## ON THE ORDERING OF THE t = 2 BEST OF n ITEMS

USING BINARY COMPARISONS

bу

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Technical Report No. 113

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#### 1. Introduction

The problem considered is that of ranking the t best (i.e., largest) of n unequal numbers (or objects with respect to an associated scalar such as weight) when only binary errorless comparisons are allowed. In some applications these n numbers are unknown but in others, e.g., the "sorting problem", the numbers are actually known. Here a machine (or a person) starts with a sequence of n numbers in random order and uses only binary comparisons to put them all in (say) ascending order. In the application to aligning n tennis players according to ability, we call this a "tournament problem". We assume that the players have unequalability (or skill), that the better player always wins, and that the relation "better than" is transitive. If we have n unequal weights and a simple balance that only allows one weight on each pan, then this problem (of ordering the n weights) is called a "weighing problem". From the point of view of questionnaire theory (which emphasizes the graph-theoretic and information-theoretic nature of the problem), this is called the problem of 'tri'. These are clearly all the same problem, corresponding to t = n (or equivalently t = n - 1) in our formulation, and we prefer to call it the "Steinhaus expectation problem" for t = n - 1 because of the early interest Steinhaus showed in a related minimax problem (see below).

It is assumed that the n numbers are initially in random order, i.e., either their order has been randomized or we are willing to assume this. To explain our goal consider the number T of binary comparisons (or tests) required for n = 3. Already T is not constant (T = 2 or 3) and, from the initial random order, we obtain the expectation  $E\{T \mid n = 3\} = 8/3$  for

#### TREVOURSE OR

miniman problem (see below). For the many Languages of the early interest Steinmans, showed in a noloted cun Tournillation, and we presell to golf it the Breinberts or sectation problem? the same problem, corresponding to c = n (or equivalently c = n - 1) in of the problem), this he ealded the groblem of 'trif'. These ere clearly all Tiedouy. (which cromingsizes time in spiretic or exic information at one the nectic actual an called a liveigning problem. From the potat of view of questionneine allows one weight on each pant then this problet (of budering the n weights) εκαυριμένου, επίτες με προκές της προκική νουθές και επίτερες οι επίτερες σελέπες συμφ the become player always wind, sand that the believing "become indicated as problem". We seeme that she players have uniquelaislaty (or skill), that aligning at gennis pleyers according to bidliky, we acil this a "tournement companients to put them all in (sey) asceduting of m. In the application to stants with a sequence of in Eurosers in rendom ovder and vess only binary (normal elia), augigem e anom elamilisticom la completa anomem en elamina en elamina en elamina elamina elamina gypt took toka teacae in gundera at estimumm one the cases, a see, call the sactions  $h_{\mathbb{R}^2}$  notified, when conth existly consists some constitutions are satisfied. The sourof the emedical maniport (or options that it responds to an employed respect their Are problem countained in this of realistic the feet (1..., despect))

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the optimal procedure. Our main goal is to find a procedure R which minimizes this expectation. Several new procedures are introduced in this paper, all with expectations below that of the Steinhaus procedure defined below. Some of these have values smaller than any procedure known to the author and some are conjectured to be optimal.

Another goal of this paper is to find a procedure R which minimizes the maximum number of the test required to guarantee that we can order the t best of n numbers; we refer to this as the "Steinhaus minimax problem".

The expectation and minimax goals are not unrelated and for small values of n we can find procedures both E-optimal (i.e., with smallest expectation) and M-optimal (i.e., with smallest maximum).

Steinhaus [23] gives a basic fully-inductive procedure  $R_c$  for the minimax goal. In the 1950 edition of this book he conjectures that this procedure is optimal for all n but this is deleted in a later edition and in another book [24] on problems a counterexample is explicitly worked out for n = 5. Although the procedure  $R_c$  is at the "bottom" of our list of procedures for t = n - 1 (it has the largest expectation and the largest maximum length among all the procedures in the table section 5), it represents an important standard for comparison partly because it is both E-optimal and M-optimal among the fully inductive procedures [10] and partly because more is known about its properties. Kislicyn found general bounds for the expectation under  $R_{_{\mathbf{S}}}$  in [14] and derived an asymptotic expression for the same expectation in [15]. Although this procedure  $R_s$  is widely known in Computer Science (it is called Binary Insertion or TID or Ranking by Insertion or Binary Search by different authors), it is remarkable how many writers in this field assume either explicitly as on page 236 of [13] or implicitly that  $R_s$  is either E-optimal or M-optimal (or both) and are not familiar with other work in this area.

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Another important procedure for both the M-goal and the E-goal is the semi-inductive procedure  $R_F$  of Ford and Johnson [8], although the paper is only concerned with the minimax problem. In fact, the procedure  $R_F$  is E-optimal for  $n \le 5$  and the expected values for moderate n (calculated by A. Hadian and the author) were found to be smaller than any others found in print at the start of this investigation. Cesari [4] and Hadian [10] have modified the procedure  $R_F$  for  $n \ge 6$  to obtain a smaller expectation without changing the M-value.

Picard [17] has given a procedure for n=6 (and t=5) which is both E-optimal and M-optimal. His approach through questionnaire theory combines a graph-theoretic and an information-theoretic analysis, which he applies to many interesting search problems.

For the sake of completeness we should also mention the related papers of Bose and Nelson [1] and Hibbard [11] (see also the references in the latter) but, because they apply restrictions on the number of locations in a computer that can be used or because their criterion is slightly different from our T or because their results are not in contention with ours, we omit their procedures in our comparisons. Also our problem is related to that of merging ordered strings of numbers into a single string, if the criterion is simply the number T of binary comparisons required and not the total number of key-transfers as in Burge [2]. In the latter paper it was empirically observed that our procedures were equally good under his (key-transfer) criterion but that his procedure was inferior under our T-criterion.

The main emphasis in this paper is on the use of 2 ideas for a testing procedure, namely pairing and expected uncertainty. Our entropy procedure  $R_{\underline{E}}$  selects at each stage the comparison that maximizes the expected reduction in entropy due to a single comparison. Equivalently it chooses the comparison that results in the smallest amount of uncertainty (or yields the maximum

Another important procedure for both the M-such and the M-such and the M-such is the central distribution of Mark and Johnson [3], although the paper is early sequenced with the minimum problem. In fact, the procedure  $R_{\mu}$  is M-special for a 1 5 and the expected values for moderate a (calculated by inciden and the author) were found to be smaller than any others dound in paint of the chart of this author investigation. Cosent [4] and without another the incommon is for a 2 to obtain a smaller expectation without charteing the N-velve.

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amount of information). By introducing certain types of pairing for the early comparisons the procedure can be greatly simplified and in some instances actually improved. The idea of expected entropy was used for the grouptesting problem by Sobel and Groll [22] and has also been used for other search problems by F. Dubail [7], who has called it "generalized entropy."

Our main interest is in one-step entropy procedures. A fairly obvious generalization of  $R_E$ , say  $R_{E,g}$  which selects the comparison that maximizes the expected reduction in entropy in the next g tests  $(g \ge 1)$  can also be considered, as it is in  $\lfloor 22 \rfloor$  for the group-testing problem. All our procedures are such that they can make use of any a priori knowledge about the initial ordering as well as a posteriori knowledge gained at each stage.

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The procedure  $R_E = R_{E,1}$  (the pure one-step entropy procedure) gives optimal expectation results for small values of  $n(n \le 6)$  for t = 2 and also for t = n - 1) wherever optimal procedures are known. In addition each of the three entropy procedures consistently improves on known results for moderate values of  $n \cdot c$ . In fact, it turns out to be interesting to find instances where  $R_E$  is not optimal. All our empirical results are consistent with a conjecture that an E-optimal procedure can be obtained from the procedure  $R_E$  or from the family  $R_{E,g}$  with a moderately small value of  $g \cdot c$ .

The case t=2 will actually be treated first in this paper, before the case of t=n-1, because it is a simpler problem and at the same time it exhibits the complexities associated with the case of general t  $(1 \le t \le n-1)$ .

The case of small t has a slight history of its own starting with Lewis Carroll's essay [3] on the faulty manner (Cup System) of awarding the second prize in a lawn tennis tournament in his day. He points out that if players are eliminated after 1 loss then there is a high probability of not

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The uncocdure  $E_{g_1} = N_{g_2,h}$  (if a pure one-stay introffy encocdure) gives optimish superfiction because were evalues of  $n(n+\delta)$  for t=0 and also for t=n-h) wherever optimish procedures are ensure. In evolution each of the finite cattory procedures consistently deproves on most results for moderness values of  $u_1$ . In fact, its land, for equality to be interesting to find fine and only one optimished. All our explanation we expend that a conjecture that a procedure can be obtained from the procedure  $N_{g_1}$  or from the finitely  $N_{g_2}$  while a modernessly analyse of  $N_{g_1}$ .

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The and off small : has a challet history of the own scanding which having Carrollic edsor [3] on the featily manner (Guy System) of evanding the second prime in a lawn tennis tenning the proper and the following the players and eliminated effort I doer then there is a high probability of not

finding the correct second-best player. For example, with n=8 under complete pairing (or so-called knock-out tournament that pairs off all the non-losers) the second best player has probability 3/7 of being in the same group of four as the best player and hence of not receiving the second prize.

The case t=2 is discussed by Steinhaus [23] and the papers of J. Schrier [19] & J. Slupecki [20] are fundamental to our result that two of our procedures are M-optimal for t=2. The case t=2 has also been considered by Picard [17] and we use one of his procedures  $R_p$  in our table of comparisons. For t=2 we regard  $R_p$  as an analogue of the Steinhaus procedure  $R_S$  for t=n-1, and we only consider procedures that are at least as good as  $R_p$  for the E-goal or the M-goal.

The work of David [5], Glenn [9] and Maurice [16] deals with knock-out, round robin and double-elimination tournaments and is related to our subject but not to the present paper. In their work randomness is a result of associating more skill with a higher probability of winning. In our case the better player always wins and the randomness arises only from the initial random ordering of the n players. It is felt that a knowledge of the best procedures when there are distinct differences in skill (so that the better player always wins) should be helpful to design procedures for models which bring randomness into the observed results. A fine discussion of the work of David, Glenn, and Maurice on these types of tournaments is given in David [6].

Although no attempt is made in this paper to apply the techniques for large values of n or to find the procedure best for machine computation, the author feels that there is a challenge presented here to adapt the entropy procedure, or some modification of it, to large-scale machine computations for the large values of n. It is conjectured that the results will be substantially better than any others in print (see e.g. Bose and Nelson [1])

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even if one uses a slightly different criterion than the number of comparisons for comparing procedures.

### 2. Procedures for the Ordering Problem With t = 2

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Several procedures are introduced, all of which are new, except for the procedure  $R_p$  due to Picard [17]. One of these procedures is an adaptation to t=2 of the Ford-Johnson procedure and is denoted by  $R_p$ . One of the entropy procedures  $R_p$  is uniformly as good or better than any other procedure for all the values of n considered ( $2 \le n \le 10$ ). Based on the work of Schrier [19]& Slupecki [20], two of the procedures are shown to be M-optimal. Each procedure is briefly described in this section and a table of numerical comparisons is given; properties and derivation of results are given in Section 3.

We use the term 'fully-inductive' to indicate a scheme in which the procedure for n players depends directly on that for n - 1 players. The term 'semi-inductive' indicates that the scheme for n players depends directly on that for  $\lceil \frac{n}{2} \rceil$  players, where  $\lceil x \rceil$  is the largest integer  $\leq x$ . All logarithm in this paper are to the base 2 unless stated otherwise.

Procedure  $R_E$ : This is a one-step entropy procedure for t=2 and is based on finding the binary comparison that minimizes the expected reduction in entropy after one comparison.

Procedure  $R_{E_1}$ : Suppose  $n = 2^r + c(0 \le c < 2^r)$  and we conduct a knock-out tournament on the first (or any)  $2^r$  players. Procedure  $R_{E_1}$  starts in this way and then uses the one-step entropy method to complete the problem.

For complete pairing and  $n=2^r+c$  we also want to allow pairing among the c remaining players; we then write  $n=2^r+2^r+2^2+\ldots 2^s$   $(r_1>r_2>\ldots>r_s\geq 0)$  and perform a knock-out tournament for each of these powers of two.

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Procedure  $R_{\overline{g}}$ : This is a cho-step catropy unoccdere for z=2 and is based on Minding the pineur companison that minimize the expected veducation in entropy after one companison.

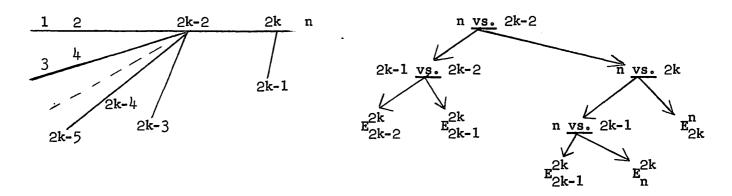
Exposedure  $R_{\rm eff}$ : Suppose  $n=C^2+c(0\le c\le 2^2)$  and we conduct a imosti-dut tounnessent on the flinct (or any)  $S^2$  players. Exposeding  $R_{\rm eff}$  starts in this way and then uses the one-step entropy matheta to complete the problem. For appoint the problem.

For complete pointing and  $n=2^{\circ}+c$  were also wind to allow pointing among the G werealizing playene; we then write  $n=2^{\circ}+2^{\circ}+\ldots$  among the G werealizing playene; we then write  $n=2^{\circ}+2^{\circ}+\ldots$   $2^{\circ}$   $(x_1>x_2>\ldots>x_s>0)$  and paralogm a knock-cut tournament for each of these powers of two.

Procedure  $R_{\underline{E}_2}$ : For this procedure we do a complete pairing and then use the one-step entropy method to complete the problem.

Procedure  $R_F$ : This is an analogue of the Ford-Johnson procedure applied to the case t=2. Suppose n=2k or 2k+1. We describe the procedure by 3 steps.

- 1. Using ordinary pairing, we pair off 2k of the players for the first k comparisons, leaving one man out if n is odd.
- 2. Use induction (with the obvious procedures for n = 2 and 3) to order the t = 2 best among the k winners in step 1.
- 3. If n is even, step 2 results in an overall best player and 2 contenders for second best; thus requiring only 1 more comparison. If n is odd, we use a diagram for the third step. Let n or 2k + 1 denote the player left out in steps 1 and 2, let 2k denote the winner in step 2, 2k-1 (resp., 2k-2) denote the contender that lost to 2k in step 1 (resp., step 2). The diagram and the continuation are given by



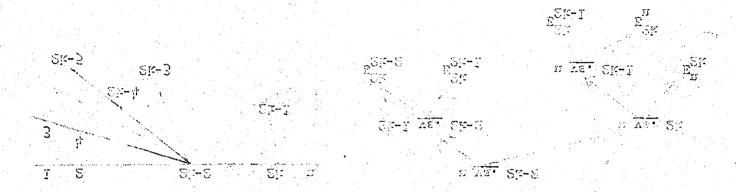
#### Figure 1

In figure 1 the left (resp. right) fork under a  $\underline{vs}$  b indicates that a loses to b (resp. a wins over b) and the endpoint  $E_b^a$  indicates the final result that a is best and b is second best.

Expositive is a for this procedure we do a complete pairing and then use the one-ster entropy method to complete the problem.

Fromedure  $\frac{1}{2}$ : This is an enalogue of the Ford-Johnson procedure sublied to the case  $\mathfrak{c}=2$ . Suppose n=2n or 2n+1. We describe the procedure by 3 steps.

- 1. Using ordinary pairing, we pair off Ot of the players for the fixet in comparisons, leaving one man out if a is odd.
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In Gigure 1 the Left (nesp. vight) fort under a  $\overline{vs}$ . 5 indicates that a lioses to 5 (neep. a wine over 5) and the endpoint  $\overline{u}_{b}^{6}$  indicates the linear result that a is best and 5 is second best.

Procedure  $R_{\underline{I}}^*$ : This is a semi-inductive procedure without pairings. Let n=2k or 2k+1 as above. We first partition the n players into 2 subsets, each of size at least k, without making any comparisons and then for  $n \ge 4$  follow the three steps:

- 1. Use induction (with the obvious procedure for n=2 and 3) to find the best player separately in each of the two subsets, keeping track of all contenders for second best.
- 2. Let the two winners play to determine the best player and put the loser (but not his inferiors) in contention for the second best. Suppose there are now  $c \ge 2$  contenders for second best.
- 3. Use any simple knock-out tournament (with exactly c 1 games) to determine the second best.

Procedure R<sub>M</sub>: For this procedure we again use the binary expansion of n and complete pairing:

- 1. Find the best one separately in each of the subsets for which  $r_i \ge 0$ .
- 2. Play the best one of the smallest subset (of size 2<sup>rs</sup>) against the winner of the second smallest subset (of size 2<sup>rs-1</sup>). Play this winner against the winner of the third smallest subset (of size 2<sup>rs-2</sup>), etc., until the best one of all n is determined. Let c denote the number of contenders for second best.
- 3. Use any simple knock-out tournament with exactly c 1 games to determind the second best.
- Procedure  $R_p$ : This fully-inductive procedure for t=2 due to Picard [17] is an analogue of the Steinhaus procedure for t=n-1. Let the players (in random order) be denoted by 1, 2, ..., n; the iterative scheme of the procedure is decribed in 3 steps:

Exocedure R.\*: This is a semi-inductive procedure whithere pointings.

Lat u = the or the i as above. We Thist profite up a propert into a subcote, each of the atliast is, whithere metring any companisons and then for u = i follow the third stops:

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- A. Elay the best one of the scallest subset (of size  $2^{-3-1}$ ). Elay this winner winner of the second smallest subset (of size  $2^{-3-1}$ ). Elay this winner equinst the winner of the think inclinating subset (of size  $2^{-3-1}$ ), etc., until the best one fight in is determined. Let a denote the number of contending for eacond bast.
- 3. Ved any chapte took tounnease with emotive e 1 gener to detain the the second best.

Procedure  $R_{ij}$ : Toke fully-inductive procedure for T=0 due to Ficard [Lij] is an analogue of the Steinhaus procedure for T=n-1. Let the players (in random evelon) be denoted by [, i, ..., n; the describe advers of the procedure is decribed in 3 steps:

1. Play 1 vs. 2 and assume 1 loses to 2.

2. Play 3 vs. the loser 1. If 3 loses then he is removed from contention. If 3 wins then 1 is removed from contention and 3 plays 2 to re-establish an ordering between the two top contenders.

3. Thus in either case we again have an ordered pair of contenders and if there are new players left we simply repeat the above scheme.

Although the procedure  $R_{p}$  is remarkable for its simplicity and amenability to anlysis and machine computation, we later show that it is inadmissible. However this procedure is useful as a standard for comparison for the E-problem and is conjectured to be optimal in the class of fully-inductive procedures for t=2.

Procedure  $R_{\underline{I}}$ : This is a semi-inductive procedure with the same first step as  $R_{\underline{I}}^*$ , which we omit. We use the obvious procedures for n=2 and 3 and assume that  $n \ge 4$  in the following steps:

- 2. Use induction on each set separately to find both the best and the second-best players. Suppose a  $\prec$  b and c  $\prec$  d are the two pairs obtained, where  $\prec$  denotes 'is inferior to'.
- 3. Play b <u>vs</u>. d and assume that d wins. Then play b <u>vs</u>. c to determine the second best. Thus for  $n \ge 4$ , step 3 consists of exactly 2 games.

Although  $R_{\overline{\mathbf{I}}}$  is quite poor in expectation we include it for purposes of comparison and to illustrate the importance of subtle differences in procedure.

- L. Play I vers and respication loses to 2.
- 8. Firty 3 vs. the locar 1. If 3 loses then he is newdyed from contention.

  Li 3 cline then L is removed from contention and 3 plays 2 to ne-establish en ordering forward the tro to to contend end.
- 3. Then in childre case we again have an ordered point of contenders and in the thora are next players leaf we simply repeat the above contains.

Although the procedure  $R_{\rm p}$  is versively for its show that he for and smen-splitty to sulfath and machine conjugation, we later show that it is in-solution. Toward this procedure is useful as a standard for comparison for the S-2robian and is comparition for the S-2robian and is comparitude; to be appried in the claim of fully-inductive procedures for a = 1.

Exequentia  $\mathbb{R}_{\frac{1}{4}}$ : While he a soun-inductive procedure with we have fine a stap of  $\mathbb{R}_{\frac{1}{4}}$ , which we state. We use the opvious procedure for n=0 and n=0 for n=0 for n=0 forms.

- D. Use induction on and the sandarion both the wat and the second-heat places. Suppose a for such a for a graph the two pains obtained, where heat contents is infinitely to.
- 3. Flay 5 vs. d and estime finat d white. Then play 5 vs. c to deferming the second bast. Thus for a 1 h, step 3 consists of emetly 2 genes. Although R is dente poor in expectation we include it for purposes of comparison and to illustrate the importance of subtle difficuences in procedure.

#### Comparison of Eight Procedures for the t=2 Ordering Problem

Lower Bounds and	Expected Values											
Procedures	<b>n=</b> 2	n=3	n=4	<b>n=</b> 5	<b>n</b> =6	n=7	n=8	<b>n=</b> 9	n=10			
LB <sup>§</sup>	1	2.584 2.500	3.917 4.000	4.922 5.000	5•773 6.500	6.488 7.500	8.380 9.00	9.057 10.000	9.668 11.000			
R <sub>E</sub> 1	1	2 <u>2</u>	4 <u>0</u>	<u>5</u> 8	6 <sup>45</sup>	7 170	90	1084	11112			
R <sub>E</sub>	1	<u>ಜ</u> ಜ <u>ಇ</u>	<u>수</u> 일	85 85 85	6 <sup>45</sup>	7 <sup>170</sup> 7 <sup>170</sup>	9 <sup>5</sup> 90	n.c. 10 <sup>84</sup>	N.C. 11 <sup>112</sup>			
-2 R <sub>F</sub>	1	2 <u>2</u>	4 <u>0</u>	<u>58</u>	6 <u>60</u>	7 <sup>176</sup>	<u>90</u>	10160	11 1008			
R <sub>I</sub> *	1	2 <u>2</u>	4 <u>0</u>	5 <u>12</u>	<u>660</u>	7 <del>180</del>	90	10420	11 <sup>1512</sup>			
$^{R}_{M}$	1	2 <u>2</u>	4 <u>0</u>	5 <u>18</u>	6 <u>60</u>	7 <del>180</del>	90	10840	11 <u>2268</u>			
R <sub>P</sub>	1	2 <u>2</u>	4 <u>1</u>	5 <sup>17</sup>	6 <u>81</u>	8 <u>39</u>	9 <u>366</u>	10 <sup>829</sup>				
R <sub>I</sub>	1	2 <u>2</u>	4 <u>0</u>	<u>5</u> 20	7 <u>30</u>	8 <del>140</del>	10 <sup>9</sup>	11840	13 <sup>1260</sup>			
D § §		3	6	30	90	210	840	1260	3780			

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141	$\mathbf{n}$	IIIH X	va	THES	5

LB <sup>##</sup>	1	3	4	6	7	8	9	11	12
R <sub>M</sub>	1	3	4	6	7	8	9	11	12
R <sub>I</sub> *	1	3	4	6	7	8	9	11	12
$^{ m R}_{ m F}$	1	3	4	6	7	9	9	11	12.
R <sub>E2</sub>	1	3	4	6	7	9	9	11	12
R <sub>E</sub> 1	1	3	4	6	8	9	9	11	12
R <sub>E</sub>	1	3	4	6	8	9	11		
$\mathbf{R}_{\mathbf{I}}$	1	3	4	6	8	9	10	12	14
R <sub>P</sub>	1	3	5	7	9	11	13	<b>1</b> 5	17

Notes § This is a lower bound for all procedures using cycle pairing.

<sup>#</sup> CLB =  $n - 2 + \frac{1}{2}[2 \log n]$  is a conjectured lower bound.

 $<sup>\</sup>S\S$  Each D is the common denominator of all underlined numerators above it.

<sup>##</sup> This M-lower bound due to Schrier is LB = n - 1 + [log (n-1)].

N.C. means not computed.

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## 3. Formulas and Properties

Since our best results are for the simplest procedures, we consider our procedures in reverse order of their appearance in Section 2.

A. Let  $f_6(n)$  denote the expected number of tests under procedure  $R_{\overline{1}}$  for t=2. From the definition, we easily obtain the recursion formulas for  $m\geq 2$ .

(3.1) 
$$f_6(2m) = 2 f_6(m) + 2$$
$$f_6(2m+1) = f_6(m) + f_6(m+1) + 2$$

with boundary conditions  $f_6(2) = 1$  and  $f_6(3) = 2\frac{2}{3}$ .

From the first equation of (3.1) we obtain by iteration for

$$n = 2m = 2^r$$
 and  $r \ge 1$ 

-1

1

X

(3.2) 
$$f_6(2^r) = 3 \times 2^{r-1} - 2$$
.

For  $n = 2^r + c$  (with  $0 \le c < 2^r$ ), we set  $f_6(n) = 3 \times 2^{r-1} - 2 + g_6(c) + k \times 2^r$  in (3.1). After using one boundary condition to show that k = 0, we obtain the simpler homogeneous formulas

(3.3) 
$$g_6(2c) = 2g_6(c)$$
$$g_6(2c+1) = g_6(c) + g_6(c+1)$$

with only one boundary condition  $g_6(1) = \frac{5}{3}$ . By iteration in (3.3) we obtain  $g_6(c) = \frac{5c}{3}$ . Hence for all  $n \ge 2$ 

(3.4) 
$$f_6(n) = f_6(2^r+c) = 3 \times 2^{r-1} - 2 + \frac{5c}{3}$$

Let  $\overline{f}_6(n)$  denote the maximum number of tests required under  $R_I$  for t=2. The equations for  $\overline{f}_6(n)$  are exactly the same as in (3.1), the only change being that the second boundary condition is now  $\overline{f}_6(3)=3$ . Repeating the above argument gives  $\overline{g}_6(c)=2c$  and hence for all  $n\geq 2$  (3.5)  $\overline{f}_6(n)=\overline{f}_6(2^r+c)=3\times 2^{r-1}-2+2c$ .

Since  $f_6(n) \ge (3n/2)$  - 2 and we later exhibit procedures of asymptotic  $(n \to \infty)$  order  $n + \log n$ , it follow that  $R_T$  is asymptotically inefficient.

Moreover, several of the other procedures are uniformly E-better (i.e., equal to or

# 3. Foundles and Inoposties

Since our best results sie for the chaplest procedures, we consider our procedures in reverse order of their enverses in Section 2.

In Lot  $\mathbb{Z}_{\mathbb{Z}_{+}}(n)$  devote the expected number of tests under procedure  $\mathbb{Z}_{\frac{1}{2}}$  for  $\mathbb{Z}_{+}$  . For  $\mathbb{Z}_{+}$  for the  $\mathbb{Z}_{+}$  for the  $\mathbb{Z}_{+}$  for  $\mathbb{Z}_{+}$  for

$$\begin{array}{rcl} & \mathcal{E}_{-}(\mathbb{Z}m) & = & \mathcal{E}_{-}(m) + \mathcal{E}_{-} \\ & & \mathcal{E}_{-}(\mathbb{Z}m+1) & = & \mathcal{E}_{-}(m) + \mathcal{E}_{-}(m+1) + \mathcal{E}_{-} \end{array}$$

with boundary conditions  $\mathbb{E}_{\mathcal{S}}(S) = 1$  and  $\mathbb{E}_{\mathcal{S}}(S) = -\frac{S}{3}$ .

Whom the first equetion of (3.1) we obtain by itemation for

$$n = 2n = 2^{2} \quad \text{and} \quad n > 1$$

(3.2) 
$$\epsilon_{ij}(s^{ij}) = 3 \times s^{ij-1} - 2.$$

For 
$$n = 2^n + c$$
 (with  $0 \le c < 2^n$ ), we set  $E_1(n) = 3 \times 2^{n-1} - 2 + c$ 

thatt he = 0, we obtain the stabler hemogeneems decadels a

$$(3.2) = 8_{5}(c) + 8_{5}(c+1)$$

which only one boundary condition  $f_{\mathcal{S}}(\mathbb{L}) = \frac{\pi}{3}$ . By fouration in (3.3) we obtain  $g_{\mathcal{S}}(z) = \frac{5c}{3}$ . Repositor of z

(3.4) 
$$\frac{1}{3}(a) = \frac{2}{3}(a^{2}+a) = 3 = 3 = a^{2} - 3 + \frac{2a}{3}$$

Under Trecedure  $\mathbb{R}_{\underline{x}}$  it is envious to note that all vandomness can be traded had: to n=3.

Let  $\overline{Z}_{\mathcal{C}}(\mathfrak{A})$  decrete initializar number of the terrequired under  $\mathfrak{A}_{\overline{1}}$  for  $\mathfrak{C}=2$ . The equations for  $\overline{Z}_{\mathcal{C}}(\mathfrak{A})$  are enactly the same as in (3.1); the only change being that the second boundary condition is now  $\overline{Z}_{\mathcal{C}}(\mathfrak{A})=3$ .

Repeating the above argument gives To (c) = Se and hence for all n 2 2

(3.5) 
$$\overline{z}_{1}(n) = \overline{z}_{2}(e^{z}+c) = 3 \times e^{z} -2 + ec.$$

Since  $E_{ij}(n)$  is (3n/2) - 2 and we later eministic precedures of regimentials. (n-n) order  $n+\log n_{ij}$  it follow that  $R_{ij}$  is esymmetrically inelationary. Moreover, forest of the other procedures and undistriby R-botter (i.e., equal to on smaller in expectation) than  $\,R_{\rm I}\,$  for all  $\,n\geq 2$  . Similar remarks hold for the maximum length.

B. Let  $f_5(n) = E\{T | R_p\}$  for t = 2. Since the j<sup>th</sup> player  $(j \ge 3)$  wins his first game (and hence/an 'extra' game) with probability 2/j, it follows that for all  $n \ge 2$ 

(3.6) 
$$f_{5}(n) = n-1 + \sum_{j=3}^{n} \frac{2}{j} = n-4 + 2 \sum_{j=1}^{n} \frac{1}{j} \approx n + 2 \log n.$$

Clearly, if players  $j=2, 3, \ldots, n$  all win, we obtain the maximum length  $\overline{f}_5(n)$ ; hence for all  $n \ge 2$ 

(3.7) 
$$\overline{f}_5(n) = 2n-3$$
.

Although  $R_p$  has a better expectation than  $R_{\rm I}$ , it has a minimax value that is much worse; these results already show up in our table for  $n \leq 10$ . For all n and asymptotically  $(n \to \infty)$  we have

(3.8) 
$$f_5(n) \le n-4 + 2(\log_e n + \gamma + \frac{1}{2n}),$$

where  $\gamma=.577...$  is Euler's constant; this can be used to show that  $f_5(n)$  is smaller than  $(\frac{3n}{2}-2)$  and hence smaller than  $f_6(n)$  for all n>2.

C. Let  $f_{\downarrow}(n) = E\{T | R_M\}$  for t = 2. Let  $n = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_s}$  in binary notation; this partitions the n players at random into s 'connected' subsets of sizes  $2^{r_1}(i=1,2,\ldots,s)$  with  $r_1 > r_2 > \ldots > r_s \geq 0$ . Inside these sets we need a total of n-s comparisons to find the s best players and between the s subsets we need an additional s-l comparisons to find the overall best player. The winner of the  $j^{th}$  subset has probability  $2^{r_j}/n$  of being the overall best. Since we do a knock-out tournament within each subset and because of the order in step 2, this winner carries along with him  $r_j$  contenders for second best from his own

smaller in supectation) than R, for all m 2. Shallar remarks boldfor the mentions Sales.

5. Let  $\mathbb{Z}_{r}(u) = \mathbb{Z}[\mathbb{Z}_{r}]$  for v = 2. Since v = 1 player  $(\cdot \cdot \cdot \cdot \cdot)$  first v = 1 player  $(\cdot \cdot \cdot \cdot \cdot)$  with v = 1 player  $(\cdot \cdot \cdot \cdot \cdot \cdot)$  for v = 1 and v = 1 player  $(\cdot \cdot \cdot \cdot \cdot \cdot \cdot)$  with probability  $2/\frac{1}{2}$  (or follows that for all v = 2)

(3.6) 
$$\mathbb{E}_{\mathbf{z}}(\mathbf{u}) = \mathbf{u} \cdot \mathbf{1} + \frac{\mathbf{u}}{2} = \mathbf{u} \cdot \mathbf{u} + 2 = \mathbf{u} \cdot \mathbf{u} + 2 = \mathbf{u} \cdot \mathbf{u} + 2 = \mathbf{u} \cdot \mathbf{u}$$

Clearify, an players , j=2 ,  $j_1,\ldots,n$  while we obtain the hate man we like with  $\overline{z}_{ij}(n)$ ; hence for all  $n\geq 2$ 

$$(3\cdot 4) = \frac{1}{2}(5) = \frac{1}{2}(5)$$

Although  $R_p$  has a better unpertablar than  $R_q$ , it has a minimum value that is much variet these results already show up in our table for a 10. For all it and asymptotically (a - + + ) we have

where Y = .577.1:18 Gulen's conjecut: this can be used to show that  $E_{j}(n)$  is smaller than  $(\frac{3n}{2}-2)$  and to see aller than  $E_{j}(n)$  for all  $n \ge 2$ .

G. Let  $E_{ij}(u) = E[T]R_{ij}$  for v = 2. Let  $v = 2^{\frac{T_{ij}}{2}} + 2^{\frac{T_{ij}}{2}} + 2^{\frac{T_{ij}}{2}} + 2^{\frac{T_{ij}}{2}}$  in binary notation; this, partitions the n players at random into s connected subsets of sizes  $2^{\frac{T_{ij}}{2}}(1 = 1, 2, \dots, s)$  with  $v_{ij} > v_{ij} > v_{$ 

subset, j - 1 more from the j-1 larger subsets and 1 -  $\delta_{js}$  from the smaller subsets; here  $\delta_{js}$  = 1 if j = s and = 0 if j < s. Thus if the j<sup>th</sup> subset produces the best one then an additional  $r_j$  + (j-1) +  $(1-\delta_{js})$  - 1 comparisons are needed. Hence for all  $n \geq 2$ 

(3.9) 
$$f_{\mu}(n) = (n-s) + (s-1) + \frac{1}{n} \sum_{j=1}^{s} (r_{j} + j - 1 - \delta_{js})^{2^{j}}$$
$$= n - 2 + \frac{1}{n} \sum_{j=1}^{s} (r_{j} + j - \delta_{js})^{2^{j}}.$$

11/2. j-1

This is not easily amenable to an asymptotic analysis; we therefore derive a lower bound for  $f_{\downarrow}(n)$  and use the maximum value as an upper bound. A lower bound is obtained by taking only the first term of the summation in (3.9). We note that  $r_1 = [\log n]$  and that  $r_1 - \delta_{1s} = [\log(n-1)]$ . Hence for all  $n \geq 2$ 

(3.10) 
$$f_{4}(n) \ge n - 2 + \frac{2^{\lceil \log n \rceil}}{n} (1 + \lceil \log(n-1) \rceil).$$

This already shows that for any sequence  $n_i$  of n-values

(3.11) 
$$f_{i}(n_{i}) \ge n_{i} - 2 + \frac{1}{2} \log(n_{i} - 1)$$

procedure.

and puts a lower bound on the possible asymptotic form of  $\,f_{\mu}(n_{_{\dot{1}}})\,$  as  $\,n_{_{\dot{1}}}\to\infty$  .

The maximum  $\overline{f}_{l_1}(n)$  required under  $R_M$  occurs when the winner of the first subset (of size 2 1) is the overall winner and hence

$$f_{l_1}(n) \leq \overline{f}_{l_1}(n) = (n-s) + (s-1) + r_1 - \delta_{1s} = n - 1 + [\log(n-1)].$$
 Since this same value was shown by Schrier [19] & Slupecki [20] to be a lower bound for the minimax value of any procedure, it follows that  $R_M$  is an M-optimal

subset;  $\frac{1}{2} - \frac{1}{4}$  nore from  $\frac{6}{3} > \frac{1}{4} - \frac{1}{4}$  larger subsets and  $\frac{1}{4} - \frac{1}{3} > \frac{1}{3}$  whose the smaller subsets; here  $\frac{6}{3} = \frac{1}{4}$  if  $\frac{1}{4} = \frac{1}{4}$  case the base one then an additional  $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{4} + \frac{1}{$ 

(8.5) 
$$\hat{z}_{\frac{1}{2}}(n) = (n-s) + (s-\frac{1}{2}) + \frac{1}{n} \cdot (s-\frac{1}{2} - s-\frac{1}{2}) 2^{\frac{n}{2}}$$
$$= n - 2 + \frac{1}{n} \cdot (s+\frac{1}{2} - s) 2^{\frac{n}{2}}.$$

This is not easily amanable to an attraction analysis; we therefore derive a lower bound for  $i_{l_1}(z)$  and see the training result as an upper bound. A lower bound is obtained by taken only the first take of the suscation in (...), we note that  $x_1 = \{\log n\}$  and that  $x_1 = \{\log n\}$ . Hence for all  $n \ge 2$ .

(j.10) 
$$\dot{z}_{i,j}(\mathbf{a}) \geq \mathbf{a} + 2 + \frac{\mathbb{E}[\log |\mathbf{a}|]}{2} (1 + \lfloor \log(2-1) \rfloor).$$

This already shows that for my sequence no of n-velues

(1.11) 
$$t_{i_1}(a_{i_1}) \ge a_{i_1} - 2) + \frac{1}{2} \log(a_{i_2} - 1)$$

and pure a lower bound on the possible asymptothe form of  $E_{ij}(\lambda_{ij})$  as

The marriages  $\overline{\mathcal{E}}_{ij}(n)$  ropy red value  $\mathbb{R}_{ij}$  observable when the variety of the first subset (of size  $2^{-1}$ ) is the overall wanner and hence

$$(2.12) \qquad \varepsilon_{1}(u) \leq \overline{\varepsilon}_{1}(u) = (2.4) + (391) + v_{1} - v_{13} = u + 1 + 100(u + 1)$$

Alace this same value was shown by subrier [17] to becallowdribound for some for che minimal value of any procedure, it follows that N<sub>ii</sub> is an Morthad procedure.

The procedure  $R_{\underline{M}}$  is also important because it attempts to solve the t=2 problem by separating the two problems of finding the best and (conditional on the extra information picked up) then finding the second best. Although this idea was also used by Picard in Section 7.3.1 of [17], it should be noted that our procedure  $R_{\underline{M}}$  is not the same as his procedure; call the latter procedure  $R_{\underline{P}_1}$ . In fact, it is fairly easy to show (details are omitted) that for any  $n\geq 2$  the procedure  $R_{\underline{P}_1}$  has expectation

(3.12a) 
$$E\{T|R_{P_1}\} = n - 1 + \frac{2}{n}(n-2) + \frac{1}{n}\sum_{j=1}^{n-2}(j-1) = \frac{3}{2}(n-1) - \frac{1}{n} \frac{3n}{2}$$

which is to be compared with the upper bound  $n + \log n$  obtained for  $R_M$  in (3.12) above. We can say that  $R_{P_1}$  is inadmissible for both the E-goal and the M-goal since  $R_M$  is at least as good for all n and, in fact, strictly better for  $n \geq 4$ . In particular, for the example with n = 5 considered by Picard,  $R_{P_1}$  gives 5.8 and 7 for the expectation and maximum, respectively, compared to 5.6 and 6 for  $R_{M^*}$ 

The problem  $R_{ij}$  is also injectant because it attempts to solve the problem by separation the two problems of finding the best and (conditional on the enters information picked up) then finding the second best. Although this idea was also used by Meand in Reaction 7. .1 of [10], it should be noted that our procedure  $R_{ij}$  is not the same as bits procedure; call the latter procedure  $R_{ij}$ . In fact, it is faitly easy to show (details are ordered) that for any  $n \ge 2$  the procedure  $R_{ij}$  has embediation

(\*.12a) 
$$\Im \left( \mathbb{T}^{2} \mathbb{Z}_{2} \right) = n - 1 + \frac{2}{n} (n-2) + \frac{1}{n} \Im \left( 1 + 1 \right) = \frac{2}{2} (n-1) + \frac{1}{n} \Re \left( \frac{n}{2} \right)$$

Which is to be compared with the upper bound n+1.8 m obtained for  $R_{\rm H}$  in ( .12) above. We can say that  $R_{\rm H}$  is inacmissible for both the E-joal and the H-goal since  $R_{\rm H}$  is at least as good for all usuad, in fact, strictly better for  $n \geq 4$ . In particular for the enable with n=0 considered by Picard,  $R_{\rm H}$  gives [.8 and [ for the expectation and infinity, respectively, compared to ), o and 6 for Eq.

D. Let  $f_3(n)$  denote  $E\{T|R_{T^*}\}$  for t=2. For  $n=2^r+c$  (with  $0 \le c < 2^r$ ); let  $g_r(n)$  denote the probability that there are r contenders for second best after step 3 of the procedure  $R_{T^*}$ . To show that  $g_r(n)+g_{r+1}(n)=1$ , assume for any  $n'<2^r$  (say, n' associated with r'<r) that the number of contenders for second best is either r' or r'+1 with probability one. Then for even n=2m as a result of step 3

$$g_{r}(2m) = g_{r-1}^{2}(m) + g_{r-1}(m)g_{r}(m) = g_{r-1}(m),$$

$$(3.13)$$

$$g_{r+1}(2m) = g_{r}^{2}(m) + g_{r-1}(m)g_{r}(m) = g_{r}(m).$$

and the sum of these two equations is again one. Also for odd n=2m+1

$$g_{r}(2m+1) = g_{r-1}(m)g_{r-1}(m+1) + g_{r-1}(m)g_{r}(m+1)(\frac{m}{2m+1}) + g_{r}(m)g_{r-1}(m+1)(\frac{m+1}{2m+1})$$

$$g_{r+1}(2m+1) = g_{r}(m)g_{r}(m+1) + g_{r-1}(m)g_{r}(m+1)(\frac{m+1}{2m+1}) + g_{r}(m)g_{r-1}(m+1)(\frac{m}{2m+1})$$

and the sum is  $\{g_{r-1}(m) + g_r(m)\}\{g_{r-1}(m+1) + g_r(m+1)\} = 1$ . Since  $g_1(2) = 1$  and  $g_2(2) = 0$ , this result must hold for all  $n \ge 2$ .

Since r is determined by m we now write g(m) without the subscript r and obtain from (3.13) and (3.14) (and the result just proved)

$$(3.15) \quad g(2m) = g(m)$$

$$g(2m+1) = \begin{cases} \frac{mg(m) + (m+1)g(m+1)}{2m+1} & \text{if } m+1 \text{ is not a power of 2} \\ \frac{mg(m)}{2m+1} & \text{if } m+1 \text{ is a power of 2}, \end{cases}$$

where g(2) = 1 and  $g(3) = \frac{1}{3}$ . It is easily checked that for

0. Let  $\hat{x}_1(n)$  denote  $\hat{x}(\hat{x}_1^*\hat{x}_1^*)$  for t=2. For  $n=2^5+3$  (we)

5. (a) denote the precability that there are xcontenders for second best after stap in of the precading  $\hat{x}_1^*$ . To sow

5. (a)  $+\hat{x}_{k+1}(n) = \hat{x}_k$  assume for any  $n \in \mathbb{R}^n$  (say, n' associates with x' in that the vertical of contenders for second best is either x' or x' + 1 with probability one. Then for even n = 2n as a result of step

$$g_{\pi}(2n) = g_{\pi-1}^{\mathcal{Z}}(n) + g_{\pi-1}^{\mathcal{Z}}(n)g_{\pi}(n) = g_{\pi-1}(n),$$

$$(g_{\pi}, g_{\pi})$$

$$g_{\pi+1}(2m) = g_{\pi}^{\mathcal{Z}}(n) + g_{\pi-1}(n)g_{\pi}(m) = g_{\pi}(m),$$

and the sum of thase two empations is again one. Theo for old n = 210 + 1

$$\mathcal{Z}_{\mathcal{L}}(2v+\tilde{x}) = \mathcal{Z}_{\mathcal{L}-\tilde{x}}(w)\mathcal{Z}_{\mathcal{L}-\tilde{x}}(v+\tilde{x}) + \mathcal{Z}_{\mathcal{L}-\tilde{x}}(w)\mathcal{Z}_{\mathcal{L}}(v+\tilde{x})(\frac{v+\tilde{x}}{2v+\tilde{x}})$$

$$+ \mathcal{Z}_{\mathcal{L}}(w)\mathcal{Z}_{\mathcal{L}-\tilde{x}}(v+\tilde{x})(\frac{v+\tilde{x}}{2v+\tilde{x}})$$

$$\mathcal{Z}_{\mathcal{L}+\tilde{x}}(2v+\tilde{x}) = \mathcal{Z}_{\mathcal{L}}(w)\mathcal{Z}_{\mathcal{L}}(v+\tilde{x}) + \mathcal{Z}_{\mathcal{L}-\tilde{x}}(v)\mathcal{Z}_{\mathcal{L}-\tilde{x}}(v+\tilde{x})(\frac{v+\tilde{x}}{2v+\tilde{x}})$$

$$+ \, \mathcal{Z}_{\underline{x}}(\cdot, \cdot) \mathcal{Z}_{\underline{x} - \underline{x}}(\cdot, +\underline{x}) (\frac{2n+\underline{x}}{2})$$

and the sum is  $\{g_{T-1}(u) + g_T(u)\}\{g_{T-1}(u+1) + g_T(u+1)\} = 1$ , since  $g_{\frac{1}{2}}(2) = 1$  and  $g_{\frac{1}{2}}(2) = 0$ , this result use hold for all  $n \geq 2$ .

Since r is determined by m = rox we the r(u) without the

Since r is determined by in we now write g(in) rithout the subscript r and obtain from (4.14) and (4.14) (and the result just probed)

$$g(2n) = r(n)$$

$$\frac{\log(n) + (n+1) \cdot c(n+1)}{2n+1} \quad \text{i.i. a+1 is not a rower of 2}$$

$$g(2n+1) = \frac{\log(n)}{2n+1} \quad \text{if a+1 is a power of 2},$$

where g(2) = 1 and  $g(1)' = \frac{1}{3}$ . It is easily checked that for

 $2^r \le n \le 2^{r+1}$  and  $r \ge 1$  the solution is

(3.16) 
$$g(n) = \frac{2^{r+1}-n}{n}$$
.

Since it takes exactly n-1 comparisons to find the best and an additional r-1 or r comparisons with probabilities g(n) and 1-g(n), respectively, for the second best, we have from (3.16) for  $2^r \le n < 2^{r+1}$  and  $r \ge 1$ 

(3.17) 
$$f_3(n) = n - 1 + (r-1)g(n) + r\{1 - g(n)\} = n + r - \frac{2^{r+1}}{n}$$
$$= n + [\log n] - \frac{2^{1+[\log n]}}{n}.$$

The smallest value we add to n-1 in the above is  $r-1=[\log n]-1$  and the largest is  $r-\delta_{1s}=[\log(n-1)];$  hence

(3.18) 
$$n-2+[\log n] \le f_3(n) \le n-1+[\log(n-1)].$$

Thus  $f_3(n)$  is of asymptotic  $(n \to \infty)$  form  $n + \log n$ . By the same argument as in (3.18) the maximum length is

(3.19) 
$$\overline{f}_3(n) = n - 1 + [\log(n-1)].$$

4

Since the minimax value has to be at least this by Schreier's result, it follows that procedures  $R_{1}$  and  $R_{M}$  are both M-optimal. E. Let  $f_{2}(n) = E\{T|R_{F}\}$  for t=2; for this procedure we start with the minimax problem and  $\overline{f}_{2}(n)$ . It follows directly from the details of step 3 (see Section 2) that according as n=2m is even or n=2m+1 is odd, respectively, we have

(3.20) 
$$\overline{f}_{2}(2m) = \overline{f}_{2}(m) + m + 1$$

$$\overline{f}_{2}(2m + 1) = \overline{f}_{2}(m) + m + 3,$$

 $2^{x} \le n < 2^{x+1}$  and  $x \ge 1$  the solution is

$$(z, \overline{z}, z) \qquad g(z) = \frac{2^{z+\overline{z}} - z}{z}.$$

Since it takes exactly n-1 comparisons to find the best and an additional v-1 or v comparisons with probabilities g(u) and 1-g(u), respectively, for the second best, we have from (3.1.) for  $2^{r} \le n \le 2^{r+1}$  and  $r \ge 1$ 

(3.17) 
$$E_3(n) = n - 1 + (n-1)g(n) + r[1 - g(n)] = n + r - \frac{2^{n+1}}{n}$$
$$= n + [\log n] - \frac{2^{1+(\log n)}}{n}.$$

The smallest value we add to n-1 in the above is  $\tau - 1 = [\log n] - 1$  and the largest is  $v - \delta_{1s} = [\log (n-1)]$ ; hence

(3.13) 
$$n-2+[\log n]-1 \le f(n) \le n-1+[\log(n-1)].$$

Thus  $\hat{x}_j(n)$  is of asymptotic  $(n \to \omega)$  for  $n+\log n$ . By the sure argument as in  $(x,i\beta)$  the minimum value is

$$([.15])$$
  $\overline{(n)} = n - 1 + [log(n-1)].$ 

Affine the minimum value has to be at Laast this by Sobreher's result, it follows that procedures  $R_{\overline{L}}$ , and  $R_{\overline{L}}$  are both M-optimal.

3. Let  $L_{\overline{L}}(u) = E[T]R_{\overline{L}}$  for t=2; for this procedure we start with the minimum problem and  $\overline{L}_{\overline{L}}(u)$ . It follows directly from the details of step 3 (see Section 2) that according as u=2m is even or u=2m+1 is odd, respectively, we have

where  $\overline{f}_2(2) = 1$  and  $\overline{f}_2(3) = 3$ , Letting  $g(n) = \overline{f}_2(n+1) - \overline{f}_2(n)$  and setting  $\overline{f}_2(1) = -1$  gives

$$g(2m) = 2$$
(3.21)
 $g(2m + 1) = g(m) - 1,$ 

where g(1) = 2 is the only boundary condition. Clearly

(3.22) 
$$g(4m + 1) = g(2m) - 1 = 1 = g(3)$$
.

If m = 2c + 1 is an odd integer then by (3.21) and (3.22)

(3.23) 
$$g(4m - 1) = g(8c + 3) = g(4c + 1) - 1 = 0 = g(7)$$
.

In general, if  $m = 2^{p-2}d$  where d is odd and  $p \ge 2$ , then by iteration and (3.23)

(3.24) 
$$g(4m-1) = g(2^{p}d-1) = g(2^{p-1}d-1) = g(4d-1) - (p-2) = 2 - p.$$

A single expression for  $\overline{f}_2(n)$  for both odd and even n can now be obtained by summing the values g(j) (j=1,2,...,n-1) and  $\overline{f}_2(1)$ . In a straightforward manner we obtain

(3.25) 
$$\overline{f}_{2}(n) = n - \left(\frac{1 + (-1)^{n}}{2}\right) + \sum_{\substack{j=1 \ j=1}}^{\lfloor \log \frac{n}{2} \rfloor} (2 - j) \left[\frac{n + 2^{j}}{2^{j+1}}\right]$$
$$= \left[\frac{5n - 2(-1)^{n}}{4}\right] - \sum_{\substack{j=1 \ j=1}}^{\lfloor \log \frac{n}{8} \rfloor} j \left[\frac{n + 2^{j+2}}{2^{j+1}}\right].$$

It is not clear how to show that this is of asymptotic form  $n + \log n$  because of the appearance of  $(\log n)^2$  in the asymptotic analysis. However, for  $n = 2^r$  it is easy to show (we omit the details) that  $\overline{f}_2(2^r) = 2^r + r - 2$ . It follows from Schrier's result that for  $n = 2^r$  we need at least

where  $\overline{x}_2(2) = 1$  and  $\overline{x}_2(1) = 1$ , Letting  $g(n) = \overline{x}_2(n+1) - \overline{x}_2(n)$  and satting  $\overline{x}_2(1) = -1$  gives

$$g(2n) = 2$$
  
 $g(2n + 1) = g(n) - 1,$ 

where g(1) = 2 is the only boundary condition. Clearly

$$(3.22) \qquad g(4m+1) = g(2m) + 1 = 1 = g(3).$$

If a = 2c + 1 is an odd integer than by (3.21) and (3.22)

(3.23) 
$$g(2x - 1) = g(3x + 3) = g(2x + 1) - 1 = 0 = g(7)$$
.

In general, if  $m = 2^{n-2}i$  where d is odd and  $t \ge 2$ , then by iteration and (4.2.).

$$(5,2k) \qquad g(kv - 1) = g(2^2dv - 1) = g(2^{p-1}dv - 1) = g(kdv - 1) - (2^p - 2) = 2^p - p.$$

A simple expression for  $\vec{f}_2(z)$  for both odd and even in can not be obtained by smaller the values of(i) ( = i, 2,..., n-1) and  $\vec{f}_2(i)$ . In a straightforward camer we obtain

(1.2.) 
$$\bar{z}_{2}(n) = n - (\frac{1+(-1)^{n}}{2}) + \sum_{j=1}^{\lfloor \frac{n+2^{-j}}{2} \rfloor} (2-j) + \frac{n+2^{-j}}{2^{j+1}}$$

$$= \left[\frac{(n-2(-1)^{n})}{n}\right] - \frac{(10\sqrt{n})^{n}}{(n-2)^{n}} \cdot \frac{(n+2)^{n+2}}{2^{n+2}}.$$

It is not clear how to show that this is of asymptotic form  $n + \log n$  because of the appearance of (log n)? in the asymptotic enalysis. However, for  $n = 2^{n}$  it is easy to show (we contribe details) that  $\overline{F}_{2}(2^{n}) = 2^{n} + n - 2$ . It follows from debrier's result that he need at least

(3.26) 
$$n-1+[\log(n-1)]=2^r-1+[\log(2^r-1)]=2^r+r-2$$

comparisons and hence it follows that procedure  $R_F$  (for t=2) is M-optimal for n equal to a power of 2.

We note from the table that procedure  $R_{\overline{F}}$  has a slight inefficiency for  $n = 7 = 2^3 - 1$ . This gets magnified for n = 15, 31 and 63 and it is quite surprising to find that  $\overline{f}_2(n)$  is not monotonic; in fact  $\overline{f}_{2}(15) = 19 > \overline{f}_{2}(16) = 18$ . This means that in a tournament with n = 15players it would be better (in the minimax sense) to introduce a fictitious 16th player (say, a beneficent deity) who always loses and hence never is selected to be best or second best. For n = 62 we could use 2 such deities since  $\overline{f}_2(62) = 69$ ,  $\overline{f}_2(63) = 71$  and  $\overline{f}_2(64) = 68$ . This lack of monotonicity did not occur with our previous procedures and is conjectured not to occur for any of the entropy procedures. It also serves to prove that  $R_{\overline{F}}$  is not M-optimal for t = 2 and as our table shows it is also not E-optimal. However the analogous procedure for the complete ranking problem (t = n - 1) is quite efficient and was shown [8] to be M-optimal for  $n \le 11$  and for n = 20, 21; S. Johnson (personal communication) states that it was also shown by M. Wells by machine methods to be Mooptimal for n = 12.

For both t=2 (and t=n-1) the procedure  $R_F$  is also of interest for its relatively low expectation. To find an exact expression for the expectation  $f_2(n)$  we return to step 3 for odd n=2k+1 and compute the probabilities associated with the tree for  $R_F$  in Section 2. The total number of equally likely cases after step 2 is

(3.27) 
$$\binom{4}{2}\binom{6}{2}\dots\binom{2k-4}{2}(2k-3)(n-2)n = \frac{(2k+1)!}{k(k+1)2^k} = C(say).$$

For the first comparison (after step 2) these are split into

$$(5.25) = 2 - 1 + (100)(n-1) = 2^{2} - 1 + (10)(2^{2}-1) = 2^{2} + 2 - 2$$

comparisons and hence it follows that procedure  $R_{\tilde{g}}$  (for t=2) is N-optimal for a squal to a power of 2.

្នំ ប្រទ**ុំ ស្ត្រី ( )** ស្ត្រី ( ) ស្ communication) states that it was also shown by it, itelia by anchine wethour [8] to be M-optimal for  $n \le M$  and for n = 20, 21; 4. Johnson (personal complate ranking problem (t = n - 1) is take efficient and was shown shows he is also mot beorething. Tomever the analogous (procedure for the serves to prove that  $\mathbb{R}_{y}$  . Is not M-optimal for  $\mathbb{R}=2$  , and as out table conjectured not to occur for any of the entropy procedures. It also lack of womotomicity did not occur with our previous procedures and is use 2 such deteries some  $\mathbb{Z}_2(\mathbb{S}^2) = \mathbb{S}_2(\mathbb{S}^2) = \mathbb{T}_2$  and  $\mathbb{Z}_2(\mathbb{S}^2) = \mathbb{S}$ . This bende never is selected to be best or second best. For a = 32 as could fictitious loth player (say, a beneficent deaty) sho always loses and players it would be better (in the admissa sease) to introduce a  $f_2(15) = 19$  )  $f_2(10) = 19$ . This reads that in a courmanishe with n = 1. it is quite suppristing to find that  $\frac{\pi}{2}(n)$  is not mandtonic; in incr for  $n=7=2^2-1$ . This gets washinked for n=19, of and de note from the table that procedume M, has a slidgit inedfichenny

For both t=2 (and t=n-1) the procedure  $\mathbb{R}_{p}$  is also of interest for its relatively low empectation. To find an ardet empression for the empectation  $f_{2}(n)$  we return to step 3 for odd n=2k+1 and compute the probabilities associated with the tree for  $\mathbb{R}_{p}$  in Section 2. The total number of equalit likely cases after step 2 is

(3.27) 
$$\binom{k}{2}\binom{6}{2}\dots\binom{2k-k}{2}(2k-3)(n-2)n = \frac{(2n+1)}{k(k+1)2!} = 3(say).$$

Fast the finist companison (after stay 2) these are splitt into

(3.28) 
$$C_1 = {\binom{4}{2}} \dots {\binom{2k-4}{2}} (2k-3)(n-3)(n-1)$$
 and  $C - C_1$ 

cases for the left and right fork, respectively, thus yielding the probabilities

(3.29) 
$$p_1 = \frac{(n-3)(n-1)}{n(n-2)}$$
 and  $p_2 = \frac{2n-3}{n(n-2)}$ .

Similarly the two probabilities for the one remaining fork in our tree for  $R_{_{\rm F}}$  (in Section 2) are easily computed to be

(3.30) 
$$p_{21} = \frac{n-1}{2n-3}$$
 and  $p_{22} = \frac{n-2}{2n-3}$ .

Hence the expectation associated with step 3 for n odd is

(3.31) 
$$2(p_1 + p_2 p_{22}) + 3p_2 p_{21} = 2 + \frac{n-1}{n(n-2)} < 3$$

instead of the 3 used in the 2<sup>nd</sup> equation of (3.20). Thus we have to substract  $1 - (n_i-1)/n_i(n_i-2)$  for each odd integer  $n_i \ge 3$  that appears in the sequence n, [n/2], [n/4],...; suppose there are t such integers  $n_1, n_2, \ldots, n_t$ . Then our result is

(3.32) 
$$f_2(n) = \overline{f}_2(n) - t + \sum_{i=1}^{t} \frac{n_i - 1}{n_i(n_i - 2)}$$
,

where  $\overline{f}_2(n)$  is given by (3.56).

Although it is not proved that  $f_2(n)$  is strictly increasing in n, this does appear to be true by the table in Section 2 and by specific calculations for n = 15, 16, 62, 63, and 64. In particular, we note that for  $n = 2^r$  we obtain from (3.32)

(3.33) 
$$f_2(2^r) = \overline{f}_2(2^r) = 2^r + r - 2.$$

It has not been proved that  $2^r + r - 2$  is a lower bound for  $E\{T \mid n = 2^r\}$  for all procedures, but this is conjectured to be true. It is not too

$$(5.23)$$
  $c_1 = {h \choose 2} \dots {2k-h \choose 2} (2k-3)(n-3)(n-1) \text{ and } c_1 = c_1$ 

cases, for the left and right forth, respectively, thus gielding the gropabilities

(5.29) 
$$p_{\chi} = \frac{(n-3)(n-1)}{n(n-2)}$$
 and  $p_{\chi} = \frac{2n-g}{n(n-2)}$ .

Similarly the two probabilitiess for the one recaining fork in our treafor  $\mathbb{R}_{\frac{n}{n}}$  (in Section 2) are easily computed to be

(30) 
$$F_{21} = \frac{n-1}{2n-5}$$
 and  $F_{22} = \frac{n-2}{2n-5}$ .

Hence the expectation associated with step 3 for n old is

$$(\text{S.SI}) \qquad 2(p_{\frac{1}{\lambda}}+p_{2}p_{22}) + 2p_{21} = 2 + \frac{n-1}{n(n-2)} < 3,$$

Instead of the 3 used in the  $2^{10}$  equation of (1.23). Thus we have to substract  $1 - (n-1)/n_4(n-2)$  for each old integer  $n_4 \ge -$  that appears in the sequence  $n_1 = (n/2)$ , [n/2], [n/2], ...; suppose there are the sub-integers  $n_1 + n_2 + \cdots + n_k$ . That our result is

(1.2) 
$$\varepsilon_2(n) = \overline{\varepsilon}_2(n) - \varepsilon + \frac{\varepsilon}{z_1} \frac{n_2-1}{n_2(n_2-2)}$$

where  $\tilde{x}_{g}(\tilde{n})$  is given by (3.50).

Although it is not proved that  $\hat{x}_{2}(u)$  is strictly increasing in u, this does appear to be true by the table in section 2 and by specific calculations for  $u=1\beta$ , io, 2, 6, and 3. In particular, we note that for  $u=2^{v}$  we obtain from (3.2)

$$(3.33)$$
  $\tilde{\epsilon}_{2}(2^{x}) = \bar{\epsilon}_{2}(2^{x}) = 2^{x} + x - 2.$ 

It has not been proved that  $2^T + r - 2$  is a lower bound for  $\mathbb{R}[T] n \neq 2^T$ . For all procedures, but take is conjectured to be true. It is not too

difficult to show that this lower bound holds among all procedures in certain classes (e.g., the class with the property that the best one is selected in the first n-1 comparisons) but the general result is still outstanding.

It can be shown that the procedure  $R_F$  is the best one given that the first two steps of  $R_F$  are to be used, namely ordinary pairing and (semi-) induction on the winners; such results are considered by Hadian [10]. F. Let  $f_6(n) = E\{T|R_E\}$  for t=2. For the entropy procedures we have no exact formulas for all n and hence less complete results. The major evidence of the efficiency of these procedures lies in the numerical results and comparisons. We describe in some detail the procedure  $R_E$  for n=6. The table in Section 2 shows that for  $n\leq 10$  our best results are consistently obtained by one of the three entropy procedures. In particular  $R_E$  appears to be the best of all.

Without exact formulas we cannot prove that the expectation under  $R_E$  has the same asymptotic form  $n+\log n$  as under procedure  $R_{1}^*$  but this is conjectured to be true. In the next section we derive lower bounds for the expectation under  $R_{E}$  and  $R_{E_{2}}$ . In the table in Section 2, there are given values of  $n-2+\frac{1}{2}[2\log n]$ , which is conjectured to be a lower bound for all procedures for t=2.

For n=6 we now illustrate in detail one step in the calculations for  $R_E$ . It was previously found that the procedure tests 1 vs. 2 and (for all  $n \ge 4$ ) then 3 vs. 4 and then (assuming even numbers are the winners) 2 vs. 4. After this a complete pairing (defined below) procedure tests 5 vs. 6 and as shown in the next section this reduces the entropy by 2(2n-3)/n(n-1), which equals 6/10 for n=6. We wish to show that this tests is not used by  $R_E$ , since the test 5 vs. 2 gives

difficult to show that this lower bound holds among all procedures in certain classes (e.g., the class with the property that the best one is selected in the first n - 1 comparisons) but the general result is still outstanding.

At can be shown that the procedure  $\mathbb{R}_{R}$  is the best one given that the first two steps of  $\mathbb{R}_{R}$  are to be used, namely ordinary pairing and (semi-) induction on the winners; such results are considered by Redian (10]. W. Let  $\mathcal{E}_{G}(\mathbf{n}) = \mathcal{E}[\mathbb{T}_{|R|}]$  for  $\mathbf{t} = 2$ . For the entropy procedures we have no exact formulas for all  $\mathbf{n}$  which hence less complete results. The najor evidence of the efficiency of these procedures likes in the numerical results and comparisons. We describe in some detail the procedure  $\mathbb{R}_{R}$  for  $\mathbf{n} = 6$ . The table in section 2 slows that for  $\mathbf{n} \leq 10$  our best results are consistently obtained by one of the fluxes entropy procedures. In particular  $\mathbb{R}_{R}$  appears to be the best of all.

Without exact forwhas we sample prove that the expectation which  $R_{g_{g_{1}}}$  has the same cayaptetic form  $n+\log n$  as under procedure  $R_{g_{2}}$  and the confectured to be true. In the next seption we derive hower bounds for the expectation moder  $R_{g_{1}}$  and  $R_{g_{2}}$ . In the table in dection 2, there are given values of  $n-2+\sqrt[3]{2}$  for n, which is conjectured to be a lower bound for all procedures for 6=2.

For n=0 we now illustrate in detail one step in the calculations for  $R_{2}$ . It was previously found that the procedure tests di  $\tau s$ , 2 and (for all  $n \geq h$ ) then 3 vs. 4 and then (assuming even numbers are the winners) 2 vs. 4. After this a complete paining (defined below) frobaltre tests 3 vs. 6 and as shown in the next section this reduces the entropy by 2(2n-3)/n(n-1), which equals 3/n = 3. But will to show that this tests not used by  $R_{2}$ , since the test 3 vs. 2 gives

a larger reduction in entropy. The expected uncertainty  $E\{U\}$  after 2 vs. 4 (assuming 4&2 are the winners) is easily shown by direct calculation, or by (4.7) and (4.8) below, to be

(3.34) 
$$E\{U\} = \log 30 - \frac{32}{15} = 2.773...$$

The probability that 5 loses to 2 (resp., wins over 2) at this stage after 4 beats, 2 is easily seen to be 8/15 (resp., 7/15).

If 5 loses to 2 then we are left with the following sets of possible (true) states of nature:

1 subset (called  $D_2^{4}$ ) with 24 cases,

3 subsets (called  $D_3^{l_1}$ ,  $D_{l_2}^{l_3}$ ,  $D_{l_4}^{l_5}$ ) with 8 cases each.

The total number of cases for the left fork is 48.

If 5 wins over 2 then we are left with the cases:

2 subsets (called  $D_{4}^{5}$ ,  $D_{5}^{4}$ ) with 12 cases each,

3 subsets (called  $D_3^4$ ,  $D_4^6$  and  $D_6^4$ ) with 4 cases each,

2 subsets (called  $D_5^6$  and  $D_6^5$ ) with 3 cases each.

The total number of cases for the right fork is 42. Here  $D_{\bf i}^{\bf j}$  indicates the possible decision that  $\bf j$  is best and  $\bf i$  is second best. Hence the expected uncertainty after  $\bf 5$  vs.  $\bf 2$  is

(3.35) 
$$\mathbb{E}\{\mathbb{U}\} = \frac{8}{15} \{\frac{1}{2} \log 2 + \frac{1}{2} \log 6\} + \frac{7}{15} \{\frac{4}{7} \log \frac{7}{2} + \frac{2}{7} \log \frac{21}{2} + \frac{1}{7} \log 14\}$$

$$= \frac{1}{5} + \frac{2}{5} \log 3 + \frac{7}{15} \log 7 = 2.144.$$

Hence the reduction in entropy is the difference 2.773 - 2.144 = 0.629, which is greater than the reduction 0.6 obtained by the test 5 vs. 6. This result only held for n = 6 and in fact 5 vs. 6 gives a bigger reduction in entropy for all  $n \ge 7$ .

The final tree obtained for n = 6 under  $R_{\rm F}$  is:

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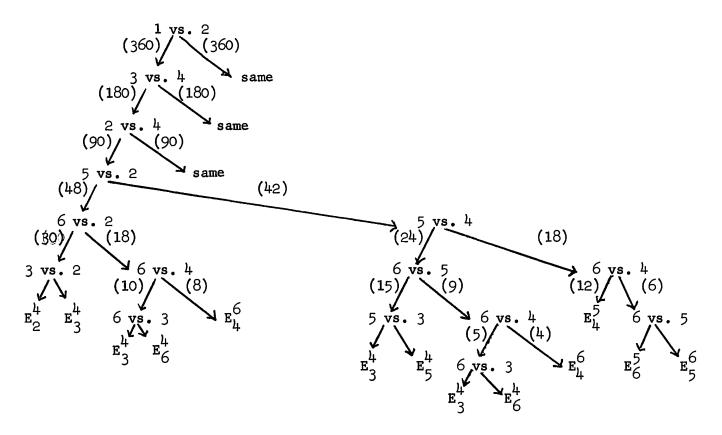
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2.11.1 5.17.5 क्रांत साहाजकुत है है है। वे सम्बद्धांत्रा है reduceron

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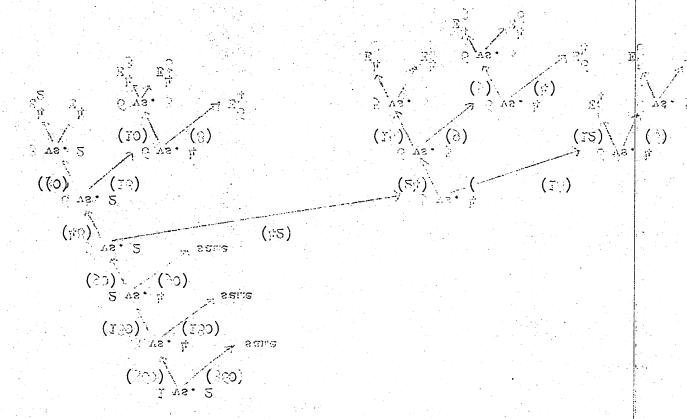
F<sub>S</sub> render ្រ ៕ ដ The final tree obtained for



Here the word 'same' indicates a repetition of the corresponding left fork. The numbers in parentheses show the partition of the original 6! = 720 cases or states of nature and are useful in computing the expectation. The symbol  $E_{\bf i}^{\bf j}$  indicates an endpoint where the decision  $D_{\bf i}^{\bf j}$ , that  $\bf j$  is best and  $\bf i$  is second best, is made.

No other procedure was found that had a smaller expectation for n = 6 but three of our procedures have a maximum length of 7.

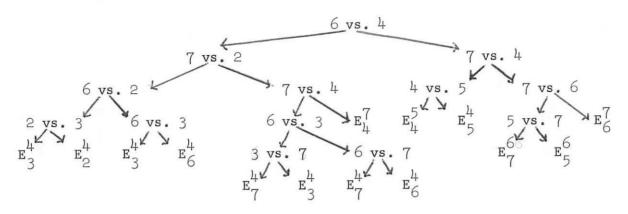
For  $n \le 4$  the entropy procedures are the optimal procedures in common with 3 of the other procedures. For n=5 they coincide with the procedures  $R_F$  giving an expectation of  $5\frac{4}{15}$  and a maximum length of 6. For n=7 we use complete pairing, i.e., 1 vs. 2, 3 vs. 4, 2 vs. 4 and 5 vs. 6. By our convention the number with the higher power of 2 is the winner. The rest of  $R_E$  (as well as  $R_E$  and  $R_E$ ) is given by:



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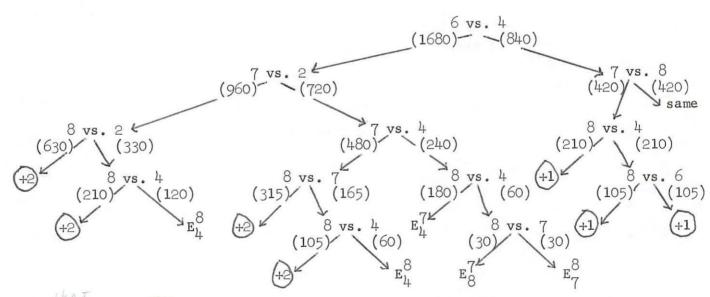
For  $n \le 4$  the entropy procedures are the optimal procedures in common with  $\beta$  of the other procedures. For  $n = \beta$  they coincide with the procedures  $R_F$  giving an expectation of  $\beta \frac{k_F}{15}$  and a canimum length of  $\beta$ . For  $n = \beta$  we use complete pairing, i.e., lars,  $2, \beta$  vs.  $k_F$  2 vs.  $k_F$  and  $\beta$  vs.  $\beta$ . By our convention the number with the higher power of 2 is the winner. The rest of  $R_F$  (as well as  $R_F$  and  $R_F$ ) is given by:



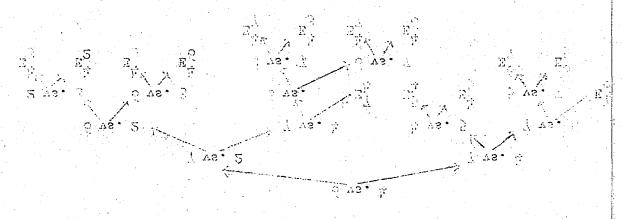
Here the expectation is  $4 + 3\frac{17}{21} = 7\frac{17}{21}$  and the maximum length is 4 + 5 = 9.

It is interesting to note that  $R_E$  tests 6 vs. 4 (after 1 vs. 2, 3 vs. 4, 2 vs. 4 and 5 vs. 6) for all  $n \geq 7$  whereas  $R_{E_1}$  and  $R_{E_2}$  both test 7 vs. 8 (and then 6 vs. 8 and then 4 vs. 8) for all  $n \geq 8$ . Hence for  $n \geq 8$  the procedure  $R_E$  differs from both  $R_{E_1}$  and  $R_{E_2}$  and is more difficult to obtain.

For n=8 the continuation for procedure  $R_E$  (after 1 vs. 2, 3 vs. 4, 2 vs. 4 and 5 vs. 6) was found to be:



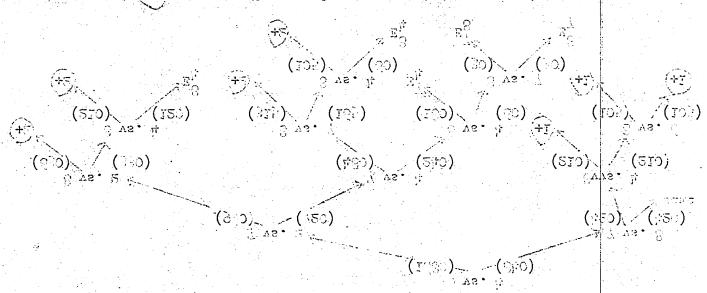
Here that symbol (+j) denotes the exactly j more obvious comparisons are needed to complete the procedure. In this instance the expectation is readily computed to be  $9\frac{1}{168}$  and the maximum length is 11.



Here the expectation is  $\frac{1}{2} + \frac{17}{21} = 7 \frac{17}{21}$  and the variance length is  $\frac{1}{4} + \frac{1}{6} = 9$ .

It is interesting to note that  $R_{ij}$  tests 0 vs. 4 (after 1 vs. 2, 3 vs. 4, 2 vs. 4 and 5 vs. 3) for all  $n \ge 7$  whereas  $R_{ij}$  and  $E_{ij}$  both test 7 vs. 3. (and then 6 vs. 3 and then 4 vs. 3) for all  $n \ge 3$ . Nonce for  $n \ge 3$  the procedure  $R_{ij}$  differs from both  $R_{ij}$  and is note difficult to obtain.

For n = 3 the continuation for procedure  $R_3$  (after 1 vs. 2; rs. 4; 2 vs. 4; and 1 vs. 6) was found to be:



Here the symbol (+i) denotes the enactly j more obvious comparisons are needed to complete the procedure. In this instance the expectation is readily computed to be  $9\frac{1}{100}$  and the maximum length is 11.

For n=8 the procedure  $R_{E_1}$  (which is the same as  $R_{E_2}$  for  $n=2^r$ , any integer r) only requires 9 comparisons on the average and has a maximum length of 10. It is conjectured that procedure  $R_{E_1}$  will continue to be as good or better than  $R_{E_1}$  for all larger values of n.

## 4. Cycle-Pairing, Complete Pairing and Ordinary Pairing.

Ordinary pairing means of course that k comparisons are made when n=2k or 2k+1. A knock-out tournament for getting the best player when  $n=2^r$  consists of ordinary pairing of all those players that won in the previous round. Hence the number of rounds is r and the total number of comparisons is n-1. To define complete and cycle pairing, we make use of the

<u>Lemma</u>: If the highest power of 2 that factors into n! is p, i.e.,  $n! = 2^p(2c+1)$  with  $c \ge 0$  an integer, and the integer s is defined by writing n in binary notation as

(4.1) 
$$n = 2^{r_1} + 2^{r_2} + ... + 2^{r_s},$$

where  $r_1 > r_2 > \dots > r_s \ge 0$ , then

(4.2) 
$$p = n - s = \sum_{i=1}^{s} {r_i \choose 2^i - 1} = \sum_{j=1}^{r_1} \left[ \frac{n}{2^j} \right].$$

<u>Proof:</u> Using induction on n, the inference from n to n + 1 for even n is obvious since p is not changed and n (resp., s) increases (resp., decreases) by one. If n is odd then  $r_s = 0$ . Suppose n + 1 replaces  $1 + 2 + \ldots + 2^j$  by  $2^j$ . Then p is increased by j, s is decreased by j - 1 and n is of course increased by one. Since the result also holds for n = 1, the result is proved. The proof of the last equality in (4.2) is omitted. From the first summation in (4.2) we see that p is exactly the number of comparisons needed to find the

For n=0 the procedura  $R_{\rm p}$  (which is the same as  $R_{\rm p}$  for  $n=2^{\rm p}$ , any integer r) only requires 9 comparisons on the everage and has a maximum length of 10. It is conjectived that procedure  $R_{\rm p}$  will continue to be as good or better than  $R_{\rm p}$  for all larger values of n.

## A. Oyole-Pairing, Complete Pairing and Oxdinary Pairing.

Ordinary painting seads of course that k comparisons are made when n=2lt or 2lt+1. A knock-out continuant for getting the best player when  $n=2^{lt}$  consists of ordinary painting of all those players that won in the previous round. Hence the number of rounds -x and the total number of comparisons is -x. To define confide and eyele painting, we make use of the

Lemma: If the highest power of 2, that factors into  $n_*$  is  $p_*$  i.a.,  $n_*^* = 2^p(2c+1)$  with  $c \ge 0$  in integer, and the integer is is defined by writing  $n_*$  in binary notation as

$$(4.1)$$
  $a = 2^{\frac{r_1}{4}} \cdot 2^{\frac{r_2}{4}} \cdot ... + 2^{\frac{r_3}{3}}$ 

where  $x_1 > x_2 > \dots > x_s \ge 0$ , then

(4.2) 
$$y = n - s = \frac{s}{n-1}(2^{n-1}) = \frac{1}{n-1}$$
.

Froof: Using induction on m, the inference from n to n + 1 for even n is obvious since p is not changed and n (resp., s) increases (resp., decreases) by one. If n is odd then  $r_s = 0$ . Suppose n + 1 replaces 1 + 2 + ... + 2 by 2 . Then p is increased by j, s is decreased by j - 1 and n is of course increased by one. Since the result also holds for n = 1, the result is proved. The proof of the last equality in (k.2) is omitted. From the first summation in (4.2) is see that p is exactly the number of comparisons needed to find the

best player in each of the s subsets of sizes  $2^{i}(i = 1, 2, ..., s)$ .

For complete pairing we form the s subsets defined by (4.1) and do the p comparisons needed to find the best player in each subset; this type of pairing is used in  $R_M$  and  $R_{E_2}$ . For cycle pairing we only do the  $r_1$ -1 pairings needed to find the best player in the largest subset of size  $2^{r_1}$ , as defined in (4.1). Of course for  $n=2^{r_1}$  these two concepts coincide.

Since we are conjecturing that among the E-optimal procedures there is a cycle-pairing procedure, it is of interest to let  $R_{\rm c}$  denote any cycle-pairing procedure and see what properties it has; this is the aim of the present section.

For  $n=2^r$  the cycle-pairing (as well as the complete-pairing) procedure gives us after n-1 comparisons the best player and exactly r contenders for second best. Since we need exactly r-1 further comparisons for finding the second best, it follows that for any procedure r0 with r1 with r2 with r3 with r4 comparisons

(4.3) 
$$E\{T|R_c\} = 2^r + r - 2 = \max\{T|R_c\}.$$

We now obtain a lower bound for each of the three types of pairing. If there is a cycle pairing procedure among the E-optimal procedures, then the lower bound for any  $R_{\text{C}}$  should also be a lower bound for all procedures.

We define a comparison  $C_j$  [ or  $C_j$  (a vs. b)] to be of level j if the 2 players a and b each have  $2^{j-1}-1$  inferiors, the two sets proven are disjoint, and each of the 2 players has no/superiors (j = 1, 2,...,[log n]). We want to prove a result about the reduction in entropy for any comparison of level j, regardless of where it occurs in our procedure. First take j = 1; consider  $C_1$  (a vs. b) and assume that we may or may not have some incomplete knowledge from comparisons among the remaining n-2 players.

best pluyer in each of the s subsets of sizes  $2^{-1}(i=1,2,\ldots,s)$ .

For complete putring we form the s subsets defined by (1.1) and do the p comparisons needed to find the best player in each subset; this type of pairing is used in  $\mathbb{R}_+$  and  $\mathbb{R}_+$ . For excile pairing we only do the x - 1 pairings needed to find the best player in the largest subset of size  $2^{-1}$ , as defined in (4.1). Of course for  $n = k^{-1}$  these two concepts coincide.

Ainse we are conjecturing that along the S-optimal procedures there is a cycle-pairing procedure, it is of interest to let  $\mathbb{R}_3$  denote any cycle-pairing procedure and see what properties it has; this is the aim of the present section.

For  $n = 2^n$  the office-pairing (as well as the complete-pairing)

procedure gives us after n-1 contarisons the gest player and exactly

recontenders for second best. Since we need exactly n-1 further

comparisons for dividing the second best, it follows that for any procedure  $\overline{n}_{\alpha}$  with  $n=2^{n}$ 

(3.3) 
$$\mathbb{E}[\mathbf{T}]\mathbf{R}_{\mathbf{c}}^{-1} = 2^{\mathbf{T}} + \mathbf{r} - 2 = \max_{\mathbf{c}} \{\mathbf{T}\}\mathbf{R}_{\mathbf{c}}^{-1}$$

We now obtain a lower joinn for each of the three types of paining.

If there is a cycleepaining procedure with the B-optimal procedures, then the lower bound for any R<sub>Q</sub> should also be a lower bound for all procedures. We define a commertson or of the side 
Life the 2 players a and be and heave 2 = 1 inferiors, the two sets prover being disjoint, and each of the 2 players has polisheriors (f = 1, 2,..., fog a]). The want to prove a result about the reduction in entropy for any comparison of level f, rejections of where it occurs in our procedure. First take f = 1; consider C<sub>1</sub> (a ys. 6) and assume that we say or may not have some incomplete impalage from comparisons among the repaining n-2 players.

<u>Lemma</u>: The reduction in entropy  $r_1$  (a vs. b) due to the 1st. level comparison  $C_1$  (a vs. b) is given by

(4.4) 
$$r_1 (a \text{ vs. b}) = \frac{2(2n-3)}{n(n-1)}$$
,

regardless of the knowledge previously obtained about ordering that affects only the remaining n-2 players.

common what affects and

Proof: For convenience, take a = 2k + 1 and  $b = 2k + 2 \le n$  and assume the previous knowledge concerns only players 1, 2,..., 2k. Consider any definite order, say  $1 \le 2 \le ... \le 2k$ , for these 2k players (where  $\le$  means is inferior to); the same argument holds for any such fixed order. The remaining subsets of possible states of nature corresponding to the possible decisions  $D_i^j$  (i,  $j \ge 2k$ ),  $D_i^{2k}$ ,  $D_{i}^i$  (i > 2k) and  $D_{2k-1}^{2k}$  and the number of cases (or the relative probability) for each, before the comparison  $C_1$  is made, are as follows:

(n-2k)(n-2k-1) subsets with (n-2)!/(2k)! cases in each (4.5) 2(n-2k) subsets with (n-2)!/(2k-1)! cases in each, 1 subset with (n-2)!/(2k-2)! cases in it.

The probability of each subset is simply the number of cases in it divided by the total number of possible cases; by (4.5) this total is n!/(2k)! Hence the uncertainty  $E_1\{U\}$  before making the comparison  $C_1$  is

(4.6) 
$$E_{1}\{U\} = \frac{(n-2k)(n-2k-1)}{n(n-1)} \log n(n-1) + \frac{(2n-4k)2k}{n(n-1)} \log \frac{n(n-1)}{2k}$$

$$+ \frac{2k(2k-1)}{n(n-1)} \log \frac{n(n-1)}{2k(2k-1)} = \log n(n-1) - \frac{2k(2n-2k-1)}{n(n-1)} \log 2k$$

$$- \frac{2k(2k-1)}{n(n-1)} \log (2k-1).$$

After making the comparison  $C_1$  and assuming by our convention that 2k+1 loses to 2k+2, the subsets in the first two rows of (4.5) which put 2k+1

Leans: The reduction in entropy  $x_{\frac{1}{2}}$  (a vs. b) due to the lst. Level concarison  $C_{\frac{1}{2}}$  (a.vs. b) is given by

$$(a,b) = \frac{2(2a-1)}{n(2a-1)}.$$

regandless of the imowhedge previously obtained about the ordering lof inche only the remaining n - 2 players.

Froof: For convenience, take a = 2k + 1 and  $b = 2k + 2 \le n$  and assume the previous knowledge concerns players(it. 21..., .2k. Considering quy definite order, say  $1 \le 2 \le \dots \le 2k$ ; for these 2k players (where we may as inferior to); the same argument holds for any such fined order. The remaining subsets of possible states of nature corresponding to the possible decisions of (1,  $j \ge 2k$ ),  $\sum_{i=1}^{2k} j_i (1 > 2k)$  and  $b_{2k-1}^{2k}$  and the number of cases (or the relative brobability) for each, before the comparison  $C_j$  is gade, are as follows:

(n-2k)(n-2k-1) subvats with (n-2)./(2k)! cases in each (k,j) 2(n-2k) subsets with (i+2)!/(2k-1)! cases in each, 1 subset with (n-2)!/(2k-2)! cases in it.

The probability of each subset is simply the number of cases in it divided by the total number of possible cases; by ( $\langle \cdot, \cdot \rangle$ ) this total is n!/(2k)! Hence the uncertainty  $\mathbb{E}_{\frac{1}{2}}[U]$  before making the comparison  $C_{\frac{1}{2}}$  is

$$\begin{array}{ll} (u,j) & \quad \overline{u_1} \cdot \overline{u_1} & = \frac{(u-2k)(n-2k-1)}{n(n-1)} \cdot \overline{10g_1} \cdot \overline{u(n-1)} + \frac{(2n-k\epsilon)\pi\epsilon}{n(n-1)} \cdot \overline{10} \cdot \overline{u(n-1)} \\ & \quad + \frac{2k(2k-1)}{n(n-1)} \cdot \overline{10g_2} \cdot \overline{u(n-1)} = 10g_1 \cdot \overline{u(n-1)} - \frac{2k(2n-2k-1)}{n(n-1)} \cdot \overline{u(n-1)} \\ & \quad - \frac{2k(2k-1)}{n(n-1)} \cdot \overline{10g_2} \cdot \overline{u(n-1)}. \end{array}$$

After matring the comparison  $C_{\frac{1}{2}}$  and assuming by our convention that  $2^{\frac{1}{2}+1}$  loses to  $2^{\frac{1}{2}+2}$ ; the subsets in the first two rows of  $(\frac{1}{2},\frac{1}{2})$  which put  $2^{\frac{1}{2}+1}$ 

in the first or second place have to be treated separately and we then have the five types:

2n-4k-3 subsets with (n-2)!/(2k)! cases in each,
(n-2k-2)(n-2k-3) subsets with (n-2)!/2(2k)! cases in each,

(4.7) 2 subsets with (n-2)!/(2k-1)! cases in each,
2n-4k-4 subsets with (n-2)!/2(2k-1)! cases in each,
1 subset with (n-2)!/2(2k-2)! cases in it.

From (4.7) we find that the total number of cases is n!/2(2k)! Since the complementary result, 2k+1 wins over 2k+2, gives rise to a symmetrical set of results, it follows that the expected uncertainty  $E_2\{U\}$  after the comparison  $C_1$  can now be obtained from (4.7). By straightforward algebra as in (3.35) and subtraction from (4.6), we obtain the desired result

(4.8) 
$$E_2\{U\} - E_1\{U\} = \frac{2(2n-3)}{n(n-1)}$$
.

If we average this result over various possible fixed values of 1, 2,..., 2k then we obtain the same result (4.7) for any partial knowledge about the players 1, 2,..., 2k (or any subset thereof) and this proves our result.

A similar calculation for any comparison  $C_j$  of level  $j(j=1,2,...,[\log n])$  gives the more general result

(4.9) 
$$r_j(a \text{ vs. b}) = \frac{2^{j}(2n-1-2^{j})}{n(n-1)}$$
.

Since the proof is quite similar to the lemma above, we omit the proof of (4.9) for j > 1.

For  $n=2^r+c$   $(0 \le c < 2^r)$  any cycle-pairing procedure  $R_C$  has at least  $2^{r-1}$  pairings of level 1, at least  $2^{n-2}$  pairings of level 2,..., at least 1 pairing of level r. We can assume that it has exactly these

in the fixst or second place have to be treated separately and we then have the fixe types:

2n-4n-3 subsets with (n-2), (2n); cases in each, (n-2n-2)(n-2n-3) subsets with (n+2), (2n); cases in each, (4,7) 2 subsets with (n-2), (2n-1); cases in each, (n-4n-1) subsets with (n-2), (2n-1); cases in each, (n-2), subset with (n-2), (2n-2); cases in it.

From (4.7) we find that the total number of cases is ni/2(4k); Since the complementary result, 2k+1 wins over 2k+2. gives the to a symmetrical set of results, it follows that the appeared uncertainty  $\mathbb{F}_2[0]$  after the comparison  $\mathbb{F}_1$  can pay be obtained from (4.7). By straightforward algebra as in (4.3) and subtaction in n (4.4), he obtain the desired result

$$(5,0) \qquad \mathbb{E}_{2}[0] = \mathbb{E}_{1}[0] = \frac{2(2n-1)}{n(n-1)}$$

If we average this result over various possible fixed values of 1, 2,..., 2k then we obtain the same result (4...) don any partial imoviedge about the players 1, 2,..., 2k (or any subset thereof) and this proves our result.

A similar calculation for any comparisor 3 of level [( = 1,2..., log n])

 $(3.9) v_{3}(a vs. b) = \frac{2^{\frac{3}{2}(2n-1-2^{\frac{3}{2}})}}{n(n-1)}.$ 

gives the more general result

or (4.9) For 1 > 1.

Since the proof is quite similar to the land above, we omit the proof

For  $n=2^{r}+c$  ( $0\leq c<2^{r}$ ) any cycle-pairin, procedure  $R_{d}$  has utilizate  $2^{r-1}$  pairings of level 1, at least  $2^{n-2}$  pairings of level 2..., at least 1 pairing of level r+1. We can assume that it has exactly these

numbers of pairings among the first  $2^r$ -1 comparisons. Then the reduction in entropy due to these comparisons is, using (4.9),

(4.10) 
$$Q = \sum_{j=1}^{r} \frac{2^{j}(2n-1-2^{j})2^{r-j}}{n(n-1)} = 2^{r} \{\frac{(2n-1)r-2^{r+1}+2}{n(n-1)}\}.$$

Let  $T_1=2^r-1$  denote the number of these comparisons and  $T_2$  denote the remaining, so that  $T=T_1+T_2$ . Since the total uncertainty at the outset is  $\log n(n-1)$  and 1 is an upper bound for the reduction in entropy for all steps (in particular, for those after the first  $T_1$ ), it follows that

(4.11) 
$$Q + (1 \times E\{T_2\}) \ge \log n(n-1).$$

Hence, with the help of (4.11), we obtain the desired lower bound for any cycle-pairing procedure  $R_{_{\rm C}}$ 

(4.12) 
$$E\{T|R_c\} = 2^r - 1 + E\{T_2\} \ge 2^r - 1 + \log n(n-1) - 2^r\{\frac{(2n-1)r-2^{r+1}+2}{n(n-1)}\}.$$

Of course, for  $n = 2^r$  we obtain an improvement by using (4.3).

The corresponding result for complete pairing is obtained by using (4.1) and noting that the first p=n-s comparisons consist of  $\lfloor n/2^j \rfloor$  comparisons  $C_j$  of level j ( $j=1,2,\ldots,r_1$ ). Hence we replace Q in (4.11) by

$$(4.13) Q_1 = \sum_{j=1}^{r_1} \frac{2^{j}(2n-1-2^{j})}{n(n-1)} \left[ \frac{n}{2^{j}} \right] \leq \frac{(2n-1)r_1-2^{r_1+1}}{n-1}.$$

By a similar argument to that above we have for any procedure that uses complete pairing (such as  $R_{\underline{M}}$ )

(4.14) 
$$E\{T\} \ge n - s + \log n(n-1) - \sum_{j=1}^{r_1} \frac{2^{j}(2n-1-2^{j})}{n(n-1)} \left[\frac{n}{2^{j}}\right],$$

where the sum can be bounded as in (4.13) for asymptotics; here again we get an improvement for  $n = 2^r$  by using  $2^r + r - 2$  from (4.3).

innotes of pairings a one the first  $2^{2}-1$  comparisons. Then the reduction in entropy the to these comparisons is, using (4.9),

$$(4.10) \qquad v = \frac{r^{-2}(2n-1-2^{-1})2^{r-1}}{r(n-1)} = 2^{r} \cdot \frac{(2n-1)r-2^{r+1}+2}{n(n-1)}.$$

Let  $\mathbb{R}_{\underline{1}}=2^{n}-1$  denote the number of these comparisons and  $\mathbb{R}_{\underline{2}}$  denote the remaining, so that  $\overline{\mathbf{r}}=\overline{\mathbf{r}}_{\underline{1}}+\overline{\mathbf{r}}_{\underline{2}}$ . Since the total uncertainty at the outset is log n(n-1) and 1 is an upper bound for the reduction in entropy for ell steps (in particular, for those after the first  $\overline{\mathbf{r}}_{\underline{1}}$ ), it follows that

$$(\#,11) \qquad +\left(1\times \mathbb{E}\left(\mathbb{T}_{2}\right)\right)\geq 10\cdot \mathbb{E}(\mathbb{A}-1).$$

Hence, with the below of (4.11), we obtain the desired lower bound for any cycle-parting procedure  $R_2$ 

$$(4.12) \qquad \mathbb{E}\{\mathbb{T}[\mathbb{R}_{c}] = 2^{r} - 1 + \mathbb{E}\{\mathbb{T}_{c}\} - 2^{r} - 1 +$$

Of course, for  $n=2^r$  we obtain an fabrovement by reamy (4.3).

The corresponding result, for complete pairing is obtained by using (h,1) and noting that the first p=n+s conjurtations consist of [n/2] compurisons  $\mathfrak{E}_j$  of level  $\{(i=1,2,\ldots,r_1)$ . Hence we regions in (h,11) by

$$(3,15) \qquad \frac{z_1}{z} = \frac{2 \cdot (2z_1 + 2^{\frac{1}{2}})}{z_1(z_1 - 1)} \cdot \frac{z_2}{z_1} = \frac{(2z_1 - 1)z_1 + 2^{-\frac{1}{2}} + 2}{z_1 - 1} .$$

The shallar arginest to that above we have for any processing that uses complete pairing (seen as  $x_{M}$ )

$$(4,14) \qquad \mathbb{E}[\mathbb{T}] \geq n + s + \log n(n-1) - \frac{r_1}{1+1} \frac{2(2n-1-2^{\frac{3}{2}})}{2^{\frac{3}{2}}}],$$

where the sum can be bounded as in (4.17) for asymptotics; here again we get an improvement for  $n=2^n$  by using  $2^n+\kappa-2$  from (k,.).

For ordinary pairing we use  $[\frac{n}{2}]$  pairings of level 1 only and find in a similar manner that for any procedure that uses ordinary pairing (such as  $R_F$ )

(4.15) 
$$E\{T\} \ge \log n(n-1) + \frac{(n-2)(n-3)}{n(n-1)} \left[\frac{n}{2}\right].$$

Since ordinary pairing and cycle pairing are both part of complete pairing, it follows that the lower bound in (4.14) is not less than those in (4.12) and (4.15). However since we conjecture that there is a cyclepairing procedure among those that are E-optimal, the lower bound in (4.12) is of more interest; it is given in the table in Section 2 without the improvement for  $n = 2^r$ .

Although the lower bound in (4.12) is asymptotically  $(r \to \infty)$  equal to n for  $n = 2^r$ , it should be pointed out that for  $n = 3 \times 2^{r-1}$  the asymptotic  $(r \to \infty)$  value is only  $\frac{2}{3}(n + \log n) + \mathcal{O}(1)$ .

For ordinary pairing we use  $\lceil \frac{n}{2} \rceil$  pairings of level 1 only and find in a similar manner that for any procedure that uses ordinary pairing (such as  $R_p$ )

(4.15) 
$$E\{T\} \ge \log n(n-1) + \frac{(n-2)(n-3)}{n(n-1)} [\frac{n}{2}].$$

Since ordinary pairing and cycle pairing are both part of couplete pairing; it follows that the lower bound in (4.1h) is not less than those in (4.1h) and (4.1h). However since we conjecture that there is a cycle-pairing procedure about those this are E-optimal, the lower bonns in (4.12) is of more interest; it is given in the table in Section 2 without the improvement for  $n = 2^{r}$ .

Although the lower bound in (4.12) is asymptotically  $(x \to \infty)$  equal to a for  $a = 2^r$ , it should be pointed out that for  $a = x \cdot 2^{r-1}$  the asymptotic  $(x \to \infty)$  value is only  $\frac{2}{r}(a + 100 \text{ n}) + \frac{r}{r}(1)$ .

### 5. Procedures for the Ordering Problem with t = n - 1.

1

Several procedures are introduced all of which are new, except for procedure  $R_S$  due to Steinhaus [23] and  $R_F$  due to Ford and Johnson [8]. Our main interest is in the concept of the maximum expected reduction in entropy in g steps for small positive integers g. It is shown in Section 6 that for g=1 this maximum is achieved by finding the comparison that partitions all the remaining possible states of nature (or cases) into two sets which are (as close as possible to being) equal in size. For the g-step (expected) entropy procedure we wish to make the  $2^g$  subsets (as far as possible) equal in size in the sense of maximizing  $-(p_1\log p_1 + p_2 \log p_2 + p_2) + p_3 + p_4 + p_4 + p_5 + p_5 + p_6 + p_6$  where  $p_1 = C_1/T$ ; where  $p_1 = C_1/T$  where  $p_2 = C_1/T$  is the number of cases in the  $p_1$  subset and  $p_2 = C_1/T$  is the concept of complete pairing (explained in Sections 2 and 4) also enters in all of the new procedures. The word 'expected' in referring to entropy procedures is dropped after section 5.

The procedure  $R_N$  uses the idea of inserting units into a 'main chain' and it changes the unit to be inserted when there is evidence that 'noise' is entering the procedure. The concept of noise, the criterion for noticing its presence, and its relation to the expectation are discussed in Section 6.

Procedure  $R_E^*$ : This is essentially a 1-step entropy procedure for t=n-1, i.e., it is based on finding the binary comparison that minimizes the expected reduction in entropy after one comparison. At some isolated points we allow the use of 2-step or 3-step entropy without a formalized rule explaining when the higher-step entropies will be used. Complete pairings is used for the first p comparisons.

# 5. Proce three for the Ordering Problem with the nit 1.

Envisor Representation are introduced all of which are new, exectly for procedure  $\Gamma_S$  due to Stainbrad and introduced all of which are new, exectly for a state to stain the concept of the national procedure  $\Gamma_S$  due to Ford and Johnson [8]. Our value interest is a state of the concept of the national procedure  $\Gamma_S$  in the national states for  $\Gamma_S$  in the state of the concentrating possible states of national in the concentration too neck which are (at case,) know the season which are (at close as possible to value) equal in that the factor procedure we wish to sake the  $\Gamma_S$  interest (at factor  $\Gamma_S$ ) where  $\Gamma_S$  is concentrated as a concentration of the concentration of the concentration of access to the  $\Gamma_S$  are concept as a confined to the factor  $\Gamma_S$  and concentrated of cases at the  $\Gamma_S$  and concept of complete pointing (at latter in Scottone total number of cases, find concept of complete pointing (at latter in Scottone  $\Gamma_S$  and  $\Gamma_S$ ) also entergoined in all of the procedure of season of cycles ages and  $\Gamma_S$  and concepts in all of the brackers and  $\Gamma_S$  and  $\Gamma_S$  and  $\Gamma_S$  are concepts of complete for the world concentration. The brackers is a color of cycles in the color of cycles and  $\Gamma_S$  and  $\Gamma_S$  are colored as the cycles and  $\Gamma_S$  and cycles and  $\Gamma_S$  and cycles are cycles and  $\Gamma_S$  and  $\Gamma_S$  are cycles and  $\Gamma_S$  are cycles and  $\Gamma_S$  and  $\Gamma_S$  are c

The procedure R. the size is of the first time of the first is evicance that inchest for entering the procedure. The concept of notice, for entering the procedure, The concept of notices for notices for entering the first relation to the expectation resistance in Section is

Procedure R.\*: This is essentially a 1-ster univery procedure for c = n - 1, i.e., it is besed on finding the binary communion that minimizes the errested reduction in cutropy after one comparison. All some isolated points we allow the use of a-ctop or 3-step entropy midious a fermediated while explaining that the higher-step entropies will be used. Complete ratificatings a need for the first final Priconferisons.

Procedure R<sub>E</sub>: This is a pure 1-step entropy procedure which also uses complete pairing for the first p comparisons. Higher-step entropies are used to decide between two comparisons only when the 1-step entropy reductions are equal.

Procedure R<sub>N</sub>: This procedure also uses complete pairing for the first p comparisons; this establishes a 'main chain' (denoted by the powers of 2 under our convention). After that, units are inserted in the main chain, i.e., we only compare a unit off the main chain with a unit on the main chain. We continue to try to insert the unit chosen until either it is inserted or there is evidence that noise (denoted by N) is entering the procedure (A criterion for this is given). The decision, as to which unit should be inserted and what comparison to make, is sequential and based on 1-step entropy considerations, i.e., given the present state of knowledge, we select the comparison that maximizes the expected reduction in entropy due to the next comparison only.

It should be clear from the above procedures that further improvement through the use of higher-step entropies is thought to be possible, but this requires extra computation and has not been investigated.

Procedure  $R_G$ : This procedure is based on first ordering separately the s subsets formed by complete pairing and then using the 1-step entropy criterion for merging these ordered subsets, each of size equal to a power of 2. To get something different than  $R_E$  for  $n=2^r$ , we assume that each of the two halves of size  $2^{r-1}$  must be separately ordered and then merged.

The table below shows the numerical results for these procedures and compares them with those for  $R_S$  and  $R_F$ . Important omissions from this table are the optimal procedures of C. Picard [17] for  $n \le 6$  and a procedure

9.

Freeedure Rg: This is a pure-fets; entroug procedure which also uses complete patring for the finat F companison. Migher-step entronges are used to decide between two companisons will when the i-step entropy reductions are equal.

Procedure N.; This procedure also uses complete; sixing for the Zixet F compensations; this octabilishes a 'main distin' (docudes fr the powers of thise our convention). After that, units are inserted in the rain chain, i.e., we only compare a unit off the main distin rath a finite on the main chain, i.e., continue to try to insert the unit on the main! chines it is imported or continue to try to insert the unit on the main! chines it is imported or continue to the roles disting the procedure (AA extremion for this is given), "The foliat is of to which white should be insertioned and what comparts to make, is sequential, and bread on I-sign entropy constraints that reministers the careers trated and in entropy out that reministers that animates the expected veduction in entropy out to the tenth of the comparts on that reministers the careers reduction in entropy out to that

It should be clear from the grove prodedures that further amonoment through the nee of higher-step entropies to thought to be possible, but this requires entry computation can have not been anyonething.

Expressions  $E_G$ : This proceeding is proced on there or leading seperately the enthants of the formed by complete political and then using the in-atem enthants and tender of the mention these ordered subsets, deal of the equal to a perec of 2. To get remaining different from L. for n = C, we exerted find each of the fact halves of the  $C^{r-1}$  such be definitely existed and then meaged.

The depth below chose the numberther results for these precedures and compered then with those for  $\mathbb{R}$  on  $\mathbb{R}_{p}$ . Important outsatons from this tests are the obtained precedures of  $\mathbb{R}_{p}$ . Therefore the note of an energy of the process.

## Comparison of Six Procedures for the Complete Ordering Problem

Lower Bound and		Expected Values										
Procedure	s	- 2	- 1		n=6		n=8	<b>7</b> 0	- 10			
	<b>n=</b> 2	n=3	n=4	<b>n</b> =5		n=7		n=9	n=10			
LB	1	2 <u>2</u>	4 <del>2</del>	6 <del>14</del>	926	12 118	15 118	18 <sup>1574</sup>	2111966			
$R_{E}^{*}$	1	2 <u>2</u>	42	6 <del>14</del>	926	12121	15 <sup>121</sup>	18 <sup>1592</sup>	N.C.			
$R_{\overline{N}}$	1	2 <u>2</u>	4 <u>2</u>	6 <del>14</del>	926	12-122	15 <sup>122</sup>	18 <sup>1608</sup>	N.C.			
R <sub>E</sub>	1	<u>2</u> 2	42	6 <del>14</del>	926	12123			N.C.			
$R_{ m F}$	1	2 <u>2</u>	42	6 <del>14</del>	927	12 144	15 <sup>144</sup>	18 <sup>1656</sup>	21 12060			
$^{\mathrm{R}}_{\mathrm{G}}$	1	2 <u>2</u>	42	7 <del>1</del>	9 <u>30</u>	12 150		18 <sup>1737</sup>	21 13725			
$R_S$	1	2 <sup>2</sup>	4 <u>2</u>	7 <sup>1</sup>	9 <u>33</u>	12 186	15 <sup>186</sup>	18 <sup>2304</sup>	223015			
Column Denominat (D)	or —	3	3	<b>1</b> 5	45	315	315	2835	14175			
Noise Units (NU) (Noise N = NU/D)												
R <sub>E</sub> *	0	0	0	0	0	3	3	18	N.C.			
$^{R}$ N	0	0	0	0	0	4	4	34	N.C.			
$R_{ m E}$	0	0	0	0	0	5	5	50	N.C.			
$R_{\mathbf{F}}$	0	0	0	0	1	26	26	82	94			
$^{\mathrm{R}}$ G	0	0	0	2	4	32	59	163	1759			
$R_{S}$	0	0	0	2	7	68	68	730	5224			
(Min., Max.) of the Number T of Comparisons under R												
MLB	1	3	5	7	10	13	16	19	22			
R <sub>E</sub> *	(1,1)	(2,3)	(4,5)	(6,7)	(9,10)	(11,13	(14,16	)(18,19)	N.C.			
$^{R}_{N}$	(1,1)	(2,3)	(4,5)	(6,7)	(9,10)	(11,13	(11,13)(14,16		N.C.			
$R_{E}$	(1,1)	(2,3)	(4,5)	(6,7)	(9,11)	(11,13	(14,16	(18,20)	N.C.			
$R_{\overline{F}}$	(1,1)	(2,3)	(4,5)	(6,7)	(8,10)	(10,13	(13 <b>,</b> 16	)(16,19)	(19,22)			
$R_{G}$	(1,1)	(2,3)	(4,5)	(6,8)	(8,11)	(11,14	·) <mark>(14,17</mark>	)(17,20)	(20,23)			
$_{\rm R_S}$	(1,1)	(2,3)	(4,5)	(6,8)	(8,11)	(10,14	(13,17	)(16,21)	(19,25)			

# Comparison of 18: Precedures for the Complete Ordering Problem

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. "		)(I:,e)								the same of the sa
	(19,21)	(19,61)	)(1,1	(10,14	(3,11)	(:	(3,4)	(د, ن)	(1.1)	

of Cesari [4] for n=7 which has only 3 units of noise; no rule for general n is given in their work. The lower bound LB in the table is defined by

(5.1) LB = 
$$r + \frac{2c}{n!}$$

X

X

where r and c are defined by writing  $n! = 2^r + c$  ( $0 \le c < n!$ ); this result comes from the work of Huffman [12] and was applied to this problem independently by Kislicyn [14] and through questionnaire theory by Picard [17]. The corresponding minimax lower bound MLB = 1 + [log n!] for the  $n \ge 3$  in the minimax problem was used by Ford and Johnson [8] and is also discussed by Steinhaus [25].

### 6. Properties and Proofs.

We define the 'Halving Procedure' as one which always selects a comparison that makes the resulting two sets of cases (as far as possible) equal in size. Let T denote the total number of possible states of nature at any stage and let x and y = T - x denote the partition resulting from some comparison.

Lemma 1: The halving procedure and the 1-step entropy procedure are equivalent.

Proof: The reduction in entropy at any stage is given by

(6.1) 
$$\log T - \frac{x}{T} \log x - \frac{y}{T} \log y = -(p_x \log p_x + p_y \log p_y)$$

where  $p_x = \frac{x}{T}$  and  $p_y = \frac{y}{T}$ . It is well known that that right side of (6.1) is maximized by setting  $p_x = p_y$  or x = y and this proves the result.

Of course, if we could always partition the states of nature exactly in half then we would have an optimal solution. All our difficulties arise from the fact that this halving is not always possible. On the other hand

of Cesari [4] for n=7 witch best only a units of noise; no wile for ageneral in its given in their works. The lower bound ID in the right to weathnest by

where x and x are defined by writing  $v_* = 2^n + x$  (0 is  $(n_*)$ ; this result comes from the work of Inffigure [12] and was applied to this problection incompanion by Mishitzyn [14] and through these formules theory by Frografian corresponding minimax hower found  $[43] = 1 + \{10, n_*\}$  for the n in the minimax problem was used by Ford and To ason [1] and is also discussed by Steinhard [2].

# G. Properties and Propfs.

the define the Malvin, Procedure es sur abjulations selects a comparison that rakes the resulting two sets of cases (as far as possible) equal in size, her I denote the total number of vossible states of another at any stage and let it said y = I + I denote the partition resulting from some comparisons.

Leading il: The impositing procedure and the 1-step entropy procedure are configurated.

Proof: The reduction in sutropy at any state is given by

(i.i.) Log 
$$T = \frac{2}{T}$$
 Log  $X = T$  for  $\frac{2}{T}$  and  $Y = T$  got  $Y = T$ 

where  $p_{x} = \frac{x}{2}$  and  $p_{y} = \frac{y}{2}$ . It is well impose that that there is a deck (0.1) is markwised by setting  $\hat{p}_{x} = \hat{p}_{y}$  on n = y and this propes the result.

Of course, in we could always partition the states of wether enacting in half then we would have an officeal solution. All out difficulties arise from the fact that this half is not always possible. On the other hand

it is not necessary to partition the set exactly in half to get an optimal breakdown. We now give some results about this point.

Let H(T) denote the expected number of comparisons required when there are T possible states of nature. Let x denote the smaller of the two subset sizes that result from some comparison; suppose we could choose any subset size x we wish. Then H(1) = 0 and for T > 2

(6.2) 
$$H(T) = 1 + \min_{1 \le x \le T/2} \left\{ \frac{x}{T} H(x) + \frac{(T-x)}{T} H(T-x) \right\}.$$

Let h(y) = yH(y). Then (6.2) takes the simpler form

(6.3) 
$$h(T) = T + \min_{1 \le x \le T/2} \{h(x) + h(T-x)\}.$$

Define r and c by writing  $T = 2^r + c$  where  $0 \le c < 2^r$ . It can be readily proved as in lemma 2 of [21] that the minimum in (6.3) is attained at x = T/2 and that an exact expression for H(T) for all T > 0 is

(6.4) 
$$H(T) = r + \frac{2c}{T} = r + \frac{2}{T}(T - 2^r).$$

X

The following result was found to be quite useful in searching for procedures with less noise and in particular it is used in the definition of procedure  $R_N$ . It corresponds to lemma 3 of [21] but it should be noted that because of different boundary conditions the result is completely different from that in the above-mentioned lemma.

Lemma 2: For any  $T \ge 2$  an integer y will yield the minimum in (6.3) if and only if there is no power of 2 strictly between y and T - y.

Proof: Let h(x; T) denote the sum in braces in (6.3); because of the symmetry about x = T/2, we assume  $x \le T - x$ . Consider different possible inequalities between x, T - x and the power of 2 that is closest to their average T/2.

ne is not necessary to partition the set enactly in bail to yet an optimal court and primal point. We now give some results about this point.

Let R(T) denote the expected number of compactsons required then there are T possible states of matrixe. Let x denote the smaller of the two smaller is the two smaller states that result from some compactson; suppose we could shoose any subset gize x we wast show. Then R(A) = 0 and for  $T \geq 2$ 

$$\mathbb{I}(\bar{x}) = \bar{x} + Min \quad \frac{n}{n} \mathbb{I}(x) + \frac{(n-1)}{n} \mathbb{I}(\bar{x}-z).$$

Let II(y) = yII(y). Then (5.2) takes the simpler form

(0.4) 
$$h(T) = T + \min_{1 \le x \le T/\Sigma} \{h(x) + h(T-x)\}.$$

Darline x and c Syntricing  $T = S^2 + c$  where  $N = c < S^2$ . It can be read by proved as in large 2 of [UD] that the windows in (0.5) is attained at x = T/2 and that an exact expression for T(T) for all  $T \ge 0$  is

(6.9) 
$$\mathbb{I}(T) = r + \frac{2c}{T} = r + \frac{2}{r}(T - 2^{r}).$$

The following result was found to be quite usafel in secrebing for procedure with less noise and in particular at is usafin the definition of procedure  $R_{\rm lf}$ . It corresponds to lemma and fill some it should be noted that because of different boundary conditions for result is completely different from that in the above-mentioned lemma.

Lemma 2: For any  $T \geq 2$  an integer y will yield the minimum in (0.3) if and only if there is no power of 2 strictly between y and T - y.

Proof: Let h(n; T) denote the sum in braces in (0.7); because of the symmetry about x = T/2, we assume n = T - n. Consider different possible inequalities between n = T - n and the power of 2 that is closest to their average T/2.

Case 1:  $2^{r-1} \le x \le T-x \le 2^r$ .

Then, letting r(x) denote the r-value for x, r(x) = r(T-x) = r-1 and to check the equality in (6.3) we use (6.4) and compute

(6.5) 
$$T + h(x; T) = T + (r-1)x + 2(x-2^{r-1}) + (r-1)(T-x) + 2(T-x-2^{r-1})$$
$$= rT + 2(T-2^r) = h(T).$$

Hence the minimum in (6.3) is attained for such values of x.

Case 2A:  $2^{s-1} \le x < 2^s$ ,  $2^r < T-x$  for  $1 \le s \le r$ , and

Case 2B:  $2^{s-1} \le x < 2^s$ ,  $2^r = T-x$  for  $1 \le s < r$ .

Then r(x) = s-1 and r(T-x) = r and a similar computation gives for both Cases 2A and 2B

(6.6) 
$$T + h(x; T) = T + x(s-1) + 2(x-2^{s-1}) + (T-x)r + 2(T-x-2^r)$$
$$= h(T) + (T-x-2^r) + (2^s-x)t + 2^s(2^t-t-1) > h(t),$$

since  $t = r-s \ge 0$  and (hence)  $2^t-t-1 \ge 0$ . In Case 2A (resp., Case 2B) strict inequality follows from the fact that  $T - x - 2^r > 0$  (resp.,  $(2^s-x)t > 0$ ). Hence the minimum in (6.3) cannot be achieved for such values of x.

Case 3: 
$$2^{s-1} \le x < 2^s$$
,  $2^{r-1} < T-x < 2^r$  for  $s \le r-1$ .

Here r(x) = s-1, r(T-x) = r-1 and r-s > 0. As above, we obtain

(6.7) 
$$T + H(x; T) = T + x(s-1) + 2(x-2^{s-1}) + (T-x)(r-1) + 2(T-x-2^{r-1})$$
$$= h(T) + 2^{s}(2^{t}-t-1) + (2^{s}-x)t > h(T)$$

since t = r-s > 0 and  $2^{t}-t-1 \ge 0$ . Thus the minimum in (6.3) cannot be achieved for such x values.

Then, letting x(x) denote the r-value for x, x(x) = x(x-x) = x-1 and to chack the equality in  $(0, \cdot)$  we use (0, i) and compute

(4.1) 
$$T + h(x; T) = T + (x-1)n + 2(x-2^{x-1}) + (x-1)(x-x) + 2(x-2^{x-1})$$
$$= xT + 2(T-2^x) = h(T).$$

Hence the chrimme in (5.3) is attained for such values of  $\pi$ .

Case 24:  $2^{s-1} \le x \le 2^s$ ,  $2^x : T-x$  for  $1 \le s \le r$ , and

Case 25:  $2^{s-1} \le x \le 2^s$ ,  $2^x = T-x$  for  $1 \le s \le r$ ,

Then  $v(\pi) = s-1$  and  $v(\pi-\pi) = r$  and a similar comparation gives for both Cases 21 and 23

since  $\epsilon = r-z = 0$  and (hence),  $2^{n}-\epsilon-1 \geq 0$ . In Case 25 (resp., Case 25) strict inequality follows from the fact that  $2^{n}-z = 2^{n} > 0$  (resp.,  $(z^{3}-z)\epsilon > 0$ ). Hence the nation in (0.,) cannot be achieved for such values of z.

Case 4: 
$$2^{3+1} \le \pi < 2^3$$
,  $2^{n-1} < \pi_{-n} < 2^n$  for a vet.

Here H=(x) = s-1, v(x-x) = v-1 and x-s = 0. As above, we obtain

(0.7) 
$$T + II(z; T) = T + x(s-1) + 2(x-2^{s-1}) + (T-x)(z-1) + 2(T-x-2^{t-1})$$
$$= h(T) + 2^{s}(2^{2}-z-1) + (2^{s}-x)z > h(T)$$

since t = r - s = 0 and  $2^{\frac{c}{2} + t - 1} \ge 0$ . Thus the minimum in  $(0, \cdot)$  connot be achieved for such a values.

Case 4:  $x = 2^s$ ,  $2^{r-1} < T-x < 2^r$  for  $s \ge r-2$ . Here r(x) = s, r(T-x) = r-1 and  $t = r-s \ge 2$ . As above we obtain

(6.8) 
$$T + H(x; T) = T + s2^{s} + (T-2^{s})(r-1) + 2(T-2^{s}-2^{r-1})$$
$$= h(T) + 2^{s}(2^{t}-t-1) > h(T)$$

since  $2^{t}$ -t-1 > 0 for  $t \ge 2$ ; the minimum in (6.3) is again not achieved. Since these four cases exhaust the possible relations between x, T-x and the power of 2 closest to their average T/2, the lemma is proved.

It follows from this lemma that in selecting a comparison at any stage of a procedure we can determine, by looking at the 2 resulting subset sizes (and their relation to the power of 2 closest to their average), whether or not this particular comparison is introducing an inefficiency (which we call noise) into the procedure. This is exactly the criteria that was used in the procedure  $R_N$ . It should be mentioned that lemma 2 is related to the theorem of Sandelius [18] which uses a different approach and does not get our later results.

We are also interested in the amount of noise brought into the procedure, especially when there is exactly one power of 2 strictly between the two subset sizes. For Cases 2A, 2B, 3 and 4 this corresponds to t=0, 1, 1 and 2, respectively. For Case 2A the amount added to h(T) is  $T-x-2^T$  and

(6.9) 
$$T-x-2^r < 2^s-x \text{ since } s = r \text{ and } T < 2^{r+1}.$$

For Cases 2B and 3 the amount added to h(T) is  $2^{S}-x$  and

(6.10)  $2^s - x \le T - x - 2^s$  since s = r - 1 and  $T \ge 2^r$ . For Case 4 the amount added to h(T) is  $2^s = 2^{r-2}$  and

Gase is:  $x = S^2$ ,  $S^{n-k} < x + x < S^n$  for s = x + S.

Here  $x(x) = s_1 \cdot x(x + x) = x + 1$  and  $x = x + s \ge S$ . As above denoted in

$$(5.0) \quad \mathbf{T} + \mathbf{H}(\mathbf{x}; \, \mathbf{T}) = \mathbf{T} + \mathbf{s}^2 + (\mathbf{T} + \mathbf{2}^3)(\mathbf{v} - \mathbf{1}) + 2(\mathbf{T} + \mathbf{2}^3 + \mathbf{2}^{\mathbf{T} + \mathbf{1}})$$

$$= \mathbf{v}(\mathbf{T}) + 2^3(2^2 - \mathbf{v} - \mathbf{1}) > \mathbf{h}(\mathbf{T})$$

since  $2^{\epsilon}$ - $\epsilon$ -1>0 for  $\epsilon \geq 2$ ; the diminute in  $(\cdot, \varepsilon)$  is again not achieved, since these four cases enhance the possible relativity between x; T- $\varepsilon$  and the power of 2 closert to their average 1/2, the lemma is proved.

It follows from this lesses that in selecting a comparison at any stage of a procedure we can determine by looking at the 2 resulting subset sizes (and their refation to the power of 2 closest to their average), whether or not this particular comparison is introducing an inedital ency (which we call noise) into the procedure. This is startly the printer that a subset is the crossion of Sandelins [1,] which uses a different approach and does not get unitaries results.

The axe also inferested in the imports of notes for the into the procedure, especially when their is exactly one power of R othership between the two subset sizes, both Cases 22 Atin 23 and able to be corresponds to t = 0, t, t and t, respectively. For Case 2A the anomic alded to f(t) is f(t) is f(t) = and

For Cases 2B and 3 the thought added to  ${
m b}(T)$  is  $2^3-\pi$  and

$$(0.10)$$
  $S^{5}$ - $\pi \le T$ - $\pi$ - $S^{5}$  since  $s = \pi$ - $1$  and  $T \ge S^{2}$ . For Caus  $h$  the assumbtadded to  $h(T)$  is  $S^{3} = S^{T+2}$  and

(6.11) 
$$2^{r-2} < T-x-2^{r-1}$$
 since  $x = 2^{r-2}$  and  $T > 2^{r-1}$ .

Hence we have proved the following

<u>Lemma 3</u>: The noise N due to a comparison with exactly one power of 2 strictly between the subset sizes,  $T_1 < 2^a < T_2 = T - T_1$ , is simply the minimum distance to this power of 2, i.e.,

(6.12) 
$$N = Min(2^a - T_1, T_2 - 2^a).$$

The contribution of this noise N to the expectation if we start with T cases is then N/T; if we start with any larger number D of cases (D > T) then this contribution is to be multiplied by the probability T/D of entering this part of the tree. Hence the overall contribution to the expectation for this arbitrary comparison is N/D. This latter result which we just proved can be regarded as a corollary to lemma 3, but its usefulness is such that we prefer to write it as a theorem below. Let the noisy nodes of a tree have noises  $N_1, N_2, \ldots, N_w$ ; we call a noisy node simple if the two subset sizes obtained by that comparison have exactly one power of 2 between them. The common expected value of any noiseless tree (i.e., one with no noisy nodes) that starts with n possible states of nature is H(n). Then we have the Theorem: For any procedure R which has only noiseless nodes and simple noisy nodes the expectation is given by

(6.13) 
$$E\{T|R\} = H(n) + \frac{1}{n} \sum_{i=1}^{w} N_i,$$

where H(n) is given in (6.4) and the  $N_i$  are given by (6.12). This result enables one to keep track of the expectation of a procedure (or the expected length of the tree) while the procedure is still being constructed. Clearly it is quite useful in searching for the existence or non-existence of noiseless trees. It was used for most of our computations in the table

(5.11) 
$$2^{r-2} \le T-x-2^{r-1}$$
 since  $x = 2^{r-2}$  and  $T > 2^{r-1}$ .

Lenne we have proved the following Lenne : The noise H due to a companison with exactly one power of

Lemma 3: The noise H due to a comparison with exactly one power of 2 structly between the subset sizes,  $x_1 < 2^a - x_2 = x - x_1$ , is simply the minimum distance to this power of 2, i.e.,

(5.12) 
$$\mathbf{W} = \mathrm{Min}(\mathbf{2}^{\mathbf{a}} - \mathbf{T}_{1}, \ \mathbf{T}_{2} - \mathbf{2}^{\mathbf{a}}).$$

The contribution of this roise N to the expectation if we start with T cases is then N/T; if we start with any larger number P of cases (D>T) then this contribution is to be well third by the probability T/D of entering this part of the true. Hence the overall contribution to the expectation for this antitiorary comparison is N/D.

This latter result which we just now admen be regarded as a conditary to below. Let the massimate N but its assimilars is such that we prefet to write it as a chaore, below. Let the nowy nodes of a tree bave roines  $N_1, N_2, \ldots, N_r$ ; we call a notey node timple if the two spicest sizes obtained by that doracison have exactly one power of a detream that. The domain ampended value of any noiseless thus (1.4.1) one with no noisy nodes) that starts with a possible states of native is N(n). Then we have the

$$(6.13) \qquad \mathbb{E}\{\mathbf{r}[R] = \mathbf{H}(n) + \frac{n}{n} \sum_{i=1}^{N} \mathbb{R}_{i},$$

notes the expectation to given by

where \$\tilde{a}(a)\$ is given in (0.4) and the \$\tilde{a}\$, are given by (0.12). This result enables one to keep track of the emportation of a processor. (or the expected length of the tree) while the procedure is still bring constances. Obtainly it is quite useful in scarching for the existence or non-emistence of noiseless trues. It was used for most of our computations in the table.

above and also in the footnotes below the procedures listed below.

The above analysis is of general interest for our search problem and is not to be associated only with the entropy procedures. For example, the formula in (6.4) also applies to the Steinhaus procedure  $R_S$ . Since the Steinhaus procedure makes the individual insertions without noise, it follows that H(i) is the expected number of comparisons necessary to insert an item into a chain of length i. It easily follows, using (6.4), that the expectation under  $R_S$  for n units with  $2^r \le n < 2^{r+1}$  is given by

(6.14) 
$$E\{T | R_S\} = \sum_{i=2}^{n} H(i) = r(n+1) + 2(n-2^r) - \sum_{j=2}^{n} \frac{2^{1+[\log j]}}{j}$$

A similar expression was obtained by Trybuła (personal communication); the asymptotic properties have been investigated by Kislicyn [14] and Hadian [10].

The procedure  $R_F$  was defined in [8] and developed by means of separate recursion formulas for odd and even values of n which involve complicated sums; no explicit expression for the minimax integer U(n) under  $R_F$  was given. A single explicit expression for U(n) for all  $n \geq 1$  is

(6.15) 
$$U(n) = j(n + \frac{1}{2}) - \frac{1}{3}(2^{j}-1) - \frac{1}{6}(\frac{1+(-1)^{j+1}}{2})$$

where  $j = [\log(\frac{3n+2}{2})]$ . This form also has the advantage that it quickly gives an asymptotic  $(n \to \infty)$  evaluation for U(n), namely

(6.16) 
$$U(n) = jn - \frac{2^{j+2}}{3} + \frac{1}{2} \log n + O(1)$$

where j = j(n) is defined above. The results (6.14) and (6.15) are derived by Hadian in [10].

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The above analysis is of general interest for our agency included on the notice of the figure of acceptable of (x, b) also applies to the designant procedure is since the obtaining procedure is since the obtaining procedure is a since the obtaining procedure is a since the obtaining procedure is a contract of the fallows that if (x, b) is the expected attribute of respective in their factors of the first since (x, b) is the expect of the first since (x, b).

$$(5;24) \qquad \mathbb{E}[x][x] = (4;x) = x(x) + c(x - x) + c(x - x) = x(x) $

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consider a converse designates for eac and even rations of a relation through the constitution of the constant  $\mathcal{U}(n)$  consider  $\mathcal{U}_{\mathbf{i}}$  was given. A single original representation for  $\mathcal{U}(n)$  for all n

$$(2.25) 1(2) = 2(n + \frac{5}{4}) - \frac{2}{3}(5, -1) - \frac{5}{3}(\frac{1}{4}(-\frac{7}{4})_{\frac{1}{2}+\frac{7}{4}})$$

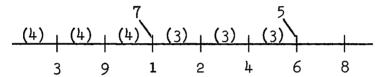
where  $f = [Log(\frac{-n+c}{p})]_*$  that four also accoming a vertical form of particular to a constant of the covening of the following form of particular to the covening of th

$$(5.15) \quad V(n) = 3n - \frac{3^{3+2}}{2} + \frac{1}{2} \log n + O(1)$$

Finise f = f(n) as fertimes theorem who re-under (2.11) and (2.11) and (2.11).

### 7. Remarks about the Table and the Trees.

The trees below represent only a small sample of the trees constructed for the table in Section 5. Only the more involved trees with the most novel results are given. No tree was found that gives better (i.e., quieter) results than the modified entropy procedure  $R_{\rm E}$ . However there is reason to believe that higher-step entropy procedures may improve some of our results. This is based on the fact that in several situations that arise the 2-step entropy is a clear improvement on the 1-step entropy; we give one illustration that arises under  $R_{\rm E}$  for n=9. After 7 comparisons one of the nodes of the tree has associated with it 21 possible states of nature which we represent by the diagram:



The slanting lines indicate that 5 belongs somewhere below 6 and 7 belongs somewhere below 1. If we insert 5 first it has 6 spaces in which to go and the number of cases (or relative probability) for each is shown by the number in parentheses. The 1-step entropy procedure requires that we compare 5 with 1 to obtain the (12, 9) split rather than 5 vs. 9 which gives a (13, 8) split. However the 2-step entropy procedure compares the four-way split (6, 6, 3, 6) (which has a unit of noise) for the former start with the four-way split (4, 4, 7, 6) for the latter start. The (4, 4, 7, 6) split is preferred under 2-step entropy since its 2-step reduction in entropy is

(7.1) 
$$\frac{8}{21} \log \frac{21}{4} + \frac{7}{21} \log \frac{21}{7} + \frac{6}{21} \log \frac{21}{6} = 1.957...,$$

compared to 1.952... for the (6, 6, 3, 6) split.

## V. Reserve about the Tabie and the Twees.

Who beets delow represent only a mail sample of the trees constructed for the table in Section 5. Only the more involved trees with the ost novel results are given. No tree was found that the detter (t.e., quieter) results than the wolfished entropy procedure R., Nortver that is reason to believe that higher-step entropy procedures may instone some of our results. This is based on the fact that in several situations that are set the 2-step entropy is a clear improvement on the 1-step entropy; we give one illustration that arises under R. for n = 9. After 7 contrations one of the nodes of the tree has associated with it 21 possible partisons one of the nodes of the tree has associated with it 21 possible states of nature which we have also the the tree has associated with it 21 possible states of nature which we appresent of the tree has associated with it 21 possible states of nature which we appresent of the tree has associated with it 21 possible.

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companed to 1.372... for the (6, 6, 6, 6) split.

The procedure  $R_{\widetilde{N}}$  represents an attempt to use our above results about noise in the construction of a procedure and the results are quite good. In fact the procedure  $R_{\widetilde{N}}$  appears to be better than the 1-step entropy procedure  $R_{\widetilde{E}}$  but not as good as the modified entropy procedure  $R_{\widetilde{E}}$ .

The symbol S in our tree denotes a branch that is symmetrical to or equivalent to another branch to its left, which is further developed. The symbol H, with the integer j on the last arrow leading to it, means that the concluding steps starting from this point are obvious noiseless insertions that requires an additional expected number H(j) of comparisons (starting at that node and including it in the count). The symbol  $H_1$  indicates that the remaining steps are not insertions but they are still obvious and noiseless so that the same result (6.4) applies; we can regard the H's and  $H_1$ 's as equivalent. The circled integers between the two forks of a noisy node is the number of noise units at that node.

It appears to be true that no noise can arise at a node that corresponds to a total of eight or fewer cases (i.e., states of nature) but this has not been proved.

None of the procedures used contained any noisy nodes that were not simple.

Each of the trees below starts after the p pairings associated with complete pairing; here p is the highest power of 2 that factors into n! Hence the total number of cases (or states of nature) at the top of the tree is  $D = n!/2^p$ , which is the common denominator in the table in Section 5.

Since there are 3 noise units the expectation for n=7 under  $R_E^*$  is  $4 + H(D) + (3/D) = 12 \frac{121}{315} = 12.384...$  For n=8 the procedure  $R_E^*$  is exactly the same except for 3 extra pairings (7 vs. 8, 6 vs. 8, and 4 vs. 8) at the outset. Hence the expectation under  $R_E^*$  for n=8 is

- ग्रेट एक स्टेडिंग अस्त (राज्य क्षेत्र) है । <u>ခုသည်စပ်ပုံသူ</u> ्डड उन्हें ತನ್ನು ಬರುವು ಸತಿವಿಧಿಕರ ತರು ಭವ್ಯಕ್ಷಣಕ್ಕೆ 003 ್ರಾಣ್ಯ ಸ್ವಾದ್ಯಾಣ್ಣ ಶ engitold. 17. 17. 00 Civa the modified ಶಿಷ್ಣಾಭಾತ a brogagnia ಸ್ತೋಂಥಲಕ mar same serger भूष अध्य सम्ब 01 [] टेल्वद to consecustages le sampeone one designation es uses 71 30% **≟०७**सर्≅ e E Signature &

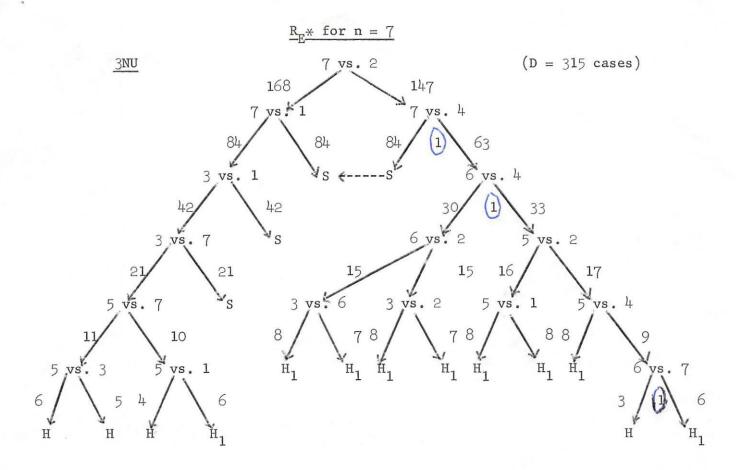
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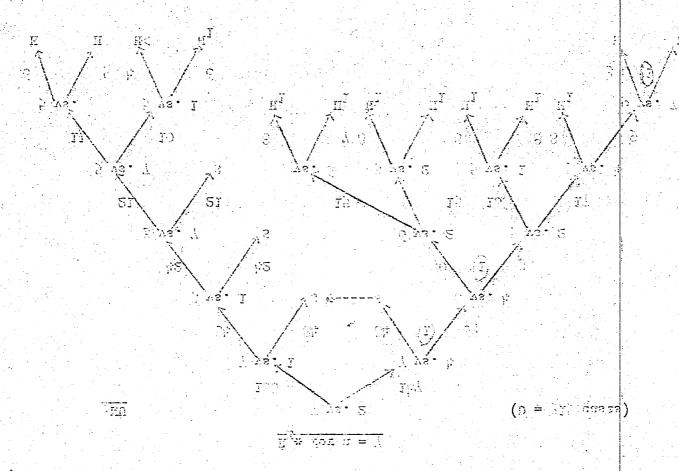
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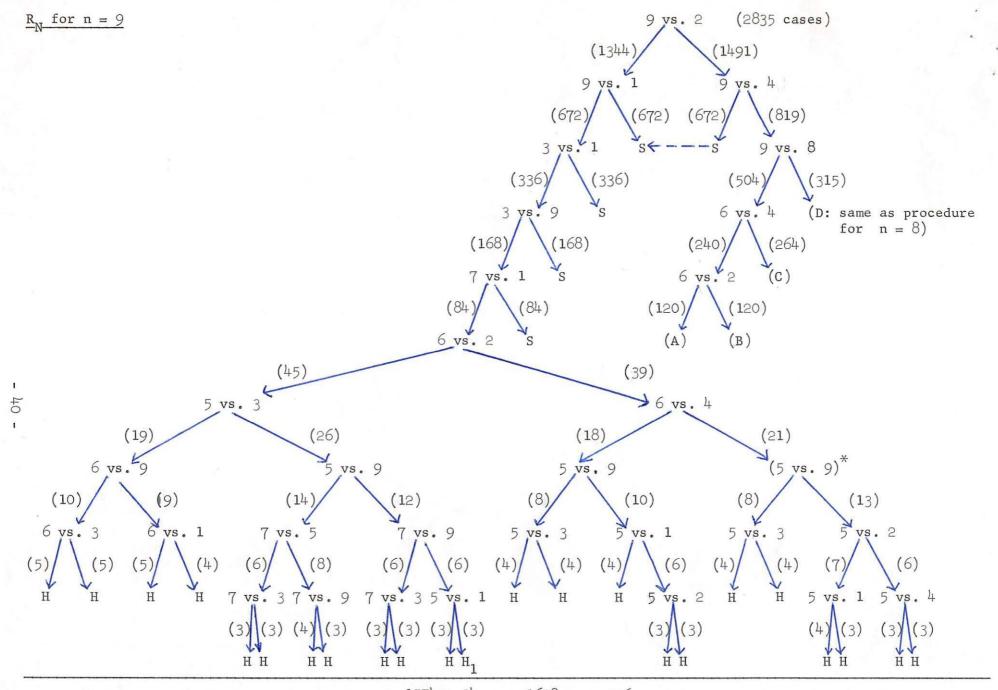
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 $15\frac{121}{315}=15.384...$  It is conjectured that these are the best possible results for n=7 and 8 but this has yet to be proved. Cesari [4] has shown that no noiseless procedure exists for n=7. With the aid of our results above one could try to show that no procedure with NU < 3 exists, but this has not been attempted.



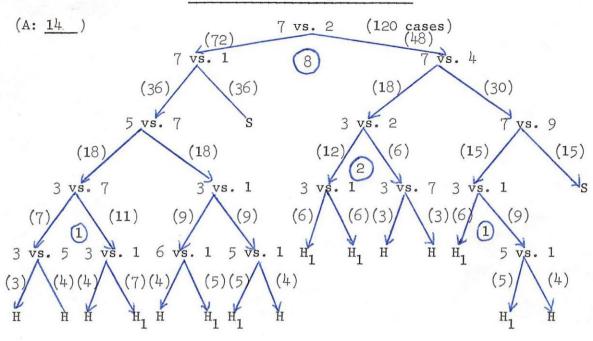
If  $\frac{121}{12n} = 15$ ,  $\frac{3h}{12n}$ , ... It is conjectured that these are the bast possible results for n=7 and  $\frac{3h}{12n}$  the bast possible results for n=7 and  $\frac{3h}{12n}$  that this has yet to be growed. Geometrially, as always about that no no select procedure exists for n=7, with the and off or results above one could try to show that no procedure with  $\overline{MU} = 1$  exists, but this has not been attentived.

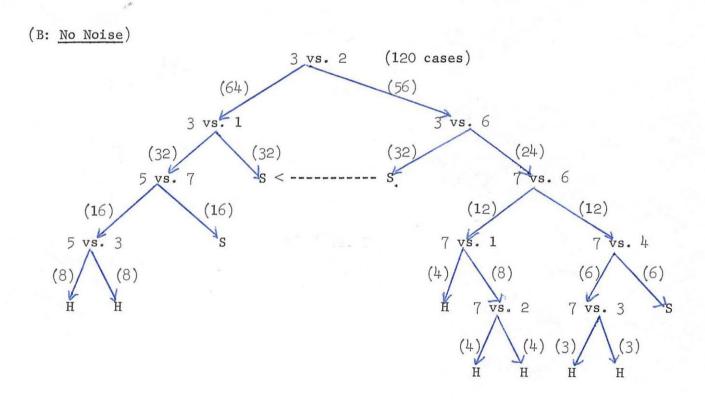


Total Noise is 34 N U and hence  $E\{T|R_N\} = 18 \frac{1574 + 34}{D} = 18 \frac{1608}{2835} = 18 \frac{536}{945} = 18.567...$  Two-step entropy was used only at (\*).

本 8.6 m (大崎東) 名 tong Dood Look U W 4 に sector LaSoT .(\*) is vino bosu The state definition of the state of the sta

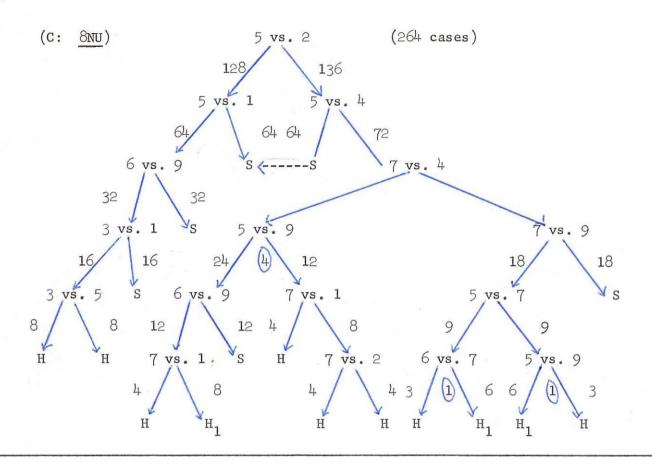
# Continuation of $R_N$ for n = 9

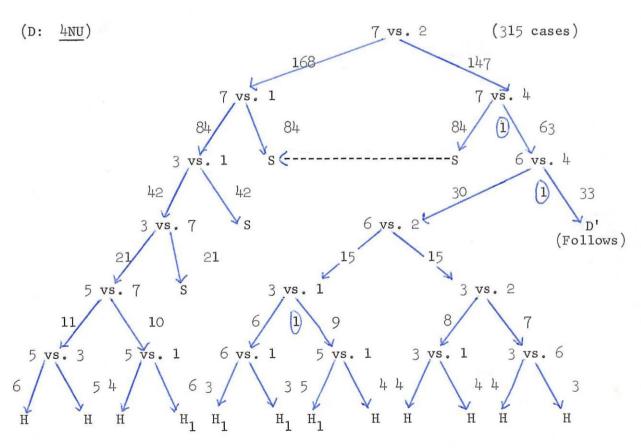


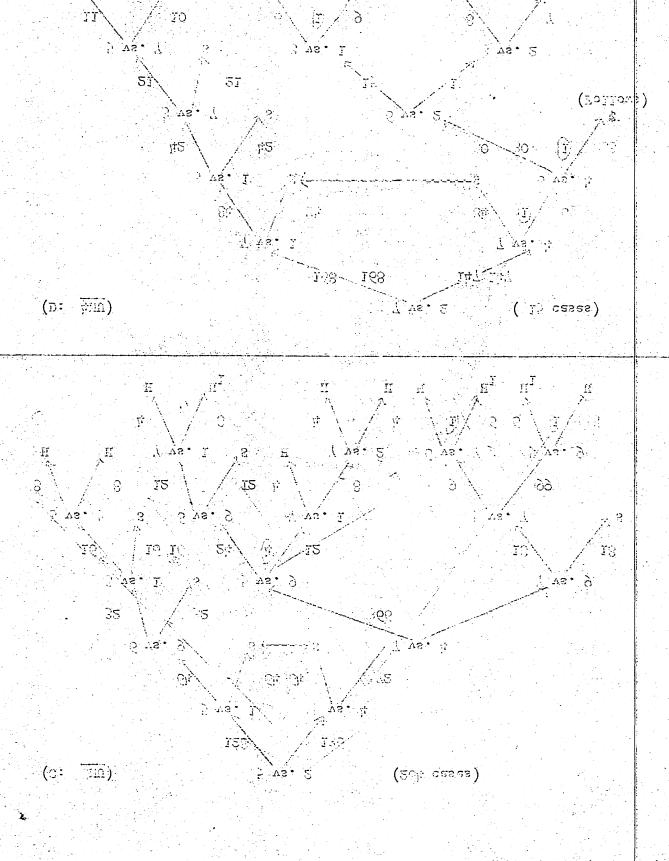


(16) (S);) (120 deses) (B: No Noise) /(3c) (V: TYMA) .. Continuation of  $R_{\overline{x}}$  for n=9

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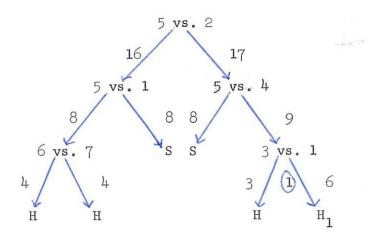


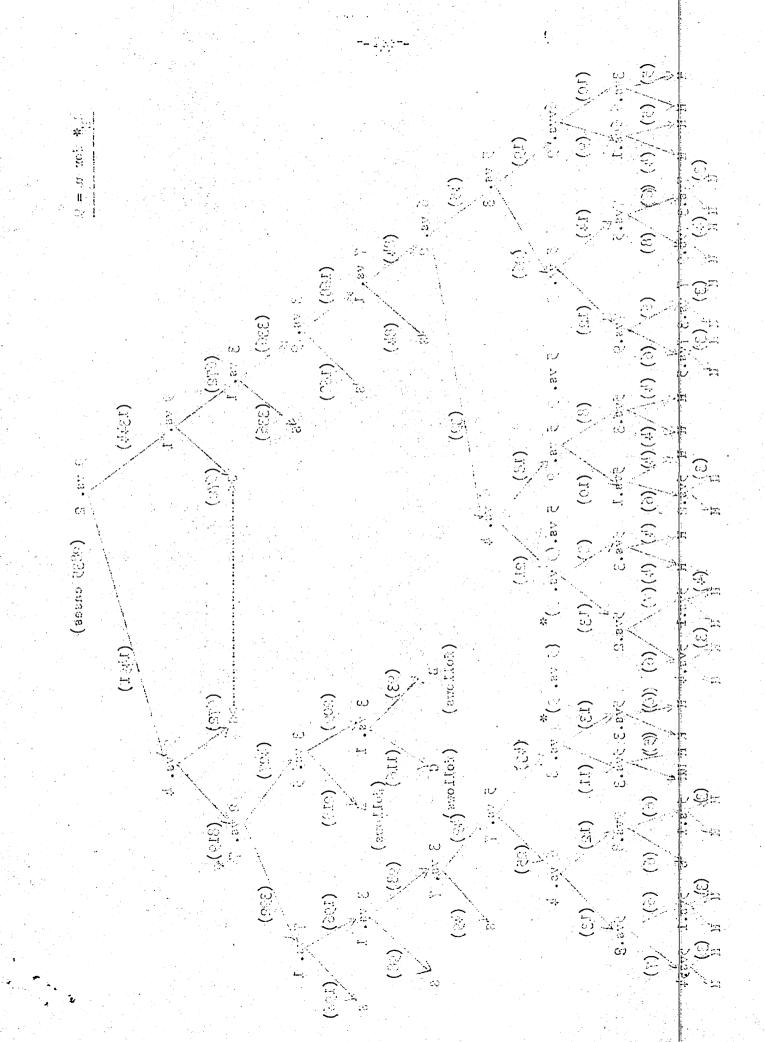




# Continuation of $R_N$ for n = 9 (D)

D':





Two-step entropy was used only at (\*) above to avoid one unit of noise, which becomes 32 because of the multiplicities (S). The total remaining number of noise units (NU) is 18 and hence  $E\{T|R_E*\}=18\frac{1574+18}{D}=18\frac{1592}{2835}=18.562...$ ; to get the result for  $R_E$  we add 18+32=50 NU and the result is  $18\frac{1624}{2835}=18.573...$ 

#### ACKOWLEDGEMENT

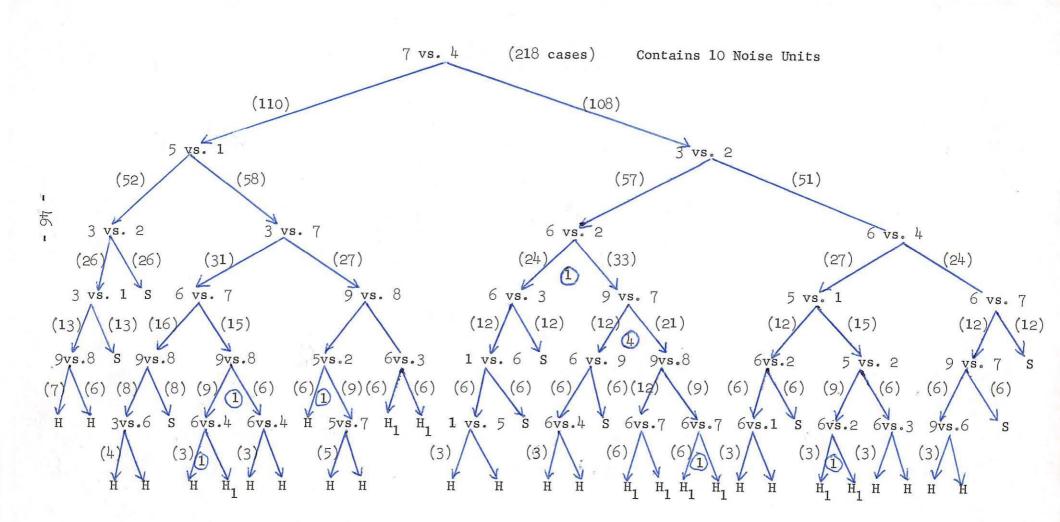
The author wishes to thank A. Hadian of the University of Minnesota for many stimulating conversations on this paper. In particular it was he who recognized that the dynamic programming approach in (6.3) could be applied to the problem of ordering all n numbers.

Two-step entropy was used only at (%) above to avoid one unit of noise, which becomes 32 because of the writiplicities (6). The total remaining number of noise units (at) is it and hence if  $\mathbb{E}_{\mathbb{R}^n} = 10^{\frac{1}{16}} + 1$  and there is  $\mathbb{E}_{\mathbb{R}^n} = 10^{\frac{1}{16}} + 2 = 10$ . We add the result for  $\mathbb{R}_{\mathbb{R}^n} = 10^{\frac{1}{16}} + 2 = 10$ . We add the result is  $\mathbb{R}_{\mathbb{R}^n} = 10^{\frac{1}{16}} + 2 = 10$ .

## VCKOMPRIGREMAL

The author wheres to thank M, Ladien of the University of Minnesots for many stinuisting converstations on this paper. In particular it was he who recognized that the dynamic projectoring are noted in (6.3) cente be explised to the projecting all n muchans.

# Continuation of $R_{E}$ for n = 9



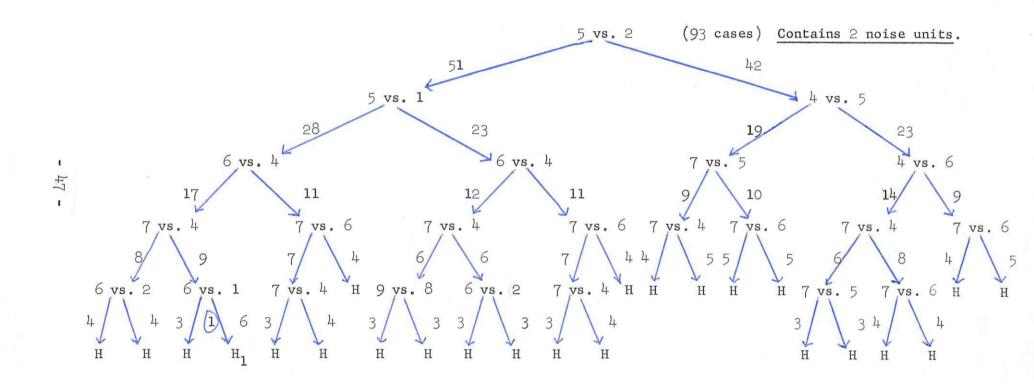
antan serielf Of enteriord (seems 213)

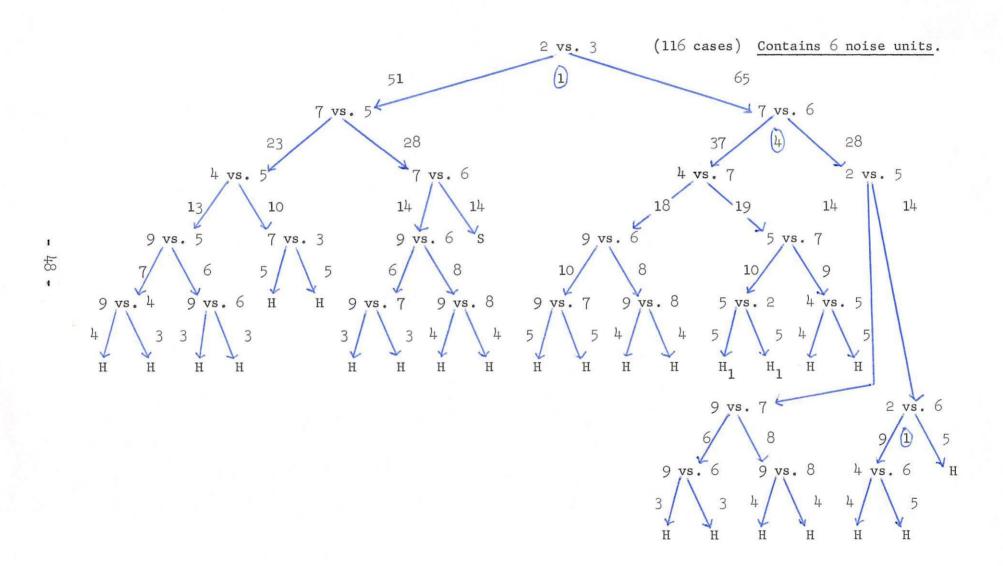
(101)

》 (記)

(30) (30) (31) (30) (30) (31)

(35) / (35)





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