A MATRIX APPROACH TO NONSTATIONARY CHAINS

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CONTENTS

1. Limitations of Stationary Chains.

The field of Markov chains applies to a wide range of subject matter. The assumption that the transition probabilities of a Markov chain be stationary leads to a rich body of theorems, which serve as a good first approximation to the real world. There are, nevertheless, very many real situations to which the model of a stationary chain (Markov chain with stationary transition probabilities) is inappropriate.

This paper develops a matrix approach [cf. Lipstein (1965, 1968)] for studying nonstationary chains. We explore the notion of causative matrix which, when multiplying a transition probability matrix, yields the immediately subsequent one. The special model involving constant causative matrix is studied in detail.

The general case of nonstationary chains has received only little attention in the literature. Hajnal (1956, 1958), Mott (1957, 1959) and Sarymsakov (1961) appear to be the only papers in English. Linnik (1948), (1949), Sarymsakov (1953), (1956), (1958) and Sarymsakov and Mustafin (1957) study the subject in Russian.

All of these papers consider nonstationary chains from a point of view different to ours. We show that the basic descriptive characteristics of nonstationary chains can be captured and described by means of a sequence of causative matrices.

Nonstationary chains are characterized by either convergent or divergent behavior. When convergent the chain is tending toward complete independence as represented by a Bernoulli process.

We contend that nonstationary chains in general can be studied more effectively through identification of a functional relationship between their causative matrices. The constant chain constitutes a special case of such a relationship. Higher order nonstationary chains may be susceptible to systematic study through the derivation of higher order causative matrices analogous to higher order differences or differentials.

A stationary chain is a stochastic process containing a finite number n of states $E_1, E_2, \ldots, E_n$ such that

(a) there is an initial distribution $(a_1, a_2, \ldots, a_n)$, where a_k is the probability that the first state is E_k, and

(b) there is a transition probability matrix

$$\underset{\sim}{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix},$$

where p_{ij} is the conditional probability of E_j occurring given that E_i is the present state. In view of (a) and (b), we have

$$0 \leq a_i \leq 1 \quad \text{and} \quad \Sigma\, a_i = 1$$

$$0 \leq p_{ij} \leq 1 \quad \text{and for all} \quad i, \ \sum_j p_{ij} = 1.$$

This transition probability is stationary when it is constant over time; that is if E_j occurred at time t and E_i at $t-1$, p_{ij} is stationary when independent of t. As defined above, a chain is both finite and discrete; it has a finite number n of states and the time intervals are discrete.

Real world situations in which the transition probabilities vary over time include epidemiology and learning theory [cf. Harary & Lipstein (1962)]. Much criticism [cf. Ehrenberg (1965), Massy & Morrison (1968), Ehrenberg (1968)] has been leveled at applying stationary chains in such situations. The change in empirical stimuli over time results in changes in the probability values. There are many other situations in which a stationary chain is applied, even though it is not realistic, since it is essentially the only way which can be handled analytically.

In a nonstationary chain, after the initial probability factor has been invoked, the probabilistic situation is completely described by a sequence of transition matrices $\underset{\sim}{P}_1$, $\underset{\sim}{P}_2$, $\underset{\sim}{P}_3$, ... Each of these matrices $\underset{\sim}{P}_t$ contains the conditional probability distributions which hold at time t, given the status at time $t-1$. In the stationary case, all these transition matrices $\underset{\sim}{P}_t$ are the same. If at least two of the transition matrices are different, then the chain is nonstationary. Another kind of special case results from those situations in which the general transition matrix at time t, namely $\underset{\sim}{P}_t$, has as its entries $p_{ij} = f_{ij}(t)$. In this case, every entry is a function of t. In this case the probabilistic behavior of the entire nonstationary process is known when f is given.

But there are many situations in which only the first few transition matrices $\underset{\sim}{P}_t$ are known and the problem is to predict the future behavior of the chain. This is the problem of real interest in studying chains, both stationary and nonstationary. In the case of stationary chains, the prediction of future behavior of the events is well known [cf. Kemeny and Snell (1960), Styan & Smith (1964)]. The graphical structure of the chain contributes additional information as mentioned in Harary and Lipstein (1962), and Harary, Norman, & Cartwright (1965). Our concern is to devise methods for a similar treatment of nonstationary chains.

2. Causative Matrices.

Consider a nonstationary chain as given by a sequence of transition matrices $\underset{\sim}{P}_1, \underset{\sim}{P}_2, \underset{\sim}{P}_3, \ldots$. In order to describe the change occurring from each of these transition matrices to the next, we introduce an accompanying sequence of matrices $\underset{\sim}{C}_1, \underset{\sim}{C}_2, \ldots$, which will be called causative matrices defined by the following equations:

$$(2.1) \qquad \underset{\sim}{P}_1\underset{\sim}{C}_1 = \underset{\sim}{P}_2,\ \underset{\sim}{P}_2\underset{\sim}{C}_2 = \underset{\sim}{P}_3, \ldots,\ \underset{\sim}{P}_t\underset{\sim}{C}_t = \underset{\sim}{P}_{t+1}, \ldots$$

Each causative matrix can be immediately expressed in terms of the transition matrices, provided they are all nonsingular:

$$(2.2) \qquad \underset{\sim}{C}_1 = \underset{\sim}{P}_1^{-1}\underset{\sim}{P}_2, \text{ and in general, } \underset{\sim}{C}_t = \underset{\sim}{P}_t^{-1}\underset{\sim}{P}_{t+1}.$$

We emphasize that the causative matrices $\underset{\sim}{C}_t$ are merely devices for describing the change involved between each transition matrix and the next one.

In these terms, a stationary chain is that special case obtained by taking every $\underset{\sim}{C}_t = \underset{\sim}{I}$, the identity matrix of order n. Of course when all the transition matrices are different, none of the causative matrices will be the identity. We note that the assumption that every $\underset{\sim}{P}_t$ is nonsingular is not a strong restriction of generality. The reason is that even a small change in the values of the entries of a singular matrix will result in nonsingularity [cf. Householder (1964)]. To illustrate, given the two stochastic matrices:

$$\underset{\sim}{P}_1 = \begin{pmatrix} .7 & 0 & .3 \\ .2 & .8 & 0 \\ .1 & 0 & .9 \end{pmatrix} \qquad \underset{\sim}{P}_2 = \begin{pmatrix} .9 & 0 & .1 \\ 0 & .8 & .2 \\ .3 & 0 & .7 \end{pmatrix}$$

we find $\underset{\sim}{C}_1 = \underset{\sim}{P}_1^{-1}\underset{\sim}{P}_2 = \begin{pmatrix} 1.2 & 0 & -.2 \\ -.3 & 1.0 & .3 \\ .2 & 0 & .8 \end{pmatrix}$.

It is not entirely coincidental that the causative matrix $\underset{\sim}{C}_1$ in this example resembles the identity matrix in the sense that the diagonal entries are near 1 and the other entries near 0.

The causative matrices $\underset{\sim}{C}_t$ have all been those which multiply the probability matrix $\underset{\sim}{P}_t$ at each time stage t on the right to obtain the next matrix $\underset{\sim}{P}_{t+1}$. There is no _a priori_ reason for choosing right multiplication for this purpose rather than left. Thus we may refer to the matrices $\underset{\sim}{C}_t$ as _right causative matrices_ and introduce the corresponding sequence $\underset{\sim}{D}_t$ of _left causative matrices_ induced by the nonstationary chain whose probability matrix sequence is $\underset{\sim}{P}_1, \underset{\sim}{P}_2, \ldots$, by the equations

(2.3) $\qquad \underset{\sim}{D}_1\underset{\sim}{P}_1 = \underset{\sim}{P}_2 , \quad \underset{\sim}{D}_2\underset{\sim}{P}_2 = \underset{\sim}{P}_3, \ldots, \underset{\sim}{D}_t\underset{\sim}{P}_t = \underset{\sim}{P}_{t+1}, \ldots.$

Analogously to (2.2) we have when each $\underset{\sim}{P}_t$ is nonsingular

(2.4) $\qquad \underset{\sim}{D}_1 = \underset{\sim}{P}_2\underset{\sim}{P}_1^{-1} , \quad \underset{\sim}{D}_2 = \underset{\sim}{P}_3\underset{\sim}{P}_2^{-1}, \ldots, \underset{\sim}{D}_t = \underset{\sim}{P}_{t+1}\underset{\sim}{P}_t^{-1} \ldots.$

The left and right causative matrices are different in general. Just as the action on a given matrix $\underset{\sim}{M}$ by a permutation matrix $\underset{\sim}{A}$ on the left, $\underset{\sim}{A}\underset{\sim}{M}$, permutes the rows of $\underset{\sim}{M}$, and on the right, $\underset{\sim}{M}\underset{\sim}{A}$, permutes its columns, a corresponding effect is found with left and right causative matrices emphasizing the rows and columns respectively. In the language of binary relations, this phenomenon is known as "directional duality" [cf. Harary, Norman and Cartwright (1965)]. An example involving 2×2 matrices will illustrate this point:

$$\underset{\sim}{P}_1 = \begin{pmatrix} .8 & .2 \\ .2 & .8 \end{pmatrix}, \quad \underset{\sim}{P}_2 = \begin{pmatrix} .9 & .1 \\ .2 & .8 \end{pmatrix}$$

$$\underset{\sim}{C}_1 = \underset{\sim}{P}_1^{-1}\underset{\sim}{P}_2 = \begin{pmatrix} 1.13 & -.13 \\ -.03 & 1.03 \end{pmatrix}; \quad \underset{\sim}{D}_1 = \underset{\sim}{P}_2\underset{\sim}{P}_1^{-1} = \begin{pmatrix} 1.17 & -.17 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, when the roles of $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_2$ are interchanged in this 2×2 example, so that

$$\underset{\sim}{P}_1 = \begin{pmatrix} .9 & .1 \\ .2 & .8 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{P}_2 = \begin{pmatrix} .8 & .2 \\ .2 & .8 \end{pmatrix}$$

we find that

$$\underset{\sim}{C}_1 = \underset{\sim}{P}_1^{-1}\underset{\sim}{P}_2 = \begin{pmatrix} .89 & .11 \\ .03 & .97 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{D}_1 = \underset{\sim}{P}_2\underset{\sim}{P}_1^{-1} = \begin{pmatrix} .86 & .14 \\ 0 & 1 \end{pmatrix}.$$

We will use right causative matrices for the remainder of this section.

By a (right) <u>constant chain</u> we will mean a nonstationary chain $\underset{\sim}{P}_1, \underset{\sim}{P}_2, \ldots$ in which all the (right) causative matrices are equal; we will call this matrix $\underset{\sim}{C}$. In this case we verify at once that

$$(2.5) \qquad \underset{\sim}{P}_{t+s} = \underset{\sim}{P}_t\underset{\sim}{C}^s, \; s = 0, 1, \ldots.$$

Since $\underset{\sim}{C} = \underset{\sim}{P}_1^{-1}\underset{\sim}{P}_2 = \underset{\sim}{P}_2^{-1}\underset{\sim}{P}_3$, it follows at once that $\underset{\sim}{P}_3$ is expressible in terms of $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_2$ by the equation

$$\underset{\sim}{P}_3 = \underset{\sim}{P}_2\underset{\sim}{P}_1^{-1}\underset{\sim}{P}_2,$$

In general, we find that

$$(2.6) \qquad \underset{\sim}{P}_{t+1} = \underset{\sim}{P}_t\underset{\sim}{P}_{t-1}^{-1}\underset{\sim}{P}_t.$$

Therefore every transition matrix $\underset{\sim}{P}_t$ of a constant chain may be expressed in terms of $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_2$ by the equations

$$(2.7) \qquad \underset{\sim}{P}_t = (\underset{\sim}{P}_2\,\underset{\sim}{P}_1^{-1})^{t-2}\,\underset{\sim}{P}_2 = \underset{\sim}{P}_2(\underset{\sim}{P}_1^{-1}\,\underset{\sim}{P}_2)^{t-2}; \quad t = 2, 3, \ldots$$

By definition of a constant chain, every transition matrix is determined as soon as $\underset{\sim}{P}_1$ and $\underset{\sim}{C}$ are given, and in fact $\underset{\sim}{P}_t = \underset{\sim}{P}_1\underset{\sim}{C}^{t-1}$ follows from (2.5). Considerable information about a constant chain can be obtained from $\underset{\sim}{C}$ alone.

To ease the algebra we set $\underset{\sim}{P}_1 = \underset{\sim}{Q}$. Then if $\underset{\sim}{P}(u, v)$ denotes the transition probability matrix from time periods u to v, we have

$$(2.8) \qquad \underset{\sim}{P}(t_o+r,\ t_o+r+1) = \underset{\sim}{Q}\,\underset{\sim}{C}^r; \quad r = 0, 1, \ldots,$$

where $\underset{\sim}{Q} = \underset{\sim}{P}_1 = \underset{\sim}{P}(t_o,\ t_o+1)$. The starting point t_o is arbitrary. Hence

$$(2.9) \qquad \underset{\sim}{P}(t_o,\ t_o+r+1) = \underset{\sim}{Q}\cdot\underset{\sim}{Q}\,\underset{\sim}{C}\cdot\underset{\sim}{Q}\,\underset{\sim}{C}^2\cdot\ldots\cdot\underset{\sim}{Q}\,\underset{\sim}{C}^r$$

$$= \prod_{s=0}^{r}(\underset{\sim}{Q}\,\underset{\sim}{C}^s) = \underset{\sim}{T}_r,$$

say, is the transition probability matrix from time periods t_o to t_o+r+1. The limiting properties of a constant chain depend on the convergence or divergence of $\underset{\sim}{C}^r$ and $\underset{\sim}{T}_r$, as $r \to \infty$. Lipstein (1965) conjectured that $\underset{\sim}{C}^r$ diverged <u>only</u> when the largest characteristic root of $\underset{\sim}{C}$ exceeded one in absolute value, and implied that if all roots of $\underset{\sim}{C}$ equalled unity, then $\underset{\sim}{C} \equiv \underset{\sim}{I}$. That this is not so will be illustrated in section 3 for the case of 2 states.

A causative matrix is similar to a stochastic matrix, in that it has unit row sums; it may, however, have negative elements.

LEMMA 2.1. Let $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ be stochastic matrices of order n. Then if $\underset{\sim}{Q}$ is nonsingular, the causative matrices

$$(2.10) \qquad \underset{\sim}{C} = \underset{\sim}{Q}^{-1}\underset{\sim}{R}, \qquad\qquad \underset{\sim}{D} = \underset{\sim}{R}\,\underset{\sim}{Q}^{-1},$$

have row sums of unity but may have negative elements.

Proof. Let $\underset{\sim}{e} = (1, 1, \ldots, 1)'$ be a column vector with each component unity. Then $\underset{\sim}{Q}\,\underset{\sim}{e} = \underset{\sim}{R}\,\underset{\sim}{e} = \underset{\sim}{e}$. Hence $\underset{\sim}{Q}^{-1}\underset{\sim}{e} = \underset{\sim}{e}$ and so $\underset{\sim}{C}\,\underset{\sim}{e} = \underset{\sim}{Q}^{-1}\underset{\sim}{R}\,\underset{\sim}{e} = \underset{\sim}{Q}^{-1}\underset{\sim}{e} = \underset{\sim}{e}$. Similarly $\underset{\sim}{D}\,\underset{\sim}{e} = \underset{\sim}{e}$. Examples of $\underset{\sim}{C}$ and $\underset{\sim}{D}$ with at least one element negative were given above, just below (2.4). (qed)

It follows from Lemma 2.1 that a causative matrix has a characteristic root of unity. When all other roots are less than one in absolute value $\underset{\sim}{C}^r$ converges to $\underset{\sim}{e}\,\underset{\sim}{\ell}'$ as $r \to \infty$, where $\underset{\sim}{\ell}'$ is the left-hand characteristic vector of $\underset{\sim}{C}$ corresponding to the unit root (the right-hand vector is $\underset{\sim}{e}$). Lipstein (1965) suggested that in such a case $\underset{\sim}{T}_r$ also converged to $\underset{\sim}{e}\,\underset{\sim}{\ell}'$. This is immediate only for $\underset{\sim}{C}$ stochastic; it may happen that $\underset{\sim}{C}$ is not stochastic but $\underset{\sim}{T}_r$ and $\underset{\sim}{\ell}'$ have all elements nonnegative. In this latter case we have been unable to prove in general that $\underset{\sim}{T}_r$ and $\underset{\sim}{C}^r$ have the same limit, since the number of component matrices in the product $\underset{\sim}{T}_r$ increases with r. In section 3 we study the situation for the two state case. The results of Hajnal (1956, 1958), Mott (1957, 1959), and Sarymsakov (1961) do not seem to assist in obtaining more general results.

Another question taken up in section 3 is when does $\underset{\sim}{T}_r$ cease to be stochastic ($\underset{\sim}{C}$ nonstochastic)? That is, what values can $\underset{\sim}{C}$ take on so that the chain may have a constant causative matrix?

3. Two-state Nonstationary Chains

In this section we assume that the nonstationary chain has two states with constant causative matrix

$$(3.1) \qquad \underset{\sim}{C} = \begin{pmatrix} u, & 1-u \\ u-\lambda, & 1-u+\lambda \end{pmatrix},$$

$$= \underset{\sim}{Q}^{-1}\underset{\sim}{R} = \begin{pmatrix} a, & 1-a \\ 1-b, & b \end{pmatrix}^{-1} \begin{pmatrix} c, & 1-c \\ 1-d, & d \end{pmatrix};$$

thus

$$(3.2) \qquad u = \frac{bc - (1-a)(1-d)}{a+b-1} = \frac{a+d-ad+bc-1}{a+b-1}$$

and

$$(3.3) \qquad \lambda = \frac{c+d-1}{a+b-1} = \frac{tr(\underset{\sim}{R})-1}{tr(\underset{\sim}{Q})-1}$$

is the non-unit characteristic root of $\underset{\sim}{C}$. It follows immediately that $\underset{\sim}{C}^s$ has characteristic roots 1 and λ^s, so that $tr(\underset{\sim}{C}^s) = 1 + \lambda^s$. Hence we may write

$$(3.4) \qquad \underset{\sim}{C}^s = \begin{pmatrix} u_s, & 1-u_s \\ u_s-\lambda^s, & 1-u_s+\lambda^s \end{pmatrix}; \qquad s = 1, 2, \ldots,$$

where we take $u_1 \equiv u$. To evaluate u_s, we find from $\underset{\sim}{C}^{s+1} = \underset{\sim}{C}\underset{\sim}{C}^s = \underset{\sim}{C}^s\underset{\sim}{C}$ that

$$(3.5) \qquad u_{s+1} = u_s-\lambda^s+u\lambda^s = u-\lambda+\lambda u_s; \qquad s = 1, 2, \ldots.$$

When $\lambda = 1$, we obtain $u_{s+1} = u_s-1+u$, while otherwise

$$(3.6) \qquad u_s = [u-\lambda+\lambda^s(1-u)]/(1-\lambda).$$

Hence

$$(3.7) \qquad u_s = \begin{cases} s(u-1)+1; & \lambda = 1 \\ \dfrac{u-\lambda+\lambda^s(1-u)}{1-\lambda}; & \lambda \neq 1 \end{cases}.$$

From this we obtain:

THEOREM 3.1. The limit of u_s is given by

$$\lim_{s \to \infty} u_s = \frac{u - \lambda}{1 - \lambda}; \quad -1 < \lambda < 1$$

$$= 1; \ \lambda = 1, \ u = 1$$

$$= \pm\infty \text{ or is undefined, otherwise.}$$

THEOREM 3.2. When $\lambda = -1$, u_s is both Cesaro- and Euler-summable to $\frac{1}{2}(u+1)$.

Proof. Let

$$t_n = \frac{1}{n} \sum_{s=1}^{n} u_s = \frac{1}{2n} \sum_{s=1}^{n} (u + 1 + (-1)^s(1-u)).$$

Then

$$t_n = \frac{u+1}{2} + \frac{(1-u)}{2n} \sum_{s=1}^{n} (-1)^s$$

$$= \frac{u+1}{2}; \ n \text{ even}$$

$$= \frac{u+1}{2} - \frac{(1-u)}{2n}; \ n \text{ odd.}$$

Thus $t_n \to \frac{1}{2}(u+1)$ as $n \to \infty$, so u_s is Cesaro-summable to $\frac{1}{2}(u+1)$.

Let

$$w_n = \frac{1}{2} \sum_{s=0}^{n} \binom{n}{s} k^{n-s}(1-k)^s[u + 1 + (-1)^{s+1}(1-u)], \text{ for some } 0 < k < 1.$$

Then

$$w_n = \frac{u+1}{2} + \frac{(1-u)}{2} \sum_{s=0}^{n} \binom{n}{s} k^{n-s}(1-k)^s(-1)^{s+1}$$

$$= \frac{u+1}{2} - \frac{(1-u)}{2} (2k-1)^n.$$

Since $0 < 2k-1 < 1$, $w_n \to \frac{1}{2}(u+1)$ as $n \to \infty$, so u_s is Euler-summable to $\frac{1}{2}(u+1)$. (qed)

THEOREM 3.3. <u>When</u> $\lambda = +1$, $u \neq 1$, u_s <u>is neither Cesaro- nor Euler-summable</u>.

<u>Proof</u>. Let $t_n = \frac{1}{n}\sum_{s=1}^{n} u_s = \frac{1}{n}\sum_{s=1}^{n}(s(u-1)+1)$. Then $t_n = 1 + \frac{(n+1)(u-1)}{2}$ which diverges as $n \to \infty$. Let

$$w_n = \sum_{s=0}^{n}\binom{n}{s}k^{n-s}(1-k)^s[(s+1)(u-1)+1]; \; 0 < k < 1.$$

Then

$$w_n = u + (u-1)\sum_{s=1}^{n}\binom{n}{s}sk^{n-s}(1-k)^s$$

$$= u + (u-1)[\sum_{t=0}^{m}\binom{m}{t}k^{m-t}(1-k)^t](1-k)n; \; m = n-1, \; t = s-1,$$

which diverges as $n \to \infty$. (qed)

COROLLARY 3.1. <u>The limit of</u> $\underset{\sim}{C}^s$ <u>is given by</u>

$$\lim_{s\to\infty} \underset{\sim}{C}^s = \underset{\sim}{e}(u-\lambda,\ 1-u)/(1-\lambda),$$

<u>when</u> $-1 < \lambda < 1$.

<u>Proof</u>. Directly from Theorem 3.1 and (3.7). (qed)

COROLLARY 3.2. <u>When</u> $\lambda = -1$, $\underset{\sim}{C}^s$ <u>is Cesaro- and Euler-summable to</u>

$\frac{1}{2}\underset{\sim}{e}(u+1,\ 1-u)$.

COROLLARY 3.3. <u>When</u> $\lambda = +1$, $\underset{\sim}{C}^s$ <u>is neither Cesaro- nor Euler-summable unless</u> $u = 1$ <u>in which case</u> $\underset{\sim}{C} \equiv \underset{\sim}{I}$.

We now turn out attention to the limit of (3.6) as r tends to ∞. We obtain

THEOREM 3.4. When $-1 < \lambda < +1$ and $-1 \leq a+b-1 < 1$, the limit

$$\lim_{r\to\infty} \underset{\sim}{P}(t, t+r+1) = \prod_{s=0}^{\infty} (\underset{\sim}{QC}^s) = \lim_{s\to\infty} \underset{\sim}{C}^s = \underset{\sim}{e}(u-\lambda, 1-u)/(1-\lambda). \tag{3.8}$$

Proof. From (3.2) and (3.7),

$$\underset{\sim}{QC}^s = \begin{bmatrix} u_s + \lambda^s(a-1), & 1 - u_s - \lambda^s(a-1) \\ u_s - b\lambda^s, & 1 - u_s + b\lambda^s \end{bmatrix} \tag{3.9}$$

$$= \underset{\sim}{e}(u_s, 1-u_s) + \lambda^s\binom{1-a}{b}(-1, 1)$$

$$= \underset{\sim}{A}_s + \lambda^s\underset{\sim}{B}, \text{ say.}$$

Then

$$\underset{\sim}{A}_s\underset{\sim}{A}_t = \underset{\sim}{A}_t \quad \text{which is idempotent rank one}$$

$$\underset{\sim}{BA}_s = \underset{\sim}{0}$$

$$\underset{\sim}{A}_s\underset{\sim}{B} = (b - u_s(a+b-1))\underset{\sim}{e}(-1, 1) \tag{3.10}$$

$$\underset{\sim}{B}^s = (a+b-1)^{s-1}\underset{\sim}{B} \qquad (s = 1, 2, \ldots). \tag{3.11}$$

When $-1 < a+b-1 < 1$, then

$$\lim_{s\to\infty} \underset{\sim}{B}^s = \underset{\sim}{0}.$$

We claim that

$$\underset{\sim}{P}(t, t+r+1) = \prod_{s=0}^{r} (\underset{\sim}{QC}^s) = \sum_{t=0}^{r+1} \lambda^{\frac{1}{2}t(2r-t+1)}\underset{\sim}{A}_{r-t}\underset{\sim}{B}^t, \tag{3.12}$$

where $\underset{\sim}{A}_{-1} \equiv \underset{\sim}{I}$, $u_0 \equiv 1$.

To prove (3.12) we see that for $r = 1, 2, 3$ we have

$$(\underset{\sim}{A}_0 + \underset{\sim}{B})(\underset{\sim}{A}_1 + \lambda\underset{\sim}{B}) = \underset{\sim}{A}_1 + \lambda\underset{\sim}{A}_0\underset{\sim}{B} + \lambda\underset{\sim}{B}^2 \quad (r = 1)$$

$$(\underset{\sim}{A}_0 + \underset{\sim}{B})(\underset{\sim}{A}_1 + \lambda\underset{\sim}{B})(\underset{\sim}{A}_2 + \lambda^2\underset{\sim}{B}) = \underset{\sim}{A}_2 + \lambda^2\underset{\sim}{A}_1\underset{\sim}{B} + \lambda^{1+2}\underset{\sim}{A}_0\underset{\sim}{B}^2 + \lambda^{1+2}\underset{\sim}{B}^3 \quad (r = 2)$$

$$\begin{aligned}(\underset{\sim}{A}_0 + \underset{\sim}{B})(\underset{\sim}{A}_1 + \lambda\underset{\sim}{B})(\underset{\sim}{A}_2 + \lambda^2\underset{\sim}{B})(\underset{\sim}{A}_3 + \lambda^3\underset{\sim}{B}) = \underset{\sim}{A}_3 &+ \lambda^3\underset{\sim}{A}_2\underset{\sim}{B} + \lambda^{2+3}\underset{\sim}{A}_1\underset{\sim}{B}^2 \\ &+ \lambda^{1+2+3}\underset{\sim}{A}_0\underset{\sim}{B}^3 \\ &+ \lambda^{1+2+3}\underset{\sim}{B}^4 \quad (r = 3).\end{aligned}$$

Since $\sum_{k=0}^{r} k - \sum_{k=0}^{r-t} k = \frac{1}{2}[r(r+1) - (r-t)(r-t+1)] = \frac{1}{2}t(2r-t+1) \qquad (\sum_{k=0}^{-1} k \equiv 0)$,

we can now prove (3.12) by induction by verifying that

$$\sum_{t=0}^{r+1} \lambda^{\frac{1}{2}t(2r-t+1)}\underset{\sim}{A}_{r-t}\underset{\sim}{B}^t(\underset{\sim}{A}_{r+1} + \lambda^{r+1}\underset{\sim}{B})$$

$$= \underset{\sim}{A}_{r+1} + \lambda^{r+1}\underset{\sim}{A}_r\underset{\sim}{B} + \sum_{t=1}^{r+1} \lambda^{\frac{1}{2}t(2r-t+1)+r+1}\underset{\sim}{A}_{r-t}\underset{\sim}{B}^{t+1}$$

$$= \sum_{u=0}^{r+2} \lambda^{\frac{1}{2}u(2r-u+3)}\underset{\sim}{A}_{r-u+1}\underset{\sim}{B}^u; \; u = t+1.$$

We investigate the limit of (3.12) by rewriting it as

$$\underset{\sim}{A}_r + \sum_{t=1}^{r+1} \lambda^{\frac{1}{2}t(2r-t+1)}\underset{\sim}{A}_{r-t}\underset{\sim}{B}^t. \tag{3.13}$$

From (3.9) and Theorem 3.1 we have that $\underset{\sim}{A}_r \to e(u-\lambda, 1-u)/(1-\lambda)$ as $r \to \infty$. To establish the full result we need only prove that the second term in (3.13) converges to zero. We may write it as

$$\sum_{t=1}^{r+1} \lambda^{\frac{1}{2}t(2r-t+1)}\underset{\sim}{A}_{r-t}\underset{\sim}{B}\,(a+b-1)^{t-1} \tag{3.14}$$

using (3.11). Whence using (3.10) we may write it as

$$(3.15)\qquad [\sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t-1}(b - u_{r-t}(a+b-1))]\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}$$
$$+ \lambda^{\frac{1}{2}r(r+1)}(a+b-1)^{r}\underset{\sim}{B}.$$

The last term clearly converges to zero since $-1 < \lambda < 1$ and $-1 < a+b-1 < 1$. To establish the full result we need only now prove that the expression in square brackets converges to zero. We have

$$\left|\sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t-1}(b-u_{r-t}(a+b-1))\right| \le \sum_{t=1}^{r} |\lambda|^{\frac{1}{2}t(2r-t+1)}|a+b-1|^{t-1}b$$
$$+ \sum_{t=1}^{r} |\lambda|^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t}\,\frac{(|u-\lambda|+|\lambda|^{r-t}|1-u|)}{1-\lambda}$$
$$< |\lambda|^{r}b\sum_{t=1}^{r}|a+b-1|^{t-1} + \frac{(|\lambda|^{r}|u-\lambda|+|\lambda|^{2r-1}|1-u|)}{1-\lambda}\sum_{t=1}^{r}|a+b-1|^{t}$$
$$= \frac{|\lambda|^{r}b(1-|a+b-1|^{r})}{1-|a+b-1|} + \frac{(|\lambda|^{r}|u-\lambda|+|\lambda|^{2r-1}|1-u|)[|a+b-1|-|a+b-1|^{r+1}]}{(1-\lambda)(1-|a+b-1|)}$$
$$\to 0 \quad \text{as} \quad r \to \infty. \qquad \text{(qed)}$$

COROLLARY 3.4. <u>When</u> $\lambda = 1$ <u>and</u> $u = 1$, <u>then</u> $\underset{\sim}{C} = \underset{\sim}{I}$ <u>and when</u> $-1 < a+b-1 < 1$, <u>then</u>

$$\lim_{r\to\infty} \underset{\sim}{P}(t,\ t+r+1) = \prod_{s=0}^{\infty} \underset{\sim}{Q} = \lim_{r\to\infty} \underset{\sim}{Q}^{r+1} = \underset{\sim}{e}(1-b,\ 1-a)/(2-a-b)\ .$$

COROLLARY 3.5. <u>When</u> $\lambda = 1$ <u>and</u> $u \neq 1$ $[-1 < a+b-1 < 1]$ <u>then</u> $\underset{\sim}{P}(t,\ t+r+1)$ <u>diverges</u>.

<u>Proof</u>. In this case we can write (3.14) as

$$(3.16)\qquad \underset{\sim}{A}_r + \sum_{t=1}^{r+1} \underset{\sim}{A}_{r-t}\underset{\sim}{B}_t = \underset{\sim}{A}_r + \sum_{t=1}^{r+1} \underset{\sim}{A}_{r-t}(a+b-1)^{t-1}\underset{\sim}{B}$$
$$= \underset{\sim}{A}_r + (a+b-1)^{r}\underset{\sim}{B} + \sum_{t=1}^{r}(a+b-1)^{t-1}(b-(a+b-1)((u-1)(r-t)+1))\underset{\sim}{e}(-1,1).$$

The (1, 1) element is from (3.9):

$$r(u-1)+1+(a+b-1)^r(a-1) + (a-1)\sum_{t=1}^{r}(a+b-1)^{t-1} + (u-1)\sum_{t=1}^{r}(a+b-1)^t(r-t)$$

$$= r(u-1)\sum_{t=0}^{r}(a+b-1)^t + (a-1)\sum_{t=0}^{r}(a+b-1)^t - (u-1)\sum_{t=1}^{r}t(a+b-1)^t + 1$$

$$= \frac{[r(u-1)+(a-1)](1-(a+b-1)^{r+1})(2-a-b)+(2-a-b)^2-(u-1)(a+b-1+r(a+b-1)^{r+2}-(r+1)(a+b-1)^{r+1})}{(2-a-b)^2}$$

$$= (2-a-b)^{-2}\{(2-a-b)(1-b)-(a-1)(a+b-1)^{r+1}(2-a-b)$$

$$+ (u-1)[r(2-a-b)-r(a+b-1)^{r+1}(2-a-b)-a-b+1-r(a+b-1)^{r+2}$$

$$+ (r+1)(a+b-1)^{r+1}]\}$$

$$= \frac{(1-b)+r(u-1)}{2-a-b} + \frac{(a+b-1)^{r+1}((1-a)(2-a-b)+(u-1))+(1-a-b)(u-1)}{(2-a-b)^2} \to \infty.$$

(qed)

COROLLARY 3.6. <u>When</u> $a + b = 2$ $(a+b-1 = 1)$ <u>then</u> $\underset{\sim}{Q} = \underset{\sim}{I}$ <u>and when</u> $-1 < \lambda < 1$, <u>the limit</u>

$$\lim_{r\to\infty} \underset{\sim}{P}(t,\ t+r+1) = \prod_{s=0}^{\infty} \underset{\sim}{C}^s = \lim_{r\to\infty} \underset{\sim}{C}^{r(r+1)/2} = \underset{\sim}{e}(u-\lambda,\ 1-u)/(1-\lambda).$$

COROLLARY 3.7. <u>When</u> $a + b = 0$ $(a+b-1 = -1)$ <u>then</u> $a = b = 0$ <u>and when</u> $-1 < \lambda < 1$, <u>the limit</u>

$$\lim_{r\to\infty} \underset{\sim}{P}(t,\ t+r+1) = \prod_{s=0}^{\infty} \underset{\sim}{QC}^s = \lim_{s\to\infty} \underset{\sim}{C}^s = \underset{\sim}{e}(u-\lambda,\ 1-u)/(1-\lambda).$$

<u>Proof</u>. Theorem 3.4 holds through (3.13) which we can write as

$$(3.17) \qquad \underset{\sim}{A}_r + \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)} \underset{\sim}{A}_{r-t} \underset{\sim}{B}^t + \lambda^{\frac{1}{2}r(r+1)} \underset{\sim}{B}^{r+1}.$$

The first term of (3.17), is by (3.9) equal to $\underset{\sim}{e}(u_r, 1-u_r)$ and the $(1, 1)$ element of this converges to $(u-\lambda)/(1-\lambda)$ as required. We now prove that the other terms in (3.17) converge to zero and the corollary is established. The third term of (3.17), using (3.11), is

$$(3.18) \qquad \lambda^{\frac{1}{2}r(r+1)}(-1)^r\begin{pmatrix}-1, & 1\\ 0, & 0\end{pmatrix}$$

and the $(1, 1)$ element thereof clearly converges to zero. The second term of (3.17), using (3.11) and (3.10), is

$$(3.19) \qquad \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}\underset{\sim}{A}_{r-t}\underset{\sim}{B}^t = \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}(-1)^{t-1}\underset{\sim}{A}_{r-t}\underset{\sim}{B}$$

$$= \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}(-1)^{t-1}u_{r-t}\underset{\sim}{e}(-1, 1).$$

The $(1, 1)$ element is

$$(3.20) \qquad \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}(-1)^t(u-\lambda+\lambda^{r-t}(1-u))/(1-\lambda)$$

$$= \frac{u-\lambda}{1-\lambda}\sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}(-1)^t + \frac{1-u}{1-\lambda}\sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)+r-t}(-1)^t.$$

Let $s_r = \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)}(-1)^t$. Then $s_1 = -\lambda$, $s_2 = -\lambda^2(1-\lambda) = -\lambda^2(1+s_1)$ and

$$s_r = \lambda^r \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)-r}(-1)^t = \lambda^r(-1+\sum_{t=2}^{r} \lambda^{\frac{1}{2}t(2r-t+1)-r}(-1)^t)$$

$$= \lambda^r(-1) + \sum_{u=1}^{r-1} \lambda^{\frac{1}{2}u(2r-u-1)}(-1)^{u+1}$$

since $u = t-1$. Thus

$$(3.21) \qquad s_r = -\lambda^r(1 + s_{r-1}),\ r = 2, 3, \ldots.$$

Now $|s_1| = \lambda$, $|s_2| = |\lambda|^2|1+s_1| \leq |\lambda|^2 + |\lambda|^3 \leq 2|\lambda|^2$, and

$$(3.22) \qquad |s_r| \leq |\lambda|^r + |\lambda|^r|s_{r-1}|$$

so that by induction

$$(3.23) \qquad |s_r| \leq r|\lambda|^r.$$

Consider the infinite series $\sum_{r=1}^{\infty} r|\lambda|^r$. Then by d'Alembert's test this series is convergent, since

$$\lim \frac{(r+1)|\lambda|^{r+1}}{r|\lambda|} = \lim [\frac{r+1}{r}]|\lambda| = |\lambda| < 1 .$$

[cf. Example p. 44 of Hyslop (1954)]. Hence $r|\lambda|^r \to 0$ and so $|s_r| \to 0$ [cf. Hyslop (1954) p. 30 Theorem 8]. Thus the first term of (3.20) converges to zero. The second term is $(1-u)/(1-\lambda)$ times

$$(3.24) \qquad \sum_{t=1}^{r} \lambda^{\frac{1}{2}t(2r-t+1)+r-t}(-1)^t = -\sum_{u=2}^{r+1} \lambda^{\frac{1}{2}u(2r-u+1)}(-1)^u, \; u = t+1$$

and so also converges to zero. This completes the proof. (qed)

CORPOLLARY 3.8. <u>When</u> $a + b = 2$ $(a+b-1 = 1, a = 1, b = 1)$ <u>then</u> $\underset{\sim}{Q} = \underset{\sim}{I}$, <u>and the limit</u>

$$(3.25) \qquad \prod_{s=0}^{\infty} \underset{\sim}{QC}^s = \lim_{r\to\infty} \underset{\sim}{C}^{r(r+1)/2}.$$

<u>When</u> $\lambda = 1$, $\underset{\sim}{C} = \underset{\sim}{I}$ <u>and</u> <u>(3.25)</u> $\equiv \underset{\sim}{I}$. <u>When</u> $\lambda = -1$; $c, d = 0$; $\underset{\sim}{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ <u>and</u> $\prod_{s=0}^{r} \underset{\sim}{QC}^s = \underset{\sim}{P}(t, t+r+1)$ <u>is Euler- and Cesaro-summable to</u> $\frac{1}{2}\underset{\sim}{ee}'$.

COROLLARY 3.9. <u>When</u> $a + b = 0$ $(a+b-1 = -1;\ a, b = 0)$ <u>then</u> $\underset{\sim}{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $u = 1 - d$, $\lambda = 1 - c - d$. <u>When</u> $\lambda = 1$, $\underset{\sim}{C} = \underset{\sim}{I}$, <u>the limit</u>

$$\lim_{r\to\infty} \prod_{s=0}^{r} \underset{\sim}{Q}\underset{\sim}{C}^{s} = \lim_{r\to\infty} \underset{\sim}{Q}^{r(r+1)/2},$$

and $\prod_{s=0}^{r} \underset{\sim}{Q}\underset{\sim}{C}^{s}$ is Euler- and Cesaro-summable to $\frac{1}{2}\underset{\sim}{e}\underset{\sim}{e}'$. When $\lambda = -1$, $\underset{\sim}{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underset{\sim}{Q}$, $\prod_{s=0}^{r} \underset{\sim}{Q}\underset{\sim}{C}^{s} = \prod_{s=0}^{r} \underset{\sim}{C}^{s+1}$ and $\prod_{s=0}^{r} \underset{\sim}{Q}\underset{\sim}{C}^{s}$ is Euler- and Cesaro-summable to $\frac{1}{2}\underset{\sim}{e}\underset{\sim}{e}'$.

COROLLARY 3.10. When $-1 < a+b-1 < 1$ and $\lambda = -1$, then $\prod_{s=0}^{r} \underset{\sim}{Q}\underset{\sim}{C}^{s}$ is Cesaro-summable to $\underset{\sim}{e}(w, 1-w)$, where

$$w = \frac{\frac{1}{2}(c+d)(d-b) + (1-d)}{1 + (c+d-1)^2}.$$

Proof.

$$\prod_{s=0}^{r} \underset{\sim}{Q}\underset{\sim}{C}^{s} = \underset{\sim}{A}_r + \sum_{t=1}^{r+1} (-1)^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t-1}\underset{\sim}{A}_{r-t}\underset{\sim}{B} \tag{3.26}$$

$$= \underset{\sim}{e}(u_r, 1-u_r) + \sum_{t=1}^{r} (-1)^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t-1}\underset{\sim}{A}_{r-t}\underset{\sim}{B}$$

$$+ (-1)^{\frac{1}{2}r(r+1)}(a+b-1)^{r}\underset{\sim}{B}$$

where

$$u_r = \tfrac{1}{2}(1+u+(-1)^r(1-u)),$$

$$\underset{\sim}{B} = \begin{pmatrix} 1-a \\ b \end{pmatrix}(-1, 1)$$

$$\underset{\sim}{A}_{r-t}B = (b-u_{r-t}(a+b-1))\underset{\sim}{e}(-1, 1).$$

The last term of (3.26) converges to 0 as $r \to \infty$. The (1, 1) element of the remaining terms in (3.26) is

$$u_r - \sum_{t=1}^{r} (-1)^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t-1}(b - u_{r-t}(a+b-1)) \tag{3.27}$$

$$= \tfrac{1}{2}(u+1) + \tfrac{1}{2}(-1)^r(1-u) - b \sum_{t=1}^{r} (-1)^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t-1}$$

$$+ \tfrac{1}{2}(u+1) \sum_{t=1}^{r} (-1)^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t}$$

$$+ \tfrac{1}{2}(1-u) \sum_{t=1}^{r} (-1)^{\frac{1}{2}t(2r-t+1)+r-t}(a+b-1)^{t}$$

$$= \tfrac{1}{2}(u+1) + (\tfrac{1}{2}(u+1)(a+b) - (b+u)) \sum_{t=1}^{r} (-1)^{\frac{1}{2}t(2r-t+1)}(a+b-1)^{t-1} \tag{3.28}$$

$$+ \tfrac{1}{2}(1-u)(-1)^{\frac{1}{2}r(r+1)}(a+b-1)^r.$$

The last term converges to 0 as $r \to \infty$. It remains to consider the second term in (3.28). Let

$$s_r = \sum_{t=1}^{r} (-1)^{\frac{1}{2}t(2r-t+1)}\mu^{t-1}; \quad \mu = a+b-1. \tag{3.29}$$

Then

$$s_r = (-1)^r - \mu - \mu^2 s_{r-2}; \quad r \geq 3, \tag{3.30}$$

since

$$s_r = (-1)^r + (-1)^{r+(r-1)}\mu + (-1)^{r+(r-1)+(r-2)}\mu^2 + \ldots + (-1)^{r+\ldots+1}\mu^{r-1}$$

$$= (-1)^r + \mu(-1)^{r+(r-1)}(1+\mu[(-1)^{r-2}+(-1)^{(r-2)+(r-3)}\mu$$

$$+ \ldots + (-1)^{(r-2)+\ldots+1}\mu^{r-3}])$$

$$= (-1)^r - \mu - \mu^2 s_{r-2}.$$

Let $t_r = s_{2r}$ and $v_r = s_{2r-1}$, $r = 1, 2, \ldots$. Then t_r and v_r are convergent sequences with limits t and v, say. From (3.30) we see that $t = (1-\mu)/(1+\mu^2)$ and $v = -(1+\mu)/(1+\mu^2)$. Hence s_r is Cesaro-summable to $(u+v)/2 = -\mu/(1+\mu^2)$. Thus (3.28) is Cesaro-summable to

$$\tag{3.31} \tfrac{1}{2}(u+1) - \frac{\mu}{(1+\mu^2)} [\tfrac{1}{2}(u+1)(a+b) - (b+u)].$$

With $\lambda = -1$, $a + b - 1 = 1 - c - d$. Hence

$$\tag{3.32} \begin{aligned} b + u &= b + \frac{bc - (1-a)(1-d)}{a + b - 1} \\ &= \frac{b(1-c-d) + bc - (1-a)(1-d)}{a + b - 1} \\ &= 1 - d. \end{aligned}$$

Also $u + 1 = 2 - d - b = a + c$. Substituting in (3.31), we get

$$\tag{3.33} \tfrac{1}{2}(a+b) - \frac{\mu}{1+\mu^2} [\tfrac{1}{2}(a+c)(a+b) - (1-d)].$$

Simplifying (3.33), we obtain

$$\tag{3.34} \frac{1}{1+\mu^2} [\tfrac{1}{2}(a+c)(1-\mu) + \mu(1-d)]$$

which reduces to

$$\tag{3.35} \frac{\tfrac{1}{2}(c+d)(d-b) + (1-d)}{1 + (c+d-1)^2}. \qquad \text{(qed)}$$

We may summarize the above results on the limiting behavior of $\underset{\sim}{C}^s$ and $\underset{\sim}{T}_r = \prod_{s=0}^{r} (\underset{\sim}{Q}\underset{\sim}{C}^s)$ in the following two tables:

TABLE 3.1. Limiting Behavior of $\underset{\sim}{C}^s$

λ	$\lim u_s = (\underset{\sim}{C}^s)_{11}$	$\lim \underset{\sim}{C}^s$
(-1, +1)	$(u-\lambda)/(1-\lambda)$ [Theorem 3.1]	$\underset{\sim}{e}(u-\lambda,\ 1-u)/(1-\lambda)$ [Corollary 3.1]
+1 (u = 1)	1 [Theorem 3.1]	$\underset{\sim}{C} \equiv \underset{\sim}{I}$ [Corollary 3.3]
+1 ($u \neq 1$)	none [Theorem 3.3]	none
-1	* $\frac{1}{2}(u+1)$ [Theorem 3.2]	*$\frac{1}{2}\underset{\sim}{e}(u+1,\ 1-u)$ [Corollary 3.2]

*Cesaro/Euler sum.

TABLE 3.2. Limiting Behavior of $\underset{\sim}{T}_r = \prod_{s=0}^{r}(\underset{\sim}{Q}\underset{\sim}{C}^s)$

λ	$-1 < a+b-1 < 1$	a, b = 0	a, b = 1
(-1, +1)	$\underset{\sim}{e}(u-\lambda,\ 1-u)/(1-\lambda)$ [Theorem 3.4]	as at left [Corollary 3.7]	as at left [Corollary 3.6]
+1	$\underset{\sim}{e}(1-b,\ 1-a)/(2-a-b)$; u = 1 [Corollary 3.4] divergent; $u \neq 1$ [Corollary 3.5]	*$\frac{1}{2}\underset{\sim}{e}\underset{\sim}{e}'$ [Corollary 3.9]	$\underset{\sim}{T}_r \equiv \underset{\sim}{I}$ [Corollary 3.8]
-1	*$\underset{\sim}{e}(w,\ 1-w)$ [Corollary 3.10]	*$\frac{1}{2}\underset{\sim}{e}\underset{\sim}{e}'$ [Corollary 3.9]	*$\frac{1}{2}\underset{\sim}{e}\underset{\sim}{e}'$ [Corollary 3.8]

*Cesaro/Euler sum.

The notion of a chain with constant causative matrix is appropriate only when $\underset{\sim}{T}_r = \prod_{s=0}^{r} (\underset{\sim}{QC}^s)$ is stochastic. When $\underset{\sim}{C}$ is itself stochastic it follows directly that $\underset{\sim}{T}_r$ is also stochastic. We find, however, that $\underset{\sim}{T}_r$ may be stochastic without $\underset{\sim}{C}$ having all elements nonnegative. We study the situation for the case of two states.

The product matrix $\underset{\sim}{T}_r = \prod_{s=0}^{r} (\underset{\sim}{QC}^s)$ has a limit in the two-state case if and only if $-1 < \lambda < 1$ (Theorem 3.4), or $\lambda = 1$ <u>and</u> $u = 1$. The latter case will be disregarded in what follows since then $\underset{\sim}{C} \equiv \underset{\sim}{I}$ and the chain is stationary. We enquire first for conditions that $\underset{\sim}{T}_r$ has a stochastic limit. From Theorem 3.4 the limit is

$$\underset{\sim}{e}(u-\lambda,\ 1-u)/(1-\lambda) \tag{3.36}$$

where $-1 < \lambda < 1$. Thus (3.36) is stochastic if and only if $0 \le (u-\lambda)/(1-\lambda) \le 1$, or

$$\lambda \le u \le 1. \tag{3.37}$$

We now find the condition that $\underset{\sim}{C}$ is stochastic. From (3.1) we require $0 \le u \le 1$ and $0 \le u-\lambda \le 1$. That is

$$\max\ (0,\ \lambda) \le u \le \min\ (1,\ 1+\lambda). \tag{3.38}$$

When $0 \le \lambda < 1$, (3.38) is the same as (3.37) so that $\prod_{s=0}^{\infty} (\underset{\sim}{QC}^s)$ is stochastic if and only if $\underset{\sim}{C}$ is stochastic $(0 \le \lambda < 1)$.

When $-1 < \lambda \le 0$, the situation differs. From (3.38) we obtain

$$0 \le u \le 1+\lambda \tag{3.39}$$

and (3.37) may hold without (3.39) being satisfied. That is, $\prod_{s=0}^{\infty} (\underset{\sim}{QC}^s)$ is stochastic but $\underset{\sim}{C}$ is not, provided $-1 < \lambda \le 0$ and

$$-1 < \lambda \le u < 0 \quad \text{or} \quad 0 < 1+\lambda < u \le 1. \tag{3.40}$$

We now show that (3.36) is stochastic whenever $-1 < \lambda \le 0$. We need the following:

LEMMA 3.1. Whenever $\lambda < 0$, the inequality

$$(3.41) \qquad \lambda \le u \le 1,$$

holds, where λ is the non-unit characteristic root of $\underset{\sim}{C}$ and u its leading element.

Proof. From (3.2) and (3.3), we have that

$$(3.42) \quad u = \frac{bc - (1-a)(1-d)}{a+b-1}$$

$$= \frac{bc - (1-a)(1-d) + c(a-1) + c(1-a)}{a + b - 1}$$

$$= c + (1-a)\lambda.$$

Similarly we may write

$$(3.43) \qquad u = b\lambda + (1-d).$$

Thus $u = \lambda + c - a\lambda \ge \lambda$, and $u = 1 - d + b\lambda \le 1$, when $\lambda < 0$, from (3.43). Hence the result. (qed)

Since (3.41) and (3.37) are the same, we find that for $-1 < \lambda \le 0$, the limit (3.36) is always stochastic. In addition we find the following more powerful result:

LEMMA 3.2. Whenever $-1 < \lambda \le 0$, the matrix $\underset{\sim}{QC}^s$ is stochastic for all $s = 0, 1, 2, \ldots$, where λ is the non-unit characteristic root of $\underset{\sim}{C}$.

Proof. Using (3.9) it suffices to prove that

$$(3.44) \qquad 0 \le u_s + \lambda^s(a-1), \quad u_s - b\lambda^s \le 1.$$

Since λ is symmetric in a and b it suffices to show the first of the inequalities in (3.44). From (3.7) and (3.42) we may write this as

$$0 \le c - \lambda a + \lambda^s(a-c) \le 1 - \lambda . \tag{3.45}$$

The middle quantity may be written $c(1-\lambda^s) - a\lambda(1-\lambda^{s-1})$ which is component-wise nonnegative. To see the right-hand side of (3.45) we note that $c(1-\lambda^s) - a\lambda(1-\lambda^{s-1}) \le 1 - \lambda^s - a\lambda(1-\lambda^{s-1}) = 1 - \lambda + \lambda(1-a)(1-\lambda^{s-1}) \le 1 - \lambda$. (qed)

We summarize the above results as:

THEOREM 3.5. *Let* λ *be the non-unit characteristic root of the causative matrix* $\underset{\sim}{C}$. *Whenever* $0 \le \lambda < 1$, $\prod_{s=0}^{\infty} (\underset{\sim}{QC}^s)$ *is stochastic if and only if* $\underset{\sim}{C}$ *is. Whenever* $-1 < \lambda \le 0$, $\underset{\sim}{QC}^s$ *is stochastic for all* $s = 0, 1, \ldots,$ *and so are* $\prod_{s=0}^{r} (\underset{\sim}{QC}^s)$ *and* $\prod_{s=0}^{\infty} (\underset{\sim}{QC}^s)$, *but* $\underset{\sim}{C}$ *is then only stochastic provided*

$$0 \le u \le 1 + \lambda, \tag{3.46}$$

where u *is the leading element of* $\underset{\sim}{C}$.

From the above we note that when $0 \le \lambda < 1$, $\prod_{s=0}^{\infty} (\underset{\sim}{QC}^s)$ is not stochastic whenever $\underset{\sim}{C}$ is not stochastic. This occurs if and only if

$$0 < \lambda < 1 < u \quad \text{or} \quad 0 < u < \lambda < 1. \tag{3.47}$$

In these cases we find that $\underset{\sim}{QC}^s$ tends monotonically to a matrix which is not stochastic and which is also the limit of $\underset{\sim}{T}_r$. We can, therefore, find the largest value of s such that $\underset{\sim}{QC}^s$ is a stochastic matrix.

THEOREM 3.6. *Let* λ *be the non-unit characteristic root of the causative matrix* $\underset{\sim}{C}$, *and* u *its leading element. Then* $\underset{\sim}{QC}^s$ *tends monotonically to a limit matrix which is not stochastic, and* $\underset{\sim}{QC}^s$ *is a stochastic matrix, provided*

$$\lambda^s(c-a) \ge b\lambda - d; \quad 0 < \lambda < 1 < u, \tag{3.48}$$

$$(3.49) \qquad \lambda^s(d-b) \geq a\lambda - c; \qquad 0 < u < \lambda < 1,$$

when $a + b - 1 > 0$, where $\underset{\sim}{Q}$ has diagonal elements a and b and $\underset{\sim}{QC}$ has diagonal elements c and d, and provided

$$(3.50) \qquad \lambda^s(b-d) \geq b\lambda - d; \qquad 0 < \lambda < 1 < u,$$

$$(3.51) \qquad \lambda^s(a-c) \geq a\lambda - c; \qquad 0 < u < \lambda < 1,$$

when $a + b - 1 < 0$.

Proof. When $0 < \lambda < 1 < u$, we have from (3.42) that $1 < u = c + (1-a)\lambda < c + 1 - a$. Hence $c > a$, and similarly from (3.43) we find $b > d$. The (1,1) and (2,1) elements of $\underset{\sim}{QC}^s$ from (3.9) and (3.7), are respectively

$$(3.52) \qquad \frac{u - \lambda - \lambda^s(c-a)}{1 - \lambda}; \qquad \frac{u - \lambda - \lambda^s(b-d)}{1 - \lambda},$$

which both increase monotonically to $[(u-\lambda)/(1-\lambda)] > 1$ when $u > 1$. When $a + b - 1 > 0$, $\lambda < 1$ implies $c + d < a + b$ so that $c - a < b - d$. In this case the first quantity in (3.52) is the larger and does not exceed 1 provided (3.48) holds. When $a + b - 1 < 0$, the second quantity in (3.52) is the larger and does not exceed 1 provided (3.50) holds. The proof of (3.49) and (3.51) follows similarly. (qed)

To compute values of c and d such that (3.48) through (3.51) hold for fixed a, b and $s = 2, 3, \ldots$, we transform the inequalities to the (λ, u)-plane. For (3.48) we obtain

$$(3.53) \qquad u \leq 1 + \frac{\lambda^s(1-a)(1-\lambda)}{1 - \lambda^s},$$

when $a + b - 1 > 0$ and $0 < \lambda < 1 < u$. When $s = 1$ (3.53) reduces to $u \leq 1 + \lambda(1-a)$ or $c \leq 1$ using (3.42). Similarly for (3.49) we obtain

$$(3.54) \qquad u \geq \lambda - \frac{\lambda^s(1-b)(1-\lambda)}{1 - \lambda^s},$$

when $a + b - 1 > 0$ and $0 < u < \lambda < 1$. When $s = 1$ (3.54) reduces to $u \geq b\lambda$ or $d \leq 1$ using (3.43).

To find the values of c and d for fixed a,b and $s = 2, 3, \ldots$, we increment λ from 0 to 1 and find the appropriate bound for u. We then solve for c and d using

$$c = u - (1-a)\lambda, \tag{3.55}$$

$$d = 1 - u + b\lambda, \tag{3.56}$$

which follow directly from (3.42) and (3.43) respectively.

We illustrate the above relationships for the particular case of $a = .6$ and $b = .9$. In Figure 3.1 we show the unit square in the (c,d) plane truncated by the diagonal lines denoting $\lambda = -1$ $(c+d = \frac{1}{2})$ and $\lambda = +1$ $(c+d = 1\frac{1}{2})$. The lines $u = 0$ $(4d + 9c = 4)$, $u = 1$ $(4d + 9c = 9)$, $u = 1+\lambda$ $(6d+c = 1)$, and $u = \lambda$ $(6d+c = 6)$ form a parallelogram 1 within the above region, with corners (1,0), (.4,.1), (.6,.9), and (0,1). Within this parallelogram $\underset{\sim}{C}$ is stochastic and so is $\underset{\sim}{QC}^s$ for all $s = 1, 2 \ldots$. The region below and to the left of the parallelogram but above and to the right of $\lambda = -1$ forms two triangles 2a and 2b in which $\underset{\sim}{C}$ is not stochastic but $\underset{\sim}{QC}^s$ is stochastic for all $s = 1, 2, \ldots$. The reflection of these two triangles about $c+d = 1$ gives the remaining regions 3a and 3b where $\underset{\sim}{QC}^s$ converges; the convergence, however, while monotonic, is to a nonstochastic limit and so in triangles 3a and 3b is stochastic only for s such that (3.48) and (3.49) are satisfied. The triangle 3a is enlarged in Figure 3.2, where the bounds such that $\underset{\sim}{QC}^s$ is stochastic are shown for $s = 2, 3, 4, 10$. In the region to the right of the curve labelled $s = s_0$, but within the triangle, $\underset{\sim}{QC}^s$ is stochastic only for $s = 1, 2, \ldots, s_0-1$. We note that $\underset{\sim}{QC}^2$ is not stochastic for just over half the region given by the triangle, while $\underset{\sim}{QC}^{10}$ is not stochastic for almost all the region.

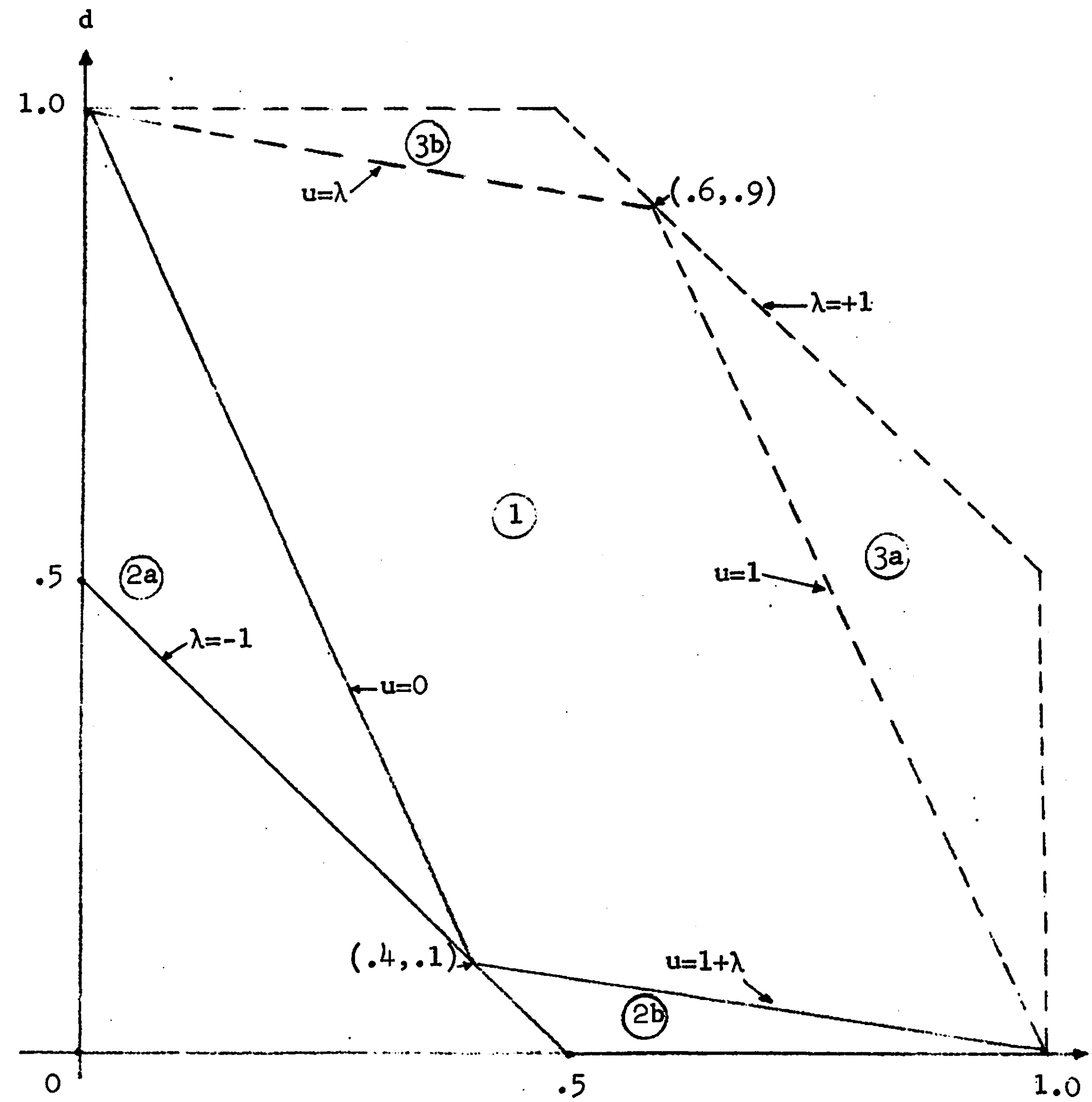
The curves in Figure 3.2 were found by computing (3.53), (3.55) and (3.56). The limit point at $\lambda = 1$ was found by substituting $1 + \lambda + \ldots + \lambda^{s-1} = (1-\lambda^s)/(1-\lambda)$ into (3.53) and then setting $\lambda = 1$ to yield

$$(3.57) \qquad u \leq 1 + (1-a)/s.$$

The corresponding values of c and d follow from (3.55) and (3.56) as

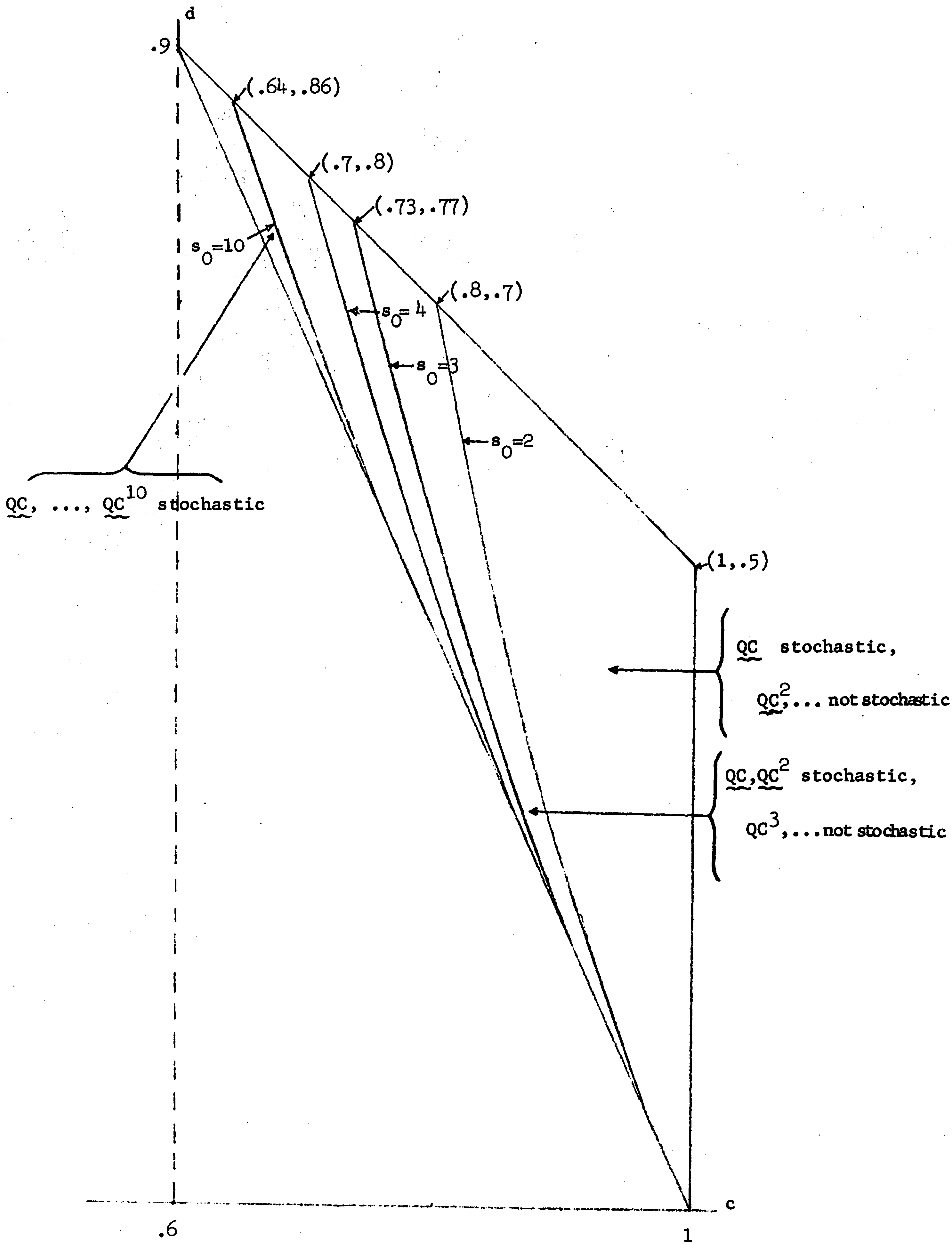
$$(3.58) \qquad c = a + (1-a)/s; \qquad d = b - (1-a)/s.$$

Figure 3.1. Regions in the (c,d) plane for stochastic $\underset{\sim}{QC}^s$.



Region	$\underset{\sim}{C}$	$\underset{\sim}{QC}^s$ stochastic
1	stochastic	all s
2a, 2b	not stochastic	all s
3a, 3b	not stochastic	s bounded by (3.48/49)

Figure 3.2. A region in (c,d) plane where $\underset{\sim}{QC}^s$ is stochastic only for $s < s_0$.



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