September 15, 1958

A NOTE ON THE LAGRANGIAN SADDLE-POINTS[1]

Leonid Hurwicz[2]

Hirofumi Uzawa[3]

Technical Report No. 1

School of Business Administration

University of Minnesota

Minneapolis, Minnesota

1/ This paper will appear as a chapter in Studies in Linear and Non-Linear Programs by Keneth Arrow, Leonid Hurwicz, and Hirofumi Uzawa, to be published by the Stanford University Press.

2/ Work conducted in part under contract Nonr2582(00), Task NR 042-200

3/ Work conducted in part under contract N6onr-25133, Task NR-047-004.

The work conducted in part under Office of Naval Research contracts at the University of Minnesota and Stanford University.

A NOTE ON THE LAGRANGIAN SADDLE-POINTS

by Leonid Hurwicz and Hirofumi Uzawa [1]

1. Introduction.

1.1 In the present note we obtain a theorem on programming in linear spaces, designed to handle a class of cases not covered by Theorem V.3.1 of Chapter 4 in [4]. (This theorem will be referred to below as Theorem V.3.1.) The latter theorem guarantees the existence of Lagrangian multipliers (functionals) for a class of extremization problems under rather mild restrictions on the nature of spaces and functions considered, but it requires the order-defining cones ("positive orthants") to possess interiors. This requirement is satisfied by the positive orthants in such spaces as (m) and (M)[2], but it is not satisfied in many important cases; in particular, it fails to be satisfied in such spaces as (ℓ_p), (L_p), (s), (S).[3] It is natural to inquire whether Lagrangian multipliers might exist in situations where the positive orthant has no interior. The present note provides only a partial answer to this question. It is shown that the assumption of a non-empty interior cannot be completely dispensed with, but it can be weakened in such a way as to yield Lagrangian saddle-points for a class of cases including the (ℓ_p) spaces. On the other hand, it is found that such saddle-points need not exist in the (S) or (s) spaces. With regard to the (L_p) spaces the question is still open.

1.2. We are concerned with the problem of extremization (more specifically, maximization) under constraints in certain linear topological spaces.

Our maximization problem is stated in terms of a linear system $\mathcal{X}$ and two linear topological spaces $\mathcal{Y}$, $\mathcal{Z}$, with ordering relations defined on $\mathcal{Y}$ and $\mathcal{Z}$ respectively by the convex cones P_y and P_z. A point $y_o \in \mathcal{Y}$ is said to be <u>maximal</u> <u>over</u> the set $Y \subseteq \mathcal{Y}$, if $y_o \in Y$ and, for each $y \in Y$, $y \geq y_o$ implies $y \leq y_o$ [4].

Given two concave functions, $f: X \to \mathcal{Y}$ and $g: X \to \mathcal{Z}$, X a convex subset[5] of $\mathcal{X}$, we shall say that x_o <u>maximizes</u> $f(x)$ <u>subject to</u> $x \in X$ <u>and</u> $g(x) \geqq 0$ if $f(x)$ is maximal over the set $f[X \cap g^{-1}(P_z)]$; i.e., if $x_o \in X$, and $g(x_o) \geqq 0$, and $f(x) \not\geq f(x_o)$ for all $x \in X$ satisfying $g(x) \geqq 0$.

We are interested in conditions under which the Lagrangian expression

$$(1) \qquad \Phi(s, z^*; y^*) = y^*[f(x)] + z^*[g(x)]$$

(where y^* and z^* are respectively continuous linear functionals over $\mathcal{Y}$ and $\mathcal{Z}$) has a (non-negative) saddle-point, i.e., the following conditions are satisfied: there exists continuous linear functionals y^*_o and z^*_o such that

$$(2.1) \qquad y^*_o \geq 0 \text{ and } z^*_o \geqq 0,$$

$$(2.2) \qquad \Phi(x, z^*_o; y^*_o) \leqq \Phi(x_o, z^*_o ; y^*_o) \leqq \Phi(x_o, z^* ; y^*_o)$$

for all $x \in X$ and all $z^* \geqq 0$.[6]

In Theorem V.3.1, it was shown that the Lagrangian Φ has such a saddle-point if

(N') the convex cones P_y and P_z both have non-empty interiors, and

(R') for some $x_* \in X$, $g(x_*)$ is an interior point of P_z.[7]

Theorem V.3.1 remains valid if (R') is replaced by the condition

(R'') for any linear functional $z^* \geq 0$, there exists a point x_{z^*} such that $z^*(x_{z^*}) > 0$.

(R'') is meaningful even when the cone P_z has no interior, and the question arises as to whether the condition (N') could be weakened or modified.

1.3. At this point we shall give two examples to show that the condition (N') cannot be completely dispensed with, even in such specialized classes of spaces as complete metric or locally convex.

First, consider the case in which $\mathcal{Y}$ is the set of all reals (with the natural topology) and $\mathcal{Z}$ is the space (S), the set of all measurable functions defined on $0 \leqq t \leqq 1$. It is known[8] that there is no non-null continuous linear functional on the space (S). It then follows from a theorem of Klee[9] that there is no non-null linear non-negative functional on (S); cf. [6], p. 267, (4.1). Hence, in (S), the regularity condition (R'') is (vacuously) satisfied.[10] Now suppose that the Langrangian Φ has a saddle-point at (x_o, z^*_o); then $z^*_o = 0$ (since there are no other non-negative functionals) and, since $y^*_o \geqq 0$, we must have $f(x) \geqq f(x_o)$ for all $x \in X$. But if X is the set of all non-negative reals, $f(x) = x$, and $g(x) = a - xb$, with $\underline{a}$ and $\underline{b}$ in (S), $a(t) = 2$ for all $\underline{t}$ in $[0,1]$, $b(t) = 1$ for all $\underline{t}$ in $[0,1]$, then $x_o = 2$, yet $f(3) > f(2)$. Hence the Lagrangian has no saddle-point here.

Second, consider the space (s), the set of all numerical sequences $a = (a_1, a_2, \ldots)$. Let $\mathcal{X}$ and $\mathcal{Z}$ be each an (s) space, $\mathcal{Y}$ the set of reals, and let

$$f(x) = \sum_{i=1}^{\infty} 2^{-i} x_i (1+x_i)^{-1} \quad , \; g(x) = b - x \, ,$$

$b = (1/2, 1/2, \ldots)$, $X = P_z$ = the set of all numerical sequences with non-negative coordinates. Then $x_o = b$ and $f(x_o) = 0$. Now suppose that the Lagrangian Φ has a saddle-point. It is known[11] that every continuous functional z^* on the space (s) is of the form

$$z^*(z) = \sum_{i=1}^{n} z^*_i \, z_i$$

where $\underline{n}$ is a finite integer and $z = (z_1, z_2, \ldots)$. If Φ has a saddle-point at (x_o, z^*_o), we must have

$$f(x) + \sum_{i=1}^{n} z^*_{oi} \cdot (1/2 - x_i) \leqq 0 \text{ for all } \underline{x} \text{ in (s)}.$$

But let $x = (x_1, x_2, \ldots)$, with $x_i = 1/2$ for all $i \neq n+1$ and $x_{n+1} = 1$. Then

$$f(x) + \sum_{i=1}^{n} z^*_{oi} \cdot (1/2 - x_i) = 2^{-(n+1)} \cdot (1/2 - 1/3) > 0,$$

thus contradicting the required inequality. Hence the Lagrangian saddle-point does not exist here either.

1.4 In attempting to extend the theorem on Lagrangian saddle-points, we may note that it requires the existence of non-null linear continuous non-negative functionals on the product space $\mathcal{Y} \times \mathcal{Z}$. One natural approach is through certain theorems guaranteeing the continuity of any linear non-negative functional. Some of these theorems, unfortunately, require that the positive orthant possess interior, and are, therefore, of no value for our purpose. But the theorems of Nachbin ([8], p. 464, Theorem 1) and Klee (Lemma A, cf. footnote 9 above) do not call for the existence of the interior and can be used here.

Since both (S) and (s) satisfy all of the conditions of Lemma A,[12] the preceding examples show that the hypotheses of Lemma A (together with the convexity of X, the concavity of f and g, and the regularity condition (R'')) are not sufficient to guarantee the existence of linear (even non-continuous) functionals satisfying the saddle-point inequalities (2). Hence, even in spaces where the hypotheses of Lemma A are satisfied, we must supply some counterpart of the condition (N') above which required that the positive orthants have interiors.

Now an examination of the proof of Theorem V.3.1 shows that (N') was used to establish the existence of a hyperplane (i.e., a continuous linear functional) separating two convex sets. But under the hypotheses of Lemma A we only need to prove the existence of a linear functional (i.e., a maximal linear variety) separating the two sets, since the continuity then follows from non-negativeness. The existence of such a separating functional is purely algebraic property and hence the conditions under which such a separation can be performed are also algebraic. These are stated in an earlier paper of Klee ([5], p.456,(8.10))[13/]:

> Two disjoint convex sets can be separated by a linear functional provided that either the linear system (vector space) is finite-dimensional or one of the sets has a non-empty core.

The core of a set S in a linear system L is defined as follows: a point $\underline{x}$ in L belongs to the core of X, to be denoted by X_o, if and only if

> given any $z \in L$, $z \neq x$, there exists a point $y' = \lambda' x + 1\ (1-\lambda')z$, $0 < \lambda' < 1$, such that all points $y = \lambda x + (1-\lambda)z$, $0 < \lambda < \lambda'$, belong to S.

I.e., $\underline{x}$ belongs to S if and only if, in every direction of the linear system, there is a segment originating at $\underline{x}$.

In every linear topological space, the core of a set includes its interior, but the converse is not always true. In certain spaces, as we shall see below, the assumption that the core is non-empty is weaker than the assumption that the interior is non-empty, and it becomes natural to replace the condition (N') (stating that the interiors are non-empty) by the condition

(N'') the cores of P_y and P_z are non-empty.

Under these circumstances it turns out [14/] that the regularity condition (R'') is equivalent to

(R''') for some $x_* \in X$, $g(x_*)$ is in the core of P_z.

Hence the conditions (N'') and (R''') are the precise counterparts of (N') and (R'), with 'core' replacing 'interior'.

1.5 Applying Lemma A to the linear functional obtained by the method of Theorem V.3.1, we find that the conclusions of Theorem V.3.1 remain valid with 'core' replacing 'interior' in its assumptions, provided we assume in addition that the spaces $\mathcal{Y}$ and $\mathcal{Z}$ are complete metric linear and that the cones P_y and P_z are closed and span their respective spaces. A precise statement of this result is found in 2.1 below.

Now all of the following spaces are complete metric linear and have closed space-spanning positive orthants: (S), (s), (L_p), $p > 0$, (ℓ_p), $p > 0$. Hence the possibility of applying the theorem of 2.1. below depends on whether their positive orthants have non - empty cores. It turns out that the spaces (S), (s), (L_p), have positive orthants with empty cores.[15] For (S), (s), and (L_p), $p < 1$, this was to be expected, since we know that the theorem of 2.1 fails in these spaces. With regard to the spaces (L_p) the question of the existence of Lagrangian saddle-points is still unresolved. But it turns out that the positive orthant of bounded functions on any domain has a non-empty core[16] and this includes all the (ℓ_p) spaces, $p > 0$. The fact that (ℓ_p), $p < 1$, are covered is rather interesting, since it shows that local convexity is not necessary; on the other hand, the fact that (s) does not, in general, have Lagrangian saddle-points, shows that local convexity is not sufficient in a complete metric space with a closed space-spanning positive orthant whose core is empty, even when (R'') holds.

2. <u>A Theorem on Lagrangian Saddle-Points</u>.

2.1 <u>Theorem</u>. Let $\mathcal{X}$ be a linear system, $\mathcal{Y}$ and $\mathcal{Z}$ complete metric linear spaces; let P_y and P_z be closed convex cones, with non-empty cores, in $\mathcal{Y}$ and $\mathcal{Z}$ respectively, and let $P_y - P_y$ and $P_z - P_z$ have interiors; let X be a (fixed) convex subset of $\mathcal{X}$, <u>f</u> a concave function on X to $\mathcal{Y}$, <u>g</u> a concave function on X to $\mathcal{Z}$, and

(R''') for some x_* X, $g(x_*)$ is in the core of P_z.

Under these assumptions, if x_o maximizes $f(x)$ subject to $x \in X$, $g(x) \geqq 0$, then there exist continuous linear functionals $y^*_o \gneqq 0$ and $z^*_o \geqq 0$ such that the Lagrangian expression $\Phi(x,z^*;y^*)$ defined by (1) has a saddle-point at (x_o, z^*_o) for all $x \in X$ and $z^* \geqq 0$; i.e., Φ satisfies the inequalities (2).

2.2 Proof of the Theorem.

Let $\mathcal{W}$ be the topological product space $\mathcal{Y} \times \mathcal{Z}$ and consider the subset of $\mathcal{W}$ defined by

(3) $A = \{ w = (y,z) : y \in \mathcal{Y},\ z \in \mathcal{Z},\ y \leqq f(x),\ z \leqq g(x)$, for some $x \in X \}$. Because of the concavity of the functions f and g and the convexity of the set X, the set A is also convex.

Now suppose it can be shown that there exists a hyperplane supporting the set A at the point $w_o = (f(x_o), 0_z)$, 0_z being the origin of $\mathcal{Z}$, i.e., a non-null linear continuous functional w^*_o such that, for some real number α,

(4) $$w^*_o(w_o) = \alpha \qquad \text{and } w^*_o(w) \leqq \alpha \qquad \text{for all } w \in A.$$

Let $w^*_o = (y^*_o, z^*_o)$. Then, by the reasoning used in the proof of Theorem V.3.1, we may conclude that $y^*_o \geqq 0$ and $z^*_o \geqq 0$, and that the Lagrangian expression $\Phi(x,z;y^*_o)$ satisfies the inequalities in (2) above. Furthermore, the regularity condition (R''') implies that $y^*_o \gneqq 0$. For suppose that $y^*_o = 0$. Since w^*_o is non-null, it follows that $z^*_o \gneqq 0$. Let z_1 be a point in $\mathcal{Z}$ such that $z^*_o(z_1) < 0$. But (2) implies that

$$z^*_o\,[g(x)] \leqq 0 \quad \text{for all } x \in X;$$

hence, in particular, we have

$$z^*_o[g(x_*)] \leqq 0$$

for the element $x_* \in X$ whose existence is postulated by the condition (R''').

Therefore, letting $z_\lambda = \lambda g(x_*) + (1-\lambda)z_1$, we have

$$z_0^*(z_\lambda) < 0 \quad \text{for all} \quad 0 \leq \lambda < 1.$$

But $g(x_*)$, by hypothesis, is in the core of P_z; hence $z \in P_z$ for sufficiently small $\lambda > 0$ and hence $z_0^*[z_\lambda] \geqq 0$ if λ is small enough; this contradiction shows that $y_0^* \geqq 0$.

Therefore, it only remains to be shown that a w_0^* with the specified properties does exist. To establish the existence of w_0^*, we first define the sets

(5) $$\left\{ A_1 = \left\{ w=(y,z) : y \leq f(x), \quad z \leqq g(x), \text{ for some } x \in X \right\} \right.,$$

(6) $$B = \left\{ w=(y,z) : y \geqq f(x_0), \ z \geqq 0 \right\}.$$

The set B is convex with a non-empty core and disjoint from the convex set A_1. Hence, by Lemma B (cf. footnote 13), there exists a maximal linear variety, i.e., an additive homogeneous non-null functional, $h(w) = \alpha$, such that

(7) $$h(w') \leqq \alpha \leqq h(w'') \quad \text{for all } w' \in A_1, \ w'' \in B.$$

It follows from (7) that the image h(B) is not the whole space of reals, and since B is closed convex, h linear, and the space complete metric linear, we may apply Lemma A (cf. footnote 9). It follows that h is continuous; we shall, therefore, denote it by w_0^*.

Because of the continuity of w_0^*, we have

(8) $$w_0^*(w) \leqq \alpha \qquad \text{for all } w \in \bar{A}_1,$$

where $\bar{A}_1$ is the closure of A_1. Therefore, if we can show that

(9) $$A \subseteq \bar{A}_1,$$

it will follow that

(10) $$w_0^* \leqq \alpha \qquad \text{for all } w \in A.$$

In order to establish (9), let $w = (y,z) \in A$. For any $y_1 \geqq 0$ and $\varepsilon > 0$, we have

$$w_\varepsilon = (y - \varepsilon y_1, z) \in A_1$$

and $\lim_{\varepsilon \to 0} w_\varepsilon = w$. Hence $w \bar{A}_1$.

Finally, since $w_o \in A \cap B$, (7) implies

(11) $$w_o^*(w_o) = \alpha \quad .$$

Hence we have established that w_o^* is a continuous linear functional with (10) and (11) satisfied, so that the set $\{ w : w_o^*(w) = \alpha \}$ is a hyperplane supporting the set A at the point w_o. This completes the proof of the Theorem.

2.3 In this section, we investigate the relationships between the various regularity conditions. The conditions to be examined are:

(R') There exists $x_* \in X$ such that $g(x_*)$ is in the interior of P_z.

(R_C'') For any non-null non-negative linear <u>continuous</u> functional z^*, there exists $x_{z^*} \in X$ such that

(12) $$z^*(x_{z^*}) > 0 .$$

(R'') For any non-null non-negative linear functional z^* there exists $x_{z^*} \in X$ such that (12) is satisfied.

(R''') There exists $x_* \in X$ such that $g(x_*)$ is in the core of P_z.

It will first be shown that, in any topological space, the following implications hold:

$$(R') \Rightarrow (R''') \Rightarrow (R'') \Rightarrow (R_C'') \quad .$$

(a) $(R') \Rightarrow (R''')$. <u>Proof</u>. By (R'), $g(x_*)$ is in the interior of P_z for some $x_* \in X$. But, in any linear topological space, every point of the interior is a point of the core (cf Klee [5] ,p.445,(2.1)). Hence $g(x_*)$ is in the core of P_z and (R''') holds.

(b) $(R''') \Rightarrow (R'')$. <u>Proof</u>. We show that, for any $z^* \gneq 0$, $z^*(z_*) > 0$ for $z_* = g(x_*)$. For suppose the latter statement to be false. Then, for some $z_o^* \gneq 0$, $z_o^*(z_*) = 0$. Now there exists an element z_1 such that $z_o^*(z_1) < 0$. Hence $z_o^*(z_\lambda) < 0$ for any $z = \lambda z_* + (1-\lambda) z_1$, $0 \leq \lambda < 1$. On the other hand, since z_* is in the core of P_z, $z_\lambda \in P_z$ for sufficiently small $\lambda > 0$, so that $z_0^*(z_\lambda) \geqq 0$. This contradiction completes the proof.

(c) $(R'') \Rightarrow (R''_C)$. <u>Proof</u>. If a z^* qualifies under (R''_C), it also qualifies under (R'').

Thus we see that (R''_C) is no more restrictive than any of the other regularity conditions. (R''_C) would have been sufficient in the proof of our Theorem where (R''') was used. However, under the other assumptions made, this would not have added generality to our result. For

(d) $(R''_C) \Rightarrow (R'')$ if the space is complete metric linear and P_z closed and space spanning[17] (i.e., such that $P_z - P_z = \mathcal{Z}$),

(e) $(R'') \Rightarrow (R''')$ if the core of P_z is non-empty.

<u>Proof</u>. (d) follows directly from Lemma A. To prove (e), we show that if (R''') is false while the core of P_z is non-empty, (R'') must be false too. Let (R''') be false. Consider the set $A = \{z: z \leqq g(x) \text{ for some } x \in X\}$. Since X is convex and g concave, A must be convex. Now suppose it has been shown that A does not intersect the core of P_z. Then Lemma B provides the required non-negative functional. Hence it remains to be shown that the intersection of A with the core of P_z is empty.

Suppose that the intersection is not empty, so that there is an element $\underline{z}$ in the core of P_z such that $z = g(x)$ for some $x \in X$. Now, for any $0 \leqq \lambda \leqq 1$, $(1-\lambda)\ [g(x)-z]$ is in P_z by definition of A. Also, if λ is sufficiently small, $(1-\lambda)z + \lambda y$ is in P_z for any $\underline{y}$. Hence, for any $\underline{y}$ and all sufficiently small λ, the sum of these two vectors must also be in the convex cone P_z. In view of the identity

$$(1-\lambda)g(x) + \lambda y = (1-\lambda)z + (1-\lambda)[g(x)-z] ,$$

the vector $(1-\lambda)g(x) + \lambda y$ is in P_z for any $\underline{y}$ and all sufficiently small λ. The latter statement means that g(x) is in the core of P_z and hence contradicts the falsity of (R''').

We may also note that

(f) $(R''') \nRightarrow (R')$ if P_z has a non-empty interior. <u>Proof.</u> When a convex set in a linear topological space has an interior, its core equals the interior (cf. Klee [5], p. 448, (4.5) and (p 445, (2.1)).

Hence, in view of (d_1) (cf. footnote 17), (e), and (f), the conditions (R') and $R''_C)$ are equivalent in any topological linear space if P_z has a non-empty interior.

3. <u>Applications.</u>

3.1 Let T be an arbitrary set and $\mathcal{B}(T)$ the set of all bounded real-valued functions defined on T; i.e.,

(13) $$\mathcal{B}(T) = \left\{ x=(x(t)) : \sup_{t \in T} |x(t)| < \infty \right\} .$$

$\mathcal{B}(T)$ is a linear system with usual addition and scalar multiplication. Let $\mathcal{L}$ be any linear system in $\mathcal{B}(T)$, and P the positive orthant of $\mathcal{L}$; i.e.,

(14) $$P = \left\{ x = (x(t)) : x \in \mathcal{L},\ x(t) \geqq 0 \text{ for all } t \in T \right\} .$$

We may note that a function positively bounded from below belongs to the core of P ; i.e., if x = (x(t)) in P satisfies

(15) $$\inf_{t \in T} x(t) = m_x > 0 ,$$

then <u>x</u> is in the core of P. In fact, for any point z = (z(t)) in $\mathcal{L}$, let

(16) $$\lambda_o = (1 + M_z/m_x)^{-1}$$

where $M_z = \sup_{t \in T} | z(t) | < \infty$. Then we have

(17) $$0 < \lambda_o < 1 \quad \text{and } M_z/m_x = \lambda_o^{-1} - 1 .$$

Hence, for any λ such that $0 < \lambda \leqq \lambda_o$, we have

$$(18) \qquad (1-\lambda)x(t) + \lambda z(t) \geqq (1-\lambda)m_x - \lambda|z(t)|$$
$$\geqq \lambda m_x [(1-\lambda)/\lambda - M_z/m_x]$$
$$\geqq \lambda m_x [(1-\lambda)/\lambda - (1-\lambda_o)/\lambda_o]$$
$$\geqq 0 .$$

This inequality shows that, for any $\underline{z}$ in $\mathcal{L}$, there exists a positive number λ_o such that $(1-\lambda)x + \lambda z$ is in P for all $0 < \lambda < \lambda_o$. This is equivalent to saying that $\underline{x}$ is in the core of the positive orthant P.

We also note that any function $x=(x(t))$ in $\mathcal{L}$ is represented as the difference of two functions belonging to P:

$$(19) \qquad x = x^+ - x^- ,$$

where $x^+(t) = \max [x(t), 0]$, $x^-(t) = \max [-x(t), 0]$ for all $t \in T$.

The conditions that the positive orthant P have a non-empty core and that the set P - P have a non-empty interior are therefore satisfied for each of the following spaces:

(m), (M), (ℓ_p) with $p > 0$.

Also, in each of these spaces the positive orthant is closed. Hence our theorem (2.1 above) may be applied to these types of linear spaces, while Theorem V.3.1 would not have been applicable in the (ℓ_p) spaces, since their positive orthants have empty interiors. It should be pointed out that, for these spaces, our theorem could also have been proved by using Nachbin's Theorem 1 in [8] instead of Lemma A.

3.2 In this section we show that, in certain spaces, the positive orthant has no core. This makes it impossible to apply the theorem of 2.1, but does not necessarily imply the non-existence of a Lagrangian saddle-point.

(a) <u>The space</u> (s). This is the space of all numerical sequences $x = (x_1, x_2, \ldots)$. Denote by P its positive orthant, i.e., the set of all sequences $x = (x_1, x_2, \ldots)$ such that $x_i \geqq 0$ for $i=1,2,\ldots$. Suppose that a certain element $\underline{y} = (y_1, y_2, \ldots)$ of P is in the core P_o of P. Then $y_i > 0$ for all $\underline{i}$ and given any element z of the space, there must exist a number $0 < \lambda' \leq 1$, such that all vectors of the form $\lambda y + (1-\lambda)z$ belong to P, provided $0 < \lambda < \lambda'$. In particular, this would have to be true for z such that $z_i = -iy_i$ for all i. But then

$$\lambda y_i - (1-\lambda)(iy_i) \geq 0 \quad \text{for all } i = 1,2,\ldots,$$

provided λ is small enough, so that

$$i \leq \lambda/(1-\lambda) \quad \text{for } i=1,2,\ldots,$$

which is impossible. Hence y is not in the core P_0; since it was picked arbitrarily, P_0 must be empty.

(b) The space (S). This is the space of all measurable functions defined on the interval [0,1]. We proceed in a manner analogous to that for the space (s). Given an element y in the positive orthant P, i.e., a measurable function $y(t)$, $y(t) \geq 0$, $0 \leq t \leq 1$, we take for z the function specified by the condition $z(t) = -t^{-1}y(t)$; z is measurable since it is the product of $(y(t))$ and $((-t^{-1}))$ which are measurable. This choice of z shows that y is not in P_0 and hence the core of P is empty.

(c) The space L. This is the space of the equivalence classes of all (absolutely) Lebesque integrable functions defined on the interval [0,1], i.e., L_1. To begin with, let us show that a bounded function $y(t)$ cannot be in the core of the positive orthant. Thus suppose that $y(t)$ is a bounded non-negative function and take as z the function

$$z(t) = -t^{-(1/2)}$$

(This is legitimate since z is integrable.) Now for any λ, however small, there is a positive number t_λ such that[18/]

$$\lambda y(t_\lambda) + (1-\lambda)z(t_\lambda) < 0 \text{ for all } t \text{ in } (0,t_\lambda).$$

Hence a bounded function cannot be an element of the core. By a similar procedure, one can show that a function bounded on some open interval (a,b) within [0,1] cannot be in the core: here we define z(t) as $-(t-a)^{-(1/2)}$ for t in (a,1) and zero in [0,a] .

Now a non-negative integrable function need not be bounded on an open interval, but it must be bounded on a set of positive measure (cf. Titchmarsh [11], pp. 341-2). Let E be such a set. Then there will exist a point a in [0,1) such that the intersection of E with the interval [a,b) is of positive measure for all b a (cf. Saks [9], pp 128-131). Here again z(t), defined as in the preceding case, shows that an arbitrarily chosen non-negative integrable y(t) is not in the core of the positive orthant, since, for any $\lambda > 0$, there exists a set E_λ of positive measure with $\lambda y(t) + (1-\lambda)z(t) < 0$ for $t \in E_\lambda$. Hence the core is empty.

(d) Other L_p spaces. The proof that the core is empty proceeds along lines analogous to those of the proof for L.

3.3 Among applications of particular interest to the economist, it may be mentioned that Debreu's Theorem 2 in [3], asserting that a Pareto Optimum is a Valuation Equilibrium, can be modified in the following way: Debreu's assumption that, for infinite-dimensional spaces, the production set has an interior would be weakened to requiring that Y have a core, while the space would be assumed complete linear metric and orderings in terms of suitable convex cones would be introduced.

The University of Minnesota and Stanford University.

REFERENCES

[1] Banach, S., Théorie des Opérations Linéaires, Hafner Publishing Co., N. Y., 1932.

[2] Bourbaki, N., Espaces Vectoriels Topologiques, Chapters I and II, Livre V, Premiere Partie, Elements de Mathématique, Hermann & Cie, 1953.

[3] Debreu, G., "Valuation Equilibrium and Pareto Optimum," Proc. Nat. Acad. Sci., U.S.A., vol. 40, 1954, pp. 588-592.

[4] Hurwicz, L., "Programming in Linear Spaces," Chapter 4 in Studies in Linear and Non-Linear Programming, by K. J. Arrow, L. Hurwicz and H. Uzawa, Stanford University Press, 1958.pp

[5] Klee, V. L., Jr., "Convex Sets in Linear Spaces," Duke Mathematical Journal, vol. 18, 1951, pp. 443-466

[6] Klee, V. L., Jr., "Boundedness and Continuity of Linear Functionals," Duke Math. J., Vol. 22, 1955, pp.263-269.

[7] LaSalle, J. P., "Pseudo-Normed Linear Spaces," Duke Math.J., Vol 18, 1941, pp. 131-136.

[8] Nachbin, L. "On the Continuity of Positive Linear Transformations," Proc.Int. Congr.of Math., No. 6, vol. 1 (1950), pp 464-465.

[9] Saks, S., Theory of the Integral, Second Revised Edition, Warszawa-Lwow 1937.

[10] Stone, M. H., Convexity, The University of Chicago, 1946 (Lecture notes prepared by H. Flanders).

[11] Titchmarsh, The Theory of Functions, Second Edition, Oxford University Press, 1939.

FOOTNOTES

1/ We wish to acknowledge the support received from the Office of Naval Research.

We are grateful to Professor S. Karlin, of Stanford University, for a suggestion concerning the regularity condition in Theorem V.3.1, Ch. 4 of [4] which provided a stimulus for this note, and also to Professor G. K. Kalisch, of the University of Minnesota, for clarification of the problem discussed in section 3.2 (c) of the present note.

2/ Cf. Banach [1], pp. 10-11.

3/ Cf. Banach [1], pp. 9-12.

4/ We write $y' \geqq y''$ if and only if $y' - y'' \in P_y$; $y' \geq y''$ means that $y' \geqq y''$ but $y'' \not\geqq y'$. (I.e., y_o is maximal over Y if y_o if $y_o \in Y$ and $y \geq y_o$ for all $y \in Y$.) The same notational principles are followed in other spaces.

5/ X is usually a convex cone, typically the positive orthant.

6/ $y^* \geqq 0$ means that $y^*(y) \geqq 0$ for all $y \geqq 0$; $y^* \geq 0$ means that $y^* \geqq 0$ but $-y^* \not\geqq 0$, so that $y^* \geq 0$ implies $y^* \neq 0$; the notation for other spaces is similar.

7/ In Chapter 4 of [4] this has been called the Slater regularity condition.

8/ Cf. Banach [1], p.234, and LaSalle [7], p.134.

9/ Klee [6], p.266: "(3.4) If E is a complete metric linear space, C is a closed convex subset of E such that C-C has non-empty interior, and <u>f</u> is a linear functional on E from which $fC \neq E$, then <u>f</u> is continuous." For the sake of brevity, this result will be referred to as <u>Lemma A</u>.

It may be noted that if C - C = E (which we express by "C spans E") then C-C necessarily has a non-empty interior.

10/ The situation is the same in the L_p spaces for $0 < p < 1$. Cf. Klee [6], p 267.

11/ Cf. Banach [1], p. 50.

12/ (S) and (s) are what Banach calls F-spaces. Cf Banach [1], p. 35.

13/ This result will be referred to as <u>Lemma B</u>. Cf. also Stone, [10], Part 3.

14/ See section 2.3 below.

15/ See section 3.2 below.

16/ See section 3.1 below.

17/ (d_1) : $(R''_C) \Longrightarrow (R'')$ in any linear topological space if P_z has a non-empty interior. Cf. Bourbaki [2], p. 52, Prop. 16.

18/ Let $y(t) = M$ for all $t \in [0,1]$. Then $\lambda y + (1-\lambda)(-t^{-(1/2)}) \geqq 0$ implies $t \geqq (1/k^2)$, $k = (1-\lambda)/(\lambda M)$. Hence we may take any $0 < t_\lambda < (1/k^2)$.

School of Business Administration
University of Minnesota

BASIC DISTRIBUTION LIST FOR UNCLASSIFIED TECHNICAL REPORTS

for contract Nonr 2582(00), Task NR 042-200

ADDRESS		ADDRESS	
Head, Statistics Branch Office of Naval Research Washington 25, D. C.	2	Professor W. G. Cochran Department of Statistics Harvard University Cambridge, Massachusetts	1
Commanding Officer Office of Naval Research Branch Office Navy #100 Fleet Post Office New York, New York	2	Professor Benjamin Epstein Applied Mathematics & Statistics Lab. Stanford University Stanford, California	1
ASTIA Document Service Center Arlington Hall Station Arlington 12, Virginia	5	Professor W. Hirsch Institute of Mathematical Sciences New York University New York 3, New York	1
Office of Technical Services Department of Commerce Washington 25, D. C.	1	Dr. Paul G. Hoel Department of Mathematics University of California Los Angeles 24, California	1
Technical Information Officer Naval Research Laboratory Washington 25, D. C.	6	Professor Harold Hotelling Associate Director Institute of Statistics University of North Carolina Chapel Hill, North Carolina	1
Professor T. W. Anderson Department of Mathematical Statistics Columbia University New York 27, New York	1	Professor L. Hurwicz School of Business Administration University of Minnesota Minneapolis 14, Minnesota	1
Professor Z. W. Birnbaum Laboratory of Statistical Research Department of Mathematics University of Washington Seattle 5, Washington	1	Professor Leo Katz Department of Statistics Michigan State University East Lansing, Michigan	1
Professor A. H. Bowker Applied Mathematics & Statistics Lab. Stanford University Stanford, California	1	Professor Oscar Kempthorne Statistics Laboratory Iowa State College Ames, Iowa	1
Professor Ralph A. Bradley Department of Statistics & Statistical Laboratory Virginia Polytechnic Institute Blacksburg, Virginia	1	Dr. Carl F. Kossack Statistical Laboratory Engineering Administration Building Purdue University Lafayette, Indiana	1
Professor Herman Chernoff Applied Mathematics & Statistics Lab. Stanford University Stanford, California	1		

ADDRESS	
Professor Gerald J. Lieberman Applied Mathematics and Statistics Laboratory Stanford University Stanford, California	1
Professor William G. Madow Department of Statistics Stanford University Stanford, California	1
Professor J. Neyman Department of Statistics University of California Berkeley 4, California	1
Professor Herbert Robbins Mathematical Statistics Department Columbia University New York 27, New York	1
Professor Murray Rosenblatt Department of Mathematics Indiana University Bloomington, Indiana	1
Professor L. J. Savage Statistical Research Laboratory Chicago University Chicago 37, Illinois	1
Dr. Herbert Solomon Teachers College Department of Statistics Columbia University New York 27, New York	1
Professor Frank Spitzer Department of Mathematics University of Minnesota Minneapolis 14, Minnesota	1
Professor S. S. Wilks Department of Mathematics Princeton University Princeton, New Jersey	1

ADDRESS	
Professor Evan J. Williams Institute of Statistics State College Section North Carolina State College Raleigh, North Carolina	1
Professor J. Wolfowitz Department of Mathematics Cornell University Ithaca, New York	1