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Four dimensional $\mathcal{N} = 2$ theories from six dimensions

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Four dimensional $\mathcal{N} = 2$ theories from six dimensions

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To Amma, Thatha and Patti.

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Four dimensional $\mathcal{N} = 2$ theories from six dimensions

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By formulating the six dimensional $(0, 2)$ superconformal field theory $\mathcal{X}[\mathfrak{j}]$ on a Riemann surface decorated with certain codimension two defects, a multitude of four dimensional $\mathcal{N} = 2$ supersymmetric field theories can be constructed. In this dissertation, various aspects of this construction are investigated in detail for $\mathfrak{j} = A, D, E$. This includes, in particular, an exposition of the various partial descriptions of the codimension two defects that become available under dimensional reductions and the relationships between them. Also investigated is a particular observable of this class of four dimensional theories, namely the partition function on the four sphere and its relationship to correlation functions in a class of two dimensional non-rational conformal field theories called Toda theories. It is pointed out that the scale factor that captures the Euler anomaly of the four dimensional theory has an interpretation in the two dimensional language, thereby adding one of the basic observables of the 4d theory to the 4d/2d dictionary.

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Outline

Chapters 1-2 are introductory and are primarily a review of existing knowledge in this field. The selection of results reviewed in these Chapters is idiosyncratic to the needs of the following chapters. Chapters 3-5 detail results from my original research. A more elaborate outline of Chapters 3-5 is also provided at the end of Chapter 2.

A large part of the material in Chapters 3-5 has appeared in the following two papers,

- Aswin Balasubramanian, "Describing codimension two defects", *JHEP07(2014)095* [14]
- Aswin Balasubramanian, "The Euler anomaly and scale factors in Liouville/Toda CFTs", *JHEP04(2014)127* [15]

Parts of the work will also appear in

- Aswin Balasubramanian, "Codimension two defects and the representation theory of Weyl groups (in preparation)", *Contribution to proceedings of String-Math 2014*.

In addition to this, the dissertation also includes some previously unpublished material.

Chapter 1

Four dimensional $\mathcal{N} = 2$ theories

1.1 Introduction

One of the major motivations to study supersymmetric field theories in four dimensions is the possibility that many observables in these theories can be computed exactly. Such a luxury is not available for the non supersymmetric quantum field theories, atleast with our current understanding. Essential to any such exact result is the ability to transcend the traditional perturbative frameworks in which QFTs are usually defined. A result that remains valid outside the domain of validity of the perturbative schemes is termed 'non-perturbative'. Certain phenomenon are termed 'non-perturbative' and this reflects the fact they are not visible in any perturbative formalism. Electric-Magnetic duality of the kind that is considered here is one such phenomenon.

The fact certain observables in supersymmetric field theories can be computed to all orders in perturbation theory indicates the somewhat special 'simplicity' that comes with supersymmetry. This simplicity exists already with the minimal amount of supersymmetry in four dimensions corresponding to $\mathcal{N} = 1$ theories where, for example, the superpotential obeys non-renormalization theorems (see [169] for a modern treatment). With more supersymmetry, stronger statements become possible. On some occasions, such as in theories with more than four supercharges, non-perturbative statements become possible. The maximally supersymmetric theory in four dimensions is $\mathcal{N} = 4$ SYM and this theory is, in many ways, the simplest QFT. This is the four dimensional QFT that is most amenable to exact analysis. The subject matter of this dissertation involves theories with eight supercharges and $\mathcal{N} = 2$ supersymmetry in four dimensions. Certain three dimensional theories with eight supercharges will also play an important role. The theories with eight supercharges allow for richer variation in non-perturbative behaviour (when compared to the theories with sixteen supercharges) while still being amenable to a substantial amount of exact analysis. Other important motivations for the study of supersymmetry have been, historically, the potential relevance of low energy supersymmetry as a phenomenological tool and the fact that supersymmetric quantum field theories arise from limits of string/M theory.

A general strategy to describe the low energy behaviour of $\mathcal{N} = 2$ theories was provided in [167, 166]. This involves the specification of an algebraic curve that has come to be called the Seiberg-Witten curve, together with a differential called the Seiberg-Witten differential. The algebraic curve and the differential encode much information about the infrared physics of $\mathcal{N} = 2$ and the combined data is referred to as the 'Seiberg-Witten solution' of $\mathcal{N} = 2$ theories. In recent years, the class of $\mathcal{N} = 2$ theories for which the Seiberg-Witten strategy can be realized has grown substantially. This has led to new insights into the dynamics of $\mathcal{N} = 2$ theories in four dimensions. The new insights have been possible thanks to two major advancements. The first is an improved understanding of defect operators in supersymmetric quantum field theories of various dimensions. The second is the unraveling of a web of connections between aspects of $\mathcal{N} = 2$ theories and myriad ideas in modern mathematics. These mathematical ideas happen to serve, quite well, the needs of a $\mathcal{N} = 2$ field theorist. A part of this interaction between physical and mathematical ideas involves geometric approaches to representation theory and this figures prominently in this dissertation.

The remainder of this introductory chapter (the first of two) reviews basic elements of four dimensional $\mathcal{N} = 2$ theories. In addition, the Seiberg-Witten solution of some elementary examples in this class of theories is discussed. In the following chapter, an introduction to the recent advances in the study of a large class of $\mathcal{N} = 2$ theories is provided. The six dimensional $(0, 2)$ SCFT plays an important role in many of these considerations. Hence, the second introductory chapter is focused more directly on the construction of four dimensional theories starting from six dimensions.

To end this Introduction, here are a few references where Seiberg-Witten theory is discussed in greater detail. For some textbook treatments of Seiberg-Witten theory, see [196, 56] and for other useful reviews of the subject, see [106, 132, 126, 159, 24]. For a review of Seiberg-Witten theory from a more modern perspective, see [186]. For some overviews of recent developments in this field, the interested reader is referred to [148, 149, 147, 184, 185].

1.2 $\mathcal{N} = 2$ theories in four dimensions

The $\mathcal{N} = 2$ algebra in four dimensions has the following form,

$$\{Q^I_\alpha, Q^{\dagger J}_{\dot\beta}\} = \delta^{IJ} P_\mu \sigma^\mu_{\alpha\dot\beta}, \tag{1.1}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon^{IJ}\epsilon_{\alpha\beta} Z, \tag{1.2}$$

where Z is a complex number called the central charge of the algebra since it commutes with all the generators. Theories with $\mathcal{N} = 2$ symmetry in four dimensional include both Lagrangian and non-Lagrangian field theories [1]. But, to orient oneself about the physics of $\mathcal{N} = 2$ theories, it is helpful to look at the Lagrangian theories in the family. Recall the $\mathcal{N} = 1$ superfield formalism. Let Φ be a chiral superfield, V a vector superfield and W_α the field strength defined as,

$$W_\alpha = \frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V \tag{1.3}$$

Expanding Φ, W_α in components,

$$\Phi = \phi + \psi_\alpha \Theta^\alpha + F\Theta_\alpha\Theta^\alpha \tag{1.4}$$

$$W_\alpha = \lambda_\alpha + F_{(\alpha\beta)}\Theta^{\alpha\beta} + D\Theta_\alpha + \dots . \tag{1.5}$$

[1]In general, the quip that a field theory is 'non-Lagrangian' should be taken to imply that there is no known Lagrangian description.

The simplest gauge theory with $\mathcal{N} = 2$ supersymmetry is the pure YM theory build out of a single $\mathcal{N} = 1$ vector multiplet (A_μ, λ_α) and a single $\mathcal{N} = 1$ chiral multiplet $(\phi, \tilde{\lambda}_\alpha)$ that transforms in the adjoint of the gauge group G. In terms of $\mathcal{N} = 1$ superfields, the Lagrangian for this theory can be written as

$$\mathcal{L}_{\text{vector}} = \frac{\Im(\tau)}{4\pi} \int d^4\Theta \mathrm{Tr} \Phi^\dagger e^{[V,.]} \Phi + \int d^2\Theta \frac{-i}{8\pi} \tau \mathrm{Tr} W_\alpha W^\alpha + \text{cc}, \tag{1.6}$$

where τ is the complexified coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{1.7}$$

This theory possesses a $SU(2)_R$ symmetry. This acts by rotations on $\lambda_\alpha, \tilde{\lambda}_\alpha$.

In order to add matter to the pure $\mathcal{N} = 2$ YM theory, one can add copies of the $\mathcal{N} = 2$ hypermultiplet. This contains two copies of the $\mathcal{N} = 1$ chiral multiplet, $Q = (\psi, \phi), \tilde{Q}^\dagger = (\tilde{\psi}^\dagger, \tilde{\phi}^\dagger)$ in conjugate representations of the gauge group G. Let us pick $G = SU(N)$ and take N_f hypers in the fundamental representation. The Lagrangian for the hypermultiplets is of the form (with gauge indices supressed)

$$\begin{aligned}\mathcal{L}_{\text{hyper}} = \int d^4\Theta (Q^\dagger e^V Q_i + \tilde{Q}^i e^{-V} \tilde{Q}_i^\dagger) \quad &+ \quad \left(\int d^2\Theta \tilde{Q}^i \Phi Q_i + \text{cc} \right) \\ &+ \quad \left(\sum_i \int d^2\Theta m_i \tilde{Q}^i Q_i + \text{cc} \right).\end{aligned} \tag{1.8}$$

The hypermultiplet Lagrangian above also has a manifest $SU(2)_R$ symmetry that acts by rotations on $Q, \tilde{Q}^\dagger$. The full Lagrangian for the theory with gauge fields and matter (in some representation of the gauge group) is of the form,

$$\mathcal{L}_{gauge-theory} = \mathcal{L}_{vector} + \mathcal{L}_{hyper} \tag{1.9}$$

1.2.1 Vacuum moduli spaces

To become oriented about the vacuum structure of $\mathcal{N} = 2$ theories, it is useful to consider (again) the gauge theories in the family. The Lagrangian is as in (1.9). The classical vacuum moduli spaces are determined by the zeros of the potential,

$$V = \frac{1}{2}(D)^2 + F^{i\dagger}F_i, \tag{1.10}$$

where $F_i = \partial W/\partial\phi_i$ and $D^a = \phi_i^\dagger (T^a)^i_j \phi^j$. Since the potential is a sum of squares, one can analyze the equations obtained by setting the individual terms to zero. Setting the D-term to zero yields,

$$\begin{aligned} [\Phi^\dagger, \Phi] &= 0 \\ \mid Q_j Q^{\dagger j} - \tilde{Q}^{\dagger j}\tilde{Q}_j \mid_{trace-free} &= \nu. \end{aligned} \tag{1.11}$$

Setting the F-term to zero yields,

$$\begin{aligned} \mid Q^j \tilde{Q}_j \mid_{trace-free} &= \rho \\ Q^j_a \mu^i_j + \Phi^b_a Q^i_b &= 0 \\ \tilde{Q}^a_j \mu^j_i + \tilde{Q}^b_i \Phi^a_b &= 0. \end{aligned} \tag{1.12}$$

The space of solutions obtained by setting $Q, \tilde{Q} = 0$ and keeping $\Phi \neq 0$ is called the Coulomb branch of vacua $\mathcal{B}$. Geometrically, the Coulomb branch is a (rigid) special Kahler manifold [76, 45]. The classical Coulomb branch has a special point where $\Phi = 0$ and the full non-abelian symmetry is restored. This description of the Coulomb branch remains valid in the high energy regime of the quantum theory (UV). As one goes to lower energies (IR), the metric on the Coulomb branch receives quantum corrections. The full quantum Coulomb branch could, in general, also have singularity structure that is quite different from the one for the classical Coulomb branch.

The other extreme alternative of setting $\Phi = 0, \mu^i_j = 0$ and allowing $Q, \tilde{Q} \neq 0$ yields what is known as the Higgs branch $\mathcal{H}$. In a theory with fundamental hypers, the Higgs branch carries no residual gauge symmetry. The defining equations of a Higgs branch (from 1.13) define a hyper-kahler manifold by way of a hyper-kahler quotient construction. Classically, the Higgs branch intersects the Coulomb branch when $Q, \tilde{Q} = 0$.

In more general cases, there may not be a true Higgs branch but only a 'maximally' Higgsed branch, where some residual $U(1)^k$ gauge symmetry might remain. Such a branch is sometimes referred to as a 'Kibble branch' in the literature. To simplify terminology, the term Higgs branch will henceforth be used to encompass these cases as well. This more general notion is more convenient in the context of generalized S-duality and Gaiotto gluing.

When the masses μ^i_j are non-zero, some of the directions in the Higgs branch will get lifted. For arbitrary values of μ^i_j, the entire Higgs branch will be lifted. More generally, one can consider cases where both Φ and $Q, \tilde{Q}$

are non-zero (say, with masses turned to zero). These would parameterize 'mixed' branches $\mathcal{K}_\alpha$. The most general structure can then be schematically described as,

$$\mathcal{M}_{vac} = \mathcal{B} \cup (\cup_\alpha \mathcal{K}_\alpha) \cup \mathcal{H}. \tag{1.13}$$

The patterns of intersection between the different branches can be quite intricate in general and can be subject to change under quantum corrections. See [8, 9] for a study of Higgs branches in several $\mathcal{N} = 2$ gauge theories.

Coulomb branch of the quantum theory

One of the important features of this theory is that moduli spaces of vacua persist in the quantum theory. For the Coulomb branch, this can be seen from the Lagrangian in (1.6) that no potential term can be generated for ϕ. At an arbitrary point of the Coulomb branch, the scalar field in the vector multiplet takes a non-zero expectation value. At such a point, the gauge group is broken to $U(1)^{rank(G)}$. The effective action for the low energy theory at a generic point on the Coulomb branch is then constrained by the fact that it has to one for a $\mathcal{N} = 2$ theory of $rank(G)$ $U(1)$ vector multiplets. The Lagrangian for such a theory is of the form,

$$\frac{1}{8\pi} \int d^4\Theta K(\bar{a}_i, a_j) + \int d^2\Theta \frac{-i}{8\pi} \tau^{ij}(a) W_{\alpha,i} W_j^\alpha + \text{cc}, \tag{1.14}$$

with the quantities $K(\bar{a}, a)$ and $\tau(a)$ being related through a locally holomorphic function $F(a)$ in the following fashion,

$$\tau^{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} \tag{1.15}$$

$$a_D^i = \frac{\partial F}{\partial a_i} \tag{1.16}$$

$$K = i(\bar{a}_D^i a_i - \bar{a}_i a_D^i). \tag{1.17}$$

The undetermined function F is called the prepotential and (a, a_D) constitute special co-ordinates for the Coulomb branch. The Kahler potential has a simple expression in terms of these special co-ordinates. Determining $F(a)$ for a given UV theory amounts to providing the solution to the problem of describing the IR behaviour of the theory on the Coulomb branch. Seiberg & Witten provided a general strategy for writing down the effective action for $U(1)$ gauge fields. An important insight here involves the electric-magnetic duality for the $U(1)$ theory and the description of a special set of co-ordinates on the Coulomb branch. This data is captured most succinctly by an algebraic curve together with a differential whose periods give the special co-ordinates at various points of moduli space. The form of this solution will be explored in more detail in the rest of the section.

Higgs branch of the quantum theory

The determination of the Higgs branch is somewhat simplified by the fact that the geometry of a classical Higgs branch is not corrected quantum mechanically. However, this simple picture is complicated by the possibilities of purely 'quantum' Higgs branches and the fact that in the case of a non-Lagrangian theory, there exists no general prescription that determines the geometry of the Higgs branch.

1.2.2 Beta function

The running of the complex coupling constant in a generic $\mathcal{N} = 2$ gauge theory is conveniently expressed in a renormalization scheme where the superpotential remains a holomorphic function of the chiral superfields. Consider a theory with gauge group G and with hypermultiplets in the representation R. The running of τ is then of the form,

$$\tau(\Lambda) = \tau_{UV} - \frac{K}{2\pi i}\log\frac{\Lambda}{\Lambda_{UV}} + \dots, \tag{1.18}$$

where $K = 2C(Ad) - C(R)$, with $C(r)$ for any rep denoting defined by,

$$\text{tr}(T^a)_r(T^b)_r = C(r)\delta^{ab}. \tag{1.19}$$

The correction to τ_{UV} in (1.18) is a one-loop effect. An n loop contribution to (1.18) would be of the form $Im(\tau)^{(1-n)}$. But, such a term is holomorphic in τ only for $n = 1$. So, the requirement of homolomorphy renders the beta function in (1.18) one loop exact. Therefore, any further terms appearing in the $(\dots)$ should be of non-perturbative origin. Including the general form of these non-perturbative corrections, one can write

$$\tau(\Lambda) = \tau_{UV} - \frac{K}{2\pi i}\log\frac{\Lambda}{\Lambda_{UV}} + \sum_i \alpha_i e^{-\frac{8\pi i}{g^2}} \tag{1.20}$$

Using the relationship between the τ (close to the UV) and Λ, the non-perturbative terms can be written as an expansion in Λ^4,

$$\tau(\Lambda) = \tau_{UV} - \frac{K}{2\pi i}\log\frac{\Lambda}{\Lambda_{UV}} + \sum_i \tilde{\alpha}_i \left(\frac{\Lambda^4}{\Lambda_{UV}}\right)^i \tag{1.21}$$

From the above consideration, it is clear that for the perturbative beta function to be zero in Lagrangian theories, one needs $K = 0$. Let the field content correspond to that of a vector multiplet in the adjoint of G and hypers in the representations R and $\bar{R}$. Then the following condition should be obeyed,

$$\sum_{\alpha \in \text{weights(Ad)}} (\alpha . a_E)^2 = \sum_{w \in \text{weights(R)}} (w . a_E)^2 .$$

Theories in which such a condition is obeyed are *superconformal*. A particular example of such a theory is the $SU(2)$ theory with $N_f = 4$. This theory is a helpful prototype for the considerations of the next chapter. But, we proceed first to the Seiberg-Witten solution for the Coulomb branch of the pure $SU(2)$ theory and then proceed to describe the case of the SCFT. These are the simplest Lagrangian theories for which the Seiberg-Witten solution is known. It should be noted here that in subsequent chapters, several non-Lagrangian theories will also figure prominently. The fundamental ideas behind describing a Seiberg-Witten solution extend to these cases as well.

1.3 Seiberg-Witten solution for the pure $SU(2)$ theory

In the original work of [166], the solution to the pure $SU(2)$ is described in terms of the following curve,

$$y^2 = x^3 - 2ux^2 + \Lambda^4 x, \tag{1.22}$$

with the SW differential being given by dx/y. The SW curve above is a family of elliptic curves parameterized by the Coulomb branch parameter u. On the Coulomb branch $\mathcal{B}$ of this theory, three points exist where the curve is singular. Denote these special points by $u = (+\Lambda^2, -\Lambda^2, \infty)$. Denote the two kinds of cycles on the elliptic curve as A cycles and B cycles. When u is varied in a loop around one of the singular points, one gets back the same torus but with a different basis of cycles that can be denoted as (A', B') cycles. The matrix that implements this base chance is a monodromy matrix M,

$$\begin{pmatrix} B \\ A \end{pmatrix} = M \begin{pmatrix} B' \\ A' \end{pmatrix}, M \in SL(2, \mathbb{Z}) \tag{1.23}$$

The monodromy matrix M is different at the three special values for u. Denote these monodromy matrices by $M_\infty, M_{+\Lambda^2}, M_{-\Lambda^2}$. They, however, obey the following obvious constraint,

$$M_{+\Lambda^2} = M_\infty M_{-\Lambda^2}. \tag{1.24}$$

In the case of the pure $SU(2)$ theory, the monodromy matrices can be deduced from (1.22) to be

$$\begin{aligned} M_\infty &= \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} \\ M_{+\Lambda^2} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ M_{-\Lambda^2} &= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \end{aligned} \tag{1.25}$$

It can be checked that (1.24) is obeyed in this case. The strategy employed in [166] was to start with the knowledge of M_∞ and then work towards a curve that gave a consistent picture with the known S-duality of the $\mathcal{N} = 2$ $U(1)$ theory and the behaviour of particles which saturate the BPS bound,

$$M \geq\mid Z \mid \tag{1.26}$$

Particles which saturate the above bound form *short* multiplets of the centrally extended $\mathcal{N} = 2$ algebra. Since the number of degrees of freedom contained in such a multiplet can not change abruptly, the formulas determining their masses are guaranteed to hold non-perturbatively. The special coordinates (a, a_D) that entered the description of the low-energy effective action are determined by the following period integrals,

$$a = \int_A \frac{dx}{y}, \tag{1.27}$$

$$a_D = \int_B \frac{dx}{y}. \tag{1.28}$$

The co-ordinates a, a_D enter the formula for the mass of BPS particles in the following way,

$$M =\mid ma + na_D \mid, \tag{1.29}$$

where (m, n) are electric and magnetic charges. As emphasized earlier, the above formula for the mass of a BPS particle holds even in the quantum regime. The functions $a(u)$ and $a_D(u)$ vary such that (1.29) is always obeyed. Around a singular point on the Coulomb branch where particles of charge (m, n) are massless, the local monodromy can be written as,

$$M_{m,n} = \begin{pmatrix} 1 + mn & -m^2 \\ n^2 & 1 - mn \end{pmatrix}. \tag{1.30}$$

Now, comparing with (1.26), one observes that the particles becoming massless at $u = +\Lambda^2, -\Lambda^2$ are, respectively those with charges $(0, 1), (2, 1)$. Appropriately, the former is called the 'monopole' point while the latter is called the 'dyon' point.

The singular points on the quantum Coulomb branch are indicated in an accompanying figure (Fig 1.2). Also included is the classical picture of the Coulomb branch (Fig 1.1).

1.3.1 Rewriting the SW solution : a first take

An alternative way to write the Seiberg-Witten curve is as a branched cover of a 'UV curve' [87]. This alternative form already appears, for many examples in [142, 58]. It also appears in [200]. It is suitable to call this a 'branched curve' form of the Seiberg-Witten curve. Let us illustrate this by writing the SW curve for the pure $SU(2)$ theory as,

$$\Lambda^2 z + \frac{\Lambda^2}{z} = -x^2 - u, \tag{1.31}$$

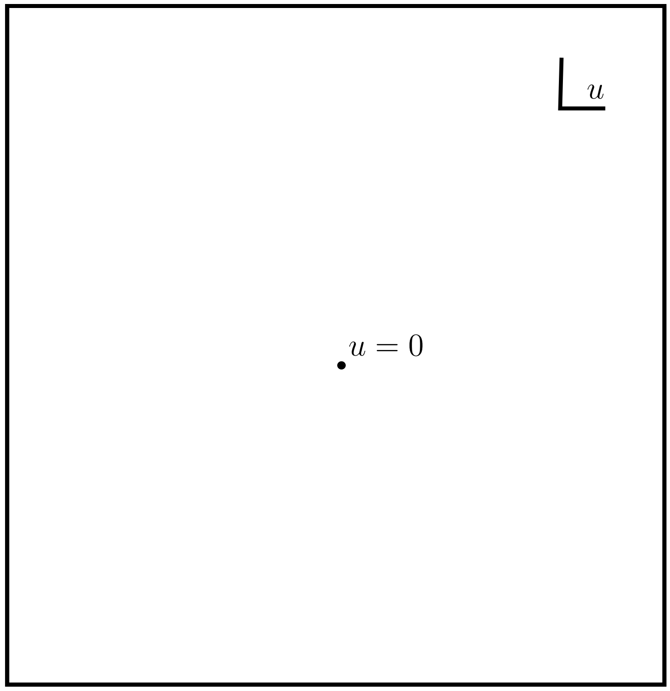

Figure 1.1: The classical Coulomb branch of the pure $SU(2)$ theory

while the SW differential is $\lambda = xdz/z$. The space parameterized by z is called the 'UV curve'. It turns out that a slight variant of the 'branched curve' form is most convenient to make explicit the connection to certain integrable systems. In this sense, the branched curve form serves as a useful intermediate step.

1.3.2 Rewriting the SW solution : a second take

With a slight redefinition of the variables, one can bring the 'branched curve' form of the Seiberg-Witten curve to a canonical form that makes it convenient to see the connection to an associated integrable system called

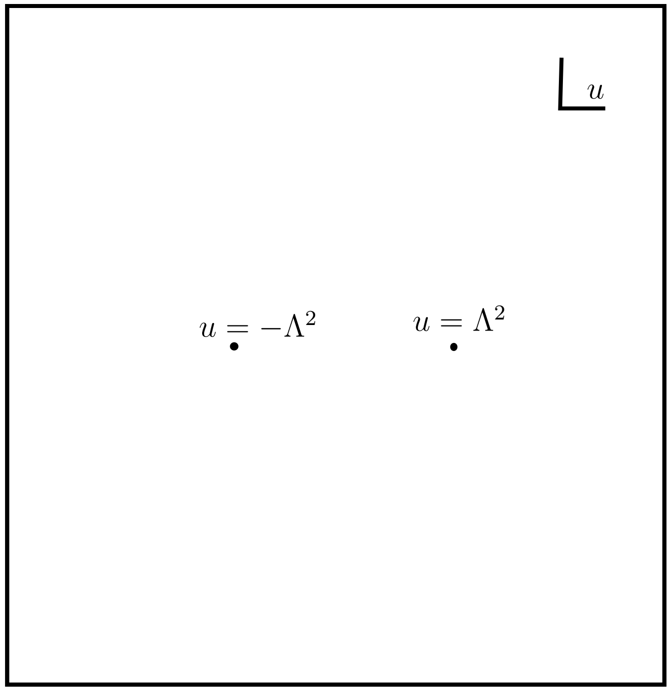

Figure 1.2: The quantum Coulomb branch of the pure $SU(2)$ theory

the Hitchin system. The Seiberg-Witten curve is the *spectral curve* of the associated Hitchin integrable system. In the particular case of the pure $SU(2)$ theory, this redefinition is quite elementary. For the theories with hypermultiplets, where masses can enter the picture, the redefinition offers a more visible increase in clarity. Following [87, 86], one can rewrite the SW curve for the pure SU(2) in the following canonical form [87],

$$\lambda^2 - \phi_2(z) = 0, \phi_2(z) = \left(\frac{\Lambda^2}{z} + u + \Lambda^2 z\right)\frac{dz^2}{z^2}, \tag{1.32}$$

with the SW differential given by $\lambda = ydz$. Here, (y, z) are co-ordinates in $T^{\star}C$, where C is the UV curve. The convenience of the canonical form is that it can readily be seen to be of the form $det(\lambda - \Phi) = 0$. Thus, it is tempting to also call this the 'spectral curve form' of the SW curve. The connection between the Seiberg-Witten solution and an associated Hitchin system has numerous consequences and will be a dominant underlying theme in many of the considerations that follow. This connection is discussed briefly in this chapter (see the definition of theories of class $\mathcal{S}$ below). A more detailed discussion follows in the next chapter.

1.4 Seiberg-Witten solution of the $SU(2), N_f = 4$ theory

Having studied the pure $SU(2)$ theory, we now turn to the $SU(2)$ theory with $N_f = 4$. As seen earlier, this theory is conformal. When the hypermultiplet masses are set to non zero values μ_i, the SW curve for this theory is given by (in the 'branched curve' form),

$$\frac{(x-\mu_1)(x-\mu_2)}{z} + f(x-\mu_3)(x-\mu_4)z = x^2 - u. \tag{1.33}$$

To see the connection to an underlying integrable system of Hitchin type, it is convenient to rewrite the above curve in the canonical form (related to above one by a variable transformation),

$$\lambda^2 - \phi_2(z) = 0, \tag{1.34}$$

where

$$\phi_2(z) = \frac{P(z)}{(z-1)^2(z-q)^2}\frac{dz^2}{z^2}, \tag{1.35}$$

where $P(z)$ is a polynomial that contains the dependence on the μ_i. A schematic of the quantum Coulomb branch is given in Fig 1.3 (with masses non-zero) and Fig 1.4 (with zero masses).

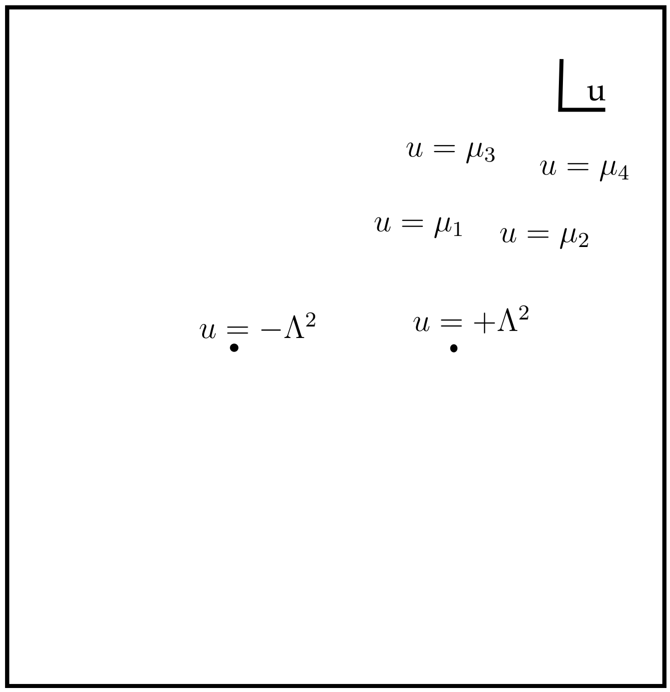

Figure 1.3: The quantum Coulomb branch of the pure $SU(2)$ theory with masses $\mu_i \neq 0$

1.4.1 Mass deformations and the flavor symmetry

The flavor symmetry F of the theory acts on the Higgs branch. When an arbitrary mass deformation is allowed, the Higgs branch is lifted. A remnant of the action of the Flavor symmetry is seen in action of the Weyl group of the flavor symmetry group on the mass parameters. In effect, the

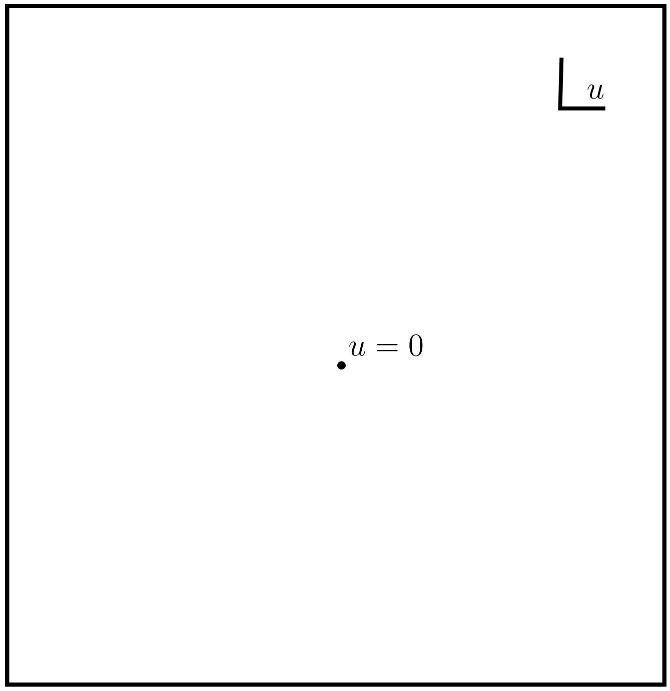

Figure 1.4: The quantum Coulomb branch of the pure $SU(2)$ theory with masses $\mu_i = 0$.

group $W(F)$ acts on the space of deformations of a geometry, in this case the special Kahler geometry of the Coulomb branch of a 4d $\mathcal{N} = 2$ field theory.

1.4.2 Retrieving the asymptotically free theories

Once the mass deformed version of the Seiberg-Witten solution is known for a $\mathcal{N} = 2$ SCFT, one can take some of the mass parameters to be infinite to obtain the asymptotically free theories. This can also be seen clearly from the schematic of the u plane for the $SU(2)$ theory with $N_f = 4$ (in the limit where $\mu_i \to \infty$).

1.4.3 Breaking to $\mathcal{N} = 1$

When a $\mathcal{N} = 2$ theory (say with a dynamical scale Λ) is deformed by a soft breaking term $W_{tree} = \frac{m}{2}\text{tr}(\Phi^a\Phi^a)$ for $m < \Lambda$, the moduli spaces of vacua are lifted and one gets a $\mathcal{N} = 1$ theory with a finite number of vacua. These vacua can be identified with the singular points on the Coulomb branch of the original $\mathcal{N} = 2$ theory. At these vacua, $\mathcal{N} = 1$ theories experience confinement due to the forming of a condensate of magnetically charged particles. This realizes, in a concrete manner, 't-Hooft's picture of confinement as a magnetic version of the Higgs mechanism in many $\mathcal{N} = 1$ theories.

The analysis of the dynamics of $\mathcal{N} = 1$ theories obtained by a soft breaking of the $\mathcal{N} = 2$ theory with a known Seiberg-Witten solution goes back to [166]. As a sample of the new insights into the construction and dynamics of $\mathcal{N} = 1$ theories in light of recent developments, see [12, 95, 84].

1.5 Seiberg-Witten solution, the Hitchin system and class $\mathcal{S}$

The somewhat special nature of the Hitchin system and the techniques, not all of which are part of the traditional toolkit of a quantum field theorist, that become available when a particular $\mathcal{N} = 2$ theory is associated to Hitchin system(s) have motivated the following definition of a subset of four dimensional $\mathcal{N} = 2$ theories called theories of class $\mathcal{S}$.

Definition 1. Theories of class $\mathcal{S}$: These are four dimensional $\mathcal{N} = 2$ field theories that possesses a Coulomb branch that can be described using a

Hitchin system associated to a Riemann surface $C_{g,n}$, together with data that specifies the behaviour of the fields of the Hitchin system at the n singularities. The Seiberg-Witten curve of such a theory is the spectral curve associated to the Hitchin system and the Seiberg-Witten differentials are the conserved Hamiltonians of the integrable system.

The connections between the Seiberg-Witten solution and integrable systems were first observed in [98, 142, 58]. The language of Hitchin systems appears first in [58] in the form of certain special cases. The connection between the more general Hitchin system and a vast class of $\mathcal{N} = 2$ theories appears in [87].

A few closely related themes of research are worth mentioning at this point. Firstly, it is *not* true that the Seiberg-Witten solution for an arbitrary $\mathcal{N} = 2$ theory has a relationship to the Hitchin system. There is, however, still an 'integrable system' hovering around the (rigid) special Kahler geometry of the Coulomb branch of many $\mathcal{N} = 2$ theories. A second theme that is common across both classes of theories is that the associated integrable systems admit a two-parameter deformation that is sometimes referred to as a 'doubly *quantum*' integrable system. Physically, this roughly corresponds to formulating[2] the four dimensional theory on an $\Omega_{\epsilon_1,\epsilon_2}$ background. The conjecture of Alday-Gaiotto-Tachikawa (which is discussed further in the section below and in Chapter 4) can be understood to be a special case of this

[2]Note that this is not a very precise notion since it is far from clear as to what formulating one of the non-Lagrangian theories on an Omega background ($\Omega_{\epsilon_1,\epsilon_2}$) means.

general theme. A specialization of this two parameter deformation arises in the works of Nekrasov-Shatashvili [157], Nekrasov-Pestun [156] and this includes cases where the associated 'quantum' integrable system is not of Hitchin type. It is notable that quantized versions of the Hitchin integrable system also play a role in certain approaches to the Geometric Langlands Program. This connection is outlined further in the next chapter.

1.6 Under the hood

In the rest of the current chapter, certain questions relevant to the physics of $\mathcal{N} = 2$ theories that are either constrained/determined by the Seiberg-Witten solution are discussed. One of the remarkable features of the SW solution is that it encodes in an efficient way atleast a part of the answers to these questions.

1.6.1 The BPS particle spectrum

Recall that in the maximally supersymmetric $\mathcal{N} = 4$ case, the ability to understand the spectrum of BPS particles and the fact that the spectrum was in conjunction with the expectations of Montonen-Olive S-duality constituted major evidence in favor of the S-duality proposal [172, 171]. Sen's constructions in these papers were also central to the resurgence of interest in dualities in the study of non-perturbative aspects of four dimensional quantum field theories.

In the context of $\mathcal{N} = 2$ theories and their description by Seiberg-

Witten theory, a corresponding question to ask is how the spectrum of one particle BPS states varies as one moves to different points on the Coulomb branch. The monodromy data that is part of the SW solution demands a certain behaviour of this spectrum near the singular points of the Coulomb branch. A useful way to analyze the behaviour of this spectrum is to calculate the following index (the second helicity super-trace)

$$\Omega(\gamma; u) \equiv -\frac{1}{2}\mathrm{Tr}_{H_{BPS}}(-1)^{2J_3}(2J_3)^2, \tag{1.36}$$

where γ is the charge of the BPS state, u parameterizes the Coulomb branch and J_3 is a rotation generator in $SO(3)$, the little group associated to massive particles in four dimensions. The index defined above is piecewise constant as one varies u and jumps across walls of marginal stability. The jumps $\Delta\Omega$ across such walls are now understood to be described by Wall Crossing Formulas (WCF) that originally appeared in the study of Donaldson-Thomas invariants attached to certain Calabi-Yau three folds. A physical interpretation of this formula can be obtained by considering the four dimensional theory formulated on $\mathbb{R}^3 \times \mathbb{S}^1_R$. The low energy theory is described by a $3d$ sigma model with a hyper-Kahler target space. When the four dimensional Coulomb branch is described as the base of a Hitchin system, the target space of the 3d sigma model obtained upon compactification is the total space of the Hitchin system. The smoothness of this metric (calculated at finite R) determines the wall crossing behaviour in the 4d Coulomb branch [88].

1.6.2 Seiberg-Witten solution and the instanton expansion

Recall from an earlier discussion that the complex coupling constant has the following expansion in an $\mathcal{N}=2$ theory,

$$\tau(\Lambda)=\tau_{UV}-\frac{K}{2\pi i}\log\frac{\Lambda}{\Lambda_{UV}}+\sum_i \alpha_i e^{-\frac{8\pi i}{g^2}}. \tag{1.37}$$

An expansion similar to the above one exists for the prepotential $F(a)$ as well (recall that $\tau=\partial^2 F/\partial a^2$). The non-perturbative terms in the expansion are determined by the Seiberg-Witten solution in a somewhat indirect but calculable manner (see [143] for the original work and an appendix of [186] for a review). It is an interesting question as to whether such non-perturbative data can be calculated directly from the non-abelian UV theory by direct instanton calculations. In several cases, this is actually possible along the lines of [114]. More generally, a technique of formulating the four dimensional theory on a rigid supergravity background called the $\Omega_{\epsilon,-\epsilon}$ background allowed Nekrasov-Okounkov to arrive at the Seiberg-Witten solution for several $\mathcal{N}=2$ theories [155].

A version of the Omega background (more precisely, it is $\Omega_{\epsilon,\epsilon}$) of Nekrasov et al also appears in Pestun's computation of the full non-perturbative partition function for several Lagrangian $\mathcal{N} = 2$ theories formulated on the four sphere via localization techniques [160]. This is also the setting for the AGT conjecture and forms the subject matter of Chapter 4 where a more detailed discussion is provided.

Chapter 2

The view from six dimensions

A central theme that will be exploited in the rest of this dissertation is the fact that several four dimensional $\mathcal{N} = 2$ theories admit a construction starting from six dimensions. The starting point in six dimensions is one of the interacting $\mathcal{N} = (0, 2)$ supersymmetric SCFT(s). Such SCFTS have an A,D,E classification. The construction of the corresponding superconformal algebra goes back to [152]. The actual construction of such theories is much more recent [198, 183]. The constructions proceed by considering various limits in String/M theory where the gravitational degrees of freedom are decoupled and a local quantum field theory[1] describes the remaining degrees of freedom[170].

In this Chapter, several results about the six dimensional theory and its behaviour under dimensional reductions are recalled. This class of theories has been called $\mathfrak{X}[\mathfrak{j}]$ in recent literature and this nomenclature will be adopted in what follows. It is sometimes convenient to just talk of 'the theory $\mathfrak{X}[\mathfrak{j}]$' but such usage should be understood to always refer to the entire family.

[1] In should be noted that there exist limits in String/M theory that yield non-gravitational theories that are non-local. An example of this is *little string theory*.

This theory is superconformal and has $osp(6,2|4)$ as its superconformal algebra. This algebra has the following form,

$$\{Q^i_\alpha, Q^j_\beta\} = \eta^{ij}(\gamma^m)_{\alpha\beta}P_m. \tag{2.1}$$

There is a centrally extended version of the above algebra [115],

$$\{Q^i_\alpha, Q^j_\beta\} = \eta^{ij}(\gamma^m)_{\alpha\beta}P_m + (\gamma^m)_{\alpha\beta}Z^{[ij]}_m + (\gamma^{mnp})_{\alpha\beta}Z^{(ij)}_{mnp} \tag{2.2}$$

This form of the algebra is important to understand the presence of 1/2 BPS defect operators in the theory. The first type of central term allows for the existence of 1/2 BPS strings (with a two dimensional world volume) while the second kind of central terms allows for the existence of 1/2 BPS codimension two defects (with a four dimensional world volume) [50]. The latter objects will play a significant role in the discussions that follow.

The basic representation associated to the above algebra is the tensor multiplet. Such a multiplet consists of self-dual, closed three-form $H_{\mu\nu\rho}$ ($\mu,\nu,\rho = 0\ldots5$), five scalars X^k ($k = 1\ldots5$) and sixteen fermions ψ^i_α ($i = 1\ldots4, \alpha = 1\ldots4$). The field H is the curvature associated to the two-form B which has an abelian gauge symmetry,

$$B'_{\mu\nu} = B_{\mu\nu} + (\partial_\mu\chi_\nu - \partial_\nu\chi_\mu). \tag{2.3}$$

One can construct free and interacting theories in six dimensions using the abelian tensor multiplets. On the other hand, Theory $\mathcal{X}[\mathfrak{j}]$ is, heuristically, a 'non-abelian' version of such theories. But, no known construction of this theory using such a non-abelian version of the tensor multiplet exists in the literature.

2.1 Constructions of theory $\mathcal{X}[\mathfrak{j}]$

The only known constructions of this theory arise as limits of string/M theory. For $\mathfrak{j}$ classical, the theory can be viewed as the theory on a stack of M5 branes. In this picture, $M2$ branes ending on the $M5$ branes (placed in $\mathbb{R}^5$ for $\mathfrak{j} = A_n$ and in $\mathbb{R}^5/\mathbb{Z}_2$ for $\mathfrak{j} = D_n$) become light as the $M5$ coincide and provide the light degrees of freedom that live on the brane[183]. An alternative construction using type II string theory on an ADE singularity allows one to obtain the cases for exceptional $\mathfrak{j}$ as well [197]. Neither of these constructions provide a conventional description of the theory in terms of Lagrangians and Action principles. In fact, the behaviour of the theory under dimensional reductions gives reason to believe that such a description can not exist. One can consider various dimensional reduction schemes to study this six dimensional theory. But, certain simple schemes outline the surprising properties of this theory in an obvious way. For example,

- Compactification on a circle $\mathbb{S}^1_R$ of radius R yields 5d maximally supersymmetric Yang-Mills theory with gauge group G with a gauge coupling that grows with the radius R. This can be understood from the fact that M theory on a $\mathbb{R}^{1,9} \times \mathbb{S}^1_R$ reduces to Type IIA string theory on $\mathbb{R}^{1,9}$ for small values of the radius R. $M5$ branes that wrap the circle are identified with $D4$ branes in the type IIA theory. The world volume theory on the stack of $D4$ branes is 5d maximally supersymmetric Yang-Mills with gauge group G.

- Compactification on a torus $\mathbb{T}^2$ yields 4d maximally supersymmetric Yang-Mills theory with gauge group G and with coupling τ that equals the complex structure of the torus. This can be seen by extending the above discussion to the case with an additional circle $\tilde{\mathbb{S}}^1$ and using the T-duality between type IIA and type IIB when both theories are compactified on $\tilde{\mathbb{S}}^1$. The $D4$ brane compactified on $\tilde{\mathbb{S}}^1$ is identified with the D3 brane of type IIB string theory. The world volume theory on a stack of $D3$ branes is $\mathcal{N} = 4$ SYM.

Among the other peculiar properties of this theory is the fact that in the classical cases, the number of degrees of freedom in the large rank limit behaves as $rank(\mathfrak{j})^3$ (see [107] for an argument for type A and [204] for type D) as opposed to $rank(\mathfrak{j})^2$ in conventional Lagrangian theories based on a gauge group.

Now, a few of the other attempts at constructing the six dimensional theory and some of their successes will be noted briefly without attempting a more complete discussion. In the cases where $\mathfrak{j} = A_n, D_n$, there exists an alternative approach to understanding the six dimensional theory using the AdS/CFT correspondence. For example, in the case of $\mathfrak{sl}_N$, the AdS dual description is via M-theory on $AdS_7 \times S^4$. The rank$(\mathfrak{j})^3$ growth of number the degrees of freedom has been derived using this realization [110]. An approach using a discrete light cone gauge formulation of the six dimensional theory has been used in [3] to obtain a realization of the chiral primaries as certain special co-ordinates in an associated moduli-space.

2.1.1 Moduli space of vacua

These theories have a moduli space of vacua (called a "Coulomb branch" by a slight extension of the usual notion of a Coulomb branch) where the theory is described by interacting abelian tensor multiplets. The space of vacua is parameterized by the vevs of the scalars in the abelian tensor multiplet. There are $5 \times \text{rank}(\mathfrak{j})$ of them and the vacuum moduli space has the following structure,

$$\mathcal{M}_{vac} = \mathbb{R}^5 \otimes \mathfrak{h}[\mathfrak{j}]/W[\mathfrak{j}]. \tag{2.4}$$

2.2 Theory $\mathcal{X}[\mathfrak{j}]$ as a relative field theory

Unlike quantum gauge theories that genuinely depend on a gauge group G, the theory $\mathcal{X}[\mathfrak{j}]$ is dependent only on the choice of a lie algebra $\mathfrak{j}$. Further, the theory defined on a six manifold does not have a partition function, but instead has a partition vector [202, 199]. In all of these respects, the six dimensional theory exhibits some unusual properties. When considering the dimensional reduction of such a theory to one lower dimension, a more conventional dependence on a gauge group G emerges [77]. If the compactification involves a twist by an outer automorphism of the Dynkin diagram associated to $\mathfrak{j}$, then G is the compact group associated to the 'folded' Dynkin diagram. If the twist is trivial, the G is identical to J, the compact lie group associated to $\mathfrak{j}$.

This state of affairs is somewhat reminiscent of the dependence of the *chiral* WZW model in 2d on a lie algebra. Much like the six dimensional theory, the chiral WZW model also lacks a partition function. Both of these theories can be understood as being 'relative field theories' (see [77]). In the case of the 2d WZW model, a dependence on a group G emerges in the WZW/Chern-Simons connection where the 2d theory is realized as the theory on the boundary of a three manifold on which Chern Simons theory with gauge group G is defined. The analogy between the six dimensional theory and the WZW model can be used in other contexts as well. Say, for example, in the study of defect operators of the six dimensional theory (See discussion in the next section).

2.3 Supersymmetric defect operators

From the supersymmetry algebra, it is clear that the six dimensional theory has BPS strings and BPS three-brane defects. The latter defects are alternatively called codimension two defects, reflecting the fact that they are four dimensional defects in an ambient six dimensional theory. This latter name has been more common in the recent literature and will be adopted here. The codimension two defect operators of the six dimensional theory play a crucial role in the construction of $\mathcal{N} = 2$ theories in four dimensions. The four dimensional world volume of these defects is expected to support a 4d $\mathcal{N} = 2$ theory. One would like to know how these degrees of freedom on the defect [2] can be coupled to the bulk degrees of freedom. Since the six dimensional theory lacks a description in terms of Lagrangians and classical fields, there has been no realization of the above goal.

An alternate strategy to study these defect operators is to consider the six dimensional theory together with a single defect operator under various dimensional reduction schemes. This allows one to relate the existence of the defect operators in six dimensions to the existence of defect operators in certain lower dimensional field theories. The latter scenario often allows for more detailed analysis. For example, reducing to four dimensional affords a link with the defect operators of $\mathcal{N} = 4$ SYM with gauge group G. The modern viewpoint on classifying defect operators in such Lagrangian field theories is to proceed by describing the behaviour of the bulk fields near the world volume of the defect [121] together with some additional data describing the coupling of the bulk fields to the degrees of freedom that live on the defect. This point of view has allowed detailed investigations of defects of various dimensions in $\mathcal{N} = 4$ SYM (For example, see [99, 201, 92]).

Codimension two defects and the Weyl group $W[g]$

One can also try and understand some properties of these defects by extending the analogy between the six dimensional theory and the 2d WZW

[2] When the word 'defect' is used without further clarification, it can be assumed to correspond to the codimension two defect of the six dimensional theory or some dimensional reduced form of it.

model. In the discussion that follows, it is best to think of the case of chiral non-compact WZW models based on the lie algebra of a *complex* group $G_{\mathbb{C}}$. The codimension two defects of the six dimensional theory are somewhat analogous to the primaries of the chiral WZW model. The traditional way of classifying primaries in the WZW model is to look for operators whose classical limit is of the form $V_j = e^{2(j,\phi)}$ where $j \in \mathfrak{h}^*$. There is a $\hat{g}$ Verma module associated to this primary. The most general such primary corresponds to the case of a principal series representation of $G_{\mathbb{C}}$ and in this case, there are no null vectors in the associated Verma module.

For special values of j, such null vectors can appear. Now, it is important to note that specifying the value of j is a highly redundant way to tag a primary. This is due to the fact that any Weyl reflection of j would correspond to the same primary. So, it would be more convenient to have certain properties associated to the primary V_j which do not change under Weyl reflections. Luckily, such quantities do exist. These are given by the values of the quadratic Casimir and other higher Casimir [3] operators (of degrees equal $e_i + 1$, where e_i are the exponents of $\mathfrak{g}$). Let us denote these by $\mathcal{I}_k$ where $\mathcal{I}_2$ is the quadratic Casimir. For the most general WZW primary, these quantities are all independent. However, once null vector relations appear in the Verma module, the invariants then obey certain relations. The

[3] By a theorem that is independently due to Chevalley and Harish-Chandra, these Weyl invariant functions generate the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

following map,

$$\text{null vectors} \leftrightarrow \text{relations between } \mathcal{I}_k \tag{2.5}$$

can be quite complicated to describe in general. But, it is clear that such a map exists. A priori, the above discussion may have nothing to say about the problem of having to describe codimension two defects of the six dimensional theory. The latter are four dimensional objects and their description is expected to mimic the complexities of a 4d $\mathcal{N} = 2$ field theory. This is significantly more data than what accompanies a zero dimensional object like a primary in a WZW model. But, it turns out that there is atleast a pattern that persists between the two examples. As will be shown in the next chapter, there is a map analogous to the one above,

$$\text{4d Higgs branch } \leftrightarrow \text{4d Coulomb branch} \tag{2.6}$$

that behaves in a very similar way to (2.5). In the above map, the 4d Higgs branch and the 4d Coulomb branch are to be understood as data that are strictly *local* to a single codimension two defect. This analogy with null vectors/invariants can actually be made precise by understanding the AGT primary map where a primary in a Toda theory is assigned to a given codimension two defect. The largest such primary corresponds to a 'self-dual principal series' representation and corresponds to the defect with the largest Higgs and Coulomb branches.

This primary does not contain any null vectors in the corresponding $\mathcal{W}$ algebra Verma module. Smaller primaries, on the other hand, obey certain null vector relations. The data that specifies the null vector relations is most directly related to the Higgs branch attached to the defect. The Higgs branch is a hyper-kahler space that can be described as a Slodowy slice in the nilpotent cone $\mathcal{N}_{\mathfrak{g}}$. On the other hand, attached to the primary are certain invariants Δ_k. The pattern of relations among Δ_k controls the contribution of the primary to the bootstrap problem (equivalently, to the space of conformal blocks in this non-rational CFT). By the AGT dictionary, this is related to the local contribution to the Coulomb branch. Such data is, most naturally, associated to certain nilpotent orbits in $\mathfrak{g}^\vee$. A detailed picture relating these descriptions is the main subject of Chapter 3 (see the beginning of Chapter 5 for a summary of the results of Chapter 3). The discussion above using the WZW model should be seen as an analogy. One way to make it more precise would be through an understanding of the map between primaries in gauged WZW models and Toda models. A discussion of this map for type A appears in Chapter 4.

2.4 Compactification on $\mathcal{C}_{g,n}$

Now, we discuss the general construction of theories of class $\mathcal{S}$. One begins with the six dimensional theory with a collection of n codimension two defects $\{D_i\}$. Now, formulate the theory on $\mathbb{R}^{1,3} \times C_{g,n}$ together with a partial twist so that eight of the original sixteen supersymmetries can be preserved under compactification [86]. The codimension two defects are taken to span all of $\mathbb{R}^{1,3}$ and live at the punctures on the Riemann surface.

In a limit where the area of the Riemann surface is taken to zero, this yields a four dimensional theory with $\mathcal{N} = 2$ supersymmetry. The Riemann surface C is identified with the 'UV curve' from the previous Chapter.

The Seiberg-Witten curve that describes the Coulomb branch of this theory is the spectral curve of the Hitchin system associated to $C_{g,n}$. For a complete specification of this data, one needs to specify the singularity structure of the fields in the associated Hitchin system at the punctures. The nature of the defect D_i determines the singularity structure for the Hitchin fields. The connection to the Hitchin system can be best understood upon a further compactification of the four dimensional theory on $\mathbb{S}^1$ and then by interchanging[4] the order of compactifications [87]. The class of defects can be classified by the nature of the singularity in the associated Hitchin system. A deeper investigation into the properties of each defect turns out to involve simultaneous descriptions that also use Nahm boundary conditions (or a dual Hitchin system) and Toda primary operators. These aspects are taken up in greater detail in the next chapter. But, we note here the following taxonomy of defect operators.

- Tame (regular) defects : There are the codimension two defects for which the associated Hitchin singularity is a simple pole.

- Wild (irregular) defects : These are the codimension two defects for which the associated Hitchin singularity is a pole of higher order.

[4]In any such operation, one is using a QFT analog of Fubini's theorem.

2.4.1 Flavor symmetries and defects

An important insight in Gaiotto's construction is the idea that subgroups of the flavor symmetries of the four dimensional theory obtained by compactification from six dimensions can be viewed as being *attached* to a particular codimension two defect operator. For example, the $SU(2), N_f = 4$ theory can be obtained from six dimensions using the A_1 theory and compactifying it on a two sphere with four defect insertions. The global symmetry of this theory is $SO(8)$. In the six dimensional construction, one views each defect insertion as carrying a $SU(2)$ subgroup of the $SO(8)$ flavor symmetry group. So, the construction makes manifest a $SU(2)\times SU(2)\times SU(2)\times SU(2)$ subgroup of the full $SO(8)$ flavor symmetry. In the most general cases outside of j $= A_1$, the defect operators can be classified using the structure theory of nilpotent orbits of a complex semi-simple lie algebra. For a large class of such regular defects, the flavor symmetry group can also be identified using this theory (see chapter 3 for more on this). If D_i are the defects and $F(D_i)$ are the associated flavor symmetries. Then, the four dimensional theory obtained on compactifying on a Riemann surface $\mathcal{C}_{g,n}$ with the defects D_i has a global symmetry that is atleast $\prod_i F(D_i)$. In certain special cases, the global symmetry is enhanced to a larger group (as in the $SO(8)$ case above).

2.4.2 Gaiotto Gluing and generalized S-duality

The six dimensional construction also affords a beautiful geometrical picture of the generalized S-duality that the $\mathcal{N} = 2$ SCFTs obey. Specific instances of this duality were constructed by Seiberg-Witten[167] and Argyres-Seiberg[10]. Gaiotto's construction from six dimensions allows a vast expansion of the available examples where the nature of this generalized S-duality can be analyzed.

Recall that $\mathcal{N} = 2$ SCFTs have a set of marginal coupling constants. This space is identified with the compactified moduli space of the punctured Riemann surface $\overline{M}_{g,n}$. Different factorization limits of the punctured Riemann surface (alternatively, different boundary point of $\overline{M}_{g,n}$) correspond to *different* limits (potentially Lagrangian) of the same underlying quantum field theory. In each such limit, a weakly coupled gauge group appears. But, the 'matter' multiplets to which it is coupled may not be conventional $\mathcal{N} = 2$ matter which admits a description in terms of Lagrangians. The gauge coupling constant τ of the gauge group that appears in such a limit is related to the plumbing fixture parameter q of the long cylinder connecting the two factorized halves of the Riemann surface,

$$q = e^{2\pi i\tau} \tag{2.7}$$

This is schematically expressed in Fig (2.1). In the figure, $q = e^{2\pi i\tau}, q' = e^{2\pi i\tau'}$ and the groups G and G' are different in the most general cases. Even in the cases where $G = G'$, the weakly coupled gauge fields in one limit of the UV curve are not the same as the weakly coupled gauge fields in another limit of the UV curve. The association of a weakly coupled gauge group G with gluing data is sometimes called the Gaiotto gluing conjecture.

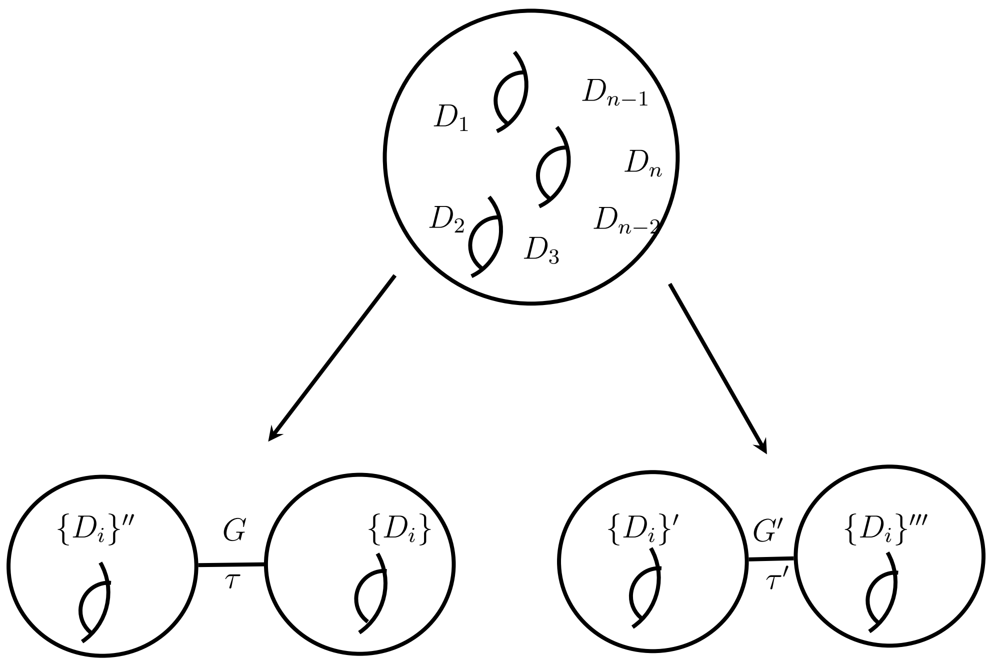

Figure 2.1: A schematic of generalized S-duality for theories of class $\mathcal{S}$.

2.4.3 Examples

S-duality of $\mathcal{N} = 2, N_f = 4$ theory

The $\mathcal{N} = 2, N_f = 4$ theory is obtained from six dimensions using the A_1 theory and four defect insertions on the four punctured sphere $C_{0,4}$. The generalized S-duality of this theory can be understood using different factorizing limits of the four puncture sphere (depicted in Fig below). Denote the flavor groups attached to the defect by $SU(2)_a, SU(2)_b, SU(2)_c, SU(2)_d$. The flavor symmetry groups get permuted under the S-duality transformations. This reflects the action of the $SL(2,\mathbb{Z})$ symmetry by triality on the three eight dimensional representations of the flavor symmetry group $SO(8)$ (See Fig 2.2).

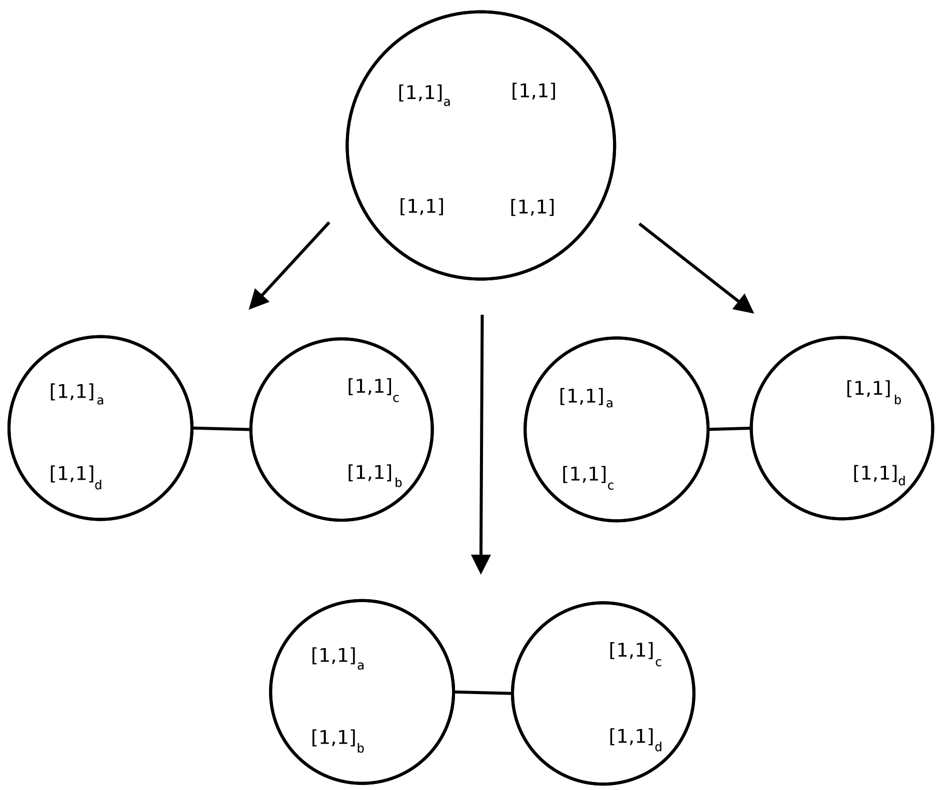

Figure 2.2: S-duality of $SU(2), N_f = 4$ theory realized as different limits of the UV curve $C_{0,4}$

Argyres-Seiberg duality

The case of Argyres-Seiberg duality corresponds to a construction using the A_2 theory together with a couple (each) of the regular and sub-regular (in this case, it is the same as minimal) defects taken on a four punctured sphere $C_{0,\{2,2\}}$. The associated global symmetries are $SU(3)$ for the regular defects and $U(1)$ for the minimal defects. This gives a net Flavor symmetry group of $SU(3) \times U(1) \times SU(3) \times U(1)$. This corresponds well with the fact that two of the degenerating limits in the UV curve of this theory admit a description in terms of conventional Lagrangians. This is the direct higher rank generalization of the SCFT of the previous section : the $SU(3)$ theory with $N_f = 6$. The third limit, however, does not have such a Lagrangian description. It turns out to be a $SU(2)$ gauge theory coupled to a fundamental hyper and a $SU(2)$ gauging of the E_6 SCFT of Minahan-Nemeschansky [146]. This pattern of generalized S-duality is depicted in Fig 2.3.

This example clearly demonstrates that the answer to the question "What is the non-perturbative physics of a gauge theory ?" is not even necessarily in the form of another gauge theory. For the class of $\mathcal{N} = 2$ theories that arise from six dimensions, this is the generic situation. The cases where there is a Lagrangian description in every corner of the coupling constant moduli space correspond to special situations.

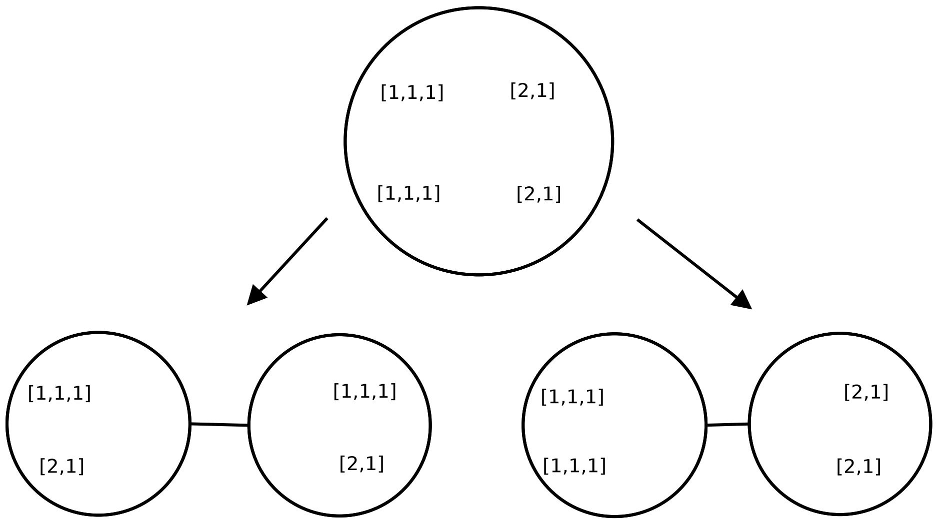

Figure 2.3: Argyres-Seiberg duality realized as different limits of the UV curve $C_{0,\{,2,2\}}$

The list of $\mathcal{N} = 2$ theories that can be obtained by constructions from six dimensions is quite vast and this includes infinitely many Lagrangian field theories and infinitely many non-Lagrangian field theories. A classification program has been carried out for low rank $\mathfrak{j}$ in a series of works [37, 38, 40, 42, 41].

However, there exist many $\mathcal{N} = 2$ theories for which a six dimensional construction is not available. The Seiberg-Witten geometry for some of these theories can still be obtained by other techniques (See [156] and [124] for a window into several such cases). For certain other theories, the solution remains unknown (see [23] for such a list).

2.5 The Hitchin system

In the rest of the chapter, some useful properties of Hitchin systems will be recalled. As discussed earlier, the relationship between Hitchin systems and $\mathcal{N}=2$ theories is a crucial part of several considerations in Chapters 3-5. The G - Hitchin system on a Riemann surface C is governed by the following Yang-Mills-Higgs type equations [112],

$$F_A + [\phi, \phi^*] = 0, \tag{2.8}$$

$$\bar{\partial}_A \phi = 0. \tag{2.9}$$

where F_A is the curvature of a connection A in a G bundle and ϕ is a Higgs field in the adjoint representation of G. The above equations form an elliptic system and they can be understood as the dimensional reduced version of the self-dual Yang-Mills equations in four dimensional Euclidean space.

The space of solutions to the above equation is denoted by $\mathcal{M}_H(C, G)$. This is a hyper-kahler manifold. In one of its complex structures (usually denoted by I), the moduli space can be described as the space of Higgs bundles (the above description). In the other complex structures J, K, the natural description of this moduli space is quite different (see below).

The hyper-kahler nature of the Hitchin moduli spaces affords the following three viewpoints (see [108, 28] for recent surveys),

- Dolbeaut viewpoint ($\mathcal{M}_H \cong \mathcal{M}_{dol}$) : In complex structure I, the Hitchin moduli space is the moduli space of Higgs bundles. In other words, this is the space of pairs (A, ϕ) where A, ϕ are the usual fields of the Hitchin system.

- de-Rham viewpoint ($\mathcal{M}_H \cong \mathcal{M}_{DR}$) : In complex structure J, the Hitchin moduli space is the moduli space of flat connections on a holomorphic vector bundle. This can be seen by building a connection $\mathcal{A} = A + \phi/\zeta + \bar{\phi}\zeta$, $\zeta \in \mathbb{C}^*$. The Hitchin equations reduce to a flatness constraint for $\partial + \mathcal{A}$.

- Betti viewpoint ($\mathcal{M}_H \cong \mathcal{M}_B$) : In complex structure K, the moduli space is the space of conjugacy classes of representations of the fundamental group, Hom$(\pi_1(C), G)/G$. This realization is also termed a character variety. $\mathcal{M}_B$ is isomorphic to $\mathcal{M}_{DR}$ (in a complex analytic sense) by the Riemann-Hilbert correspondence which is the relationship between systems of partial differential equations with specified singularity structure and the possible monodromy data for their solutions.

The Hitchin map takes a Hitchin pair (A, ϕ) to the characteristic polynomials $det(\lambda - \Phi)$,

$$f_H : (A, \phi) \to det(\lambda - \Phi). \tag{2.10}$$

The Hitchin map is a natural generalization (a spectral curve version) of the Chevalley restriction map that takes an element of the lie algebra to its characteristic polynomial (equivalently, the unordered set of its eigenvalues),

$$f_C : \mathfrak{g} \to \mathfrak{h}/W. \tag{2.11}$$

The Hitchin moduli space can be viewed as the following fibration,

$$\mathcal{M}_H \to^{f_H} \mathcal{B}, \tag{2.12}$$

where $\mathcal{B}$ is an affine space called the Hitchin base. The fiber $f_H^{-1}(u)$ for a generic $u \in \mathcal{B}$ is an abelian variety. This presentation makes it obvious that the Hitchin system has the further feature of being an *algebraic integrable system*.

The Hitchin system associated to C has a spectral curve[5] $\Sigma \subset T^*C$ defined by $det(\lambda - \phi) = 0$, where ϕ is a one form built out of the Casimirs $Tr(\phi)^k$. For theories of class $\mathcal{S}$, this spectral curve is identified with the Seiberg-Witten curve of the associated $\mathcal{N} = 2$ theory. Let (y, z) be coordinates for T^*C with z parameterizing C. The canonical differential $\lambda = ydz$ on T^*C restricted to Σ is the SW differential. Further, the Casimirs parameterize the base $\mathcal{B}$ of the Hitchin system. $\mathcal{B}$ is identified with the four dimensional Coulomb branch.

[5]Mathematically, for Hitchin systems outside of type A, it is sometimes convenient to think of a 'cameral cover' instead of a spectral cover. But, this finer point is ignored here.

2.5.1 Singularities of the Hitchin system and defects

The construction of solutions to (2.9) can be generalized to the setting of a Riemann surface with punctures $C_{g,n}$. Allowing for this generalization is crucial in the physical context for this allows one to retrieve some of the basic examples of $\mathcal{N} = 2$ theories from six dimensions. At the n punctures, the fields of the Hitchin system (A, ϕ) have singularities.

The relationship between singularities of Hitchin systems and codimension two defects is best observed by considering a class $\mathcal{S}$ construction (as discussed earlier in this chapter) and further reducing the four dimensional theory on a circle $\mathbb{S}^1$ to reduce to three dimensions. Now, invert the order of reductions from six dimensions. That is, reduce first on $\mathbb{S}^1$ (a longitudinal circle for the defect) and *then* on C. The reduction on $\mathbb{S}^1$ gives 5d SYM with gauge group G along with a codimension two defect of this theory. Focusing on the behaviour close to a single defect insertion, and considering the compactification of the 5d theory on $C \setminus \{\cdot\}$, one obtains the Hitchin equations formulated on C with certain specified singularity conditions at the defect insertion $\{\cdot\}$ [87].

In the mathematical literature, the cases with simple pole singularities are called tame singularities and the ones with higher pole singularities are called wild singularities. In physical language, this corresponds to two different classes of codimension two defects,

- Regular defects corresponding to the case of tame singularities (simple poles).

- Irregular defects corresponding to the case of wild singularities (higher poles).

The examples in the previous section correspond to cases where only regular defects were considered. The regular defects have associated flavor symmetries (as discussed earlier in the section) and correspondingly, certain mass deformation parameters. These mass parameters are eigenvalues of the matrix valued residues at the simple pole for the field ϕ. When these mass parameters are set to zero, the theories constructed using the regular codimension two defects yield certain SCFTs of class $\mathcal{S}$. The $SU(2)$, $N_f = 4$ theory is a particular example of such an SCFT. The most general such SCFT, however, is of non-Lagrangian type. By turning on mass deformation parameters and taking the limit where some/all of the mass parameters are taken to be infinite, one can obtain the asymptotically free theories of class $\mathcal{S}$. In this limit, the regular singularities collide and become wilder singularities. Incorporating the case of wild defects is also essential in obtaining the Argyres-Douglas class of SCFTs and their higher rank generalizations.

For some mathematical background on the moduli spaces of Hitchin systems with regular singularities (treated as parabolic Higgs bundles), see [175, 27]. For constructions of the Hyper-kahler structure on such moduli spaces, see [129, 154]. For the corresponding theory in the case of wild singularities, see [26]. In the physical context, consequences in the case of wild singularities have been explored in [201, 87].

The above results suffice for the purposes of Chapters 3-5. However, some further themes regarding Hitchin systems are explored in the rest of the section in the hope that they serve as useful additional background and possibly as motivation for future work extending Chapters 3-5.

2.5.2 Reduction of the Hitchin system to Nahm equations

A construction of Gukov-Witten (Section 3.8 of [99]) shows that Hitchin equations (formulated on a space of two real dimensions) reduce to Nahm equations (on a one dimensional space) under an $\mathbb{S}^1$ invariance condition. Consequently, solutions of Hitchin equations with singularities descend to solutions of the resulting Nahm equations with pole boundary conditions. This fact can be useful in extending several results in this dissertation. Here, a brief explanation of how this extension can be achieved is given.

In subsequent chapters, a Nahm system associated to a complex lie algebra $\mathfrak{g}$ and a Hitchin system associated to its Langlands dual $\mathfrak{g}^\vee$ will play important roles. One way to understand this Nahm system is as part of the specification of a boundary condition for 5d SYM with gauge group G (see Section 3.2.3 for example). Viewed from six dimensions, this scenario arises when the theory $\mathcal{X}[\mathfrak{j}]$ together with a single defect is taken on $\mathbb{R}^{2,1} \times H \times \mathbb{S}^1$, where H is a half-cigar [6]. Denote the circle of the half-cigar by $\tilde{\mathbb{S}^1}$. The

[6]Here, the term half-cigar denotes a circle fibered over $\mathbb{R}^+$ such that the fiber shrinks to zero size at the origin. This geometry is referred to as a 'cigar' in most of the physics

codimension two defect is taken to wrap $\mathbb{R}^{1,2} \times \tilde{\mathbb{S}}^1$ and is placed at the tip of H. Upon compactifying on the transverse circle $\mathbb{S}^1$, one obtains the five dimensional scenario outlined above.

Now, consider replacing the $\mathbb{R}^{1,2} \times H \times \mathbb{S}^1$ by $\mathbb{R}^{1,1} \times \hat{\mathbb{S}}^1 \times H \times \mathbb{S}^1$ and reduce to *two* dimensions by starting again in six dimensions and reducing first on $\hat{\mathbb{S}}^1$ and then on $\mathbb{R}^{1,1} \times \tilde{\mathbb{S}}^1$. Note that in this setup, the defect continues to wrap the first three co-ordinates and the $\tilde{\mathbb{S}}^1$ of the half-cigar H. So, the two circles being reduced on are ones which the defect wraps.

Local to the defect, one now obtains the Hitchin equations for $\mathfrak{g}$, now formulated on $\mathbb{R}^+ \times \mathbb{S}^1$. Note that both of these are directions transverse to the original defect in six dimensions. The defect is now described by singularities for this set of Hitchin equations. Requiring that this construction of the defect lifts to the 6d construction imposes an $\mathbb{S}^1$ invariance condition. Equivalently, compactify further on $\mathbb{S}^1$ (transverse to the defect) and require that the order of reductions does not matter. So, such defects are, on the one hand, identified with the pure Nahm boundary conditions (5d viewpoint) and as circle invariant singularities in a Hitchin system (2d viewpoint). These Nahm and Hitchin systems are now for the *same* lie algebra $\mathfrak{g}$. The existence of such a common description for the $\mathbb{S}^1$ invariant defects is not surprising given the construction of Gukov-Witten recalled above. Employing this connection, one could translate the statements made in the

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literature even though the second end of a cigar is nowhere to be seen.

Nahm$[\mathfrak{g}]$: *Hitchin*$[\mathfrak{g}^\vee]$ setting to ones in a *Hitchin*$[\mathfrak{g}]$: *Hitchin*$[\mathfrak{g}^\vee]$ setting.

2.5.3 Connections to the Geometric Langlands Program

It turns out that a quantized version of the Hitchin system plays an important role in an approach to the geometric Langlands program (GLP) initiated by Beilinson-Drinfeld [17]. In recent years, yet another approach to the GLP has been initiated by Kapustin-Witten [123] and this takes as its starting point the S-duality of $\mathcal{N}=4$ SYM. The two approaches are, heuristically, expected to be related by differing dimensional reductions from the six dimensional theory with the AGT correspondence playing a mediating role. In order to clarify this, consider an arbitrary theory of class $\mathcal{S}$ and compactify this theory further by formulating it on $\mathbb{T}^2 = S^1 \times \tilde{S}^1$. The resulting two dimensional theory is a 2d sigma model with $(4,4)$ supersymmetry. Viewed from the vantage point of the six dimensional theory, this amounts to a net compactification scheme of taking $\mathfrak{X}[\mathfrak{j}]$ (together with some defects) on $C_{g,n} \times T^2$. Now, consider changing the order of compactifications. That is, compactify first on T^2 to go from six to four dimensions. This yields $\mathcal{N}=4$ SYM potentially with some defect(s). To get to two dimensions, one further compactifies the $\mathcal{N}=4$ theory on $C_{g,n}$. This is precisely the setup considered by Witten & collaborators in the gauge theory approach to the GLP [123, 99, 81, 91, 79]. In this approach, dual (G and $G^\vee$) Hitchin fibrations over a common base $\mathcal{B}$ and the *mirror symmetry* between the two fibrations plays a central role. While the subsequent chapters in this dissertation do not make a direct connection with either of the approaches to the GLP in their full glory, unmistakable elements of the Langlands philosophy thread through the various considerations.

2.5.4 Motivic properties

A feature of the Hitchin system that captures the physics of $\mathcal{N} = 2$ theories under discussion is that many of its properties are of 'motivic' origin. This means that the corresponding properties speak to aspects of the underlying polynomial equations and are independent of the field of definition of such equations. In other words, if one is interested in a property of the Hitchin system that can be expressed in terms of purely algebraic data, one can seek an answer for such a question in a setting very different from the world of Riemann surfaces (corresponding to the field of definition being $\mathbb{C}$). For example, one can define a Hitchin fibration for curves over finite fields. This sets up the possibility of a back and forth of ideas between the different settings. Problems and tools originally developed in one setting often allow an extension to others. For an example of such a transfer of techniques from the geometric to the arithmetic side, see [151] and for a work that uses arithmetic methods to achieve geometric ends, see [109].

2.6 Outline for Chapters 3-5

With the preliminaries in place, the motivation and the results of the subsequent chapters can be outlined in greater detail. As seen earlier, an understanding of the properties of certain codimension two defect operators of the six dimensional theory is crucial to understanding the constructions of 4d $\mathcal{N} = 2$ field theories. The various available descriptions of such defects and the maps between these are the subject matter of Section 3. The overall picture emerging from these descriptions is summarized with the help of several detailed tables in Section 5. The focus of Section 4 is on the partition function of the four dimensional theories on a four sphere. In particular, the role of the scale factor in the dictionary relating such partition functions to correlation functions in certain two dimensional non-rational CFTs is explained. The existence of such a dictionary is part of a large program that has come to be called the 'Alday-Gaiotto-Tachikawa' conjecture. Chapters 3 and 4 additionally contain more detailed introductions to their respective subject material.

Chapter 3

Describing codimension two defects

3.1 Introduction

The study of defect operators in quantum field theories has a long history and has received closer attention in recent years. Apart from exposing deep connections to representation theory, such studies turn out to be useful in the understanding of various non-perturbative dualities. The six dimensional SCFT $\mathcal{X}[\mathfrak{j}]$ has played a special role in some of the recent developments along this theme. As discussed earlier chapters, the theory lacks an intrinsic description in terms of classical fields, Lagrangians and action principles and thus precludes much direct investigation. Yet, under various dimensional reductions, this theory can be better understood. The specific objects that would be the focus of this chapter are certain 1/2 BPS codimension two defects of theory $\mathcal{X}[\mathfrak{j}]$. More generally, the objects of interest are certain four dimensional $\mathcal{N} = 2$ SCFTs (and their massive deformations) that can be built out of the codimension two defects[1]. For a large class of regular (twisted or untwisted) codimension two defect of $\mathcal{X}[\mathfrak{j}]$, we have (following [39] and the general lesson from [120]),

[1]Henceforth, any invocation of the term 'codimension two defect' should be taken to mean 'codimension two defects of theory $\mathcal{X}[\mathfrak{j}]$'.

- An associated nilpotent orbit in $\mathfrak{g}$ called the Nahm orbit ($\mathcal{O}_N$). This arises as a Nahm type boundary condition in 4d $\mathcal{N} = 4$ SYM with gauge group [2] G on a half space (or equivalently a boundary condition for 5d SYM with gauge group G on a half space times a circle S),

- An associated nilpotent orbit in Langlands/GNO dual $\mathfrak{g}^\vee$ called the Hitchin orbit ($\mathcal{O}_H$) with some further discrete data that can be captured by specifying a subgroup of $\overline{A}(\mathcal{O}_H)$, where $\overline{A}(\mathcal{O}_H)$ is Lusztig's quotient of the component group of the centralizer of the corresponding nilpotent element (identified upto $\mathfrak{g}^\vee$- conjugacy). This arises as a codimension two defect for 5d SYM with gauge group $G^\vee$ on a half space times a circle $\tilde{S}$,

- A semi-degenerate primary of the Toda[$\mathfrak{g}$] theory that is given by the specification of a set of null vectors in the corresponding W-algebra Verma module.

Here, $\mathfrak{g}$ is an arbitrary simple lie algebra. For the untwisted defects, the lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{j}$ and thus simply laced. For the twisted sector defects, $\mathfrak{g}$ is the lie algebra corresponding to the folded Dynkin diagram [3]. In particular, the twisted sector defects require the cases where $\mathfrak{g}$ is

[2] The gauge group G is compact. But it turns out that the defects of concern are classified by nilpotent orbits in the complexified lie algebra $\mathfrak{g}_\mathbb{C}$, which will still denote by $\mathfrak{g}$ to simplify notation.

[3] The naming of lie algebras $\mathfrak{j}$ and $\mathfrak{g}$ is consistent with how they appear in [39].

non-simply laced. This set of regular defects will be called the CDT class of defects in the rest of the Chapter.

The availability of these multiple descriptions is convenient since different aspects of the defects become manifest when expressed in each of these terms. However, one would expect that each one of these constitute a partial description of a given codimension two defect. This chapter concerns the relationship between these three descriptions. A dictionary between the Hitchin data and the Nahm data has already been provided in [39] for arbitrary $\mathfrak{g}$ and the discussion here hopes to complement the one provided in [39]. Further, the relationship of this data to that of a Toda semi-degenerate primary is explained for a particular subset of defects that correspond to the Nahm data being a nilpotent orbit of principal Levi type. The relevant set of Toda operators were obtained in the work of [120] for type A. In type A, all non-zero nilpotent orbits are principal Levi type. So, the setup here covers all of them. Outside of type A, there are nontrivial orbits that occur as non-principal orbits in Levi subalgebras. Extending the Toda part of the dictionary to such Nahm orbits would be an interesting problem.

The task that is accomplished here is modest if viewed in the larger scheme of things and the results only point to a need for more detailed investigations into the connections between geometric representation theory and the construction of class $\mathcal{S}$ theories. It should be mentioned here that almost all of the mathematical considerations in this chapter arise from well known results and can be found in the existing literature. The one exception is a certain property that is discussed in Section 5 that places the '*Higgs branch Springer invariant*' on a different footing from what one may call a '*Coulomb branch Springer invariant*'. Further, it is hoped that the presentation of the known mathematical results is in a language that is friendly to physicists. The placing of these results in a physical framework yields some new insights into the physics and is also likely to motivate future investigations.

The plan of the Chapter is as follows. Section 3.2 offers a review of some dimensional reduction schemes used in the study of codimension two defects. Section 3.3 reviews the set of boundary conditions studied by Gaiotto-Witten and action of S-duality on certain classes of these boundary conditions. Section 3.4 collects results from the mathematical literature on order reversing duality maps and the closely related representation theory of Weyl groups. In Section 3.5, a way to relate the Hitchin and Nahm descriptions is provided using properties of the Higgs branch associated to the defect. This reproduces the setup of [39] and provides a physical framework for some defining properties of the order reversing duality used in [39]. Equivalently, this provides the S-duality map for the subset of boundary conditions in $\mathcal{N} = 4$ SYM that correspond to the CDT class of codimension two defects. In Section 3.7, a map is constructed between the set of codimension two defects and the set of semi-degenerate primary operators in Toda theory for the cases where the Nahm orbit is of principal Levi type.

In Chapter 5, the results in Section 3.5 and Section 3.7 are combined and the complete setup relating Toda, Nahm and Hitchin data is presented. Numerous realizations of this setup are collected in the tables in Section 5.2. Sections 3.5,3.7 form the core of the Chapter. It is worth emphasizing that much of the tight representation theoretic structures become obvious only with the compiling of detailed tables for various cases. Such tables are contained in Chapter 5. The arguments in Sections 3.5,3.7 apply for all simple $\mathfrak{g}$. So, the tables include data for the non-simply laced $\mathfrak{g}$ as well. These are relevant for local properties of the twisted defects of the theory $\mathcal{X}[\mathfrak{j}]$, $\mathfrak{j} \in A, D, E$ and for S-duality of boundary conditions between $\mathcal{N} = 4$ SYM with non-simply laced gauge groups G and $G^{\vee}$, where $\mathfrak{g}$ is the lie algebra corresponding to the folded Dynkin diagram [39]. However, there is a feature of the setup in the non-simply laced cases that raises some puzzles about the case for arbitrary $\mathfrak{g}$. This is discussed in Section 3.5.4 of this chapter and in Chapter 5.

Displaying information in the tables in a succinct way requires the introduction of some notation for nilpotent orbits and irreducible representations of Weyl groups. This is introduced in Appendices A, B. Also included are two appendices that provide a short summary of the Borel-de Seibenthal method (Appendix C) to find all possible centralizers of semi-simple elements and the Macdonald-Lusztig-Spaltenstein induction method (Appendix D).

3.2 Codimension two defects under dimensional reductions

Let us take the theory $\mathfrak{X}[\mathfrak{j}]$ on various six manifolds M_6 with the required partial twists to preserve some of the supersymmetries. For the current purposes, it is helpful to recall a small subset of the various reduction schemes that are helpful while studying the supersymmetric defect operators in this theory. Each scheme will be summarized by a dot $(\cdot)$ and dash $(\leftrightarrow)$ table. Unless specified otherwise, the co-ordinate labels in such tables are in the obvious order implied by the notation for the manifold M_6.

3.2.1 $\mathbb{R}^{3,1} \times C_{g,n}$

Consider the theory $\mathfrak{X}[\mathfrak{j}]$ formulated on $\mathbb{R}^{3,1} \times C_{g,n}$ where $C_{g,n}$ is a Riemann surface of genus g in the presence of n codimension two defects $\mathcal{O}_i$. When the area of the Riemann surface tends to zero, an effectively four dimensional $\mathcal{N} = 2$ field theory is obtained [200, 86].

	1	2	3	4	5	6
$\mathcal{O}_i$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\cdot$	$\cdot$

The coupling constant moduli space of such theories is the moduli space of the Riemann surface with punctures. The low energy effective action of $\mathcal{N} = 2$ theories in four dimensions is captured by the Seiberg-Witten solution. For these theories obtained from six dimensions, the SW solution is identified with an algebraic complex integrable system associated to the Riemann surface $C_{g,n}$ called the Hitchin system. In particular, the SW curve is identified with the spectral curve of the Hitchin system and the SW differentials are the conserved "Hamiltonians" of the same.

3.2.2 $\mathbb{R}^{2,1} \times \mathbb{S}^1 \times C_{g,n}$

Following [87], one can seek a description of the codimension two defect in terms of a Hitchin system using a compactification on $\mathbb{R}^{2,1} \times \mathbb{S}^1 \times C_{g,n}$, with a codimension two defect wrapping the circle $\mathbb{S}^1$.

	1	2	3	4	5	6
$\mathcal{O}_1$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\cdot$	$\cdot$

The nature of the defect is captured by the singularity structure of the Higgs fields near the location of the defect on C. When the Higgs field has a simple pole,

$$\phi(z) = \frac{\rho}{z} + \ldots, \tag{3.1}$$

it corresponds to the tamely ramified case and corresponding defects are called regular defects. For regular defects with no mass deformations, the residue at the simple pole (ρ) is a nilpotent element of the lie algebra $\mathfrak{j}$. The nature of the defect depends only the nilpotent orbit to which element ρ belongs. While prescribing the behaviour in (3.1) is sufficient to identify a defect (upto perhaps some additional discrete data), we will momentarily see that pairs of nilpotent orbits are in some ways a more efficient description of a given codimension two defect. When the poles for the Higgs field occur at higher orders, it corresponds to the case of wild ramification and the corresponding defects are called irregular defects [201, 87].

3.2.3 $\mathbb{R}^{2,1} \times H \times \mathbb{S}^1$

To see that a pair of nilpotent orbits are relevant for the description of a single codimension two defect, follow [39] and formulate $\mathcal{X}[\mathfrak{j}]$ on $\mathbb{R}^{2,1} \times H \times \mathbb{S}^1$. Here, H is a half-cigar which can be thought of as a circle ($\tilde{S}_1$) fibered over a semi-infinite line. Here again, consider the reduction with a single defect $\mathcal{O}_1$ (along with, maybe, a twist that allows for non-simple laced gauge groups to appear in five and four dimensions). The fifth co-ordinate refers to the co-ordinate along $\tilde{S}_1$.

	1	2	3	4	5	6
$\mathcal{O}_1$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\cdot$	$\leftrightarrow$	$\cdot$

Upon dimensional reduction in the fifth and six dimensions, this setup reduces to the one considered by Gaiotto-Witten [92] in their analysis of supersymmetric boundary conditions in $\mathcal{N} = 4$ SYM on a half-space. Performing a reduction first on $\mathbb{S}^1$ gives us 5d SYM with gauge group G and a codimension one defect. Further reducing on $\tilde{\mathbb{S}}^1$ gives 4d SYM with gauge group G on a half-space and 1/2 BPS boundary condition that is labeled by a triple $(\mathcal{O}, H, \mathcal{B})$, where $\mathcal{O}$ is a nilpotent orbit, H is a subgroup of the centralizer of the $\mathfrak{sl}_2$ triple associated to the nilpotent orbit $\mathcal{O}$ and $\mathcal{B}$ is a three dimensional boundary SCFT. Interchanging the order of dimensional reductions, one gets 4d SYM with gauge group $G^\vee$ on a half space with a dual boundary condition $(\mathcal{O}', H', \mathcal{B}')$. In the case of $\mathfrak{g} = A_{N-1}$, nilpotent orbits have a convenient characterization in terms of partitions of N. An order reversing duality on nilpotent orbits plays an important role in the description of the

S-duality of boundary conditions. This duality acts as an involution only in the case of A_{n-1} and fails to be an involution in the other cases. This failure to be an involution leads to a much richer and complex structure than the case for type A. This more general order reversing duality will hover around much of the considerations in the rest of the Chapter and will be discussed in greater detail in subsequent sections.

3.2.4 $\mathbb{R}^{1,1} \times \mathbb{R}^2 \times \mathbb{T}^2$

	1	2	3	4	5	6
$\mathcal{O}_1$	$\cdot$	$\cdot$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$

Here, let us consider the reduction with a single defect $\mathcal{O}_1$ on $\mathbb{R}^{1,1} \times \mathbb{R}^2 \times \mathbb{T}^2$ such that the defect wraps the $\mathbb{T}^2$ [39] (again, possibly with a twist). The theory in four dimensions is now $\mathcal{N} = 4$ SYM with gauge group G and a surface operator inserted along a surface $\mathbb{R}^2 \subset \mathbb{R}^{1,3}$. This is the kind of setup considered in [99]. The S-dual configuration is then a surface operator in $\mathcal{N} = 4$ SYM with gauge group $G^\vee$.

3.2.5 Associating invariants to a defect

Under various duality operations, it may turn out that the most obvious description of a given codimension two defect is quite different. So, it is helpful to associate certain invariants to a given defect which can be calculated independently in the various descriptions. If the defect comes associated with non-trivial moduli spaces of vacua, then a basic invariant is the dimension of these moduli spaces. For the codimension two defects in question, one can associate, in general, a Higgs branch dimension and a graded Coulomb branch dimension. These will correspond to the local contributions to the Higgs and Coulomb branch dimensions of a general class $\mathcal{S}$ theory built out of these defects.

In the work of [39], the graded coulomb branch dimension played an important role in the interpretation of the role played by an order reversing duality that related the two descriptions of these four dimensional defects in their realizations as boundary conditions for $\mathcal{N}$=4 SYM. Here, a complementary discussion that relies crucially on properties of the Higgs branch will be provided. To this end, associate an invariant to the defect that will be called the *Higgs branch Springer invariant*. This will be an irreducible representation of the Weyl group $W[\mathfrak{g}](\simeq W[\mathfrak{g}^\vee])$ and can be calculated on both sides of the S-duality for boundary conditions in $\mathcal{N} = 4$ SYM. This will turn out to be a more refined invariant than just the dimension of the Higgs branch. The discussion will also have the added advantage that it provides a physical setting for certain *defining* properties of the order reversing duality map as formulated in [179] (and used in [39]). Associated to this invariant is a number that will be called the Sommers invariant $\tilde{b}$ highlighting the fact it plays a crucial role in [179]. Its numerical value equals the quaternionic Higgs branch dimension.

3.2.6 An invariant via the Springer correspondence

This invariant is attached to the defect by considering the Springer resolution of either the nilpotent cone $\mathcal{N}^\vee$ or $\mathcal{N}$ (depending on which side of the duality the invariant is being calculated). The discussion in this section will be somewhat generic and is meant to give an introduction to the Springer correspondence. The calculation of the invariant is deferred to a later section. For some expositions of the theory behind the Springer resolution, see [116, 43, 52]. The explicit description of what is known as the Springer correspondence can be found in [35].

Now, consider the nilpotent variety $\mathcal{N}$ and how the closures of other nilpotent orbits sit inside the nilpotent variety $\mathcal{N}$. This leads to a pattern of intricate singularities. For example, in the case of closure of the subregular orbit $\overline{\mathcal{O}}^{sr}$ inside $\mathcal{N}[\mathfrak{g}]$ for $\mathfrak{g} \in A, D, E$, we get the Kleinien singularities $\mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $SU(2)$. Such finite subgroups also have a similar A,D,E classification. A well known fact is that these singularities admit canonical resolutions. For types B_n, C_n, G_2, F_4, one can still obtain a very explicit description of these singularities by considering the $A_{2n-1}, D_{n+1}, D_4, E_6$ singularities with some additional twist data [177]. The deeper singularities of the nilpotent variety, however, do not have such a direct presentation. There is however a general construction due to Springer which is a simultaneous resolution of all the singularities of the Nilpotent variety. It enjoys many interesting properties and plays a crucial role in the study of the representation theory of $G_{\mathbb{C}}$. It is constructed in the following way. Consider pairs $(e, \mathfrak{b})$ where e is a nilpotent element and $\mathfrak{b}$ is a Borel subalgebra containing e. This space of pairs is called the Springer variety $\tilde{\mathcal{N}}$. It is also canonically isomorphic to $T^*\mathcal{B}$, the co-tangent bundle to the Borel variety. The Borel variety $\mathcal{B}$ is the space of all Borel subalgebras in $\mathfrak{g}$ and is also called the flag manifold since elements of the Borel variety stabilize certain sequences of vector spaces of increasing dimension ('flags'). The condition that a non-zero nilpotent element e should belong to $\mathfrak{b}$ leads to a smaller set of Borel subalgebras that will be denoted by $\mathcal{B}_e$. This is a subvariety of the full Borel variety. The subvariety so obtained depends only on the orbit to which e belong. So, a more convenient notation is $\mathcal{B}_{\mathcal{O}}$, where $\mathcal{O}$ is a nilpotent orbit containing e. Now, consider the map that just projects to one of the factors in the pair $\mu : (e, \mathfrak{b}) \to e$. When e to allowed take values in arbitrary nilpotent orbits, the map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ provides a simultaneous resolution of the singularities of $\mathcal{N}$. For e being the zero element, the fiber over e, $\mu^{-1}(0)$ is the full Borel variety. And, $\text{dim}(\mathcal{B}) = \frac{1}{2}\text{dim}(\mathcal{N})$. For more general nilpotent elements, this dimension formula is modified to (see [181, 35])

$$\text{dim}(\mathcal{B}_{\mathcal{O}}) = \frac{1}{2}(\text{dim}(\mathcal{N}) - \text{dim}(\mathcal{O})). \tag{3.2}$$

Resolutions in which the fibers obey the above relationship belong to a class of maps called semi-small resolutions. In other words, the Springer resolution of the nilpotent cone is a semi-small resolution [32]. Apart from constructing the resolution, Springer also showed that the Weyl group acts on the cohomology ring of the fiber $\mathcal{B}_{\mathcal{O}}$. This action commutes with the action of the component group $A(\mathcal{O})$ which acts just by permuting the irreducible components of $\mathcal{B}_{\mathcal{O}}$. In particular, the top dimensional cohomology $H^{2k}(\mathcal{B}_{\mathcal{O}}, \mathbb{C})$ (with $k = \dim_{\mathbb{C}}(\mathcal{B}_{\mathcal{O}})$) decomposes in the following way as a $W[\mathfrak{g}] \times A(\mathcal{O})$ module,

$$H^{2k}(\mathcal{B}_{\mathcal{O}}, \mathbb{C}) = \bigoplus_{\chi \in Irr(A(\mathcal{O}))} V_{\mathcal{O},\chi} \otimes \chi \tag{3.3}$$

where χ is an irreducible representation of the $A(\mathcal{O})$ and $V_{\mathcal{O},\chi}$ is an irreducible representation of the Weyl group. The component group $A(\mathcal{O})$ is defined as $C_G(e)/C_G(e)^0$, where $C_G(e)$ is the centralizer of the e in group $G_{\mathbb{C}}$ and $C_G(e)^0$ is its connected component. The groups $A(\mathcal{O})$ are known for any nilpotent orbit $\mathcal{O}$ and can be obtained from the mathematical literature [44, 178]. When the decomposition in (3.3) involves nontrivial χ, there are non-trivial local systems associated to the nilpotent orbit and $V_{\mathcal{O},\chi}$ corresponds to one of these local systems. In the classical cases, $A(\mathcal{O})$ is either trivial or the abelian group $(S_2)^n$ for some n. In type A, the component group is always trivial. In the exceptional cases, $A(\mathcal{O})$ belongs to the list S_2, S_3, S_4, S_5. While S_2, S_3 occur as component groups for numerous orbits in the exceptional cases, the groups S_4 and S_5 correspond to unique nilpotent orbits in F_4 and E_8 respectively.

In most cases, all irreducible representations of $A(\mathcal{O})$ appear in the above direct sum (3.3). In cases where this does not occur, the number of missing representations is always one and the pair $(\mathcal{O}, \chi)$ is called a *cuspidal* pair. Such cuspidal pairs are classified and a generalization due to Lusztig incorporates these pairs as well into what is called the generalized Springer correspondence (see [174] for a review). One can further show that all irreps of $W[\mathfrak{g}]$ occur as part of the summands like (3.3) for some unique pair $(\mathcal{O},\chi)$. The irreps of $W[\mathfrak{g}]$ which occur with the trivial representation of $A(\mathcal{O})$ (in other words, those that correspond to some pair $(\mathcal{O},1)$) are sometimes called the Orbit representations of $W[\mathfrak{g}]$ [4].

Let $Irr(W)$ be the set of all irreducible representation of $W[\mathfrak{g}]$ and let $[\mathcal{O}]$ be the set of all nilpotent orbits in $\mathfrak{g}$ and $[\tilde{\mathcal{O}}]$ be the set of all pairs $(\mathcal{O},\chi)$, where χ is an irreducible representation of $A(\mathcal{O})$. The nature of the decomposition in (3.3) defines an injective map,

$$Sp[\mathfrak{g}] : Irr(W) \rightarrow [\tilde{\mathcal{O}}]. \tag{3.4}$$

This injective map is called the Springer correspondence. A specific instance of this map will be denoted by $Sp[\mathfrak{g},r] : r \mapsto (\mathcal{O},\chi)$ for a unique pair $(\mathcal{O},\chi) \in [\tilde{\mathcal{O}}]$.

When the inverse exists, it will be denoted by $Sp^{-1}[\mathfrak{g},(\mathcal{O},\chi)]$ or (when $\chi=1$) $Sp^{-1}[\mathfrak{g},\mathcal{O}]$. The following two instances of the Springer map hold for all $\mathfrak{g}$. Let $\mathcal{O}^{pr}$ and $\mathcal{O}^{0}$ denote the principal orbit and the zero orbit respectively. Then,

$$Sp^{-1}[\mathfrak{g},\mathcal{O}^{pr}] = \mathrm{Id} \tag{3.5}$$

$$Sp^{-1}[\mathfrak{g},\mathcal{O}^{0}] = \epsilon, \tag{3.6}$$

[4] This terminology however is not uniformly adopted. The name Springer representation is also used sometimes as an alternative.

where Id, ϵ refer (respectively) to the trivial and the sign representations of $W[\mathfrak{g}]$. This is the Springer correspondence in Lusztig's normalization. In [35], the Springer correspondence is described in this normalization. Many geometric notions that one may associate with the theory of nilpotent orbits like partial orders, induction methods, duality transformations, special orbits, special pieces etc. have algebraic analogues in the world of Weyl group representations. The two worlds interact via the Springer correspondence.

In the context of understanding properties of codimension two defects, an interest in the Springer correspondence can be justified in the following way. For the class of defects under discussion, there is an associated Higgs branch moduli space which admits at least two different descriptions. One of them is as the space of solutions to Nahm equations with a certain boundary condition. This involves a nilpotent orbit in $\mathfrak{g}$ that will be called the Nahm orbit $\mathcal{O}_N$. The second realization is obtained as the Higgs branch of theory $T^\rho[G]$. In either case, an invariant to the defect can be assigned using the Springer correspondence. In the former case, the association is somewhat direct once the Nahm orbit $\mathcal{O}_N$ is known. In the latter case, this invariant will satisfy a non-trivial compatibility condition with properties of the Springer fiber over another nilpotent orbit $\mathcal{O}_H$ (the Hitchin orbit in $\mathfrak{g}^\vee$) that goes into the description of the Coulomb branch of $T^\rho[G]$. Requiring that this consistency condition hold for all defects will turn out to determine the pairs $(\mathcal{O}_N, \mathcal{O}_H)$ that can occur in the description of the defect. The ability to do so is completely independent of the availability of brane constructions and this allows one to understand the exceptional cases as well. Explaining how this can be done would be the main burden of the following two sections.

This ends the somewhat brief introduction to classical Springer theory. Here, it is interesting to note that the relationship between the classical Springer theory discussed above (and its generalizations) and Hitchin systems have been explored recently in the context of the geometric Langlands program [99, 150, 18].

3.2.7 An invariant via the Kazhdan-Lusztig Map

An alternative to using the Springer correspondence to define an invariant for a co-dimension two defect would be to consider the Kazhdan-Lusztig map which provides an injection from the set of nilpotent orbits in $\mathfrak{g}$ to the set of conjugacy classes in $W[\mathfrak{g}]$. This is, in a sense, a dual invariant to the one provided by considering the Springer correspondence. In the context of the four dimensional defects of the theory $\mathcal{X}[\mathfrak{j}]$, one could consider the compactification scheme of Section 3.2.4. The resulting four dimensional picture would involve $\mathcal{N} = 4$ SYM with a surface operator, similar to the setup considered in [100]. There, it was necessary to match the local behaviour of polar polynomials formed out of the Higgs field in an associated Hitchin system on the G & $G^{\vee}$ sides for the determination of the S-duality map. It was argued in [100] that the KL map offered a compact way to implement this check. Here, this invariant will not play a central role. But, it will feature in a discussion of a possible extension of the setup provided in Chapter 5.

3.3 S-duality of Gaiotto-Witten boundary conditions

Recall that Gaiotto-Witten constructed a vast set of 1/2 BPS boundary conditions for $\mathcal{N}=4$ SYM on a half space [92]. The most general boundary condition in this set can be described by a triple $(\mathcal{O}, H, \mathcal{B})$. Here, $\mathcal{O}$ is a nilpotent orbit. By the Jacobson-Morozov theorem, to every nilpotent orbit $\mathcal{O}$ is an associated $\mathfrak{sl}_2$ embedding $\rho_{\mathcal{O}} : \mathfrak{sl}_2 \to \mathfrak{g}$. H is a subgroup of the centralizer of $\mathfrak{sl}_2$ triple associated to $\mathcal{O}$ and $\mathcal{B}$ is a three dimensional SCFT living on the boundary that has a H symmetry. This data is translated to a boundary condition as below,

- Impose a Nahm pole boundary condition that is of type $\rho_{\mathcal{O}}$,
- At the boundary, impose Neumann boundary conditions for gauge fields valued in the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$,
- Gauge the H symmetry of three dimensional boundary $\mathcal{B}$ and couple it to the corresponding four dimensional vector multiplets.

In talking about these boundary conditions, it is very helpful to always think of some special cases. Take $\{\mathcal{O}^0, \mathcal{O}^m, \mathcal{O}^{sr}, \mathcal{O}^{pr}\}$ to refer respectively to {the zero orbit, the minimal orbit, the sub-regular orbit,the principal orbit }. The principal orbit is sometimes called the regular orbit in the literature but in the discussions here, only the former name will appear. For the subgroup H, take $\{Id\}$ to denote the case where the gauge group is completely Higgsed at the boundary and $\{G\}$ to be case where it is not Higgsed. For the boundary field theory[5] $\mathcal{B}$, the value $\varnothing$ corresponds to the case where there is no boundary field theory that is coupled to the bulk vector multiplets. A class of boundary theories named $T^{\rho}[G]$ played an important role in the discussion of S-dualities in [91] and cases where $\mathcal{B} = T^{\rho}[G]$ will turn out to be important in the current discussion as well.

The Higgs and Coulomb branches of these theories are certain subspaces [6] inside the Nilpotent cones $\mathcal{N}$ and $\mathcal{N}^{\vee}$. For much of what follows, various notions associated with the structure theory of nilpotent orbits in complex semi-simple Lie algebras will be routinely invoked. Accessible introductions to these aspects can be found in [44, 145].

With these preliminaries established, one can now look at how S-dualities act on some of the simplest boundary conditions. For example, consider the triple $(\mathcal{O}^0, Id, \varnothing)$ that corresponds to the Dirichlet boundary conditions for the gauge fields and $(\mathcal{O}^0, G, \varnothing)$ corresponds to Neumann boundary conditions for the gauge fields. One of the important features of the GW set of boundary conditions is that it is *closed* under S-duality. But, the simple minded boundary conditions recounted above get mapped to

[5]Elsewhere in the dissertation, the symbol $\mathcal{B}$ has been used to also refer to the four dimensional Coulomb branch. This clash in notation is regretted but it should be clear from the context as to what $\mathcal{B}$ refers to.

[6] *strata* would, technically, be a more accurate term.

non-trivial boundary conditions. The S-dual of $(\mathcal{O}^0, Id, \varnothing)$ in a theory with gauge group G is the boundary condition $(\mathcal{O}^0, G^\vee, T[G])$ in a theory with gauge group $G^\vee$. On the other hand, the dual of $(\mathcal{O}^0, G, \varnothing)$ is $(\mathcal{O}^{pr}, Id, \varnothing)$. One strong evidence in favor of the identification of S-duality between these boundary conditions is the fact that dimensions of the vacuum moduli space of $\mathcal{N} = 4$ SYM with these boundary conditions happen to match on both sides. In the two cases considered above, the moduli space is the nilpotent cone $\mathcal{N}$ in the first case and a point in the second case. These occurrences of the S-duality map [7] are listed in table 3.1.

Table 3.1: S-duality of boundary conditions in $\mathcal{N} = 4$ SYM

$G - \mathcal{N} = 4$ SYM	$G^\vee - \mathcal{N} = 4$ SYM	Vacuum moduli space
$(\mathcal{O}^0, G, \varnothing)$	$(\mathcal{O}^{pr}, Id, \varnothing)$	$\cdot$
$(\mathcal{O}^0, Id, \varnothing)$	$(\mathcal{O}^0, G^\vee, T[G])$	$\mathcal{N}$
$(\mathcal{O}^\rho, Id, \varnothing)$	$(\mathcal{O}^0, G^\vee, T^\rho[G])$	$\mathcal{S}^\rho \cap \mathcal{N}$

We will not be needing the constructions of Gaiotto-Witten in their full generality. The cases that will be of direct relevance to discussions here correspond to the ones with a pure Nahm pole boundary condition and its S-dual case of a Neumann boundary condition along with a coupling to a three dimensional theory $T^\rho[G]$ and certain deformations thereof. In the rest

[7] We are concerned here just with the $\mathbb{Z}_2$ subgroup of the full S-duality group that acts on the coupling constant as $\tau^\vee = -1/n_r\tau$, where n_r is the ratio of lengths of the longest root to the shortest root.

of the section, we will look closely at duality between $(\mathcal{O}^\rho, Id, \varnothing)$ in the theory with gauge group G and $(\mathcal{O}^0, G^\vee, T^\rho[G])$ in the theory with gauge group $G^\vee$. An important point to note here is that the specification of the boundary condition on the $G^\vee$ is incomplete without a description of how the theory $T^\rho[G]$ is coupled to boundary multiplets. In the adopted conventions, the Higgs branch of $T[G]$ will have a G global symmetry, while the Coulomb branch has a $G^\vee$ global symmetry. So, the natural way to couple $T^\rho[G]$ would be to gauge the global symmetry on the Coulomb branch[8] and couple it to the boundary vector multiplets of the $G^\vee$ theory. The Higgs branch of $T^\rho[G]$ is now understood to be the vacuum moduli space of the full four dimensional theory with this boundary condition. As one may guess, understanding this instance of the duality map requires a careful study of the moduli spaces of Nahm equations under different pole boundary conditions and the theories $T^\rho[G]$ and their vacuum moduli spaces. Some of the main elements of such a study are outlined in the rest of the Section.

3.3.1 Moduli spaces of Nahm equations

Various aspects of Nahm equations and their moduli space of solutions are reviewed in [92]. For some other useful works which elucidate Nahm equation from different points of view, see [55, 11].

[8]The symmetries on the Coulomb branch are not obvious in any Lagrangian description of $T^\rho[G]$. So, a more practical way to describe this coupling is to use the description of this branch as the Higgs branch of the mirror theory $T_{\rho^\vee}[G]$. But, to simplify things, all statements here are made with the theories $T^\rho[G]$.

In the setting of boundary conditions of $\mathcal{N} = 4$ SYM [92], Nahm boundary conditions arise as a generalization of the usual Dirichlet boundary conditions. Recall that there are six real scalar fields in this theory. Let $\overrightarrow{X}$ be the triplet for which Nahm type boundary conditions conditions are imposed. Formulate the theory on $\mathbb{R}^3 \times \mathbb{R}^+$ and let y be a co-ordinate along $\mathbb{R}^+$ with $y = 0$ being the boundary. Let ρ be a $\mathfrak{sl}_2$ embedding, $\rho : \mathfrak{sl}_2 \to \mathfrak{g}$. Then, the boundary conditions are of the form

$$\frac{dX^i}{dy} = \epsilon_{ijk}[X^i, X^j] \tag{3.7}$$

$$X^i = \frac{t^i}{y}, y \to 0 \qquad (i = 1, 2, 3). \tag{3.8}$$

with t^i being a $\mathfrak{sl}_2$ triple associated to $\rho(e, f, h)$, (e, f, h) being the standard triple. The first part is the usual Nahm equation while the second part of the boundary condition modifies it to a Nahm pole boundary condition. When ρ is the zero embedding, this reduces to the case of a pure Dirichlet boundary condition. Following the works of Kronheimer [131], it is known that solutions to (3.8) is a hyper-kahler manifold. Denote this by $\mathcal{M}_\rho(\overrightarrow{X}_\infty)$, where $\overrightarrow{X}_\infty$ are the values of $\overrightarrow{X}$ at $y \to \infty$. When $\overrightarrow{X}_\infty = 0$, $\mathcal{M}_\rho(\overrightarrow{X}_\infty)$ is a singular space. Some special cases are

- ρ is the zero embedding. Here, $\mathcal{M}_\rho(0)$ is the nilpotent variety $\mathcal{N}$ of G.
- ρ is the sub-regular embedding. In this case, $\mathcal{M}_\rho(0)$ is a singularity of the form $\mathbb{C}^2/\Gamma$.
- For ρ being the principal embedding, $\mathcal{M}_\rho(0)$ is just a point.

In the more general cases, $\overrightarrow{X}_\infty$ is a non-zero semi-simple element and one obtains a resolution/deformation of the singular space. In this more general case, $\overrightarrow{X}_\infty \in \mathfrak{t}^3/W$, where W is the Weyl group. Specializing to $\overrightarrow{X}_\infty = (i\tau, 0, 0)$, one gets a resolution of the moduli space of solutions in one of the complex structures. It turns out that many of the ideas in the setup just reviewed play an important role in geometric representation theory. From a purely complex point of view, these moduli spaces have been studied in the works of Grothendieck-Brieskorn-Slodowy [177, 176]. The general solution to Nahm pole boundary conditions is in fact best described as the intersection $\mathcal{S}^\rho \cap \mathcal{N}$ where $\mathcal{S}^\rho$ is the Slodowy slice that is transverse (in $\mathfrak{g}$) to the nilpotent orbit ρ. The realization of these spaces as solutions to Nahm equations gives a new hyper-kahler perspective.

3.3.2 Springer resolution of Slodowy slices

Consider the Springer resolution μ discussed in Section 3.2.6. As already noted, this resolution is semi-small. Now, consider the preimage of $\mathcal{S} = \mathcal{S}^\rho \cap \mathcal{N}$ under μ, given by $\tilde{\mathcal{S}} = \mu^{-1}(\mathcal{S})$. It can be shown that $\dim(\tilde{\mathcal{S}}) = \dim(\mathcal{N}) - \dim(\mathcal{O}_N)$ (all dimensions are complex dimensions unless stated otherwise). The Springer fiber $\mathcal{B}_N = \mu^{-1}(e)$, where e is a representative of $\mathcal{O}_N$ is a space of dimension $\dim(\mathcal{B}_N) = \frac{1}{2}(\dim(\mathcal{N}) - \dim(\mathcal{O}_N))$. Further, $\mathcal{B}_N$ is a Lagrangian sub-manifold of $\tilde{\mathcal{S}}$ and can be obtained as a homotopy retract of $\tilde{\mathcal{S}}$ [43, 96]. In particular, $H^*(\tilde{\mathcal{S}}) = H^*(\mathcal{B}_N)$. Slodowy's construction naturally endows an action of the Weyl group on $H^*(\tilde{\mathcal{S}})$ as the monodromy representation. This then endows a Weyl group action on $H^*(\mathcal{B}_N)$. It is known that this action matches with the one from Springer's construction [176] (in Lusztig's normalization). In particular, $H^{\text{top}}(\mathcal{B}_N)$ is a $W[\mathfrak{g}] \times A(\mathcal{O}_N)$ module. In light of the fact that the moduli space of solutions is actually a hyper-Kahler manifold, it is natural to associate to it a quaternionic dimension. Let $\dim_{\mathbb{H}}(\mathcal{S}^\rho \cap \mathcal{N})$ be the quaternionic dimension. Then, the dimension formulas immediately imply

$$\dim_{\mathbb{H}}(\mathcal{S}^\rho \cap \mathcal{N}) = \dim_{\mathbb{C}}(\mathcal{B}_N). \tag{3.9}$$

It is convenient to note the above relation since $\dim_{\mathbb{C}}(\mathcal{B}_N)$ is often readily available in the mathematical literature on Springer resolutions.

3.3.3 Vacuum moduli spaces of $T^\rho[G]$

The $T^\rho[G]$ theories are certain 3d $\mathcal{N} = 4$ SCFTs that play an important role in the description of S-duality of boundary conditions for $\mathcal{N} = 4$ SYM. For G classical, Gaiotto-Witten provide brane constructions in type IIB string theory (following the setup of [104]) to describe the boundary conditions. In particular, their setup provides a brane construction of many of the three dimensional theories $T^\rho[G]$. An example of such a brane construction for $G = SU(N)$ is given in Fig 3.1. For G exceptional, the theories $T^\rho[G]$ exist although brane constructions are no longer available. There are however some general features that are expected to be shared by all $T^\rho[G]$. Most notable among this is the fact that the vacuum moduli spaces of these theories arise as certain subspaces of $\mathcal{N} \times \mathcal{N}^\vee$, where $\mathcal{N}$ is the nilpotent cone for the

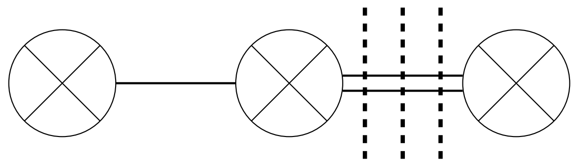

Figure 3.1: Brane realization of $T[SU(3)]$. The D5 linking numbers are $l_i = (2,2,2)$ and the NS5 linking numbers are $\tilde{l_i} = (1,1,1)$

lie algebra $\mathfrak{g}$ while $\mathcal{N}^\vee$ is the nilpotent cone associated to the dual lie algebra $\mathfrak{g}^\vee$. More concretely [91, 39] let $(\mathcal{O}_N, \mathcal{O}_H)$ denote a pair of nilpotent orbits in $\mathfrak{g}, \mathfrak{g}^\vee$. The Higgs branch of $T^\rho[G]$ is a hyper-kahler manifold of complex dimension $\dim(\mathcal{N}) - \dim(\mathcal{O}_N)$ and the Coulomb branch of $T^\rho[G]$ is another hyper-kahler manifold of dimension $\dim(\mathcal{O}_H)$. It follows that for the corresponding four dimensional theory[9] on the co-dimension two defect, the dimensions of the Higgs branch and the Coulomb branch dimension are $\dim(\mathcal{N}) - \dim(\mathcal{O}_N)$ and $\frac{1}{2}(\dim(\mathcal{O}_H))$ respectively.

3.3.4 Resolution of the Higgs branch

Recall that under the conventions adopted, the theory $T^\rho[G]$ appears on the side of the duality with 4d SYM for gauge group $G^\vee$ and its Coulomb branch is a nilpotent orbit in $\mathfrak{g}^\vee$. Upon coupling to the boundary gauge fields, the Higgs branch of the theory is identified as the vacuum moduli space of the 4d theory with a boundary. The equivalence between this

[9]Recall $T^\rho[G]$ is obtained by compactifying the four dimensional $\mathcal{N} = 2$ codimension two defect theory on a circle and hence has a Higgs branch of the same dimension and a Coulomb branch that is twice the dimension of the 4d Coulomb branch.

Higgs branch and the presentation of the space as $\mathcal{S}^\rho \cap \mathcal{N}$ is a highly non trivial assertion but one that can not be checked directly since an independent prescription for the Higgs branch does not exist for arbitrary $T^\rho[G]$. In the discussion here, it will be taken for granted that the S-dual boundary condition for a Nahm pole boundary condition should indeed involve one of the theories $T^\rho[G]$. Under this assumption, it will be possible to determine which of the $T^\rho[G]$ arise as part of the dual boundary condition to a particular Nahm boundary condition. Now, associated to the theory $T^\rho[G]$ are certain Fayet - Iliopoulos (FI) parameters $\overrightarrow{\zeta}$. The Springer resolution of the Higgs branch of $T^\rho[G]$ can be understood to arise from giving particular non-zero values to some of the FI parameters [91]. Although an explicit description of this geometry is not available, one expects this to match the $\mathfrak{g}$ description where the resolution parameters entered the Nahm description as $\overrightarrow{X}_\infty$. The upshot of the argument here is that it makes sense to attach a Springer invariant to the resolved Higgs branch of $T^\rho[G]$. In Section 3.5, it will be seen that requiring that the Springer invariant obtained from the $\mathfrak{g}$ and $\mathfrak{g}^\vee$ descriptions match is a strong constraint on the relationship between $\mathcal{O}_H$ and $\mathcal{O}_N$. The next section sets the ground by introducing several mathematical notions that are critical for Section 3.5.

3.4 Duality maps and Representations of Weyl groups

3.4.1 Various duality maps

Order reversing duality maps turn out to play an important role in understanding the physics of $T^\rho[G]$ theories and hence of the associated co-dimension two defects. But, there are different order reversing duality maps in the mathematical literature and it is helpful to know certain defining features of these maps to understand the nature of their relevance to the physical questions. To this end, here is a quick review of the available duality maps. Let us define the following. The set of all nilpotent orbits in $\mathfrak{g}$ will be denoted by $[\mathcal{O}]$. The set of all nilpotent orbits in $\mathfrak{g}^\vee$ will be denoted by $[\mathcal{O}^\vee]$. The special orbits within these two sets will be denoted by $[\mathcal{O}_{sp}]$, $[\mathcal{O}^\vee_{sp}]$. The notation $[\overline{\mathcal{O}}]$ refers to all pairs $(\mathcal{O}, C)$ where $\mathcal{O} \in [\mathcal{O}]$ and C is an conjugacy class of the group $\bar{A}(\mathcal{O})$. This group $\bar{A}(\mathcal{O})$ is a quotient (defined by Lusztig) of the component group $A(\mathcal{O})$ of the nilpotent orbit $\mathcal{O}$. The following order reversing duality maps have been constructed in the mathematical literature.

The duality map	Its action
Lusztig-Spaltenstein	$d_{LS} : [\mathcal{O}] \to [\mathcal{O}_{sp}]$
Barbasch-Vogan	$d_{BV} : [\mathcal{O}] \to [\mathcal{O}^\vee_{sp}]$
Sommers	$d_S : [\overline{\mathcal{O}}] \to [\overline{\mathcal{O}^\vee}_{sp}]$
Achar	$d_A : [\overline{\mathcal{O}}] \to [\overline{\mathcal{O}^\vee}]$

Each of these maps invert the partial order on the set of nilpotent orbits. For example, the principal orbit is always mapped to the zero orbit and the zero orbit is always mapped to the principal orbit. The name 'order-reversing duality' is meant to highlight this fact. The Lusztig-Spaltenstein map is explicitly detailed in [181] and is the only order-reversing duality map that strictly stays within $\mathfrak{g}$ and does not pass to the dual lie algebra. In this sense, it occupies a different position from the other three maps. The order reversing map of Sommers [179] (further elaborated upon in [1] and extended by Achar in [2]) is defined [10] by combining the duality construction due to Lusztig-Spaltenstein [181] and a map constructed by Lusztig in [139]. The duality map of Barbasch-Vogan [16] arises from the study of primitive ideals in universal enveloping algebras (equivalently of Harish-Chandra modules) and can be thought of as a special case of the duality maps due to Sommers and Achar.

Everytime an order reversing duality map is used, it will be explicitly one of the maps summarized in the table above. The order reversing duality that is used in [39] is the Sommers duality map d_S. If one forgets the additional discrete data associated to the special orbit that arises on the $\mathfrak{g}^\vee$ side, this reduces to the duality map of Barbasch-Vogan, d_{BV}. In [39], the name *Spaltenstein dual* is used for describing a duality map that passes to the dual lie algebra. This terminology is potentially confusing if one wants to compare with the mathematical literature and will not be adopted here. All of these maps are easiest to describe when their domain is restricted to

[10] One could equivalently view the Sommers map as being defined in the opposite direction, $d_S : [\overline{\mathcal{O}^\vee}_{sp}] \to [\mathcal{O}]$. The way it is written here is the direction in which it is invoked in [39].

just the special orbits. It is an important property of the maps that they act as involutions on the special orbits. Considering the case of special orbits in $\mathfrak{g} = \mathfrak{so}_8$, $\mathfrak{g}^\vee = \mathfrak{so}_8$. In this case, all the above maps coincide and their action is best seen as the unique order reversing involution acting on the closure diagram for special orbits.

$$
\begin{array}{ccc}
 & [7,1] & \\
 & | & \\
 & [5,3] & \\
\diagup & | & \diagdown \\
[4^2]' & [5,1^3] & [4^2]'' \\
\diagdown & | & \diagup \\
 & [3^2,1^2] & \\
\diagup & | & \diagdown \\
[2^4]' & [3,1^5] & [2^4]'' \\
\diagdown & | & \diagup \\
 & [2^2,1^4] & \\
 & | & \\
 & [1^8] &
\end{array}
$$

Figure 3.2: Hasse diagram describing the closure ordering for special nilpotent orbits in $\mathfrak{so}_8$.

As one further remark, let us note here a particular subtlety. Even in scenarios where d_{LS} and d_{BV} have identical domain and image, they could disagree. For example, in the case of $\mathfrak{g} = F_4$, $\mathfrak{g}^\vee = F_4$. So, the domain and the image for d_{LS} are identical to that for d_{BV}. But, d_{LS} and d_{BV} disagree for certain nilpotent orbits (see the Hasse diagram for F_4 in [39]).

An important feature of all the duality maps is their close interaction with the Springer correspondence and consequently with the representation theory of Weyl groups. In fact, some of the maps are defined using the Springer correspondence. So, any attempt to gain a deeper understanding of how the duality maps work is aided greatly by a study of the representation theory of Weyl groups. In the rest of the section, some of the elements of this theory are recounted.

3.4.2 Families, Special representations and Special orbits

Let $Irr(W)$ denote the set of irreducible representation of the Weyl group W. There is a distinguished subset of $Irr(W)$ called special representations that are well behaved under a procedure known as truncated induction (or j induction, see Appendix D) and duality. To explain this, denote the set of special representations by S_W. Now, let s_p be a special representation of a parabolic subgroup W_p. Requiring that the identity representation be special and considering all parabolic subgroups of a Weyl group and proceeding inductively, define s to be special if $s = j_{W_p}^{W}(s_p)$ for some parabolic subgroup W_p and additionally $s' = i(s)$ is also special. Here, $i(s)$ refers to Lusztig's duality which in almost all cases acts as tensoring by the sign representation. The exceptions are certain cases in E_7 and E_8 which will be discussed at a later point (See Section 5.2.7). Proceeding in this fashion, Lusztig determined the set of all special representations in an arbitrary Weyl group in [137].

Another important notion that is defined inductively is that of a cell module[11]. This is a not-necessarily irreducible module of W that, again, has some very nice properties under induction and duality. The trivial representation Id is defined to be a cell module by itself. One arrives at the other cell modules in the following way. Let c be a cell module of $Irr(W)$ and c_p be a cell module of a parabolic subgroup W_p of W. Consider their behaviour under two operations for arbitrary subgroups W_p,

$$c' = \epsilon \otimes c, \tag{3.10}$$

$$c'' = Ind_{W_p}^{W}(c_p), \tag{3.11}$$

where Ind is the usual induction (in the sense of Frobenius) from a parabolic subgroup. Requiring that the above two operations always yield another cell module determines all the cell modules in $W[\mathfrak{g}]$ for every $\mathfrak{g}$. The structure of these cell modules has what may seem like a surprising property. Each cell module has a *unique* special representation as one of its irreducible summands. Additionally, the representations that occur as part of a cell module that contains a special representation s occur *only* in the cell modules that contain s as the special representation. This structure suggests a certain partitioning of $Irr(W)$ [138]. It is of the following form [12],

$$Irr(W) = \coprod_s f_s \tag{3.12}$$

[11] An equivalent term is that of a 'constructible representation' but the term cell module will be preferred.

[12] There is an equivalent partitioning of Weyl group representations using the idea of a two-cell of the finite Weyl group. Henceforth, the term family will be used uniformly.

where s is a special representation. An irrep r occurs in the family f_s if and only if it occurs in a cell module along with the special representation s. In type A, all representations are special and hence the above partitioning reduces to the statement that each irreducible representation of $W(A_n)$ belongs to a separate family in which it is the only constituent. This simple structure however does not hold for Weyl groups outside of type A. The general case includes non-special representations which occur as constituents of some of the families f_s. So, a typical family contains a unique special representation (which can be used to index the family as in 3.12) and a few non-special representations. Associated to each family are the cell modules in which the representation s occurs as the special summand. As an example of a family with more than one constituent, consider the unique non-trivial family in D_4 (see Appendix B.1.3 for the notation adopted),

$$f_{([2,1],[1])} = \{([2,1],[1]),([2^2],-),([2],[1^2])\}. \tag{3.13}$$

The special representation in this family is given by $([2,1],[1])$ and the cell modules that belong to this family are

$$c_1 = ([2,1],[1]) \oplus ([2^2],-), \tag{3.14}$$

$$c_2 = ([2,1],[1]) \oplus ([2],[1^2]). \tag{3.15}$$

To every irreducible representation of a Weyl group, Lusztig assigns a certain invariant such that it is constant within a family and unique to it. Its value is equal to the dimension of the Springer fiber associated to the special element in a given family. For the family in the example discussed above, the a value is 3 and it is the unique family in $W(D_4)$ that has $a = 3$. Here, it is appropriate to also note that one of the earliest characterizations of *special orbits* was via the Springer correspondence. A nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is special if and only if $Sp^{-1}[\mathfrak{g}, \mathcal{O}]$ is a special representation of the Weyl group. Alternatively, a non-special orbit $\mathcal{O}$ is the one for which $Sp^{-1}[\mathfrak{g}, \mathcal{O}]$ yields a non-special irrep of W. Note that some irreps correspond under the Springer correspondence to non-trivial local systems on $\mathcal{O}$. So, not every non-special representation is associated to a non-special orbit. For example, in D_4,

$$Sp[D_4, ([2^2], -)] = ([3, 2^2, 1], 1) \tag{3.16}$$

$$Sp[D_4, ([2], [1^2])] = ([3^2, 1^2], \psi_2), \tag{3.17}$$

where ψ_2 is the sign representation of S_2, the component group of $[3^2, 1^2]$. In the first case above, the Springer correspondence assigns a non-special representation to a non-special orbit while in the second case, it assigns a non-special representation a non-trivial local system on a special orbit. The structure of the cell modules can now be seen as

$$c_1 = \text{special orbit rep} \oplus \text{non-special orbit rep} \tag{3.18}$$

$$c_2 = \text{special orbit rep} \oplus \text{non-orbit rep}.$$

For all families with three irreducible representations, the cell structure follows an identical pattern to the one just discussed. The special orbit together with all the non-special orbits to which the Springer correspondence assigns (when the orbits are taken with the trivial representation of the component groups) Weyl group irreps that are in the same family as that of the special representation (assigned to the special orbit by Sp^{-1}) form what is called a *special piece* [135]. Geometrically, it is the set of all orbits which are contained in the closure of the special orbit $\mathcal{O}$ but are not contained in the closure of any other special orbit $\mathcal{O}'$ that obeys $\mathcal{O}' < \mathcal{O}$ in the closure ordering on special orbits. Note that in the example above, there is a cell module which contains all the Orbit representations corresponding to the special piece. The tables in Chapter 5 show, explicitly, that this pattern persists for every special piece in low rank classical cases and all the exceptional cases. That this pattern actually persists for every special piece can be shown using certain results in [139] (the summary of results at the end of pg. xiii and the beginning of pg. xv are most pertinent here)[13]. Further, the relevant results in [139] also imply that the number of orbits in the special piece is equal to the number of irreducible representations of the finite group $\bar{A}(\mathcal{O}^\vee)$ for some special orbit $\mathcal{O}^\vee$ in the dual lie algebra. A weaker statement that the Orbit representations of a special piece belong to the same family is available in [135].

For larger families, the overall structure of cell modules is substantially more complicated than (3.19). For example, consider the family in $W(E_8)$ that contains the special representation $\phi_{4480,16}$ [35],

[13] I thank G. Lusztig for correspondence on these matters.

$$f_{\phi_{4480,16}} = \{\phi_{4480,16}, \phi_{7168,17}, \phi_{3150,18}, \phi_{4200,18}, \phi_{4536,18}, \phi_{5670,18}, \phi_{1344,19}, \phi_{2016,19}, \phi_{5600,19}, \phi_{2688,20}, \phi_{420,20}, \phi_{1134,20}, \phi_{1400,20}, \phi_{1680,22}, \phi_{168,24}, \phi_{448,25}, \phi_{70,32}\}.$$

This family has $a = 16$ and has a total of 17 irreps which organize themselves into the following seven cell modules,

$$\begin{aligned}
c_1 &= \phi_{4480,16} \oplus \phi_{7168,17} \oplus \phi_{3150,18} \oplus \phi_{4200,18} \oplus \phi_{1344,19} \oplus \phi_{2016,19} \oplus \phi_{420,20} \\
c_2 &= \phi_{4480,16} \oplus \phi_{7168,17} \oplus \phi_{3150,18} \oplus \phi_{4200,18} \oplus \phi_{5670,18} \oplus \phi_{1344,19} \oplus \phi_{5600,19} \oplus \phi_{1134,20} \\
c_3 &= \phi_{4480,16} \oplus \phi_{7168,17} \oplus 2\phi_{4200,18} \oplus \phi_{4536,18} \oplus \phi_{5670,18} \oplus \phi_{1344,19} \oplus \phi_{5600,19} \oplus \phi_{1400,20} \oplus \phi_{168,24} \\
c_4 &= \phi_{4480,16} \oplus \phi_{7168,17} \oplus \phi_{3150,18} \oplus \phi_{4536,18} \oplus 2\phi_{5670,18} \oplus 2\phi_{5600,19} \oplus \phi_{1134,20} \oplus \phi_{1680,22} \oplus \phi_{448,25} \\
c_5 &= \phi_{4480,16} \oplus \phi_{7168,17} \oplus 3\phi_{4536,18} \oplus 3\phi_{5670,18} \oplus 2\phi_{5600,19} \oplus 2\phi_{1400,20} \oplus 3\phi_{1680,22} \oplus \phi_{448,25} \oplus \phi_{70,32} \\
c_6 &= \phi_{4480,16} \oplus 2\phi_{7168,17} \oplus \phi_{3150,18} \oplus \phi_{4200,18} \oplus \phi_{4536,18} \oplus \phi_{5670,18} \oplus \phi_{2016,19} \oplus \phi_{5600,19} \oplus \phi_{2688,20} \\
c_7 &= \phi_{4480,16} \oplus 2\phi_{7168,17} \oplus \phi_{4200,18} \oplus 2\phi_{4536,18} \oplus 2\phi_{5670,18} \oplus 2\phi_{5600,19} \oplus \phi_{2688,20} \oplus \phi_{1400,20} \oplus \phi_{1680,22}.
\end{aligned} \tag{3.19}$$

Here again, c_1 is the collection of all Orbit representations in the family and the corresponding orbits form a special piece (see the table for E_8 in Section 5.8). The patterns in the other cell modules for this family are not very obvious.

In the following sections, the various notions introduced in this section will play an important role. For a more detailed exposition of the theory of Weyl group representations, see [139, 35].

3.5 Physical implications of duality maps

3.5.1 CDT class of defects via matching of the Springer invariant

Recall from the discussion of S-duality of 1/2 BPS boundary conditions in $\mathcal{N} = 4$ SYM that the vacuum moduli space of the theory on a half space has two different realizations. One is its realization in the G description and the other is its realization in the $G^\vee$ description. For the examples considered, the former was as a solution to Nahm equations with certain pole boundary conditions. The solution is in general of the form $\mathcal{S}^\rho \cap \mathcal{N}$, where ρ is a nilpotent orbit in $\mathfrak{g}$. On the $G^\vee$ side, this space is realized as the Higgs branch of theory $T^\rho[G]$. Recall that the Higgs branch is a (singular) hyper-kahler space. So, the above statement in particular means that the metric on the moduli space is the same in both realizations. There is, at present, no known way to check this equality for arbitrary cases. However, there is strong evidence that the above identification holds for all $\mathcal{O}^\rho$ in any simple $\mathfrak{g}$.

The S-duality map however would be incomplete if one could not say something about what the Coulomb branch of $T^\rho[G]$ should be. It is the Coulomb branch of $T^\rho[G]$ that is gauged and coupled to the boundary gauge fields on the $G^\vee$ side. In [91], in the case of type A_n, it is shown that the Coulomb branch of $T^\rho[G]$ is given by a nilpotent orbit in $\mathfrak{g}^\vee = A_n$ whose partition type is P^T, the transpose of the partition type P of the orbit ρ. Geometrically, transposition on the partition type acts as an order reversing duality on the set of nilpotent orbits taken with the partial order provided by their closure ordering[44]. So, in the more general cases, one can guess that something similar to the case of A_n prevails and description of the Coulomb branch of $T^\rho[G]$ will involve an order reversing duality between the data on the $\mathfrak{g}$ and the $\mathfrak{g}^\vee$ sides. Before the more general case is discussed, consider the case of $\mathfrak{g} = su(N)$ and a hypothetical scenario where one did not know that the right S-duality map between boundary conditions picks out the $T^\rho[SU(N)]$ that has a Coulomb branch given by a dual nilpotent orbit as the correct theory to couple at the boundary in the description of the S-dual of Nahm pole boundary condition of type partition type P. If, however, one is convinced that the boundary condition on the $G^\vee$ side should involve one of the $T^\rho[G]$ theories, then there is a unique theory whose Higgs branch matches the dimension of $\mathcal{S}^\rho \cap \mathcal{N}$. This theory would be the obvious candidate for the boundary theory on the $G^\vee$ side. And this theory has as its Coulomb branch the nilpotent orbit P^T. One could call this argument *dimension matching*, for merely requiring that the dimensions of the moduli space in its two realizations match turns out to completely specify the duality map. Outside of type A, the above argument can't be carried out directly for there are different $T^\rho[G]$ that have Higgs branches of the same dimension.

Additionally, for certain G in the classical types, the quivers that describe $T^\rho[G]$ turn out to be 'bad' in the sense of [91]. This complicates the description of the IR limit of the associated brane configurations. Moreover, when G is of exceptional type, a quiver description of the three dimensional theory is no longer available. In this context, it is convenient to use a more refined invariant which will be called the *Higgs branch Springer invariant*. It has the advantage of being calculable for all G and can distinguish $T^\rho[G]$ that have Higgs branches of the same dimension. The point of view pursued here is that once the interaction between the representation theory and the vacuum moduli spaces of $T^\rho[G]$ is understood for G classical (where brane constructions are available), then the available results from representation theory can be used to understand cases for which there is no brane construction available. Such a point of view is additionally supported by the fact that the corresponding representation theoretic results are highly constrained and enjoy a degree of uniqueness. This is also the point of view adopted in [39] whose setup is what we are seeking to arrive at, albeit by a different route.

Let us now proceed to associate a Higgs branch Springer invariant on both sides of the S-duality map and require that they match. The irrep that occurs in this matching will be called $\bar{r}$. It seems suitable to call this check for the S-duality map as *Higgs branch Springer invariant matching*, or *$\bar{r}$-matching* for short. This invariant $\bar{r}$ is calculated on the $\mathfrak{g}$ in a straightforward manner,

$$\bar{r} = Sp^{-1}[\mathfrak{sl}_N, \mathcal{O}_N]. \tag{3.20}$$

From the brane constructions, we know that nilpotent orbits that enter the description of the Higgs and Coulomb branches of $T^{\rho}[SU(N)]$ are related by an order reversing duality between the nilpotent orbits. The analogue of an order reversing duality at the level of Weyl group representations is tensoring by the sign representation ϵ. And, indeed, one sees that the $\bar{r}$ obtained as in (3.20) above obeys

$$\bar{r} = \epsilon \otimes Sp^{-1}[\mathfrak{sl}_N, \mathcal{O}_H]. \tag{3.21}$$

Alternatively, one can *require* that

$$Sp^{-1}[\mathfrak{sl}_N, \mathcal{O}_N] = \epsilon \otimes Sp^{-1}[\mathfrak{sl}_N, \mathcal{O}_H] \tag{3.22}$$

and this, in turn, determines $\mathcal{O}_N$ for a given $\mathcal{O}_H$.

Now, it is natural to try and generalize this for other $\mathfrak{g}$. For arbitrary $\mathfrak{g}$, the Springer correspondences in $\mathfrak{g}^\vee$ and $\mathfrak{g}$ would give irreps of $W[\mathfrak{g}^\vee]$ and $W[\mathfrak{g}]$. Since there is a canonical isomorphism between the two, it is natural to parameterize the irreps of the two Weyl groups in a common fashion (see Appendix B and [35]). This would also allow one to formulate a 'matching' argument along the lines of 3.22. This does turn out to be hugely helpful as this simple-minded generalization specifies the duality map in numerous cases. Let us for a moment consider case where Hitchin data is $(\mathcal{O}_H, 1)$. Merely requiring that

$$Sp^{-1}[\mathfrak{g}, \mathcal{O}_N] = \epsilon \otimes Sp^{-1}[\mathfrak{g}^\vee, \mathcal{O}_H], \tag{3.23}$$

one can obtain the order reversing duality map for all $\mathcal{O}_N$ special except for the cases discussed in Section 5.2.7. One can handle all the cases uniformly by replacing the RHS in (3.23) with the unique special representation in the family of $\epsilon \otimes Sp^{-1}[\mathfrak{g}^\vee, \mathcal{O}_H]$. This version of the duality operation that implements a fix for the 'exceptional' (in the sense of Section 5.2.7) cases is due to Lusztig. In the discussion below, the duality operation will continue to the represented as tensoring by sign with the understanding that, if needed, the above fix can always be applied to the definition.

Now, consider the following equivalent formulation of Eq (3.23),

$$\boxed{Sp^{-1}[\mathfrak{g}, \mathcal{O}_N] = Sp^{-1}[\mathfrak{g}, d_{LS}(\mathcal{O}_H)]}, \tag{3.24}$$

where d_{LS} is the Lusztig-Spaltenstein order reversing duality map that stays within the lie algebra $\mathfrak{g}$. The equivalence of the above formulation to Eq (3.23) follows from a property of the map d_{LS} when acting on special orbits,

$$Sp^{-1}[\mathfrak{g}, d_{LS}(\mathcal{O})] = \epsilon \otimes Sp^{-1}[\mathfrak{g}, \mathcal{O}]. \tag{3.25}$$

From (3.24), we get the order reversing duality for the cases where $\mathcal{O}_N$ is *special*. For the other cases, one has to formulate a more sophisticated argument. Before we get to that, let us try to understand how the Springer invariant can be calculated when we allow for a particular symmetry breaking deformation in the bulk on the $\mathfrak{g}^\vee$ side.

The boundary condition on the $\mathfrak{g}^\vee$ side involves $\mathcal{N} = 4$ SYM on a half space with a coupling to a three dimensional theory $T^\rho[G]$ that lives on the boundary. Now, deform this boundary condition by giving a vev to the adjoint scalars of the bulk theory. Let this vev be some semi-simple element $m \in T^\vee$. Now, in the $m \to \infty$ limit, the bulk symmetry is broken from $G^\vee$ to $L^\vee$, where $l^\vee$ is a subalgebra that arises as the centralizer $Z_{\mathfrak{g}^\vee}(m)$. Pick m such that a representative $e^\vee$ of the Coulomb branch orbit $\mathcal{O}_H$ is a distinguished nilpotent element in $\mathfrak{l}^\vee$. Taking the $m \to \infty$ limit gives a boundary condition in $\mathcal{N} = 4$ SYM with gauge group $L^\vee$ with the theory at the boundary being $T^{\tilde{\rho}}[L]$, where $\tilde{\rho}$ refers to a nilpotent orbit $\mathcal{O}^{\tilde{\rho}}$ in $\mathfrak{l}$, the Langlands dual of $\mathfrak{l}^\vee$. Let us call such a deformation of the boundary condition on the $G^\vee$ side a *distinguished symmetry breaking* (d.s.b),

$$(\mathcal{O}^0, G^\vee, T^\rho[G]) \longrightarrow_{d.s.b} (\mathcal{O}^0, L^\vee, T^{\tilde{\rho}}[L]). \tag{3.26}$$

The above deformation can be done for any boundary condition of the form $(\mathcal{O}^0, G^\vee, T^\rho[G])$ in $\mathcal{N} = 4$ SYM. When $\mathfrak{l}^\vee$ is a Levi subalgebra, this procedure, in a sense, reproduces the Bala-Carter classification of nilpotent orbits in $\mathfrak{g}^\vee$ (see Appendix A and [35]). Let us briefly restrict to the case where $\mathfrak{l}^\vee$ is indeed a Levi subalgebra. In what follow, it is helpful to note that every distinguished orbit is special and d_{LS} always acts as an involution on special orbits. Now, associate an irrep of $W[\mathfrak{l}^\vee]$ to the Coulomb branch of $T^{\tilde{\rho}}[L]$ in the following way,

$$s = Sp^{-1}[\mathfrak{l}^\vee, d_{LS}(\mathcal{O}_H^{\mathfrak{l}^\vee})], \tag{3.27}$$

where d_{LS} is the duality map that stays within $\mathfrak{l}^\vee$. Now, it turns out that the following is always true,

$$\bar{r} = j_{W[\mathfrak{l}^\vee]}^{W[\mathfrak{g}^\vee]}(s), \tag{3.28}$$

where $\bar{r}$ is *Higgs branch Springer invariant* defined earlier and the operation $j_{W[\mathfrak{l}^\vee]}^{W[\mathfrak{g}^\vee]}$ refers to Macdonald-Lusztig-Spaltenstein induction from irreps of the Weyl subgroup $W[\mathfrak{l}^\vee]$ to the parent Weyl group $W[\mathfrak{g}^\vee]$ (See Appendix D). The j induction procedure is sometimes also called truncated induction. It plays a critical role in the interaction of Springer theory with induction within the Weyl group and especially in isolating how the $W[\mathfrak{g}^\vee]$ module structure of $H^{\text{top}}(\mathcal{B})$ can be induced from a $W[\mathfrak{l}^\vee]$ module structure. More generally, the cohomology in lower degrees also obey certain induction theorems (see, for example [136, 193]). For the current purposes (associating a Springer invariant to the defect), only the structure of $H^{\text{top}}(\mathcal{B})$ is relevant and hence (3.28) is sufficient.

Now, (3.28) allows us to rewrite the matching condition (3.24) as

$$\boxed{\begin{aligned} s &= Sp^{-1}[\mathfrak{l}^\vee, d_{LS}(\mathcal{O}_H^{\mathfrak{l}^\vee})] && (3.29a)\\ Sp^{-1}[\mathfrak{g}, \mathcal{O}_N] &= j_{W[\mathfrak{l}^\vee]}^{W[\mathfrak{g}^\vee]}(s) && (3.29b)\end{aligned}}$$

The above matching condition *determines* the pairs $\mathcal{O}_N, \mathcal{O}_H$ for $\mathcal{O}_N$ being a special orbit. Different $\mathcal{O}_N$ arise on the $\mathfrak{g}$ side when the various non-conjugate Levi subalgebras $\mathfrak{l}^\vee$ are considered on the $\mathfrak{g}^\vee$ side. The nilpotent orbit $\mathcal{O}^{\tilde{\rho}}$ that appears in (3.26) can now be identified by the condition $Sp^{-1}[\mathfrak{l}, \mathcal{O}^{\tilde{\rho}}] = s$.

Apart from this highly constraining structure, the matching condition (3.29) additionally enjoys the following beautiful feature. In order to extend the domain of the duality map to include cases where $\mathcal{O}_N$ is non-special, all that one has to do is to allow for $\mathfrak{l}^\vee$ to be an arbitrary centralizer and not just a Levi subalgebra. These more general centralizers are what are called pseudo-Levi subalgebras in [179]. These are classified by the Borel-de Seibenthal procedure which proceeds by enumerating the non-conjugate subsets of the set of extended roots associated to $\mathfrak{g}^\vee$ (See Appendix C).

So, by allowing $\mathfrak{l}^\vee$ to be a pseudo-Levi subalgebra in which a representative $e^\vee$ of the Hitchin orbit O_H is distinguished, one obtains an order reversing duality map that recovers the entire CDT class of defects. Here, it is worth noting that a combinatorial shadow of the cohomological matching condition (3.29) is the fact that there is a bijection between the set of Sommers pairs $(e^\vee, \mathfrak{l}^\vee)$, where $e^\vee$ is a representative of a special orbit in $\mathfrak{g}^\vee$ and the set of all nilpotent orbits in $\mathfrak{g}$. This pattern reappears in many non-trivial relationships that tie representation theoretic constructions associated to $\mathfrak{g}$ and $\mathfrak{g}^\vee$ under the broader Langlands philosophy [134].

Now, by Sommers' extension of the Bala-Carter theorem [178], this more refined data on the Hitchin side is actually equivalent to specifying $(\mathcal{O}_H, C)$ where C is a conjugacy class in $\bar{A}(\mathcal{O}_H)$. $\bar{A}(\mathcal{O}_H)$ is always a Coxeter group. Within this Coxeter group, there is a well defined way to translate data of the form $(\mathcal{O}_H, C)$ to something of the form $(\mathcal{O}_H, \mathcal{C})$ [1], where $\mathcal{C}$ is the Sommers-Achar subgroup of $\bar{A}(\mathcal{O}_H)$ (in the notation and terminology of [39]). For non-special Nahm orbits, this subgroup $\mathcal{C}$ enters the description of the Coulomb branch data in a crucial way as explained in [39]. One also observes that the map between Hitchin and Nahm data offers the following distinction between special and non-special Nahm orbits in the language of boundary conditions for $\mathcal{N} = 4$ SYM. When $\mathcal{O}_N$ is special, the distinguished symmetry breaking deformation on the $G^\vee$ side produces a theory on the boundary whose Coulomb branch is a distinguished orbit in a Levi subalgebra $\mathfrak{l}^\vee$. On the other hand, when $\mathcal{O}_N$ is non-special, the distinguished symmetry breaking deformation on the $G^\vee$ side produces a theory on the boundary whose Coulomb branch is a distinguished orbit in a pseudo-Levi subalgebra $\mathfrak{l}^\vee$ that is not a Levi subalgebra. The description given here is the exact definition of the map in [179] [14]. Here, the definition is placed in a physical context.

3.5.2 Local data

Once the dictionary between the Nahm/Hitchin data is established, one has the following immediate consequences for some of the local prop-

[14] To avoid confusion, it is useful to note that in the notation adopted here, nontrivial local systems appear on the $\mathfrak{g}^\vee$ side, while they appear on the $\mathfrak{g}$ side in Sommers' notation.

erties of the codimension two defects [39],

$$\dim_{\mathbb{H}}(\text{Higgs branch }) = \frac{1}{2}\Big(\dim(\mathcal{N}) - \dim(\mathcal{O}_N)\Big), \quad (3.30)$$

$$\dim_{\mathbb{C}}(\text{Coulomb branch}) = \frac{1}{2}\dim(\mathcal{O}_H). \quad (3.31)$$

Further, the contributions to the trace anomalies a, c and the flavor central charge k can also be determined as outlined nicely in [39]. Before turning to the Toda description, here are some further comments which future work can presumably clarify.

3.5.3 Novel nature of the matching conditions

In the discussion in the early part of this Section, a particular symmetry breaking deformation is applied to the four dimensional theory that was called distinguished symmetry breaking. In fact, outside of type A, this was an essential part of the matching constraint on the duality map if one seeks solutions with $\mathcal{O}_N$ being non-special. It is worthwhile to highlight that one is able to retrieve the Springer invariant for the undeformed theory (UV) from the Springer invariant for the deformed theory (IR) by using the truncated induction procedure.

This structure of the matching conditions suggests that one should think of the family of defect theories $T^{\rho}[G]$ (or alternatively, the boundary conditions of the 4d $\mathcal{N} = 4$ SYM) 'inductively'. In other words, to understand $T^{\rho}[G]$, one first understands $T^{\tilde{\rho}}[L]$ for $L^{\vee}$ being certain subgroups of $G^{\vee}$ and then proceed by induction on semi-simple rank of G to cover all the cases. The procedure to find all solutions to the matching condition proceeds exactly in this fashion.

To further this point of view, it would be interesting to explore the relationship between other calculable observables of these theories under operations that are analogues of truncated induction. In this direction, it is notable that there have been recent advances in the understanding of the Hilbert Series and $\mathbb{S}^3$ partition functions of 3d $\mathcal{N} = 4$ theories (see, for example [122, 103, 47, 54, 46]).

3.5.4 The appearance of endoscopic data

Let the connected component of the centralizer in the group $G^\vee$ of the semi-simple element $m^\vee$ be $L^\vee$. The complex lie algebra associated to this group is the pseudo-Levi subalgebra $\mathfrak{l}^\vee$. Now, upon taking Langlands duals, one observes that $\mathfrak{l}$ is not necessarily a subalgebra of $\mathfrak{g}$. Data of this form occurs for specific choices of the semi-simple element $m^\vee$. Such cases are termed[15] 'elliptic endoscopic'. The general method to compute all cases of 'elliptic endoscopy' is using the Borel-de Siebenthal algorithm (See Appendix C). Here is a simple example of such an occurrence.

[15]More concretely, the corresponding group $L_{\mathbb{C}}$ would be an elliptic endoscopic group for $G_{\mathbb{C}}$. Such scenarios play an important role in the framework of geometric endoscopy explored in [81].

An example of elliptic endoscopic data

Take $G^\vee = SO(2n+1)$. The connected component of the centralizer of a semi-simple element $M = m \ \text{diag}(1, -1, -1, \ldots, -1) \in G^\vee$ is the group $L^\vee = SO(2n)$. Its lie algebra is $\mathfrak{l}^\vee = \mathfrak{so}(2n)$ and this is an example of a pseudo-Levi subalgebra that is not a Levi-subalgebra. Taking Langlands duals at the level of lie algebras, $\mathfrak{g} = \mathfrak{sp}(2n), \mathfrak{l} = \mathfrak{so}(2n)$. $\mathfrak{so}(2n)$ is not a subalgebra of $\mathfrak{sp}(2n)$.

The general pattern here suggests that there is a relationship between the Slodowy slices $\mathcal{S}^\rho \cap \mathcal{N}_\mathfrak{g}$ and $\mathcal{S}^{\tilde{\rho}} \cap \mathcal{N}_\mathfrak{l}$ (understood to be Higgs branches of $T^\rho[G]$ and $T^{\tilde{\rho}}[L]$ or Coulomb branches of their corresponding 3d mirror theories) even when the geometry of nilpotent orbits in $\mathfrak{g}$ is wildly different from that in $\mathfrak{l}$. This relationship should, in a sense, be a 'dual' of the relationship offered by distinguished symmetry breaking on the $\mathfrak{g}^\vee$ side.

3.6 Mass deformations for regular defects

Here, the general picture for understanding mass deformation is outlined. Denote by F the flavor symmetry group associated to a regular defect. This is a connected, reductive group. It arises as the centralizer of the $\mathfrak{sl}_2$ triple $(e, f, h) \in \mathfrak{g}$ that is associated to the Nahm orbit $\mathcal{O}_N$. Consider the maximal torus of the flavor symmetry group, $\mathbb{T}(F)$. Let $\mu \in \mathfrak{h}(F)$ be a semi-simple element. It follows that $[\mu, e] = 0$. Now, denote the dual element in $\mathfrak{h}^\star$ as $\mu^\vee$. One expects that $\mu^\vee$ acts as a mass deformation. In other words, there is a semisimple orbit in $\mathfrak{g}^\vee$ (denoted by $\rho(\mu^\vee)$) such that in the $\mu^\vee \to 0$ limit, one obtains the nilpotent orbit corresponding to $e^\vee$. In other words, the Higgs field in the Hitchin singularity for the undeformed defect has the following behaviour,

$$\phi = \frac{\rho(e^\vee)}{z} + \ldots, \tag{3.32}$$

where $\rho_{e^\vee}$ is a representative of the nilpotent orbit associated to $e^\vee$. The mass deformed defect is then given by,

$$\phi = \frac{\rho(\mu^\vee)}{z} + \ldots \tag{3.33}$$

where $\rho(\mu^\vee)$ is a representative of the semi-simple orbit associated to $\mu^\vee$ and further, $\lim_{\mu^\vee \to 0}(\rho(\mu^\vee)) = \rho(e^\vee)$.

A scenario where the above setup can be realized, atleast in principle, is when orbit $\mathcal{O}_H$ is an *induced* orbit (in the sense of Lusztig-Spaltenstein induction [140]). But, there exist special orbits which are *not* induced. There are called *rigid* special orbits. Their existence suggests that one should look at an 'affine analogue' of orbit induction for a setup of the above form to be realized. Of particular interest would be cases where the special orbit that is part of the Hitchin data has a non-trivial special piece associated to it. This is left for future work.

3.7 The part about Toda

In light of the observations of AGT-W [6, 203], it is expected that the sphere partition function of a theory of class $\mathcal{S}$ (built using codimension two defects of $\mathcal{X}[\mathfrak{j}]$ as in 3.2.1) can be expressed as a correlator in a two dimensional Toda CFT of type $\mathfrak{g}$. Let us briefly recall some facts about Toda CFTs. They are described by the following Lagrangian on a disc with a curvature insertion at infinity,

$$S_T = \frac{1}{2\pi}\int \sqrt{\hat{g}}d^2z\left(\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \sum_{i=1}^{\mathrm{rank}(\mathfrak{g})} 2\pi\Lambda e^{2b(e_i,\phi)}\right) + \frac{1}{\pi}\int (Q,\phi)d\theta + (\ldots), \tag{3.34}$$

where $e_i \in \mathfrak{h}^*$ are the simple roots of the root system associated to $\mathfrak{g}$, $\phi \in \mathfrak{h}$ is the Toda field and $Q = b + b^{-1}$. A special case of Toda[$\mathfrak{g}$] is Liouville CFT. It corresponds to the case $\mathfrak{g} = A_1$. Recall that the chiral algebra of Liouville CFT is the Virasoro algebra. The chiral algebra of the more general Toda[$\mathfrak{g}$] theories are certain affine $\mathcal{W}$ algebras. These theories have conserved currents $\mathcal{W}^k(z)$ of integer spins k. The spectrum of values $\{k-1\}$ in a particular Toda[$\mathfrak{g}$] theory is equal to the set of exponents of the lie algebra $\mathfrak{g}$. The unique spin 2 conserved current in this set is the stress tensor $\mathcal{W}^2(z) = T(z)$.

The $\mathcal{W}$-algebras that arise in such theories have the additional property that they can be obtained by a Hamiltonian reduction procedure from affine Lie algebras which arise as the chiral algebras of non-compact WZW models. This procedure admits a generalization for every $\sigma : \mathfrak{sl}_2 \to \mathfrak{g}$ and this allows one construct other W algebras. When σ is taken to be principal, then one obtains the usual Toda[$\mathfrak{g}$] theories. It is only the Toda[$\mathfrak{g}$] theories that will concern us in what follows since this is the setting for the direct generalizations of [6, 203] to arbitrary theories of class $\mathcal{S}$. While Toda theories exist for both simply laced and non-simply laced $\mathfrak{g}$, the discussion that follows will be confined to the case $\mathfrak{g}(\cong \mathfrak{j}) \in A, D, E$. If one were to consider the twisted defects and seek a Toda interpretation for them, an adaptation of much of the arguments below for $\mathfrak{g} \in B, C, F_4, G_2$ would likely be relevant.

When trying to build an understanding of the AGT conjecture for an arbitrary theory of class $\mathcal{S}$, a good starting point is to have the following local-global setup in mind,

- *Local aspects of the AGT conjecture* : This is the claim that the regular codimension two defects of the $\mathcal{X}[\mathfrak{g}]$ admit a description in terms of certain primary operators of the principal Toda theory of type $\mathfrak{g}$. Let us call this part of the AGT dictionary the *primary map* $\wp$. This map is a bijection from the set of defects to the set of semi-degenerate states (borrowing terminology from [120]) in the Toda theory and concerns data that is local to the codimension two defect insertion on the Riemann surface $C_{g,n}$ and does not involve the Riemann surface in any way.

- *Global aspects of the AGT conjecture* : If the description of the four dimensional theory involves compactification of $\mathcal{X}[\mathfrak{g}]$ on $C_{g,n}$, then the sphere partition function (including non-perturbative contributions) of this theory is obtained by a Toda correlator on $C_{g,n}$ with insertions of the corresponding primary operators of Toda theory at the n punctures. The identification of the corresponding Toda primary is done according to the map $\wp$. The identification of the conformal block with the instanton partition function is a crucial ingredient in the global AGT conjecture. Checks of the conjecture for the sphere partition function in cases of arbitrary $\mathfrak{g}$ are available in specific corners of the coupling constant moduli space where Lagrangian descriptions become available for the four dimensional theories[6, 203].

In the discussion above, a choice was made to restrict to four dimensional SCFTs obtained by the compactification from six dimensions involving just the regular defects. But, it is interesting to note that the formalism associated to the AGT conjecture can also be extended to the cases where SCFTs are built out of irregular defects[16] as in [29, 90, 118] and certain aspects extend to the case of asymptotically free theories (See, for example [85, 125]). There exist generalizations which involve partition functions in the presence of supersymmetric loop and surface operators of the 4d theory (See, for example [5, 63, 62] and [7]). Some of the mathematical implications that follow from the observations of AGT have been explored in [33, 153, 165, 144]. For a more complete review of the literature, consult [184].

The global AGT conjecture suggests that the OPE of codimension two defects of the six dimensional theory is controlled by the $\mathcal{W}$-algebra symmetry of the Toda theory. While this is powerful as an organizing idea, it is particularly hard to proceed in practice as the non-linear nature of $\mathcal{W}$

[16] The terminology of regular and irregular defects is from [201, 87].

algebras complicates their representation theory. In the discussion that follows, the goal is only to establish the *primary map* for as many defects as possible in arbitrary $\mathfrak{g}$. In particular, global aspects of the AGT conjecture or any of its generalizations are not analyzed (except for a discussion about scale factors).

3.7.1 The primary map $\wp$

In the original work of AGT, this map was obtained for the case of A_1. There is just a single nontrivial codimension two defect [17] in this case. So, the map is particularly straightforward to describe. After setting the radius of the four sphere to be unity (see Chapter 4 for how the radius dependence on the overall partition function can be analyzed), this map can be described as

$$\wp : [1^2]_N \to e^{2\alpha\phi} \mid \alpha = Q/2 + im, \tag{3.35}$$

where ϕ is the Liouville field. In the map above, the Nahm orbit is used to identify the defect operator. The defect could have alternatively been identified by the Hitchin orbit associated to it, namely the orbit $[2]_H$. But, it will turn out that the Nahm orbit is the one that is convenient for obtaining the generalization of this for arbitrary $\mathfrak{g}$. So, it is convenient to use it to tag a particular codimension two defect. Two important aspects of the above map are

[17] The trivial defect (the defect corresponding to the principal Nahm pole) is always mapped to the identity operator on the 2d CFT side.

- A precise identification of $\Re(\alpha)$
- An identification of $\Im(\alpha)$ with im where m is a mass deformation parameter.

An identification similar to the one above for the mass parameter m exists for the Coulomb branch modulus a. In both of these cases, a distinguished real subspace of the $\mathcal{N} = 2$ theory's parameters is picked out in writing the map to the corresponding Liouville primary.

To extend these argument to higher rank cases, a natural thing to try and obtain is a generalization of the primary map $\wp$ that is in the same form. Say,

$$\wp : \mathcal{O}_N \to e^{(\alpha,\phi)} \mid \alpha = \Re(\alpha) + \Im(\alpha), \tag{3.36}$$

with some prescribed conditions on $\Re(\alpha)$ and $\Im(\alpha)$ that depend on $\mathcal{O}_N$. Here, $\phi \in \mathfrak{h}$ is the Toda field and it is a r-dimensional vector of scalar fields where r is the rank of $\mathfrak{g}$ and $\alpha \in \mathfrak{h}^*$ is the Toda momentum. The relevant primaries for the case of A_n were identified in [120] (a precise formulation in terms of the Nahm orbit data is provided below). The general picture is that $\wp$ maps the zero Nahm orbit to the maximal puncture while the other Nahm orbits are mapped to certain semi-degenerate primary operators in the corresponding Toda theory. The principal Nahm orbit is mapped to the identity operator. The semi-degenerate primaries of [120] contain null vectors at level-1 with the exact number and nature of these null vectors depending on the associated Nahm orbit. Combinatorially, specifying the level-1 null vectors amounts to specifying a certain subset of the simple roots in the root system associated to A_n. One gets the relationship to the Nahm orbit by noticing a very natural connection between subsets of simple roots and nilpotent orbits in A_n. This connection is offered by the Bala-Carter classification of nilpotent orbits in $\mathfrak{g}$. For a quick summary of the work of Bala-Carter, see Appendix A and for a more detailed[18] account, see [44, 35, 25]. For the current purposes, the important fact will be that the Bala-Carter classification amounts to specifying a pair $(\mathfrak{a}, e)$ where $\mathfrak{a}$ is the semi-simple part of Levi subalgebra of $\mathfrak{g}$ and e is a distinguished nilpotent element in that subalgebra.[19]

Levi subalgebra are naturally classified by non-conjugate subsets of the set of simple roots. When e is principal nilpotent in a Levi subalgebra, the corresponding orbit is called principal Levi type [20]. It turns out that all the non-zero orbits in type A are principal Levi type. In particular, the combinatorial data associated to a Nahm orbit by the Bala-Carter theory is precisely the subset of simple roots corresponding to the subalgebra $\mathfrak{a}$. Once the combinatorial data is placed in the setting of nilpotent orbits, a reasonable generalization would be to consider all principal Levi type orbits in

[18] I thank Birne Binegar for correspondence and for sharing some related unpublished work.

[19] The Levi subalgebra occurring in this discussion should not be confused with the Levi subalgebra $\mathfrak{l}^\vee$. The former is a subalgebra of $\mathfrak{g}$ and arises as part of the Nahm data while the latter is a subalgebra of $\mathfrak{g}^\vee$ and is part of the Hitchin data.

[20] Interestingly, certain finite $\mathcal{W}$ algebras associated to nilpotent orbits of principal Levi type also play an important role in the mathematical approach to a variant of the original setup of AGT [33], extended to arbitrary $\mathfrak{g}$.

arbitrary $\mathfrak{g}$. The combinatorial data assigned to such orbits is always a subset of the simple roots of the root system associated to $\mathfrak{g}$. Additionally, let F denote the reductive part of the centralizer of the triple (e, f, h) associated to the Nahm orbit. This is the *global symmetry* associated to the Higgs branch of the codimension two defect, or equivalently of $T^{\rho}[G]$ [39]. Now, the mass deformation parameters of $T^{\rho}[G]$ (and hence of the defect) are valued in a Cartan subalgebra of $\mathfrak{f}$. In particular, the number of such linearly independent parameters is equal to rank($\mathfrak{f}$). For any non-zero orbit of principal Levi type, this quantity is necessarily non-zero. It is a general property that

$$\text{rank}(\mathfrak{f}) = \text{rank}(\mathfrak{g}) - \text{rank}(\mathfrak{a}). \tag{3.37}$$

Now, consider a Toda primary with momentum $\alpha \in \Lambda^{+}$ that obeys

$$\begin{aligned} (\Re(\alpha), e_i) &= 0, \\ 0 \leq \Re(\alpha) &\leq Q\rho, \\ \Im(\alpha) &= 0, \end{aligned} \tag{3.38}$$

where e_i is any simple root in the root system corresponding to the subalgebra $\mathfrak{a}$ and ρ is the Weyl vector of $\mathfrak{g}$ and the relation $\leq$ is in the partial order on the set of dominant weights Λ^{+}. Imposing the above conditions would also mean, in particular, that $(\alpha, \rho_{\mathfrak{a}}) = 0$, where $\rho_{\mathfrak{a}}$ is the Weyl vector of the subalgebra $\mathfrak{a}$. When the Nahm orbit associated to codimension two defect is principal Levi type, I argue that (3.38) provides the right Toda primary in the massless limit. A piece of evidence that supports such a statement is the following. Let us write $\Re(\alpha)$ as a combination of the fundamental weights of $\mathfrak{g}$

$$\Re(\alpha) = a_i \omega_i, \tag{3.39}$$

where $a_i \neq 0$ and $\{\omega_i\}$ is some subset of the fundamental weights. Now, deform the Toda momentum such that it acquires an imaginary part given by

$$\Im(\alpha) = m_i \omega_i, \tag{3.40}$$

so that $(\alpha, e_i) = 0$ holds for all e_i being simple roots of $\mathfrak{a}$. The m_i introduced above are the mass parameters that one would associate with the codimension two defect. And the total number of such linearly independent parameters will equal the number of fundamental weights occurring in (3.39) and this is equal to precisely rank($\mathfrak{f}$), as expected. For type A, the above procedure reproduces the semi-degenerate primaries considered in [120] [21]. For non-zero orbits that are not principal Levi type, one natural guess is that the level-1 null vectors that are imposed are still given by the set of simple roots that one associates to the Bala-Carter Levi. In these cases, a nilpotent representative will correspond to a non-principal distinguished nilpotent orbit in $\mathfrak{a}$. This corresponds to picking a further subset of the simple roots of $\mathfrak{a}$. This additional combinatorial data may presumably be translated to null vector conditions at higher level, but this needs to be made precise. The case of non-principal Levi type orbits for which rank($\mathfrak{f}$) is zero would

[21] This point was made in [15] using the Dynkin weight h of the Nahm orbit.

be particularly interesting since the mere existence of such cases challenges the wisdom that $\Im(\alpha)$ should give rise to an associated mass deformation. In $\mathfrak{g} = E_8$, for example, all orbits that are distinguished in $\mathfrak{a} = E_8$ have rank$(\mathfrak{f}) = 0$. To give some idea about how many of the nilpotent orbits in $\mathfrak{g}$ tend to be of principal Levi type, the data for certain low rank $\mathfrak{g}$ is displayed in Table 3.2.

It should be mentioned here that one can device some local checks of the map $\wp$ that are sensitive to the Coulomb branch data. In [120], it was checked that the behaviour of the Seiberg-Witten curve near the punctures is reproduced in a 'semi-classical' limit of the Toda correlators together with insertions of the currents $\mathcal{W}^k(z)$. This is really a direct check on the local contribution to the Coulomb branch from a Toda perspective. Here, the map between the Nahm and Hitchin data obtained in the previous section already provides a candidate for the local contribution to the Coulomb branch from a Toda primary whose Nahm orbit is principal Levi type. But, a direct check of this assertion would be more pleasing.

3.7.2 Local contributions to Higgs and Coulomb branch dimensions

As just discussed, once the relation between the Nahm data and the Toda primary is known, one can use the dictionary between the Nahm/Hitchin data to associate a Hitchin orbit to a Toda primary. With this, the effective contribution to the local Higgs branch and the local Coulomb branch from a particular Toda primary can be inferred. From the tinkertoy constructions

Table 3.2: Nilpotent orbits of principal Levi type in certain Lie algebras

$\mathfrak{g}$	# of Nilpotent orbits	# of principal Levi orbits
A_4	7	7
B_4	13	12
C_4	14	12
D_4	12	11
E_6	21	17
E_7	45	32
E_8	70	41
F_4	16	12
G_2	5	4

[39], the following expressions are known for $n_h - n_v$ (the total quaternionic Higgs branch dimension) and d (the total Coulomb branch dimension) in terms of the Nahm and Hitchin orbit data for each defect $(\mathcal{O}_H^i, \mathcal{O}_N^i)$,

$$(n_h - n_v) \quad = \quad \sum (n_h - n_v)^i + (n_h - n_v)^{\text{global}} \tag{3.41}$$

$$d \quad = \quad \sum_i d^i + d^{\text{global}} \tag{3.42}$$

with

$$(n_h - n_v)^i \quad = \quad \frac{1}{2}\Big(\text{dim}(\mathcal{N}) - \text{dim}(\mathcal{O}_N^i)\Big) = \text{dim}(\mathcal{B}_N^i) \tag{3.43}$$

$$d^i \quad = \quad \frac{1}{2}\text{dim}(\mathcal{O}_H^i) \tag{3.44}$$

and

$$(n_h - n_v)^{\text{global}} \quad = \quad (1-g)\text{rank}(\mathfrak{g}) \tag{3.45}$$

$$d^{\text{global}} \quad = \quad (g-1)\text{dim}(\mathfrak{g}) \tag{3.46}$$

3.7.3 Local and Global contributions to Scale factors in Toda theories

As a simple illustration of the local-global interplay, one can consider how the scale factor in the sphere partition function that captures the Euler anomaly of the four dimensional theory is calculated. From a purely four dimensional perspective, the Euler anomaly is very well understood in the tinkertoy constructions. In following chapter, the radius dependent factor in the sphere partition function is made explicit and the relation to a corresponding scale factor in the two dimensional CFT is pointed out.

Chapter 4

Euler anomaly and scale factors in Liouville/Toda theories

4.1 Introduction

In several investigations of the dynamics of theories of class $\mathcal{S}$ (introduced in Chapter 1), it has become increasingly clear that various observables of this class of theories admit an efficient description using the language of two dimensional physics. A particular example of such an observable is the partition function of the four dimensional theory defined on a sphere ($Z_{\mathbb{S}^4}$). Following Pestun's evaluation of the partition function for a subset of class $\mathcal{S}$ theories theories via localization [160] and the construction of these theories using the $(0,2)$ six dimensional theory SCFT $\mathfrak{X}[\mathfrak{j}]$ [86, 87], AGT noticed the remarkable fact that the partition functions in type[1] $\mathfrak{g} = A_1$ coincide with certain correlators in a particular Liouville conformal field theory [6]. They further conjectured (see also [203] in this regard) that an analogous relationship exists for partition functions of various higher rank theories and corresponding Toda correlators. Many checks of this proposal are available in specific corners of the moduli space where the four dimen-

[1] The lie algebras $\mathfrak{j}, \mathfrak{g}$ have the same interpretation as in the earlier chapters.

sional theories admit a Lagrangian description as a weakly coupled gauge theory along with conventional matter multiplets. At other corners of the moduli space (which happen to be the vast majority), one runs into the following predicament. On the four dimensional side, the localization techniques do not extend as there is no known Lagrangian description. On the two dimensional side, a complete analytical understanding of the corresponding Toda correlators is missing. One of the initial motivations for the work in this chapter was to partly alleviate this situation by pointing out that the AGT dictionary can very easily be expanded to include an observable that is much better understood, namely the Euler anomaly of the four dimensional SCFT. Borrowing ideas from the tinkertoy constructions, I propose a framework for calculating this dependence. This framework is of independent interest and can potentially shine light on certain aspects of the tinkertoy constructions. While the Chapter is confined to theories of type A_n, the results from the previous chapter can be used to extend it to arbitrary type.

4.2 Partition function on $\mathbb{S}^4$ and the Euler anomaly

For any four dimensional theory that is defined on a four sphere, it is expected that the logarithm of the sphere partition function has a divergent piece that is proportional to the Euler anomaly a [34]. This is an important observable for any CFT since it is a measure of the massless degrees of freedom in the CFT. In [34], it was also conjectured that such a measure exists at all points along a renormalization group flow and that its value strictly decreases as more degrees of freedom are integrated out. A version of this conjecture has recently been proved in [128]. The goal here is to focus on the class $\mathcal{S}$ SCFTs and make the dependence on the Euler anomaly manifest in their sphere partition functions. We will begin by considering the case of conformal class $\mathcal{S}$ theories with Lagrangian descriptions. A definition of these theories on the round four sphere and a localization scheme to evaluate the partition function of the theory so defined[2] was described by Pestun [160]. This construction was recently extended to the case of the more general case of an ellipsoid $\mathbb{S}^4_b$ [102]. In much of the literature on the AGT conjecture, the dependence of the partition function on the Euler anomaly is not made explicit[3]. In the original work of [160], this was not necessary as the corresponding factors in the partition function cancel in the calculation of expectation values of BPS Wilson and 't-Hooft loop operators[4]. For the purposes of this work, it would be important to make this dependence explicit. The considerations in this Chapter will be restricted to analyzing the case of a round sphere.

While the focus here is solely on the physical $\mathcal{N} = 2$ theories, it is interesting to note that the dependence made explicit here has a cousin in the world of topological QFTs obtained from twisting the Lagrangian $\mathcal{N} = 2$ theories. In the evaluation of their partition functions on a general four

[2] See also [74] and [66] on the question of defining such theories on curved manifolds.
[3] For considerations of similar issues in three dimensions, see [117].
[4] I thank V.Pestun for a discussion.

manifold (with non-zero Euler characteristic χ and signature σ), the measure in the path integral has an explicit dependence on the anomaly parameters a, c [173].

4.2.1 Localization on the four sphere

For a general superconformal $\mathcal{N} = 2$ theory with matter in representation W of the gauge group G taken on a sphere $\mathbb{S}^4$ of radius R_0, the one loop functional determinant around the locus of classical solutions on which the theory localizes was evaluated in [160]. It takes the following form,

$$Z^W_{1-loop} = \frac{\prod_{\alpha\in\text{weights(Ad)}}\prod_{n=1}^{\infty}((\alpha.a_E)^2+\mu^2n^2)^n}{\prod_{w\in\text{weights(W)}}\prod_{n=1}^{\infty}((w.a_E)^2+\mu^2n^2)^n}.$$

The hypermultiplet masses have been set to zero and $\mu = R_0^{-1}$. Let us focus our attention on a prototypical infinite product that occurs in these determinants and go through with the steps of regularizing it. We choose the one in the numerator of the example just studied and rewrite it as

$$\prod_{n=1}^{\infty}((\alpha.a_E)^2+\mu^2n^2)^n = \prod_{n=1}^{\infty}(i(\alpha.a_E)+\mu n)^n(-i(\alpha.a_E)+\mu n)^n.$$

Each factor can further be rewritten as

$$\prod_{n=1}^{\infty}(i(\alpha.a_E)+\mu n)^n = \frac{\prod_{n,m\in N^2}(i(\alpha.a_E)+\mu m+\mu n)}{\prod_{n\in N}(i(\alpha.a_E)+n\mu)}, \tag{4.1}$$

where N^2 is the set of all (m, n) such that $m, n \in \mathbb{N} = 0, 1, 2, \ldots$. The form of the infinite product in the numerator is very suggestive of a regularizing scheme using the Barnes double zeta function ζ_2^B. For the denominator, the

Hurwitz zeta function seems like the appropriate choice. Let us recall the sum representation for ζ_2^B,

$$\zeta_2^B(s,x;a,b) = \sum_{m=0,n=0}^{m=\infty,n=\infty} (x+am+bn)^{-s}.$$

$\zeta_2^B(s,x)$ can be analytically continued to a meromorphic function which has poles when $x = -n_1 a - n_2 b$. We can use ζ_2^B to regulate infinite products using the following (formal) identity

$$\prod_{n,m\in N_0} (x+ma+nb) = e^{-\zeta_2^{B'}(0,x;a,b)}.$$

Before the products in this problem are regularized, it is helpful to note that under a scaling transformation that takes $(x,a,b) \to (kx,ka,kb)$, the new regularized product is related to old product in the following way (the additional steps are reviewed in Appendix E.1)

$$\prod_{n,m=0}^{\infty} (k(x+ma+nb)) = k^{\zeta_2^B(0,x;a,b)} e^{-\zeta_2^{B'}(0,x;a,b)}.$$

Similar equations hold for the Hurwitz zeta function. Now, using $x = i(\alpha.a_E), k=\mu, a=1, b=1$, (4.1) is regularized to

$$\frac{\prod_{n,m\in N^2}(i(\alpha.a_E)+\mu m+\mu n)}{\prod_{n\in N}(i(\alpha.a_E)+n\mu)} = \mu^{\zeta_2^B(0,i(\alpha.a_E);1,1)-\zeta^H(0,i(\alpha.a_E))} e^{-\zeta_2^{B'}(0,i(\alpha.a_E);1,1)+\zeta^{H'}(0,i(\alpha.a_E))}.$$

Further noting that

$$\begin{aligned} \zeta_2^B(0,x;1,1) &= \frac{5}{12} - x + \frac{x^2}{2}, \\ \zeta^H(0,x) &= \frac{1}{2} - x, \end{aligned}$$

and

$$e^{-\zeta_2^{B'}(0,x;1,1)+\zeta^{H'}(0,x;1,1)} = G(1+x),$$

where $G(z)$ is the Barnes G function, [5] the regularized product becomes

$$\mu^{-\frac{1}{12}+\frac{\alpha.a_E^2}{2}} G(1+\frac{i\alpha.a_E}{\mu}).$$

Thus the total contribution from each root in (4.2.1) is

$$\mu^{-\frac{1}{6}+(\alpha.a_E^2)^2} G(1+\frac{i\alpha.a_E}{\mu}) G(1-\frac{i\alpha.a_E}{\mu}).$$

Regulating each piece in a similar way and defining $H(z) = G(1+z)G(1-z)$,

$$Z^W_{1-loop} = \frac{\prod_{\alpha\in\text{weights(Ad)}} \mu^{-1/6} H(\frac{i\alpha.a_E}{\mu})}{\prod_{w\in\text{weights(W)}} \mu^{-1/6} H(\frac{iw.a_E}{\mu})}.$$

In the above step, the expression has been simplified using the condition for vanishing beta function

$$\sum_{\alpha\in\text{weights(Ad)}} (\alpha.a_E)^2 = \sum_{w\in\text{weights(W)}} (w.a_E)^2.$$

Let us specialize to the case of $G = SU(N)$ and $N_f = 2N$. This gives,

$$Z^{N_f=2N}_{1-loop,SU(N)} = \mu^{\frac{1}{6}(N^2+1)} \frac{\prod_{\alpha\in\text{weights(Ad)}} H(\frac{i\alpha.a_E}{\mu})}{\prod_{w\in\text{weights(W)}} H(\frac{iw.a_E}{\mu})}. \tag{4.2}$$

The μ dependent factor in front of the product of H functions in (4.2) will play an important role in the identification of the Euler anomaly in the next section.

[5]For a summary of properties of the Barnes function and other special functions that appear in this Chapter, see Appendix E.2.

4.2.2 The Euler anomaly

All the necessary tools required bring out the dependence of the sphere partition function on the Euler anomaly are now assembled. From [160], the general form of the partition function (including non-perturbative contributions) is

$$Z_{\mathbb{S}^4} = \int_{\mathbf{a}\in\mathfrak{g}} d\mathbf{a} e^{-S_{cl}(\mathbf{a},\mu)} Z_{1-loop}(\mathbf{a},\mu)|Z_{inst}(\mathbf{a},\mu)|^2,$$

where $S_{cl} = \frac{8\pi(\mathbf{a},\mathbf{a})}{g^2\mu^2}$ and Z_{1-loop} is given by (4.2). Z_{inst} is the Nekrasov partition function defined on a $\Omega_{\epsilon_1,\epsilon_2}-$ background with $\epsilon_1 = \epsilon_2 = \mu$. This can be reduced to an integral over the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$

$$Z_{\mathbb{S}^4} = \int_{\mathbf{a}\in\mathfrak{h}} d\mathbf{a}\mathbf{V}(\mathbf{a}) e^{-S_{cl}(\mathbf{a},\mu)} Z_{1-loop}(\mathbf{a},\mu)|Z_{inst}(\mathbf{a},\mu)|^2, \tag{4.3}$$

where $\mathbf{V}(\mathbf{a})$ is the Vandermonde determinant. It is now convenient to change variable from $\mathbf{a}$ to $\tilde{\mathbf{a}} = \mathbf{a}/\mu$. Note here that the form of S_{cl} and Z_{inst} are such that they are independent of μ when expressed in terms of $\tilde{\mathbf{a}}$. So, the integral in the new variables is

$$Z_{\mathbb{S}^4} = \mu^{(N^2-1)+\frac{1}{6}(N^2+1)} \int_{\tilde{\mathbf{a}}\in\mathfrak{h}} d\tilde{\mathbf{a}}\mathbf{V}(\tilde{\mathbf{a}}) e^{-S_{cl}(\tilde{\mathbf{a}})} Z_{1-loop}(\tilde{\mathbf{a}})|Z_{inst}(\tilde{\mathbf{a}})|^2. \tag{4.4}$$

The exponent of μ in the above expression can be identified as $4a$ where a is the Euler anomaly of the theory. This factor should be proportional to χa where χ is the Euler characteristic of the curved manifold on which the theory is defined. To fix conventions concretely, one can follow [65] and set

$$Z^{-1}\mu\frac{\partial Z}{\partial \mu} = -\int dx^4 \langle T^j_j \rangle = 2\chi a. \tag{4.5}$$

In a theory with N_S real scalars, N_F Dirac fermions and N_V vector fields, a (as normalized above) is given by

$$a = \frac{1}{360}(N_S + 11N_F + 62N_V). \tag{4.6}$$

Recall that a $\mathcal{N} = 2$ vector multiplet is the equivalent of a vector field, two real scalars and a single Dirac fermion and that a $\mathcal{N} = 2$ hypermultiplet is the equivalent of four real scalars and one Dirac fermion. So, for a $\mathcal{N} = 2$ theory with n_v vector multiplets and n_h hyper multiplets,

$$4a = n_v + \frac{n_h - n_v}{6}. \tag{4.7}$$

From (4.4), calculate

$$Z^{-1}\mu\frac{\partial Z}{\partial \mu} = (N^2 - 1) + \frac{N^2 + 1}{6}. \tag{4.8}$$

and note that the result equals $4a$ for the theory. Noting that $\chi(\mathbb{S}^4) = 2$, this indeed matches with (4.5). For Lagrangian theories (like the ones considered so far), parameterizing a by n_v, n_h is the most obvious choice for these correspond to the number of vector multiplets and the number of hypermultiplets. Often, this is used for arbitrary theories with the understanding that it is just a convenient parameterization of the trace anomalies. It is then appropriate to call n_h and n_v the effective number of hypermultiplets and vector multiplets. The formula for the other trace anomaly c is given by

$$c = \frac{n_v}{4} + \frac{n_h - n_v}{12}. \tag{4.9}$$

For a general class $\mathcal{S}$ theory obtained by taking theory $\mathcal{X}[\mathfrak{g}]$ on $C_{g,n}$, the quantities n_v and $n_h - n_v$ are related to the dimensions of vacuum moduli spaces in a simple fashion. Let d_k denote the graded Coulomb branch dimension, that is the number of Coulomb branch operators of degree k. n_v is given by

$$n_v = \sum_k (2k-1)d_k. \tag{4.10}$$

$(n_h - n_v)$ on the other hand is equal to the quaternionic Higgs branch dimension when there is such a branch. For theories without a true Higgs branch, one can still associate a maximally Higgsed branch whose quaternionic dimension is $n_h - n_v$ upto some abelian vector multiplets[89],

$$\text{dim}_{\mathbb{Q}}(\mathcal{H}) = n_h - n_v + g \ \ \textbf{rank}(\mathfrak{g}). \tag{4.11}$$

The total n_h and n_v for any theory is computed as in [39],

$$n_h = \sum_i n_h^i + n_h^{global}, \tag{4.12}$$

$$n_v = \sum_i n_v^i + n_v^{global}, \tag{4.13}$$

where the global contributions [6] are given by [19, 4]

$$\begin{aligned} n_h^{global} &= \frac{4}{3}(g-1)\hat{h}(\text{dim}G), \\ n_v^{global} &= (g-1)(\frac{4}{3}\hat{h}\text{dim}G + \text{rank}G), \end{aligned} \tag{4.14}$$

[6]The central charge of the Toda CFT of type $\mathfrak{g}$ also has a similar presentation owing to the fact that it too can be obtained from the anomaly polynomial in six dimensions[30, 4].

where $\hat{h}$ is the dual Coxeter number and n_h^i, n_v^i are the local contributions from a codimension two defect. In the rest of the Chapter, the goal will be to understand how the Euler anomaly (4.8) is encoded in the Liouville/Toda correlators assigned to a general class $\mathcal{S}$ theory of type $\mathfrak{g} = A_n$.

4.3 Scale factors in Liouville correlators

In this section, the prefactor that encodes the Euler central charge is shown to have a natural role in Liouville theory. It will be identified with the scale factor for the stripped correlator. A plausible path integral argument for how this scale factor arises is provided for the simplest case of a three point function and will be used to get some intuition for the appearance of such a factor. For higher point functions, such a luxury does not exist and one would have to resort to calculating them directly from the scaling behaviour of the Υ functions that occur in the DOZZ formula.

Recall that Liouville field theory on a Riemann surface C is defined by the following action (written with an unconventional normalization, $\phi = \hat{\phi}/6$ where $\hat{\phi}$ is the Liouville field in the usual normalization),

$$S_L = \frac{1}{72\pi} \int \sqrt{\hat{g}} d^2 z \left(\frac{1}{2} \hat{g}^{ab} \partial_a \phi \partial_b \phi + 3Q\hat{R}\phi + 2\pi\Lambda e^{2b\phi} \right), \tag{4.15}$$

where z is a complex co-ordinate on the C. This theory is conformal upto a $c-$ number anomaly. While the observables of the theory depend only on the conformal class of the metric g on C, it is often convenient to perform calculations by choosing a particular reference metric $\hat{g}$ in the same conformal class as g. The action above is written in terms of this reference metric. The physical metric is given by $g_{ab} = e^{\frac{2\hat{\phi}}{Q}}\hat{g}_{ab}$. The stress energy tensor for this theory is a shifted version of that for a free theory :

$$T(z) = -(\partial\hat{\phi})^2 + Q\partial^2\hat{\phi} \tag{4.16}$$

and the central charge is given by

$$c = 1 + 6Q^2. \tag{4.17}$$

Let us now formulate this theory on the Euclidean two sphere. Here, g is taken to be the usual round metric and $\hat{g}$ as a flat metric. Calculations with the reference metric are to be done with the understanding that there is an operator insertion at infinity that encodes the curvature of the physical metric. A way to demand this is through a boundary condition for the field ϕ

$$\phi = -2Q\log(R/R_0) + \mathcal{O}(1), \tag{4.18}$$

where R (=$\sqrt{z\bar{z}}$) is the distance measured in the flat reference metric. The parameter R_0 is introduced here for purely dimensional reasons. Its role in the overall scheme of things will become more transparent as we proceed. Now, a way to restrict to an integration over only fields that obey (4.18) is to write the Liouville action on a disc of radius R along with boundary term that implements the curvature boundary condition and a field independent term that keeps the action finite in the $R \to \infty$ limit[7].

$$S_{L,\text{disc}} = \frac{1}{72\pi}\int_D \sqrt{\hat{g}}d^2z\left(\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi+2\pi\Lambda e^{2b\phi}\right)+\frac{Q}{12\pi}\int_{\partial D}\phi d\theta+\frac{1}{3}Q^2\log(R/R_0). \tag{4.19}$$

The above action is invariant under a conformal transformation of the metric combined with a corresponding shift in the Liouville field,

$$\begin{aligned} z' &= w(z), \\ \phi'(z) &= \phi(z) - \frac{Q}{2}\log\left(\frac{\partial w}{\partial z}\right)^2. \end{aligned}$$

Note that last term plays an important role in ensuring invariance under this transformation and further, it also guarantees that the action is finite [105, 206].

According to the AGT correspondence, the partition function of a A_1 class $\mathcal{S}$ theory on the round sphere is identified with a corresponding $n-$point correlator in the $c = 25$ Liouville CFT (upto some factors). Recall that these theories are obtained by compactifying theory $\mathcal{X}[\mathfrak{g}]$ on a Riemann surface $C_{g,n}$ of genus g in the presence of n codimension two defects whose locations on C are given by n punctures. The AGT correspondence assigns to this theory a Liouville correlator $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$ where $\mathcal{O}_i = e^{2\alpha_i\phi}$. The Liouville momenta are related to the mass deformation parameters of the 4d theory as $\alpha_i = Q/2 + im_i$. One of the simplest examples of this 4d-2d dictionary is illustrated by the case of a sphere with three punctures. This

[7]Henceforth, such a limit will be assumed whenever Liouville/Toda actions on the disc are considered.

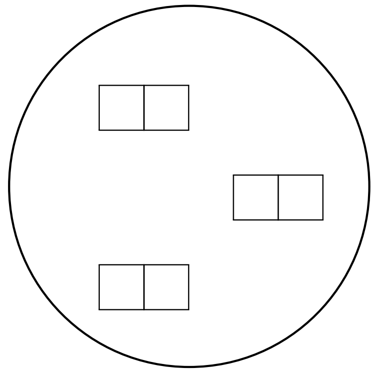

Figure 4.1: A_1 theory on a sphere with three punctures

corresponds to a theory of four free hypermultiplets. On the Liouville side, the correlator is known to take the following form,

$$V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])} = C(\alpha_1,\alpha_2,\alpha_3)|z_{12}|^{-2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{-2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{-2(\Delta_2+\Delta_3-\Delta_1)}.$$

where $C(\alpha_1,\alpha_2,\alpha_3)$ is given by,

$$\begin{aligned} C(\alpha_1,\alpha_2,\alpha_3) &= \left[\pi\Lambda\gamma(b^2)b^{2-2b^2}\right]^{(Q-\sum_i\alpha_i)/b} \times \\ & \frac{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)}. \end{aligned}$$

The notation introduced here for the correlator is done with a view towards the higher rank cases. The $\mathfrak{sl}_2$ refers to the fact that Liouville CFT can be obtained from the $SL(2,\mathbb{R})$ WZW model under a gauging labeled by the principal embedding of $\mathfrak{sl}_2 \to \mathfrak{sl}_2$ and the $[1^2]$ refers to the partition of $2 = 1+1$ that corresponds to the only non-trivial regular puncture coming from a codimension two defect of the A_1 theory[8]. The Λ dependent factors

[8]Going forward, the notation $V[\mathfrak{g}]_{g,[\ldots]}$ will be used to denote a correlator in the Toda

that occur in the above formula follow from an analysis of scaling properties of Liouville correlators [127, 57, 49]. The complete formula was proposed in [60, 206] along with some evidence for why this is true. It was then derived by Teschner using a recursion relation [191]. Now, introduce a quantity that will be called the stripped correlator,

$$\hat{V}[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])} = \frac{V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])}}{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)} \tag{4.20}$$

It is the quantity $\hat{V}[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])}$ that seems most appropriate to identify as the partition function of four hypermultiplets. One expects that this quantity should posses an anomalous scaling term just like the one calculated in the previous section. And it indeed does have such an anomaly term and it matches exactly with that for a theory of four hypermultiplets ($n_h = 4, n_v = 0$). This can be seen by noting the scaling behaviour of the Υ function (See Appendix B),

$$\Upsilon(\mu x; \mu\epsilon_1, \mu\epsilon_2) = \mu^{2\zeta_2^B(0,x;\epsilon_1,\epsilon_2)}\Upsilon(x;\epsilon_1,\epsilon_2). \tag{4.21}$$

There are a total of $\Upsilon(x)$ factors in the denominator of $\hat{V}[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])}$ whose arguments take the value $x = 1$ in the $m_i \to 0$ limit of $b = 1$ Liouville theory. From Appendix B, note that $2\zeta_2^B(0,1;1,1) = -1/6$. This implies (in the $m_i \to 0$ limit),

$$\hat{V}[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])} = \mu^{4/6}\hat{V}[\mathfrak{sl}_2]^{R_0=1}_{0,([1^2],[1^2],[1^2])}. \tag{4.22}$$

theory labeled by a principal embedding of $\mathfrak{sl}_2 \to \mathfrak{g}$ on a genus g surface with punctures which are labeled by some representation theoretic data contained in the [. . .].

The factor $\mu^{4/6}$ matches with μ^{4a} for this theory and is thus in keeping with expectations. The dependence on the parameter $\mu = R_0^{-1}$ is usually suppressed when the Liouville correlators are analyzed. It had been additionally brought out here for it serves the useful purpose of encoding the Euler anomaly of the associated 4d SCFT which in this case is a trivial theory of four free hypermultiplets. For an exception on this matter, see [59] where additional dimensionful parameters appear in the expression for the Liouville correlator $V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])}$. However, note that the exponent of the additional dimensionful parameter in [59] is independent of the operator insertions. This wont be the true in what follows. The exponent of μ will have an important (and very subtle) dependence on the number and type of operator insertions. It turns out that for the case of the three point function, there is a plausible argument where the path integral description can be used to obtain the dependence on μ. Consider,

$$V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])} := \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int d[\phi] e^{-S_{L,\text{disc}}} \prod_{i=1}^{3} e^{2\alpha_i \phi}. \tag{4.23}$$

Let us restrict ourselves to the case that corresponds to setting all the hypermultiplet masses m_i to zero. Note that a primary operator $e^{2\alpha_i\phi}$ modifies the boundary condition close to the insertion to $\phi = 2\Re(\alpha)\log(r_i/R_0)$. To keep the action finite, one needs to introduce additional terms that are local to the punctures,

$$S_{L,\text{disc}} = \frac{1}{72\pi}\int_D \sqrt{\hat{g}}d^2z\left(\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi \quad + \quad 2\pi\Lambda e^{2b\phi}\right) + \frac{Q}{12\pi}\int_{\partial D}\phi d\theta + \frac{1}{3}Q^2\log(R/R_0)$$
$$+ \quad \sum_{i=1}^{3}\left(- \frac{\Re(\alpha)}{6\pi}\int_{\partial c_i}\phi d\theta - \frac{2}{3}\Re(\alpha)^2\log(R/R_0)\right).$$

For a translationally invariant measure $d[\phi]$, the R_0 contributions arise directly from the integrand. The global contribution is from the boundary term in S_{cl} that is associated to the curvature insertion and is given by $(R_0)^{+Q^2/3}$. For the punctures, $\Re(\alpha_i) = Q/2$. So, each such operator insertion contributes $(R_0)^{-Q^2/6}$. Collecting these gives,

$$V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])} = \mu^{Q^2/6}V[\mathfrak{sl}_2]^{R_0=1}_{0,([1^2],[1^2],[1^2])}. \tag{4.24}$$

For the case of a round sphere, we have $Q = 2$ and this implies $Q^2/6 = 2/3$. This is identified with the quantity $4a(= n_h/6)$ for a theory of four free hypermultiplets while R_0 is identified with the radius of the four sphere that was used as background for defining the partition function of the theory. Here, a comment on the unconventional normalization in $S_{L,disc}$ is required. The normalization of ϕ was chosen such that the dependence of μ for the three point function agrees with the corresponding value for $4a$. Equivalently, one could have picked the this factor such that the n_h value for a single full puncture equals 4. But, once it has been fixed, there are no free parameters. There will be similar choice of normalization later when the local contributions to these scale factors from are considered from a WZW point of view.

The calculation above reproduces the scale factor in (4.22). When the scale factor is calculated from the Υ functions, the exact origin of the μ parameter is somewhat obscured by the regularization that is implicit in final form the DOZZ result. The path integral sheds *some* light on how the scale factor enters into the picture via regularization. But, this is still incomplete since no such argument seems to be available readily for higher point functions. From (4.21), it is also clear that the overall scale factor is sensitive to the analytical structure of the correlator. This relationship is most straightforward when a correlator that corresponds to a free 4d theory is considered. In this case, the scale factor is purely from the n_h contributions. The number of polar divisors in the correlation function is equal to n_h. In the example just considered, the number of polar divisors for the DOZZ three point function is 4 and this indeed matches with the n_h for a theory of four hypermultiplets.

A point worth emphasizing here is that the AGT primary map, namely the relation $\alpha_i = Q/2 + m_i$, is written after a dimensionful scale (the radius of the four sphere) is set to be unity. The goal of making the Euler anomaly explicit can alternatively be stated as that of making the dependence on this scale explicit in the correlators.

4.3.1 Higher point functions

Once the three point function is known, the higher point functions for Liouville can be obtained by the bootstrap procedure. This entails picking a factorization limit for the higher point function and writing the $n-$point function as an integral/sum over states in the $3g-3+n$ factorization channels with the integrand being built out of the $2g-2+n$ three point functions and appropriate conformal blocks. Confirming that the analytical structure of the resulting $n-$point functions is in keeping with the *a priori* expectations (say, from a path integral point of view) involves a delicate interplay between the DOZZ three point function, the conformal block and the representation theory of the Virasoro algebra [188] (See Appendix F for a short review). When there are enough punctures on both sides of the channel, the channel state is a primary with a momentum of the form $\alpha = Q/2 + i\mathbb{R}^+$ [161, 168]. The correlation functions built in the above fashion are also required to obey the generalized crossing relations. This imposes a highly nontrivial constraint on the three point function. For the case of Liouville, it has been checked that the DOZZ proposal does satisfy these constraints [162, 101]. Let us proceed now by looking at some examples of how the scale factor can be calculated for these higher point functions.

4.3.1.1 $V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])}$

This is the correlator corresponding to $\mathcal{N}=2$ SYM with gauge group $SU(2)$ and $N_f = 4$. The flavor symmetry for this theory is $SO(8)$. The theory has four mass deformation parameters which can each be assigned to a $SU(2)$ flavor subgroup of $SO(8)$. These mass parameters will be related

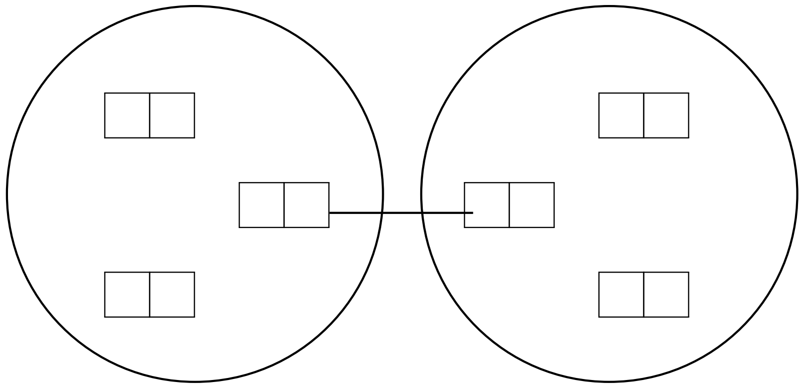

Figure 4.2: A_1 theory on a sphere with four punctures in a degenerating limit.

to the Liouville momenta in the following fashion

$$\alpha_i = \frac{Q}{2} + m_i.$$

The eigenvalues of the mass matrix are $m_1 + m_2$, $m_1 - m_2$, $m_3 + m_4$ and $m_3 - m_4$.

To write down the four point function in Liouville theory, α_i, α are initially taken to lie on the physical line. That is, $\alpha_i = Q/2 + is_i^+, \alpha = Q/2 + is^+$ for $s_i^+, s^+ \in \mathbb{R}^+$. The four point function can then be written as

$$Z_{S^4} = V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_{\alpha \in \frac{Q}{2}+is^+} d\alpha C(\alpha_1, \alpha_2, \alpha) C(Q-\alpha, \alpha_3, \alpha_4) \mathcal{F}_{12}^{34}(c, \Delta_\alpha, z_i) \mathcal{F}_{12}^{34}(c, \Delta_{Q-\alpha}, \bar{z}_i).$$

In writing this, the fact that when $\alpha \in \frac{Q}{2} + is$, $\bar{\alpha} = Q - \alpha$ has been used. Now, using the symmetry of the entire integrand under the Weyl reflection

$\alpha \to Q - \alpha$, one can unfold the integral to one over $\mathbb{R}$. This gives

$$V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])} \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{1}{2}\int_{\alpha \in \frac{Q}{2}+is} d\alpha C(\alpha_1, \alpha_2, \alpha) C(Q-\alpha, \alpha_3, \alpha_4) \mathcal{F}^{34}_{12}(c, \Delta_\alpha, z_i) \mathcal{F}^{34}_{12}(c, \Delta_{Q-\alpha}, \bar{z}_i),$$

where $s \in \mathbb{R}$. As with the three point function, let us defined the stripped four point function,

$$\hat{V}[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])} = \frac{V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])}}{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)\Upsilon(2\alpha_4)} \tag{4.25}$$

To calculate the overall R_0 dependence, the anomalous terms from the Υ factors should be collected. A simple variable change collects the extra factors from the integration over channel momenta and the conformal blocks. The contribution from the eight polar divisors in the integrand is also straightforward to calculate and is equivalent to the contribution from the denominator in (4.2.1). As for the term $\Upsilon(2\alpha)\Upsilon(2Q-2\alpha)$, this can be rewritten terms of the H function in order to make the Vandermonde factor explicit (as in [6]). Let us note here the steps involved,

$$\Upsilon(2\alpha)\Upsilon(2Q-2\alpha) = \Upsilon(Q+2ia)\Upsilon(Q-2ia) \tag{4.26}$$

$$= \frac{1}{\Gamma_2(Q+2ia)\Gamma_2(-2ia)} \frac{1}{\Gamma_2(Q-2ia)\Gamma_2(2ia)} \tag{4.27}$$

Recalling the following property (Appendix B) of the digamma function,

$$[\Gamma_2(x+\epsilon_1+\epsilon_2)\Gamma_2(x)]^{-1} = x[\Gamma_2(x+\epsilon_1)\Gamma_2(x+\epsilon_2)] \tag{4.28}$$

and applying it to case of $\epsilon_1 = b, \epsilon_2 = 1/b$,

$$\begin{aligned}\Upsilon(2\alpha)\Upsilon(2Q-2\alpha) &= (2ia)^2[\Gamma_2(b+2ia)\Gamma_2(b^{-1}+2ia)]^{-1}[\Gamma_2(b-2ia)\Gamma_2(b^{-1}-2ia)] \\ &= -4a^2 H(2ia)H(-2ia).\end{aligned} \tag{4.29}$$

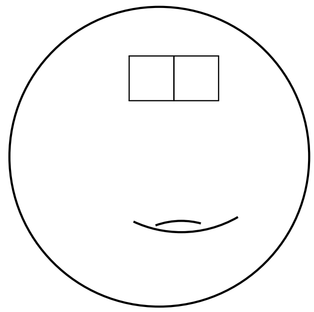

Figure 4.3: A_1 theory on a torus with one puncture

The above factor taken together with the single $\Upsilon(b)$ that remains in $\hat{V}[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])}$ provide the numerator in the expression for Z_{1-loop} (4.2.1) together with Vandermonde factor. The calculation of the scale factor is thus reduced the calculation that we already performed. So, we have

$$\hat{V}[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])} = \mu^{23/6}\hat{V}[\mathfrak{sl}_2]^{R_0=1}_{0,([1^2],[1^2],[1^2],[1^2])}. \tag{4.30}$$

The exponent of R_0 can be interpreted as $4a$ and this indeed matches (4.8) for $N = 2$.

4.3.1.2 $V[\mathfrak{sl}_2]_{1,([1^2])}$

For an arbitrary mass deformation, this theory corresponds to $\mathcal{N} = 2^*$ SYM with $SU(2)$ gauge group with a hypermultiplet in the adjoint and one free hypermultiplet. The corresponding Liouville correlator can be expressed in terms of the one point conformal block for the torus.

$$V[\mathfrak{sl}_2]_{1,([1^2])}(\alpha_1) = \int_{\alpha\in Q/2+s} d\alpha C(Q-\alpha,\alpha_1,\alpha)\mathcal{F}_{\alpha_1}(\Delta_\alpha,q)\mathcal{F}_{\alpha_1}(\Delta_{Q-\alpha},\bar{q}).$$

The stripped correlator in the $g = 1$ case is defined as

$$\hat{V}[\mathfrak{sl}_2]_{1,([1^2])}(\alpha_1) = \frac{V[\mathfrak{sl}_2]_{1,([1^2])}(\alpha_1)}{\Upsilon(2\alpha_1)}. \tag{4.31}$$

Calculating the R_0 dependence as in the case of the four point function,

$$\hat{V}[\mathfrak{sl}_2]_{1,([1^2])} = \mu^{19/6}\hat{V}[\mathfrak{sl}_2]^{R_0=1}_{1,([1^2])}. \tag{4.32}$$

Ignoring the contribution of a decoupled hypermultiplet(with $4a = 1/6$) gives the expected answer that $4a = 3$ for the $N = 2^*$ theory.

For higher point functions on arbitrary surfaces, one proceeds in a similar manner by defining the stripped correlator as

$$\hat{V}_{g,[\ldots]}(\alpha_1, \alpha_2 \ldots \alpha_n) = \frac{V_{g,[\ldots]}(\alpha_1, \alpha_2 \ldots \alpha_n)\Upsilon(b)^{g-1}}{\prod_i \Upsilon(2\alpha_i)}, \tag{4.33}$$

where $V_{g,[\ldots]}(\alpha_1, \alpha_2 \ldots \alpha_n)$ is the Liouville correlator built out of $(2g - 2 + n)$ DOZZ three point functions and $(3g - 3 + n)$ factorizing channels. Calculating the contributions to the scale factor directly from (4.33),

$$\begin{aligned} 4a &= (2g-2+n)\left(3\frac{5}{6} - \frac{1}{6} + 4\frac{1}{6}\right) + (3g-3+n) - \frac{5}{6}n - \frac{1}{6}(g-1) \\ &= \frac{53}{6}(g-1) + \frac{19n}{6}. \end{aligned} \tag{4.34}$$

From (4.13), $n_h = 8(g-1) + 4n$, $n_v = 9(g-1) + 3n$ and one sees immediately that $4a$ calculated above satisfies

$$4a = n_v + \frac{n_h - n_v}{6}. \tag{4.35}$$

4.3.2 Liouville theory from a gauged WZW perspective

Before proceeding to discuss the higher rank generalizations, it is useful to recast the scale factor calculations in an alternate language. It is well known that classical Liouville theory can be obtained via a Hamiltonian reduction starting from the $SL(2,\mathbb{R})$ WZW model. A quantum version of this reduction, which has been the subject of a rich variety of studies from various different points of view (for instance, see [73], [61, 83] and [164, 111, 163, 97]) is then expected to yield Liouville conformal field theory. This point of view is powerful since it permits an easy generalization to higher rank cases where a non rational CFT with W-symmetry is obtained for every inequivalent (upto $\mathfrak{g}$ conjugacy) $\sigma : \mathfrak{sl}_2 \to \mathfrak{sl}_N$. In the $\mathfrak{g} = \mathfrak{sl}_2$ case considered here, the only non-trivial embedding is the principal embedding and this corresponds to Liouville CFT. With this in mind, let us look at how the spectrum of primaries in Liouville can be related to a set of WZW primaries. In its Wakimoto realization, this model is realized in terms of a scalar field ϕ and bosonic ghosts β, γ with the following bosonization rules

$$J^{+} = -\beta(z)\gamma(z)^2 + \alpha\gamma(z)\partial\phi(z) + k\partial\gamma(z), \tag{4.36}$$

$$J^{3}(z) = \beta(z)\gamma(z) - \frac{\alpha}{2}\partial\phi(z), \tag{4.37}$$

$$J^{-}(z) = \beta(z). \tag{4.38}$$

with $\alpha^2 = 2k-4$. Now, consider the primary field $P(j)$ whose free field realization is $\gamma^{-j}\overline{\gamma}^{-j}e^{(j+1)\phi}$. This operator is identified with a Liouville primary of the form $e^{\alpha\phi}$ (upto a scale that will be fixed momentarily) where $\alpha = -jb$. Naively, the conformal dimensions of the primaries match. That is,

$$\Delta_\alpha = \Delta_j - j, \tag{4.39}$$

where $\Delta_\alpha = \alpha(Q-\alpha)$ and $\Delta_j = -\frac{j(j+1)}{k-2}$ with the identification $b^2 = k-2$. In early investigations of these gauged WZW models, it was shown that the two and three point functions of Liouville can be obtained exactly under the above identification of primaries along with (4.39) holding [61, 83].

One of the advantages of the WZW prescription is that the classical solutions are perfectly regular. In the WZW language, there is no singularity in the local solution near the insertion of the puncture and consequently, there are no regularizing terms in the classical action. So, where does the dependence of R_0 arise ? I argue that it arises from carefully considering the dimensionful factors that enter in the relationship between the Liouville and WZW primaries. First, in the gauged WZW model, one works with an 'improved' stress energy tensor

$$\hat{T}(z) = T(z)_{WZW} - \partial J^3(z), \tag{4.40}$$

so that the constraint $J^- = 1$ can be imposed without breaking conformal invariance. Under the improved stress energy tensor, the primary $P(j)$ has a shifted dimension $\hat{\Delta} = \Delta_j - j - j$. To keep the map between primaries intact along with relation $\Delta_\alpha = \Delta_j - j$, a scale factor that offsets the shift in dimension of $P(j)$ should be included

$$e^{\alpha\phi} \equiv (R/R_0)^{+j}\gamma^{-j}\overline{\gamma}^{-j}e^{(j+1)\phi}. \tag{4.41}$$

A further redefinition of ϕ is needed in order to match the normalization used in the previous section. It is chosen such that $j = -2(= -4(\rho,\rho))$ corresponds to the full puncture for a $b = 1$ theory with $n_h/6 = -2/3$. In this normalization,

$$e^{\alpha\tilde{\phi}} \equiv (R/R_0)^{+j/3}\gamma^{-j/3}\overline{\gamma}^{-j/3}e^{(j+1)\tilde{\phi}}. \tag{4.42}$$

4.4 Scale factors in Toda correlators I : Primaries and free theories

In the Toda case, the WZW approach is much more convenient to capture the local n_h contributions to the scale factor since a Toda action with appropriate boundary terms is not readily available for an arbitrary codimension two defect. However, the global n_h contribution will always be computed using the curvature insertion in the Toda action. This asymmetric treatment is purely one of convenience. A complete understanding of boundary actions in Toda theory might be a way to obtain a more uniform treatment [71].

The most general Toda theory of type A can be obtained by a gauging of the $SL(n,\mathbb{R})$ WZW model. Unlike the case of A_1, the higher rank cases offer more than one ways of gauging the $SL(n,\mathbb{R})_L \times SL(n,\mathbb{R})_R$ symmetry such that conformal invariance in preserved[22, 21]. An optimal way to index inequivalent Toda theories is by associating a $\mathfrak{sl}_2$ embedding $\sigma : \mathfrak{sl}_2 \to \mathfrak{sl}_N$ for every such gauging [72, 51]. Each of the theories obtained by a nontrivial embedding σ has a $\mathcal{W}-$ symmetry whose chiral algebra is called a $\mathcal{W}-$ algebra. This algebra is a non-linear extension of the Virasoro symmetry by currents $\{\mathbf{W}^i(z)\}$ of spin $i(>2)$. The unique spin 2 current in the chiral algebra is the stress energy tensor $\mathbf{T}(z) \equiv \mathbf{W}^2(z)$.

As with $Sl(2,\mathbb{R})$, consider the Wakimoto realization of the $SL(n,\mathbb{R})$ model with the required number of β, γ, ϕ fields. The following constraints are imposed [72]

$$J(x) = Ke + j(x), \tag{4.43}$$

$$\tilde{J}(x) = -Kf + \tilde{j}(x). \tag{4.44}$$

where e, f, h are the images of the standard $\mathfrak{sl}_2$ generators and $j(x) \in \mathfrak{g}_{\geq 0}$ and $\tilde{j} \in \mathfrak{g}_{\leq 0}$ [9].

When the grading is even, the system of constraints is first class. When the grading has odd pieces, then at first sight, the system is not first class. One can introduce auxiliary fields (as in [21]) or consider a grading by a different element M such that $[M,h]=0, [M,e]=2e, [M,f]=-2f$ [72]. In the latter case, it is possible to define a new set of constraints (now first class) equivalent to the original.

In this Chapter, only the theories obtained by the principal embedding will be considered. It has the following action on the disc (written in the same unconventional normalization that was used for the Liouville

[9] K is a potentially dimensionful constant.

case),

$$S_{T,\text{disc}} = \frac{1}{72\pi}\int_D \sqrt{\hat{g}} d^2z \left(\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi \;\; + \;\; \sum_{i=1}^{n-1} 2\pi\Lambda e^{2b(e_i,\phi)}\right) + \frac{1}{6\pi}\int_{\partial D}(Q,\phi)d\theta + \frac{2}{3}(Q,Q)\log(R/R_0). \tag{4.45}$$

The conformal transformations that leave the above action invariant (classically) are

$$\begin{aligned} z' &= w(z), \\ \phi' &= \phi - Q\rho\log\left(\frac{\partial w}{\partial z}\right)^2, \end{aligned}$$

and the field ϕ obeys the following boundary condition at the boundary of the disc

$$\phi = -Q\rho\log(R/R_0) + \mathcal{O}(1). \tag{4.46}$$

The chiral algebra for this theory is generated by the currents $\{\mathbf{W}^i(z)\}$ of spin $i = 2\ldots n-1$. The spins of the currents are identified with the exponents of the group $SL(n,\mathbb{R})$. The global n_h contribution arises from the boundary term due to the curvature insertion (specializing to $Q = 2$ and generalizing the relevant boundary term for a surface of arbitrary genus), we get the n_h dependent contribution to $4a^{global}$,

$$(4a)^{global}_{n_h} = \frac{8}{3}(g-1)(\rho,\rho). \tag{4.47}$$

Now, using $(4a)^{global}_{n_h} = n_h/6$), it follows that

$$n_h^{global} = 16(g-1)(\rho,\rho). \tag{4.48}$$

This matches with (4.14) once we use the Freudenthal-de Vries formula for (ρ, ρ). We will now proceed to analyze an interesting family of primary operators also indexed by inequivalent embeddings of $\rho : \mathfrak{sl}_2 \to \mathfrak{g}$. In type A, the identification of these primaries has been done in [120]. Following [120], these states are referred to as semi-degenerate primaries. They will be related to certain primaries in the WZW model. To go beyond just calculating the n_h contributions, it will also be useful to associate an irreducible representation of the Weyl group to each of those operators.

4.4.1 Toda primaries from a gauged WZW perspective

The set of semi-degenerate primaries relevant for the AGT correspondence was constructed in [120] by applying the screening operators $\mathcal{S}_i^{(\pm)}$ to Toda primary whose momentum satisfies certain conditions. The screening operators have the following form

$$\mathcal{S}_i^{(\pm)} = \int \frac{dz}{2\pi i} e^{(\beta e_i \cdot \phi)}, \tag{4.49}$$

where e_i are the simple co-roots of $\mathfrak{sl}_N$. Requiring that these operators have $\Delta = 1$ forces β to be either $\beta_+ = -b$ or $\beta_- = -1/b$. The screening operators have the special property that they commute with the generators of the $\mathcal{W}$ algebra. That is $[W_l^k, S_i^{\pm}] = 0$. Now, the state $(\mathcal{S}^{\pm})^{n^{\pm}} |\alpha - n_{\pm}\beta_{\pm} e_i >$ either vanishes identically or has a null state at level $n_+ n_-$. For the latter to happen, the α have to satisfy

$$(\alpha, e_i) = (1 - n_j^+)\alpha_+ + (1 - n_j^-)\alpha_-, \tag{4.50}$$

for some j. If the null vectors are taken to appear at level one, the above condition is simplified to

$$(\alpha, e_i) = 0, \tag{4.51}$$

for e_i being some subset of simple co-roots. Having recalled the construction in [120], we proceed to obtain these primaries in the gauged WZW setting. The proposed map is the following

$$e^{(\alpha,\phi)} \equiv (R/R_0)^{8(\rho,\rho)-4(\rho,h)+\frac{1}{2}\dim\mathfrak{g}_1^h} \prod_i \gamma_i \prod_k \bar{\gamma}_k \times e^{(j+2\rho,\phi)}, \tag{4.52}$$

for some specific choices of α (and consequently of j). The different semi-degenerate states are obtained for the choices of α outlined in [120]. For the case $b = 1$, the set in [120] can be obtained by setting $\alpha = 2\rho - \lambda$ where λ is twice the Weyl vector of a subalgebra of $\mathfrak{sl}_N$. The spin j in the WZW primary is obtained using $j = -\alpha$. The justification for the scale factor in the above map is similar in spirit to the one encountered in the case of Liouville (see Section 4.3.2) but the details are complicated by the wider variety of semi-degenerate state that are available in the higher rank Toda theories. This requires the introduction of some representation theoretic notions.

First, note that considerations of scaling in Toda theory involve more possibilities in that one has to first pick a weight vector and consider scaling in the direction of that weight vector. The maximal puncture is the one that is not invariant under a scaling along any weight vector. In other words, for a maximal puncture, there is no $\lambda \in \Lambda$ such that $(\alpha, \lambda) = 0$. For other smaller punctures, there always exists such a λ and the 'smallness' of the puncture is related to how 'big' the λ is. The scare quotes are included to highlight that this notion of small/big is not rigorous since two sets (the set of regular punctures and the set of weight vectors) admit only a poset structure and it may turn out that certain pairs do not have an order relationship. The h in the above formula is obtained in the following way. Take the subalgebra $\mathfrak{l}$ of $\mathfrak{sl}_N$ for which λ is twice the Weyl vector ($2\rho_{\mathfrak{l}}$). Let e_i be a set of simple co-roots for this subalgebra. Impose the null vector conditions (4.51) for this set. Now, consider orbit of λ under $W[\mathfrak{sl}_N]$. There is a unique element $h = w\lambda$ for $w \in W[\mathfrak{sl}_N]$ and $h \in \Lambda^+$, the set of dominant weights of $\mathfrak{g}$. This dominant weight is the Dynkin element (See Appendix for explanation of this terminology) of a nilpotent orbit in $\mathfrak{sl}_N$. Such orbits are classified by partitions of N. One can translate between the different quantities in the following way. Given a partition $[n_1 n_2 \dots n_k]$ such that $\sum n_i = N$, write λ as $(-n_1+1, -n_1+3, \dots n_1-1, -n_2+1, \dots n_2-1 \dots - n_k+1, \dots n_k-1)$. Reordering the elements of λ such that they are non-decreasing gives us h, the Dynkin element.

The element h occurs as the semi simple element in the $\mathfrak{sl}_2$ triple $\{e, f, h\}$ associated to the corresponding embedding. The lie algebra $\mathfrak{g}$ has a natural grading defined by the h eigenvalue

$$\mathfrak{g} = \oplus_j \mathfrak{g}_i = \oplus_{j<0} \mathfrak{g}_i + \mathfrak{g}_0 + \oplus_{i>0} \mathfrak{g}_i. \tag{4.53}$$

We can now turn to the interpretation of the scale factor in (4.52). Consider the special case : j such that h is trivial ($\lambda = 0$). This corresponds to a 'maximal' puncture. As with the case of Liouville, the necessity of using a modified stress tensor $\hat{T}^{\rho}(z)$ (ρ denotes the fact that this is the stress tensor for the principal Toda theory) introduces extra contributions to the scaling dimension. To avoid spoiling the relationship $\Delta_\alpha = \Delta_j - (j, 2\rho)$, there is a need to introduce a scale factor of the form $(R/R_0)^{4(\rho,\rho)}$. When h is non trivial, there are some scalings for which the primary is invariant (as opposed to transforming by a scale factor). Local to the primary insertion, associate a $\mathfrak{sl}_2$ embedding with Dynkin element h and consider the spectrum of γ fields associated to this grading. Their dimensions are given by how they behave under a scaling defined by $\hat{T}^{h}(z)$. If the embedding is even ($dim\mathfrak{g}_{\pm i} = 0$ for i odd), one would like to remove the contribution to the scaling dimension from the corresponding set of γ fields. When the embedding is not even, this procedure will work if a grading under a different element M is considered. This M is such that it provides an even grading while obeying $[e, M] = 2e, [f, M] = -2f, [h, M] = 0$ [72]. Under the new grading, the dimension of $\mathfrak{g}_{\geq 2}$ increases by $\frac{1}{2}\text{dim}\mathfrak{g}_1$. So, a full accounting of the dimensional factors produces the exponent of R/R_0 in (4.52). As with the Liouville case, ϕ needs to be normalized such that $h = 0$ produces the correct n_h contribution from a full puncture. In this normalization,

$$e^{(\alpha,\phi)} \equiv (R/R_0)^{\frac{8(\rho,\rho)-4(\rho,h)+\frac{1}{2}\text{dim}\mathfrak{g}_1}{6}} \prod_i \gamma_i \prod_k \bar{\gamma}_k \times e^{(j+2\rho,\phi)}. \tag{4.54}$$

The exponent of R_0 is recognized as the local contribution to $n_h/6$

from CDT [39][10]. One would like to believe that the other local properties ascribed to this class of codimension two defects of the six dimensional theories should also have a description in terms of properties of the corresponding semi-degenerate operators in Toda theory. In order for this dictionary to be built further, it is important to associate to every semi-degenerate primary a unique irrep of the Weyl group.

4.4.2 Toda primaries and representations of Weyl groups

In this section, a representation of the Weyl group $W[\mathfrak{sl}_N] = S_N$ will be associated to every semi-degenerate primary in an A_n Toda theory. Recall from the previous section that the momentum of a general semi-degenerate primary obeys $(\alpha, e_i) = 0$ for $i = 1 \ldots k$. The e_i are a subset of the set of simple co-roots Π. In the current case, they form a subsystem[11]. Denote this set by S_N. Denote by S_N^+, the set of positive root of this subsystem. Let Λ^+ be the set of positive roots for $\mathfrak{g}$. Note here that when h is zero, S_N^+ is empty and when h is the Dynkin element of the principal nilpotent orbit, S^+ is Λ^+

.

Using this data, one can obtain a unique irreducible representation of the Weyl group by a construction due to MacDonald [141] [12]. The co-root system lives naturally in $\mathfrak{h}$. Each co-root can be thought of as a linear

[10] A clarification regarding the notation is in order. What is called $\dim\mathfrak{g}_1$ here is the same as $\dim\mathfrak{g}_{1/2}$ of [39]. The difference in notation arises from the choice of normalization of h.

[11] More accurately, a conjugacy class of subsystems.

[12] See the text [35] for an elaborate discussion of this construction and its generalization due to Lusztig and Lusztig-Spaltenstein.

functional on $\mathfrak{h}^*$. Now, construct the following rational polynomial on $\mathfrak{h}^*$,

$$\pi = \prod_{e_\alpha \in S_N^+} e_\alpha. \tag{4.55}$$

Using this, construct a subalgebra of the symmetric algebra($\mathfrak{S}$) on $\mathfrak{h}^*$ by considering all polynomials $\mathcal{P} = w\pi$. This subalgebra is a $W-$ module and in fact, furnishes an irreducible representation of the Weyl group.

It turns out that *all* irreps for Weyl groups of types A, B, C can be obtained by considering the various inequivalent subsystems.

The contribution to the total Coulomb branch dimension of the four dimensional theory from a primary that is labeled by a Nahm orbit $\mathcal{O}_N$ is actually related to the dimension of a dual orbit [39]. This formula can be rewritten in terms of the cardinality of the set S_n^+ in the following way

$$d = |\Delta^+| - |\Delta^+_{S_N}| = \frac{1}{2}\text{dim}\mathcal{O}_{P^t}. \tag{4.56}$$

where P is the partition type associated to the Nahm orbit $\mathcal{O}_N$ and P^t is the transpose partition. Let ϕ_i be the generators of the full symmetric algebra. Let us additionally note here the formula

$$n_v = \sum_i [2\text{deg}(\phi_i) - 1] - \sum_{h>0} [2h-1] = 2(2\rho, 2\rho - h) + \frac{1}{2}(\text{rank}\mathfrak{g} - \text{dim}\mathfrak{g}_0^h). \tag{4.57}$$

This quantity is called n_v since it will turn out to be the contribution of the codimension two defect to the effective number of vector multiplets. To give a flavor for the values n_h, n_v in the various cases, the properties of Toda semi-degenerate states for the A_2, A_3 theories in are collected in Tables 4.4.2

h	Nahm Orbit	Hitchin Orbit	Toda momentum(α)	n_h	n_v
$(0,0,0)$	$[1^3]$	$[3]$	$2(\omega_1+\omega_2)$	16	13
$(1,0,-1)$	$[2,1]$	$[2,1]$	$3\omega_1$	9	8
$(2,0,-2)$	$[3]$	$[1^3]$	0	0	0

Table 4.1: Semi-degenerate states in A_2 Toda theory.

h	Nahm Orbit	Hitchin Orbit	Toda momentum(α)	n_h	n_v
$(0,0,0,0)$	$[1^4]$	$[4]$	$2(\omega_1+\omega_2+\omega_3)$	40	34
$(1,0,0,-1)$	$[2,1^2]$	$[3,1]$	$3\omega_2+2\omega_1$	30	27
$(1,1,-1,-1)$	$[2,2]$	$[2,2]$	$4\omega_2$	24	22
$(2,0,0,-1)$	$[3,1]$	$[2,1^2]$	$4\omega_1$	16	15
$(3,1,-1,-3)$	$[4]$	$[1^4]$	0	0	0

Table 4.2: Semi-degenerate states in A_3 Toda theory.

and 4.4.2. In the tables, the fundamental weights are denoted by ω_i and the nomenclature of a 'Nahm Orbit' and a 'Hitchin Orbit' is in continuation of Chapter 3.

4.4.3 Toda, Nahm and Hitchin descriptions

Recall that in the CDT description [39] of this class of regular codimension two defects, a pair of nilpotent orbits $(\mathcal{O}_N, \mathcal{O}_H)$ play a central role[13] in the description of a *single* defect. In denoting $\mathcal{O}_N$ as the 'Nahm data' and $\mathcal{O}_H$ as the 'Hitchin data', I have continued to use the terminology of the previous Chapter. In what follows, only the case of type A will be considered, with the understanding the it can be extended to other cases when the asso-

[13] In cases outside of type A, there is also a discrete group.

ciated Nahm data is a principal Levi orbit. The map from the Nahm data to the corresponding Toda primary was called the AGT primary in Chapter 3,

$$\mathcal{P} : \mathcal{O}_N \to e^{2(\alpha,\phi)} \tag{4.58}$$

In the setup here, the h from the previous sections is associated to the 'Nahm Data'. The relationship of the Nahm datum to the Hitchin datum is explained in Chapter 3 using an invariant constructed using the Springer correspondence. A composition of the AGT primary map together with map between the Nahm and Hitchin data provides a relationship between the Toda primary and the associated Hitchin singularity.

Recall from the previous Chapter that the quantities $n_h - n_v$ and $a(r)$ also have a direct interpretation in Springer theory,

$$n_h - n_v = \dim_{\mathbb{C}}(\mathcal{B}_N), \tag{4.59}$$

$$a(r) = \dim_{\mathbb{C}}(\mathcal{B}_H), \tag{4.60}$$

where $\mathcal{B}_N$ and $\mathcal{B}_H$ are Springer fibers associated to the Nahm orbit (denoted by a $\mathcal{O}_N$) in $\mathfrak{g}$ and Hitchin orbit (denoted by $\mathcal{O}_H$) in $\mathfrak{g}^\vee$ respectively.

4.4.4 Examples of free theories : A_2 tinkertoys

The overall scale factor calculation from a Toda perspective is much simplified when the corresponding 4d theory is just a free theory of hypermultiplets. These are the theories for which the total n_v is zero. Recall that this quantity is defined as

$$n_v = \sum_i n_v^i + n_v^{\text{global}}, \tag{4.61}$$

where n_v^{global} is defined as

$$n_v^{global} = (1-g)(\frac{4}{3}\hat{h}\text{dim}(G) + \text{rank}(G)), \tag{4.62}$$

and n_v^i is given by 4.57. In the tinkertoy terminology, these are called free fixtures [36]. Let us consider one of the free fixtures that occur in the A_2 theory and understand how the n_h contribution to the scale factor is encoded in the corresponding Toda correlator. Specializing the Toda action on a disc to this case,

$$\begin{aligned} S_{T,\text{disc}} = \frac{1}{72\pi}\int_D \sqrt{\hat{g}}d^2z\bigg(\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi \quad &+ \quad \sum_{i=1}^{i=2} 2\pi\Lambda e^{2b(e_i,\phi)}\bigg) \\ &+ \quad \frac{1}{6\pi}\int_{\partial D}(Q,\phi)d\theta + \frac{2}{3}(Q,Q)\log(R/R_0). \end{aligned} \tag{4.63}$$

There are two regular punctures to consider when dealing with the A_2 family of theories of class S. The root space is two dimensional and is spanned by the simple roots $\vec{e_1}, \vec{e_2}$. The roots are normalized so that the the entries in scalar product matrix $K_{i,j} = (\vec{e_i}, \vec{e_j})$ are given by $K_{ii} = 2, K_{12} = K_{21} = -1$. The set of positive roots is $\vec{e} > 0 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ where $\vec{e_3} = \vec{e_1} + \vec{e_2}$. The fundamental weights are $\vec{\omega}_1, \vec{\omega}_2$ and they obey $(\vec{\vec{\omega}}_i, \vec{e}_j) = \delta_{ij}$. As usual, $\vec{\rho}$ is half the sum of positive roots and h_i (the weights of the fundamental representation) are given by

$$h_1 = \vec{\omega}_1, \tag{4.64}$$

$$h_2 = h_1 - e_1, \tag{4.65}$$

$$h_3 = h_2 - e_2. \tag{4.66}$$

The maximal puncture corresponds to a Toda primary $\mathcal{O}^{max}_{\vec{p}} = \exp(\vec{p}.\vec{\phi})$ where $\vec{p}$ is valued in the dual of the lie algebra. Writing $\vec{p} = \alpha_1\vec{\omega_1} + \alpha_2\vec{\omega_2}$, it is seen that a general Toda primary has two complex numbers as parameters. In the A_2 Toda case, there is yet another puncture which corresponds to $\mathcal{O}^{min}_{\vec{p}} = \exp(\vec{p}.\vec{\phi})$ where $\vec{p}$ in constrained to $\vec{p} = \chi\vec{\omega}_2$ (or equivalently $\chi\omega_1$).

4.4.4.1 $V[\mathfrak{sl}_3]_{0,([2,1],[1^3],[1^3])}$

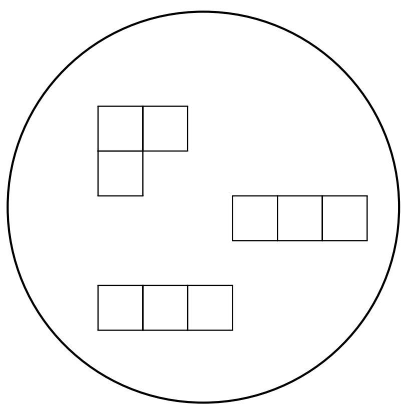

Figure 4.4: A_2 theory on a sphere with one minimal and two maximal punctures

The three point function with one argument taking a semi-degenerate value was obtained in [69]. It is given by

$$V[\mathfrak{sl}_3]_{0,([2,1],[1^3],[1^3])} = C(\alpha_1,\alpha_2,\alpha_3)|z_{12}|^{-2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{-2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{-2(\Delta_2+\Delta_3-\Delta_1)},$$

where

$$\begin{aligned} C(\chi\vec{\omega_2},\vec{p_1},\vec{p_2}) &= \left[\pi\Lambda\gamma(b^2)b^{2-2b^2}\right]^{(Q-\sum_i\alpha_i)/b} \times \\ & \frac{\Upsilon(b)^{n-1}\Upsilon(\chi)\prod_{\vec{e}>0}\Upsilon((\vec{Q}-\vec{p_1}).\vec{e})\Upsilon((\vec{Q}-\vec{p_2}).\vec{e})}{\prod_{i=1,j=1}^{i=3,j=3}\Upsilon\left(\frac{\rho}{2}+(\vec{p_1}-\vec{Q}).\vec{h_i}+(\vec{p_2}-\vec{Q}).\vec{h_j}\right)}. \end{aligned}$$

As was the case with the three punctured sphere in the Liouville case, the poles comes from the Υ functions in the denominator and these correspond to the screening conditions. For the A_2 case, there are two primitive screening conditions

$$(\rho\vec{\omega}_2+\vec{p_2}+\vec{p_3}).\vec{\omega}_1 = \Omega_{m,n}, \tag{4.67}$$

$$(\rho\vec{\omega}_2+\vec{p_2}+\vec{p_3}).\vec{\omega}_2 = \Omega_{m.n}, \tag{4.68}$$

and the rest are obtained by applying the two Weyl relations and identifying screening conditions that differ only by an overall Weyl reflection. The two reflections act by

$$\sigma_1 : \vec{p} \to ((2\vec{Q}-\vec{p}).\vec{e_1})\vec{e_1}, \tag{4.69}$$

$$\sigma_2 : \vec{p} \to ((2\vec{Q}-\vec{p}).\vec{e_2})\vec{e_2}. \tag{4.70}$$

where $\vec{Q} = Q\vec{\rho}$ and $Q = b + b^{-1}$ as before. The number of distinct screening conditions agrees with the assignment $n_h = 9$ for this fixture. As with

Liouville correlators, we define a stripped version,

$$\hat{V}[\mathfrak{sl}_3]_{0,([2,1],[1^3],[1^3])} = \frac{V[\mathfrak{sl}_3]_{0,([2,1],[1^3],[1^3])}}{\Upsilon(b)^{n-1}\Upsilon(\chi)\prod_{\vec{e}>0}\Upsilon((\vec{Q}-\vec{p_1}).\vec{e})\Upsilon((\vec{Q}-\vec{p_2}).\vec{e})}. \quad (4.71)$$

The scale factor for the stripped correlator comes from combining the anomalous scaling of the nine Υ functions that enforce the screening conditions. This gives,

$$\hat{V}[\mathfrak{sl}_3]_{0,([2,1],[1^3],[1^3])} = \mu^{9/6}\hat{V}[\mathfrak{sl}_3]^{R_0=1}_{0,([2,1],[1^3],[1^3])} \quad (4.72)$$

The argument can also be inverted in the sense that the knowledge of the scale factor for the stripped correlator corresponding to a free theory can be used to predict the analytical structure (=number of polar divisors) of the corresponding Toda three point function. Two such families are discussed below as examples. It is worth emphasizing that this is by no means an exhaustive list.

4.4.5 Families of free fixtures and corresponding Toda correlators

In the literature on Toda theories, the only correlation functions for which the analytical structure is explicitly known is the Fateev-Litvinov family [69]. These correspond to the family of free fixtures that will be called f_N. They correspond to N^2 free hypermultiplets transforming in the $(N, \bar{N})$ representation of the flavor symmetry group. This data is reflected in the fact that the FL family of Toda correlators have N^2 polar divisors with the exact same representation structure. That this should be the case could have been inferred from knowing the scale factor assigned to this correlator and deducing the value of n_h from that. Recall that for a free theory, $n_h = 24a$. The conjecture is that n_h is the number of polar divisors for the corresponding Toda correlator. For the Toda correlators corresponding to other families of free fixtures, corresponding results do not seem to be available in the literature. But, knowing the corresponding scale factor values along with the representation data [36] , the analytical form of these correlators can be conjectured. This can be done for any family of free fixtures using the following formula

$$n_h = \sum_i n_h^i - 16(\rho, \rho), \tag{4.73}$$

where n_h^i is the contribution from each primary insertion and can be deduced from the scale factor in (4.54). The last term is the global contribution from the sphere with ρ denoting the Weyl vector. Let us now look at a couple of examples to understand what is meant by families of free theories.

4.4.5.1 f_n

This is the Fateev-Litvinov family corresponding to N^2 polar divisors. This does correspond to the n_h value associated to this free fixture. In the uniform notation used for Toda correlators, this would correspond to $V[\mathfrak{sl}_N]_{0,([2,1^{N-1}],[2,1^{N-1}],[N])}$.

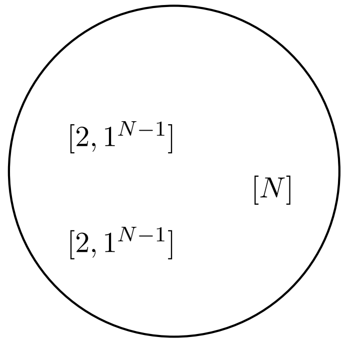

Figure 4.5: The f_N family of free fixtures corresponding to the Fateev-Litvinov family of Toda three point functions.

4.4.5.2 g_n

This is a new family $V[\mathfrak{sl}_N]_{0,([2^2,1^{N-2}],[3,2,1^{N-2}],[N])}$ of three point functions for which the analytical structure can be conjectured based on the Tinkertoy constructions. This family has $n_h = \frac{1}{6}N^3 - \frac{3}{2}N^2 + \frac{28}{3}N - 10$ and this number should equal the number of polar divisors (built out of Υ functions as in the case of f_N). From a purely Toda perspective, requiring that the poles arise only from the screening conditions (and its Weyl reflections) for this correlator should lead to the same result.

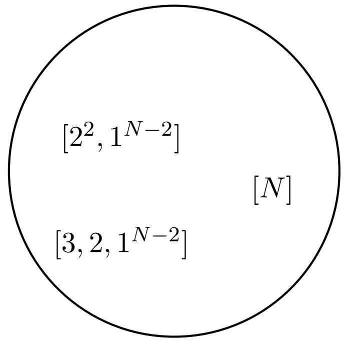

Figure 4.6: The g_N family of free fixtures corresponding to a family of Toda three point functions.

4.5 Scale factors in Toda correlators II : Interacting theories

4.5.1 Factorization in Toda theories

Apart from observing that $Z_{\mathbb{S}^4}$ matches with the Liouville correlators, AGT also noted that the identities satisfied by CFT correlators with single $T(z)$ insertions can be understood as a deformed version of the Seiberg-Witten curve. For example, $\mathbf{T}(z)$ insertions in Liouville correlators on the sphere obey the following identity,

$$\langle \mathbf{T}(z) \prod_i \mathcal{O}_i(z_i) \rangle = \sum_i \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{\mathcal{L}_{-1}}{z - z_i} \right) \langle \prod_i \mathcal{O}_i(z_i) \rangle.$$

These are what are called the conformal Ward identities. An immediate consequence of this is that correlation functions of descendants (defined to be states obtained by acting on $\mathcal{O}_i$ by modes of $T(z)$ or $\bar{T}(\bar{z})$) are fully determined in terms of the correlation functions of the primaries.

Let us now define the following quadratic differential,

$$\phi_2(z)dz^2 = -\frac{\langle \mathbf{T}(z) \prod_i \mathcal{O}_i(z_i) \rangle}{\langle \prod_i \mathcal{O}_i(z_i) \rangle}.$$

In a suitable limit, the conjecture [6] is that

$$\phi_2(z)dz^2 \to \phi_2^{SW}.$$

In the general Toda case, the full chiral algebra has more such identities that arise from insertions of the higher spin tensors $\mathbf{W}^n(z)$, $n > 2$. However, the $\mathcal{W}$-Ward identities fail to determine the correlation functions with descendants completely in terms of the correlators of primaries. One can define a number that quantifies the nature of this failure. This number turns out to be related to the total Coulomb branch dimension. As an example, consider the three point in A_2 Toda theory together with all its descendants.

$$D(V_{0,\{0,3\}}) = \langle \prod_{i=1}^{3} D_i \mathcal{O}_{\vec{p}(z_i)} \rangle,$$

where D_i is a product of the modes of the operators $T(z)$ and $W_3(z)$. The primaries obey

$$\begin{aligned} \mathbf{T}(z)\mathcal{O}(w) &= \frac{\Delta \mathcal{O}(w)}{(z-w)^2} + \frac{\partial \mathcal{O}(w)}{(z-w)} + \text{non-singular} \\ \mathbf{W}^3(z)\mathcal{O}(w) &= \frac{\Delta^{(3)}\mathcal{O}(w)}{(z-w)^3} + \frac{W^{(3)}_{-1}\mathcal{O}(w)}{(z-w)^2} + \frac{W^{(3)}_{-2}\mathcal{O}(w)}{(z-w)} + \text{non-singular}. \end{aligned}$$

Observe that $D(V_{0,\{0,3\}})$ obeys a set of local ward identities. These can be obtained by inserting $\int_\infty f_k W_k(z) = 0$ into the correlator where f_s is a meromorphic function with poles at $z = z_i$. Using the local ward identities, all correlators in the family can be written in terms of those of the form $D_0(V_{0,\{0,3\}})$ where $D_0 = \{L_{-1}, W_{-1}, W_{-2}\}$. The total number of linearly independent correlators in the set $D_0(V_{0,\{0,3\}})$ is nine (three D_0s for each primary). Imposing the global ward identities further constrains this set of correlators. The total number of global ward identities is 8 in the W_3 case. This shows that W-symmetry fails to determine the correlators of descendants completely in terms of that of the primaries. A representative of the set of correlation functions than cannot be linearly related to $V_{0,\{,0,3\}}$ is $\langle W_{-1}^k \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle$. Let us assign Coulomb branch dimension as $d = 9 - 8 = 1$ to this family. It is easy to see that when one of the primaries is semi-degenerate, the total Coulomb branch dimension is zero. This is because the null vector takes the following form

$$(L_{-1} - \frac{3}{2} W_{-1}) |\mathcal{O}_1\rangle = 0.$$

This can be used to turn the W_{-1} to a L_{-1}. So, the family $D_0(V_{0,\{1,2\}})$ actually has no Coulomb branch (Coulomb branch dimension is zero). Using the spectrum of semi-degenerate operators in Toda theory and null vector conditions that they obey, this dimension can be calculated for any such family. This matches the corresponding 4d field theory's Coulomb branch dimension. For a similar count of equations, see [71] and [130]. One can also define a finer quantity, namely the graded Coulomb branch d_k. This is related to the quantity called n_v by the following formula

$$n_v = \sum_k (2k - 1) d_k. \tag{4.74}$$

Recall here the definition of n_v^{global},

$$n_v^{global} = (1-g)(\frac{4}{3}\hat{h}\text{dim}(G) + \text{rank}(G)), \tag{4.75}$$

where $\hat{h}$ is the dual Coxeter number and $G = SU(N)$ for all cases considered here. Some practice with the appearance of smaller gauge groups in the various limits of the corresponding 4d theories leads us to propose the following criteria for a full factorization in Toda theory. This corresponds to the appearance of an $SU(N)$ gauge group in the four dimensional theory. Take the degeneration limit where punctures α_i appear one side of the channel and punctures β_j appear on the other side. Construct the following quantities,

$$X_\alpha \equiv \sum_i n_v^{\alpha_i} + n_v^{max} + n_v^{global,g=0}, \tag{4.76}$$

$$X_\beta \equiv \sum_i n_v^{\beta_i} + n_v^{max} + n_v^{global,g=0}. \tag{4.77}$$

If and only if $X_\alpha, X_\beta \geq 0$, there is full factorization for the Toda[14] correlator. Exactly which subgroup appears as the gauge group in a channel where one of the quantities X_α, X_β become negative requires more detailed analysis involving the exact Toda correlators. This seems possible to carry out only in a limited number of cases (see example below). On the four dimensional side, this data has been determined in [36] using constraints that come from requiring Coulomb branch diagnostics like the graded d_k to match in all

[14]For the case of Liouville, this reduces to the familiar condition for a macroscopic state to occur in the factorization channel[161, 168].

factorization limits. A physical interpretation of this phenomenon using the properties of the Higgs branch has been given in [89].

4.5.1.1 A conjecture

With the experience of examples worked out so far and based on the general physical expectation that the Euler anomaly should be encoded as a scale factor in the sphere partition function of any conformal class $\mathcal{S}$ theory, one can formulate the following conjecture.

Conjecture 1. *Let* $\hat{V}[\mathfrak{g}]_{g,(\{\mathcal{O}_N^i\})}$ *be the stripped Toda correlator corresponding to the sphere partition function of class* $\mathcal{S}$ *SCFT (with mass deformation parameters* m_i*) obtained by taking theory* $\mathfrak{X}[\mathfrak{g}]$ *on Riemann surface of genus* g *with* n *punctures along with* n *codimension two defects (with Nahm labels* $\{\mathcal{O}_N^i\}, i = 1 \ldots n$*) placed at the punctures. Let the Euler anomaly of the SCFT be* a *and the inverse radius of the four sphere on which the SCFT is formulated be* μ*. Then,* $\hat{V}[\mathfrak{g}]_{g,(\{\mathcal{O}_N^i\})}$=$\mu^{4a}(\hat{V}[\mathfrak{g}]_{g,(\{\mathcal{O}_N^i\})})_{R_0=1}$ *in the* $m_i \to 0$ *limit, irrespective of the factorization limit in which the scale factor is calculated.*

The stripped correlator $\hat{V}$ in the general case is defined to be

$$\hat{V}[\mathfrak{g}]_{g,(\{\mathcal{O}_N^i\})} = \frac{V[\mathfrak{g}]_{g,(\{\mathcal{O}_N^i\})}\Upsilon(b)^{rank(\mathfrak{g})(g-1)}}{\prod_i D_i^0}, \tag{4.78}$$

where $\prod_i D_i^0$ is the collection of all factors in the correlator $V[\mathfrak{g}]_{g,(\{\mathcal{O}_N^i\})}$ that become identically zero in the $m_i \to 0$ limit. In certain familiar cases, the factors D_i^0 have an expression in terms of Υ functions. In the more general cases, the inverse of the stripped correlator may be best viewed as an iterated residue [15],

$$\hat{V}[\mathfrak{g}]^{-1}_{g,(\{\mathcal{O}^i_N\})} = \frac{\text{IRes}\left(V[\mathfrak{g}]^{-1}_{g,(\{\mathcal{O}^i_N\})}\right)_{m_i\to 0}}{\Upsilon(b)^{rank(\mathfrak{g})(g-1)}}. \tag{4.79}$$

Following the intuition from the path integral argument for the three point function in the Liouville case, one expects that the parameter μ can be understood to be the dimensionful parameter that enters in the definition of the regularized correlator. When the correlator is such that every factorization limit involves a channel with $X_\alpha, X_\beta \geq 0$, it is immediate that the scale factor is independent of the limit in which it is evaluated. When this is not the case, the above statement is a non-trivial constraint on the nature of the state appearing in the factorization channel (For such an example, see Section 4.5.3.1 below). The above conjecture is stated for arbitrary $\mathfrak{g}$ since it is expected to hold in all the cases. This Chapter provides a list of concrete examples in the case $\mathfrak{g} = A_n$. The setup outlined in Chapter 3 and summarized in Chapter 5 allows an extension of this conjecture to the cases outside of type A when all the defects used in the class $\mathcal{S}$ construction are of principal Levi type.

4.5.2 Examples : Theories with a known Lagrangian

4.5.2.1 $V[\mathfrak{sl}_3]_{0,([2,1],[2,1],[1^3],[1^3])}$ in its symmetric limit

Since the most general three point function is not known in closed form, this four point function is written in the factoring limit that allows

[15] $IRes(\ldots)_{m_i\to 0} = Res(Res(\ldots)_{m_1\to 0})_{m_2\to 0}$ and so on.

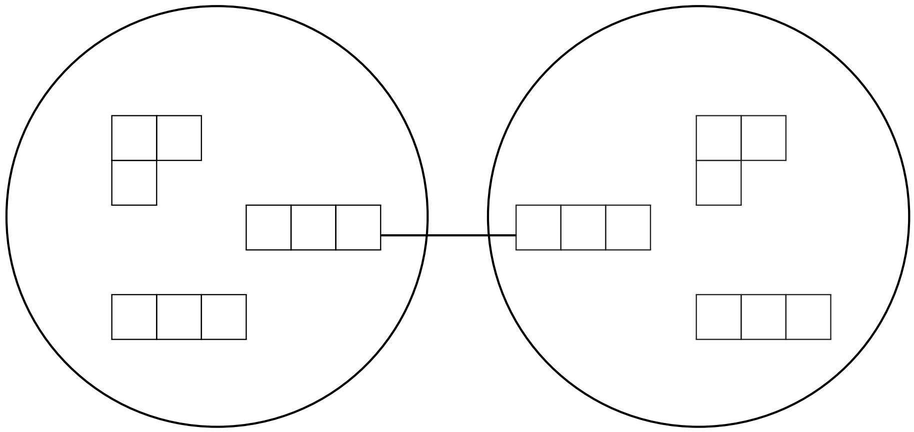

Figure 4.7: The A_2 theory on a sphere with two minimal and two maximal in the symmetric limit.

us to express it in terms of the f_N family of three point functions in the following way

$$V[\mathfrak{sl}_3]_{0,([2,1],[2,1],[1^3],[1^3])}(\rho\vec{\omega_2},\sigma\vec{\omega_2},\vec{p_1},\vec{p_2})$$
$$= \int_{\vec{p}\in\vec{Q}+i(s_1^+\vec{\omega_1}+s_2^+\vec{\omega_2})} d\vec{p}C(\rho\vec{\omega_2},\vec{p_1},\vec{p})C(\vec{Q}-\vec{p},\vec{p_2},\sigma\vec{\omega_2})$$
$$\mathcal{F}_{\mathfrak{sl}_3}\left[\begin{array}{cc}\vec{p_1} & \vec{p_2}\\ \chi\vec{\omega_2} & \sigma\vec{\omega_2}\end{array}\right](\vec{\alpha},z_i)\mathcal{F}_{\mathfrak{sl}_3}\left[\begin{array}{cc}\vec{p_1} & \vec{p_2}\\ \chi\vec{\omega_2} & \sigma\vec{\omega_2}\end{array}\right](\vec{Q}-\vec{\alpha},\bar{z}_i),$$

where the three point function belong to the Fateev-Litvinov family f_N. The dependence of the conformal blocks on the momenta is through the dimensions $\Delta_{\vec{p}}, \Delta_{\vec{p}}^{(3)}$. These are given by

$$\Delta_{\vec{p}} = \frac{(2\vec{Q}-\vec{p}).\vec{p}}{2}, \tag{4.80}$$

$$\Delta_{\vec{p}}^{(3)} = i\sqrt{\frac{48}{22+5c}}(\vec{p}-\vec{Q},h_1)(\vec{p}-\vec{Q},h_2)(\vec{p}-\vec{Q},h_3). \tag{4.81}$$

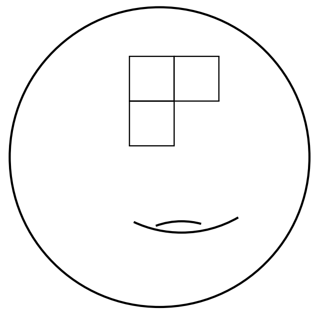

Figure 4.8: A_2 theory on a torus with one minimal puncture

Proceeding as in the case of the four point function for Liouville, one can rewrite Υ functions in the numerator in terms of the H functions making the Vandermonde explicit. This gives an integration of the form $\int da_1 da_2 (a_1^2 + a_2^2)(a_1^4 + a_2^4)$ implying $n_v = 8$ (as expected for a gauge theory with gauge group $SU(3)$). Defining $\hat{V}[\mathfrak{sl}_3]_{0,([2,1],[2,1],[1^3],[1^3])}$ as in (4.78) and collecting the anomalous scaling factors,

$$\hat{V}[\mathfrak{sl}_3]_{0,([2,1],[2,1],[1^3],[1^3])} = \mu^{29/3}(\hat{V}[\mathfrak{sl}_3]_{0,([2,1],[2,1],[1^3],[1^3])})_{R_0=1}. \tag{4.82}$$

The value of $4a$ is correctly reproduced.

4.5.2.2 $V[\mathfrak{sl}_3]_{1,([2,1])}$

This is the correlator that pertains $Z_{\mathbb{S}^4}$ of $SU(3)$ gauge group with an adjoint hypermultiplet and a free hyper. It has the following expression,

$$V[\mathfrak{sl}_3]_{1,([2,1])}(\chi\vec{\omega}_2) = \int d\vec{p} \frac{\Upsilon(b)^{n-1}\Upsilon(\rho)\prod_{\vec{e}>0}\Upsilon((\vec{Q}-\vec{p}).\vec{e})\Upsilon((\vec{Q}+\vec{p}).\vec{e})}{\prod_{i=1,j=1}^{i=3,j=3}\Upsilon\left(\frac{\rho}{2}+(\vec{p}-\vec{Q}).\vec{h_i}+(\vec{Q}-\vec{p}).\vec{h_j}\right)} \mathcal{F}^{g=1}_{\mathfrak{sl}_3}[\chi\omega_2,\vec{p}].$$

Again, defining $\hat{V}[\mathfrak{sl}_3]_{1,([2,1])}$ following (4.78) and collecting anomalous scale factors,

$$\hat{V}[\mathfrak{sl}_3]_{1,([2,1])} = \mu^{49/6}(\hat{V}[\mathfrak{sl}_3]_{1,([2,1])})_{R_0=1}. \tag{4.83}$$

Ignoring the contribution from the decoupled abelian vector multiplets reproduces the expected value for $4a$.

4.5.3 Examples : Theories with no known Lagrangian description

4.5.3.1 $V[\mathfrak{sl}_3]_{0,([2,1],[2,1],[1^3],[1^3])}$ in its asymmetric limit

Let us now consider this correlator in the limit where two minimal punctures are on one side and the two maximal punctures are on the other side of the factorization channel. The duality between the corresponding four dimensional theories (that arise in the two limits) was discovered by Argyres-Seiberg [10].

In this limit, $X_\alpha < 0, X_\beta > 0$. So, the condition for a full factorization is not satisfied. In its other limit, we have already seen that this theory has $n_h = 18, n_v = 8$ (with the corresponding implications for the three point functions appearing in the symmetric limit). To understand the asymmetric limit, let us write the four point function in the following form

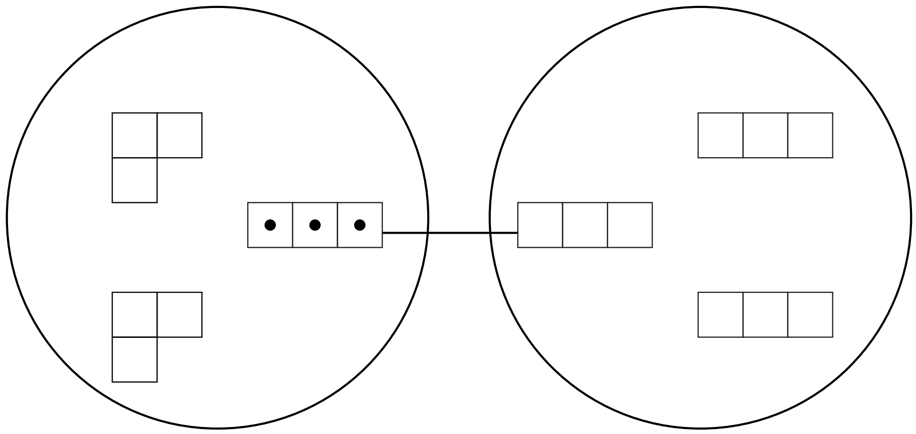

Figure 4.9: The A_2 theory on a sphere with two minimal and two maximal in the asymmetric limit.

$$V[\mathfrak{sl}_3]_{0,([2,1],[2,1],[1^3],[1^3])}(\rho\vec{\omega_2},\sigma\vec{\omega_2},\vec{p_1},\vec{p_2})$$
$$= \int_{\vec{p}\in\vec{Q}+i(s_1^+\vec{\omega_1}+s_2^+\vec{\omega_2})} d\vec{p}C(\rho\vec{\omega_2},\sigma\vec{\omega_2},\vec{p})C(\vec{Q}-\vec{p},\vec{p_2},\vec{p_1})$$
$$\mathcal{W}_2\begin{bmatrix}\vec{p_1} & \vec{p_2}\\ \chi\vec{\omega_2} & \sigma\vec{\omega_2}\end{bmatrix}(\vec{\alpha},z_i)\mathcal{W}_2\begin{bmatrix}\vec{p_1} & \vec{p_2}\\ \chi\vec{\omega_2} & \sigma\vec{\omega_2}\end{bmatrix}(\vec{Q}-\vec{\alpha},\bar{z}_i).$$

Here, the three point function $C(\rho\vec{\omega_2},\sigma\vec{\omega_2},\vec{p})$ can be understood as a limit of the Fateev Litvinov family f_N where one of the maximal punctures is made minimal. When this is done, the three point function becomes identically zero except when the following condition is obeyed [119, 64],

$$w - w_1 + w_2 + \frac{3}{2}\left(\frac{w_1}{\Delta_1} - \frac{w_2}{\Delta_2}\right)(\Delta - \Delta_1 - \Delta_2) = 0. \qquad (4.84)$$

In the above equation the cubic invariant is referred to as w instead of $\Delta^{(3)}$ to avoid confusion with the subscripts. The above condition restricts the channel momentum to a one dimensional subspace of the most general macroscopic Toda state. This corresponds to the choice of a $SU(2)$ subgroup. After canceling factors between the numerator and the denominator of the f_N correlator (specialized to $N = 3$) and using the properties of the Υ functions, the measure for the channel integral is seen to be of the form $a^2 da$. One would like to account for the scale factor in this limit. The n_h contribution is easy to account for since this arises only from the local contributions of the punctures and the global contribution of the sphere. n_v on the other hand is non-trivial. From the factorization channel, we get $n_v = 3$ (as opposed to $n_v = 8$ from the factorization channel in the symmetric limit). This implies that the stripped three point function corresponding to three maximal punctures has a scale factor that corresponds to $n_h = 16, n_v = 5$.

This discussion aims to be nothing more than a poor substitute for an analysis of the factorization problem in Toda theories. It was included to provide an example of how the accounting for the scale factor could be different in the various factorization limits. It is examples like this that make the conjecture in Section 4.5.1.1 a non-trivial constraint on Toda factorization.

4.5.3.2 $V[\mathfrak{sl}_3]_{0,([1^3],[1^3],[1^3])}$

Not much is known in closed form for this correlator (Fig 4.5.3.2). Integral expressions for this correlator are available under some special limits.

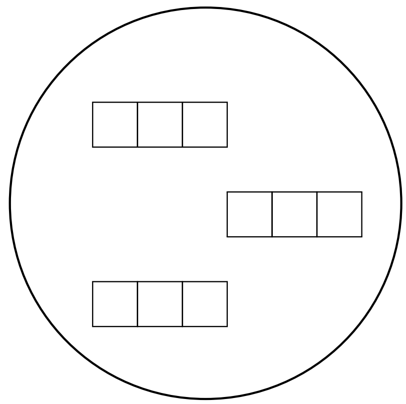

Figure 4.10: A_2 theory on a sphere with three maximal punctures

See [69, 70] for the state of the art on Toda computations. Note that this is the correlator corresponding to the partition function on $\mathbb{S}^4$ of the $\mathcal{T}_3$ theory. This correlator arises in a 'decoupling limit' of the previous example where two minimal punctures are collided and replaced with a maximal puncture. As discussed, the scale factor for the stripped correlator in this case should correspond to $n_h = 16, n_v = 5$.

4.6 Summary

In this Chapter, it is argued that the Euler anomaly of a 4d SCFT belonging to class $\mathcal{S}$ is encoded in the scale factors of the corresponding stripped Liouville/Toda correlators. This factor is always of the form μ^{4a} where a is the Euler anomaly and the quantity μ can be identified with the inverse radius of the four sphere on which the theory is formulated. The quantity a has a parameterization in terms of quantities n_h, n_v (given in 4.8).

The parameterization of a by n_h and n_v is convenient since the two types of contributions to a arise differently in the Liouville/Toda context[16],

- The local n_h contribution arises from the scale factors in the relationship between the Toda and WZW primaries while the global n_h factor arises from the boundary term associated to the curvature insertion in the Toda action on the disc,
- The n_v contributions arise from every factorization channel (when there is one) and from the 'strongly coupled' SCFTs. The contribution from the former is straightforward to pin down while the latter is known by requiring consistency with crossing symmetries (S-dualities in the four dimensional context).

The above setup should be contrasted with how these quantities are calculated in the four dimensional context in (4.13). Requiring that they agree is then a non-trivial constraint on Toda factorization and a conjecture was outlined to this effect in Section 4.5.1.1. When the total n_v contribution is zero, the corresponding four dimensional theory is a free theory with n_h hypermultiplets. The relationship between the scale factor in such theories and the analytical structure of the Toda correlator allows one to make predictions for the number of polar divisors in certain Toda correlators. Some examples of this were outlined in Section 4.4.5.

[16]This is obviously so in the 4d theories with Lagrangian description. So, it is perhaps not a surprising feature.

As briefly alluded to in the Introduction to this Chapter, the 4d/2d relationship for the class of theories studied here has attracted attention recently from various different points of view. It is natural to consider the connections of those with the setup of this Chapter. The conjecture that is provided for the scale factor should follow automatically if crossing symmetry for Toda theories is proved. In the case of Liouville CFT, this was done in [162] using the theory of infinite dimensional representations of the quantum group $\mathcal{U}_q[\mathfrak{sl}_2]$. So, one would expect that the theory of infinite dimensional representations of more general quantum groups, especially those that correspond to representations of the modular double (see [82] for some recent mathematical developments) would be relevant for the study of quantum Toda field theory. A closely related point of view would be the one from quantum Teichmuller theory for Liouville [189, 195] and generalizations thereof, namely that of higher Teichmuller theories [113, 27, 75]. The partition functions analyzed here have also been described from the point of view of topological strings [194]. Yet another connection to explore in detail would be that between the setup considered here and the geometric Langlands program with tame ramification [17, 123, 80, 99, 78, 190, 187]. But, these are left for future considerations.

Notation

All of the notation that is relevant for Chapter 5 is collected here.

$[\mathcal{O}_N]$	Set of nilpotent orbits in $\mathfrak{g}$.
$[\mathcal{O}_H]$	Set of special nilpotent orbits in $\mathfrak{g}^\vee$.
$[\mathcal{O}_H^{\mathfrak{l}^\vee}]$	Set of special nilpotent orbits in $\mathfrak{l}^\vee \subset \mathfrak{g}^\vee$.
$\mathfrak{l}^\vee$	A pseudo-Levi subalgebra of $\mathfrak{g}^\vee$
$\mathfrak{l}$	Langlands dual of $\mathfrak{l}^\vee$. May not be a subalgebra of $\mathfrak{g}$.
$\mathfrak{a}$	Semi-simple part of the Levi subalgebra (of $\mathfrak{g}$) that is part of BC label for $\mathcal{O}_N$.
$A(\mathcal{O}_H)$	Component group of the Hitchin nilpotent orbit.
$\bar{A}(\mathcal{O}_H)$	Lusztig's quotient of the component group.
ψ_H	Irrep of $\bar{A}(\mathcal{O}_H)$.
$\mathcal{C}_H$	Sommers-Achar subgroup of $\bar{A}(\mathcal{O}_H)$. It is such that $j_{C_H}^{\bar{A}(\mathcal{O}_H)}(\text{sign}) = \psi_H$.
$Irr(W)$	Set of irreducible representations of the Weyl group W of $\mathfrak{g}$.
$Irr(W^\vee)$	Set of irreducible representations of the Weyl group $W^\vee$ of $\mathfrak{g}^\vee$.
$\bar{r}$	An irreducible representation of the Weyl group $W[\mathfrak{g}]$.
r	The irrep $\bar{r}$ tensored with the sign representation.
f_r	The family to which the irrep r belongs.
$Sp[\mathfrak{g}]$	Springer's injection from $Irr(W)$ to pairs $(\mathcal{O}, \psi)$, where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$ and ψ is a representation of its component group
$Sp^{-1}[\mathfrak{g}]$	Inverse of Springer's injection. This maps acts only on the subset of $(\mathcal{O}, \psi)$ which occurs in the image of $Sp[\mathfrak{g}]$.
$j_{W'}^{W}(r_{W'})$	The truncated induction procedure of MacDonald-Lusztig-Spaltenstein.
n_h	Contribution to effective number of hypermultiplets.
n_v	Contribution to effective number of vector multiplets.
d	Contribution to the total Coulomb branch dimension.
$\mathcal{B}_N$	Springer fiber associated to the Nahm orbit.
$\mathcal{B}_H$	Springer fiber associated to the Hitchin orbit.
$a(f_r)$	Lusztig's invariant. Its value is the same for any irrep in a given family. This equals $\dim_{\mathbb{C}}(\mathcal{B}_H)$.
$\tilde{b}(\bar{r})$	Sommers' invariant. This equals $\dim_{\mathbb{C}}(\mathcal{B}_N)$.

Chapter 5

The setup relating Toda/Nahm/Hitchin descriptions

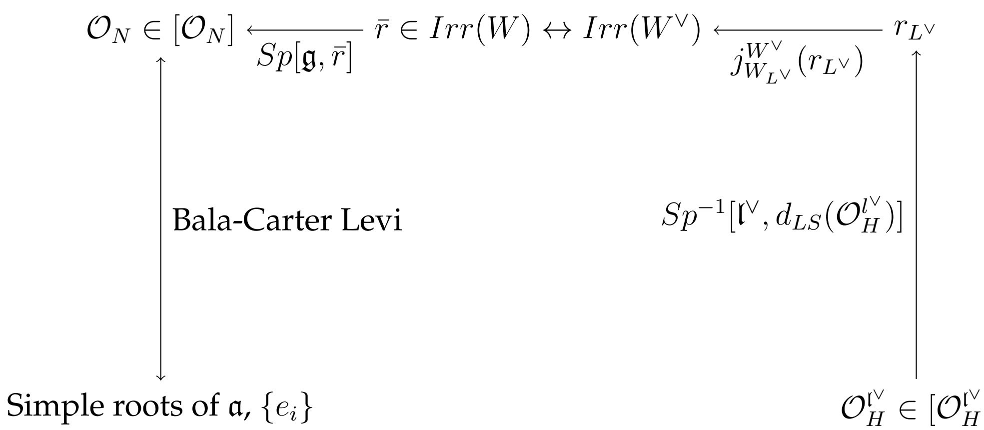

Figure 5.1: The setup

5.1 Introduction

In this Chapter, the constructions of Chapter 3 are summarized. The set of maps relating the Toda/Nahm/Hitchin descriptions is summarized in Fig 5. Some of the interesting physical quantities can be obtained from the above figure in the following way,

$$\begin{aligned} \text{simple roots for } \mathfrak{a}, \{e_i\} &\implies \{\text{level 1 null vectors for a Toda primary}\}, \\ (n_h - n_v) &= \frac{1}{2}\Big(\dim(\mathcal{N}) - \dim(\mathcal{O}_N)\Big) = \tilde{b}(\bar{r}), \\ d &= \frac{1}{2}\dim(\mathcal{O}_H) = \mid \Lambda^+ \mid - a(f_r). \end{aligned} \tag{5.1}$$

The identification of the Toda primary is taken to be for just the cases where $\mathcal{O}_N$ is principal Levi type. These quantities enter the description of the four dimensional theory (obtained via the class $\mathcal{S}$ constructions) and its partition function on a four sphere.

Note the asymmetric nature of the setup. The asymmetry arises from the fact that in the CDT description of these defects, in cases outside type A, the Hitchin side involves only special orbits in $\mathfrak{g}^\vee$ with an additional datum involving subgroups of their component groups while the Nahm side involves all possible nilpotent orbits in $\mathfrak{g}$ along with the trivial representation of their component groups[1].

The two relations in (5.1) giving the local contributions to the Higgs and Coulomb branch dimension hold for *all* cases. Also included in the tables is the representation r obtained by tensoring $\bar{r}$ with the sign representation and the value of Lusztig's invariant $a(f_r)$ for the family containing the irrep r. For the defects whose Nahm data is a special orbit, the irrep

[1]An expanded set of regular defects might allow one to think about the $\mathfrak{g}$ and $\mathfrak{g}^\vee$ descriptions of the defect in a more symmetric way. However, that possibility is not explored here.

r is the Orbit representation associated to the corresponding Hitchin orbit. For defects with non-special orbits as Nahm data, the irrep $Sp^{-1}[(\mathcal{O}_H, \psi_H]$ (when it exists) turns out to be a different non-special irrep belonging to the same family as r. It is notable that in these cases, the irrep r is not one of the Springer reps associated to non-trivial local systems on the Hitchin orbit. The general pattern (observed by calculations in classical lie algebras of low rank and all exceptional cases) is that there exist a cell module $c_1'(= \epsilon \otimes c_1)$ belonging to the family that contains r and the unique special representation in the family together with other such r ($= \epsilon \times \bar{r}$) arising from all the non-special orbits in the same special piece [2]. The representations associated to the non-trivial local systems on $\mathcal{O}_H$ occur as summands in cell modules that are strictly different from c_1'. It isn't clear if this is a known result. In any case, it is clear that a r matching argument using a Springer invariant seems out of reach for the Coulomb branch data. However, intuitively, one expects that the Coulomb branch considerations in [39] and the Higgs branch $\bar{r}$ matching argument provided here should be part of one unified setup. In this context, associating certain other invariants like the conjugacy class of the Weyl group to the Coulomb branch data might be helpful. Achieving this would also seem relevant to developing a direct Coulomb branch check for the Toda primary for arbitrary $\mathfrak{g}$.

The part of the setup that provides the dictionary between Hitchin/Nahm

[2]It is interesting that in recent work [133], finite W-algebra methods are used to study certain properties of cell modules in a given family/two-cell.

descriptions can be extended in a straightforward way to the case where $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are non simply laced (with relevance for the twisted defects of the six dimensional theory and for S-duality of boundary conditions in $\mathcal{N} = 4$ SYM with non-simply laced gauge groups). But, there is a new feature in these cases. The Langlands dual of the pseudo-Levi subalgebra $\mathfrak{l}^\vee$ which is denoted by $\mathfrak{l}$ is no longer guaranteed to be a subalgebra of $\mathfrak{g}$. The general procedure to find all possible centralizers of semi-simple elements in a complex lie algebra is to follow the Borel-de Seibenthal algorithm. Following this algorithm, one immediately recognizes the inevitability of the situation where $\mathfrak{l} \not\subseteq \mathfrak{g}$ (See Appendix C).

5.2 Tables

These detailed tables are included so that the reader can get some appreciation for the details of how the order reversing duality map works. The reader is especially encouraged to check these tables by following the map from one side to the other for a few scattered examples from the simply laced and non-simply laced cases.

Some of the calculations involved in compiling the tables were done using the CHEVIE package for the GAP system [93]. Consulting the standard tables in Carter's book is also essential. The partitioning of the Weyl group representations into families is provided in Carter [35]. The Cartan type of the pseudo-Levi subalgebra $\mathfrak{l}^\vee$ that arises on the $\mathfrak{g}^\vee$ side is included as part of the tables for some simple cases. For the exceptional cases, it can be obtained from [179]. The data collected in the tables is available in the mathematical literature often very explicitly or perhaps implicitly. It is hoped that the details help those who are not familiar with this literature. What is new is the physical interpretation of some defining features of the order reversing duality map.

In the tables for F_4, E_6, E_7, E_8, the duality map for special orbits is detailed first and then separate tables are devoted for the non-trivial special pieces. The special orbits that are part of non-trivial special pieces thus occur in both tables.

In the non-simply laced cases, the number d corresponds to a part of the local contribution to the Coulomb branch dimension. There is an additional contribution that comes from the fact that the nilpotent orbits for G non-simply laced arise actually from the twisted defects of the six dimensional theory [39].

The tables themselves were generated in the following way. The data for the columns $\mathcal{O}_N, \tilde{b}, \bar{r}, (\mathcal{O}_H, C_H)$ follows directly from the data that is used in the description of the $\bar{r}$-matching. The irrep r is obtained by tensoring $\bar{r}$ by the sign representation. The column $a(f_r)$ is Lusztig's invariant attached to the family to which the representation r belongs. It is equal to the dimension of the Springer fiber associated to the Hitchin orbit.

5.2.1 Simply laced cases

5.2.2 A_3

$| \Lambda^+ |= 6$

Table 5.1: Order reversing duality for $A_3 = \mathfrak{su}(4)$

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$	$l^\vee$
$[1^4]$	6	$[1^4]$	$[4]$	0	6	$[4]$	A_3
$[2, 1^2]$	3	$[2, 1^2]$	$[3, 1]$	1	5	$[3, 1]$	A_2
$[2, 2]$	2	$[2, 2]$	$[2, 2]$	2	4	$[2, 2]$	$A_1 + A_1$
$[3, 1]$	1	$[3, 1]$	$[2, 1^2]$	3	3	$[2, 1^2]$	A_1
$[4]$	0	$[4]$	$[1^4]$	6	0	$[1^4]$	$\varnothing$

Families with multiple irreps

None

5.2.3 D_4

$| \Lambda^+ |= 12$

Table 5.2: Order reversing duality for $D_4 = \mathfrak{so}_8$

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$	$l^\vee$
$[1^8]$	12	$[1^4].-$	$[4].-$	0	12	$[7,1]$	D_4
$[2^2,1^4]$	7	$[1^3].[1]$	$[3].[1]$	1	11	$[5,3]$	D_4
$[2^4]^I$	6	$([1^2].[1^2])'$	$([2].[2])'$	2	10	$[4^2]^I$	A_3
$[2^4]^{II}$	6	$([1^2].[1^2])''$	$([2].[2])''$	2	10	$[4^2]^{II}$	A_3
$[3,1^5]$	6	$[2,1^2].-$	$([3,1].-)$	2	10	$[5,1^3]$	A_3
$[3,2^2,1]^\star$	4	$[2^2].-$	$[2^2].-$	3	9	$[3^2,1^2], S_2$	$4A_1$
$[3^2,1^2]$	3	$[2,1].[1]$	$[2,1].[1]$	3	9	$[3^2,1^2]$	A_2
$[5,1^3]$	2	$[3,1].-$	$[2,1^2].-$	6	6	$[3,1^5]$	$2A_1$
$[4^2]^I$	2	$([2].[2])'$	$([1^2].[1^2])'$	6	6	$[2^4]^I$	$2A_1$
$[4^2]^{II}$	2	$([2].[2])''$	$([1^2].[1^2])''$	6	6	$[2^4]^{II}$	$2A_1$
$[5,3]$	1	$[3].[1]$	$[1^3].[1]$	7	5	$[2^2,1^4]$	A_1
$[7,1]$	0	$[4].-$	$[1^4].-$	12	0	$[1^8]$	$\varnothing$

The Nahm orbits $[3,2^2,1]$ and $[3^2,1^2]$ are part of the only non-trivial special piece for D_4.

Families with multiple irreps

Family f	$a(f)$
$\{([2,1],[1]),([2^2],-),([2],[1^2])\}$	3

5.2.4 E_6

$| \Lambda^+ |= 36$

Table 5.3: Order reversing duality for special orbits in E_6

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H)$
0	36	$\phi_{1,36}$	$\phi_{1,0}$	0	36	E_6
A_1	25	$\phi_{6,25}$	$\phi_{6,1}$	1	35	$E_6(a_1)$
$2A_1$	20	$\phi_{20,20}$	$\phi_{20,2}$	2	34	D_5
A_2	15	$\phi_{30,15}$	$\phi_{30,3}$	3	33	$E_6(a_3)$
$A_2 + A_1$	13	$\phi_{64,13}$	$\phi_{64,4}$	4	32	$D_5(a_1)$
$A_2 + 2A_1$	11	$\phi_{60,11}$	$\phi_{60,5}$	5	31	$A_4 + A_1$
$2A_2$	12	$\phi_{24,12}$	$\phi_{24,6}$	6	30	D_4
A_3	10	$\phi_{81,10}$	$\phi_{81,6}$	6	30	A_4
$D_4(a_1)$	7	$\phi_{80,7}$	$\phi_{80,7}$	7	29	$D_4(a_1)$
A_4	6	$\phi_{81,6}$	$\phi_{81,10}$	10	24	A_3
D_4	6	$\phi_{24,6}$	$\phi_{24,12}$	12	26	$2A_2$
$A_4 + A_1$	5	$\phi_{60,5}$	$\phi_{60,11}$	11	25	$A_2 + 2A_1$
$D_5(a_1)$	4	$\phi_{64,4}$	$\phi_{64,13}$	13	23	$A_2 + A_1$
$E_6(a_3)$	3	$\phi_{30,3}$	$\phi_{30,15}$	15	21	A_2
D_5	2	$\phi_{20,2}$	$\phi_{20,20}$	20	16	$2A_1$
$E_6(a_1)$	1	$\phi_{6,1}$	$\phi_{6,25}$	25	11	A_1
E_6	0	$\phi_{1,0}$	$\phi_{1,36}$	36	0	0

Table 5.4: Order reversing duality for nontrivial special pieces in E_6

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$
$3A_1$	16	$\phi_{15,16}$	$\phi_{15,4}$	3	33	$E_6(a_3), S_2$
A_2	15	$\phi_{30,15}$	$\phi_{13,3}$	3	33	$E_6(a_3)$
$2A_2 + A_1$	9	$\phi_{10,9}$	$\phi_{10,9}$	7	29	$D_4(a_1), S_3$
$A_3 + A_1$	8	$\phi_{60,8}$	$\phi_{60,8}$	7	29	$D_4(a_1), S_2$
$D_4(a_1)$	7	$\phi_{80,7}$	$\phi_{80,7}$	7	29	$D_4(a_1)$
A_5	4	$\phi_{15,4}$	$\phi_{15,16}$	15	21	A_2, S_2
$E_6(a_3)$	3	$\phi_{30,3}$	$\phi_{30,15}$	15	21	A_2

Families with multiple irreps

Family f	$a(f)$
$\{\phi_{30,3}, \phi_{15,4}, \phi_{15,5}\}$	15
$\{\phi_{80,7}, \phi_{60,8}, \phi_{90,8}, \phi_{10,9}, \phi_{20,10}\}$	7
$\{\phi_{30,15}, \phi_{15,16}, \phi_{15,17}\}$	3

5.2.5 E_7

$|\Lambda^+| = 63$

Table 5.5: Order reversing duality for special orbits in E_7

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H)$
0	63	$\phi_{1,63}$	$\phi_{1,0}$	0	63	E_7
A_1	46	$\phi_{7,46}$	$\phi_{7,1}$	1	62	$E_7(a_1)$
$2A_1$	37	$\phi_{27,37}$	$\phi_{27,2}$	2	61	$E_7(a_2)$
A_2	30	$\phi_{56,30}$	$\phi_{56,3}$	3	60	$E_7(a_3)$
$(3A_1)''$	36	$\phi_{21,36}$	$\phi_{21,3}$	3	60	E_6
A_2+A_1	25	$\phi_{120,25}$	$\phi_{120,4}$	4	59	$E_6(a_1)$
A_2+2A_1	22	$\phi_{189,22}$	$\phi_{189,5}$	5	58	$E_7(a_4)$
A_2+3A_1	21	$\phi_{105,21}$	$\phi_{105,6}$	6	57	A_6
A_3	21	$\phi_{210,21}$	$\phi_{210,6}$	6	57	$D_6(a_1)$
$2A_2$	21	$\phi_{168,21}$	$\phi_{168,6}$	6	57	D_5+A_1
$D_4(a_1)$	16	$\phi_{315,16}$	$\phi_{315,7}$	7	56	$E_7(a_5)$
$(A_3+A_1)''$	20	$\phi_{189,20}$	$\phi_{189,7}$	7	56	D_5
$D_4(a_1)+A_1$	15	$\phi_{405,15}$	$\phi_{405,8}$	8	51	$E_6(a_3)$
A_3+A_2	14	$\phi_{378,14}$	$\phi_{378,9}$	9	54	$D_5(a_1)+A_1$
D_4	15	$\phi_{105,15}$	$\phi_{105,12}$	12	51	A_5''
$A_3+A_2+A_1$	13	$\phi_{210,13}$	$\phi_{210,10}$	10	53	A_4+A_2
A_4	13	$\phi_{420,13}$	$\phi_{420,10}$	10	53	$D_5(a_1)$
♠ A_4+A_1	11	$\phi_{510,11}$	$\phi_{510,12}$	12	51	A_4+A_1
$D_5(a_1)$	10	$\phi_{420,10}$	$\phi_{420,13}$	13	50	A_4
A_4+A_2	10	$\phi_{210,10}$	$\phi_{210,13}$	13	50	$A_3+A_2+A_1$

(..contd)

(Table 5.5 continued)

A_5''	12	$\phi_{105,12}$	$\phi_{105,15}$	15	48	D_4
$D_5(a_1)+A_1$	9	$\phi_{378,9}$	$\phi_{378,14}$	14	49	A_3+A_2
$E_6(a_3)$	8	$\phi_{405,8}$	$\phi_{405,15}$	15	48	$D_4(a_1)+A_1$
D_5	7	$\phi_{189,7}$	$\phi_{189,20}$	20	43	$(A_3+A_1)''$
$E_7(a_5)$	7	$\phi_{315,7}$	$\phi_{315,16}$	16	47	$D_4(a_1)$
D_5+A_1	6	$\phi_{168,6}$	$\phi_{168,21}$	21	42	$2A_2$
$D_6(a_1)$	6	$\phi_{210,6}$	$\phi_{210,21}$	21	42	A_3
A_6	6	$\phi_{105,6}$	$\phi_{105,21}$	21	42	A_2+3A_1
$E_7(a_4)$	5	$\phi_{189,5}$	$\phi_{189,22}$	22	41	A_2+2A_1
$E_6(a_1)$	4	$\phi_{120,4}$	$\phi_{120,25}$	25	38	A_2+A_1
E_6	3	$\phi_{21,3}$	$\phi_{21,36}$	36	27	$(3A_1)''$
$E_7(a_3)$	3	$\phi_{56,3}$	$\phi_{56,30}$	30	33	A_2
$E_7(a_2)$	2	$\phi_{27,2}$	$\phi_{27,37}$	37	26	$2A_1$
$E_7(a_1)$	1	$\phi_{7,1}$	$\phi_{7,46}$	46	17	A_1
E_7	0	$\phi_{1,0}$	$\phi_{1,63}$	63	0	0

Table 5.6: Order reversing duality for nontrivial special pieces in E_7

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$
$3A_1'$	31	$\phi_{35,31}$	$\phi_{35,4}$	3	60	$E_7(a_3), S_2$
A_2	30	$\phi_{56,30}$	$\phi_{56,3}$	3	60	$E_7(a_3)$
$4A_1$	28	$\phi_{15,28}$	$\phi_{15,7}$	4	59	$E_6(a_1), S_2$
$A_2 + A_1$	25	$\phi_{120,25}$	$\phi_{120,4}$	4	59	$E_6(a_1)$
$A_3 + 2A_1$	16	$\phi_{216,16}$	$\phi_{216,9}$	8	55	$E_6(a_3), S_2$
$D_4(a_1) + A_1$	15	$\phi_{405,15}$	$\phi_{405,8}$	8	55	$E_6(a_3)$
$D_4 + A_1$	12	$\phi_{84,12}$	$\phi_{84,15}$	13	50	A_4, S_2
$D_5(a_1)$	10	$\phi_{420,10}$	$\phi_{420,13}$	13	50	A_4
$(A_5)'$	9	$\phi_{216,9}$	$\phi_{216,19}$	15	48	$D_4(a_1) + A_1, S_2$
$E_6(a_3)$	8	$\phi_{405,8}$	$\phi_{405,15}$	15	48	$D_4(a_1) + A_1$
D_6	4	$\phi_{35,4}$	$\phi_{35,31}$	30	33	A_2, S_2
$E_7(a_3)$	3	$\phi_{56,3}$	$\phi_{56,30}$	30	33	A_2

(..contd)

(Table 5.6 continued)

$2A_2 + A_1$	18	$\phi_{70,18}$	$\phi_{70,9}$	7	56	$E_7(a_5), S_3$
$(A_3 + A_1)'$	17	$\phi_{280,17}$	$\phi_{280,8}$	7	56	$E_7(a_5), S_2$
$D_4(a_1)$	16	$\phi_{315,16}$	$\phi_{315,7}$	7	56	$E_7(a_5)$
$A_5 + A_1$	9	$\phi_{70,9}$	$\phi_{70,18}$	16	47	$D_4(a_1), S_3$
$D_6(a_2)$	8	$\phi_{280,8}$	$\phi_{280,17}$	16	47	$D_4(a_1), S_2$
$E_7(a_5)$	7	$\phi_{315,7}$	$\phi_{315,16}$	16	47	$D_4(a_1)$

Families with multiple irreps

Family f	$a(f)$
$\{\phi_{56,3}, \phi_{35,4}, \phi_{21,6}\}$	3
$\{\phi_{120,4}, \phi_{105,5}, \phi_{15,7}\}$	4
$\{\phi_{405,8}, \phi_{216,9}, \phi_{189,10}\}$	8
$\{\phi_{420,10}, \phi_{336,11}, \phi_{84,12}\}$	10
♠$\{\phi_{512,11}, \phi_{512,12}\}$	11
$\{\phi_{420,13}, \phi_{336,14}, \phi_{84,15}\}$	13
$\{\phi_{405,15}, \phi_{216,16}, \phi_{189,17}\}$	15
$\{\phi_{120,25}, \phi_{105,26}, \phi_{15,28}\}$	25
$\{\phi_{56,30}, \phi_{35,31}, \phi_{21,33}\}$	30
$\{\phi_{315,7}, \phi_{280,8}, \phi_{70,9}, \phi_{280,9}, \phi_{35,13}\}$	7
$\{\phi_{315,16}, \phi_{280,17}, \phi_{70,18}, \phi_{280,18}, \phi_{35,22}\}$	16

5.2.6 E_8

$| \Lambda^+ |= 120$

Table 5.7: Order reversing duality for special orbits in E_8

$\mathcal{O}_N$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$\mathcal{O}_H$
0	120	$\phi_{1,120}$	$\phi_{1,0}$	0	120	E_8
A_1	91	$\phi_{8,91}$	$\phi_{8,1}$	1	119	$E_8(a_1)$
$2A_1$	74	$\phi_{35,74}$	$\phi_{35,2}$	2	118	$E_8(a_2)$
A_2	63	$\phi_{112,63}$	$\phi_{112,3}$	3	117	$E_8(a_3)$
$A_2 + A_1$	52	$\phi_{210,52}$	$\phi_{210,4}$	4	116	$E_8(a_4)$
$A_2 + 2A_1$	47	$\phi_{560,47}$	$\phi_{560,5}$	5	115	$E_8(b_4)$
A_3	46	$\phi_{567,46}$	$\phi_{567,6}$	6	114	$E_7(a_1)$
$2A_2$	42	$\phi_{700,42}$	$\phi_{700,6}$	6	114	$E_8(a_5)$
$D_4(a_1)$	37	$\phi_{1400,37}$	$\phi_{1400,7}$	7	113	$E_8(b_5)$
$D_4(a_1) + A_1$	32	$\phi_{1400,32}$	$\phi_{1400,8}$	8	112	$E_8(a_6)$
$A_3 + A_2$	31	$\phi_{3240,31}$	$\phi_{3240,9}$	9	111	$D_7(a_1)$
$D_4(a_1) + A_2$	28	$\phi_{2240,28}$	$\phi_{2240,10}$	10	110	$E_8(b_6)$
A_4	30	$\phi_{2268,30}$	$\phi_{2268,10}$	10	110	$E_7(a_3)$
D_4	36	$\phi_{525,36}$	$\phi_{525,12}$	12	108	E_6
♠$A_4 + A_1$	26	$\phi_{4096,26}$	$\phi_{4096,12}$	11	109	$E_6(a_1)+ A_1$
$A_4 + 2A_1$	24	$\phi_{4200,24}$	$\phi_{4200,12}$	12	108	$D_7(a_2)$
$A_4 + A_2$	23	$\phi_{4536,23}$	$\phi_{4536,13}$	13	107	$D_5 + A_2$
$D_5(a_1)$	25	$\phi_{2800,25}$	$\phi_{2800,13}$	13	107	$E_6(a_1)$
$A_4 + A_2 + A_1$	22	$\phi_{2835,22}$	$\phi_{2835,14}$	14	106	$A_6 + A_1$
$D_4 + A_2$	21	$\phi_{4200,21}$	$\phi_{4200,15}$	15	105	A_6
$D_5(a_1) + A_1$	22	$\phi_{6075,22}$	$\phi_{6075,14}$	14	106	$E_7(a_4)$

(..contd)

(Table 5.7 continued)

$E_6(a_3)$	21	$\phi_{5600,21}$	$\phi_{5600,15}$	15	105	$D_6(a_1)$
D_5	20	$\phi_{2100,20}$	$\phi_{2100,20}$	20	100	D_5
$E_8(a_7)$	16	$\phi_{4480,16}$	$\phi_{4480,16}$	16	104	$E_8(a_7)$
$D_6(a_1)$	15	$\phi_{5600,15}$	$\phi_{5600,21}$	21	99	$E_6(a_3)$
$E_7(a_4)$	14	$\phi_{6075,14}$	$\phi_{6075,22}$	22	98	$D_5(a_1) + A_1$
A_6	15	$\phi_{4200,15}$	$\phi_{4200,21}$	21	99	$D_4 + A_2$
$A_6 + A_1$	14	$\phi_{2835,14}$	$\phi_{2835,22}$	22	98	$A_4 + A_2 + A_1$
$E_6(a_1)$	13	$\phi_{2800,13}$	$\phi_{2800,25}$	25	95	$D_5(a_1)$
$D_5 + A_2$	13	$\phi_{4536,13}$	$\phi_{4536,23}$	23	97	$A_4 + A_2$
$D_7(a_2)$	12	$\phi_{4200,12}$	$\phi_{4200,24}$	24	96	$A_4 + 2A_1$
♠$E_6(a_1) + A_1$	11	$\phi_{4096,11}$	$\phi_{4096,27}$	26	94	$A_4 + A_1$
E_6	12	$\phi_{525,12}$	$\phi_{525,36}$	36	84	D_4
$E_7(a_3)$	10	$\phi_{2268,10}$	$\phi_{2268,30}$	30	90	A_4
$E_8(b_6)$	10	$\phi_{2240,10}$	$\phi_{2240,28}$	28	92	$D_4(a_1) + A_2$
$D_7(a_1)$	9	$\phi_{3240,9}$	$\phi_{3240,31}$	31	89	$A_3 + A_2$
$E_8(a_6)$	8	$\phi_{1400,8}$	$\phi_{1400,32}$	32	88	$D_4(a_1) + A_1$
$E_8(b_5)$	7	$\phi_{1400,7}$	$\phi_{1400,37}$	37	83	$D_4(a_1)$
$E_8(a_5)$	6	$\phi_{700,6}$	$\phi_{700,42}$	42	78	$2A_2$
$E_7(a_1)$	6	$\phi_{567,6}$	$\phi_{567,46}$	46	74	A_3
$E_8(b_4)$	5	$\phi_{560,5}$	$\phi_{560,47}$	47	73	$A_2 + 2A_1$
$E_8(a_4)$	4	$\phi_{210,4}$	$\phi_{210,52}$	52	68	$A_2 + A_1$
$E_8(a_3)$	3	$\phi_{112,3}$	$\phi_{112,63}$	63	57	A_2
$E_8(a_2)$	2	$\phi_{35,2}$	$\phi_{35,74}$	74	46	$2A_1$
$E_8(a_1)$	1	$\phi_{8,1}$	$\phi_{8,91}$	91	29	A_1
E_8	0	$\phi_{1,0}$	$\phi_{1,120}$	120	0	0

Table 5.8: Order reversing duality for nontrivial special pieces in E_8

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$
$3A_1$	64	$\phi_{84,64}$	$\phi_{84,4}$	3	117	$E_8(a_3), S_2$
A_2	63	$\phi_{112,63}$	$\phi_{112,3}$	3	117	$E_8(a_3)$
$4A_1$	56	$\phi_{50,56}$	$\phi_{50,8}$	4	116	$E_8(a_4), S_2$
$A_2 + A_1$	52	$\phi_{210,52}$	$\phi_{210,4}$	4	116	$E_8(a_4)$
$A_2 + 3A_1$	43	$\phi_{400,43}$	$\phi_{400,7}$	6	114	$E_8(a_5), S_2$
$2A_2$	42	$\phi_{700,42}$	$\phi_{700,6}$	6	114	$E_8(a_5)$
$D_4 + A_1$	28	$\phi_{700,28}$	$\phi_{700,16}$	13	107	$E_6(a_1), S_2$
$D_5(a_1)$	25	$\phi_{2800,25}$	$\phi_{2800,13}$	13	107	$E_6(a_1)$
$2A_3$	26	$\phi_{840,26}$	$\phi_{840,14}$	12	108	$D_7(a_2), S_2$
$A_4 + 2A_1$	24	$\phi_{4200,24}$	$\phi_{4200,12}$	12	108	$D_7(a_2)$
A_5	22	$\phi_{3200,22}$	$\phi_{3200,16}$	15	105	$D_6(a_1), S_2$
$E_6(a_3)$	21	$\phi_{5600,21}$	$\phi_{5600,15}$	15	105	$D_6(a_1)$
$D_5 + A_1$	16	$\phi_{3200,16}$	$\phi_{3200,22}$	25	95	$E_6(a_3), S_2$
$D_6(a_1)$	15	$\phi_{5600,15}$	$\phi_{5600,21}$	25	95	$E_6(a_3)$

(..contd)

(Table 5.8 continued)

D_6	12	$\phi_{972,12}$	$\phi_{972,32}$	30	90	A_4, S_2
$E_7(a_3)$	10	$\phi_{2268,10}$	$\phi_{2268,30}$	30	90	A_4
A_7	11	$\phi_{1400,11}$	$\phi_{1400,29}$	28	92	$D_4(a_1) + A_2, S_2$
$E_8(b_6)$	10	$\phi_{2240,10}$	$\phi_{2240,28}$	28	92	$D_4(a_1) + A_2$
D_7	7	$\phi_{400,7}$	$\phi_{400,43}$	42	78	$E_8(a_5), S_2$
$E_8(a_5)$	6	$\phi_{700,6}$	$\phi_{700,42}$	42	78	$E_8(a_5)$
E_7	4	$\phi_{84,4}$	$\phi_{84,64}$	63	57	A_2, S_2
$E_8(a_3)$	3	$\phi_{112,3}$	$\phi_{112,63}$	63	57	A_2
$A_3 + A_2 + A_1$	29	$\phi_{1400,29}$	$\phi_{1400,11}$	10	110	$E_8(b_6), S_2$
$D_4(a_1) + A_2$	28	$\phi_{2240,28}$	$\phi_{2240,10}$	10	100	$E_8(b_6)$
$2A_2 + A_1$	39	$\phi_{448,39}$	$\phi_{448,9}$	7	113	$E_8(b_5), S_3$
$A_3 + 2A_1$	38	$\phi_{1344,38}$	$\phi_{1344,38}$	7	113	$E_8(b_5), S_2$
$D_4(a_1)$	37	$\phi_{1400,37}$	$\phi_{1400,8}$	7	113	$E_8(b_5)$
$2A_2 + 2A_1$	36	$\phi_{175,36}$	$\phi_{175,12}$	8	112	$E_8(a_6), S_3$
$A_3 + 2A_1$	34	$\phi_{1050,34}$	$\phi_{1050,10}$	8	112	$E_8(a_6), S_2$
$D_4(a_1) + A_1$	32	$\phi_{1400,32}$	$\phi_{1400,8}$	8	112	$E_8(a_6)$
$E_6 + A_1$	9	$\phi_{448,9}$	$\phi_{448,39}$	37	83	$D_4(a_1), S_3$
$E_7(a_2)$	8	$\phi_{1344,8}$	$\phi_{1344,38}$	37	83	$D_4(a_1), S_2$
$E_8(b_5)$	7	$\phi_{1400,7}$	$\phi_{1400,37}$	37	83	$D_4(a_1)$

(..contd)

(Table 5.8 continued)

$A_4 + A_3$	20	$\phi_{420,20}$	$\phi_{420,20}$	16	104	$E_8(a_7), S_5$
$D_5(a_1) + A_2$	19	$\phi_{1344,19}$	$\phi_{1344,19}$	16	104	$E_8(a_7), S_4$
$A_5 + A_1$	19	$\phi_{2016,19}$	$\phi_{2016,19}$	16	104	$E_8(a_7), S_3 \times S_2$
$E_6(a_3) + A_1$	18	$\phi_{3150,18}$	$\phi_{3150,18}$	16	104	$E_8(a_7), S_3$
$D_6(a_2)$	18	$\phi_{4200,18}$	$\phi_{4200,18}$	16	104	$E_8(a_7), S_2 \times S_2$
$E_7(a_5)$	17	$\phi_{7168,17}$	$\phi_{7168,17}$	16	104	$E_8(a_7), S_2$
$E_8(a_7)$	16	$\phi_{4480,16}$	$\phi_{4480,16}$	16	104	$E_8(a_7)$

Families with multiple irreps

Family f	$a(f)$
$\{\phi_{112,3}, \phi_{84,4}, \phi_{28,8}\}$	3
$\{\phi_{210,4}, \phi_{160,7}, \phi_{50,8}\}$	4
$\{\phi_{700,8}, \phi_{400,7}, \phi_{300,8}\}$	8
$\{\phi_{2268,10}, \phi_{972,12}, \phi_{1296,13}\}$	10
$\{\phi_{2240,10}, \phi_{1400,11}, \phi_{840,13}\}$	10
♠$\{\phi_{4096,11}, \phi_{4096,12}\}$	11
$\{\phi_{4200,12}, \phi_{3360,13}, \phi_{840,14}\}$	13
$\{\phi_{2800,13}, \phi_{700,16}, \phi_{2100,16}\}$	16
$\{\phi_{5600,15}, \phi_{3200,16}, \phi_{2400,17}\}$	16
$\{\phi_{5600,21}, \phi_{3200,22}, \phi_{2400,23}\}$	22
$\{\phi_{4200,24}, \phi_{3360,25}, \phi_{840,31}\}$	25
$\{\phi_{2800,25}, \phi_{700,28}, \phi_{2100,28}\}$	28
♠$\{\phi_{4096,26}, \phi_{4096,27}\}$	26
$\{\phi_{2240,28}, \phi_{1400,29}, \phi_{840,31}\}$	29
$\{\phi_{2268,30}, \phi_{972,32}, \phi_{1296,33}\}$	32
$\{\phi_{700,42}, \phi_{400,43}, \phi_{300,44}\}$	43
$\{\phi_{210,52}, \phi_{160,55}, \phi_{50,56}\}$	55
$\{\phi_{112,63}, \phi_{84,64}, \phi_{28,68}\}$	64
$\{\phi_{1400,7}, \phi_{1344,8}, \phi_{448,9}, \phi_{1008,9}, \phi_{56,19}\}$	7
$\{\phi_{1400,8}, \phi_{1050,10}, \phi_{1575,10}, \phi_{175,12}, \phi_{350,14}\}$	8
$\{\phi_{1400,32}, \phi_{1050,34}, \phi_{1575,34}, \phi_{175,36}, \phi_{350,38}\}$	32
$\{\phi_{1400,37}, \phi_{1344,38}, \phi_{448,39}, \phi_{1008,39}, \phi_{56,49}\}$	37
$\{\phi_{4480,16}, \phi_{7168,17}, \phi_{3150,18}, \phi_{4200,18}, \phi_{4536,18}, \phi_{5670,18},$ $\phi_{1344,19}, \phi_{2016,19}, \phi_{5600,19}, \phi_{2688,20}, \phi_{420,20}, \phi_{1134,20},$ $\phi_{1400,20}, \phi_{1680,22}, \phi_{168,24}, \phi_{448,25}, \phi_{70,32}\}$	16

5.2.7 A comment on exceptional orbits

The families marked with a ♠ are the only families with just two irreps. There is one such family in E_7 and two such families in E_8. The orbits for whom the associated Orbit representation is one of these are referred to as exceptional orbits. They are known to have somewhat peculiar properties among all nilpotent orbits (See Carter[35] Prop 11.3.5 and [20, 48]). The special representations that occur in these families are the only ones which do not give another special representation when tensored with the sign representation. They are also known to posses some special properties from the point of view of the representation theory of Hecke algebras. These are the only cases among where $\mathcal{O}_N$ is a special orbit and $Sp[r] \neq \mathcal{O}_H$. Another way to view this anomalous situation would be to say that the natural partial ordering on special representations [3] of the Weyl group is reversed by a tensoring with sign in all cases except these. There is a version of this inversion map due to Lusztig (denoted earlier in Chapter 3 by $i(r)$), which remedies these anomalous cases by assigning the special representation in the family of $\epsilon \otimes r$ to be $i(r)$.

In this context, it is important to note that there are subtler partial orders that are defined by Achar [2] and Sommers [180] which when transferred to Irr(W) may enable the treatment of these cases on a more equal footing with every other instance of duality. From a physical standpoint, it

[3]This can be obtained by transferring the closure ordering on the set of Special orbits to the set of Special representation.

would be interesting to know if these subtler partial orders are related to the partial order implied by the possible Higgsing patterns of the corresponding three dimensional $T[G]$.

5.2.8 Non-simply laced cases

5.2.9 $\mathfrak{g} = B_3$, $\mathfrak{g}^\vee = C_3$ and $\mathfrak{g} = C_3$, $\mathfrak{g}^\vee = B_3$

$\mid \Lambda^+ \mid = 9$

Table 5.9: Order reversing duality for $\mathfrak{g} = B_3$, $\mathfrak{g}^\vee = C_3$

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$
$[1^7]$	9	$-.[1^3]$	$[3].-$	0	9	$[6]$
$[2^2,1^3]$	5	$-.[2,1]$	$[2,1].-$	1	8	$[4,2]$, S_2
$[3,1^4]$	4	$[1].[1^2]$	$[2].[1]$	1	8	$[4,2]$
$[3,2^2]$	3	$[1^2].[1]$	$[1].[2]$	2	6	$[3^2]$
$[3^2,1]$	2	$-.[3]$	$[1^3].-$	4	5	$[2^2,1^2]$, S_2
$[5,1^2]$	1	$[2].[1]$	$[1].[1^2]$	4	5	$[2^2,1^2]$
$[7]$	0	$[3].-$	$-.[1^3]$	9	0	$[1^6]$

Families with multiple irreps

Family f	$a(f)$
$[2].[1], -.[3], [2,1].-$	1
$[1].[1^2], [1^3].-, -.[2,1]$	4

Table 5.10: Order reversing duality for $\mathfrak{g} = C_3$, $\mathfrak{g}^\vee = B_3$

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$
$[1^6]$	9	$-.[1^3]$	$[3].-$	0	9	$[7]$
$[2, 1^4]$	6	$[1^3].-$	$-.[3]$	1	8	$[5, 1^2], S_2$
$[2^2, 1^2]$	4	$[1].[1^2]$	$[2].[1]$	1	8	$[5, 1^2]$
$[2^3]$	3	$[1^2].[1]$	$[1].[2]$	2	7	$[3^2, 1]$
$[3^2]$	2	$[1].[2]$	$[1^2].[1]$	3	6	$[3, 2^2]$
$[4, 1^2]$	2	$[2, 1].-$	$-.[2, 1]$	4	5	$[3, 1^4], S_2$
$[4, 2]$	1	$[2].[1]$	$[1].[1^2]$	4	5	$[3, 1^4]$
$[6]$	0	$[3].-$	$-.[1^3]$	9	0	$[1^7]$

5.2.10 G_2

$$| \Lambda^+ | = 6$$

Table 5.11: Order reversing duality for $\mathfrak{g}_2$

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$
1	6	$\phi_{1,6}$	$\phi_{1,0}$	0	6	G_2
A_1	3	$\phi''_{1,3}$	$\phi''_{1,3}$	1	5	$(G_2(a_1), S_3)$
$\tilde{A}_1$	2	$\phi_{2,2}$	$\phi_{2,2}$	1	5	$(G_2(a_1), S_2)$
$G_2(a_1)$	1	$\phi_{2,1}$	$\phi_{2,1}$	1	5	$(G_2(a_1), 1)$
G_2	0	$\phi_{1,0}$	$\phi_{1,6}$	6	0	1

Families with multiple irreps

Family f	$a(f)$
$\{\phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}\}$	1

5.2.11 F_4

$$| \Lambda^+ | = 24$$

Table 5.12: Order reversing duality for special orbits in F_4

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H)$
0	24	$\phi_{1,24}$	$\phi_{1,0}$	0	24	F_4
$\tilde{A}_1$	13	$\phi_{4,13}$	$\phi_{4,1}$	1	23	$F_4(a_1)$
$A_1 + \tilde{A}_1$	10	$\phi_{9,10}$	$\phi'_{9,2}$	2	22	$F_4(a_2)$
[4] $\star A_2$	9	$\phi''_{8,9}$	$\phi''_{8,3}$	3	21	B_3
$\star\tilde{A}_2$	9	$\phi'_{8,9}$	$\phi'_{8,3}$	3	21	C_3
$F_4(a_3)$	4	$\phi_{12,4}$	$\phi_{12,4}$	4	20	$F_4(a_3)$
$\star B_3$	3	$\phi''_{8,3}$	$\phi''_{8,9}$	9	15	A_2
$\star C_3$	3	$\phi'_{8,3}$	$\phi'_{8,9}$	9	15	$\tilde{A}_2$
$F_4(a_2)$	2	$\phi_{9,2}$	$\phi_{9,10}$	10	14	$A_1 + \tilde{A}_1$
$F_4(a_1)$	1	$\phi_{4,1}$	$\phi_{4,13}$	13	11	$\tilde{A}_1$
F_4	0	$\phi_{1,0}$	$\phi_{1,24}$	24	0	0

Families with multiple irreps

Family f	$a(f)$
$\{\phi_{4,1}, \phi'_{2,4}, \phi_{2,4}\}$	1
$\{\phi_{4,13}, \phi'_{2,16}, \phi''_{2,16}\}$	13
$\{\phi_{12,4}, \phi_{16,5}, \phi'_{6,6}, \phi''_{6,6}, \phi'_{9,6}, \phi''_{9,6}, \phi'_{4,7}, \phi''_{4,7}, \phi_{4,8}, \phi'_{1,12}, \phi''_{1,12}\}$	4

[4]These instances (marked with a $\star$) of the duality map are a bit subtle. Although the Weyl group of the dual is isomorphic in a canonical way to the original, there is an exchange of the long root and the short root. The notation for $\bar{r}$ incorporates this exchange.

Table 5.13: Order reversing duality for non trivial special pieces in F_4

$(\mathcal{O}_N)$	$\tilde{b}$	$\bar{r}$	r	$a(f_r)$	d	$(\mathcal{O}_H, C_H)$
A_1	16	$\phi''_{2,16}$	$\phi'_{2,4}$	1	23	$(F_4(a_1), S_2)$
$\tilde{A}_1$	13	$\phi_{4,13}$	$\phi_{4,1}$	1	23	$F_4(a_1)$
$A_2 + \tilde{A}_1$	7	$\phi''_{4,7}$	$\phi''_{4,7}$	4	20	$(F_4(a_3), S_4)$
$A_1 + \tilde{A}_2$	6	$\phi'_{6,6}$	$\phi'_{6,6}$	4	20	$(F_4(a_3), S_3)$
B_2	6	$\phi''_{9,6}$	$\phi''_{9,6}$	4	20	$(F_4(a_3), S_2 \times S_2)$
$C_3(a_1)$	5	$\phi_{16,5}$	$\phi_{16,5}$	4	20	$(F_4(a_3), S_2)$
$F_4(a_3)$	4	$\phi_{12,4}$	$\phi_{12,4}$	4	20	$F_4(a_3)$

Appendix

Appendix A

Nilpotent orbits in complex lie algebras

Summarize the parameterization of nilpotent orbits in type A, B, C, D. The dimension of such an orbit that corresponds to a partition (of a suitable type) of N that is given by $[n_i]$. Let its transpose partition be $[s_i]$. Let r_k be the number of times the number k appears in the partition $[n_i]$. Such an orbit will be denoted by $\mathcal{O}_{n_i}$. Its dimension is given by [44],

$$\begin{aligned}
\dim(\mathcal{O}_{n_i}) &= \dim(\mathfrak{g}) - \left(\sum_i s_i^2 - 1\right) && \text{for} \quad \mathfrak{g} = A_n \\
\dim(\mathcal{O}_{n_i}) &= \dim(\mathfrak{g}) - \frac{1}{2}\left(\sum_i s_i^2 - \sum_{i\in\text{odd}} r_i\right) && \text{for} \quad \mathfrak{g} = B_n, D_n \\
\dim(\mathcal{O}_{n_i}) &= \dim(\mathfrak{g}) - \frac{1}{2}\left(\sum_i s_i^2 + \sum_{i\in\text{odd}} r_i\right) && \text{for} \quad \mathfrak{g} = C_n
\end{aligned}$$

In the exceptional cases, the dimensions of the orbits can be obtained from the tables in [35, 44] (also reproduced in [39]). The closure ordering on the nilpotent orbits plays an important role in many considerations and this is often described by a Hasse diagram. It is often to instructive to look at the Hasse diagrams for just the special nilpotent orbits for the order reversing dualities act as an involution on this subset of orbits. In the exceptional cases, such diagrams are available in the Appendices of [39]. There were originally determined by Spaltenstein in [181].

Bala-Carter theory

A efficient classification system for nilpotent orbits was provided by the classification theorem of Bala-Carter. Their fundamental insight was to look for distinguished nilpotent orbits in the semi-simple part of a Levi subalgebra $\mathfrak{l}$ of a complex lie algebra $\mathfrak{g}$. Since the semi-simple parts characterize the Levi subalgebras, the BC classification is sometimes just described by a pair $(e, \mathfrak{l})$ where e is a representative of a nilpotent orbit in $\mathfrak{g}$ and $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$. A classification of all such pairs amounts to a classification of the set of all nilpotent orbits in $\mathfrak{g}$. Levi subalgebras themselves are classified by subsets of the set of simple roots. By providing a classification of all distinguished nilpotent elements in all Levi subalgebras, Bala-Carter effectively provided a classification scheme for all nilpotent orbits. This complements the classification by partition labels in the classical cases and is somewhat indispensable in the exceptional cases for which there is no partition type classification. When Bala-Carter labels are specified for a nilpotent orbit, the capitalized part of the label identifies a parabolic subalgebra $\mathfrak{p}$ whose Levi part is Levi subalgebra $\mathfrak{l}$. If there is a further Cartan type label enclosed within parenthesis, this denotes a non-principal nilpotent orbit in that Levi subalgebra. If there is no further label attached, then it is a principal nilpotent orbit in the Levi subalgebra $\mathfrak{l}$. For example, $E_6(a_1)$ and D_5 are the BC labels for two different nilpotent orbits in E_6. The former is not principal Levi type while the latter is.

While it is not absolutely necessary, it is sometimes instructive to assign BC labels to nilpotent orbits in the classical cases as well. So, it is useful to summarize it here (see [13, 158] for more in this regard). Let $[n_i]$ be the partition describing a classical nilpotent orbit ρ and let $\mathfrak{l}$ be the Bala-Carter Levi [1]

- type A : For the orbit corresponding to the partition $[n_i]$, $\mathfrak{l}$ is of Cartan type $A_{n_1-1} + A_{n_2-1} + \ldots$

- type B, D : If n_i are all distinct and odd, then ρ is distinguished in $\mathfrak{l} = B_n \text{or} D_n$, where $2n+1 = \sum_i n_i$ or $2n = \sum_i n_i$. For every pair of n_i that are equal (say to n), add a factor of A_{n-1} to $\mathfrak{l}$ and form a reduced partition with the repeating pair removed. Proceed inductively, till the reduced partition is empty. If the final partition is a [3], then add a factor $\tilde{A}_1$. It follows that the principal Levi type orbits have BC labels of the form $A_{i_1} + A_{i_2} + \ldots + \tilde{A}_1$ or $A_{i_1} + A_{i_2} + \ldots + B_n/D_n$.

- type C : If n_i are all distinct and even, then ρ is distinguished in $\mathfrak{l} = C_n$, where $2n = \sum_i n_i$. For every pair of n_i that are equal (to n, say), add a factor of $\tilde{A}_{n-1}$ to $\mathfrak{l}$ and form a reduced partition with the repeating pair removed. Proceed inductively, till the reduced partition is empty. If the final partition is a [2], then add a factor of A_1. This implies

[1] No relationship is implied here to any of the subalgebras in the main body of the dissertation. Bala-Carter theory will be used on both $\mathfrak{g}$ and $\mathfrak{g}^\vee$ sides and the corresponding notation is introduced therein.

the principal Levi type orbits have BC labels $\tilde{A}_{i_1} + \tilde{A}_{i_2} + \ldots + A_1$ or $\tilde{A}_{i_1} + \tilde{A}_{i_2} + \ldots + C_n$.

In the exceptional cases, the nilpotent orbits that are principal Levi type are immediately identifiable for they are always written in terms of their BC labels.

Appendix B

Representation theory of Weyl groups

B.1 Irreducible representations of Weyl groups

Here, the notation that is used in [35] to describe irreducible representations of Weyl groups is summarized. In the classical cases, there are certain combinatorial criteria for an irrep to be a special representation and for a set of representation to fall in the same family. These are also reviewed briefly. A general feature obeyed by all Weyl groups is that the trivial representation and the sign representation are special and consequently, they fall into their own families.

B.1.1 type A_{n-1}

The irreducible representation of $W[A_n] = S_n$ are given by partitions of n. The convention is that $[n]$ corresponds to the trivial representation while $[1^n]$ corresponds to the sign representation. All irreducible representations are special and they occur in separate families.

B.1.2 type B_n & C_n

The irreducible representations are classified by two partitions $[\alpha].[\beta]$ where $[\alpha]$ and $[\beta]$ are each partitions of p, q such that $p + q = n$. To each such pair of partitions $[\alpha].[\beta]$, associate a symbol in the following way.

- For each ordered pair $[\alpha].[\beta]$, enlarge α or β by adding trailing zeros if necessary such α has one part more than β.

- Then consider the following array :

$$\begin{pmatrix} \alpha_1 & & \alpha_2+1 & & \dots & & \alpha_{m+1}+m \\ & \beta_1 & & \beta_2+1 & \dots & \beta_m+(m-1) & \end{pmatrix}$$

- Apply an equivalence relation on such arrays in the following fashion :

$$\begin{pmatrix} 0 & & \lambda_1+1 & & \dots & & \lambda_m+1 \\ & 0 & & \mu_1+1 & \dots & \mu_m+1 & \end{pmatrix} \sim \begin{pmatrix} 0 & & \lambda_1 & & \dots & & \lambda_m \\ & 0 & & \mu_1 & \dots & \mu_m & \end{pmatrix}$$

- Each pair $[\alpha].[\beta]$ then provides a unique equivalence class of arrays. Let a representative for such an array be
$\begin{pmatrix} 0 & & \lambda_1 & & \dots & & \lambda_m \\ & 0 & & \mu_1 & \dots & \mu_m & \end{pmatrix}$

- This is the *symbol* for the corresponding irreducible representation.

Two irreps $[\alpha].[\beta]$ and $[\alpha'].[\beta']$ fall in the same family if and only if their symbols are such that their symbols contains the same $\{\lambda_i, \mu_i\}$ (treated as unordered sets). Within the set of all irreps that fall in a family, there is a unique irrep whose for which the associated symbol satisfies an ordering property :

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \dots \mu_m \leq \lambda_{m+1}. \tag{B.1}$$

This unique representation within the family is the special representation.

B.1.3 type D_n

The irreducible representations are classified again by pairs of partitions $[\alpha][\beta]$, with α, β being partitions of p, q such that $p+q=n$ but with one additional caveat. If $\alpha = \beta$, then there are two irreducible representations corresponding to this pair $([\alpha].[\alpha])'$ and $([\alpha].[\alpha])''$. Now, associate a symbol to this irrep by the following steps

- Write $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$ as non-decreasing strings of integers. Add a few leading zeros if needed such that α, β have the same number of parts. Now, consider the array $\begin{pmatrix} \alpha_1 & \alpha_2+1 & \ldots & \alpha_m+m-1 \\ \beta_1 & \beta_2+1 & \ldots & \beta_m+m-1 \end{pmatrix}$

- Impose the following equivalence relation on such arrays

$$\begin{pmatrix} 0 & \lambda_1+1 & \lambda_2+1 & \ldots & \lambda_m+1 \\ 0 & \mu_1+1 & \mu_2+1 & \ldots & \mu_m+1 \end{pmatrix} \sim \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_m \\ \mu_1 & \mu_2 & \ldots & \mu_m \end{pmatrix}$$

- Each $[\alpha].[\beta]$ now determines a unique equivalence class of such arrays. A representative of that equivalence class is the symbol of the irrep.

Two irreps $[\alpha].[\beta]$ and $[\alpha'].[\beta']$ ($\alpha \neq \beta, \alpha' \neq \beta'$) fall in the same family if their symbols are such that the λ_i, μ_i occurring in them are identical (when treated as unordered sets). Within such a family, there is a unique irrep whose symbol satisfies the following ordering property,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \ldots \lambda_m \leq \mu_m \qquad \text{or} \qquad \mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \ldots \mu_m \leq \lambda_m. \tag{B.2}$$

This unique irrep would be the special representations in that family. Irreps corresponding to labels of type $([\alpha].[\alpha])'$ and $([\alpha].[\alpha])''$ are always special and hence occur in their own families.

As an example of the application of the method of symbols, the irreps of D_4 and their corresponding symbols are noted in a table.

Table B.1: Symbols for irreducible representations of $W(D_4)$

$[\alpha].[\beta]$	Symbol
$[4].[-]$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
$[3,1].[-]$	$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$
$[2,2].[-]$	$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$
$[2,1^2].[-]$	$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$
$[1^4].[-]$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix}$
$[3].[1]$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
$[2,1].[1]$	$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$
$[1^3].[1]$	$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}$
$[2].[2]$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$
$[2].[1^2]$	$\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$
$[1^2].[1^2]$	$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

Table B.2: Character table for $W(D_4)$

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}
$[-].[1^4]$	1	1	1	-1	-1	-1	1	1	1	1	1	-1	-1
$([11].[11])'$	3	-1	3	-1	1	-1	3	-1	-1	0	0	-1	-1
$([11].[11])''$	3	-1	3	-1	1	-1	-1	3	-1	0	0	1	-1
$[1].[1^3]$	4	0	-4	-2	0	2	0	0	0	1	-1	0	0
$[1^2].[2]$	6	-2	6	0	0	0	-2	-2	2	0	0	0	0
$[1].[21]$	8	0	-8	0	0	0	0	0	0	-1	1	0	0
$[-].[2,1^2]$	3	3	3	-1	-1	-1	-1	-1	-1	0	0	1	1
$[2].[2]$	3	-1	3	1	-1	1	3	-1	-1	0	0	1	-1
$[2].[2]$	3	-1	3	1	-1	1	-1	3	-1	0	0	-1	-1
$[-].[2^2]$	2	2	2	0	0	0	2	2	2	-1	-1	0	0
$[1].[3]$	4	0	-4	2	0	-2	0	0	0	1	-1	0	0
$[-].[1,3]$	3	3	3	1	1	1	-1	-1	-1	0	0	-1	-1
$[-].[4]$	1	1	1	1	1	1	1	1	1	1	1	1	1

As can be seen from the symbols, the only non-trivial family in the case of D_4 is $\{([2,1],[1]), ([2^2],-), ([2],[1^2])\}$.

It is also useful to have the character table of $W(D_4)$ (see Table B.2) since it be used to compute tensor products with the sign representation.

where the conjugacy classes c_i are

$$c_1 = 1^4.-$$

$$c_2 = 11.11$$

$$c_3 = -.1^4$$

$$c_4 = 21^2.-$$

$$c_5 = 1.21$$

$$c_6 = 2.1^2$$

$$c_7 = (2^2.-)'$$

$$c_8 = (2^2,-)''$$

$$c_9 = (-.22)$$

$$c_{10} = 31.-$$

$$c_{11} = -.31$$

$$c_{12} = (4.-)'$$

$$c_{13} = (4.-)''$$

B.1.4 Exceptional cases

The irreps will be denoted by $\phi_{i,j}$, where i is the degree and j is the b value of the irreducible representation. In the non-simply laced cases of G_2 and F_4, there might be more than one representation with same degree and b value. When this occurs, the two representations are distinguished by denoting them as $\phi'_{i,j}$ and $\phi''_{i,j}$ respectively. For example, G_2 has $\phi'_{1,3}$ and $\phi''_{1,3}$.

Here, note that these two labels will be interchanged if we were to exchange the long root and the short root of G_2. The sign and the trivial representation can be identified in this notation as being the ones with the largest b value and zero b value respectively. To give a flavor for this notation in action, here is the character table for $W[G_2]$. The special representation are $\phi_{1,0}, \phi_{2,1}, \phi_{1,6}$. Every other representation (together with $\phi_{2,1}$) is a member of the only non-trivial family in $W[G_2]$.

Table B.3: Character table for $W(G_2)$

	1	$\tilde{A}_1$	A_1	G_2	A_2	$A_1 + \tilde{A}_1$
$\phi_{1,0}$	1	1	1	1	1	1
$\phi_{1,6}$	1	-1	-1	1	1	1
$\phi'_{1,3}$	1	1	-1	-1	1	-1
$\phi''_{1,3}$	1	-1	1	-1	1	-1
$\phi_{2,1}$	2	0	0	1	-1	-2
$\phi_{2,2}$	2	0	0	-1	-1	2

There is an interesting duality operation on the set of irreducible representations of the Weyl group. For the most part, this acts as tensoring by the sign representation. An important property of the special representations of a Weyl group is that they are closed under this duality operation. This can be readily seen to be true by looking at the character tables.

Appendix C

The method of Borel-de Siebenthal

The Borel-de Seibenthal algorithm [31] can be used to obtain all possible subalgebras that arise as the connected, reductive parts of centralizers of semi-simple elements in Lie algebras (See [178, 179] and references therein). The algorithm comes down to finding non-conjugate subsystems of the set of extended roots of the Lie algebra. Let π denote the set of simple roots and Π the corresponding Dynkin diagram. Now, adjoin the lowest root to π and form $\tilde{\pi}$, the set of extended roots. Associated to this is the extended Dynkin diagram $\tilde{\Pi}$. The extended Dynkin diagrams formed by this procedure are collected in Fig C.2. Now, form a sub diagram (possibly disconnected) by removing a node of $\tilde{\Pi}$ and all the lines connecting it. The resulting diagram corresponds to a centralizer. One can proceed by removing more nodes and lines to get all possible centralizers. There is a subset of them whose diagrams can also be obtained by considering just sub diagrams of Π. These corresponds to the centralizers of semi-simple elements that are also Levi. The more general centralizers are called pseudo-Levi in Chapter 3 (following [179]). There, pseudo-Levi subalgebras of $\mathfrak{g}^\vee$ play an important role and these are denoted by $\mathfrak{l}^\vee$. Among the pseudo-Levi subalgebras $\mathfrak{l}^\vee$ that fail to be Levi subalgebras, a particularly interesting class are the ones for which their Langlands dual $\mathfrak{l}$ fails to be a subalgebra of $\mathfrak{g}$ (the Langlands dual of $\mathfrak{g}^\vee$). It follows immediately from the Borel-de Seibenthal procedure that such a scenario can occur only for $\mathfrak{g}$ being non-simply laced. Some examples of these more interesting occurrences are collected here.

C.1 Centralizer that is not a Levi

Consider the extended Dynkin diagram for D_4 and denote it by $\tilde{\Pi}(D_4)$. There is a sub diagram which is of Cartan type $4A_1$ that does not arise as a sub diagram of $\Pi(D_4)$. This corresponds to a pseudo-Levi subalgebra that is not a Levi subalgebra.

C.2 Pseudo-Levi $\mathfrak{l}^\vee$ such that Langlands dual $\mathfrak{l} \not\subseteq \mathfrak{g}$

Consider the extended Dynkin diagram for $\mathfrak{g}^\vee = B_{n+1}$ given by $\tilde{\Pi}(B_{n+1})$. There is a sub diagram which corresponds to a centralizer $\mathfrak{l}^\vee$ of Cartan type D_n. Taking Langlands duals, one gets $\mathfrak{g} = C_{n+1}$ and $\mathfrak{l} = D_n$. But, D_n is not a subalgebra of B_{n+1}.

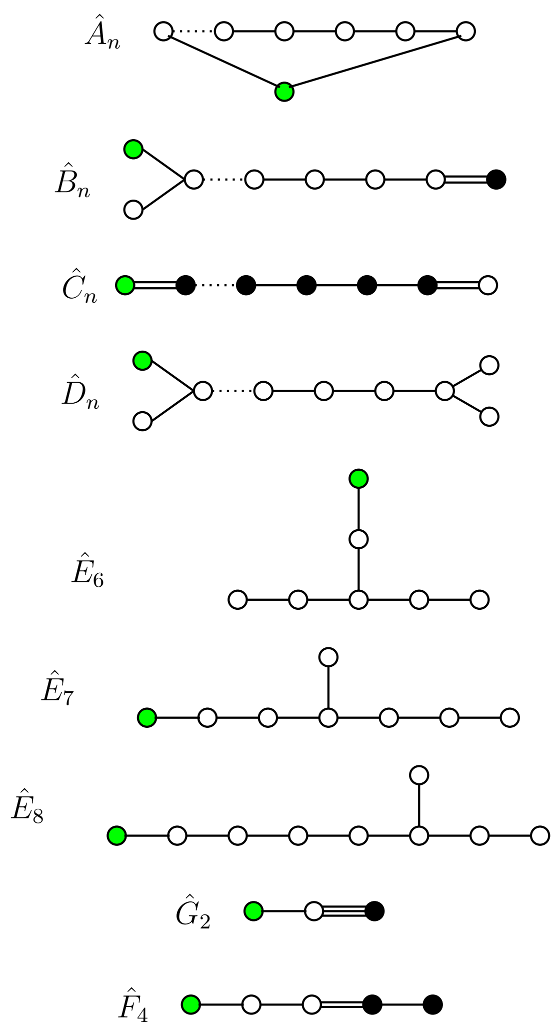

Figure C.1: Extended Dynkin diagrams

Appendix D

MacDonald-Lusztig-Spaltenstein (j-) induction

This is a general procedure that can be used to generate irreducible representations of a Weyl group $W[\mathfrak{g}]$ from irreducible representations of parabolic subgroups W_p. One can use this method to generate a large number of the irreducible representations of $W[\mathfrak{g}]$. In types A,B,C, one can actually generate *all* of them by j-induction. In other types, there is often quite a few irreducible representations that can't be obtained by j induction. A special case of this method that involves induction only from the sign representation of the parabolic subgroup W_p was developed originally by MacDonald[141].

D.1 MacDonald induction

Let W_p be a parabolic subgroup of the Weyl group $W[\mathfrak{g}]$. This is equivalent to saying that W_p is the Weyl group of a Levi subalgebra of $\mathfrak{g}$. Then, consider the positive root e_α in the root system corresponding to W_p. The positive roots are linear functionals on $\mathfrak{h}$. Form the following rational polynomial,

$$P = \prod_{e_\alpha > 0} e_\alpha. \tag{D.1}$$

Let w be an element of the Weyl group $W[\mathfrak{g}]$. Consider the algebra formed by all polynomials of the form $w(P)$. This is a subalgebra of the symmetric algebra and is naturally a $W[\mathfrak{g}]$ module. In fact, it furnishes an irreducible representation of the Weyl group $W[\mathfrak{g}]$. By choosing different subgroups W_p, one obtains different irreps of $W[\mathfrak{g}]$. This is a special case of j induction where one uses the sign representation of the smaller Weyl group to induce from. Within the notation of the more general j-induction, the MacDonald method would correspond to $j_{W_p}^{W}(sign)$.

D.2 MacDonald-Lusztig-Spaltenstein induction

The generalization of the MacDonald method to what is called j induction was provided by Lusztig- Spaltenstein in [140]. What follows is a very brief review. See [35, 94] for more detailed expositions.

Let V be a vector space on which $W[\mathfrak{g}]$ acts by reflections. Let W_r now be any reflection subgroup of $W[\mathfrak{g}]$. Let V^{W_r} be the subspace of V fixed by reflections in W_r. There is a decomposition $V = \bar{V} \oplus V^{W_r}$. Consider the space of homogeneous polynomial functions on $\bar{V}$ of some degree d and denote it by $P_d(\bar{V})$. Let r' be *any* univalent irrep of W_r. This means that r' occurs with multiplicity one in $P_d(\bar{V})$ for some d. The $W[\mathfrak{g}]$ module generated by r' is irreducible and univalent and it denoted by $j_{W_r}^{W}(r')$. When, r' is the sign representation and W_r is the Weyl group of Levi subalgebra (= a parabolic subgroup), this reduces to the MacDonald method.

The action of j induction is most transparent in type A. For types B, C, D, it can still be described by suitable combinatorics. However, in practice, it is most convenient to use packages like CHEVIE to calculate j induction. Below, some sample cases are recorded.

D.2.1 j-induction in type A

In type A, one can get all irreducible representations using j induction of the sign representation from various parabolic subgroups. The various Levi subalgebras in type A have a natural partition type classification and consequently, so do their Weyl group. Let W_P be a parabolic subgroup of partition type P. Let, P^T be the transpose partition. Then, $j_{W_P}^W = P^T$, where P^T is the partition label for the irreducible representations of S_n.

D.2.2 Example : j-induction in A_3

Here is a detailed example of j induction in action for type A. Introduce the following subgroups of the Weyl group S_4 by their Deodhar-Dyer labels (which are used in CHEVIE to index reflection subgroups). The label is of the form $[r_1, r_2 \ldots]$ and corresponds to a subset of the set of positive roots (in the ordering used by CHEVIE). By a theorem of Deodhar & Dyer [53, 67], this is a characterization of non-conjugate reflection subgroups.

Subgroup	Deodhar-Dyer label	Cartan type of assoc. subalgebra
$W_{[4]}$	$[r_1, r_2, r_3]$	A_3
$W_{[3,1]}$	$[r_1, r_2]$	A_2
$W_{[2,2]}$	$[r_1, r_3]$	$A_1 + A_1$
$W_{[2,1^2]}$	$[r_1]$	A_1
$W_{[1^4]}$	$[\varnothing]$	$\varnothing$

Denote the irreducible representation of $W = S_4$ by the usual partition labels ($[1^4]$ is the sign representation while $[4]$ is the identity representation). Applying j-induction using the sign representation in each of the subgroups above, one gets

$$\begin{aligned} j^W_{W_{1,2,3}}(sign) &= [1^4] \\ j^W_{W_{1,2}}(sign) &= [2,1^2] \\ j^W_{W_{1,3}}(sign) &= [2,2] \\ j^W_{W_1}(sign) &= [3,1] \\ j^W_{W_\varnothing}(sign) &= [4] \end{aligned}$$

D.2.3 Example : j-induction in D_4

Introduce the following subgroups of $W(D_4)$ using Deodhar-Dyer labels,

Subgroup	Deodhar-Dyer label	Cartan type
$W_{1,2,3,4}$	$[r_1, r_2, r_3, r_4]$	D_4
$W_{2,3,4}$	$[r_1, r_3, r_4]$	A_3
$W_{1,3,4}$	$[r_2, r_3, r_4]$	A_3
$W_{1,2,3}$	$[r_1, r_2, r_3]$	A_3
$W_{1,2,4,12}$	$[r_1, r_2, r_4, r_{12}]$	$4A_1$
$W_{1,3}$	$[r_1, r_3]$	A_2
$W_{3,10}$	$[r_3, r_{10}]$	$2A_1$
$W_{1,12}$	$[r_1, r_{12}]$	$2A_1$
$W_{1,2}$	$[r_1, r_2]$	$2A_1$
W_1	r_1	A_1
$W_\varnothing$	$[\varnothing]$	$\varnothing$

One obtains the following results useful for j-induction,

$$\begin{aligned}
j^W_{W_{1,2,3,4}}(sign) &= [1^4].- \\
j^W_{W_{1,2,3,4}}([1^3].[1]) &= [1^3].[1] \\
j^W_{W_{2,3,4}}(sign) &= ([1^2].[1^2])' \\
j^W_{W_{1,3,4}}(sign) &= ([1^2].[1^2])'' \\
j^W_{W_{1,2,3}}(sign) &= ([2].[1^2])'' \\
j^W_{W_{1,2,4,12}}(sign) &= [2^2].- \\
j^W_{W_{1,3}}(sign) &= [2,1].[1] \\
j^W_{W_{1,2}}(sign) &= [3,1].- \\
j^W_{W_{3,10}}(sign) &= ([2].[2])' \\
j^W_{W_{1,4}}(sign) &= ([2].[2])'' \\
j^W_{W_1}(sign) &= [3].[1] \\
j^W_{W_\varnothing}(sign) &= [4].-
\end{aligned}$$

The choice of the subgroups and the resulting irreps is no accident. The irreducible representations obtained here by j induction are precisely the Orbit representations for D_4 and they occur as $\bar{r}$ in Table 5.2.

D.2.4 Example : j-induction in G_2

As a final example of j induction, here are some results for G_2 that are relevant for the compiling of Table 5.11. Introduce the following subgroups of $W(G_2)$.

Subgroup	Deodhar-Dyer label	Cartan type
$W_{1,2}$	$[r_1, r_2]$	G_2
$W_{2,3}$	$[r_2, r_3]$	A_2
$W_{2,6}$	$[r_2, r_6]$	$A_1 \times A_1$
W_1	$[r_1]$	A_1
$W_\varnothing$	$[\varnothing]$	$\varnothing$

With this, one can note the following instances of j induction,

$$j^W_{W_{1,2}}(sign) = \phi_{1,6}$$

$$j^W_{W_{2,3}}(sign) = \phi''_{1,3}$$

$$j^W_{W_{2,6}}(sign) = \phi_{2,2}$$

$$j^W_{W_1}(sign) = \phi_{2,1}$$

$$j^W_{W_\varnothing}(sign) = \phi_{1,0}$$

The instances of j induction were again chosen such that the result is an Orbit representation of G_2. An important observation due to Lusztig is that for any arbitrary Weyl group, the Orbit representations can always be obtained by j induction.

Appendix E

Functional determinants and Special functions

E.1 Behaviour of functional determinants under scaling

Zeta function regularization is often used in the determination of functional determinants. The general strategy is the following. Let A be the operator of interest. Forming a zeta function using the eigenvalues A :

$$\zeta^A(s) = \sum_n (\lambda_n)^{-s}.$$

This is typically convergent for $s > \sigma$ for some $\sigma \in \mathbb{R}$. In many cases, this function can be analytically continued to arbitrary values of s upto some poles that are away from $s = 0$. This allows us to write the product of eigenvalues (formally) as

$$\zeta^{A'}(0) = -\log(\prod_n \lambda_n).$$

Inverting this identity give us the regularized value for $\det(A)$

$$\det(A) = \prod_n \lambda_n = e^{-\zeta^{A'}(0)}.$$

Such regularizations often find use in problems that involve evaluating Gaussian path integrals on curved manifolds. In such cases, A is typically an elliptic or a transversally elliptic operator that occurs in the quadratic part of the action.

Let us now consider a scale transformation that changes the metric as $\tilde{g} = k^{-1}g$ and leads to a change in the eigenvalues as $\tilde{\lambda_n} = k\lambda_n$. The zeta function built out of $\tilde{\lambda_n}$ is related to the original one by

$$\zeta^{A_k}(s) = k^{-s}\zeta^A(s).$$

Writing a regularized form of $\det(A_k)$ in terms of the original zeta function now requires an additional (=anomalous) term in the analogue of (E.1),

$$\zeta^{A'}(0) - (\log k)\zeta^A(0) = -\log(\prod_n \tilde{\lambda_n}).$$

Inverting this,

$$\det(\tilde{A}) = \prod_n \tilde{\lambda_n} = k^{\zeta^A(0)} e^{-\zeta^{A'}(0)}.$$

Factors of the form $k^{\zeta^A(0)}$ play an important role in Chapter 4.

E.2 Special function redux

Some properties of the special functions that are used in Chapter 4 are collected here. For a more detailed treatment of the analytical properties of these functions and a summary of the identities they obey, see [182]. The Barnes double zeta function and the Hurwitz zeta function have the following sum representations

$$\zeta_2^B(s; a, b, x) = \sum_{m,n=0} (am + bn + x)^{-s}, \tag{E.1}$$

$$\zeta^H(s, x) = \sum_{m=0} (n + x)^{-s}. \tag{E.2}$$

The derivatives at $s = 0$ of these zeta functions are related to $\Gamma_2(x)$ and $\Gamma(x)$ in the following way,

$$\zeta_2'^B(0; a, b, x) = \log(\Gamma_2(x; a, b)) + \text{const}, \tag{E.3}$$

$$\zeta_H'(0, x) = \log(\Gamma(x)) + \text{const}. \tag{E.4}$$

The Υ function that is often used in Liouville/Toda theory is defined as

$$\Upsilon(x; b, b^{-1}) = \frac{1}{\Gamma_2(x; b, b^{-1})\Gamma_2(Q - x; b, b^{-1})}, \tag{E.5}$$

where $Q = b + b^{-1}$. The derivative of the Υ function at $x = 0$ also plays an important role in the DOZZ/FL correlators. It is given by,

$$\Upsilon_0 = \frac{d\Upsilon(x)}{dx}|_{x=0} = \Upsilon(b), \tag{E.6}$$

where the final equality follows from the asymptotic properties of $\Upsilon(x)$ [192, 68]. Under a scaling transformation, $\Upsilon(x)$ has the following behaviour (this follows from the discussion in Section 2) ,

$$\Upsilon(\mu x; \mu\epsilon_1, \mu\epsilon_2) = \mu^{2\zeta_2^B(0,x;\epsilon_1,\epsilon_2)}\Upsilon(x; \epsilon_1, \epsilon_2), \tag{E.7}$$

with

$$\zeta_2^B(0, x; \epsilon_1, \epsilon_2) = \frac{1}{4} + \frac{1}{12}\left(\frac{\epsilon_1}{\epsilon_2}\right) - \frac{x}{2}\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right) + \frac{x^2}{2\epsilon_1\epsilon_2}. \tag{E.8}$$

As a shorthand, let us summarize the above scaling behaviour by saying that the scale factor for $\Upsilon(x, \epsilon_1, \epsilon_2)$ (denoted by $\mu[\Upsilon(x, \epsilon_1, \epsilon_2)]$) is $2\zeta_2^B(0, x; \epsilon_1, \epsilon_2)$. The Barnes G function (for $b = 1$) can be related to the double gamma function defined above using (see Prop 8.5 in [182])

$$G(1 + x) = \frac{\Gamma(x)}{\Gamma_2(x; 1, 1)} \tag{E.9}$$

Rewriting the above relationship in terms of derivatives of the Barnes double zeta and the Hurwitz zeta functions,

$$e^{-\zeta_2^{B'}(0,x;1,1)+\zeta^{H'}(0,x;1,1)} = G(1+x).$$

Noting that,

$$\Upsilon(\frac{Q}{2}+ix) = \frac{1}{\Gamma_2(\frac{Q}{2}+ix)\Gamma_2(\frac{Q}{2}-ix)} \tag{E.10}$$

The H function and the Υ function are related to the Barnes G function by

$$H(x) = G(1+x)G(1-x), \tag{E.11}$$

$$\Upsilon_{b=1}(x) = \frac{G(1+x)G(3-x)}{\Gamma(x)\Gamma(2-x)} \tag{E.12}$$

$$\Upsilon_{b=1}(Q/2+ix) = \frac{G(2+ix)G(2-ix)}{\Gamma(1+ix)\Gamma(1-ix)} \tag{E.13}$$

From Section 2, the scale factor for the H function (specialized to $\epsilon_1 = \epsilon_2 = 1$) is given by,

$$\mu[H(x)] = 2\zeta_2^B(0,x;1,1) - 2\zeta_H(0,x) = -\frac{1}{6} + x^2, \tag{E.14}$$

while the scale factor for the Υ function (again specialized to $\epsilon_1 = \epsilon_2 = 1$) is

$$\mu[\Upsilon(x)] = 2\zeta_2^B(0,x;1,1) = \frac{5}{6} - 2x + x^2 = -\frac{1}{6} + (1-x)^2. \tag{E.15}$$

Appendix F

Conformal Bootstrap

It is useful to recall how the conformal bootstrap procedure proceeds for Liouville theory. The basic idea is the procedure put forward in BPZ (for a detailed review, see [205]). For a modern understanding of the analytical bootstrap procedure as it is applies to the case of Liouville CFT, see [188].

Let us start with the two point function on the sphere. Conformal invariance constrains this to be of the form

$$V_{0,2} = \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle = \frac{\delta_{\alpha\beta}}{|z_1 - z_2|^\Delta}$$

The three point function is similarly constrained but not completely determined by requirements of conformal invariance.

$$V_{0,3} = C(\alpha_1, \alpha_2, \alpha_3)|z_{12}|^{-2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{-2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{-2(\Delta_2+\Delta_3-\Delta_1)}$$

The dynamics of the theory is encoded in $C(\alpha_1, \alpha_2, \alpha_3)$. The procedure of conformal bootstrap outlined in BPZ, [205] starts with the writing of the general four point function in terms of the three point functions and a special function known as the conformal block.

Let us start with a generic four point function and insert a complete set of states in between the four operators.

$$\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\mathcal{O}_{\alpha_3}\mathcal{O}_{\alpha_4}\rangle = \sum_{[\alpha]} \text{or} \int_{[\alpha]} \langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\mathcal{O}_{[\alpha]}\rangle\langle \mathcal{O}_{[\alpha]}{}^{*}\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle \tag{F.1}$$

where $[\alpha]$ denotes the conformal family associated to a primary $\mathcal{O}_\alpha$. Note that the members of the conformal family can be obtained by acting with the operators $\mathcal{L}_{-m}$ $(m > 0)$. Both symbols $\sum$ or $\int$ are included to highlight the the fact that in arbitrary cases, there may be a continuous integral and a discrete sum involved. However, it is the integral sign that is employed in Chapter 5. This is done to simplify notation.

Now, one can proceed by using the OPE between the first two operators to write the first term in the following way

$$\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2} = \int d\alpha C(\alpha_1,\alpha_2,\alpha) z^{\Delta_\alpha - \Delta_{\alpha_1} - \Delta_{\alpha_2}} \bar{z}^{\bar{\Delta}_\alpha - \bar{\Delta}_{\alpha_1} - \bar{\Delta}_{\alpha_2}} \mathcal{O}_{[\alpha]}$$

where,

$$\mathcal{O}_{[\alpha]} = \mathcal{O}_\alpha + \Omega_{12}^{\alpha,1} z L_{-1}\mathcal{O}_\alpha + \bar{\Omega}_{12}^{\alpha,1}\bar{z}\bar{L}_{-1}\mathcal{O}_\alpha + \Omega_{12}^{\alpha,\{1,1\}} z^2 L_{-1}^2 \mathcal{O}_\alpha + \dots .$$

The dynamics of the theory is encoded in the coefficients $\Omega_{12}^{\alpha,\{\dots\}}$ and $\bar{\Omega}_{12}^{\alpha,\{\dots\}}$ that appear in the above expansion. These constants obey a recursive set of linear equations which can be solved level by level. The final solution for $\Omega_{12}^{\alpha,\{\dots\}}$ at some low levels have the following form

$$\begin{aligned} \Omega_{12}^{\alpha,\{1\}} &= \frac{\Delta_\alpha - \Delta_{\alpha_1} - \Delta_{\alpha_2}}{2\Delta_\alpha}, \\ \Omega_{12}^{\alpha,\{1,1\}} &= \frac{(\Delta_\alpha - \Delta_{\alpha_1} - \Delta_{\alpha_2})(\Delta_\alpha - \Delta_{\alpha_1} - \Delta_{\alpha_2} + 1)}{4\Delta_\alpha(2\Delta_\alpha + 1)} - \frac{3}{2(\Delta_\alpha + 1)}\Omega_{12}^{\alpha,\{1\}}. \end{aligned}$$

As a simple example, consider the three point function in Liouville CFT.

F.1 $V_{(0,3)} = V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])}$

In the AGT correspondence, this is the correlator assigned to a theory of four free hypermultiplets. By DOZZ, we have

$$V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2])} = C(\alpha_1,\alpha_2,\alpha_3)|z_{12}|^{-2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{-2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{-2(\Delta_2+\Delta_3-\Delta_1)},$$

where $C(\alpha_1,\alpha_2,\alpha_3)$ is given by

$$\begin{aligned} C(\alpha_1,\alpha_2,\alpha_3) &= \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\sum_i \alpha_i)/b} \times \\ & \frac{\Upsilon(b)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)} \end{aligned}$$

Note that $\Upsilon(x)$ is an entire function except for zeros at $x = -mb-nb^{-1}$ or $x = Q+m'b+n'b^{-1}$ for $m,n,m',n' \in \mathbb{Z}^{\geq 0}$. The DOZZ three point function then has a pole when any one of the following conditions is satisfied,

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3-Q &= \Omega_{m,n}, \\ \alpha_1+\alpha_2-\alpha_3 &= \Omega_{m,n}, \\ \alpha_2+\alpha_3-\alpha_1 &= \Omega_{m,n}, \\ \alpha_3+\alpha_1-\alpha_2 &= \Omega_{m,n}, \end{aligned}$$

where $\Omega_{m,n}$ is used to denote the string of points $-mb-nb^{-1}$ and $Q+m'b+n'b^{-1}$. The set of poles matches with the screening conditions that arise from doing the path integral of the Liouville zero modes. Let us recall the general form of a screening condition for future purposes.

$$\sum_i \alpha_i + (g-1)Q = \Omega_{m,n},$$

where g is the genus and the sum is over all punctures. Starting with any one of the conditions, the other three can be obtained by single Weyl reflections $W_i : \alpha_i \to Q - \alpha_i$. Observe that overall Weyl reflections do not give a new screening condition. For example, starting with the condition $\sum \alpha - Q = \Omega_{m,n}$ and reflecting using $W : \sum \alpha \to Q - \sum \alpha$ leads to the same screening condition. This implies that the total number of screening conditions is four and not eight. Now, using the AGT primary map, the screening conditions can be rewritten in terms of the mass deformations

$$\frac{Q}{2} + m_1 + m_2 + m_3 = \Omega_{m,n}, \tag{F.2}$$

$$\frac{Q}{2} + m_1 + m_2 - m_3 = \Omega_{m,n}, \tag{F.3}$$

$$\frac{Q}{2} + m_2 + m_3 - m_1 = \Omega_{m,n}, \tag{F.4}$$

$$\frac{Q}{2} + m_3 + m_1 - m_2 = \Omega_{m,n}. \tag{F.5}$$

Observe that when any one of the hypermultiplet masses is set to zero, there is no pole since the point $Q/2$ does not belong to the string of poles $\Omega_{m,n}$ unless $Q = 0$. $Q = 0$ is possible only if $b = \pm i$. One can not naively continue the result to pure imaginary values of b since that is outside the region of analyticity of the DOZZ three point function [207, 105]. Since flat directions in the moduli space are opening up when such relations are satisfied, one would naively expect $Z_{\mathbb{S}^4}$ to diverge. But, such a direct interpretation for the pattern of divergences does not seem to be possible. The mass relations are instead encoded in the polar divisors of the integrand for $Z_{\mathbb{S}^4}$ in a Q-deformed manner. It is not immediately clear as to what physical meaning should be attributed to the lattice of poles. But, there is still something useful that one can learn from this simple example of a three point function. Namely, the number of hypermultiplets is nothing but the total number of screening conditions . This simple relation between number of screening conditions and n_h holds for all the free theories. The bootstrap program entails using insertions of complete states as in (F.1) and obtaining all higher point functions starting from the three point function. Requiring that the resulting higher point functions (on arbitrary genus surfaces) obey the crossing relations and its generalizations ends up being a very strong constraint on the three point function that it determines its analytical structure. One can work in the opposite direction as well. This would imply starting with the DOZZ three point function and then checking that the higher point functions have the required pole structure and obey crossing relations. In the example below, we will see how bootstrap produces the required pole structure as the result of an intricate interplay of various different factors. One could, ultimately, hope to understand Toda bootstrap at this level of detail.

F.2 $V_{(0,4)} = V[\mathfrak{sl}_2]_{0,([1^2],[1^2],[1^2],[1^2])}$

This is the correlator corresponding to $\mathcal{N}=2$ SYM with gauge group $SU(2)$ and $N_f = 4$. The flavor symmetry for this theory is $SO(8)$. The theory has four mass deformation parameters which can each be assigned to a $SU(2)$ flavor subgroup of $SO(8)$. These mass parameters will be related to the Liouville momenta in the following fashion

$$\alpha_i = \frac{Q}{2} + m_i$$

The eigenvalues of the mass matrix are $m_1 + m_2$, $m_1 - m_2$, $m_3 + m_4$ and $m_3 - m_4$. To write down the four point function in Liouville theory, one usually takes α_i, α to lie on the physical line. That is, $\alpha_i = Q/2 + is_i^+, \alpha = Q/2 + is^+$ for $s_i^+, s^+ \in \mathbb{R}^+$. The four point function can then be written as

$$\begin{aligned} Z_{S^4} = \quad & V_{0,4}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \\ & \int_{\alpha\in\frac{Q}{2}+is^+} d\alpha C(\alpha_1,\alpha_2,\alpha)C(Q-\alpha,\alpha_3,\alpha_4)\mathcal{F}^{34}_{12}(c,\Delta_\alpha,z_i)\mathcal{F}^{34}_{12}(c,\Delta_{Q-\alpha},\bar{z}_i) \end{aligned}$$

The fact that $\alpha \in \frac{Q}{2} + is$ implies $\bar{\alpha} = Q - \alpha$ has been used in the above equation. Now, using the symmetry of the entire integrand under the Weyl reflection $\alpha \to Q - \alpha$, the integral can be unfolded to one over $\mathbb{R}$. This gives

$$V_{0,4}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \frac{1}{2}\int_{\alpha\in\frac{Q}{2}+is} d\alpha C(\alpha_1,\alpha_2,\alpha)C(Q-\alpha,\alpha_3,\alpha_4)\mathcal{F}^{34}_{12}(c,\Delta_\alpha,z_i)\mathcal{F}^{34}_{12}(c,\Delta_{Q-\alpha},\bar{z}_i)$$

where $s \in \mathbb{R}$. Now, observe that the integrand depends just on α and not on $\bar{\alpha}$. This allows us to analytically continue the integrand to arbitrary values of α and then interpret (F.6) as a contour integral. Let us now study the analytical structure of the four point function by looking at different parts of the integrand (see [188]).

1. Although the *Vir* conformal blocks are completely constrained by symmetry, no closed form expression is known. But, its analytical properties wrt α are deduced by observing that the conformal blocks can be written as

$$\mathcal{F}(c,\Delta_i,\Delta_\alpha,z_i) = z_{13}^{-2(\Delta_1+\Delta_2+\Delta_3-\Delta_4)} z_{14}^{-2(\Delta_1+\Delta_4-\Delta_2-\Delta_3)} z_{24}^{-4\Delta_2} z_{34}^{-2(\Delta_3+\Delta_4-\Delta_1-\Delta_2)} F(c,\Delta_i,\Delta_\alpha,q)$$

   where $q = z_{12}z_{34}/z_{13}z_{24}$. $F(c,\Delta_i,\Delta_\alpha,q)$ has the following series expansion

$$F(c,\Delta_i,\Delta_\alpha,q) = q^{\Delta_\alpha-\Delta_1-\Delta_2} \sum_{i=0}^{\infty} F_i(c,\Delta_\alpha,\Delta_i) q^i$$

   Each term in the expansion can in turn be written as a ratio of two polynomials.

$$F_i = \frac{P_i(c,\Delta,\Delta_i)}{Q_i(c,\Delta)}$$

   The denominator $Q(c,\Delta_\alpha)$ is nothing but the divisor of the Kac determinant at level i. It is zero when when α takes values corresponding to degenerate representations

$$\alpha = -\frac{(m+1)b}{2} - \frac{(n+1)b^{-1}}{2}.$$

   When this condition is satisfied, there is a null vector in the Verma module at level $(m+1)(n+1)$. The zero of $Q(x,\Delta_\alpha)$ leads to a pole for $\mathcal{F}(z)$. A similar sequence of arguments show that at exactly the same values of α, $\mathcal{F}(\bar{z})$ also picks up a pole. This is because $\Delta_\alpha = \Delta_{Q-\alpha}$ and the dependence of the chiral and the anti-chiral conformal blocks on α is only through their dependence on Δ_α. So, $\mathcal{F}(z)$ and $\mathcal{F}(\bar{z})$ combine to give a double pole. However, the factor $\Upsilon(2\alpha)\Upsilon(2(Q-\alpha))$ has a double zero exactly at these values. So, they cancel.

2. The Υ functions in the denominator (from both two $C(\dots)$ factors combined) have simple poles when any one of the following conditions are satisfied

$$
\begin{aligned}
\alpha_1+\alpha_2+\alpha &= Q-\Omega_{m,n} & \alpha_1+\alpha_2+\alpha &= 2Q+\Omega_{m,n}\\
\alpha_1+\alpha_2-\alpha &= -\Omega_{m,n} & \alpha_1+\alpha_2-\alpha &= Q+\Omega_{m,n}\\
\alpha_1+\alpha-\alpha_2 &= -\Omega_{m,n} & \alpha_1+\alpha-\alpha_2 &= Q+\Omega_{m,n}\\
\alpha_2+\alpha-\alpha_1 &= -\Omega_{m,n} & \alpha_2+\alpha-\alpha_1 &= Q+\Omega_{m,n}\\
\alpha_3+\alpha_4-\alpha &= -\Omega_{m,n} & \alpha_3+\alpha_4-\alpha &= Q+\Omega_{m,n}\\
\alpha_3+\alpha_4+\alpha &= Q-\Omega_{m,n} & \alpha_3+\alpha_4+\alpha &= 2Q+\Omega_{m,n}\\
\alpha_3-\alpha-\alpha_4 &= -Q-\Omega_{m,n} & \alpha_3-\alpha-\alpha_4 &= \Omega_{m,n}\\
\alpha_4-\alpha-\alpha_3 &= -Q-\Omega_{m,n} & \alpha_4-\alpha-\alpha_3 &= \Omega_{m,n}
\end{aligned}
$$

Let us fix $\Re(\alpha_i) = Q/2$. As we will momentarily see, the integral is well defined for arbitrary values of $\Im(\alpha_i)$. One can also continue to arbitrary values of $\Re(\alpha_i)$ except when they end up satisfying a screening condition. In those cases, poles emerge because the contour gets pinched. To see these aspects, it is better to change variables. Set $\alpha_i = Q/2 + is_i$ where $s_i \in \mathbb{R}$ The above set of equations then imply strings of poles at the following values in the α-plane.

$$
\begin{aligned}
\alpha &= -\Omega_{m,n} - i(s_1 + s_2) & \alpha &= Q + \Omega_{m,n} - i(s_1 + s_2) \\
\alpha &= Q + \Omega_{m,n} + i(s_1 + s_2) & \alpha &= -\Omega_{m,n} + i(s_1 + s_2) \\
\alpha &= -\Omega_{m,n} + i(s_2 - s_1) & \alpha &= Q + \Omega_{m,n} + i(s_2 - s_1) \\
\alpha &= -\Omega_{m,n} + i(s_1 - s_2) & \alpha &= Q + \Omega_{m,n} - i(s_1 - s_2) \\
\alpha &= Q + \Omega_{m,n} + i(s_3 + s_4) & \alpha &= -\Omega_{m,n} + i(s_3 + s_4) \\
\alpha &= -\Omega_{m,n} - i(s_3 + s_4) & \alpha &= Q + \Omega_{m,n} - i(s_3 + s_4) \\
\alpha &= Q + \Omega_{m,n} + i(s_3 - s_4) & \alpha &= -\Omega_{m,n} + i(s_3 - s_4) \\
\alpha &= Q + \Omega_{m,n} + i(s_4 - s_3) & \alpha &= -\Omega_{m,n} + i(s_4 - s_3)
\end{aligned}
$$

Notice that every Υ function leads one string of left-poles (poles strictly in the region to the left of the contour) and another string of right-poles (pole strictly in the region right of the contour). It is useful to plot the poles in the α plane (See Fig F.1). The blue line indicates the position of the contour while the green lines indicate that of the poles. Note that for irrational b, all poles occur at distinct points along the line. The green lines are drawn as continuous lines just for convenience. The point on the green lines that is closest to the contour is the location of the first pole.

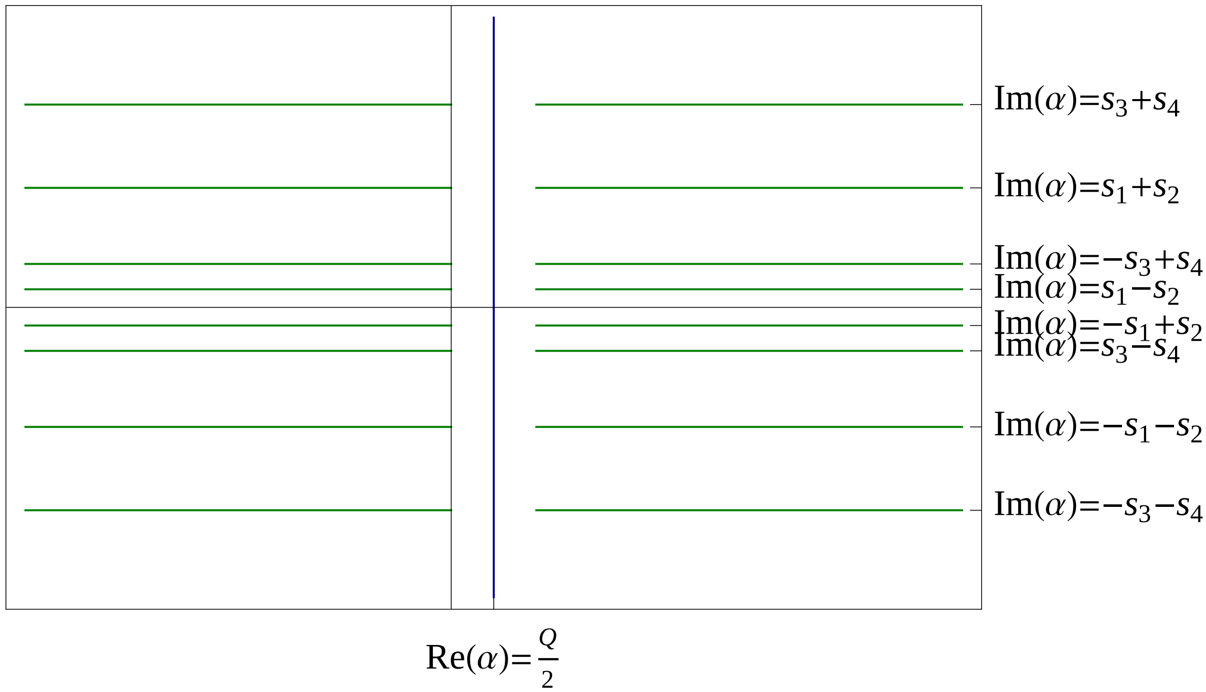

Figure F.1: Analytical structure of the integrand for $V_{0,4}$

It is useful to define an object called the set of all polar divisors of the integrand,

$$\mathcal{D}_i \equiv \{\Im(\alpha) = k | k \in \{s_1{+}s_2, -s_1{-}s_2, s_1{-}s_2, s_2{-}s_1, s_3{+}s_4, -s_3{-}s_4, s_3{-}s_4, s_4{-}s_3\}\}.$$

To define the continuation to arbitrary values of α_i, it is important to note that the poles are away from the contour as long as the following conditions are satisfied,

$$|\Re(\alpha_1 - \alpha_2)| \;<\; Q/2, \tag{F.6}$$

$$|\Re(Q - \alpha_1 - \alpha_2)| \;<\; Q/2, \tag{F.7}$$

$$|\Re(\alpha_3 - \alpha_4)| \;<\; Q/2, \tag{F.8}$$

$$|\Re(Q - \alpha_3 - \alpha_4)| \;<\; Q/2. \tag{F.9}$$

When going outside the range allowed by these inequalities, one should watch for poles to cross the contour and indent the contour correspondingly. This new contour can be rewritten as the original contour plus a finite number of circles around the poles that crossed. There are a finite number of extra terms corresponding to the residues at these poles. This prescription suffices as long as all the polar divisors $\mathcal{D}_i$ are distinct. When some of them align, the contour can get pinched when α_i takes arbitrary values. Let us called the divisors that align as $\mathcal{D}_1 \& \mathcal{D}_2$. The pinching happens when the left poles in $\mathcal{D}_1$ have moved a distance $\geq Q/2$ to the right while simultaneously, the right poles of $\mathcal{D}_2$ have moved by a distance $\geq Q/2$ to the left. If there are no new zeros emerging, such pinching leads to poles in the integral. In some cases, new zeros do emerge. The poles that arise when conditions of the form $s_i + s_i = s_i - s_j$, where (i, j) is either $(1, 2)$ or $(2, 3)$, are satisfied are canceled by the zeros of $\Upsilon(2\alpha_1), \Upsilon(2\alpha_2), \Upsilon(2\alpha_3), \Upsilon(2\alpha_4)$. But, others (say, those that follow from $s_1 + s_2 = s_3 + s_4$) will remain as poles of the integral. These are exactly the cases for which the screening condition is satisfied. As expected, the four point function has simple poles only at these values.

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