) - g(x;\psi) = g(x;\psi)\left[\frac{\{x\}}{\psi} - 1\right]$$

$$\underline{(4.3)} \qquad \frac{d}{d\psi} G(x;\psi) = (x-\{x\}-1)\, g(x-1;\psi) - (x-\{x\})\, g(x;\psi)$$

$$= g(x;\psi)\left[(x-\{x\}-1)\,\frac{\{x\}}{\psi} - (x-\{x\})\right].$$

Furthermore, letting $\Delta G(x;\psi)$ denote $G(x;\psi) - G(x-1;\psi)$ we obtain

$$\underline{(4.4)} \qquad \Delta G(x;\psi) = (x-\{x\})\, g(x;\psi) - (x-\{x\}-1)\, g(x-1;\psi) = -\frac{d}{d\psi} G(x;\psi).$$

Theorem: For $k \geq 2$, letting $\psi' = a\psi+b$ where $\psi > 0$, $a \geq 1$ and $b \geq 0$, we have the following two results

$$\underline{(4.5)} \qquad \frac{d}{d\psi} \int_0^\infty G^{k-1}(x;\psi)\, g(x;\psi)dx \quad \begin{cases} \geq 0 \text{ if } a \geq 1 \text{ and } b = 0 & \underline{\text{Case 1}} \\ \leq 0 \text{ if } a = 1 \text{ and } b \geq 0 & \underline{\text{Case 2}} \end{cases}.$$

The strict inequality holds for $a > 1$ in Case 1 and for $b > 0$ in Case 2.

Proof: Let $D(\psi)$ denote the left member of (4.5). Differentration under the integral sign in (4.5) is easily justified and, using (4.2) and (4.4), we obtain

$$\underline{(4.6)} \qquad D(\psi) = a\int_0^\infty G^{k-1}(x;\psi)[g(x-1;\psi') - g(x;\psi')]dx$$

$$- (k-1)\int_0^\infty G^{k-2}(x;\psi)\, g(x;\psi')[G(x;\psi) - G(x-1;\psi)]dx.$$

If we apply integration by parts to the first integral in (4.6) then the 'leading' term vanishes and, using (4.4) again, we obtain

$$\underline{(4.7)} \quad D(\psi) = (k-1)\int_0^\infty G^{k-2}(x;\psi)\, g(x;\psi)\, g(x;\psi')[(a-1)(x-\{x\}) + \frac{\{x\}b}{\psi\psi'}(x-\{x\}-1)]dx.$$

Since $x \geq \{x\} > x-1$ it follows that for $a \geq 1$ and $b = 0$ the integral is non-negative and that for $a = 1$ and $b \geq 0$ the integral is non-positive, which is the desired result. To prove the strict inequality we note that since $\psi > 0$ then $\psi' > 0$ also and both densities are non-degenerate. Since $x-\{x\} = 0$ only at the integers and $\{x\} = 0$ only for $0 \leq x < 1$ it follows that for $a > 1$ (and $b = 0$) the first term in (4.7) is strictly positive and that for $b > 0$ (and $a = 1$) the second term is strictly negative.

The following corollaries are now immediate consequences.

Corollary 1: If $\psi' = a\psi$ with $a > 1$ then the PCS is a strictly increasing function of ψ.

Corollary 2: If $\psi' = \psi+b$ with $b > 0$ then the PCS is a strictly decreasing function of ψ.

From corollaries 1 and 2 we can now find the pair $(\lambda,\lambda_{[k]})$ satisfying (3.9) for which the PCS is smallest and this leads to the solution of the problem. First we note that the region (3.9) is such that by increasing λ (with $\lambda_{[k]}$ fixed) we can change at least one of the two inequalities in (3.9) to an equality. By (3.6) and the form of the PCS (3.8) it is clear that any such increase in λ will not increase the PCS. Hence we can restrict our attention to points $\vec{\lambda}$ satisfying (3.7) and (3.9) with $(\lambda,\lambda_{[k]})$ on at least one of the two lines L_1, L_2 given by

$$\underline{(4.8)} \qquad \begin{aligned} L_1: &\quad \lambda_{[k]}/\lambda = r^* && (r^* > 1) \\ L_2: &\quad \lambda_{[k]}-\lambda = d^* && (d^* > 0) \end{aligned}$$

By corollary 1 the PCS is strictly increasing in λ on L_1 and by corollary 2 it is strictly decreasing in λ on L_2. It follows that the minimum is achieved where the two lines meet, i.e., at

$$\underline{(4.9)} \qquad \lambda^{o} = \frac{d^*}{r^*-1} \quad \text{and} \quad \lambda^{o}_{[k]} = \frac{r^*d^*}{r^*-1} .$$

Hence letting $\psi_o = n\lambda^{o}$ and $\psi'_o = n\lambda^{o}_{[k]}$ a solution to our problem is obtained by setting

$$\underline{(4.10)} \qquad \int_0^{\infty} G^{k-1}(x;\psi_o)\, g(x;\psi'_o)dx = P^*$$

(alternatively, (3.10), (3.11) or (3.12) could be used on the left with $\psi=\psi_o$ and $\psi'=\psi'_o$) and solving for n. The existence of this solution follows from the fact that the left member of (4.10) approaches unity as $n \to \infty$; this is shown in Section 5. The uniqueness of the solution follows from the monotonicity property in corollary 1. Denote the "decimal" solution by $n_o=n_o(k,P^*,r^*,d^*)$. In the problem under discussion we would use the smallest integer equal to or greater than n_o, i.e., the "integer" solution.

In an analogous Poisson process problem, each of k (independent) Poisson processes are observed for the same time t and the one with the most occurrences is selected as best, with ties for first place broken by randomization as above. Then the analysis remains the same as above except that n is replaced everywhere by the time variable t and here the "decimal" solution n_o (now denoted by t_o) is used.

It is clear from (4.9) and (4.10) that if we let $m=nd^*$ then the solution of (4.10), i.e., $m_o=m_o(k,P^*,r^*)$, no longer depends on d^*. This makes the problem much simpler. The experimenter can obtain m_o from a table and then use his particular d^* to find n_o from the equation $m_o=n_o d^*$. This simplifying feature also arises in the asymptotic Normal approximation discussed below.

5. Asymptotic Normal Approximation

For $\vec{\lambda}$ satisfying (3.7) and n large we have the approximation

$$(5.1) \qquad PCS \cong P\left\{\left(\frac{S_{(i)}-\psi}{\sqrt{\psi}}\right) \leq \sqrt{\frac{\psi'}{\psi}}\left[\left(\frac{S_{(k)}-\psi'}{\sqrt{\psi'}}\right) + \frac{\psi'-\psi}{\sqrt{\psi'}}\right] (i=1,2,\ldots,k-1)\right\} .$$

As $n \to \infty$ the terms deleted in forming the above approximation all tend to zero and the parenthetical quantities in (5.1) tend to independent standardized Normal chance variables. Hence, letting $\Phi(x)$ and $\varphi(x)$ denote the standardized Normal c.d.f. and density, respectively, we obtain for large n

$$(5.2) \qquad PCS \cong \int_{-\infty}^{\infty} \Phi^{k-1}\left[\sqrt{r}\,(x+b)\right] \varphi(x)dx$$

where

$$(5.3) \qquad r = \frac{\psi'}{\psi} = \frac{\lambda_{[k]}}{\lambda} \quad \text{and} \quad b = \frac{\psi'-\psi}{\sqrt{\psi'}} = \frac{\sqrt{n}\,(\lambda_{[k]}-\lambda)}{\sqrt{\lambda_{[k]}}} .$$

For the pair $(\lambda^o, \lambda^o_{[k]})$ in (4.9) which defines our "worst case" we have

$$(5.4) \qquad r = r^* \quad \text{and} \quad b = \sqrt{\frac{nd^*(r^*-1)}{r^*}} .$$

Hence the asymptotic solution is obtained by solving

$$(5.5) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}\left[\sqrt{r^*}\,(x+b)\right] \varphi(x)dx = P^*$$

for $b = b_1 = b_1(k,r^*,P^*)$ and then solving the second equation in (5.4) for $n=n_1$ (say). As in the exact case we need only table $m_1=n_1d^*$ since the solution of (5.5) in terms of m_1 does not depend on d^*. Since the left member of (5.5) is strictly increasing toward unity as $n \to \infty$ for any fixed $r^* > 1$ it follows that a unique solution in b (i.e., b_1) and hence in n (i.e., n_1) must exist

and that (5.5) holds for all $n \geq n_1$. This automatically also shows that the left member of (4.10) also tends to unity as $n \to \infty$ since the left members of (4.10) and (5.5) are asymptotically equivalent.

Tables for the right hand member of (5.2) have been computed by R. Milton but are not yet published. Some values were taken from this table to compare these asymptotic results with the exact results in Section 3. Table I at the end of this paper shows some comparisons and also provides some values of m_1 needed to carry out the ranking problem.

6. A Correction to the Normal Approximation

Using (3.4) and the results of Section 3 we can also write the PCS in terms of standardized continuous Poisson chance variables. To do this we first compute some moments of Y defined by (3.3); the central moments are close to those of the original Poisson but with interesting differences. Omitting the details which are straightforward, we obtain for the first "raw" moment and the next three central moments μ_i (i=2,3,4)

$$\underline{(6.1)} \qquad EY = \psi + \frac{1}{2}\,; \quad \mu_2 = \psi + \frac{1}{12}\,; \quad \mu_3 = \psi$$

$$\mu_4 = 3\psi^2 + \frac{19}{2}\psi + \frac{1}{80} = 3\mu_2^2 + 9\psi - \frac{1}{120}$$

Hence the cumulants $C_i(\lambda)$ (i=3,4) for the corresponding standardized chance variable Z are

$$\underline{(6.2)} \qquad C_3(\lambda) = \frac{\mu_3}{\mu_2^{3/2}} = \frac{1}{\sqrt{\psi}} \left(\frac{\psi}{\psi + \frac{1}{12}} \right)^{3/2} \approx \frac{1}{\sqrt{\psi}} = \frac{1}{\sqrt{n\lambda}}$$

$$C_4(\lambda) = \frac{\mu_4 - 3\mu_2^2}{\mu_2^2} = \frac{9\psi - \frac{1}{120}}{(\psi + \frac{1}{12})^2} \approx \frac{9}{\psi} = \frac{9}{n\lambda}$$

where the last two expressions are approximations for large values of $n\lambda$.

Thus, for large $n\lambda$, the value of $C_4(\lambda)$ is negligible compared to $C_3(\lambda)$ and, in fact, only the last expression for $C_3(\lambda)$ is used in the development below.

In terms of the density $h(z)$ and the c.d.f. $H(z)$ of Z we have

$$\underline{(6.3)} \qquad PCS = P\{Z_{(i)} < \sqrt{r_i'}\,(Z_{(k)}+b_i') \qquad (i=1,2,\ldots,p)\}$$

$$= \int_{-\infty}^{\infty} h(z) \prod_{i=1}^{p} H[\sqrt{r_i'}\,(z+b_i')]dz$$

where

$$\underline{(6.4)} \qquad r_i' = \frac{n\lambda_{[k]}+\frac{1}{12}}{n\lambda_{[i]}+\frac{1}{12}}; \quad b_i' = \frac{n(\lambda_{[k]}-\lambda_{[i]})}{\sqrt{n\lambda_{[k]}+\frac{1}{12}}}$$

If we now insert the minimizing values, λ^o and $\lambda^o_{[k]}$, for $\lambda_{[i]}$ $(i=1,2,\ldots,p)$ and $\lambda_{[k]}$ and denote the resulting values in (6.4) by r' and b' then we obtain the simpler form

$$\underline{(6.5)} \qquad PCS = \int_{-\infty}^{\infty} H^p[\sqrt{r'}\,(z+b')]\,h(z)dz.$$

For large values of $n\lambda^o$ we can drop the $\frac{1}{12}$ in (6.4) and, using r* and b as defined in (5.4), we obtain

$$\underline{(6.6)} \qquad PCS \approx \int_{-\infty}^{\infty} H^p[\sqrt{r^*}\,(z+b)]\,h(z)dz.$$

Now r* does not depend on n; let b_2 denote the exact solution obtained by setting the right hand member of (6.6) equal to P* and let b_1 denote the solution of (5.5). Finally let $b_2 = b_1+\epsilon$. After finding a correction term ϵ for b_1 we can use the second part of (5.4) with b replaced by b_2 to get an

improved solution for the "decimal" value of n, say n_2.

To derive an expression for ϵ, we first replace h(z) and H(z) in (6.6) by their Gram-Charlier (or Edgeworth) expansions, i.e.,

$$\underline{(6.7)} \qquad h(z) = \varphi(z) - C_3(\lambda^o_{[k]})\, \varphi^{(3)}(z)/3! + \ldots$$

$$\underline{(6.8)} \qquad H(z) = \Phi(z) - C_3(\lambda^o)\, \varphi^{(2)}(z)/3! + \ldots$$

where the parenthentical superscripts denote derivatives. If we substitute (6.7) and (6.8) in (6.6) and disregard all except three leading terms then we obtain the approximation

$$\underline{(6.9)} \qquad PCS \approx \int_{-\infty}^{\infty} \Phi^p(w_z)\, \varphi(z)dz - \frac{C_3(\lambda^o_{[k]})}{6} \int_{-\infty}^{\infty} \Phi^p(w_z)\, \varphi^{(3)}(z)dz$$

$$- \frac{p}{6} C_3(\lambda^o) \int_{-\infty}^{\infty} \Phi^{p-1}(w_z)\, \varphi^{(2)}(w_z)\, \varphi(z)dz$$

where $w_z = \sqrt{r^*}\,(z+b_o)$ and $\varphi^{(2)}(w_z) = (w_z^2-1)\,\varphi(w_z)$. After expanding each integral about $\epsilon=0$ and dropping second order correction terms we set the resulting approximation equal to P* and obtain

$$\underline{(6.10)} \qquad P^* = \int_{-\infty}^{\infty} \Phi^p(w'_z)\, \varphi(z)dz + \epsilon p \sqrt{r^*} \int_{-\infty}^{\infty} \Phi^{p-1}(w'_z)\, \varphi(w'_z)\, \varphi(z)dz$$

$$- \frac{C_3(\lambda^o_{[k]})}{6} \int_{-\infty}^{\infty} \Phi^p(w'_z)\, \varphi^{(3)}(z)dz - \frac{p}{6} C_3(\lambda^o) \int_{-\infty}^{\infty} \Phi^{p-1}(w'_z)\, \varphi^{(2)}(w'_z)\, \varphi(z)dz$$

where $w'_z = \sqrt{r^*}\,(z+b_1)$. Since b_1 is, by definition, the solution of (5.5) the first integral in (6.10) equals P* and this simplifies the resulting linear

equation in ϵ. After several integrations by parts and simplification we can write (6.10) in the form

$$\underline{(6.11)} \qquad \frac{\epsilon\sqrt{r^*}}{\sqrt{1+r^*}} \int_{-\infty}^{\infty} \Phi^{p-1}(w_t)\, \varphi(t)dt$$

$$= \frac{\sqrt{r^*-1}}{6\sqrt{nd^*}} \left\{ \frac{r^*(r^*-1)\, \varphi_2\varphi_3}{(r^*+1)(2r^*+1)\sqrt{3r^*+1}} \int_{-\infty}^{\infty} \Phi^{p-3}(w_t)\, \varphi(t)dt \right.$$

$$+ \frac{r^{*3/2}(7r^*+5)\, \varphi_2 b_1}{(r^*+1)^2(2r^*+1)^{3/2}} \int_{-\infty}^{\infty} \Phi^{p-2}(w_u)\, \varphi(u)du$$

$$\left. - \frac{(r^*-1)[r^*(b_1^2-1)-1]}{(r^*+1)^{5/2}} \int_{-\infty}^{\infty} \Phi^{p-1}(w_v)\, \varphi(v)dv \right\}$$

where

$$\varphi_2 = (p-1)\, \varphi\left(b_1\sqrt{\frac{r^*}{(r^*+1)(2r^*+1)}}\right); \quad \varphi_3 = (p-2)\, \varphi\left(b_1\sqrt{\frac{r^*}{(2r^*+1)(3r^*+1)}}\right)$$

$$\underline{(6.12)} \qquad w_t = t\sqrt{\frac{r^*}{3r^*+1}} + \frac{b_1\sqrt{r^*}}{3r^*+1}; \quad w_u = u\sqrt{\frac{r^*}{2r^*+1}} + \frac{b_1\sqrt{r^*}}{2r^*+1};$$

$$w_v = v\sqrt{\frac{r^*}{r^*+1}} + \frac{b_1\sqrt{r^*}}{r^*+1}$$

and the three integrals in (6.11) can all be evaluated with the aid of the same table that was mentioned in Section 5. Thus ϵ is of the form $\epsilon'/\sqrt{nd^*}$ where ϵ' depends only on r^*, b_1 and k. To get the improved solution for n we use (5.4), let $m=nd^*$, set

$$\underline{(6.13)} \qquad \frac{m(r^*-1)}{r^*} = \left(b_1 + \frac{\epsilon'}{\sqrt{m}}\right)^2$$

and solve for m. If this value of m is large, a satisfactory solution may be

obtained by setting m on the right hand side of (6.13) equal to the solution m_1 obtained from (5.5); then the solution of (6.13) can be approximated by

$$(6.14) \qquad m_2 = \left(b_1 \sqrt{\frac{r^*}{r^*-1}} + \frac{\epsilon'}{b_1} \right)^2 \approx b_1^2 \frac{r^*}{r^*-1} + 2\epsilon' \sqrt{\frac{r^*}{r^*-1}} ,$$

where the first term of the last expression is the solution m_1 of (5.5).

For small values of k there are further simplifications in (6.11). For example, if k=2 (or p=1), we can write ϵ' as

$$(6.15) \qquad \epsilon' = - \frac{[(b_1^2-1)\, r^*-1]}{6(r^*+1)^2} \frac{(r^*-1)^{3/2}}{\sqrt{r^*}} .$$

The middle expression of (6.14) then yields numerical results included in the following table (Table II).

Table II

Approximate and Exact Values of m=nd* for p=1 (k=2).

r*		P* = .90	P* = .95	P* = .99
1.25	N.A. (m_1)	14.781	24.350	48.707
	I.A. (m_2)	14.758	24.287	48.544
	E.V. (m_o)	14.764	24.303	48.584
1.50	N.A. (m_1)	8.212	13.528	27.059
	I.A. (m_2)	8.169	13.414	26.766
	E.V. (m_o)	8.180	13.442	26.837
2.00	N.A. (m_1)	4.927	8.117	16.236
	I.A. (m_2)	4.856	7.928	15.749
	E.V. (m_o)	4.875	7.973	15.859

N.A. (or m_1) = Normal approximation based on (5.4) and (5.5).

I.A. (or m_2) = Improved approximation based on the middle expression of (6.14).

E.V. (or m_o) = Exact value to 3 decimal places based on (3.12) and rounded upwards.

A larger table (Table I) comparing the Normal approximation with exact values appears at the end of this paper. In this table the maximum percentage error of the Normal approximation in all 3 parts of Table I in the range $r < 1.15$ (where the exact values are missing) is less than 1.25%.

7. Corrections based on Multivariate Hermite Polynomials

In a companion paper [3] the author has shown how a multivariate Gram-Charlier (or Edgeworth) expansion can be used to obtain correction terms to the Normal approximation (5.2). We first consider the "forward" problem of correcting the c.d.f. $F(x;\rho)$ for given values of $m(\lambda) = n(\lambda_{[k]}-\lambda)$ and $r = \lambda_{[k]}/\lambda$; then we consider the "inverse" problem of correcting the percentage point m=nd* for given values of the c.d.f. $F(x;\rho)$ and r.

To apply the results in [3] to the Poisson ranking problem we first rewrite the PCS in the form

$$\underline{(7.1)} \qquad PCS \cong P\{X_i \leq \frac{\sqrt{n}(\lambda_{[k]}-\lambda)}{\sqrt{\lambda+\lambda_{[k]}}} \qquad (i=1,2,\ldots,p)\}$$

Here the

$$\underline{(7.2)} \qquad X_i = \frac{S_{(i)}-S_{(k)}-n(\lambda-\lambda_{[k]})}{\sqrt{n(\lambda+\lambda_{[k]})}} \qquad (i=1,2,\ldots,p)$$

are p=k-1 standardized variables with a common correlation ρ. The value of ρ and the b_p-value (see (7) in [3]) are given respectively by

$$\underline{(7.3)} \qquad \rho = \frac{\lambda_{[k]}}{\lambda+\lambda_{[k]}}; \qquad b_p = \sqrt{\frac{(\lambda_{[k]}+\lambda)[\lambda+(p-1)\lambda_{[k]}]}{\lambda(\lambda+p\lambda_{[k]})}},$$

both of which depend only on r. Then $f(x;\rho)$ and $F(x;\rho)$ in [3] denote, respectively, the joint density and c.d.f. of the X_i in (7.2) and we write as as in(3.5) of [3]

$$\underline{(7.4)} \qquad f(x;\rho) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{\infty} \frac{C_{i_1,\ldots,i_p}(\rho)}{i_1!\cdots i_p!} H^*_{i_1,\ldots,i_p}(x;\rho)\,\varphi(x;\rho)$$

and $F(x;\rho)$ is obtained by a term-by-term integration of (7.4). Using (4.11) of [3], the moments of X_i required to compute the C's in (7.4) are easily found and, dropping the argument ρ of the C's and the argument λ of m, we obtain

$$C_{3,0,\ldots,0} = -\frac{\lambda_{[k]}-\lambda}{\sqrt{n}(\lambda_{[k]}+\lambda)^{3/2}} = -\frac{1}{\sqrt{m}}\left(\frac{r-1}{r+1}\right)^{3/2}$$

$$C_{2,1,0,\ldots,0} = C_{1,1,1,0,\ldots,0} = \frac{\lambda_{[k]}}{\sqrt{n}(\lambda+\lambda_{[k]})^{3/2}} = -\frac{r\sqrt{r-1}}{\sqrt{m(r+1)^3}}$$

$$\underline{(7.5)} \qquad C_{4,0,\ldots,0} = \frac{1}{n(\lambda_{[k]}+\lambda)} = \frac{1}{m}\left(\frac{r-1}{r+1}\right)$$

$$C_{3,1,0,\ldots,0} = C_{2,2,0,\ldots,0} = C_{2,1,1,0,\ldots,0}$$

$$= C_{1,1,1,1,0,\ldots,0} = \frac{\lambda_{[k]}}{n(\lambda+\lambda_{[k]})^2} = \frac{r(r-1)}{m(r+1)^2} \quad .$$

If we let

$$\underline{(7.6)} \qquad m^{(s/2)-1}C_{i_1,\ldots,i_p} = C'_{i_1,\ldots,i_p}$$

where $m=m(\lambda)$ and $s=i_1+i_2+\ldots+i_p$ and group terms in analogy with the univariate Edgeworth series, then, dropping the arguments of the H's, we can write (7.4) as

$$\underline{(7.7)} \qquad f(x;\rho) = \varphi(x;\rho)$$

$$+ \frac{\varphi(x;\rho)}{\sqrt{m}}\left\{\sum C'_{3,0,\ldots,0}H^*_{3,0,\ldots,0} + \sum C'_{2,1,0,\ldots,0}H^*_{2,1,0,\ldots,0}\right.$$

$$\left. + \sum C'_{1,1,1,0,\ldots,0}H^*_{1,1,1,0,\ldots,0}\right\} + \frac{\varphi(x;\rho)}{m}\left\{\sum C'_{4,0,\ldots,0}H^*_{4,0,\ldots,0}\right.$$

$$+ \Sigma\, C'_{3,1,0,\ldots,0} H^*_{3,1,0,\ldots,0} + \Sigma\, C'_{2,2,0,\ldots,0} H^*_{2,2,0,\ldots,0}$$

$$+ \Sigma\, C'_{2,1,1,0,\ldots,0} H^*_{2,1,1,0,\ldots,0} + \Sigma\, C'_{1,1,1,1,0,\ldots,0} H^*_{1,1,1,1,0,\ldots,0}$$

$$+ \tfrac{1}{2}[\Sigma\, C'_{3,0,\ldots,0} H^*_{3,0,\ldots,0} + \Sigma\, C'_{2,1,0,\ldots,0} H^*_{2,1,0,\ldots,0}$$

$$+ \Sigma\, C'_{1,1,1,0,\ldots,0} H^*_{1,1,1,0,\ldots,0}]^2 \} + \mathcal{O}\left(\frac{\varphi(x;\rho)}{m^{3/2}}\right)$$

where the constants $C'_{i_1,\ldots,i_p}$ depend only on r. The operation of squaring in the last part of (7.7) is symbolic with respect to the H's (not with respect to the coefficients C') in the sense that the product of two H's as well as the powers of one H are to be replaced by a single H in accordance with the rule

$$\underline{(7.8)} \qquad H^*_{i_1,\ldots,i_p} H^*_{j_1,\ldots,j_p} \to H^*_{i_1+j_1,\ldots,i_p+j_p} .$$

As an illustrative example we consider the case p=2 (or k=3) and m=nd*=10 and r*=1.5. The integrated form of (7.6) is denoted by $F(x;\rho)$ where x is a vector with equal components and the common value, from the right hand side of (7.1), is $x_1 = x_2 = \sqrt{m(r^*-1)/(r^*+1)} = \sqrt{2}$; the value of ρ from (7.3) is $r^*/(r^*+1) = .6$. It is shown in [3] that the leading term becomes

$$\underline{(7.9)} \qquad \int_{-\infty}^{\sqrt{2}} \cdots \int_{-\infty}^{\sqrt{2}} \varphi(t;\rho)\,dt_1 \cdots dt_2 = \int_{-\infty}^{\infty} \Phi^2\left(\frac{\sqrt{2}+w\sqrt{\rho}}{\sqrt{1-\rho}}\right) \varphi(w)\,dw$$

where φ(w) and Φ(w) denote the standardized univariate Normal density and c.d.f., respectively. (For an explanation of the remaining integrations the reader is referred to [3].) The numerical values obtained, grouped in analogy with the Edgeworth type series in (7.7), are

$$\underline{(7.10)} \qquad F(\sqrt{2},\sqrt{2};\rho=.6) \approx .87102 - .00252$$

$$= .86850 .$$

The corresponding exact answer is .86829. We note that the absolute value of the error .00021 is smaller than the corresponding correction term .00252. Additional computation for r* closer to unity gives even smaller ratios of error term to correction term and this shows numerically that the correction terms can make a non-trivial contribution when the exact value is difficult to compute. In addition, the number of zeros and the rate of increase of zeros from term to term gives us a "good idea" of the accuracy of our approximation and, in particular, of the leading term.

The inverse problem is to correct the percentage point m=nd* for given values of the c.d.f. $F(x;\rho)$ and r. We now use equation (5.12) of [3] and the results in Section 6 above. Let m_1, b_1, ϵ and ϵ' be defined as in Sections 5 and 6 and let

$$\underline{(7.11)} \qquad x_1 = \ldots = x_p = \sqrt{m}\sqrt{\frac{r^*-1}{r^*+1}} = b\sqrt{\frac{r^*}{r^*+1}} = x_o \text{ (say)}$$

denote the correct common value of x_i $(i=1,2,\ldots,p)$ such that $F(x_o,\ldots,x_o;\rho)$ is equal to a preassigned value P*. Let x_1^* denote the value (7.11) if b is replaced by b_1 (the value is the same if m is replaced by m_1). Let $V_1(x_o,m)$ denote the first correction term in (5.12) of [3], i.e.,

$$\underline{(7.12)} \qquad V_1(x_o,m) = \frac{1}{\sqrt{m}}\{pC'_{3,0,0}Q(x_o,x_o,x_o) + \binom{p}{2}C'_{2,1,0}R(x_o,x_o,x_o) + \binom{p}{3}C'_{1,1,1}S(x_o,x_o,x_o)\}$$

where the functions Q, R and S are given in (5.13), (5.15), and (5.16) of [3] and the constants C' are given by (7.5) and (7.6) above.

Our goal is to expand the integrated form of (7.7), i.e., equation (5.12) of [3], as a function of m around the point m_1 (or equivalently as a function of b around the point b_1). The major part of this expansion was already carried out in Section 6 where the leading term in (7.9) was expanded but the term $V_1(x_o,m)$ was not taken into account there. Hence starting with (6.10),

writing ϵ_1 instead of ϵ, and cancelling P* with the first integral in (6.10), we obtain for large values of m

$$\underline{(7.13)} \quad \epsilon_1 p\sqrt{r^*} \int_{-\infty}^{\infty} \Phi^{p-1}(w_z') \varphi(w_z') \varphi(z)dz - I_1 - I_2 + V_1(x_1^*,m) = 0$$

where I_1 and I_2 are the last two terms in (6.10). The solution of this linear equation in ϵ_1 gives a first order correction term for b_1 and then the method of (6.14), which involves the replacement of $V(x_1^*,m)$ by $V(x_1^*,m_1)$, gives a first order correction term for m_1. The value of ϵ_1 is now obtained from

$$\underline{(7.14)} \quad \frac{\epsilon_1\sqrt{r^*}}{\sqrt{r^*+1}} \int_{-\infty}^{\infty} \Phi^{p-1}(w_t') \varphi(t)dt$$

$$= \frac{\sqrt{r^*-1}}{6\sqrt{m_1}} \left\{ \frac{r^*(r^*-1)\,\varphi_2\varphi_3}{(r^*+1)(2r^*+1)\sqrt{3r^*+1}} \int_{-\infty}^{\infty} \Phi^{p-3}(w_t) \varphi(t)dt \right.$$

$$+ \frac{b_1\varphi_2 r^{*3/2}(7r^*+5)}{(r^*+1)^2(2r^*+1)^{3/2}} \int_{-\infty}^{\infty} \Phi^{p-2}(w_u) \varphi(u)du$$

$$- \frac{(r^*-1)[r^*(b_1^2-1)-1]}{(r^*+1)^{5/2}} \int_{-\infty}^{\infty} \Phi^{p-1}(w_v) \varphi(v)dv$$

$$+ \frac{(r^*-1)}{(r^*+1)^{3/2}} \left[a^2(r^*)\, 2u\, \varphi(u)\, \Phi(v) + 2a^2(r^*) \sqrt{\frac{r^{*2}+3r^*+1}{r^{*2}+r^*+1}}\, \varphi(u)\, \varphi(v) \right.$$

$$+ 4b_1\sqrt{\frac{r^*}{3r^*+1}}\, a(r^*)\, \varphi(u)\, \Phi(v)$$

$$\left. + [b_1^2 \frac{r^*}{r^*+1} - d(r^*)] \int_{-\infty}^{\infty} \Phi^2 \left[\frac{\sqrt{r^*}\left(w+b_1\sqrt{\frac{r^*+1}{2r^*+1}}\right)}{\sqrt{r^{*2}+r^*+1}} \right] \varphi(w)dw \right]$$

$$+\frac{3(p-1)}{2}\ \frac{r^*\sqrt{(r^*+1)}}{(2r^*+1)}\ \varphi(u)\left[u\Phi(y)+\frac{r^*\,\varphi(y)}{\sqrt{(r^*+1)(3r^*+1)}}\right]$$

$$-\frac{(p-1)(p-2)\ r^*}{\sqrt{3r^*+1}}\ \varphi(u)\ \varphi(y)\Bigg\}$$

where

(7.15) $$a(r^*)=\frac{r^*(r^{*2}+3r^*+1)}{(r^*+1)^2(3r^*+1)}\ \sqrt{2r^*+1}$$

(7.16) $$d(r^*)=\frac{(2r^*+1)(r^{*4}+4r^{*3}+10r^{*2}+6r^*+1)}{(r^*+1)^2(3r^*+1)^2}$$

(7.17) $$u=x_1^*\sqrt{\frac{1-\rho}{1+\rho}}=b_1\sqrt{\frac{r^*}{(r^*+1)(2r^*+1)}}$$

(7.18) $$v=x_1^*\sqrt{\frac{(1-\rho)(1-\rho+\rho^2)}{(1+\rho)(1+\rho-\rho^2)}}=b_1\sqrt{\frac{r^*(r^{*2}+r^*+1)}{(r^*+1)(2r^*+1)(r^{*2}+3r^*+1)}}$$

(7.19) $$y=x_1^*\sqrt{\frac{1-\rho}{[1+(p-2)\rho][1+(p-1)\rho]}}=b_1\sqrt{\frac{r^*}{[1+(p-1)r^*][1+pr^*]}}$$

(7.20) $$\varphi_2=(p-1)\,\varphi\left(\frac{b_1\sqrt{r^*}}{\sqrt{(r^*+1)(2r^*+1)}}\right)\ ;\ \varphi_3=(p-2)\,\varphi\left(\frac{b_1\sqrt{r^*}}{\sqrt{(2r^*+1)(3r^*+1)}}\right).$$

Here again, if the exact solution m_o is large we define $\epsilon_1=\epsilon_1'/\sqrt{m_1}$ and use (6.14) with ϵ_1' in place of ϵ' to solve for a new approximation m_3.

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Table I

Smallest value of m=nd* required to obtain a PCS ≥ P*.

P* = .90

r		k=2	k=3	k=4	k=5	k=10
1.01	N.A.	330.12	499.57	603.52	678.55	893.04
1.02	N.A.	165.88	250.88	303.01	340.63	448.15
1.03	N.A.	111.13	167.98	202.84	227.99	299.85
1.04	N.A.	83.761	126.54	152.75	171.67	225.70
1.05	N.A.	67.337	101.67	122.70	137.87	181.21
1.06	N.A.	56.388	85.086	102.66	115.34	151.54
1.08	N.A.	42.702	64.360	77.619	87.179	114.46
1.10	N.A.	34.490	51.924	62.591	70.279	92.213
1.15	N.A. E.V.	23.541 23.530	35.341 35.521	42.550 42.868	47.743 48.169	62.540 63.307
1.20	N.A. E.V.	18.066 18.052	27.048 27.223	32.527 32.840	36.471 36.891	47.697 48.458
1.25	N.A. E.V.	14.781 14.764	22.070 22.241	26.511 26.819	29.705 30.120	38.787 39.541
1.50	N.A. E.V.	8.2119 8.180	12.106 12.257	14.464 14.749	16.153 16.544	20.934 21.660
1.75	N.A. E.V.	6.0220 5.979	8.7759 8.912	10.433 10.701	11.617 11.989	14.952 15.656
2.00	N.A. E.V.	4.9271 4.875	7.1048 7.228	8.4080 8.662	9.3363 9.694	11.942 12.628
2.50	N.A. E.V.	3.8322 3.768	5.4241 5.528	6.3675 6.600	7.0362 7.371	8.9003 9.559
3.00	N.A. E.V.	3.2847 3.229	4.5758 4.666	5.3348 5.552	5.8700 6.188	7.3551 7.994

N.A. denotes the Normal approximation based on (5.4) and (5.5); to 3 decimal places and rounded upwards

E.V. denotes the exact value based on (3.12). For k ≥ 3 the N.A. entries may be off by at most 1 in the last digit; all other entries are accurate to the number of digits shown.

Table I (continued)

Smallest value of m=nd* required to obtain a PCS ≥ P*.

P* = .95

r		k=2	k=3	k=4	k=5	k=10
1.01	N.A.	543.81	737.86	854.22	937.45	1173.1
1.02	N.A.	273.26	370.62	429.00	470.75	588.93
1.03	N.A.	183.08	248.21	287.26	315.18	394.20
1.04	N.A.	137.98	187.01	216.39	237.39	296.83
1.05	N.A.	110.93	150.28	173.86	190.72	238.40
1.06	N.A.	92.890	125.80	145.51	159.60	199.45
1.08	N.A.	70.344	95.197	110.07	120.71	150.76
1.10	N.A.	56.816	76.833	88.811	97.367	121.55
1.15	N.A. E.V.	38.779 38.750	52.347 52.497	60.455 60.732	66.244 66.618	82.587 83.275
1.20	N.A. E.V.	29.761 29.723	40.101 40.241	46.274 46.540	50.678 51.043	63.098 63.778
1.25	N.A. E.V.	24.350 24.303	32.752 32.882	37.763 38.018	41.335 41.689	51.398 52.068
1.50	N.A. E.V.	13.528 13.442	18.044 18.130	20.722 20.932	22.625 22.932	27.961 28.582
1.75	N.A. E.V.	9.9203 9.803	13.130 13.181	15.023 15.197	16.365 16.636	20.111 20.693
2.00	N.A. E.V.	8.1166 7.973	10.666 10.687	12.162 12.306	13.220 13.460	16.163 16.713
2.50	N.A. E.V.	6.3129 6.126	8.1898 8.165	9.2826 9.379	10.051 10.244	12.175 12.677
3.00	N.A. E.V.	5.4111 5.191	6.9422 6.882	7.8272 7.888	8.4470 8.604	10.151 10.616

N.A. denotes the Normal approximation based on (5.4) and (5.5); and rounded upwards

E.V. denotes the exact value to 3 decimal places based on (3.12). For $k \geq 3$ the N.A. entries may be off by at most 1 in the last digit; all other entries are accurate to the number of digits shown.

Table I (continued)

Smallest value of m=nd* required to obtain a PCS ≧ P*.

P* = .99

r		k=2	k=3	k=4	k=5	k=10
1.01	N.A.	1087.8	1314.8	1448.5	1543.5	1810.6
1.02	N.A.	546.60	660.56	727.68	775.35	909.37
1.03	N.A.	366.20	442.48	487.40	519.30	608.96
1.04	N.A.	276.01	333.44	367.26	391.27	458.75
1.05	N.A.	221.89	268.02	295.17	314.45	368.63
1.06	N.A.	185.81	224.40	247.12	263.24	308.54
1.08	N.A.	140.71	169.88	187.04	199.22	233.43
1.10	N.A.	113.65	137.17	151.00	160.81	188.36
1.15	N.A. E.V.	77.570 77.476	93.545 93.585	102.93 103.069	109.59 109.802	128.26 128.729
1.20	N.A. E.V.	59.531 59.430	71.732 71.767	78.898 79.027	83.975 84.165	98.201 98.644
1.25	N.A. E.V.	48.707 48.584	58.643 58.653	64.473 64.577	68.602 68.782	80.159 80.592
1.50	N.A. E.V.	27.059 26.837	32.451 32.356	35.601 35.599	37.826 37.899	44.027 44.351
1.75	N.A. E.V.	19.844 19.538	23.708 23.526	25.955 25.867	27.539 27.526	31.936 32.175
2.00	N.A. E.V.	16.236 15.859	19.327 19.072	21.119 20.957	22.377 22.291	25.860 26.028
2.50	N.A. E.V.	12.628 12.135	14.932 14.560	16.257 15.980	17.185 16.985	19.735 19.793
3.00	N.A. E.V.	10.824 10.239	12.721 12.259	13.807 13.440	14.563 14.276	16.631 16.607

N.A. denotes the Normal approximation based on (5.4) and (5.5); and rounded upwards

E.V. denotes the exact value to 3 decimal places based on (3.12). For k ≧ 3 the N.A. entries may be off by at most 1 in the last digit; all other entries are accurate to the number of digits shown.