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Capacity of Interference Networks: Achievable Regions and Outer Bounds

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Capacity of Interference Networks: Achievable Regions and Outer Bounds

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To my parents

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Capacity of Interference Networks: Achievable Regions and Outer Bounds

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In an interference network, multiple transmitters communicate with multiple receivers using the same communication channel. The capacity region of an interference network is defined as the set of data rates that can be simultaneously achieved by the users of the network. One of the most important example of an interference network is the wireless network, where the communication channel is the wireless channel. Wireless interference networks are known to be interference limited rather than noise limited since the interference power level at the receivers (caused by other user's transmissions) is much higher than the noise power level.

Most wireless communication systems deployed today employ transmission strategies where the interfering signals are treated in the same manner as thermal noise. Such strategies are known to be suboptimal (in terms of achieving higher data rates), because the interfering signals generated by other transmitters have a structure to them that is very different from that of random thermal noise. Hence, there is

a need to design transmission strategies that exploit this structure of the interfering signals to achieve higher data rates. However, determining optimal strategies for mitigating interference has been a long standing open problem. In fact, even for the simplest interference network with just two users, the capacity region is unknown. In this dissertation, we will investigate the capacity region of several models of interference channels. We will derive limits on achievable data rates and design effective transmission strategies that come close to achieving the limits. We will investigate two kinds of networks - “small” (usually characterized by two transmitters and two receivers) and “large” where the number of users is large.

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Chapter 1

Introduction

An interference network is characterized by multiple transmitters communicating with multiple receivers using the same communication channel. Practical wireless networks such as the cellular networks are important examples of interference networks, where the common communication channel is the wireless channel. Wireless interference networks are known to be interference limited since the interference power (caused by signals transmitted by other users) received by each destination is at a higher level than the noise power. The capacity region of a wireless interference network is the set of data rates that different transmitter receiver pairs can achieve simultaneously, and hence determines the fundamental limits of performance of the network.

Determining the capacity region of wireless interference networks is a hard problem, because of the inherent decentralized nature of the interference network, whereby the co-operation that can be achieved between transmitters or between receivers is very limited. Most practical systems deal with interference either by treating it as noise or by separating different user's transmissions in orthogonal time/frequency/space. It is known that these transmission strategies are in general suboptimal since, interference generated by other transmitters usually have a structure to them that is significantly different from that of random thermal noise. Moreover, separating user's transmissions in orthogonal time/frequency/space affects the data rate achieved by the users, because the users are transmitting only for a limited duration or over a limited band of frequency. Hence, there is a need to design transmission strategies that exploits the structure of the interfering signals and enables simultaneous transmission by all the users. Examples of a couple of transmission strategies that do not require too much co-operation between transmitters or receivers and that are effective in mitigating interference are interference alignment and interference cancelation at receivers.

In this research, we investigate the capacity region of several classes of wireless interference networks. We derive limits on data rates that can be achieved in these networks and design effective transmission strategies that come close to achieving these limits. The rest of the chapter is organized as follows: In Section 1.1, the information theoretic model of the interference channel is introduced and the current state of this research area is summarized. The classes of interference networks that are studied in this thesis are presented in Section 1.2. The motivations for studying the selected class of interference channels are presented in Section 1.3. The thesis statement and contributions are presented in Section 1.4 and 1.5. Finally, Section 1.6 provides the organization of the rest of the thesis.

1.1 Interference Channel

In an interference channel, multiple transmitters communicate with multiple receivers using the same communication channel. Figure 1.1 depicts a general two

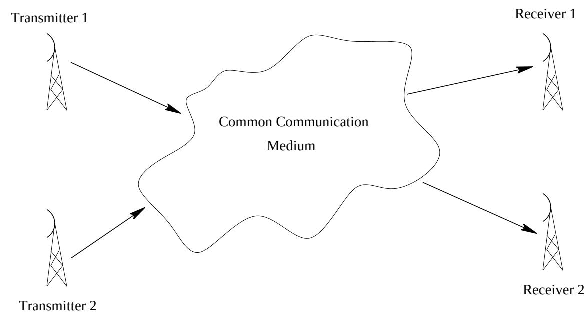


Figure 1.1: General Two User Interference Channel

user interference channel, where transmitter 1 communicates with receiver 1 and transmitter 2 communicates with receiver 2 and their transmissions interfere with each other. The capacity region of a two user interference channel is the set of data rate pairs that can be simultaneously achieved by both the users.

The interference channel was first studied from an information theoretic perspective in [1]. In [2], simple and fundamental inner and outer bounds were derived for the two user (two transmitters and two receivers) interference channel. Throughout this report, we will refer to a two user interference channel as a channel with two transmitters and two receivers as depicted in Figure 1.1.

A major breakthrough towards determining the capacity region of a two user interference channel came from the achievable rate region derived in [3], where message splitting was used as a transmission strategy. In message splitting, both the transmitters split their messages into two parts - a private part and a common part. The receivers first decode the common parts of the message transmitted by the two transmitters and then decode the private part of the message intended for them after canceling the interference caused by the common part of the messages.

In [4], an improved achievable rate region is derived by allowing each receivers to jointly decode the common messages of both transmitters and the private message of its corresponding transmitter. The achievable rate region derived in [4] is the best known achievable rate region for the two user interference channel to date. For larger interference networks with more than two users, interference alignment has been used as an effective transmission strategy in deriving order optimal achievable rate regions in [5–8].

Several outer bounds on the capacity region of interference channels have been derived over the past three decades. In [9], an outer bound for the two user discrete memoryless interference channel is derived (an interference channel is discrete memoryless if the channel inputs and outputs are discrete and the channel state is independent across time slots). In [10], an outer bound on the capacity region of a two user Gaussian interference channel is derived by allowing the two transmitters to fully co-operate with each other. In [12], Kramer derived outer bounds on Gaussian interference networks by providing extra side information to the receivers (side information is usually information about transmit signal or received signals of other transmitters and receivers). The same technique has also been used in [13–16] in deriving other outer bounds for Gaussian interference networks.

Even though interference channels have been studied for several decades, determining the capacity region of even the two user interference channel is still an open problem. The capacity region of the two user interference channel is known only for certain special cases described in [17–21]. In a recent result [13], outer bounds are derived for the two user Gaussian interference channel that differs from known inner bounds by within one bit. In other recent results [14–16], the sum capacity of the Gaussian interference channel is derived for a wide range of channel parameters (the channel gains from transmitters to receivers).

In this research, we will investigate the capacity region of several classes of interference networks. In the next section, we will introduce the different models of interference networks that we study.

1.2 Interference Network Models

The objective of this research is to derive limits on achievable data rates for several models of interference networks and design effective transmission strategies that come close to achieving the limits. The models that we will study include:

1. Cognitive Interference Networks: In a cognitive interference network, some of the transmitter nodes have some side information (about the transmit signals from other transmitters). We will study two sub-classes of cognitive interference networks.

   (a) Cognitive Radio Channel : This is a two user Gaussian interference channel with two user pairs - the licensed transmitter - receiver (tx - rx) pair and the cognitive tx - rx pair. It is assumed that the cognitive transmitter knows the message transmitted by the licensed transmitter apriori.

   (b) Cognitive Relay Network : This is a Gaussian interference network with the presence of extra cognitive relay nodes. Relay nodes serve to assist the transmitters in communicating their messages to their receivers. In this model, it is assumed that the relay nodes know the message of all the transmitters apriori.

(c) Cognitive Radio Channel in Multiple Access Networks: This is an interference network with three transmitters and two receivers. Transmitters 1 and 2 are the licensed transmitters transmitting messages in a multiple access manner to a common licensed receiver. We also have a cognitive transmitter-receiver pair communicating in the same spectrum as the licensed users. It is assumed that the cognitive transmitter knows the messages transmitted by both the licensed transmitters apriori.

(d) Cognitive Radio Channel with partial cognition: This is a two user Gaussian interference channel with two user pairs - the licensed transmitter - receiver (tx - rx) pair and the cognitive tx - rx pair. It is assumed that the cognitive transmitter knows a portion of the message transmitted by the licensed transmitter apriori.

2. Large Interference Networks: We will investigate the capacity region of k user interference channel. This is a Gaussian interference channel with k transmitters and k receivers, and each transmitter transmits an independent message to its corresponding receiver.

1.3 Motivation

The capacity region of wireless interference networks determines the set of all possible data rates that can be achieved by the users in the network. Determining the capacity region of wireless interference networks is thus an important problem. In this research, we will derive effective bounds on the capacity region of several classes of wireless interference networks. These bounds will provide insights into the capacity regions of a much wider class of wireless interference networks.

First, we will analyze the capacity region of cognitive interference networks. In the cognitive radio channel model, some transmitter nodes are cognitive and have side information on the transmission signals of other transmitters. For example, information theoretic models of cognitive radio networks assume that cognitive transmitters know the messages transmitted by other transmitters. This enables in designing transmission strategies incorporating transmitter co-operation, which is impossible to achieve in the absence of any side information. Obtaining such side information is not very impractical. In scenarios when a transmitter is located very close to another transmitter, it is possible that a transmitter is able to decode the message of the other transmitter faster than the intended receiver. The cognitive transmitter can then help the other transmitter in transmitting its information to its receiver. This model of cognitive radio channel can also serve as a new way in which software defined radios or cognitive radios can be implemented. Cognitive radios were originally thought of as devices that could communicate over the portion of the licensed spectrum unoccupied by licensed users. This model was used so that the cognitive users do not cause interference to the licensed users. By providing the cognitive radios with the message of the licensed users, we can allow the cognitive users to access the entire spectrum, while still being able to limit the interference caused to the licensed users. We also extend the channel model to study cognitive radio channels in multiple access networks. In this channel model, we have multiple licensed transmitters communicating with a common receiver in a multiple access manner. We also have a cognitive transmitter-receiver pair where the cognitive transmitter knows the messages of the licensed transmitters in an apriori manner.

Next, we will analyze cognitive relay networks. These are essentially interference networks, where additional cognitive relay nodes are deployed to assist the transmitting nodes in their communication. Relay nodes serve to increase the data rates and coverage of a network. The transmission strategies currently used in many relay based networks is the multi-hop communication, where the transmitter first transmits to the relay node and the relay node then transmits the information to the receiver. Transmission strategies that involve simultaneous transmission by both the transmitter and the relay to the receiver can lead to significantly higher data rates, particularly if the receiver is not very far from the transmitter.

In the above cognitive radio channel models, we assume that the cognitive transmitter/relay knows the messages transmitted by the other transmitters in an apriori manner. While obtaining such side information is possible in systems where the cognitive transmitter is located close to the licensed transmitters, it might not always be possible in practical networks. Hence, we study a cognitive radio channel model where the cognitive transmitter has access to only a part of the message transmitted by the licensed transmitter.

In analyzing the above mentioned cognitive interference networks, we will invariably deal with small sized networks (usually with two transmitters and two receivers). Most practical wireless interference networks are large and have many users interacting with each other over the common wireless channel. Determining the capacity region of the k user interference channel is an important step towards analyzing large wireless interference networks. In a k user interference channel, there are k transmitter-receiver pairs, and each transmitter transmits an independent message to its corresponding receiver. While a lot of research has been done on the two user interference channel, little progress has been made in analyzing the k user interference channel. The message splitting transmission strategy used in [4] for the two user interference channel is not expected to work very well for the k user case. In this dissertation, we will use lattice coding techniques to design novel interference alignment transmission strategies to determine the capacity behavior of such communication networks.

1.4 Thesis Statement

Analyzing the capacity region of a class of interference networks including various cognitive radio channel models and the general K user interference channel.

1.5 Contributions

In this research, we will derive achievable rate regions and outer bounds on the capacity region of several classes of interference channels. The contributions of this thesis are summarized below:

1. Cognitive Radio Channel : We derive an achievable rate region and an outer bound on the capacity region of a two user Gaussian cognitive radio channel, where all the transmitters and receivers have multiple antennas. This channel will be termed "Gaussian MIMO cognitive radio channel", where MIMO stands for Multiple Input and Multiple Output. In this cognitive radio channel, we assume that the cognitive transmitter knows the message of the licensed transmitter apriori. The transmission strategy used to derive the achievable rate region is based on power splitting and dirty paper coding [22] at the cognitive transmitter. The outer bound is derived through a series of channel transformations. We show that the achievable rate region and outer bound partially meet under certain channel conditions.

2. Cognitive Radio Channel in MAC Networks : We derive achievable regions and outer bounds on the capacity region of a cognitive radio channel in a multiple access network. In this channel model, we have two licensed transmitters communicating with a common receiver in a multiple access manner. We also have a cognitive transmitter-receiver pair where the cognitive transmitter knows the messages of the licensed transmitters in an apriori manner. We also derive the capacity region of such a channel model under certain

channel conditions.

3. Cognitive Relay Network : In this network setup, we study a two user Gaussian interference channel with a cognitive relay. We compute an achievable rate region by employing a transmission strategy that combines Han-Kobayashi coding scheme [4] with dirty paper coding [22]. We also derive outer bounds on the capacity region of the channel.

4. Cognitive Radio Channel with partial cognition : We consider a two user cognitive radio channel with a licensed and cognitive transmitter-receiver pair where the cognitive transmitter has access to only a portion of the message transmitted by the licensed transmitter. We derive achievable regions and outer bounds on the capacity region of such a channel model.

5. K User Interference Channel : We study the Gaussian K user interference channel model with K transmitter-receiver pairs. We use lattice coding to derive a novel interference alignment transmission strategy to analyze the capacity region of such a channel model. We derive the capacity region of such channel under certain strict symmetric channel conditions. We also determine how the sum capacity scales with increasing power in the system for a larger class of such channel models.

1.6 Organization

The rest of the dissertation is organized as follows: in chapter 2, we derive results on the capacity region of the two user MIMO cognitive radio channel. In chapter 3, we analyze the capacity region of cognitive radios in MAC networks. We study the capacity region of cognitive relay networks in chapter 4. In chapter 5, we analyze the capacity region of cognitive radio channel model with partial cognition. In chapter 6, we analyze the capacity region of K user interference channel using lattice coding schemes. Finally, we conclude in chapter 7.

Chapter 2

Capacity Region of MIMO Cognitive Radio Channel

In this chapter, we analyze the capacity region of a two user Gaussian MIMO cognitive radio channel. The cognitive radio channel we consider is a two user interference channel with a licensed transmitter-receiver pair and a cognitive transmitter-receiver pair. It is assumed that the cognitive transmitter knows the transmissions of the licensed transmitter apriori and uses this knowledge to design its own transmission signals.

2.1 Introduction

The design of radios to be "cognitive" has been identified by the Federal Communications Commission (FCC) as the next big step in better radio resource utilization [23]. The term "cognitive" has many different connotations both in analysis and in practice, but with two underlying common themes: *intelligence* built into the radio architecture coupled with *adaptivity*.

Cognitive radios have been studied under different model settings. The first models studied cognitive radios as a spectrum sensing problem [24–27]. Under this setting, the cognitive radio opportunistically uses licensed spectrum when the licensed users are sensed to be absent in that band. Problems encountered in this setup are threefold :

1. Sensing must be highly accurate to guarantee non interference with the licensed radio.

2. Control and coordination between the cognitive transmitter receiver pair is required to ensure the same spectrum is used, and finally

3. There are no QoS guarantees for the cognitive transmitter receiver pair.

Other models with different side information at the cognitive users have been studied. In [28] and [29], the authors study frequency coding by the cognitive transmitter by assuming non causal knowledge of the frequency use of the primary transmitter. Other works on this model include [30–38].

In this chapter, we study cognition from an information theoretic setting where we assume that the cognitive transmitter knows the message of the licensed transmitter apriori. Such a model is interesting for two reasons : 1) It provides an upper limit, or equivalently a benchmark on the performance of systems where the cognitive radio gains a partial understanding of the licensed transmitter and 2) It allows us to understand the ultimate limits on the cognitive transmitter by giving it maximum information and allowing it to change its transmission and coding strategy based on all the information available at the licensed user. In essence, it enlarges the possible schemes that can be implemented at the cognitive radio, and 3) It lends itself to information theoretic analysis, being a setting where such tools can be applied to determine the performance limits of the system. Many other configurations, including the interference channel setting when the cognitive transmitter does not know the message of the licensed transmitter are multi-decade long open problems.

The goal of this chapter is to study the fundamental limits of performance of cognitive radios. Along the lines of [39], we consider the model depicted in Figure 2.1. In this setting, we have an interference channel [3, 4, 11, 18], but with degraded message sets, where the transmitter with a single message is called "legacy," "primary" or "dumb" and the transmitter with both messages termed the "cognitive" transmitter. Prior work on this model for the single antenna case is in [39–43]. Recently, the capacity region of the single antenna cognitive radio channel was derived to within 1.87 bits per channel use [44, 45].

In this chapter, we study the performance of the cognitive radio model under a multiple antenna (MIMO) setting. Both the licensed and cognitive transmitter and receiver may have multiple antennas. MIMO is fast becoming the most common feature of wireless systems due to its performance benefits. Thus, it is important to study the capacity of cognitive radios under a MIMO setting. There are some instances where the methods here bears similarities with the methods used for the SISO setting. However, most of the proofs and techniques used here are distinct and considerably more involved than those used in [42]. In the SISO setting, it is possible to analyze the model for specific magnitudes of channels. This is not possible for the MIMO setting. We list some of the crucial differences between the methods used here and the methods that have been used under the SISO setting.

1. In [42], the authors obtain the outer bound using conditional entropy inequal-

ity. This method cannot be extended to the MIMO setting.

2. We obtain the outer bound through a series of channel transformations. Although the channel transformations are similar in spirit to those in [41], the actual transformations used are significantly different both in nature and in the mathematical proofs that accompany them. In [41], the authors reduce the channel to a broadcast channel where the combined transmitters have individual power constraints and the cognitive receiver has the message of the licensed user provided to it by a genie. The capacity region for such a variation of broadcast channel is not known in general. The authors solve for the capacity region of the broadcast channel using aligned channel techniques. On the other hand, we reduce the MIMO cognitive channel to a broadcast channel with sum power constraint and whose capacity region is now known [46–48]. We then use optimization techniques to compare the achievable scheme with the outer bound.

2.1.1 Main Contributions

In this chapter, our main contributions include:

1. We find an achievable region for the Gaussian MIMO cognitive channel (MCC) in a fashion analogous to [39, 41, 42].

2. We find an outer bound on the capacity region of the MCC.

3. We show that, under certain conditions (that depend on the channel parameters), the outer bound is tight for a portion of the capacity region boundary,

including points corresponding to the sum-capacity of the channel. Combining the two above, we characterize the sum capacity of this channel and a portion of its entire capacity region under certain conditions.

2.1.2 Organization

The rest of the chapter is organized as follows. We describe the notations and system model in Section 2.2. The main results are presented in Section 2.3. In Section 2.4, we present an achievable region for the Gaussian MIMO cognitive channel (MCC). An outer bound on the capacity region is shown in Section 2.5. The optimality of the achievable region for a portion of the capacity region (under certain conditions) is shown in Section 2.6. Numerical results are provided in Section 2.7. We conclude in Section 2.8.

2.2 System Model and Notation

Throughout the thesis, we use boldface letters to denote vectors and matrices. $|\mathbf{A}|$ denotes the determinant of matrix $\mathbf{A}$, while $\mathrm{Tr}(\mathbf{A})$ denotes its trace. For any general matrix or vector $\mathbf{X}$, $\mathbf{X}^\dagger$ denotes its conjugate transpose. $\mathbf{I}_n$ denotes the $n \times n$ identity matrix. $\mathbf{X}^n$ denotes the row vector $(X(1), X(2), \ldots, X(n))$, where $X(i), i = 1, 2, \ldots, n$ can be vectors or scalars. The notation $\mathbf{H} \succeq \mathbf{0}$ is used to denote that a square matrix $\mathbf{H}$ is positive semidefinite. Finally, if $\mathbf{S}$ is a set, then $\overline{S}$ denotes the closure of convex hull of S.

We consider a MIMO cognitive channel shown in Figure 2.1. Let $n_{p,t}$ and $n_{p,r}$ denote the number of transmitter and receiver antennas respectively for the licensed user. Similarly, $n_{c,t}$ and $n_{c,r}$ denotes the number of transmitter and receiver antennas for the cognitive user.

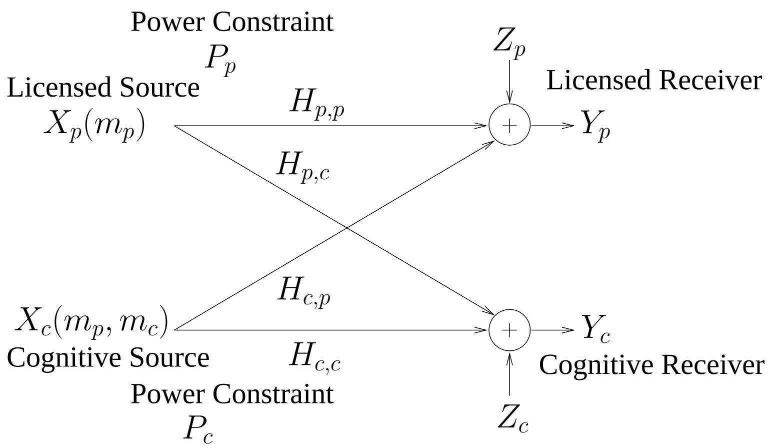


Figure 2.1: MIMO Cognitive Radio System Model

The licensed user has message $m_p \in \{1, 2, \ldots, 2^{nR_p}\}$ intended for the licensed receiver. The cognitive user has message $m_c \in \{1, 2, \ldots, 2^{nR_c}\}$ intended for the cognitive receiver as well as the message m_p of the licensed user.

The primary user encodes the message m_p into $\mathbf{X_p}^n$. Here, $\mathbf{X_p}(i)$ is a $n_{p,t}$ length complex vector. The cognitive transmitter determines its codeword $\mathbf{X_c}^n$ as a function of both m_p and m_c. Note that the cognitive transmitter wishes to communicate both m_p (to the licensed receiver) and m_c (to the cognitive receiver). The channel gain matrices are given by $\mathbf{H_{p,p}}, \mathbf{H_{p,c}}, \mathbf{H_{c,p}}$ and $\mathbf{H_{c,c}}$, and are assumed to be static. It is assumed that the licensed receiver knows $\mathbf{H_{p,p}}, \mathbf{H_{c,p}}$, the licensed transmitter knows $\mathbf{H_{p,p}}$. It is also assumed that the cognitive transmitter knows $\mathbf{H_{c,p}}, \mathbf{H_{p,c}}, \mathbf{H_{c,c}}$ and the cognitive receiver knows $\mathbf{H_{p,c}}, \mathbf{H_{c,c}}$. The received vectors of the licensed and cognitive users are denoted by $\mathbf{Y_p}^n$ and $\mathbf{Y_c}^n$ respectively.

With the above model and notations, we can describe the system at time slot i by

$$\begin{array}{c}\mathbf{Y_p}(i) = \mathbf{H_{p,p}}\mathbf{X_p}(i) + \mathbf{H_{c,p}}\mathbf{X_c}(i) + \mathbf{Z_p}(i) \\ \mathbf{Y_c}(i) = \mathbf{H_{p,c}}\mathbf{X_p}(i) + \mathbf{H_{c,c}}\mathbf{X_c}(i) + \mathbf{Z_c}(i).\end{array} \tag{2.1}$$

The additive noise at the primary and secondary receivers is denoted by ${\mathbf{Z_p}}^n$ and ${\mathbf{Z_c}}^n$ respectively. The noise vectors ${\mathbf{Z_p}}^n$ and ${\mathbf{Z_c}}^n$ are Gaussian and are assumed to be i.i.d. across symbol times and distributed according to $\mathcal{N}(0, \mathbf{I_{n_{p,r}}})$ and $\mathcal{N}(0, \mathbf{I_{n_{c,r}}})$ respectively. The correlation between ${\mathbf{Z_p}}^n$ and ${\mathbf{Z_c}}^n$ is assumed to be arbitrary. This correlation does not impact the capacity region of the system as the licensed and the cognitive decoders do not co-operate with each other. [1]

We denote the covariance of the codewords of the licensed and cognitive transmitters at time i by $\mathbf{\Sigma_p}(i)$ and $\mathbf{\Sigma_c}(i)$ respectively. Then, the transmitters are constrained by the following transmit power constraints.

$$\begin{array}{c}\sum_{i=1}^{n} \mathrm{Tr}(\mathbf{\Sigma_p}(i)) \leq nP_p \\ \sum_{i=1}^{n} \mathrm{Tr}(\mathbf{\Sigma_c}(i)) \leq nP_c.\end{array} \tag{2.2}$$

A rate pair (R_p, R_c) is said to be achievable if

1. there exists a sequence of encoding functions for the licensed and cognitive users $E_p^n : \{1, \ldots, 2^{nR_p}\} \rightarrow {\mathbf{X_p}}^n$ and $E_c^n : \{1, \ldots, 2^{nR_p}\} \times \{1, \ldots, 2^{nR_c}\} \rightarrow {\mathbf{X_c}}^n$ such that the codewords satisfy the power constraints given by (6.2),

[1] A proof of this can be obtained using steps almost exactly identical to those for the broadcast channel in [67, Exercise 15.10]

2. there exists decoding rules $D_p^n : \mathbf{Y_p}^n \to \{1, \ldots, 2^{nR_p}\}$ and $D_c^n : \mathbf{Y_c}^n \to \{1, \ldots, 2^{nR_c}\}$ such that the average probability of decoding error is arbitrarily small for suitably large values of n.

The capacity region of the Gaussian MIMO cognitive channel is the set of all achievable rate pairs (R_p, R_c) and is denoted by $\mathcal{C}_{MCC}$.

2.3 Main Results

In this section, we describe the main results of the chapter. Let $\mathbf{G} = [\mathbf{H_{p,p}} \;\; \mathbf{H_{c,p}}]$. Let $\mathcal{R}_{ach}$ denote the set described by

$$\left\{ \begin{array}{l} \Big((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \mathbf{Q}\Big) : R_p \geq 0, R_c \geq 0, \mathbf{\Sigma_p} \succeq \mathbf{0}, \mathbf{\Sigma_{c,p}} \succeq \mathbf{0}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0} \\ R_p \leq \log\left|\mathbf{I} + \mathbf{G}\mathbf{\Sigma_{p,net}}\mathbf{G}^\dagger + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}_{\mathbf{c,p}}^\dagger\right| - \log\left|\mathbf{I} + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}_{\mathbf{c,p}}^\dagger\right| \\ R_c \leq \log\left|\mathbf{I} + \mathbf{H_{c,c}}\mathbf{\Sigma_{c,c}}\mathbf{H}_{\mathbf{c,c}}^\dagger\right| \\ \mathbf{\Sigma_{p,net}} = \begin{pmatrix} \mathbf{\Sigma_p} & \mathbf{Q} \\ \mathbf{Q}^\dagger & \mathbf{\Sigma_{c,p}} \end{pmatrix} \succeq \mathbf{0}, \;\; \mathrm{Tr}(\mathbf{\Sigma_p}) \leq P_p, \;\; \mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) \leq P_c \end{array} \right\} \quad (2.3)$$

In this setting, $\mathbf{\Sigma_{p,net}}$ is a $(n_{p,t} + n_{c,t}) \times (n_{p,t} + n_{c,t})$ covariance matrix while $\mathbf{\Sigma_{c,c}}$ is a $n_{c,t} \times n_{c,t}$ covariance matrix. $\mathbf{\Sigma_p}$ and $\mathbf{\Sigma_{c,p}}$ represent principal submatrices of $\mathbf{\Sigma_{p,net}}$ of dimensions $n_{p,t} \times n_{p,t}$ and $n_{c,t} \times n_{c,t}$ respectively. The covariances matrices $\mathbf{\Sigma_p}$, $\mathbf{\Sigma_{c,p}}$ and $\mathbf{\Sigma_{c,c}}$ determine the power constraints of the system.

Let $\mathcal{R}_{in}$ be the set of rate pairs described by

$$\mathcal{R}_{in} = \overline{\left\{(R_p, R_c) : \exists\, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \mathbf{Q}, \text{ and } \Big((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \mathbf{Q}\Big) \in \mathcal{R}_{ach}\right\}} \quad (2.4)$$

Theorem 2.3.1. *The capacity region of the MCC, $\mathcal{C}_{MCC}$ satisfies*

$$\mathcal{R}_{in} \subseteq \mathcal{C}_{MCC}. \quad (2.5)$$

The proof of the theorem is given in Section 2.4. The coding strategy is based on Costa's dirty paper coding [22] [61].

We now describe an outer bound on the capacity region of the MIMO cognitive channel. Let $\alpha > 0$, $\mathbf{G}_\alpha = \left[\mathbf{H_{p,p}} \ \frac{\mathbf{H_{c,p}}}{\sqrt{\alpha}}\right]$ and $\overline{\mathbf{K}} = \begin{bmatrix} \mathbf{H_{p,p}} & \mathbf{H_{c,p}}/\sqrt{\alpha} \\ \mathbf{0} & \mathbf{H_{c,c}}/\sqrt{\alpha} \end{bmatrix}$. Let $\mathbf{\Sigma_z}$ be a covariance matrix of dimensions $(n_{p,r}+n_{c,r}) \times (n_{p,r}+n_{c,r})$ and of the form

$$\mathbf{\Sigma_z} = \begin{bmatrix} \mathbf{I}_{n_{p,r}} & \mathbf{Q_z} \\ \mathbf{Q_z}^\dagger & \mathbf{I}_{n_{c,r}} \end{bmatrix}. \tag{2.6}$$

Here, $\mathbf{Q_z}$ is a $n_{p,r} \times n_{c,r}$ matrix that makes $\mathbf{\Sigma_z}$ positive semidefinite. Let $\mathcal{R}_{conv}^{\alpha,\mathbf{\Sigma_z}}$ denote the set described by

$$\mathcal{R}_{conv}^{\alpha,\mathbf{\Sigma_z}} = \left\{ \begin{array}{l} \Big((R_p, R_c), \mathbf{Q_p}, \mathbf{Q_c}\Big) : R_p \geq 0, R_c \geq 0, \mathbf{Q_p} \succeq \mathbf{0}, \mathbf{Q_c} \succeq \mathbf{0} \\ R_p \leq \log\left|\mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_p} \mathbf{G}_\alpha^\dagger + \mathbf{G}_\alpha \mathbf{Q_c} \mathbf{G}_\alpha^\dagger\right| - \log\left|\mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_c} \mathbf{G}_\alpha^\dagger\right| \\ R_c \leq \log\left|\mathbf{\Sigma_z} + \overline{\mathbf{K}} \mathbf{Q_c} \overline{\mathbf{K}^\dagger}\right| - \log|\mathbf{\Sigma_z}| \\ \mathrm{Tr}(\mathbf{Q_p}) + \mathrm{Tr}(\mathbf{Q_c}) \leq P_p + \alpha P_c \end{array} \right\}. \tag{2.7}$$

Let $\mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}}$ denote the set of rate pairs described by

$$\mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}} = \overline{\left\{ (R_p, R_c) : \exists \mathbf{Q_p}, \mathbf{Q_c} \succeq \mathbf{0} \text{ such that } ((R_p, R_c), \mathbf{Q_p}, \mathbf{Q_c}) \in \mathcal{R}_{conv}^{\alpha,\mathbf{\Sigma_z}} \right\}}. \tag{2.8}$$

Also, let $\mathcal{R}_{out}$ be represented as

$$\mathcal{R}_{out} = \bigcap_{\Sigma_z} \bigcap_{\alpha > 0} \mathcal{R}_{out}^{\alpha}. \tag{2.9}$$

Then, the next theorem describes an outer bound on the capacity region of the MCC.

Theorem 2.3.2. *The capacity region of the MCC, $\mathcal{C}_{MCC}$ satisfies*

$$\begin{aligned} \mathcal{C}_{MCC} &\subseteq \mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}}, \forall \alpha > 0, \mathbf{\Sigma_z} \\ \mathcal{C}_{MCC} &\subseteq \mathcal{R}_{out}. \end{aligned} \tag{2.10}$$

The proof is given in Section 2.5 and proceeds by a series of channel transformations. Each channel transformation results in a new channel whose capacity region is in general a superset (outer bound) of the capacity region of the preceding channel.

Next, we discuss the optimality of the achievable region we derived and present conditions when the achievable region might meet the outer bound. Let $\mathrm{BC}(\mathbf{H_1}, \mathbf{H_2}, P)$ denote a two user MIMO broadcast channel with channel matrices given by $\mathbf{H_1}$ and $\mathbf{H_2}$ and with a transmitter power constraint of P. Let $\mathcal{C}_{BC}^{H_1,H_2,P}$ denote the capacity region of $\mathrm{BC}(\mathbf{H_1}, \mathbf{H_2}, P)$.

Let $\mathcal{R}^{\alpha}_{part,conv}$ denote the set described by

$$\left\{ \begin{array}{l} \left((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}}\right) : R_p \geq 0, R_c \geq 0, \mathbf{Q_p} \succeq \mathbf{0}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0}, \\ R_p \leq \log\left|\mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_p} \mathbf{G}_\alpha^\dagger + \frac{1}{\alpha}\mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{\Sigma}_{\mathbf{c,p}}^\dagger\right| - \log\left|\mathbf{I} + \frac{1}{\alpha}\mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}_{\mathbf{c,p}}^\dagger\right| \\ R_c \leq \log\left|\mathbf{I} + \frac{1}{\alpha}\mathbf{H_{c,c}}\mathbf{\Sigma_{c,c}}\mathbf{H}_{\mathbf{c,c}}^\dagger\right| \\ \mathrm{Tr}(\mathbf{Q_p}) + \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) \leq P_p + \alpha P_c \end{array} \right\}. \tag{2.11}$$

We let $\mathcal{R}^{\alpha}_{part,out}$ to denote the set of rate pairs described by

$$\mathcal{R}^{\alpha}_{part,out} = \overline{\left\{(R_p, R_c) : \exists \mathbf{Q_p}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0} \text{ such that } ((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}}) \in \mathcal{R}^{\alpha}_{part,conv}\right\}}. \tag{2.12}$$

Let $\mathbf{K} = [\mathbf{0}\ \mathbf{H_{c,c}}/\sqrt{\alpha}]$. We show that if the boundary of the rate region described by $\mathcal{R}^{\alpha}_{part,out}$ partially meets the boundary of the capacity region of the braoadcast channel $BC(\mathbf{G}_\alpha, \mathbf{K}, P_p + \alpha P_c)$, then the boundary of $\mathcal{R}^{\alpha}_{part,out}$ partially meets the boundary of the rate region described by $\mathcal{R}^{\alpha,\mathbf{\Sigma_z}}_{out}$ in (2.8) for some $\mathbf{\Sigma_z}$. We formally state the result in Theorem 2.3.3. For notational convenience, we denote the capacity region of $BC(\mathbf{G}_\alpha, \mathbf{K}, P_p + \alpha P_c)$ by $\mathcal{C}_{BC}^{\alpha}$.

Theorem 2.3.3. *Let $\mu \geq 1$ and $\alpha > 0$. If*

$$\max_{(R_p,R_c)\in\mathcal{R}_{part,out}^{\alpha}} \mu R_p + R_c = \max_{(R_p,R_c)\in\mathcal{C}_{BC}^{\alpha}} \mu R_p + R_c, \tag{2.13}$$

then, we have

$$\max_{(R_p,R_c)\in\mathcal{R}_{part,out}^{\alpha}} \mu R_p + R_c = \inf_{\mathbf{\Sigma_z}} \max_{(R_p,R_c)\in\mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}}} \mu R_p + R_c. \tag{2.14}$$

The proof of the theorem is described in Section 2.5. Hence, if the condition (2.13) is satisfied, the rate region described by $\mathcal{R}_{part,out}^{\alpha}$ is an outer bound on the capacity region of the MCC in terms of maximizing the μ- sum $\mu R_p + R_c$.

Let $(\hat{R}_p, \hat{R}_c)$ be a point on the boundary of the capacity region $\mathcal{C}_{MCC}$. Then, there exists a $\mu \geq 0$ such that

$$(\hat{R}_p, \hat{R}_c) = \arg \max_{(R_p,R_c)\in\mathcal{C}_{MCC}} \mu R_p + R_c.$$

The next theorem shows that if (R_p, R_c) lies on the boundary of the achievable region given by $\mathcal{R}_{in}$, then (R_p, R_c) lies on the boundary of $\mathcal{R}_{part,out}^{\alpha}$ for some $\alpha > 0$. That is, the theorem describes conditions of optimality of the achievable region $\mathcal{R}_{in}$.

Theorem 2.3.4. *For any $\mu > 0$,*

$$\max_{(R_p,R_c)\in\mathcal{R}_{in}} \mu R_p + R_c = \inf_{\alpha>0} \max_{(R_p,R_c)\in\mathcal{R}_{part,out}^{\alpha}} \mu R_p + R_c.$$

Also, there exists $\alpha^* \in (0, \infty)$, such that for any $\mu \geq 1$,

$$(R_{p,\mu}, R_{c,\mu}) = \arg \max_{(R_p,R_c)\in\mathcal{R}_{in}} \mu R_p + R_c$$

is a point on the boundary of the capacity region of the MIMO cognitive channel if the condition given by (2.13) is satisfied for α^*.

The proof of the theorem is described in Section 3.5 and is based on optimization techniques. The results in this chapter are presented in [56] [57].

2.4 Achievable Region

Proof of Theorem 2.3.1 : In this section, we show that the rate region $\mathcal{R}_{in}$ given by (5.35) is achievable on the MCC.

Encoding rule for Licensed user (E_p^n) : For every message $m_p \in \{1, \ldots, 2^{nR_p}\}$, the licensed encoder generates a n length codeword $\mathbf{X_p}^n(m_p)$, according to the distribution $p(\mathbf{X_p}^n) = \Pi_{i=1}^n p(\mathbf{X_p}(i))$, and $X_p(i) \backsim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_p})$ such that $\mathbf{\Sigma_p} \succeq \mathbf{0}$ and $\mathrm{Tr}(\mathbf{\Sigma_p}) \leq P_p$.

Encoding rule for the cognitive user (E_c^n): The cognitive encoder acts in two stages. For every message pair (m_p, m_c), the cognitive encoder first generates a codeword $\mathbf{X_{c,p}}^n(m_p, m_c)$ for the primary message m_p according to $\Pi_{i=1}^n p(\mathbf{X_{c,p}}(i)|\mathbf{X_p}(i))$, where $p(\mathbf{X_{c,p}}(i)) \backsim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_{c,p}})$ and the joint distribution of $(\mathbf{X_p}(i), \mathbf{X_{c,p}}(i))$ is given by

$$p(\mathbf{X_p}(i), \mathbf{X_{c,p}}(i)) \backsim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \mathbf{\Sigma_p} & \mathbf{Q} \\ \mathbf{Q}^\dagger & \mathbf{\Sigma_{c,c}} \end{bmatrix}\right). \tag{2.15}$$

Here, $\mathbf{Q}$ denotes the correlation between $\mathbf{X_p}(i)$ and $\mathbf{X_{c,p}}(i)$. In the second stage, the cognitive encoder generates $\mathbf{X_{c,c}}^n$ which encodes message m_c. The codeword ${\mathbf{X_{c,c}}}^n$ is generated using Costa precoding [22] by treating $\mathbf{H_{p,p}}{\mathbf{X_p}}^n + \mathbf{H_{c,c}}{\mathbf{X_{c,p}}}^n$ as non causally known interference. A characteristic feature of Costa's precoding is that ${\mathbf{X_{c,c}}}^n$ is independent of ${\mathbf{X_{c,p}}}^n$, and ${\mathbf{X_{c,c}}}^n$ is distributed as $\Pi_{i=1}^n p(\mathbf{X_{c,c}}(i))$, where $\mathbf{X_{c,c}}(i) \backsim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_{c,c}})$. Note that the codeword ${\mathbf{X_{c,p}}}^n$ is used to convey message m_p to the licensed receiver and the codeword ${\mathbf{X_{c,c}}}^n$ is used to convey message m_c to the cognitive receiver. The two codewords ${\mathbf{X_{c,p}}}^n$ and ${\mathbf{X_{c,c}}}^n$ are superimposed to form the cognitive codeword ${\mathbf{X_c}}^n = {\mathbf{X_{c,p}}}^n + {\mathbf{X_{c,c}}}^n$. It is clear that ${\mathbf{X_c}}^n$ is distributed as $\Pi_{i=1}^n p(\mathbf{X_c}(i)),\ \mathbf{X_c}(i) \backsim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_c})$, where $\mathbf{\Sigma_c} = \mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}$. The covariance matrices satisfy the constraints $\mathbf{\Sigma_{c,p}} \succeq \mathbf{0}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0}, \mathrm{Tr}(\mathbf{\Sigma_c}) \leq P_c$.

Decoding rule for the licensed receiver (D_p^n) : The licensed receiver receives $\mathbf{H_{p,p}}{\mathbf{X_p}}^n + \mathbf{H_{c,p}}({\mathbf{X_{c,p}}}^n + {\mathbf{X_{c,c}}}^n) + {\mathbf{Z_p}}^n$. It treats $\mathbf{H_{p,p}}{\mathbf{X_p}}^n + \mathbf{H_{c,p}}{\mathbf{X_{c,p}}}^n$ as the valid codeword and $\mathbf{H_{c,p}}{\mathbf{X_{c,c}}}^n + {\mathbf{Z_p}}^n$ as Gaussian noise. Taking $\mathbf{G} = [\mathbf{H_{p,p}}\ \ \mathbf{H_{c,p}}]$ and ${\mathbf{X_{p,net}}}^n = \begin{bmatrix} {\mathbf{X_p}}^n \\ {\mathbf{X_{c,p}}}^n \end{bmatrix}$, the received vector at the licensed receiver is

$$
{\mathbf{Y_p}}^n = \mathbf{G}{\mathbf{X_{p,net}}}^n + \mathbf{H_{c,p}}{\mathbf{X_{c,c}}}^n + {\mathbf{Z_p}}^n. \tag{2.16}
$$

The covariance matrix of $\mathbf{X_{p,net}}$ is denoted by $\mathbf{\Sigma_{p,net}} = \begin{bmatrix} \mathbf{\Sigma_p} & \mathbf{Q} \\ \mathbf{Q}^\dagger & \mathbf{\Sigma_{c,p}} \end{bmatrix}$, where $\mathbf{Q} = E[\mathbf{X_p}\mathbf{X}_{\mathbf{c,p}}^\dagger]$. In this setup, we use steps identical to that used for MIMO channel with colored noise in [67, Section 9.5] to show that, for any $\epsilon > 0$, there exists a block length n_1 so that for any $n \geq n_1$, the licensed decoder can recover the message m_p with probability of error $< \epsilon$ if

$$
R_p \leq \log\left|\mathbf{I} + \mathbf{G}\mathbf{\Sigma_{p,net}}\mathbf{G}^\dagger + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}_{\mathbf{c,p}}^\dagger\right| - \log\left|\mathbf{I} + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}_{\mathbf{c,p}}^\dagger\right|. \tag{2.17}
$$

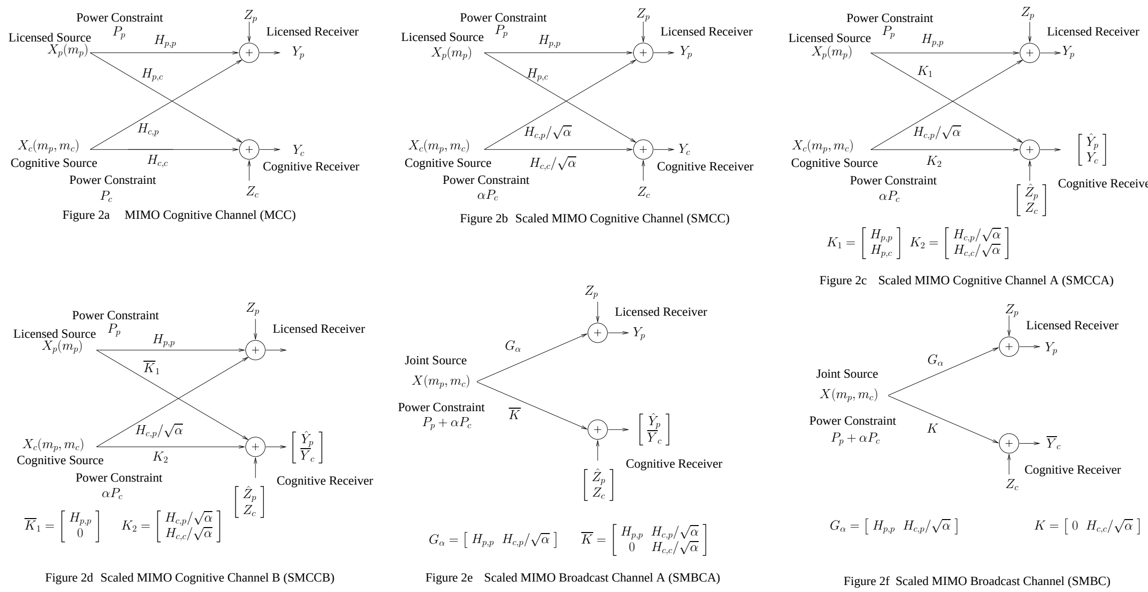
Decoding rule for the cognitive user (D_c^n) : The cognitive decoder is the Costa decoder (with the knowledge of the encoder, E_c^n). The cognitive receiver receives $\mathbf{Y_c}^n = \mathbf{H_{p,c}}\mathbf{X_p}^n + \mathbf{H_{c,c}}(\mathbf{X_{c,p}}^n + \mathbf{X_{c,c}}^n) + \mathbf{Z_c}^n$. Here, the non-causally known interference $\mathbf{H_{p,c}}\mathbf{X_p}^n + \mathbf{H_{c,c}}\mathbf{X_{c,p}}^n$ is canceled by the Costa precoder. To show this formally, we follow steps similar to Eqns (3) to (7) in [22]. We get that, for any $\epsilon_2 > 0$, there exists n_2 such that for $n \geq n_2$, the cognitive decoder can recover the message m_c with probability of error $< \epsilon_2$ if

$$R_c \leq \log \left| \mathbf{I} + \mathbf{H_{c,c}}\mathbf{\Sigma_{c,c}}\mathbf{H}^{\dagger}_{\mathbf{c,c}} \right|. \tag{2.18}$$

Note that the achievable scheme holds for all possible covariance matrices $\mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}$ that are positive semidefinite and satisfy the power constraints $\mathrm{Tr}(\mathbf{\Sigma_p}) \leq P_p$, $\mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) \leq P_c$. Hence, $\mathcal{R}_{in}$, which is the set of all achievable rate pairs described by (5.35), is achievable for any code length $n \geq \max(n_1, n_2)$.

2.5 Outer Bound on the Capacity Region

In this section, we prove that the rate region described by $\mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}}$ is an outer bound on the capacity region of the Gaussian MIMO cognitive channel. The proof proceeds by a series of channel transformations where each transformation creates an outer bound on the channel at the previous stage. At the final stage, we obtain a physically degraded broadcast channel. The capacity region of this channel is now known [46] [47] [48] and is used as the outer bound for the capacity region of the MIMO cognitive channel. Figure 2.2 depicts the various channel configurations considered, and the system equations of all the configurations. $\hat{\mathbf{Z}}_{\mathbf{p}}^n$ shown in Figures 2c, 2d and 2e has the same distribution as $\mathbf{Z_p}^n$, but has an arbitrary correlation with $\mathbf{Z_c}^n$. Before proving Theorem 2.3.2, we prove the following lemmas.



Licensed User : $Y_p = H_{p,p}X_p + H_{c,p}X_c + Z_p$

Cognitive User : $Y_c = H_{p,c}X_p + H_{c,c}X_c + Z_c$

MIMO Cognitive Channel (MCC)

Licensed User : $Y_p = H_{p,p}X_p + (H_{c,p}/\sqrt{\alpha})X_c + Z_p$

Cognitive User : $Y_c = H_{p,c}X_p + (H_{c,c}/\sqrt{\alpha})X_c + Z_c$

Scaled MIMO Cognitive Channel (SMCC)

Licensed User : $Y_p = H_{p,p}X_p + (H_{c,p}/\sqrt{\alpha})X_c + Z_p$

Cognitive User : $\hat{Y}_p = H_{p,p}X_p + (H_{c,p}/\sqrt{\alpha})X_c + \hat{Z}_p$

$Y_c = H_{p,c}X_p + (H_{c,c}/\sqrt{\alpha})X_c + Z_c$

Scaled MIMO Cognitive Channel A (SMCCA)

Licensed User : $Y_p = H_{p,p}X_p + (H_{c,p}/\sqrt{\alpha})X_c + Z_p$

Cognitive User : $\hat{Y}_p = H_{p,p}X_p + (H_{c,p}/\sqrt{\alpha}X_c + Z_p$

$Y_c = (H_{c,c}/\sqrt{\alpha})X_c + Z_c$

Scaled MIMO Cognitive Channel B (SMCCB)

Licensed User : $Y_p = \left[\, H_{p,p} \;\; H_{c,p}/\sqrt{\alpha} \,\right] X + Z_p$

Cognitive User : $\hat{Y}_p = \left[\, H_{p,p} \;\; H_{c,p}/\sqrt{\alpha} \,\right] X + \hat{Z}_p$

$Y_c = \left[\, 0 \;\; H_{c,c}/\sqrt{\alpha} \,\right] X + Z_c$

Scaled MIMO Broadcast Channel A (SMBCA)

Licensed User : $Y_p = \left[\, H_{p,p} \;\; H_{c,p}/\sqrt{\alpha} \,\right] X + Z_p$

Cognitive User : $Y_c = \left[\, 0 \;\; H_{c,c}/\sqrt{\alpha} \,\right] X + Z_c$

Scaled MIMO Broadcast Channel (SMBC)

Figure 2.2: Channel Configurations and their System Equations

Transformation 1 (MIMO Cognitive Channel (MCC) $\rightarrow$ Scaled MIMO cognitive channel) : The scaled MIMO cognitive channel is defined in Figure 2b and Figure 2.3. In this transformation, the channel matrices $\mathbf{H_{c,p}}$ and $\mathbf{H_{c,c}}$ are scaled by $1/\sqrt{\alpha}$. Also, the power constraint at the cognitive transmitter is changed to αP_c.

Lemma 2.5.1. *The capacity region of the MIMO cognitive channel is equal to the capacity region of the scaled MIMO cognitive channel (SMCC) for any* $0 < \alpha < \infty$.

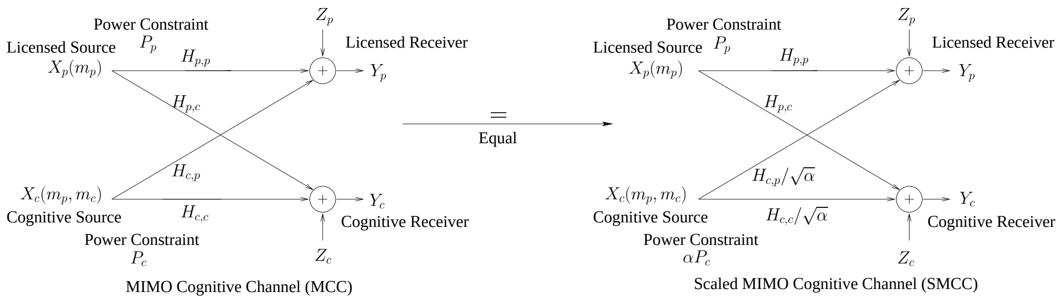


Figure 2.3: Capacity Region of MCC = Capacity Region of SMCC

Proof. Let (R_p, R_c) be a rate pair that is achievable on the MCC. That is, for all $\epsilon_1, \epsilon_2 > 0$, there exists a n and a sequence of encoder decoder pairs at the licensed and cognitive transmitter and receiver $(E_p^n : m_p \rightarrow \mathbf{X_p}^n, D_p^n : \mathbf{Y_p}^n \rightarrow \hat{m}_p, E_c^n : (m_p, m_c) \rightarrow \mathbf{X_c}^n, D_c^n : \mathbf{Y_c}^n \rightarrow \hat{m}_c)$ such that the codewords $\mathbf{X_p}^n$ and $\mathbf{X_c}^n$ satisfy the power constraints given by (6.2) and the probability of decoding error is small $(Pr(m_p \neq \hat{m}_p) \leq \epsilon_1, Pr(m_c \neq \hat{m}_c) \leq \epsilon_2)$. We use the following encoder decoder pairs at the licensed and cognitive transmitters and receivers of the scaled MIMO cognitive channel. $E_p^n : m_p \rightarrow \mathbf{X_p}^n,\ D_p^n : \mathbf{Y_p}^n \rightarrow \hat{m}_p,\ E_c^n : (m_p, m_c) \rightarrow \sqrt{\alpha}\mathbf{X_c}^n,\ D_c^n : \mathbf{Y_c}^n \rightarrow \hat{m}_c$. It follows that using these encoder and decoder pairs, the licensed and cognitive codewords satisfy the new power constraints of P_p and αP_c respectively. Also, the system equation is the same as that of the MCC and $Pr(m_p \neq \hat{m}_p) \leq \epsilon_1$ and $Pr(m_c \neq \hat{m}_c) \leq \epsilon_2$. Hence, the rate pair (R_p, R_c) is achievable on the scaled MIMO cognitive channel. Hence, the capacity region of the SMCC is a superset of the capacity region of the MCC.

Similarly, we can also establish this in the other direction, namely we can treat the MCC as the scaled version of the SMCC (scaling by $1/\alpha$). Therefore, it can be shown that the capacity region of the MCC is a superset of the capacity region of the SMCC.

Hence, the capacity region of the MCC is equal to the capacity region of the SMCC. □

Transformation 2 (scaled MIMO cognitive channel (SMCC) $\rightarrow$ scaled MIMO cognitive channel A (SMCCA)) : The scaled MIMO cognitive channel A (SMCCA) is described in Figure 2c and Figure 2.4. In this transformation, we provide a modified version of $\mathbf{Y_p}^n$, which is $\hat{\mathbf{Y}}_{\mathbf{p}}^n$ to the cognitive receiver. $\hat{\mathbf{Y}}_{\mathbf{p}}^n$ is corrupted by noise $\hat{\mathbf{Z}}_{\mathbf{p}}^n$, which has the same probability distribution as that of $\mathbf{Z_p}^n$ (i.e., complex Gaussian with zero mean and identity covariance matrix), but is permitted to be correlated with $\mathbf{Z_p}^n$ or $\mathbf{Z_c}^n$. In fact, we assume that the joint probability distribution of $(\hat{\mathbf{Z}}_{\mathbf{p}}(i), \mathbf{Z_c}(i))$ is given by

$$p(\hat{\mathbf{Z}}_{\mathbf{p}}(i), \mathbf{Z_c}(i)) = \mathcal{N}(0, \mathbf{\Sigma_z}), \tag{2.19}$$

where $\mathbf{\Sigma_z}$ has the form given by (2.6). The received vector $\hat{\mathbf{Y}}_{\mathbf{p}}^n$ is made available to the cognitive receiver by transforming the channel matrices $\mathbf{H_{p,c}}$ and $\mathbf{H_{c,c}}/\sqrt{\alpha}$ to $K_1 = \begin{bmatrix} \mathbf{H_{p,p}} \\ \mathbf{H_{p,c}} \end{bmatrix}$ and $K_2 = \begin{bmatrix} \mathbf{H_{c,p}}/\sqrt{\alpha} \\ \mathbf{H_{c,c}}/\sqrt{\alpha} \end{bmatrix}$ respectively. Hence, the received vector at the cognitive receiver is $\begin{bmatrix} \hat{\mathbf{Y}}_{\mathbf{p}}^n \\ \mathbf{Y_c}^n \end{bmatrix}$.

Lemma 2.5.2. *The capacity region of the scaled MIMO cognitive channel A (SMCCA) is a superset of the capacity region of the scaled MIMO cognitive channel (SMCC).*

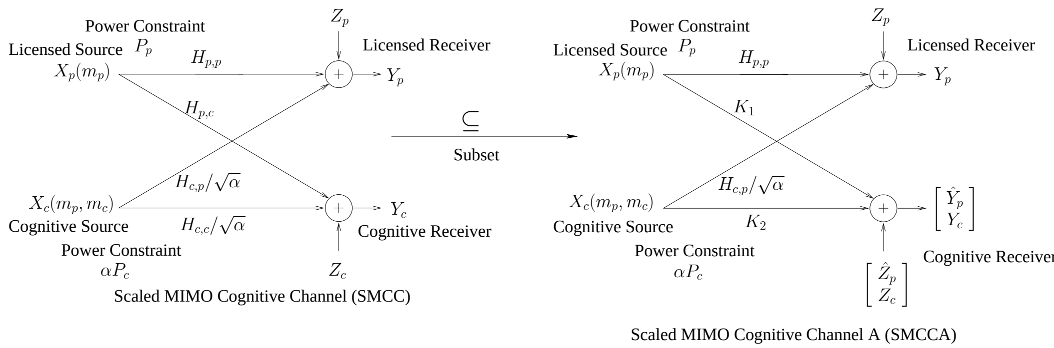


Figure 2.4: Capacity Region of SMCC $\subseteq$ Capacity Region of SMCCA

Proof. Let the rate pair (R_p, R_c) be achievable on the SMCC. That is, for all $\epsilon_1, \epsilon_2 > 0$, there exists a n and a sequence of encoder decoder pairs at the licensed and cognitive transmitter and receiver $(E_p^n : m_p \rightarrow \mathbf{X_p}^n, D_p^n : \mathbf{Y_p}^n \rightarrow \hat{m}_p, E_c^n : (m_p, m_c) \rightarrow \mathbf{X_c}^n, D_c^n : \mathbf{Y_c}^n \rightarrow \hat{m}_c)$ such that the codewords $\mathbf{X_p}^n$ and $\mathbf{X_c}^n$ satisfy the power constraints and the probability of decoding error is small $(Pr(m_p \neq \hat{m}_p) \leq \epsilon_1, Pr(m_c \neq \hat{m}_c) \leq \epsilon_2)$. In the SMCCA, we can use the same encoder decoder pair E_p^n and D_p^n at the licensed transmitter and receiver to achieve a rate R_p with probability of decoding error $< \epsilon_1$. Also, by ignoring the received vector $\hat{\mathbf{Y}}_{\mathbf{p}}^n$ at the cognitive receiver, we can use E_c^n and D_c^n at the cognitive transmitters and receivers to achieve a rate R_c with the decoding probability of error $< \epsilon_2$. Hence, the rate pair (R_p, R_c) is achievable on the scaled MIMO cognitive channel A (SMCCA). Therefore, the capacity region of the SMCCA is a superset of the capacity region of the SMCC. $\square$

Transformation 3 (scaled MIMO cognitive channel A (SMCCA) $\rightarrow$ scaled MIMO cognitive channel B (SMCCB)) : The scaled MIMO cognitive channel (B) is described in Figure 2d and Figure 2.5. The channel matrix from the licensed transmitter to the cognitive receiver is modified from $\mathbf{K_1} = \begin{bmatrix} \mathbf{H_{p,p}} \\ \mathbf{H_{p,c}} \end{bmatrix}$ to $\overline{\mathbf{K_1}} = \begin{bmatrix} \mathbf{H_{p,p}} \\ \mathbf{0} \end{bmatrix}$. Hence, the received vector at the cognitive receiver is given by $\begin{bmatrix} \hat{\mathbf{Y}}_{\mathbf{p}}^n \\ \overline{\mathbf{Y_c}^n} \end{bmatrix}$ where $\overline{\mathbf{Y_c}^n} = \frac{\mathbf{H_{c,c}}}{\sqrt{\alpha}} \mathbf{X_c}^n + \mathbf{Z_c}^n$. The intuition behind the transformation is to remove the original interference caused by the licensed transmitter to the cognitive receiver.

Lemma 2.5.3. *The capacity region of the scaled MIMO cognitive channel B (SMCCB) is equal to the capacity region of the scaled MIMO cognitive channel A (SMCCA).*

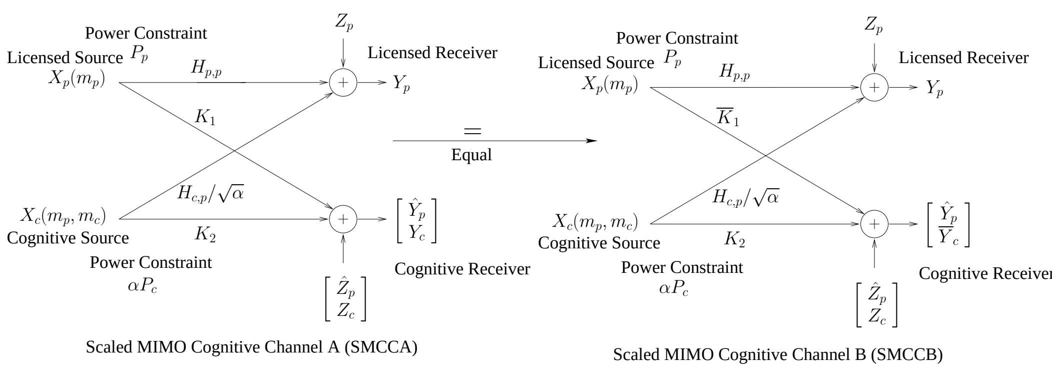


Figure 2.5: Capacity Region of SMCCA = Capacity Region of SMCCB

Proof. Let the rate pair (R_p, R_c) be achievable on the SMCCA. This implies that for every $\epsilon_1, \epsilon_2 > 0$, there exists encoder-decoder pair for the licensed user $(E_p^n(\epsilon_1), D_p^n(\epsilon_1))$ and for the cognitive user $(E_c^n(\epsilon_2), D_c^n(\epsilon_2))$ such that the probability of decoding error is less than ϵ_1 and ϵ_2 respectively for the licensed and cognitive user. Let $\delta_1, \delta_2 \in (0, 1)$. In SMCCB, the licensed user can employ $E_p^n(\min(\delta_1/2, \delta_2/2))$, $D_p^n(\min(\delta_1/2, \delta_2/2))$ to decode m_p with a probability of error $\leq \delta_1/2 < \delta_1$. The cognitive receiver uses $E_p^n(\min(\delta_1/2, \delta_2/2)),\ D_p^n(\min(\delta_1/2, \delta_2/2))$ on $\hat{\mathbf{Y}}_{\mathbf{p}}^n$ to obtain m_p with probability of error $\leq\ \delta_1/2$. The cognitive receiver can now construct ${\mathbf{X_p}}^n$ and hence $\mathbf{H_{p,c}}{\mathbf{X_p}}^n$. Thus, the cognitive receiver recovers ${\mathbf{Y_c}}^n\ =\ \mathbf{H_{p,c}}{\mathbf{X_p}}^n + \frac{\mathbf{H_{c,c}}}{\sqrt{\alpha}}\mathbf{X^n_{c,c}} + {\mathbf{Z_c}}^n$. Now, it uses, $E_c^n(\delta_2/2), D_c^n(\delta_2/2)$ to obtain m_c with probability of error $\leq\ \delta_2/2$. Clearly, the probability of error in recovering m_c is less than δ_2. Hence, the rate pair (R_p, R_c) is achievable on SMCCB. Therefore, the capacity region of SMCCB is a superset of the capacity region of SMCCA.

Let the rate pair (R_p, R_c) be achievable on SMCCB. Then, for every $\epsilon_1, \epsilon_2 > 0$, there exists encoder-decoder pair for the licensed user $(E_p^n(\epsilon_1), D_p^n(\epsilon_1))$ and for the cognitive user $(E_c^n(\epsilon_2), D_c^n(\epsilon_2))$ such that the probability of decoding error is less than ϵ_1 and ϵ_2 respectively for the licensed and cognitive user. Let $\delta_1, \delta_2 > 0$. In SMCCA, the licensed user can employ $E_p^n(\min(\delta_1/2, \delta_2/2)), D_p^n(\min(\delta_1/2, \delta_2/2))$ to decode m_p with a probability of error $\leq \delta_1/2 < \delta_1$. The cognitive user employs $E_p^n(\min(\delta_1/2, \delta_2/2)), D_p^n(\min(\delta_1/2, \delta_2/2))$ on $\hat{\mathbf{Y}}_{\mathbf{p}}^n$ to obtain m_p with probability of error $\leq \delta_2/2$. The cognitive receiver can now construct ${\mathbf{X_p}}^n$ and hence $\mathbf{H_{p,c}}{\mathbf{X_p}}^n$. Hence, the cognitive receiver subtracts $\mathbf{H_{p,c}}{\mathbf{X_p}}^n$ from ${\mathbf{Y_c}}^n$ to obtain ${\overline{\mathbf{Y}}_{\mathbf{c}}}^n$. The cognitive receiver can now use $E_c^n(\delta_2/2), D_c^n(\delta_2/2)$ to obtain m_c with probability of error $< \delta_2$. Thus, the rate pair (R_p, R_c) is achievable on SMCCA.

Therefore, the capacity region of the SMCCA is equal to the capacity region of the SMCCB. □

Transformation 4 (scaled MIMO cognitive channel (B) $\rightarrow$ scaled MIMO broadcast channel A (SMBCA)): The scaled MIMO broadcast channel A (SMBCA) is depicted in Figure 2e and Figure 2.6. We let the two transmitters to co-operate and transform it into a broadcast channel with a sum power constraint of $P_p + \alpha P_c$. The new channel matrices from the combined transmitters to the licensed and cognitive receivers are given by $\mathbf{G}_\alpha = \left[\begin{array}{cc} \mathbf{H_{p,p}} & \mathbf{H_{c,p}}/\sqrt{\alpha} \end{array} \right]$ and $\overline{\mathbf{K}} = \left[\begin{array}{cc} \mathbf{H_{p,p}} & \mathbf{H_{c,p}}/\sqrt{\alpha} \\ \mathbf{0} & \mathbf{H_{c,c}}/\sqrt{\alpha} \end{array} \right]$ respectively.

Lemma 2.5.4. *The capacity region of the scaled MIMO broadcast channel A (SMBCA) is a superset of the capacity region of scaled MIMO cognitive channel B (SMCCB).*

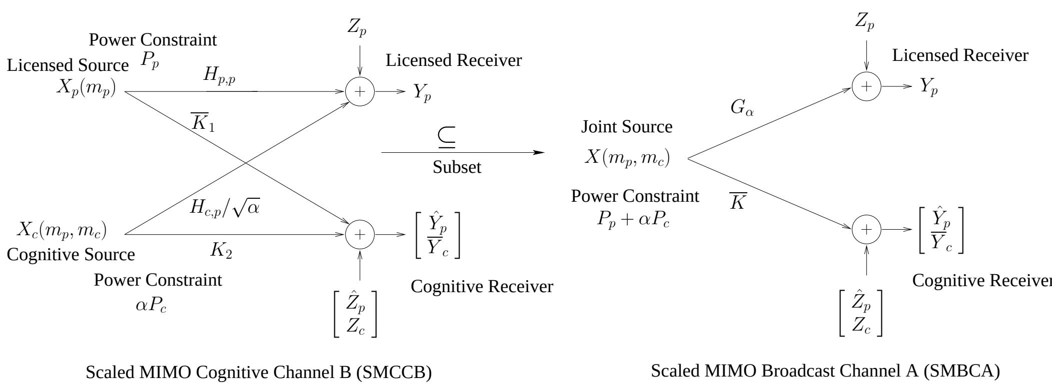


Figure 2.6: Capacity Region of SMCCB $\subseteq$ Capacity Region of SMBCA

Proof. Let the rate pair (R_p, R_c) be achievable on the SMCCB. In the SMBCA, using no collaboration between the two transmitters and using separate power constraints of P_p and αP_c respectively, we reduce the SMBCA to the SMCCB. Hence, the rate pair (R_p, R_c) is achievable on the SMBCA. Therefore, the capacity region of the SMBCA is a superset of the capacity region of the SMCCB. □

We have showed that for any $\alpha > 0$, $\mathcal{C}_{MCC} = \mathcal{C}_{SMCC} \subseteq \mathcal{C}_{SMCCA} = \mathcal{C}_{SMCCB} \subseteq \mathcal{C}_{SMBCA}$. Hence, the capacity region of the scaled MIMO broadcast channel A (SMBCA) is a superset of the capacity region of the MIMO cognitive channel (MCC).

Proof. of Theorem 2.3.2 : In the SMBCA, let $\mathbf{Q_p}$ denote the covariance matrix of the codeword for the licensed user and let $\mathbf{Q_c}$ denote the covariance matrix for the cognitive user. The SMBCA is a physically degraded broadcast channel. Hence, the capacity region of the SMBCA (as described by [46]) denoted by $\mathcal{C}_{SMBCA}$ is given by the set of rate pairs described by

$$\overline{\left\{ \begin{array}{l} (R_p, R_c) : R_p \geq 0, R_c \geq 0 \\ R_p \leq \log \left| \mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_p} \mathbf{G}_\alpha^\dagger + \mathbf{G}_\alpha \mathbf{Q_c} \mathbf{G}_\alpha^\dagger \right| - \log \left| \mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_c} \mathbf{G}_\alpha^\dagger \right| \\ R_c \leq \log \left| \mathbf{\Sigma_z} + \overline{\mathbf{K}} \mathbf{Q_c} \overline{\mathbf{K}^\dagger} \right| - \log \left| \mathbf{\Sigma_z} \right| \\ \forall \mathbf{Q_p} \succeq \mathbf{0}, \mathbf{Q_c} \succeq \mathbf{0} \text{ such that } \mathrm{Tr}(\mathbf{Q_p}) + \mathrm{Tr}(\mathbf{Q_c}) \leq P_p + \alpha P_c \end{array} \right\}}. \tag{2.20}$$

Also, this is the outer bound of the MCC. Hence, $\mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}}$ described by (2.8) is an outer bound on the capacity region of the MCC. Hence, $\mathcal{C}_{MCC} \subseteq \mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}}$. Also, $\mathcal{C}_{MCC} \subseteq \mathcal{R}_{out}$, where $\mathcal{R}_{out}$ is described in (9). □

Transformation 5 (scaled MIMO broadcast channel A (SMBCA) → scaled MIMO broadcast channel (SMBC)) : The scaled MIMO broadcast channel (SMBC) is depicted in Figure 2f and Figure 2.7. We change the received vector at the cognitive receiver from $\left[\frac{\hat{\mathbf{Y}}_\mathbf{p}^n}{\overline{\mathbf{Y}_\mathbf{c}}^n} \right]$ to $\overline{\mathbf{Y}_\mathbf{c}}^n$. This is done by changing the channel matrix from the joint transmitters to the cognitive receiver to $\mathbf{K} = \left[\begin{array}{cc} \mathbf{0} & \mathbf{H_{c,c}}/\sqrt{\alpha} \end{array} \right]$.

Lemma 2.5.5 ([62]). *The capacity region of the SMBCA is a superset of the capacity region of the scaled MIMO broadcast channel (SMBC).*

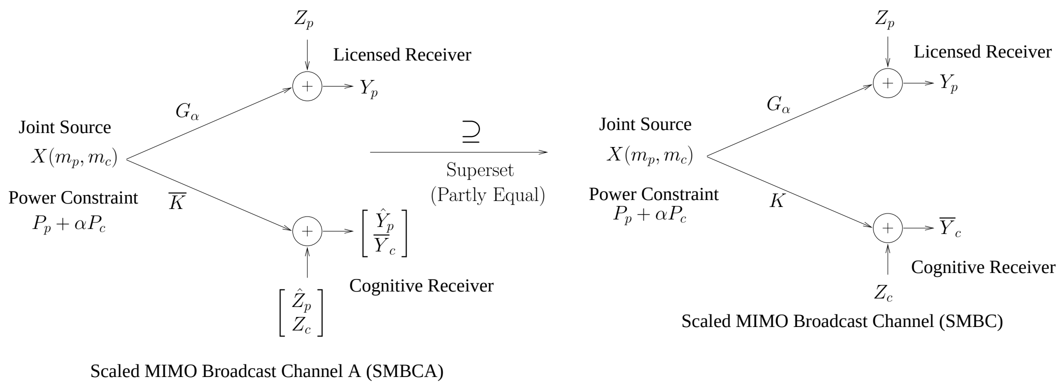


Figure 2.7: Capacity Region of SMBCA $\supseteq$ Capacity Region of SMBC

Proof. Let the rate pair (R_p, R_c) be achievable on the SMBC. That is, for all $\epsilon_1, \epsilon_2 > 0$, there exists a n and a sequence of encoder decoder pairs at the transmitter and the two receivers $(E^n : (m_p, m_c) \to \mathbf{X}^n$, $D_p^n : \mathbf{Y_p}^n \to \hat{m}_p, D_c^n : \mathbf{Y_c}^n \to \hat{m}_c)$ such that the codeword $\mathbf{X}^n$ satisfies the power constraint of $P_p + \alpha P_c$ and the probability of decoding error is small $(Pr(m_p \neq \hat{m}_p) \leq \epsilon_1, Pr(m_c \neq \hat{m}_c) \leq \epsilon_2)$.

In the SMBCA, the transmitter and the receivers use the same coding strategy. The licensed receiver can decode message m_p at a rate R_p. The cognitive receiver can ignore $\hat{\mathbf{Y}}_{\mathbf{p}}^n$ and use just $\overline{\mathbf{Y}_{\mathbf{c}}}^n$ to decode message m_c at a rate R_c. Hence, the rate pair (R_p, R_c) is achievable in the SMBCA. Hence, the capacity region of the SMBCA is in general a superset of the capacity region of the SMBC. $\square$

We describe one more lemma whose result is used in the proof of Theorem (2.3.3).

Lemma 2.5.6 ([62]). *Let $\mathcal{C}_{SMBC}$ denote the capacity region of the scaled MIMO* *broadcast channel described in Figure* 2*f. Then, for any* $\mu \geq 1$,

$$\sup_{(R_p,R_c)\in\mathcal{C}_{SMBC}} \mu R_p + R_c = \inf_{\mathbf{\Sigma_z}} \sup_{(R_p,R_c)\in\mathcal{C}_{SMBCA}} \mu R_p + R_c.$$

The proof is described in [62, Section 5.1] and is omitted here.

We now give the proof for Theorem (2.3.3).

Proof of Theorem 2.3.3 : It was shown in [46] that Gaussian codebooks (i.e., codebooks generated using i.i.d. realizations of an appropriate Gaussian random variable) achieve the capacity region for the MIMO broadcast channel. In SMBC, let $\mathbf{Q_p}$ denote the covariance of codeword $\mathbf{X}^n$ for the licensed user and $\mathbf{Q_c}$ denote the covariance matrix for the cognitive user. The covariance matrices satisfy the joint power constraint $\mathrm{Tr}(\mathbf{Q_p} + \mathbf{Q_c}) \leq P_p + \alpha P_c$. Let $\mathcal{R}^{\alpha}_{SMBC,1}$ denote the set of rate pairs described by

$$\overline{\left\{ \begin{array}{l} (R_p, R_c) : R_p \geq 0, R_c \geq 0 \\ R_p \leq \log\left|\mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_p} \mathbf{G}_\alpha^\dagger + \mathbf{G}_\alpha \mathbf{Q_c} \mathbf{G}_\alpha^\dagger\right| - \log\left|\mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_c} \mathbf{G}_\alpha^\dagger\right| \\ R_c \leq \log\left|\mathbf{I} + \mathbf{K}\mathbf{Q_c}\mathbf{K}^\dagger\right| \\ \forall \mathbf{Q_p} \succeq \mathbf{0}, \mathbf{Q_c} \succeq \mathbf{0} \text{ and } \mathrm{Tr}(\mathbf{Q_p}) + \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) \leq P_p + \alpha P_c \end{array} \right\}}. \tag{2.21}$$

Similarly, let $\mathcal{R}^{\alpha}_{SMBC,2}$ denote the set of rate pairs described by

$$\overline{\left\{ \begin{array}{l} (R_p, R_c) : R_p \geq 0, R_c \geq 0 \\ R_p \leq \log\left|\mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_p} \mathbf{G}_\alpha^\dagger\right| \\ R_c \leq \log\left|\mathbf{I} + \mathbf{K}\mathbf{Q_p}\mathbf{K}^\dagger + \mathbf{K}\mathbf{Q_c}\mathbf{K}^\dagger\right| - \log\left|\mathbf{I} + \mathbf{K}\mathbf{Q_p}\mathbf{K}^\dagger\right| \\ \forall \mathbf{Q_p} \succeq \mathbf{0}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0} \text{ such that } \mathrm{Tr}(\mathbf{Q_p}) + \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) \leq P_p + \alpha P_c \end{array} \right\}}. \tag{2.22}$$

The capacity region of SMBC, $\mathcal{C}_{SMBC}$ is the closure of the convex hull of $\mathcal{R}^{\alpha}_{SMBC,1} \cup \mathcal{R}^{\alpha}_{SMBC,2}$. That is,

$$\mathcal{C}_{SMBC} = \overline{\mathcal{R}^{\alpha}_{SMBC,1} \cup \mathcal{R}^{\alpha}_{SMBC,2}}. \tag{2.23}$$

$\mathcal{R}^{\alpha}_{SMBC,1}$ denotes the portion of the capacity region of SMBC where the licensed user's message is encoded first. That is, the cognitive receiver sees no interference. Hence, for $\mu \geq 1$, we have

$$\max_{(R_p,R_c)\in\mathcal{R}^{\alpha}_{SMBC,1}} \mu R_p + R_c = \max_{(R_p,R_c)\in\mathcal{C}_{SMBC}} \mu R_p + R_c.$$

Therefore, from Lemma 5.6, we have that for $\mu \geq 1$,

$$\max_{(R_p,R_c)\in\mathcal{R}^{\alpha}_{SMBC,1}} \mu R_p + R_c = \inf_{\mathbf{\Sigma_z}} \max_{(R_p,R_c)\in\mathcal{C}_{SMBCA}} \mu R_p + R_c.$$

We can see that, $\mathcal{R}^{\alpha}_{part,out}$ described in (2.12) is a subset of $\mathcal{R}^{\alpha}_{SMBC,1}$ formed by restricting the covariance matrix $\mathbf{Q_c}$ to have the form

$$\mathbf{Q_c} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma_{c,c}} \end{bmatrix}.$$

It can also be seen that $\mathcal{R}^{\alpha,\mathbf{\Sigma_z}}_{out}$ described in (2.8) equals $\mathcal{C}_{SMBCA}$. Hence, it follows that for any $\mu \geq 1$ and for $\alpha > 0$, if

$$\max_{(R_p,R_c)\in\mathcal{R}^{\alpha}_{part,out}} \mu R_p + R_c = \max_{(R_p,R_c)\in\mathcal{C}^{\alpha}_{BC}} \mu R_p + R_c,$$

then we have that

$$\max_{(R_p,R_c)\in\mathcal{R}^{\alpha}_{part,out}} \mu R_p + R_c = \inf_{\mathbf{\Sigma_z}} \max_{(R_p,R_c)\in\mathcal{R}^{\alpha,\mathbf{\Sigma_z}}_{out}} \mu R_p + R_c.$$

2.6 Optimality of the Achievable Region

In this section, we describe conditions under which the achievable region described by $\mathcal{R}_{in}$ in (5.35) is optimal for a portion of the capacity region. In particular, we show that if (R_p, R_c) lies on the boundary of the achievable region given by $\mathcal{R}_{in}$, then (R_p, R_c) lies on the boundary of $\mathcal{R}^{\alpha}_{part,out}$ given by (2.12) for some $\alpha > 0$. That is, for any $\mu > 0$,

$$\sup_{(R_p,R_c)\in\mathcal{R}_{in}} \mu R_p + R_c = \inf_{\alpha>0} \sup_{(R_p,R_c)\in\mathcal{R}^{\alpha}_{part,out}} \mu R_p + R_c.$$

Then there exists $\alpha^* \in (0,\infty)$ such that, for any $\mu \geq 1$, $(R_{p,\mu}, R_{c,\mu}) = \arg\max_{(R_p,R_c)\in\mathcal{R}_{in}} \mu R_p + R_c$ is a point on the boundary of the capacity region of the MIMO cognitive channel if the condition (2.13) is satisfied for α^*.

We denote by $\mathcal{R}_{ach,rate}$, the set of all $((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \mathbf{Q})$ given by

$$\left\{ \begin{array}{l} \Big((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \mathbf{Q}\Big) : R_p, R_c \geq 0, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0} \\ R_p \leq \log\left|\mathbf{I} + \mathbf{G}\mathbf{\Sigma_{p,net}}\mathbf{G}^\dagger + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}^\dagger_{\mathbf{c,p}}\right| - \log\left|\mathbf{I} + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}^\dagger_{\mathbf{c,p}}\right| \\ R_c \leq \log\left|\mathbf{I} + \mathbf{H_{c,c}}\mathbf{\Sigma_{c,c}}\mathbf{H}^\dagger_{\mathbf{c,c}}\right| \\ \mathbf{\Sigma_{p,net}} = \begin{pmatrix} \mathbf{\Sigma_p} & \mathbf{Q} \\ \mathbf{Q}^\dagger & \mathbf{\Sigma_{c,p}} \end{pmatrix} \succeq \mathbf{0} \end{array} \right\}. \tag{2.24}$$

The rate pair that maximizes $\mu R_p + R_c$ in the achievable region is given by solving the optimization problem

$$\begin{array}{c} \sup_{((R_p,R_c),\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}},\mathbf{Q})} \mu R_p + R_c \\ \text{such that } ((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \mathbf{Q}) \in \mathcal{R}_{ach,rate} \\ \mathrm{Tr}(\mathbf{\Sigma_p}) \leq P_p,\ \mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) \leq P_c \end{array}. \tag{2.25}$$

We define the functions $L(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda_1, \lambda_2)$ and $g(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}})$ as follows

$$\begin{aligned} L(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda_1, \lambda_2) &= \mu R_p + R_c - \lambda_1(\mathrm{Tr}(\mathbf{\Sigma_p}) - P_p) \\ &\quad -\lambda_2(\mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) - P_c)) \end{aligned} \tag{2.26}$$

$$g(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}) = \min_{\lambda_1\geq 0,\lambda_2\geq 0} L(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda_1, \lambda_2). \tag{2.27}$$

The optimization problem given by

$$\begin{array}{c} \max_{(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}},\mathbf{Q})} \; g(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}}) \\ \text{such that } ((R_p,R_c),\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}},\mathbf{Q}) \in \mathcal{R}_{ach,rate} \end{array} . \tag{2.28}$$

has the same optimum value as that of (2.25). This is formally stated in the lemma below.

Lemma 2.6.1. *Let M denote the optimal value of the optimization problem defined in (2.25), and U denote the optimal value of the optimization problem defined in (2.28). Then, $M = U$.*

Proof. We show that for any set of covariance matrices $(\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}})$ that do not satisfy the power constraints given by (6.2), $g(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}}) = -\infty$. The power constraints can be violated by three means :

- $\mathrm{Tr}(\mathbf{\Sigma_p}) > P_p$ and $\mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) \leq P_c$: In this case, λ_1 takes an arbitrarily large value and $\lambda_2 = 0$ to drive $g(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}})$ to $-\infty$.

- $\mathrm{Tr}(\mathbf{\Sigma_p}) \leq P_p$ and $\mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) > P_c$: In this case, $\lambda_1 = 0$ and λ_2 takes an arbitrarily large value to drive $g(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}})$ to $-\infty$.

- $\mathrm{Tr}(\mathbf{\Sigma_p}) > P_p$ and $\mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) > P_c$: In this case, λ_1 and λ_2 take arbitrarily large values to drive $g(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}})$ to $-\infty$.

When both the covariance matrices satisfy the power constraints with inequality, then $\lambda_1 = \lambda_2 = 0$. This is because, $\mathrm{Tr}(\mathbf{\Sigma_p}) - P_p$ and $\mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) - P_c$ are both negative. Hence, for any positive value of λ_1 or λ_2, $L(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}},\lambda_1,\lambda_2) \geq L(R_p,R_c,\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{\Sigma_{c,c}},0,0)$.

When one of the power constraint is satisfied with equality, say $\mathrm{Tr}(\mathbf{\Sigma_p}) - P_p = 0$ and the other power constraint is satisfied with inequality $\mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) - P_c < 0$, then, we have $\lambda_2 = 0$ and λ_1 is some real number. In any case, we still have $\lambda_1(\mathrm{Tr}(\mathbf{\Sigma_p}) - P_p) = \lambda_2(\mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) - P_c) = 0$.

Similarly, when the first constraint is satisfied with inequality, and the second constraint satisfied with equality, we have $\lambda_1 = 0$ and λ_2 is some non negative real number. We have $\lambda_1(\mathrm{Tr}(\mathbf{\Sigma_p}) - P_p) = \lambda_2(\mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) - P_c) = 0$.

Finally, if both the power constraints are satisfied with equality, λ_1 and λ_2 are some non-negative real numbers. And $\lambda_1(\mathrm{Tr}(\mathbf{\Sigma_p}) - P_p) = \lambda_2(\mathrm{Tr}(\mathbf{\Sigma_{c,p}} + \mathbf{\Sigma_{c,c}}) - P_c) = 0$.

Hence, in all the cases, the complementary slackness conditions are satisfied. Hence, the optimal solution of the optimization problem (2.28) satisfy the power constraints and the objective function reduces to that of optimization problem (2.25). Hence, both the optimization problems have the same optimal values. That is, $M = U$. □

Next, we find the optimum value of $\mu R_p + R_c$ over all the rate pairs that are in the region $\mathcal{R}^{\alpha}_{part,out}$ described by (2.12). This is done by solving the following optimization problem:

$$\begin{array}{c} \sup_{((R_p,R_c),\mathbf{Q_p},\mathbf{\Sigma_{c,c}})} \ \mu R_p + R_c \\ \text{such that } ((R_p,R_c),\mathbf{Q_p},\mathbf{\Sigma_{c,c}}) \in \mathcal{R}^{\alpha}_{part,conv,rate} \ , \\ \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) + \mathrm{Tr}(\mathbf{Q_p}) \leq \alpha P_c + P_p \end{array} \tag{2.29}$$

where $\mathcal{R}^{\alpha}_{part,conv,rate}$ is the set of quadruples $((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}})$ described by

$$\left\{ \begin{array}{l} ((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}}) : R_p, R_c \geq 0, \mathbf{Q_p}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0} \\ R_p \leq \log \left| \mathbf{I} + \mathbf{G}_\alpha \mathbf{Q_p} \mathbf{G}_\alpha^\dagger + \frac{1}{\alpha} \mathbf{H_{c,p}} \mathbf{\Sigma_{c,c}} \mathbf{H}^\dagger_{\mathbf{c,p}} \right| - \log \left| \mathbf{I} + \frac{1}{\alpha} \mathbf{H_{c,p}} \mathbf{\Sigma_{c,c}} \mathbf{H}^\dagger_{\mathbf{c,p}} \right| \\ R_c \leq \log \left| \mathbf{I} + \frac{1}{\alpha} \mathbf{H_{c,c}} \mathbf{\Sigma_{c,c}} \mathbf{H}^\dagger_{\mathbf{c,c}} \right| \end{array} \right\}. \tag{2.30}$$

We let the optimal solution of (2.29) to be denoted by $N(\alpha)$. Let $N = \min_{\alpha>0} N(\alpha)$ and

$$\alpha^* = \arg\min_{\alpha>0} N(\alpha). \tag{2.31}$$

We show in Lemma 6.2 that $\alpha^* \in (0, \infty)$ exists. Then, N is given by the optimum value of the following $\inf \sup$ optimization problem

$$N = \inf_{\alpha>0} \left\{ \begin{array}{c} \sup_{((R_p,R_c),\mathbf{Q_p},\mathbf{\Sigma_{c,c}})} \mu R_p + R_c \\ \text{such that } ((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}}) \in \mathcal{R}^{\alpha}_{part,conv,rate} \\ \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) + \mathrm{Tr}(\mathbf{Q_p}) \leq \alpha P_c + P_p \end{array} \right\}. \tag{2.32}$$

The infimum constraint $\alpha > 0$ is not a compact set. We modify the constraint on α to $\alpha \in \mathbb{R}^+ \cup \{0, \infty\}$. This is done to compactify the set by adding two extra symbols 0 and ∞. The point zero is added to make the set closed. The process of adding the point ∞ is called one point compactification. Details on one point compactification can be found in [63, Section 2.8]. The new space $\alpha \in \mathbb{R}^+ \cup \{0, \infty\}$ is compact and Hausdorff.

The optimization problem after changing the constraint set on α becomes

$$N_1 = \inf_{\alpha \in \mathbb{R}^+ \cup \{0,\infty\}} \left\{ \begin{array}{c} \sup_{((R_p,R_c),\mathbf{Q_p},\mathbf{\Sigma_{c,c}})} \mu R_p + R_c \\ \text{such that } ((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}}) \in \mathcal{R}^{\alpha}_{part,conv,rate} \\ \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) + \mathrm{Tr}(\mathbf{Q_p}) \leq \alpha P_c + P_p \end{array} \right\}. \tag{2.33}$$

We show that adding the two points 0 and ∞ to the constraint set on α does not change the optimum value of the optimization problem. This result is formally stated and proved in the following lemma.

Lemma 2.6.2. *The optimum value of the optimization problem given by (2.32), N is equal to the optimum value of the optimization problem described by (2.33), N_1. That is, $N = N_1$.*

Proof. For any $\alpha \in \mathbb{R}^+ \cup \{0, \infty\}$, we let $h(\alpha)$ to denote the value of the inner $\sup$ problem. That is,

$$\begin{array}{c} h(\alpha) = \sup_{((R_p,R_c),\mathbf{Q_p},\mathbf{\Sigma_{c,c}})} \mu R_p + R_c \\ \text{such that } ((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}}) \in \mathcal{R}^{\alpha}_{part,conv,rate} \\ \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) + \mathrm{Tr}(\mathbf{Q_p}) \le \alpha P_c + P_p \end{array} . \tag{2.34}$$

We show that $\liminf_{\alpha\to 0} h(\alpha) = \liminf_{\alpha\to\infty} h(\alpha) = \infty$.

Letting $\alpha \to 0$, we put all the power in $\mathbf{\Sigma_{c,c}}$. That is, we choose $\mathbf{\Sigma_p} = \mathbf{0}$, $\mathbf{\Sigma_{c,p}} = \mathbf{0}$, $\mathbf{Q} = \mathbf{0}$ and $\mathbf{\Sigma_{c,c}} = \frac{P_p+\alpha P_c}{n_{c,t}} \mathbf{I}_{n_{c,t}}$. Also, we take

$$R_p = 0 \text{ and } R_c = \log \left| \mathbf{I} + \frac{1}{\alpha} \frac{P_p + \alpha P_c}{n_{c,t}} \mathbf{H_{c,c}} \mathbf{H}^{\dagger}_{\mathbf{c,c}} \right| .$$

It follows from (2.30) that $((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}}) \in \mathcal{R}^{\alpha}_{part,conv,rate}$. Also, $\mathrm{Tr}(\mathbf{Q_p}) + \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) = P_p + \alpha P_c$. Hence, $((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}})$ satisfy all the necessary constraints of (2.34). Substituting these particular values of $((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}})$, we get a lower bound on $h(\alpha)$. That is,

$$\liminf_{\alpha\to 0} h(\alpha) \quad \ge \quad \liminf_{\alpha\to 0} \log \left| \mathbf{I} + \frac{1}{\alpha} \frac{P_p + \alpha P_c}{n_{c,t}} \mathbf{H_{c,c}} \mathbf{H}^{\dagger}_{\mathbf{c,c}} \right| \quad = \quad \infty. \tag{2.35}$$

Next, we look at the situation when $\alpha \to \infty$. In this case, we put all the power in $\mathbf{\Sigma_p}$. That is, we choose $\mathbf{\Sigma_p} = \frac{P_p+\alpha P_c}{n_{p,t}} \mathbf{I}_{n_{p,t}}$, $\mathbf{\Sigma_{c,p}} = \mathbf{0}$, $\mathbf{\Sigma_{c,c}} = \mathbf{0}$ and $\mathbf{Q} = \mathbf{0}$. We also choose

$$R_c = 0 \text{ and } R_p = \log \left| \mathbf{I} + \frac{P_p + \alpha P_c}{n_{p,t}} \mathbf{H_{p,p}} \mathbf{H}^{\dagger}_{\mathbf{p,p}} \right| .$$

These values of $((R_p, R_c), \mathbf{Q_p}, \mathbf{\Sigma_{c,c}})$ satisfy all the necessary constraints of (2.34). Hence, we have

$$\liminf_{\alpha\to\infty} h(\alpha) \quad \geq \quad \liminf_{\alpha\to\infty} \mu \log \left| \mathbf{I} + \frac{P_p + \alpha P_c}{n_{p,t}} \mathbf{H_{p,p}} \mathbf{H}^{\dagger}_{\mathbf{p,p}} \right| \quad = \quad \infty. \tag{2.36}$$

Hence, $h(\alpha) = \infty$ when $\alpha = 0$ or $\alpha = \infty$. Also, when $\alpha \in \mathbb{R}^+$, $h(\alpha) < \infty$. Hence, the optimum value of (2.33) is reached when α is neither 0 nor ∞. Hence, $N = N_1$. □

As $\mathbf{Q_p}$ is the covariance matrix of the codeword $\mathbf{X}(i), i = 1, \ldots, n$ for the primary user, it can be written as

$$\mathbf{Q_p} = \begin{pmatrix} \mathbf{\Sigma_p} & \mathbf{Q} \\ \mathbf{Q}^{\dagger} & \mathbf{\Sigma_{c,p}} \end{pmatrix}. \tag{2.37}$$

It is easy to see that the set $R^{\alpha}_{part,conv}$ described in (11) can also be written as

$$\left\{ \begin{array}{l} \left((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{Q}, \mathbf{\Sigma_{c,c}}\right) : R_p, R_c \geq 0, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0} \\ R_p \leq \log \left| \mathbf{I} + \mathbf{G}\mathbf{Q_p}\mathbf{G}^{\dagger} + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}^{\dagger}_{\mathbf{c,p}} \right| - \log \left| \mathbf{I} + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}^{\dagger}_{\mathbf{c,p}} \right| \\ R_c \leq \log \left| \mathbf{I} + \mathbf{H_{c,c}}\mathbf{\Sigma_{c,c}}\mathbf{H}^{\dagger}_{\mathbf{c,c}} \right| \\ \mathrm{Tr}(\mathbf{\Sigma_p}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) \leq P_p + \alpha P_c \end{array} \right\}. \tag{2.38}$$

where $\mathbf{G} = [\mathbf{H_{p,p}} \ \ \mathbf{H_{c,p}}]$. This is done by transforming $\mathbf{Q}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}$ into $\sqrt{\alpha}\mathbf{Q}, \alpha\mathbf{\Sigma_{c,p}}, \alpha\mathbf{\Sigma_{c,c}}$ respectively. We define $\mathcal{R}_{part,conv,rate}$ as the set described by

$$\left\{ \begin{array}{l} ((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{Q}, \mathbf{\Sigma_{c,c}}) : R_p, R_c \geq 0, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}} \succeq \mathbf{0} \\ R_p \leq \log \left| \mathbf{I} + \mathbf{G}\mathbf{Q_p}\mathbf{G}^{\dagger} + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}^{\dagger}_{\mathbf{c,p}} \right| - \log \left| \mathbf{I} + \mathbf{H_{c,p}}\mathbf{\Sigma_{c,c}}\mathbf{H}^{\dagger}_{\mathbf{c,p}} \right| \\ R_c \leq \log \left| \mathbf{I} + \mathbf{H_{c,c}}\mathbf{\Sigma_{c,c}}\mathbf{H}^{\dagger}_{\mathbf{c,c}} \right|, \qquad \mathbf{Q_p} = \begin{pmatrix} \mathbf{\Sigma_p} & \mathbf{Q} \\ \mathbf{Q}^{\dagger} & \mathbf{\Sigma_{c,p}} \end{pmatrix} \end{array} \right\}. \tag{2.39}$$

Hence, the optimization problem (2.33) can be written as

$$N = \inf_{\alpha \in \mathbb{R}^+ \cup \{0,\infty\}} \left\{ \begin{array}{c} \sup_{((R_p,R_c),\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{Q},\mathbf{\Sigma_{c,c}})} \mu R_p + R_c \\ \text{such that } ((R_p,R_c),\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{Q},\mathbf{\Sigma_{c,c}}) \in \mathcal{R}_{part,conv,rate} \\ \mathrm{Tr}(\mathbf{\Sigma_p}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) \leq P_p + \alpha P_c \end{array} \right\}. \tag{2.40}$$

We state the following lemma for switching $\min$ and $\max$ in minimax problems. The lemma is described and proved in Theorem 2 in [64].

Lemma 2.6.3. *(Ky-Fan's minimax switching theorem [64, Thm. 2]) Let X be a compact Hausdorff space and Y an arbitrary set (not topologized). Let f be a real-valued function on $X \times Y$ such that, for every $y \in Y$, $f(x,y)$ is lower semi continuous on X. If f is convex on X and concave on Y, then*

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \inf_{x \in X} f(x,y).^2 \tag{2.41}$$

We see that the objective function $\mu R_p + R_c$ is concave with respect to the maximizing variables $((R_p, R_c, \mathbf{Q_p}, \mathbf{\Sigma_{c,c}})$ and convex with respect to the minimizing variable α. The constraint space $\alpha \in \mathbb{R}^+ \cup \{0,\infty\}$ is compact and Hausdorff [63, Section 2.8]. Hence, all the conditions of the lemma are satisfied. Hence, by Ky-Fan's mini-max switching theorem [64], we can interchange the $\sup$ and $\inf$ without affecting the optimum value. Hence,

$$N = \sup_{((R_p,R_c),\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{Q},\mathbf{\Sigma_{c,c}}) \in \mathcal{R}_{part,conv,rate}} \begin{array}{c} \inf \\ \alpha \in \mathbb{R}^+ \cup \{0,\infty\} \\ \mathrm{Tr}(\mathbf{\Sigma_p} + \alpha \mathbf{\Sigma_{c,p}} + \alpha \mathbf{\Sigma_{c,c}}) \leq P_p + \alpha P_c \end{array} \mu R_p + R_c. \tag{2.42}$$

[2] In (49), the $\inf$ can be replaced with $\min$, but we use $\inf$ throughout to maintain continuity and to avoid confusion.

Similar to the functions L and g defined in (2.26) and (2.27), we define the functions $L_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda, \alpha)$ and $g_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \alpha)$ as follows

$$\begin{aligned} L_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda, \alpha) &= \mu R_p + R_c - \lambda(\mathrm{Tr}(\mathbf{\Sigma_p}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \\ & \quad \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) - P_p - \alpha P_c). \end{aligned} \tag{2.43}$$

$$g_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \alpha) = \inf_{\lambda \geq 0} L_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda, \alpha). \tag{2.44}$$

We define the following optimization problem

$$V = \sup_{((R_p,R_c),\mathbf{\Sigma_p},\mathbf{\Sigma_{c,p}},\mathbf{Q},\mathbf{\Sigma_{c,c}}) \in \mathcal{R}_{part,conv,rate}} \inf_{\alpha \in \mathbb{R}^+ \cup \{0,\infty\}} g_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \alpha). \tag{2.45}$$

Lemma 2.6.4. *The optimum value of optimization problem (2.42), N is equal to the optimum value of the optimization problem (2.45), V.*

Proof. The proof of the lemma is along the same lines as the proof of Lemma 2.6.1. We show that for any set of covariance matrices $\mathbf{\Sigma_p}$, $\mathbf{\Sigma_{c,p}}$ and $\mathbf{\Sigma_{c,c}}$ that do not satisfy the power constraint $\mathrm{Tr}(\mathbf{\Sigma_p}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) \leq P_p + \alpha P_c$, $g_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \alpha) = -\infty$. This is because, $\mathrm{Tr}(\mathbf{\Sigma_p}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) - P_p - \alpha P_c$ is positive, and hence, λ takes an arbitrarily high value to drive $g_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \alpha)$ to $-\infty$. Hence, the outer supremization problem ensures that the power constraint is satisfied.

Moreover, when the power constraints are satisfied with inequality, then $\mathrm{Tr}(\mathbf{\Sigma_p}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) - P_p - \alpha P_c$ is negative. Therefore, for any $\lambda > 0$, we have $L_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda, \alpha) > L_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, 0, \alpha)$. Hence,

λ will take the value zero. When the power constraint is satisfied with equality, then $\mathrm{Tr}(\mathbf{\Sigma_p}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,p}}) + \alpha \mathrm{Tr}(\mathbf{\Sigma_{c,c}}) - P_p - \alpha P_c = 0$. Then, λ will take some non negative real number. Hence, the complementary slackness condition is satisfied. Hence, the optimal solution of the optimization problem satisfy the power constraint and the objective function reduces to that of (2.42). It follows that, the optimum value of the optimization problem (2.42), N is the same as the optimum value of the optimization problem (2.45), V. □

Next, we show that the optimum value of the optimization problem (2.28), U is an upper bound on the optimal value of the optimization problem (2.45), V.

Lemma 2.6.5. *The optimal value of (2.28), U is an upper bound on the optimal value of (2.42), V.*

Proof. Both the optimization problems are $\sup \min$ problems. For any $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, we can choose $\lambda = \lambda_1$ and $\alpha = \lambda_2/\lambda_1$ so that $L_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda, \alpha) = L(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda_1, \lambda_2)$. Hence, for any $((R_p, R_c), \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}})$,

$$\inf_{\lambda \geq 0, \alpha \in \mathbb{R}^+ \cup \{0, \infty\}} L_1(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda, \alpha) \leq \inf_{\lambda_1 \geq 0, \lambda_2 \geq 0} L(R_p, R_c, \mathbf{\Sigma_p}, \mathbf{\Sigma_{c,p}}, \mathbf{\Sigma_{c,c}}, \lambda_1, \lambda_2). \tag{2.46}$$

Also, $\mathcal{R}_{part,conv,rate} = \mathcal{R}_{ach,rate}$. Hence, it follows that $V \leq U$. □

We can now prove Theorem 2.3.4.

Proof of Theorem 2.3.4 : Let $\mu \geq 1$. The proof of the theorem follows directly from Lemmas 2.6.1, 2.6.4 and 2.6.5. From Lemma 2.6.1, we have that the optimum value of the optimization problem (2.25), M equals the optimum value of optimization problem (2.28), U. From Lemma 2.6.4, we have that the optimum value of optimization problem (2.42), N equals the optimum value of the optimization problem (2.45), V. M is the solution of the optimum $\mu R_p + R_c$ over the achievable region and N is the solution of the optimum $\mu R_p + R_c$ over $\mathcal{R}^{\alpha}_{part,out}$ described in (2.12). Hence if the condition given by (2.13) is satisfied for α^* given by (2.31), $M \leq N$. From Lemma 2.6.5, we also have $V \leq U$. Hence, we have that the optimal value of the original optimization problem (2.25), M is equal to the optimal value of the optimization problem described by (2.42), N. Hence, the achievable region $\mathcal{R}_{in}$ is μ-sum optimal.

2.7 Numerical Results

In this section, we provide some numerical results on the capacity region of the MIMO cognitive channel. We consider a MIMO cognitive system where the licensed and cognitive transmitters have one antenna each, and the licensed and cognitive receivers have one and two antennas respectively. We assume that the channel coefficients are real and also restrict ourself to real inputs and outputs. We generate the channel values randomly

$$\mathbf{H_{p,p}} = 1.4435, \quad \mathbf{H_{p,c}} = \begin{bmatrix} -0.3510 \\ 0.6232 \end{bmatrix}, \quad \mathbf{H_{c,p}} = 0.799, \quad \mathbf{H_{c,c}} = \begin{bmatrix} 0.9409 \\ -0.9921 \end{bmatrix}.$$

We assume a power constraint of 5 at the licensed and cognitive transmitters. In Figure 2.8, we plot the achievable region, $\mathcal{R}_{in}$ and partial outer bounds $\mathcal{R}^{\alpha}_{part,out}$ for different values of α. Figure 2.8 shows how $\mathcal{R}^{\alpha}_{part,out}$ intersects with $\mathcal{R}_{in}$ at different

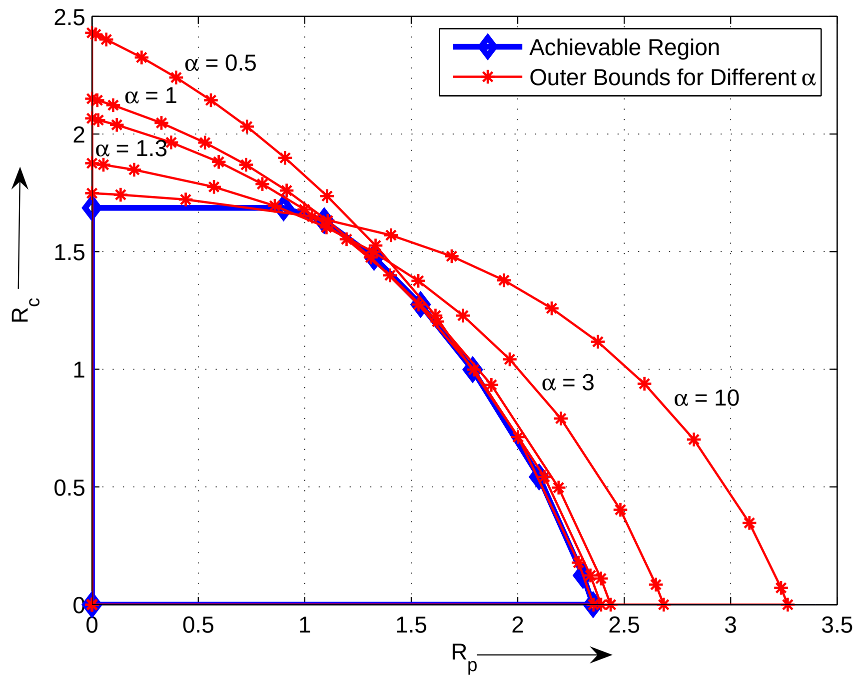


Figure 2.8: Plot of Achievable Region $\mathcal{R}_{in}$ and partial outer bounds $\mathcal{R}^{\alpha}_{part,out}$ for different values of α

points for different values of α.

Next, we find the maximum value of rate than can be supported by the licensed user in the example we considered. In both the achievable region and the outer bound, this corresponds to maximizing the μ-sum $\mu R_p + R_c$ when $\mu \rightarrow \infty$. This would correspond to using all the power to support the licensed user. Note that the maximum value of R_p in the set described by $\mathcal{R}^{\alpha}_{part,out}$ is an upper bound on the maximum value of R_p in the set $\mathcal{R}_{in}$ for all values of $\alpha > 0$, irrespective of the channel parameters.

Maximizing R_p over $\mathcal{R}_{in}$: The cognitive transmitter uses all its power for helping the licensed user. That is $\mathrm{Tr}(\mathbf{\Sigma_{c,p}}) = P_c$. This then reduces to a MIMO channel with channel matrix given by $\mathbf{G} = \left[\begin{array}{cc} \mathbf{H_{p,p}} & \mathbf{H_{c,p}} \end{array} \right]$. The licensed transmitter has a power constraint of P_p and the cognitive transmitter has a power constraint of P_c. Applying this to our example channel, we have $\mathbf{G} = \left[\begin{array}{cc} 1.4435 & 0.799 \end{array} \right]$. The optimum covariance matrix is of the form

$$\mathbf{\Sigma_{p,net}} = \left[\begin{array}{cc} 5 & 5\rho \\ 5\rho & 5 \end{array} \right],$$

where ρ is the correlation between the two transmitters. Therefore, the rate achieved by the licensed user is

$$R_p(\rho) = \frac{1}{2}\log(1 + \mathbf{G}\mathbf{\Sigma_{p,net}}\mathbf{G}^\dagger).$$

The maximum rate is attained at $\rho = 1$ and the maximum value of R_p is 2.3542.

Maximizing R_p over $\mathcal{R}^{\alpha}_{part,out}$: For a given α, this reduces to a single user MIMO channel with $\mathbf{G}_\alpha = \left[\begin{array}{cc} \mathbf{H_{p,p}} & \mathbf{H_{c,p}}/\sqrt{\alpha} \end{array} \right]$ and a sum power constraint of $P_p + \alpha P_c$. Note that, there is a significant difference between the two single user MIMO channels. The MIMO channel that we considered when solving the maximum value of R_p in the achievable region had individual power constraints at the licensed and cognitive transmitters. However, the MIMO channel we obtain when solving for the maximum value of R_p over $\mathcal{R}^{\alpha}_{part,out}$ has a sum power constraint. This is a conventional MIMO channel and the optimum covariance matrix is obtained by water-filling. For a given α, the best R_p is got by

$$\begin{array}{c} \max R_p(\alpha) = \frac{1}{2}\log|\mathbf{I} + \mathbf{G}_\alpha\mathbf{\Sigma_{p,net}}\mathbf{G}_\alpha| \\ \text{such that } \mathrm{Tr}(\mathbf{\Sigma_{p,net}}) \leq P_p + \alpha P_c. \end{array}$$

It is easy to solve this problem if we look at the flipped channel $\mathbf{G}_\alpha^\dagger$. The capacity of the flipped channel is given by

$$\begin{aligned} R_p(\alpha) &= \frac{1}{2}\log\left|\mathbf{I} + \mathbf{G}_\alpha^\dagger (P_p + \alpha P_c)\mathbf{G}_\alpha\right| \\ &= \frac{1}{2}\log\left(1 + (P_p + \alpha P_c)\mathbf{G}_\alpha \mathbf{G}_\alpha^\dagger\right). \end{aligned}$$

Note that $R_p(\alpha)$ is an outer bound on the maximum value of R_p. The best upper bound is got by minimizing over all possible values of α. The optimum value of α is got by solving a cubic equation $2(0.799)^2\alpha^3 + (0.799)^2\alpha^2 - 1.4435^2 = 0$, and its approximate value is 0.9689.

2.8 Conclusions

In this chapter, we derived an achievable region, $\mathcal{R}_{in}$ given by (5.35) and an outer bound, $\mathcal{R}_{out}^{\alpha,\mathbf{\Sigma_z}}$ given by (2.8) for the MIMO cognitive channel. We describe conditions when the achievable region is μ-sum optimal for any $\mu \geq 1$. In particular, for any $\mu \geq 1$, there exists $\alpha^* \in (0,\infty)$, such that if the region given by $\mathcal{R}_{part,out}^{\alpha^*}$ optimizes the $\mu-$ sum rate of the SMBC (for that particular α^*), then the achievable region achieves the μ-sum capacity of the MCC.

Chapter 3

Cognitive Radio in Multiple Access Networks

In this chapter, we look at cognitive radio channel in a multiple access (MAC) network. Specifically, this is an interference network with three transmitters and two receivers. Transmitters 1 and 2 are the licensed transmitters transmitting messages in a multiple access manner to a common licensed receiver. We also have a cognitive transmitter-receiver pair communicating in the same spectrum as the licensed users. It is assumed that the cognitive transmitter knows the messages transmitted by both the licensed transmitters apriori. The rest of the chapter is organized as follows. In Section 3.1, we describe the problem statement, prior work and contributions. We describe the system model in Section 3.2. In Section 3.3, we describe an outer bound on the capacity region of cognitive radio in MAC network. We describe an achievable region in Section 3.4. In Section 3.5, we show the optimality of the achievable region when the channel gain from cognitive transmitter to licensed receiver ≤ 1. We conclude in Section 3.6.

3.1 Introduction

The cognitive radio channel has been studied as a special form of interference channel where one of the transmitter (the "cognitive" transmitter) gains some knowledge about the transmissions of the other transmitter. Networks with cognitive users are gaining prominence with the development of cognitive radio technology, which is aimed at improving the spectral efficiency and the system performance by designing nodes which can adapt their strategy based on the network setup. The information theoretic model for the cognitive radio channel models the channel as a two user interference channel, where one transmitter (the cognitive transmitter) knows apriori the message transmitted by the other transmitter. Prior work on this channel model include [39–45, 57, 66]. More recently, the interference channel with a cognitive relay has been studied in [49–52, 65].

In this chapter, we study the performance limits of a cognitive radio channel in a multiple access setting. In particular, we consider a system where two primary transmitters communicate their messages to a primary receiver in a multiple access setting, and one cognitive transmitter transmits its message to a cognitive receiver. We assume that the cognitive transmitter knows apriori the messages of both the primary transmitters. We derive an outer bound on the capacity region of the cognitive radio channel in a multiple access setting (MACRC). We first derive an outer bound for the discrete memoryless channel and then show that Gaussians maximize the outer bound for the Gaussian channel when the channel gain from the cognitive transmitter to the primary receiver is "weak" (≤ 1). We also derive an achievable region for the MACRC which combines superposition and dirty paper coding techniques [22]. We show that the achievable region meets the outer bound when the cross channel gain from the cognitive transmitter to the primary receiver is weak (≤ 1). The contributions of this chapter have been presented in [53] [54].

Throughout the chapter, we denote random variables by capital letters, their realizations by lower case and their alphabets by calligraphic letters (eg. X, x and $\mathcal{X}$ respectively). We denote vectors of length n with boldface letters (e.g. $\mathbf{x}^n$), and the i^{th} element of a vector $\mathbf{x}^n$ by x_i. For any set S, $\overline{S}$ denotes the closure of the convex hull of S respectively.

3.2 System Model

In this section, we describe the system model for the cognitive radio channel in a multiple access setting (MACRC). In this system, we have two primary transmitters communicating their messages to a primary receiver in a multiple access manner, and one cognitive transmitter communicating its message to a cognitive receiver. We assume that the cognitive receiver knows apriori the messages of both the primary transmitters. The system model is described in Figure 3.1. The channel is described by $(\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{Y}, \mathcal{Y}, p(y_1, y_c|x_1, x_2, x_c))$ where $\mathcal{X}_1, \mathcal{X}_2$ denote the input alphabets of the primary transmitters, $\mathcal{X}_c$ denotes the input alphabet of the cognitive transmitter, and $\mathcal{Y}_1$ and $\mathcal{Y}_c$ denote the output alphabets of the primary and the cognitive receiver.

Transmitter i, $i \in \{1, 2\}$ has message $m_i \in \{1, 2, \ldots, 2^{nR_i}\}$ that it wishes to communicate with receiver 1 in a multiple access manner. The cognitive transmitter has message $m_c \in \{1, 2, \ldots, 2^{nR_c}\}$ that it wishes to communicate to the cognitive receiver. The cognitive transmitter has non-causal access to messages of both the primary transmitters. Let X_1, X_2, X_c and Y_1, Y_c denote the variables representing the respective channel inputs and outputs. Note that the channel input from the

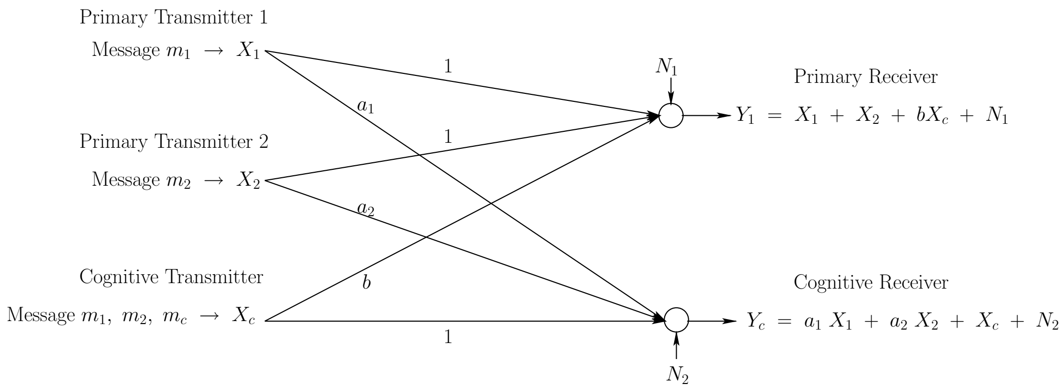


Figure 3.1: System Model of Cognitive Radio in Multiple Access Networks

cognitive transmitter (X_c) is a function of all the three messages. For the Gaussian channel, the input-output relationship can be expressed by the system equations given below:

$$\begin{aligned} Y_1 &= X_1 + X_2 + bX_c + N_1 \\ Y_c &= a_1X_1 + a_2X_2 + X_c + N_c. \end{aligned} \tag{3.1}$$

where a_1, a_2, and b represent the channel gains as shown in Figure 3.1. Throughout the chapter, we assume that the channel gains are positive, and the results can be readily extended when the channel gains are negative. N_{1i} and N_{2i} denote the additive noise at the two receivers which are i.i.d. Gaussian random variables distributed as $\mathcal{N}(0, 1)$. The channel inputs must satisfy the following power constraints:

$$\frac{1}{n}\sum_{i=1}^{n} E[X_{j,i}^2] \le P_j, \quad j \in \{1, 2, c\}. \tag{3.2}$$

A $(2^{nR_1}, 2^{nR_2}, 2^{nR_c}, n, Pe)$ code consists of message sets $M_1 = \{1, \ldots, 2^{nR_1}\}$, $M_2 = \{1, \ldots, 2^{nR_2}\}$ and $M_c \in \{1, \ldots, 2^{nR_c}\}$, three encoding functions

$$\begin{aligned} f_1 : M_1 \rightarrow \mathcal{X}_1^n,\ \ f_2 : M_2 \rightarrow \mathcal{X}_2^n, \\ f_c : M_1 \times M_2 \times M_C \rightarrow \mathcal{X}_c^n, \end{aligned} \tag{3.3}$$

and two decoding functions

$$g_1 : \mathcal{Y}_1^n \to M_1 \times M_2, \quad g_2 : \mathcal{Y}_c^n \to M_c, \tag{3.4}$$

such that the transmitted codewords $\mathbf{X}_1^n, \mathbf{X}_2^n$ and $\mathbf{X}_c^n$ satisfy the power constraints given by (4.2) and the overall decoding error probability at both the receivers is $\leq Pe$.

A rate tuple (R_1, R_2, R_c) is achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, 2^{nR_c}n, Pe^{(n)})$ codes such that $Pe^{(n)} \to 0$ as $n \to \infty$. The capacity region of the MACRC is then the set of all rate tuples (R_1, R_2, R_c) that are achievable, and is denoted by $\mathcal{C}_{MACRC}$.

3.3 Outer Bound on the Capacity Region of MACRC

In this section, we derive an outer bound on the capacity region of the MACRC when the cross channel gain from the cognitive transmitter to the primary receiver, $b \leq 1$.Let $\mathcal{P}_o$ denote the set of all probability distributions $P_o(.)$ given by

$$\begin{aligned} P_o(q, x_1, x_2, u, v, x_c) = p(q)p(x_1|q)p(x_2|q)p(u, v|x_1, x_2, q) \\ p(x_c|u, v, x_1, x_2, q). \end{aligned} \tag{3.5}$$

Let $\mathcal{R}_{out}(P_o)$ denote the set of rate tuples (R_1, R_2, R_c) given by

$$\begin{aligned} R_1 &\leq I(X_1, U; Y_1|V, X_2, Q) \\ R_2 &\leq I(X_2, V; Y_1|U, X_1, Q) \\ R_1 + R_2 &\leq I(X_1, U, X_2, V; Y_1|Q) \\ R_c &\leq I(X_c; Y_c|X_1, U, X_2, V, Q) \\ R_1, R_2, R_c &\geq 0 \end{aligned} \tag{3.6}$$

Let $\mathcal{R}_{out}$ denote the set of rate tuples given by

$$\mathcal{R}_{out} = \overline{\bigcup_{P_o(.) \in \mathcal{P}_o} \mathcal{R}_{out}(P_o)} \tag{3.7}$$

Then, the following theorem describes an outer bound on the capacity region of the discrete memoryless MACRC.

Theorem 3.3.1. *The capacity region of the discrete memoryless cognitive radio channel in a multiple access setting (MACRC) satisfies*

$$\mathcal{C}_{MACRC} \subseteq \mathcal{R}_{out}. \tag{3.8}$$

Proof. We fix a probability distribution $P_o(.) \in \mathcal{P}_o$. Then, we have

$$\begin{aligned} nR_1 &\overset{(a)}{=} H(W_1|W_2) \\ &\overset{(b)}{\leq} I(W_1; Y_1^n|W_2) + n\epsilon_n^1 \\ &\overset{(c)}{=} \textstyle\sum_{i=1}^n H(Y_{1i}|W_2, Y_1^{i-1}, X_{2i}) + n\epsilon_n^1 - \\ &\quad \textstyle\sum_{i=1}^n H(Y_{1i}|W_2, Y_1^{i-1}, W_1, X_{1i}, X_{2i}) \\ &= \textstyle\sum_{i=1}^n I(U_i, X_{1i}; Y_{1i}|V_i, X_{2i}) + n\epsilon_n^1 \end{aligned} \tag{3.9}$$

where $V_i = W_2, Y_1^{i-1}$ and $U_i = W_1, Y_1^{i-1}$. Here, (a) follows from the independence of W_1 and W_2, (b) follows from Fano's inequality and (c) follows from the fact that X_{2i} is a function of W_2.

A similar set of inequalities can be derived to show that

$$nR_2 \leq (V_i, X_{2i}; Y_{li}|U_i, X_{1i}) + n\epsilon_n^2 \tag{3.10}$$

Subsequently, we can show that

$$
\begin{aligned}
n(R_1+R_2) &= H(W_1,W_2) \\
&\leq I(W_1,W_2;Y_1^n)+n\epsilon_n^{1,2} \\
&\leq \sum_{i=1}^{n} H(Y_{1i})+n\epsilon_n^{1,2} - \qquad\qquad (3.11) \\
&\quad \sum_{i=1}^{n} H(W_1,W_2,Y_1^{i-1},X_{1i},X_{2i}) \\
&\leq \sum_{i=1}^{n} I(U_i,X_{1i},V_i,X_{2i};Y_{1i})+n\epsilon_n^{1,2}
\end{aligned}
$$

and

$$
\begin{aligned}
nR_c &\overset{(d)}{=} H(W_c|W_1,W_2,X_1^n,X_2^n) \\
&\overset{(e)}{\leq} I(W_c;Y_c^n|W_1,W_2,X_1^n,X_2^n)+n\epsilon_n^c \\
&\overset{(f)}{\leq} \sum_{i=1}^{n} H(Y_{ci}|Y_c^{i-1},W_1,W_2,X_1^n,X_2^n) \\
&\quad -\sum_{i=1}^{n} H(Y_{ci}|X_{ci},X_{1i},X_{2i})+n\epsilon_n^c \qquad\qquad (3.12) \\
&\overset{(g)}{=} \sum_{i=1}^{n} H(Y_{ci}|Y_c^{i-1},Y_1^{i-1},W_1,W_2,X_1^n,X_2^n) \\
&\quad -\sum_{i=1}^{n} H(Y_{ci}|X_{ci},X_{1i},X_{2i})+n\epsilon_n^c \\
&\leq \sum_{i=1}^{n} H(Y_{ci}|Y_1^{i-1},W_1,W_2,X_1^n,X_2^n) \\
&\quad -\sum_{i=1}^{n} H(Y_{ci}|X_{ci},X_{1i},X_{2i})+n\epsilon_n^c \\
&= \sum_{i=1}^{n} I(X_{ci};Y_{ci}|X_{1i},U_i,X_{2i},V_i)+n\epsilon_n^c
\end{aligned}
$$

where (d) follows from mutual independence between W_c, W_1 and W_2, (e) follows from Fano's inequality, (f) follows from the memoryless nature of the channel and (g) follows from the degraded nature of the channel (with the assumption that $b < 1$).

Defining Q to be the time-sharing random variable that is distributed uniformly over $\{1, 2, ..., n\}$ and defining

$$(Q, X_1, X_2, U, V, Y_1, Y_c) = (Q, X_{1,Q}, X_{2,Q}, U_Q, V_Q, Y_{l,Q}, Y_{c,Q})$$

yields the desired outer bound. □

3.4 Achievable Region for MACRC

In this section, we describe an achievable region for the MACRC. The coding strategy combines superposition and dirty paper coding techniques. Let $\mathcal{P}_i$ denote the set of probability distributions $P_i(.)$ given by

$$\begin{aligned} P_i(q, x_1, u, x_2, v, x_c, t) = p(q)p(u, x_1|q)p(v, x_2|q) \\ p(t, x_c|u, v, x_1, x_2). \end{aligned} \tag{3.13}$$

Let $\mathcal{R}_{in}(P_i)$ denote the set of rate tuples (R_1, R_2, R_c) given by

$$\begin{aligned} R_1 &\leq I(X_1, U; Y_1|V, X_2, Q) \\ R_2 &\leq I(X_2, V; Y_1|U, X_1, Q) \\ R_1 + R_2 &\leq I(X_1, U, X_2, V; Y_1|Q) \\ R_c &\leq I(T; Y_c|Q) - I(T; X_1, U, X_2, V|Q) \\ R_1, R_2, R_c &\geq 0. \end{aligned} \tag{3.14}$$

Let $\mathcal{R}_{in}$ denote the set of rate tuples (R_1, R_2, R_c) given by

$$\mathcal{R}_{in} = \overline{\bigcup_{P_i(.)\in\mathcal{P}_i} \mathcal{R}_{in}(P_i)}. \quad (3.15)$$

Then, the following theorem describes an achievable region for the MACRC.

Theorem 3.4.1. *The capacity region of the MACRC satisfies*

$$\mathcal{R}_{in} \subseteq \mathcal{C}_{MACRC}. \quad (3.16)$$

Proof. For simplicity, we shall present the coding-scheme for the degenerate case where the time-sharing random variable Q is deterministic. It should be kept in mind that the introduction of time-sharing may increase the region by convexification. We fix a $P_i(.) \in \mathcal{P}_i$ and show that the region $\mathcal{R}_{in}(P_i)$ is achievable. First, we describe codebook generation at the transmitters.

Codebook Generation: Transmitter 1 generates 2^{nR_1} vector pairs $X_1^n, U^n \sim \prod_{i=1}^n p(x_{1i}, u_i)$ and index them using $j \in \{1, \ldots, 2^{nR_1}\}$. Similarly, transmitter 2 generates 2^{nR_2} vector pairs $X_2^n, V^n \sim \prod_{i=1}^n p(x_{2i}, v_i)$ and index them using $k \in \{1, \ldots, 2^{nR_1}\}$. The cognitive transmitter generates $2^{n\overline{R}_c}$ $T^n \sim \prod_{i=1}^n p(t_i)$ and places them uniformly in 2^{nR_c} bins. We next describe the transmission strategy at the three transmitters.

Transmission strategy: Given message $m_1 \in \{1, \ldots, 2^{nR_1}\}$, transmitter 1 determines $X_1^n(m_1)$ and transmits it. Similarly, for message $m_2 \in \{1, \ldots, 2^{nR_2}\}$, transmitter 2 transmits $X_2^n(m_2)$. As the cognitive transmitter has access to messages m_1 and m_2, the cognitive transmitters determines $X_1^n(m_1), U^n(m_1), X_2^n(m_2), V^n(m_2)$.

For message $m_c \in \{1, \ldots, 2^{nR_c}\}$, the cognitive transmitter determines a sequence T^n in bin m_c such that $(T^n(w_c), X_1^n(m_1), U^n(m_1), X_2^n(m_2), V^n(m_2))$ is jointly typical. If such a T^n is located, then an X_c^n is generated according to the conditional $\prod_{i=1}^n p(x_{ci}|x_{1i}, u_i, x_{2i}, v_i)$ is generated and transmitted. We next describe the decoding strategy at the two receivers.

Reception: The primary receiver determines indices $(\hat{m}_1, \hat{m}_2)$ such that $(X_1^n(\hat{m}_1), U^n(\hat{m}_1), X_2^n(\hat{m}_2), V^n(\hat{m}_2), Y_1^n)$ is jointly typical. The cognitive receiver determines a T^n such that (T^n, Y_c^n) is jointly typical. The cognitive receiver then determines the bin index of T^n and declares that as the decoded message. We next describe the probability of error of encoding and decoding process.

Decoding Error at Primary Receiver: Let $E_{j,k}$ denote the decoding error event that $(X_1^n(j), U^n(j), X_2^n(k), V^n(k), Y_1^n)$ is jointly typical. We assume that the transmitters transmitted messages m_1 and m_2. Then the probability of decoding error is given by

$$Pe = Pr\left(E^c_{m_1,m_2} \cup \bigcup_{(j,k)\neq(m_1,m_2)} E_{j,k}\right).$$

The probability of decoding error can be upper bounded by

$$Pe \leq Pr(E^c_{m_1,m_2}) + \sum_{j\neq m_1} Pr(E_{j,m_2}) + \sum_{k\neq m_2} Pr(E_{m_1,k}) + \sum_{j\neq m_1, k\neq m_2} Pr(E_{j,k}).$$

For any $\epsilon > 0$, there exists n large enough such that the first term $Pr(E_{m_1,m_2}) \leq \epsilon$.

The other three terms can be made smaller than ϵ if

$$\begin{aligned} R_1 &\leq I(X_1, U; Y_1|X_2, V) - 3\epsilon \\ R_2 &\leq I(X_2, V; Y_1|X_1, U) - 3\epsilon \\ R_1 + R_2 &\leq I(X_1, U, X_2, V; Y_1) - 4\epsilon. \end{aligned} \tag{3.17}$$

Encoding Error at Cognitive Transmitter: An encoding error occurs at the cognitive transmitter if no T^n in bin index m_c can be found such that the sequence $(T^n, X_1^n(m_1), U^n(m_1), X_2^n(m_2), V^n(m_2))$ is jointly typical. The probability of this happening can be upper bounded by

$$Pe \leq (1 - 2^{-nI(T;X_1,U,X_2,V)})^{2^{n(\overline{R}_c - R_c)}}.$$

The probability of encoding error can be made arbitrarily small if

$$\overline{R}_c \geq R_c + I(T; X_1, U, X_2, V) \tag{3.18}$$

Decoding Error at Cognitive Receiver: The cognitive receiver determines a bin index $\hat{m}_c$ and a sequence T^n from that bin such that (T^n, Y_c^n) is jointly typical. To analyze the probability of error, we assume that the transmitter wished to communicate message m_c and no error occurred at the cognitive encoder. Then, a decoding error occurs if no T^n in bin m_c is jointly typical with Y_c^n, or if a T^n from a different bin is jointly typical with Y_c^n. The probability that no T^n in bin m_c is jointly typical with Y_c^n can be made arbitrarily small for suitably large n. The probability that a T^n from a different bin is jointly typical with Y_c^n can be made small if

$$\overline{R}_c \leq I(T; Y_c) - 3\epsilon \tag{3.19}$$

Choosing $\overline{R}_c = R_c + I(T; X_1, U, X_2, V) + \epsilon$, we get

$$R_c \leq I(T; Y_c) - I(T; X_1, U, X_2, V) - 4\epsilon. \tag{3.20}$$

Hence the region described by $\mathcal{R}_{in}$ is achievable. $\square$

3.5 Optimality of the Achievable Region

In this section, we show that for the Gaussian MACRC, when the cross channel gain from the cognitive transmitter to the primary receiver, $b \leq 1$, the achievable region described by Theorem 3.4.1 meets the outer bound described in Theorem 3.3.1. Let $\rho_1, \rho_2 \in [0, 1]$ such that $\rho_1^2 + \rho_2^2 \leq 1$. Define $\Delta = 1 - \rho_1^2 - \rho_2^2$. Define the function $L : \mathbb{R}_+ \to \mathbb{R}$ by $L(x) = \frac{1}{2}\log(1 + x)$. Let $\mathcal{R}(\rho_1, \rho_2)$ denote the set of rate tuples $(R_1, R_2, R_c) \in \mathbb{R}_+^3$ given by

$$\begin{aligned}
R_1 &\leq L\left(\frac{\left(\sqrt{P_1} + b\sqrt{P_c}\rho_1\right)^2}{1 + b^2 P_c \Delta}\right)\\
R_2 &\leq L\left(\frac{\left(\sqrt{P_2} + b\sqrt{P_c}\rho_2\right)^2}{1 + b^2 P_c \Delta}\right)\\
R_1 + R_2 &\leq L\left(\frac{\left(\sqrt{P_1} + b\sqrt{P_c}\rho_1\right)^2 + \left(\sqrt{P_2} + b\sqrt{P_c}\rho_2\right)^2}{1 + b^2 P_c \Delta}\right)\\
R_c &\leq L\left(P_c \Delta\right)
\end{aligned} \tag{3.21}$$

Let $\mathcal{R}$ denote the set of rate tuples $(R_!, R_2, R_c)$ described by

$$\mathcal{R} = \overline{\bigcup_{\rho_1, \rho_2 \in [0,1] : \rho_1^2 + \rho_2^2 \leq 1} \mathcal{R}(\rho_1, \rho_2)}. \tag{3.22}$$

Then, the following theorem describes the capacity region of the MACRC when the cross channel gain $b \leq 1$.

Theorem 3.5.1. *When the cross channel gain $b \leq 1$ in a MACRC, the capacity region of the channel is given by*

$$\mathcal{C}_{MACRC} = \mathcal{R}. \tag{3.23}$$

3.5.1 Proof of Inner Bound

Consider the achievable region given by (5.35). Take in (3.14), (X_1, X_2, X_c) jointly Gaussian with zero means and variances (P_1, P_2, P_c) respectively and where $E(X_1X_2) = 0$ and $E(X_cX_i) = \rho_i\sqrt{P_iP_c}$ for $i = 1, 2$. Choose U and V to be deterministic random variables.

The random variable T is defined as follows

$$T = X_c + \alpha_1 X_1 + \alpha_2 X_2,$$

where α_1 and α_2 are constants to be specified. It is evident that for this choice of random variables we have,

$$\begin{aligned} R_c &= I(T;Y_c) - I(T;X_1,U,X_2,V) \\ &= I(T;Y_c) - I(T;X_1,X_2) \\ &= I(T;Y_c|X_1,X_2) - I(T;X_1,X_2|Y_c) \\ &= I(X_c;Y_c|X_1,X_2) - I(T;X_1,X_2|Y_c) \end{aligned} \tag{3.24}$$

From [55, Lemma 1], there exists α_1^*, α_2^* such that $I(T;X_1,X_2|Y_c) = 0$. We choose $\alpha_1 = \alpha_1^*$ and $\alpha_2 = \alpha_2^*$. Therefore, we get

$$\begin{aligned} R_c &= I(T;Y_c) - I(T;X_1,U,X_2,V) \\ &= I(T;Y_c) - I(T;X_1,X_2) \\ &= I(X_c;Y_c|X_1,X_2,U) \\ &= L\left(P_c(1-\rho_1^2-\rho_2^2)\right). \end{aligned} \tag{3.25}$$

With these choice of random variables, we observe that

$$h(Y_1|X_2) = \frac{1}{2}\log\left(2\pi e\left(1 + P_1 + 2b\sigma_1 + b^2 P_c(1-\rho_2^2)\right)\right)$$
$$h(Y_1|X_1) = \frac{1}{2}\log\left(2\pi e\left(1 + P_2 + 2b\sigma_2 + b^2 P_c(1-\rho_1^2)\right)\right)$$
$$h(Y_1) = \frac{1}{2}\log\left(2\pi e\left(1 + P_1 + P_2 + 2b(\sigma_1+\sigma_2) + b^2 P_c\right)\right)$$
$$h(Y_1|X_1, X_2) = \frac{1}{2}\log\left(2\pi e\left(1 + b^2 P_c(1-\sigma_1^2-\sigma_2^2)\right)\right).$$

Substituting the above expressions and (3.25) into the achievable region in (5.35), it is easy to see that the achievable region matches the rate region given by $\mathcal{R}$.

3.5.2 Outer Bound

In this section, we show that Gaussians maximize the outer bound derived in Section 3.3. From Section 3.3, we have the outer bound as the union over all the rate tuples that satisfy

$$\begin{aligned}
R_1 &\leq h(Y_1|V, X_2, Q) - h(Y_1|X_1, U, X_2, V, Q) \\
R_2 &\leq h(Y_1|U, X_1, Q) - h(Y_1|X_1, U, X_2, V, Q) \\
R_1 + R_2 &\leq h(Y_1|Q) - h(Y_1|X_1, U, X_2, V, Q) \\
R_c &\leq h(Y_c|X_1, U, X_2, V, Q) - h(N_2)
\end{aligned}$$

for some $P_{Q,X_1,U,X_2,V}$ where $Y_1 = X_1 + X_2 + bX_c + N_1$, $Y_c = X_c + a_1X_1 + a_2X_2 + N_2$ and X_1 and X_2 are independent given Q. In this section, we derive the outer bound for a degenerate Q (that is, we assume that X_1 and X_2 are independent). The overall outer bound is in fact the convex hull over the entire obtained region.

Since $0 \leq I(X_c; Y_c|X_1, U, X_2, V) \leq \frac{1}{2}\log(1 + P_c)$, there exists some $\gamma \in$

$[0, 1]$ such that

$$I(X_c; Y_c | X_1, U, X_2, V) = \frac{1}{2} \log (1 + \gamma P_c),$$

and consequently

$$h(Y_c | X_1, U, X_2, V) = \frac{1}{2} \log (2\pi e(1 + \gamma P_c)). \tag{3.26}$$

Let J be a Gaussian noise with variance $1 - b^2$. Using the Entropy Power Inequality, we obtain

$$\begin{aligned} 2^{2h(Y_1|X_1,U,X_2,V)} &= 2^{2h(bX_c+N_1|X_1,U,X_2,V)} \\ &= 2^{2h(bY_c+J|X_1,U,X_2,V)} \\ &\geq 2^{2h(bY_c|X_1,U,X_2,V)} + 2^{2h(J)} \\ &= 2\pi e \left(b^2(1 + \gamma P_c) + 1 - b^2\right) \\ &= 2\pi e \left(1 + \gamma b^2 P_c\right). \end{aligned} \tag{3.27}$$

Next, we recall that for a given covariance matrix of (X_1, X_2, X_c, U, V), the conditional entropies $h(Y_1|V, X_2)$, $h(Y_1|U, X_1)$ and $h(Y_1)$ are maximized if (X_1, X_2, X_c, U, V) is a Gaussian vector. Also, we have that

$$h(Y_1|X_1, U) \leq h(Y_1|X_1) \text{ and } h(Y_1|X_2, V) \leq h(Y_1|X_2)$$

Finally, for Gaussian X_1, X_2, X_c such that X_1 and X_2 are independent and $E[X_i X_c] =$

$\rho_i\sqrt{P_iP_c}$, we observe that

$$\begin{aligned}\frac{1}{2}\log\left(2\pi e(1+\gamma P_c)\right) &= h(Y_c|X_1,U,X_2,V)\\ &= h(X_c+N|X_1,U,X_2,V)\\ &\leq h(X_c+N|X_1,X_2) \qquad (3.28)\\ &= \frac{1}{2}\log\left(2\pi e(1+\Delta P_c)\right).\end{aligned}$$

Hence, we have $\gamma \leq \Delta = 1-\rho_1^2-\rho_2^2$.

Hence, the outer bound reduces to

$$\begin{aligned}R_1 &\leq \frac{1}{2}\log\left(\frac{1+P_1+2b\sigma_1+b^2P_c(1-\rho_2^2)}{1+b^2P_c\gamma}\right)\\ R_2 &\leq \frac{1}{2}\log\left(\frac{1+P_2+2b\sigma_2+b^2P_c(1-\rho_1^2)}{1+b^2P_c\gamma}\right) \qquad (3.29)\\ R_1+R_2 &\leq \frac{1}{2}\log\left(\frac{1+P_1+P_2+2b(\sigma_1+\sigma_2)+b^2P_c}{1+b^2P_c\gamma}\right)\\ R_c &\leq \frac{1}{2}\log\left(1+\gamma P_c\right).\end{aligned}$$

where the outer bound is optimized over all $\rho_1,\rho_2 \in [0,1]$ such that $\rho_1^2+\rho_2^2 \leq 1$ and $\gamma \leq \Delta$.

We note that if one substitutes $\gamma = \Delta$ into (3.30), we get the desired region (3.22). The following lemma concludes the proof of the outer bound of Theorem 3.5.1, by showing that it is sufficient to consider $\gamma = \Delta$.

Lemma 3.5.2. *The region of all rate triples* (R_1, R_2, R_c) *such that*

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log\left(\frac{1+P_1+2b\sigma_1+b^2P_c(1-\rho_2^2)}{1+b^2P_c\gamma}\right) \\ R_2 &\leq \frac{1}{2}\log\left(\frac{1+P_2+2b\sigma_2+b^2P_c(1-\rho_1^2)}{1+b^2P_c\gamma}\right) \\ R_1+R_2 &\leq \frac{1}{2}\log\left(\frac{1+P_1+P_2+2b(\sigma_1+\sigma_2)+b^2P_c}{1+b^2P_c\gamma}\right) \\ R_c &\leq \frac{1}{2}\log\left(1+\gamma P_c\right), \end{aligned}$$

for some $(\sigma_2, \sigma_1) = (\sqrt{P_2P_c}\rho_2, \sqrt{P_1P_c}\rho_1)$ *such that* $0 \leq \rho_1^2 + \rho_2^2 \leq 1$ *and some* $\gamma \in [0, \Delta]$, $\Delta = (1-\rho_1^2-\rho_2^2)$ *remains the same if one takes* $\gamma = \Delta$ *(and therefore equal to the region (3.22)).*

Proof. Fix $R_c = \frac{1}{2}\log(1+dP_c)$. To obtain this rate, Δ cannot be smaller than d. Consider therefore $\Delta \in [d, 1]$. Denote

$$\begin{array}{ll} c(\Delta) & = L(b^2\Delta P_c) \\ f_1(\rho_1,\rho_2) & = L(P_1+2b\sigma_1+b^2P_c(1-\rho_2^2)) \\ f_2(\rho_1,\rho_2) & = L(P_2+2b\sigma_2+b^2P_c(1-\rho_1^2)) \\ f_3(\rho_1,\rho_2) & = L(P_1+P_2+2b(\sigma_1+\sigma_2)+b^2P_c) \end{array} \tag{3.30}$$

For $\gamma = \Delta$ and the rate R_c we fixed, the region becomes

$$\begin{array}{rcl} R_1 & \leq & f_1(\rho_1,\rho_2) - c(\Delta) \\ R_2 & \leq & f_2(\rho_1,\rho_2) - c(\Delta) \\ R_1+R_2 & \leq & f_3(\rho_1,\rho_2) - c(\Delta) \\ R_c & = & \frac{1}{2}\log(1+dP_c) \end{array} \tag{3.31}$$

where $\rho_1^2+\rho_2^2 = 1-\Delta$ and $\Delta \in [d, 1]$.

If we allow $\gamma \leq \Delta$, it is obvious that the optimal γ is d and the region becomes

$$\begin{array}{rcl} R_1 & \leq & f_1(\rho_1,\rho_2) - c(d) \\ R_2 & \leq & f_2(\rho_1,\rho_2) - c(d) \\ R_1 + R_2 & \leq & f_3(\rho_1,\rho_2) - c(d) \\ R_c & = & \frac{1}{2}\log(1+dP_c) \end{array} \tag{3.32}$$

where $\rho_1^2 + \rho_2^2 = 1 - \Delta$ and $\Delta \in [d, 1]$.

The regions (3.31) and (3.32) would coincide iff the optimal Δ in (3.31) as well as in (3.32) is d. We show that this is indeed the case and this establishes that the optimal γ is equal to Δ.

The optimal Δ in (3.31) is d: First, we observe that the sum of the bounds on the individual rates R_1, R_2 in (3.31) is never smaller than the sum-rate bound, that is, we establish the inequality

$$f_1(\rho_1,\rho_2) - c(\Delta) + f_2(\rho_1,\rho_2) - c(\Delta) > f_3(\rho_1,\rho_2) - c(\Delta).$$

This implies that region (3.31) is basically determined by the vertex points of pentagons. Hence, a vertex point of interest in (3.31) is determined either by the bounds on R_1+R_2 and R_1, or by the bounds on R_1+R_2 and R_2 (but not simultaneously by the two bounds on the individual rates R_1 and R_2). First, assume that the determining bounds are those of $R_1 + R_2$ and R_2. Let $\tilde{\rho}_1 \in [0, \sqrt{1-d}]$ be the correlation coefficient that achieve this vertex point, and let $\tilde{\sigma}_1$ be the corresponding correlation. It is easy to realize that for fixed ρ_1 the functions f_2, f_3 are decreasing with Δ and therefore the minimal possible Δ for this vertex point is the optimal, i.e., $\Delta = d$.

Similarly, if the determining bounds are those of $R_1 + R_2$ and R_2 we notice that for fixed ρ_2, the functions f_1, f_3 are decreasing with Δ, and therefore the optimal Δ for this vertex point is the minimal, i.e., $\Delta = d$.

The optimal Δ in (3.32) is d: We observe that the sum of the bounds on the individual rates R_1, R_2 is never smaller than the sum-rate bound in (3.32) too. That is, we have the following inequality:

$$f_1(\rho_1, \rho_2) - c(d) + f_2(\rho_1, \rho_2) - c(d) > f_3(\rho_1, \rho_2) - c(d).$$

Hence, similarly to (3.31), a vertex point of interest in (3.32) is determined either by the bounds on $R_1 + R_2$ and R_1, or by the bounds on $R_1 + R_2$ and R_2. And, similarly, the arguments

- for fixed ρ_1 the functions f_2, f_3 are decreasing with Δ
- for fixed ρ_2 the functions f_1, f_3 are decreasing with Δ,

are sufficient to prove that the optimal Δ is d.

This concludes the proof of Lemma 3.5.2 and Theorem 3.5.1. $\square$

In Figure 3.2, we depict the capacity region of MACRC when $P_1 = P_2 = P_c = 10$ and the channel gain $b = 0.5$.

3.6 Conclusions

In this chapter, we analyzed the capacity of the cognitive radio channel in a multiple access setting. We derived an outer bound on the capacity region of the MACRC when the cross channel gain from the cognitive transmitter to the primary

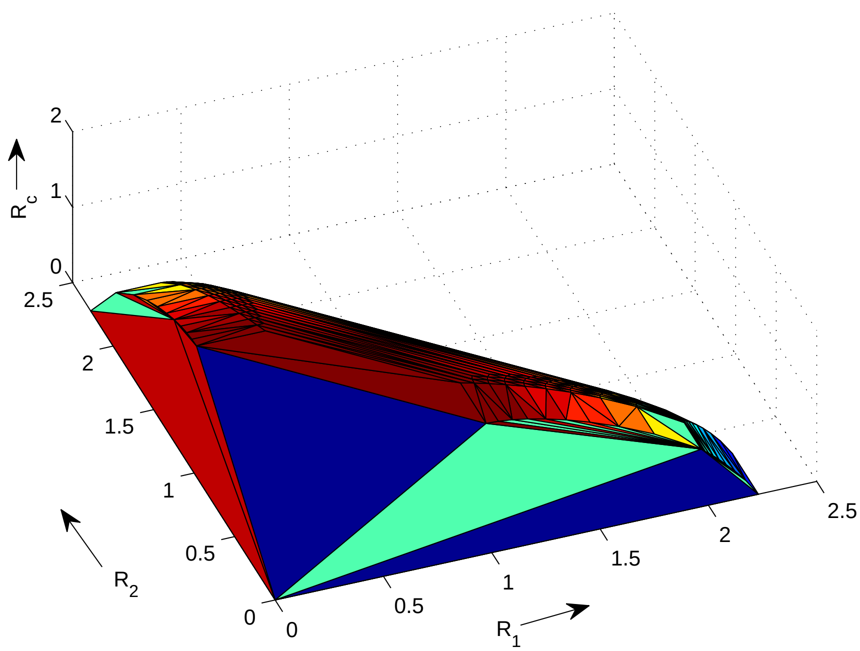


Figure 3.2: Plot of Achievable region and Outer bound for Interference channel with Cognitive Helper

receiver, $b \leq 1$. We also show that Gaussians maximize the outer bound. We derive an achievable region using superposition and dirty paper coding at the cognitive transmitter. Finally, we show that when the cross channel gain $b \leq 1$, the achievable region achieves the entire capacity region.

Chapter 4

Interference Networks with Cognitive Relay

In this chapter, we analyze the capacity region of an interference network with cognitive relay. Relay nodes serve to assist the transmitters in communicating their messages to their receivers. In this model, it is assumed that the relay nodes know the message of all the transmitters apriori. The chapter is organized as follows: The problem statement, prior work and our contributions are discussed in Section 4.1. In Section 4.2, we describe the system model. In Section 4.3, we describe a transmission strategy and a corresponding achievable region for the interference network with cognitive relay. We derive an outer bound on the capacity region of the interference network with cognitive relay in Section 4.4. We provide numerical results in Section 4.5. We conclude the chapter in Section 4.6.

4.1 Introduction

Networks with cognitive users are gaining prominence with the development of cognitive radio technology, which is aimed at improving the spectral efficiency and the system performance by designing nodes which can adapt their strategy based on the network setup. Much recent work has been focused on the two user interference channel with a cognitive transmitter [39–42, 57]. In this channel setting, one of the transmitters has non-causal access to the message transmitted by the other transmitter. In this chapter, we study a two user Gaussian interference channel in the presence of a cognitive relay (see Figure 4.1). This channel model is different from the one used in [39–42,57] in that, each transmitter has access to only their respective messages. However, we assume that there is a cognitive relay node which has non-causal access to the messages of both the transmitters. This relay node serves only to assist the two transmitters in communicating their messages to their respective receivers. An achievable region for this system is described in [49]. Other work on this channel model include [50–52].

In this chapter, we present a new achievable region for the Gaussian interference channel with a cognitive relay. This region is a generalization of the achievable region given in [49]. The coding scheme used in this chapter is a combination of the Han-Kobayashi coding scheme for the general interference channel [4] and Costa's dirty paper coding [22]. The Han-Kobayashi coding scheme was also used for the interference channel with a normal (non cognitive) relay in [51]. We perform dirty paper coding simultaneously for both the users instead of time sharing between the two users as was done in [49]. We also derive an outer bound on the capacity region of the Gaussian interference channel with a cognitive relay. The outer bound is obtained by allowing transmitter co-operation to obtain a MIMO cognitive radio channel [57]. We use the outer bound of the MIMO cognitive radio channel as the outer bound for the capacity region of the interference channel with cognitive relay. We also derive the degree of freedom (d.o.f.) region of the interference channel with cognitive relay. We show that we can achieve the full degrees of freedom of a two user no-interference channel for a large range of channel parameters. The contributions of this chapter were presented in [65].

Throughout the chapter, we denote random variables by capital letters, their realizations by lower case and their alphabets by calligraphic letters (eg. X, x and $\mathcal{X}$ respectively). We denote vectors of length n with boldface letters (e.g. $\mathbf{x}^n$), and the i^{th} element of a vector $\mathbf{x}^n$ by x_i. For any set S, $\overline{S}$ and $\mathrm{CH}(S)$ denote the closure and convex hull of S respectively. For any vector or matrix $\mathbf{A}$, $\mathbf{A}'$ denotes its transpose. $\mathrm{Tr}(\mathbf{A})$ denotes the trace of a matrix $\mathbf{A}$. We define the function $L : \mathbb{R}_+ \to \mathbb{R}$ as $L(x) = \frac{1}{2}\log(1+x)$.

4.2 System Model

We study a Gaussian interference channel with two transmitters, two receivers and a cognitive relay. The system model is described in Figure 4.1. The interference channel is described by $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r, \mathcal{Y}_1, \mathcal{Y}_2, p(y_1, y_2|x_1, x_2, x_r))$, where $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_r$ are the input alphabets associated with the two transmitters and the relay, $\mathcal{Y}_1, \mathcal{Y}_2$ are the two output alphabets. For the Gaussian channel, we assume that all the alphabets are the entire reals $\mathbb{R}$. Source i, $i = 1, 2$ has message $m_i \in \{1, \ldots, 2^{nR_i}\}$ to be communicated to destination i over n channel uses. The relay has non-causal access to both the messages m_1 and m_2 and assists the two sources. Let X_1, X_2, X_r and Y_1, Y_2 denote the random variables representing the respective channel inputs and outputs. Then, the input-output relationship can be

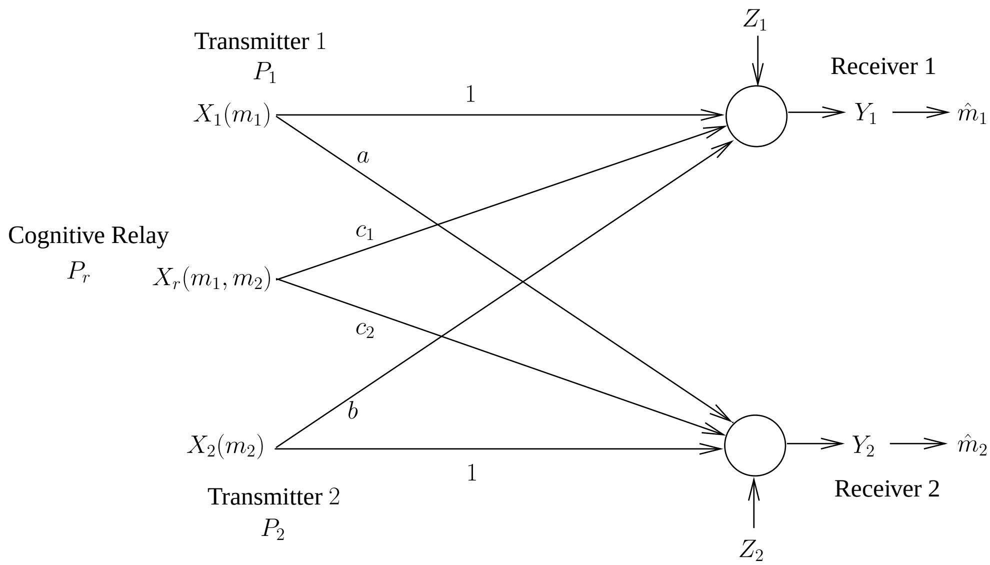


Figure 4.1: System model for Gaussian Interference Channel with Cognitive Relay.

represented by the system equations

$$\begin{aligned} Y_1 &= X_1 + bX_2 + c_1 X_r + Z_1 \\ Y_2 &= aX_1 + X_2 + c_2 X_r + Z_2, \end{aligned} \tag{4.1}$$

where a, b, c_1 and c_2 represent the channel gains as shown in Figure 4.1. Z_1 and Z_2 denote the additive noise which are i.i.d. Gaussian random variables distributed as $\mathcal{N}(0,1)$. The channel inputs must satisfy the following power constraints:

$$\frac{1}{n}\sum_{i=1}^{n} E[X_{j,i}^2] \leq P_j, \quad j \in \{1, 2, r\}. \tag{4.2}$$

A $(2^{nR_1}, 2^{nR_2}, n, Pe)$ code consists of message sets $M_1 = \{1, \ldots, 2^{nR_1}\}$ and $M_2 = \{1, \ldots, 2^{nR_2}\}$, three encoding functions

$$\begin{aligned} f_1 : M_1 \rightarrow \mathcal{X}_1^n, \;\; f_2 : M_2 \rightarrow \mathcal{X}_2^n, \\ f_r : M_1 \times M_2 \rightarrow \mathcal{X}_r^n, \end{aligned} \tag{4.3}$$

and two decoding functions

$$g_1 : \mathcal{Y}_1^n \to M_1, \quad g_2 : \mathcal{Y}_2^n \to M_2, \tag{4.4}$$

such that the transmitted codewords $\mathbf{X}_1^n, \mathbf{X}_2^n$ and $\mathbf{X}_r^n$ satisfy the power constraints given by (4.2) and an error probability $\leq Pe = \max(P_{e,1}, P_{e,2})$. For $t = 1, 2$, we have

$$P_{e,t} = \frac{1}{2^{n(R_1+R_2)}} \sum_{(m_1,m_2)} Pr[g(\mathbf{Y}_t^n) \neq m_t | (m_1, m_2) \text{ sent}]. \tag{4.5}$$

A rate pair (R_1, R_2) is achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n, Pe^{(n)})$ codes such that $Pe^{(n)} \to 0$ as $n \to \infty$. The capacity region of the interference channel with cognitive relay is then the set of all rate pairs (R_1, R_2) that are achievable, and is denoted by $\mathcal{C}_{IC}$. The d.o.f. region of the Gaussian interference channel with cognitive relay $\mathcal{D}$ is defined as

$$\mathcal{D} = \left\{ \begin{array}{l} (d_1, d_2) \in \mathbb{R}_+^2 : \forall w \in \mathbb{R}_+, \\ wd_1 + d_2 \leq \limsup_{P_1+P_2+P_r \to \infty} \\ \qquad \sup_{(R_1,R_2) \in \mathcal{C}_{IC}} \frac{wR_1+R_2}{\log(P_1+P_2+P_r)} \end{array} \right\}. \tag{4.6}$$

4.3 Achievable Region and Transmission Strategy

In this section, we describe an achievable region for the interference channel with cognitive relay and describe the corresponding transmission strategy.

Let $\mathcal{P}$ denote the set of $(P_{11}, P_{12}, P_{21}, P_{22}, P_{r11}, P_{r12}, P_{r21}, P_{r22}, P_{r3}, P_{r4})$

described by

$$\left\{ \begin{array}{c} \begin{pmatrix} P_{11}, P_{12}, P_{21}, P_{22}, P_{r11}, P_{r12}, \\ P_{r21}, P_{r22}, P_{r3}, P_{r4} \end{pmatrix} : \\ P_{11} + P_{12} = P_1 \\ P_{21} + P_{22} = P_2 \\ P_{r11} + P_{r12} + P_{r21} + P_{r22} + P_{r3} + P_{r4} = P_r \end{array} \right\}. \tag{4.7}$$

Let $P^* \in \mathcal{P}$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{-1, 1\}$. We denote $r_{11}, r_{12}, r_{21}, r_{22}$ as follows :

$$\begin{array}{c} r_{1j} = (\sqrt{P_{1j}} + \alpha_j c_1 \sqrt{P_{r1j}})^2, \;\; j \in \{1, 2\}, \\ r_{2j} = (b\sqrt{P_{2j}} + \beta_j c_1 \sqrt{P_{r2j}})^2, \;\; j \in \{1, 2\}. \end{array} \tag{4.8}$$

Let $\mathcal{R}_{i1}^{P^*}(\alpha_1, \alpha_2, \beta_1, \beta_2)$ denote the set described by

$$\left\{ \begin{array}{l} (R_{11}, R_{12}, R_{21}) : R_{11} \geq 0, R_{12} \geq 0, R_{21} \geq 0 \\ R_{11} \leq L\left(\frac{r_{11}}{1+r_{22}+c_1^2(P_{r3}+P_{r4})}\right) \\ R_{12} \leq L\left(\frac{r_{12}}{1+r_{22}+c_1^2(P_{r3}+P_{r4})}\right) \\ R_{21} \leq L\left(\frac{r_{21}}{1+r_{22}+c_1^2(P_{r3}+P_{r4})}\right) \\ R_{11} + R_{12} \leq L\left(\frac{r_{11}+r_{12}}{1+r_{22}+c_1^2(P_{r3}+P_{r4})}\right) \\ R_{11} + R_{21} \leq L\left(\frac{r_{11}+r_{21}}{1+r_{22}+c_1^2(P_{r3}+P_{r4})}\right) \\ R_{12} + R_{21} \leq L\left(\frac{r_{12}+r_{21}}{1+r_{22}+c_1^2(P_{r3}+P_{r4})}\right) \\ R_{11} + R_{12} + R_{21} \leq L\left(\frac{r_{11}+r_{12}+r_{21}}{1+r_{22}+c_1^2(P_{r3}+P_{r4})}\right) \end{array} \right\}. \tag{4.9}$$

We denote $s_{11}, s_{12}, s_{21}, s_{22}$ as follows :

$$\begin{array}{c} s_{1j} = (a\sqrt{P_{1j}} + \alpha_j c_2 \sqrt{P_{r1j}})^2, \;\; j \in \{1, 2\}, \\ s_{2j} = (\sqrt{P_{2j}} + \beta_j c_2 \sqrt{P_{r2j}})^2, \;\; j \in \{1, 2\}. \end{array} \tag{4.10}$$

$\mathcal{R}_{i2}^{P*}(\alpha_1, \alpha_2, \beta_1, \beta_2)$ denotes the set described by

$$\left\{ \begin{array}{l} (R_{12}, R_{21}, R_{22}) : R_{12} \geq 0, R_{21} \geq 0, R_{22} \geq 0 \\ R_{12} \leq L\left(\frac{s_{12}}{1+s_{11}+c_2^2(P_{r3}+P_{r4})}\right) \\ R_{21} \leq L\left(\frac{s_{21}}{1+s_{11}+c_2^2(P_{r3}+P_{r4})}\right) \\ R_{22} \leq L\left(\frac{s_{22}}{1+s_{11}+c_2^2(P_{r3}+P_{r4})}\right) \\ R_{12} + R_{21} \leq L\left(\frac{s_{12}+s_{21}}{1+s_{11}+c_2^2(P_{r3}+P_{r4})}\right) \\ R_{12} + R_{22} \leq L\left(\frac{s_{12}+s_{22}}{1+s_{11}+c_2^2(P_{r3}+P_{r4})}\right) \\ R_{21} + R_{22} \leq L\left(\frac{s_{21}+s_{22}}{1+s_{11}+c_2^2(P_{r3}+P_{r4})}\right) \\ R_{12} + R_{21} + R_{22} \leq \left(\frac{s_{12}+s_{21}+s_{22}}{1+s_{11}+c_2^2(P_{r3}+P_{r4})}\right) \end{array} \right\}. \tag{4.11}$$

Let $\overline{\alpha} = (\alpha_1, \alpha_2)$ and $\overline{\beta} = (\beta_1, \beta_2)$. Let $\mathcal{R}_{in,1}^{P*}(\overline{\alpha}, \overline{\beta})$ and $\mathcal{R}_{in,2}^{P*}(\overline{\alpha}, \overline{\beta})$ be the set of rate pairs (R_1, R_2) described by

$$\mathcal{R}_{in,1}^{P^*}(\overline{\alpha},\overline{\beta}) = \left\{ \begin{array}{l} (R_1,R_2): R_1 \geq 0, R_2 \geq 0 \\ R_1 = R_{11}+R_{12}+R_{13} \\ R_2 = R_{21}+R_{22}+R_{23} \\ (R_{11},R_{12},R_{21}) \in \mathcal{R}_{i1}^{P^*}(\overline{\alpha},\overline{\beta}) \\ (R_{12},R_{21},R_{22}) \in \mathcal{R}_{i2}^{P^*}(\overline{\alpha},\overline{\beta}) \\ R_{13} \leq L\left(\frac{c_1^2 P_{r3}}{1+c_1^2 P_{r4}}\right) \\ R_{23} \leq L(c_2^2 P_{r4}) \end{array} \right\}, \tag{4.12}$$

$$\mathcal{R}_{in,2}^{P^*}(\overline{\alpha},\overline{\beta}) = \left\{ \begin{array}{l} (R_1,R_2): R_1 \geq 0, R_2 \geq 0 \\ R_1 = R_{11}+R_{12}+R_{13} \\ R_2 = R_{21}+R_{22}+R_{23} \\ (R_{11},R_{12},R_{21}) \in \mathcal{R}_{i1}^{P^*}(\overline{\alpha},\overline{\beta}) \\ (R_{12},R_{21},R_{22}) \in \mathcal{R}_{i2}^{P^*}(\overline{\alpha},\overline{\beta}) \\ R_{13} \leq L(c_1^2 P_{r3}) \\ R_{23} \leq L\left(\frac{c_2^2 P_{r4}}{1+c_2^2 P_{r3}}\right) \end{array} \right\}. \tag{4.13}$$

Let $\mathcal{R}_{in}$ be the set of rate pairs described by

$$\mathcal{R}_{in} = \overline{\mathrm{CH}\left(\bigcup_{P^* \in \mathcal{P}} \bigcup_{\overline{\alpha},\overline{\beta}} \left(\mathcal{R}_{in,1}^{P^*}(\overline{\alpha},\overline{\beta}) \cup \mathcal{R}_{in,2}^{P^*}(\overline{\alpha},\overline{\beta}) \right) \right)}. \tag{4.14}$$

Then, the following theorem describes an achievable region for the Gaussian interference channel with cognitive relay.

Theorem 4.3.1. *The capacity region of the Gaussian interference channel with cognitive relay $\mathcal{C}_{IC}$ satisfies*

$$\mathcal{R}_{in} \subseteq \mathcal{C}_{IC}. \tag{4.15}$$

Proof of Theorem 4.3.1: We fix a $P^* \in \mathcal{P}$ where $\mathcal{P}$ is described in (4.7). We also fix $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{-1, 1\}$. We show that $\mathcal{R}_{in,1}^{P^*}(\overline{\alpha},\overline{\beta})$ is achievable. We assume that $P_{11}, P_{12}, P_{21}, P_{22} > 0$. The proof for the case when some of

$P_{11}, P_{12}, P_{21}, P_{22}$ are equal to zero is identical to the one presented here and is hence omitted.

For $i = 1, 2$, source i splits its message $m_i \in \{1, \ldots, 2^{nR_i}\}$ into 3 independent parts $(m_{i1}, m_{i2}, m_{i3}) \in \{1, \ldots, 2^{nR_{i1}}\} \times \{1, \ldots, 2^{nR_{i2}}\} \times \{1, \ldots, 2^{nR_{i3}}\}$ such that $R_{i1} + R_{i2} + R_{i3} = R_i$.

Encoding Scheme : For $i = 1, 2$, transmitter i encodes message m_{i1} into $\mathbf{X}_{i1}^n$, such that $p(\mathbf{x}_{i1}^n) = \Pi_{j=1}^n P(x_{i1,j})$, and $X_{i1,j} \backsim \mathcal{N}(0, P_{i1})$. Message m_{i2} is then encoded into $\mathbf{X}_{i2}^n$, such that $p(\mathbf{x}_{i2}^n) = \Pi_{j=1}^n P(x_{i2,j})$, and $X_{i2,j} \backsim \mathcal{N}(0, P_{i2})$. Transmitter i transmits $\mathbf{X}_i^n = \mathbf{X}_{i1}^n + \mathbf{X}_{i2}^n$.

The relay encodes message (m_{11}, m_{12}) into $\mathbf{X}_{r1}^n = \alpha_1 \sqrt{(P_{r11}/P_{11})}\mathbf{X}_{11}^n + \alpha_2 \sqrt{(P_{r12}/P_{12})}\mathbf{X}_{12}^n$, and message (m_{21}, m_{22}) into $\mathbf{X}_{r2}^n = \beta_1 \sqrt{(P_{r21}/P_{21})}\mathbf{X}_{21}^n + \beta_2 \sqrt{(P_{r22}/P_{22})}\mathbf{X}_{22}^n$. The relay node encodes message m_{13} into $\mathbf{X}_{r3}^n$ treating $(b + c_1 \beta_2 \sqrt{(P_{r22}/P_{22})})\mathbf{X}_{22}^n$ as non-causally known interference at receiver 1. That is, $\mathbf{X}_{r3}^n$ is formed using Costa's dirty paper coding [22], and is distributed as $p(\mathbf{x}_{r3}^n) = \Pi_{i=1}^n P(x_{r3,i})$ and $X_{r3,i} \backsim \mathcal{N}(0, P_{r3})$. Finally, the relay encodes message m_{23} into $\mathbf{X}_{r4}^n$ treating $(a + c_2 \alpha_1 \sqrt{(P_{r11}/P_{11})})\mathbf{X}_{11}^n + c_2 \mathbf{X}_{r3}^n$ as non-causally known interference at receiver 2. $\mathbf{X}_{r4}^n$ is distributed as $p(\mathbf{x}_{r4}^n) = \Pi_{i=1}^n P(x_{r4,i})$ and $X_{r4,i} \backsim \mathcal{N}(0, P_{r4})$. The relay transmits $\mathbf{X}_r^n = \mathbf{X}_{r1}^n + \mathbf{X}_{r2}^n + \mathbf{X}_{r3}^n + \mathbf{X}_{r4}^n$. It is to be noted that this coding scheme uses the result that the capacity region of a Gaussian broadcast channel with additive state known non-causally at the transmitter is the same as the capacity region of the same broadcast channel with no state [59].

Decoding : Receiver 1 decodes (m_{11}, m_{12}, m_{21}) jointly by treating $(b +$

$c_1\beta_2\sqrt{(P_{r22}/P_{22})})\mathbf{X}_{22}^n+c_1(\mathbf{X}_{r3}^n+\mathbf{X}_{r4}^n)+\mathbf{Z}_1^n$ as Gaussian noise. Hence, (m_{11}, m_{12}, m_{21}) can be successfully decoded at receiver 1 if $(R_{11}, R_{12}, R_{21}) \in \mathcal{R}_{i1}^{P^*}(\overline{\alpha}, \overline{\beta})$. Receiver 1 then decodes message m_{13} by treating $c_1\mathbf{X}_{r4}^n + \mathbf{Z}_1^n$ as Gaussian noise.

Receiver 2 decodes (m_{12}, m_{21}, m_{22}) jointly by treating $(a+c_2\alpha_1\sqrt{(P_{r11}/P_{11})})\mathbf{X}_{11}^n+ c_2(\mathbf{X}_{r3}^n + \mathbf{X}_{r4}^n) + \mathbf{Z}_2^n$ as Gaussian noise. Hence, (m_{12}, m_{21}, m_{22}) can be successfully decoded at receiver 2 if $(R_{12}, R_{21}, R_{22}) \in \mathcal{R}_{i2}^{P^*}(\overline{\alpha}, \overline{\beta})$. Finally, message m_{23} is decoded by treating $\mathbf{Z}_2^n$ as noise.

Hence, it follows that $\mathcal{R}_{in,1}^{P^*}(\alpha, \beta)$ is achievable. Similarly, $\mathcal{R}_{in,2}^{P^*}(\alpha, \beta)$ is also achievable. Hence, the region described by $\mathcal{R}_{in}$ is achievable for the interference channel with cognitive relay.

Remark 4.3.1. The coding scheme used to achieve the region given by $\mathcal{R}_{in}$ is a combination of Han-Kobayashi coding scheme for an interference channel [4] and Costa's dirty paper coding [22].

Remark 4.3.2. There are two main differences between the achievable region presented in this chapter and the one given in [49]. The first one is that, we incorporate message splitting and partial interference cancelation at the receiver. This strategy is motivated by the Han-Kobayashi coding scheme for the general interference channel [4]. The second major difference is, we perform dirty paper coding for both the users simultaneously and time share the order in which we perform dirty paper coding. In [49], the authors perform dirty paper coding for only one user at a time and time share between the two dirty paper coding regions.

4.4 Outer Bound on the Capacity Region of Interference Channel with Cognitive Relay

In this section, we derive outer bounds on the capacity region of the interference channel with cognitive relay. we also derive the degree of freedom region of the interference channel with cognitive relay. Let $\gamma > 0$ be any positive real number. We define the following 3×1 matrices:

$$\mathbf{G}_{1\gamma} = \begin{bmatrix} 1 & \frac{c_1}{\sqrt{\gamma}} & \frac{b}{\sqrt{\gamma}} \end{bmatrix}, \quad \mathbf{H}_{1\gamma} = \begin{bmatrix} 0 & \frac{c_2}{\sqrt{\gamma}} & \frac{1}{\sqrt{\gamma}} \end{bmatrix},$$
$$\mathbf{G}_{2\gamma} = \begin{bmatrix} \frac{1}{\sqrt{\gamma}} & \frac{c_1}{\sqrt{\gamma}} & 0 \end{bmatrix}, \quad \mathbf{H}_{2\gamma} = \begin{bmatrix} \frac{a}{\sqrt{\gamma}} & \frac{c_2}{\sqrt{\gamma}} & 1 \end{bmatrix}. \tag{4.16}$$

Consider the two 2-user Gaussian MIMO broadcast channels given in Figures 4.2 and 4.3 with three transmit antennas and one antenna at each receiver. We denote

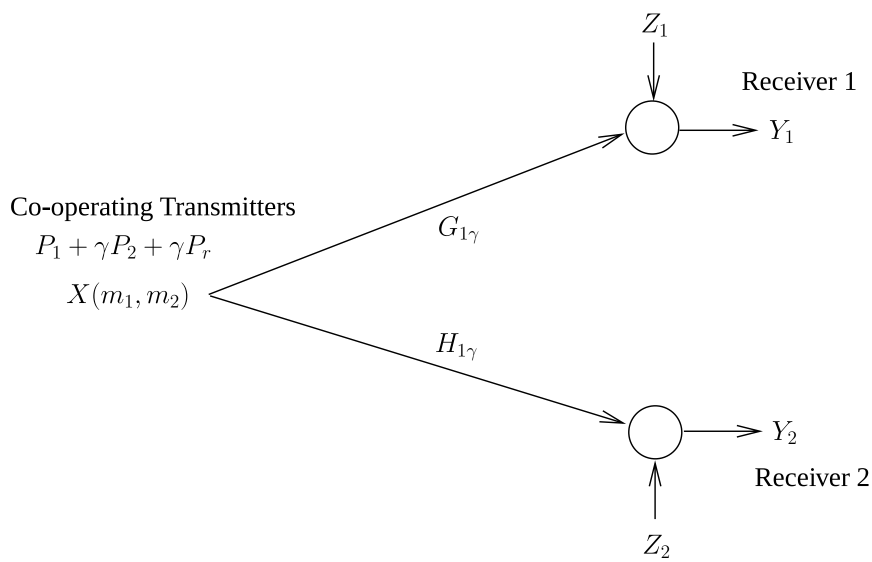


Figure 4.2: Broadcast Channel 1.

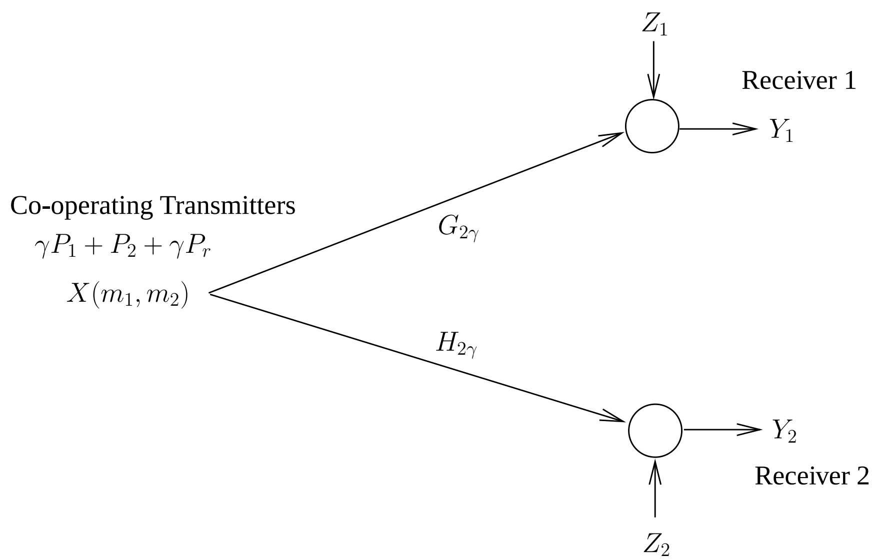


Figure 4.3: Broadcast Channel 2.

the two broadcast channels as BC_1^γ and BC_2^γ respectively. Let their capacity regions be denoted by $\mathcal{C}_{BC,1}^\gamma$ and $\mathcal{C}_{BC,2}^\gamma$ respectively. $\mathcal{R}_{BC,1}^\gamma$ represents the closure of the convex hull of the set of rate pairs described by

$$\left\{\begin{array}{l} (R_1, R_2) : R_1 \geq 0, R_2 \geq 0 \\ R_1 \leq L\left(\frac{\mathbf{G}_{1\gamma}\mathbf{\Sigma}_1\mathbf{G}_{1\gamma}^{'}}{1+\mathbf{G}_{1\gamma}(\mathbf{\Sigma}_1+\mathbf{\Sigma}_2)\mathbf{G}_{1\gamma}^{'}}\right) \\ R_2 \leq L(\mathbf{H}_{1\gamma}\mathbf{\Sigma}_2\mathbf{H}_{1\gamma}^{'}) \\ \mathbf{\Sigma}_1 \succeq \mathbf{0}, \qquad \mathbf{\Sigma}_2 \succeq \mathbf{0} \\ \mathrm{Tr}(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2) \leq P_1 + \gamma P_2 + \gamma P_r \end{array}\right\}. \tag{4.17}$$

$\mathcal{R}^{\gamma}_{BC,2}$ represents the closure of the convex hull of the set of rate pairs described by

$$\left\{ \begin{array}{l} (R_1, R_2) : R_1 \geq 0, R_2 \geq 0 \\ R_1 \leq L(\mathbf{G}_{2\gamma}\mathbf{\Sigma}_1\mathbf{G}^{'}_{2\gamma}\Big) \\ R_2 \leq L\left(\frac{\mathbf{H}_{2\gamma}\mathbf{\Sigma}_2\mathbf{H}^{'}_{2\gamma}}{1+\mathbf{H}_{2\gamma}(\mathbf{\Sigma}_1+\mathbf{\Sigma}_2)\mathbf{H}^{'}_{2\gamma}}\right) \\ \mathbf{\Sigma}_1 \succeq \mathbf{0}, \qquad \mathbf{\Sigma}_2 \succeq \mathbf{0} \\ \mathrm{Tr}(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2) \leq \gamma P_1 + P_2 + \gamma P_r \end{array} \right\}. \tag{4.18}$$

Then, we have the following lemma.

Lemma 4.4.1. *For any $\mu \geq 1$, we have*

$$\max_{(R_1,R_2)\in\mathcal{R}^{\gamma}_{BC,1}} \mu R_1 + R_2 = \max_{(R_1 m R_2)\in\mathcal{C}^{\gamma}_{BC,1}} \mu R_1 + R_2 \tag{4.19}$$

$$\max_{(R_1,R_2)\in\mathcal{R}^{\gamma}_{BC,2}} R_1 + \mu R_2 = \max_{(R_1 m R_2)\in\mathcal{C}^{\gamma}_{BC,2}} R_1 + \mu R_2. \tag{4.20}$$

The proof of the lemma follows directly from the results of [58] and is omitted here. The following theorem describes an outer bound on the capacity region of the Gaussian interference channel with cognitive relay.

Theorem 4.4.2. *Let $\mu \geq 1$. The capacity region of the Gaussian interference channel with cognitive relay, $\mathcal{C}_{IC}$ satisfies*

$$\max_{(R_1,R_2)\in\mathcal{C}_{IC}} \mu R_1 + R_2 \leq \min_{\gamma>0} \max_{(R_1,R_2)\in\mathcal{R}^{\gamma}_{BC,1}} \mu R_1 + R_2 \tag{4.21}$$

$$\max_{(R_1,R_2)\in\mathcal{C}_{IC}} R_1 + \mu R_2 \leq \min_{\gamma>0} \max_{(R_1,R_2)\in\mathcal{R}^{\gamma}_{BC,2}} R_1 + \mu R_2. \tag{4.22}$$

Proof of Theorem 4.4.2: The outer bound is obtained by allowing transmitter co-operation. We allow transmitter 2 to fully co-operate with the relay. This is done by providing transmitter 2 with message m_1 non-causally. This reduces the channel to a Gaussian MIMO cognitive channel studied in Chapter 2 [57]. Let the capacity region of the corresponding MIMO cognitive channel be denoted by $\mathcal{C}_{MCC,1}$. Then, for any $\mu \geq 1$, it is shown in [57, Theorem 3.2 and Lemma 5.6] and Chapter 2 (Theorem 2.3.2 and Lemma 2.5.6) that

$$\max_{(R_1,R_2)\in\mathcal{C}_{MCC,1}} \mu R_1 + R_2 \leq \min_{\gamma>0} \max_{(R_1,R_2)\in\mathcal{R}^{\gamma}_{BC,1}} \mu R_1 + R_2. \tag{4.23}$$

It follows that for any $\mu \geq 1$,

$$\max_{(R_1,R_2)\in\mathcal{C}_{IC}} \mu R_1 + R_2 \leq \min_{\gamma>0} \max_{(R_1,R_2)\in\mathcal{R}^{\gamma}_{BC,1}} \mu R_1 + R_2.$$

By allowing transmitter 1 to co-operate fully with the relay node, we obtain the other bound. That is, for any $\mu \geq 1$,

$$\max_{(R_1,R_2)\in\mathcal{C}_{IC}} R_1 + \mu R_2 \leq \min_{\gamma>0} \max_{(R_1,R_2)\in\mathcal{R}^{\gamma}_{BC,2}} R_1 + \mu R_2.$$

Remark 4.4.1. It is to be noted that the outer bound is not obtained by merely letting all the transmitters co-operate with a sum power constraint. In the broadcast channel in Figures 2 and 3, it can be seen that one of the channel gains is made zero. Also, the outer bound is obtained by minimizing over a series of broadcast channel with different sum power constraints and channel gains. The outer bound obtained is in general not tight, even with respect to the cognitive radio channel [42] [57], because, the non cognitive transmitter in the cognitive radio channel cannot transmit any information with respect to the message of the other transmitter.

Let $\rho_1, \rho_2 \in [-1, 1]$. Let $\mathbf{A}(\rho_1, \rho_2)$ be given by

$$A(\rho_1, \rho_2) = \begin{pmatrix} P_1 & 0 & \rho_1\sqrt{P_1 P_r} \\ 0 & P_2 & \rho_2\sqrt{P_2 P_r} \\ \rho_1\sqrt{P_1 P_r} & \rho_2\sqrt{P_2 P_r} & P_r \end{pmatrix}. \tag{4.24}$$

We define the functions $F_1(\rho_1, \rho_2)$ and $F_2(\rho_1, \rho_2)$ as

$$\begin{aligned} F_1(\rho_1, \rho_2) = L(P_1 + c_1^2 P_r(1-\rho_2^2) + 2c_1\rho_1\sqrt{P_1 P_r}) \\ F_2(\rho_1, \rho_2) = L(P_2 + c_2^2 P_r(1-\rho_1^2) + 2c_2\rho_2\sqrt{P_2 P_r}). \end{aligned} \tag{4.25}$$

The following theorem describes another outer bound on the capacity region of the interference channel with cognitive relay, $\mathcal{C}_{IC}$.

Theorem 4.4.3. *Let* $(R_1, R_2) \in \mathcal{C}_{IC}$*. Then for any* $0 \le \mu < \infty$*, we have*

$$\begin{aligned} \mu R_1 + R_2 \quad \le \quad & \max_{\rho_1,\rho_2\in[-1,1]} \mu F_1(\rho_1, \rho_2) + F_2(\rho_1, \rho_2) \\ & \textit{such that } A(\rho_1, \rho_2) \succeq 0. \end{aligned} \tag{4.26}$$

Proof of Theorem 4.4.3: The proof follows from a sequence of information theory inequalities:

$$
\begin{aligned}
n(\mu R_1 + R_2) &= \mu H(W_1|W_2) + H(W_2|W_1) && (4.27)\\
&\leq \mu I(W_1; Y_1^n|W_2) + I(W_2; Y_2^n|W_1) + n\epsilon && (4.28)\\
&= \mu h(Y_1^n|W_2) - \mu h(Y_1^n|W_1, W_2) + h(Y_2^n|W_2) && (4.29)\\
&\quad -h(Y_2^n|W_1, W_2) + n\epsilon\\
&= \mu h(Y_1^n|W_2, X_2^n) - \mu h(Y_1^n|X_1^n, X_2^n, X_r^n, W_1, W_2) + && (4.30)\\
&\quad h(Y_2^n|W_1, X_1^n) - h(Y_2^n|X_1^n, X_2^n, X_r^n, W_1, W_2) + n\epsilon\\
&\leq \mu h(Y_1^n|X_2^n) - \mu h(Y_1^n|X_1^n, X_2^n, X_r^n) + h(Y_2^n|X_1^n) - && (4.31)\\
&\quad h(Y_2^n|X_1^n, X_2^n, X_r^n) + n\epsilon\\
&= \mu h(X_1^n + c_1 X_r^n + Z_1^n|X_2^n) - \mu h(Z_1^n) + && (4.32)\\
&\quad h(X_2^n + c_2 X_r^n + Z_2^n|X_1^n) - h(Z_2^n) + n\epsilon\\
&\leq \sum_{i=1}^{n} \mu h(X_{1i} + c_1 X_{ri} - \mu Z_{1i}|X_{2i}) - \sum_{i=1}^{n} h(Z_{1i}) && (4.33)\\
&\quad + \sum_{i=1}^{n} h(X_{2i} + c_2 X_{ri} - Z_{2i}|X_{1i}) - \sum_{i=1}^{n} h(Z_{2i}) + n\epsilon
\end{aligned}
$$

where (4.28) follows from Fano's inequality and (4.31) follows because removing conditioning increases entropy. Let Q be a random variable uniformly distributed in the set $\{1, 2, \ldots, n\}$. Therefore, we have

$$
\begin{aligned}
R_1 + R_2 &\leq \mu h(X_{1Q} + c_1 X_{rQ} + Z_{1Q} | X_{2Q}, Q) - \mu h(Z_{1Q}|Q) && (4.34) \\
&\quad + h(X_{2Q} + c_2 X_{rQ} + Z_{2Q} | X_{1Q}, Q) - h(Z_{2Q}|Q) + \epsilon \\
&\leq \mu h(X_{1Q} + c_1 X_{rQ} + Z_{1Q} | X_{2Q}) - \mu h(Z_{1Q}) && (4.35) \\
&\quad + h(X_{2Q} + c_2 X_{rQ} + Z_{2Q} | X_{1Q}) - h(Z_{2Q}) + \epsilon \\
&= \mu h(X_1 + c_1 X_r + Z_1 | X_2) - \mu h(Z_1) && (4.36) \\
&\quad + h(X_2 + c_2 X_r + Z_2 | X_1) - h(Z_2) + \epsilon
\end{aligned}
$$

where (4.35) follows fom removing conditioning increase entropy and i.i.d. distribution of noise.

Let Σ denote the covariance matrix of (X_1, X_2, X_r). Σ is of the form

$$
\Sigma = \begin{pmatrix} P_1 & 0 & \rho_1 \sqrt{P_1 P_r} \\ 0 & P_2 & \rho_{r2} \sqrt{P_2 P_r} \\ \rho_{r1} \sqrt{P_1 P_r} & \rho_2 \sqrt{P_2 P_r} & P_r \end{pmatrix}. \tag{4.37}
$$

where ρ_1 denotes the correlation between X_r and X_1, and ρ_2 denotes the correlation between X_2 and X_1. The theorem follows from the result that conditional entropies with covariance constraint is maximized by Gaussian random variables.

The following theorem characterizes the d.o.f. region of the Gaussian interference channel with cognitive relay.

Theorem 4.4.4. *If $c_1 \ a \neq c_2$ and $c_2 \ b \neq c_1$, the d.o.f. region of the Gaussian interference channel with cognitive relay is*

$$
\mathcal{D}_1 = \left\{ \begin{array}{l} (d_1, d_2) \in \mathbb{R}_+^2 : \\ d_1 \leq 1, \quad d_2 \leq 1 \end{array} \right\}. \tag{4.38}
$$

If $c_1 a = c_2$ and/or $c_2 b = c_1$, then the d.o.f. region of the cognitive relay is given by

$$\mathcal{D}_2 = \left\{ \begin{array}{l} (d_1, d_2) \in \mathbb{R}_+^2 : \\ d_1 + d_2 \leq 1 \end{array} \right\}. \tag{4.39}$$

Proof of Theorem 4.4.4: We first consider the case when $c_1 a \neq c_2$ and $c_2 b \neq c_1$. We describe an outer bound on the d.o.f. region. We allow all the three transmitters to co-operate and obtain a two user broadcast channel with 3 antennas at the transmitter and 1 antenna at each receiver. The d.o.f. region of the broadcast channel is equal to the region described by (4.38). Hence, the region described by $\mathcal{D}_1$ is an outer bound on the d.o.f. region of the Gaussian interference channel with cognitive relay.

We now show that the d.o.f. region $\mathcal{D}_1$ is achievable by interference cancelation. For $i = 1, 2$, transmitter i chooses its transmit codeword X_i according to the distribution $X_i \backsim \mathcal{N}(0, Q_i), Q_i \leq P_i$. The relay transmits $X_r = \lambda_1 X_1 + \lambda_2 X_2$. Hence, we must have $\lambda_1^2 Q_1 + \lambda_2^2 Q_2 \leq P_r$. We choose $ambda_2 = -\frac{b}{c_1}$ and $\lambda_1 = -\frac{a}{c_2}$, to cancel out the interference at each receiver. To satisfy the power constraints, we choose $Q_i = \min(\frac{P_r}{2\lambda_i^2}, P_i), i = 1, 2$. We then achieve the point $(d_1, d_2) = (1, 1)$. Hence, the region $\mathcal{D}_1$ is achievable.

Next, we consider the case when $c_1 a = c_2$ and/or $c_2 b = c_1$. The region given by $\mathcal{D}_2$ is achievable by time sharing. When $c_1 a = c_2$, using arguments similar to those used in [18], we can show that receiver 2 can decode both the messages m_1 and m_2 successfully, and that is the optimal strategy for receiver 2. Hence, $d_1 + d_2 \leq 1$ is an upper bound on the d.o.f. region. The proof is similar for the case when $c_2 b = c_1$.

4.5 Numerical Results

In this section, we provide some numerical results on the capacity region of the two user Gaussian interference channel with a cognitive relay. We consider an example system, where $a = b = 2$, $c_1 = 1.5, c_2 = 0.75$. We take all power constraints to be equal to 10 (i.e., $P_1 = P_2 = P_r = 10$). Figure 4.4 plots the achievable region $\mathcal{R}_{in}$ described in (4.14), and the outer bounds in Theorem 2 and Lemma 2. The plot shows the performance improvements over the achievable region by [49] and the gap between the achievable region and the outer bounds.

4.6 Conclusions

In this chapter, we derived a new achievable region for the two user Gaussian interference channel with a cognitive relay. The achievable region is a generalization of the region given in [49]. In Theorems $4.4.2$ and $4.4.3$, we derive outer bounds on the capacity region of the interference channel with cognitive relay. We also derive the d.o.f. region of the channel setting and show that we can achieve the full degrees of freedom of a two user no-interference channel for a large range of channel parameters.

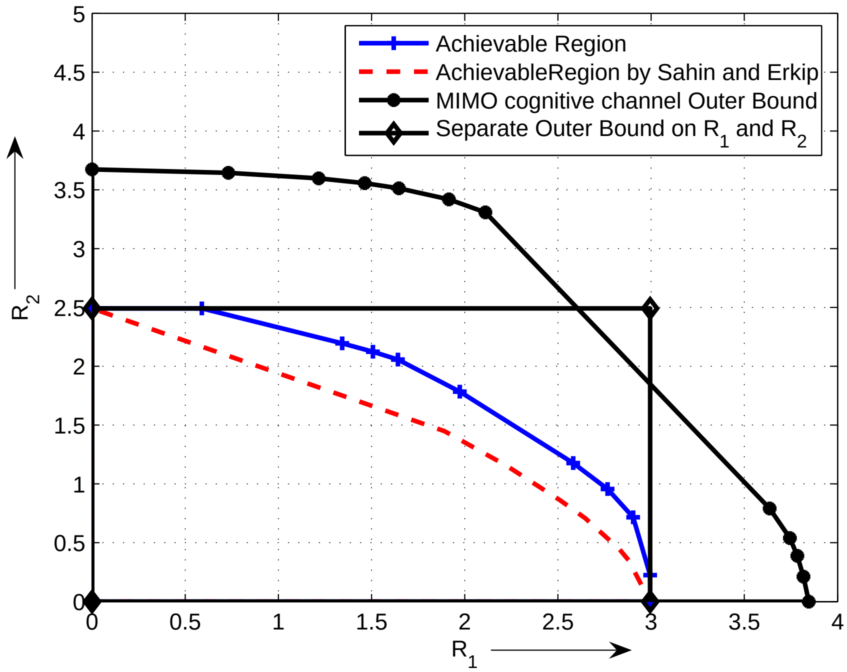


Figure 4.4: Plot of Achievable region and Outer bound for Interference channel with Cognitive Helper

Chapter 5

Cognitive Radio Channel with Partial Cognitionl

In this chapter, we study the cognitive radio channel when the cognitive (or secondary) transmitter has only a partial knowledge of the message transmitted by the licensed (or primary) transmitter. This models a much more practical model of cognitive radio. We restrict the amount of information that the cognitive radio can possess. The rest of the chapter is organized as follows: In Section 5.1, we describe the problem statement and our contributions. We describe the system model in Section 5.2. In Section 5.3, we describe an outer bound on the capacity region of partial cognitive radio channel. We describe an achievable region in Section 5.4. We conclude in Section 5.5.

5.1 Introduction

The cognitive radio channel has been studied by several researchers over the past decade. Most of the work has focused on two scenarios :

1. Underlay model where the cognitive transmitter has no information on the transmissions of the licensed transmitter and has to satisfy an interference constraint at the licensed receiver using either channel knowledge or spectrum information.

2. Overlay model where the cognitive transmitter has full knowledge of the transmissions of the licensed transmitter and it uses this side information to design its transmit strategy.

For more background on the cognitive radio models and prior work, we refer the readers to earlier chapters. This chapter considers a cognitive radio channel model where the cognitive transmitter is not fully cognitive of the other transmitter's message set. In this setting, the cognitive radio has access only to a portion of the message. As this portion varies from nothing to everything, the channel model includes the interference channel (IFC), and IFC with fully-degraded message set as special cases. This channel is referred to as an interference channel with a partially cognitive transmitter. Note that this channel model is motivated by practical constraints, where the cognitive transmitter is only able to garner limited information about the legitimate transmitter's message.

The interference channel with a partially cognitive transmitter has already been studied in [76] with a specific focus on strong interference settings. Results on degree of freedom and sum capacity of symmetric channel is available in [77, 78]. This chapter focuses on the weak interference settings. Specifically, we derive an outer bound on the capacity region of this channel for both the discrete memoryless and Gaussian cases when the interference from the cognitive transmitter to the legitimate receiver is "weak". Subsequently, we show for the Gaussian case that Gaussian distributions satisfying the constraints on the inputs/auxiliary random variables optimizes the outer bound. We also derive an achievable region for the Gaussian partially cognitive-radio channel using a combination of superposition and dirty paper coding. Note that the achievable region described in this chapter can be readily extended to discrete memoryless channel. The results in this chapter are presented in [79] [80]. The results of this chapter are joint work with Goochul Chung, a Ph.D. student in the Department of Electrical and Computer Engineering. Goochul Chung derived the outer bound on the capacity region of partial cognitive radio channel and I derived the acievable region for the partial cognitive radio channel. For the sake of completeness of the chapter, we present the outer bound along with the achievable region.

5.2 System Model and Preliminaries

Throughout this chapter, random variables are denoted by capital letters, and their realizations by the corresponding lower-case letters. X_m^n denotes the random vector $(X_m, ..., X_n)$, X^n denotes the random vector $(X_1, ..., X_n)$, and $X^{n \setminus m}$ denotes the random vector $(X_1, ..., X_{m-1}, X_{m+1}, ..., X_n)$. Also, for any set S, $\overline{S}$ denotes the convex hull of S, and $\widetilde{S}$ means the complementary set of S. Finally, the notation $X \Rightarrow Y \Rightarrow Z$ is used to denote that X and Z are conditionally independent given Y.

5.2.1 Discrete Memoryless Partially Cognitive Radio Channels

A two user interference channel as in Figure 5.1 is a quintuple $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2, p)$, where $\mathcal{X}_1, \mathcal{X}_2$ are two input alphabet sets; $\mathcal{Y}_1, \mathcal{Y}_2$ are two output alphabet sets; $p(y_1, y_2|x_1, x_2)$ is a transition probability. Since we confine channel to be mem-

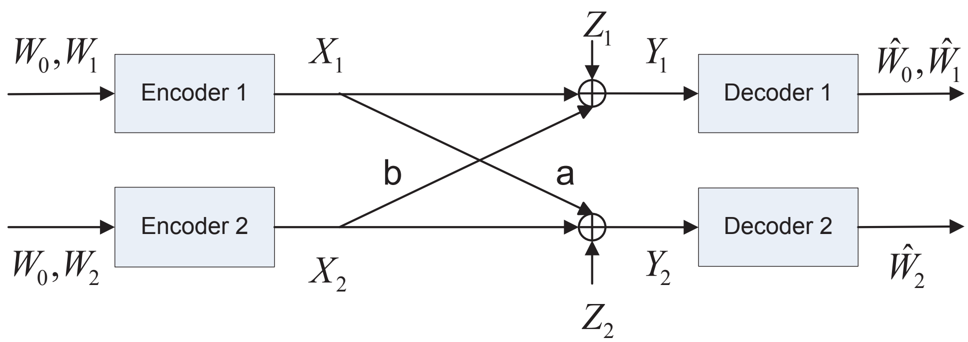


Figure 5.1: The discrete memoryless partially cognitive radio model

oryless, the transition probability of y_1^n, y_2^n given x_1^n, x_2^n is

$$p(y_1^n, y_2^n | x_1^n, x_2^n) = \prod_{i=1}^{n} p(y_{1,i}, y_{2,i} | x_{1,i}, x_{2,i}).$$

This channel model is similar to that of an interference channel with the difference being the message sets at each transmitter. Transmitter 1 is the legitimate user, who communicates messages from the sets $W_0 \in \{1, ..., M_0\}$ and $W_1 \in \{1, ..., M_1\}$ to Receiver 1, the legitimate receiver. Transmitter 2, the cognitive transmitter communicates a message $W_2 \in \{1, ..., M_2\}$ to Receiver 2, the cognitive receiver. The unique feature of this channel is that the realization of W_0 is known to *both* transmitters 1 and 2, which allows partial and unidirectional cooperation between the transmitters. An $(R_0, R_1, R_2, n, P_{e,0}, P_{e,1}, P_{e,2})$ code is any code with the rate vector (R_0, R_1, R_2) and block size n, where $R_t \triangleq \log(M_t)/n$ bits per usage for $t = 0, 1, 2$. As discussed above, W_0 and W_1 are the messages that Receiver 1 must decode with (average) probabilities of error of at most $P_{e,0}, P_{e,1}$ respectively, and W_2 is the message that Receiver 2 must decode with an error probability of at most $P_{e,2}$. Rate triplet (R_0, R_1, R_2) is said to be achievable if the error probabilities $P_{e,t}$

for $t = 0, 1, 2$ can be made arbitrarily small as the block size n grows. The capacity region of the interference channel with partially cognitive transmitter is the closure of the set of all achievable rate triplets (R_0, R_1, R_2).

Throughout this chapter, we have a restriction on the pair (R_0, R_1), such that $R_1 \geq \mu R_0$ for some positive number μ. This restriction is to ensure that optimization of rate regions does not drive the rate R_1 to zero, which results in a fully cognitive solution. This goal and restriction apply to both discrete memoryless and gaussian channel which follows.

5.2.2 Gaussian Partially Cognitive Radio Channel

In the Gaussian IFC, input and output alphabets are the reals $\mathbb{R}$, and outputs are the linear combination of the inputs and additive white Gaussian noise. A Gaussian IFC model in Figure 5.2 is characterized mathematically as follows:

$$\begin{aligned} Y_1 &= X_1 + bX_2 + Z_1, \\ Y_2 &= aX_1 + X_2 + Z_2, \end{aligned} \tag{5.1}$$

where a and b are real numbers and Z_1 and Z_2 are independent, zero-mean, unit-variance Gaussian random variables. Further, each transmitter has a power constraint

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_{t,i}^2] \leq P_t, t = 1, 2.$$

In the next section, we describe the outer bound on the capacity region for these channels under "weak" interference.

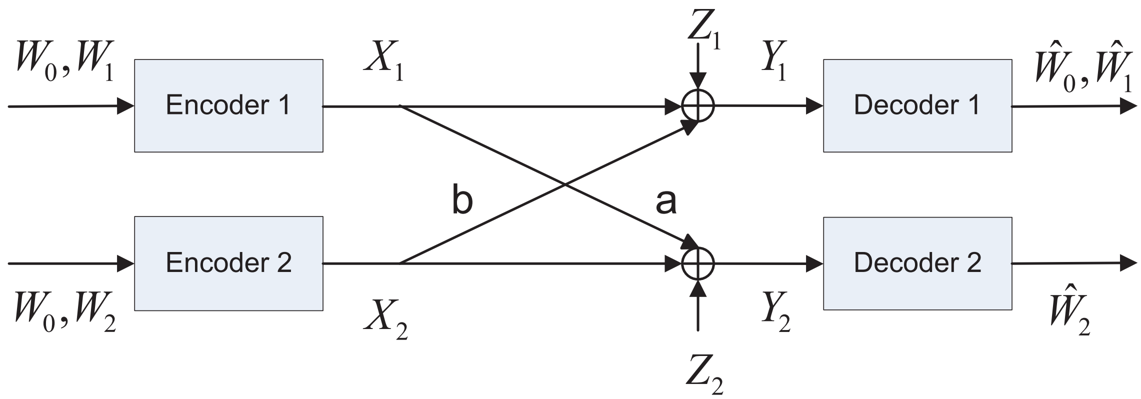


Figure 5.2: The Gaussian partial cognitive radio channel

5.3 The Outer Bound region

We first derive an outer bound on the capacity region of the discrete memoryless partial cognitive radio channel under a weak interference condition.

5.3.1 Discrete Memoryless Partially Cognitive Radio Channels

For a discrete memoryless channel, under the condition

$$X_2|X_1 \Rightarrow Y_2|X_1 \Rightarrow Y_1|X_1, \tag{5.2}$$

we say that the legitimate receiver is observing weak interference [42]. In this setting, we present the outer bound in the following theorem:

Theorem 5.3.1. *The convex closure of the following inequalities defines an outer* *bound on the capacity region of "weak" partially cognitive radio channels:*

$$R_0 \leq I(U, X_1; Y_1|V), \tag{5.3}$$

$$R_1 \leq I(X_1; Y_1|X_2), \tag{5.4}$$

$$R_0 + R_1 \leq I(U, X_1; Y_1), \tag{5.5}$$

$$R_2 \leq I(X_2; Y_2|U, X_1), \tag{5.6}$$

$$R_1 \geq \mu R_0, \tag{5.7}$$

for any $p(u, v)p(x_1|u, v)p(x_2|u)$ *such that:*

1. V *and* X_2 *are independent.*
2. X_1 *is a function of* U *and* V.
3. $(U, V) \Rightarrow (X_1, X_2) \Rightarrow (Y_1, Y_2)$.

Proof: First, we borrow the lemma from [81] which is used in constituting the outer bound.

Lemma 5.3.2 ([81])**.** *The following forms a Markov chain for the partially cognitive radio channel:*

$$(W_0, W_t) \Rightarrow (W_0, X_t) \Rightarrow Y_t, \tag{5.8}$$

where $t = 1, 2$.

We start the main proof by verifying the outer bound for R_0, R_1, and R_2.

We have

$$
\begin{aligned}
nR_0 &= H(W_0|W_1) \\
&\leq I(W_0; Y_1^n|W_1) + n\epsilon_0 \\
&= \sum_{i=1}^{n} [H(Y_{1,i}|Y_1^{i-1}, W_1) - H(Y_{1,i}|Y_1^{i-1}, W_0, W_1)] + n\epsilon_0 \\
&\leq \sum_{i=1}^{n} [H(Y_{1,i}|W_1) - H(Y_{1,i}|Y_1^{i-1}, X_1^{n\backslash i}, W_0, W_1, X_{1,i})] + n\epsilon_0 \\
&\overset{(a)}{\leq} \sum_{i=1}^{n} [H(Y_{1,i}|W_1) - H(Y_{1,i}|Y_2^{i-1}, X_1^{n\backslash i}, W_0, W_1, X_{1,i})] + n\epsilon_0 \\
&\overset{(b)}{=} \sum_{i=1}^{n} [H(Y_{1,i}|V_i) - H(Y_{1,i}|U_i, V_i, X_{1,i})] + n\epsilon_0 \\
&= \sum_{i=1}^{n} I(U_i, X_{1,i}; Y_{1,i}|V_i) + n\epsilon_0,
\end{aligned}
$$

where (a) results from the conditional Markov chain for the weak interference channel, $X_2|X_1 \Rightarrow Y_2|X_1 \Rightarrow Y_1|X_1$ in (5.2). (b) results from identifying auxiliaries $U_i = (Y_2^{i-1}, X_1^{n\backslash i}, W_0)$ and $V_i = W_1$.

$$
\begin{aligned}
nR_1 &= H(W_1) \\
&\leq I(W_1; Y_1^n) + n\epsilon_0 \\
&= I(W_1; Y_1^n|X_2^n) + n\epsilon_0 \\
&= \sum_{i=1}^{n} [H(Y_{1,i}|Y_1^{i-1}, X_2^n) - H(Y_{1,i}|Y_1^{i-1}, X_2^n, W_1)] + n\epsilon_0 \\
&\leq \sum_{i=1}^{n} [H(Y_{1,i}|X_{2,i}) - H(Y_{1,i}|X_{1,i}, X_{2,i})] + n\epsilon_0 \\
&= \sum_{i=1}^{n} I(Y_{1,i}; X_{1,i}|X_{2,i}) + n\epsilon_0,
\end{aligned}
$$

and

$$
\begin{aligned}
nR_2 &= H(W_2|W_0) \\
&\le I(W_2;Y_2^n|W_0) + n\epsilon_2 \\
&\le I(W_2;Y_2^n, X_1^n|W_0) + n\epsilon_2 \\
&\overset{(a)}{=} I(W_2;Y_2^n|X_1^n, W_0) + n\epsilon_2 \\
&= H(Y_2^n|X_1^n, W_0) - H(Y_2^n|X_1^n, W_0, W_2) + n\epsilon_2 \\
&\overset{(b)}{\le} H(Y_2^n|X_1^n, W_0) - H(Y_2^n|X_1^n, W_0, X_2^n) + n\epsilon_2 \\
&\overset{(c)}{\le} \sum_{i=1}^{n}[H(Y_{2,i}|U_i, X_{1,i}) - H(Y_{2,i}|U_i, X_{1,i}, X_{2,i})] + n\epsilon_2 \\
&= \sum_{i=1}^{n} I(X_{2,i};Y_{2,i}|U_i, X_{1,i}) + n\epsilon_2,
\end{aligned}
$$

where (a) is due to the independence of W_2 and X_1^n, (b) is from Lemma 5.3.2 $(W_0, W_2) \Rightarrow (W_0, X_2^n) \Rightarrow (Y_2^n)$, and (c) comes from the same definition above of $U_i = Y_2^{i-1}, X_1^{n\backslash i}, W_0$.

Next, we prove the outer bound for the sum rate $R_0 + R_1$. We have

$$
\begin{aligned}
nR_0 + nR_1 &= H(W_0, W_1) \\
&\leq I(W_0, W_1; Y_1^n) + n\epsilon_1 \\
&= H(Y_1^n) - H(Y_1^n | W_0, W_1) + n\epsilon_1 \\
&\overset{(a)}{=} H(Y_1^n) - H(Y_1^n | W_0, X_1^n) + n\epsilon_1 \\
&= \sum_{i=1}^{n} [H(Y_{1,i} | Y_1^{i-1}) - H(Y_{1,i} | Y_1^{i-1}, X_1^{n\setminus i}, W_0, X_{1,i})] + n\epsilon_1 \\
&\overset{(b)}{=} \sum_{i=1}^{n} [H(Y_{1,i} | Y_1^{i-1}) - H(Y_{1,i} | Y_2^{i-1}, X_1^{n\setminus i}, W_0, X_{1,i})] + n\epsilon_1 \\
&\overset{(c)}{\leq} \sum_{i=1}^{n} [H(Y_{1,i}) - H(Y_{1,i} | U_i, X_{1,i})] + n\epsilon_1 \\
&= \sum_{i=1}^{n} I(U_i, X_{1,i}; Y_{1,i}) + n\epsilon_1 .
\end{aligned}
$$

(a) results from $(W_0, W_1) \Rightarrow (W_0, X_1^n) \Rightarrow (Y_1^n)$ (Lemma 5.3.2), (b) results from $X_2 \Rightarrow Y_2 \Rightarrow Y_1$, given X_1 in (5.2), and (c) results from the definition above of $U_i = Y_2^{i-1}, X_1^{n\setminus i}, W_0$. Note that the choice of auxiliary random variables automatically satisfies the constraints imposed on them in Theorem 5.3.1.

5.3.2 Gaussian Partially Cognitive Radio Channel

For the Gaussian case, the weak interference constraint can be interpreted as the requirement of $b < 1$ in (1). With the condition, $b < 1$, the conditional Markov chain for the weak interference channel, $X_2 \Rightarrow Y_2 \Rightarrow Y_1$, given X_1 in (5.2) is satisfied. Thus, similar proof ensures the outer bound for the rate region defined in Theorem 5.3.1 to be valid for the Gaussian partially cognitive radio channel. Next, we establish three lemmas that is essential in proving the optimality of a jointly Gaussian input distribution for the region defined in Theorem 5.3.1.

Lemma 5.3.3 (Lemma 1 in [82])**.** *Let $X_1, X_2, ..., X_k$ be arbitrarily distributed zero-mean random variables with covariance matrix K, and $X_1^*, X_2^*, ..., X_k^*$ be zero mean Gaussian distributed random variables with the same covariance matrix K. Let S be any subset of $\{1, 2, ..., k\}$ and $\widetilde{S}$ be its complement. Then,*

$$h(X_S|X_{\widetilde{S}}) \leq h(X_S^*|X_{\widetilde{S}}^*). \tag{5.9}$$

Lemma 5.3.4. *Let X_1, X_2, V be an arbitrarily distributed zero-mean random variables with covariance matrix K, where X_2 and V are independent of each other. Let X_1^*, X_2^*, V^* be the zero mean Gaussian distributed random variables with the same covariance matrix as X_1, X_2, V. Then,*

$$\mathbb{E}[X_1X_2] = \mathbb{E}[X_1^*X_2^*|V^*]. \tag{5.10}$$

Without loss of generality X_1^* can be written as $X_1^* = W^* + cV^*$, where W^* is the zero mean Gaussian random variable independent of V^*. Then

$$\begin{aligned}
\mathbb{E}[X_1X_2] &= \mathbb{E}[X_1^*X_2^*] \\
&= \mathbb{E}[\mathbb{E}[X_1^*X_2^*|V^*]] \\
&= \mathbb{E}[\mathbb{E}[(W^* + cV^*)X_2^*|V^*]] \\
&= \mathbb{E}[\mathbb{E}[W^*X_2^*|V^*]] + c\mathbb{E}[\mathbb{E}[V^*X_2^*|V^*]] \\
&\overset{(a)}{=} \mathbb{E}[X_1^*X_2^*|V^*] + c\mathbb{E}[V^*\mathbb{E}[X_2^*]] \\
&\overset{(b)}{=} \mathbb{E}[X_1^*X_2^*|V^*],
\end{aligned}$$

where (a) results from the independence of X_2^* and V^*. And, (b) results from the fact that X_2^* is zero mean.

Lemma 5.3.5. *Random variables in Lemma 5.3.4, X_1^*, X_2^*, and V^* satisfy the following equation:*

$$\mathbb{E}[X_1^* X_2^* | V^*] \leq (\mathbb{E}[(X_1^*)^2 | V^*])^{\frac{1}{2}} (\mathbb{E}[(\mathbb{E}[X_2^* | X_1^*])^2])^{\frac{1}{2}}.$$

Proof: Note that

$$\begin{aligned}
\mathbb{E}[X_1^* X_2^* | V^*] &\overset{(a)}{=} \mathbb{E}[\mathbb{E}[X_1^* X_2^* | V^*, X_1^*]] \\
&\overset{(b)}{=} \mathbb{E}[X_1^* \mathbb{E}[X_2^* | V^*, X_1^*] | V^*] \\
&\overset{(c)}{\leq} (\mathbb{E}[(X_1^*)^2 | V^*])^{\frac{1}{2}} (\mathbb{E}[(\mathbb{E}[X_2^* | V^*, X_1^*])^2])^{\frac{1}{2}} \\
&\overset{(d)}{\leq} (\mathbb{E}[(X_1^*)^2 | V^*])^{\frac{1}{2}} (\mathbb{E}[(\mathbb{E}[X_2^* | X_1^*])^2])^{\frac{1}{2}},
\end{aligned}$$

where (a) comes from the law of iterated expectations, (b) from the independence of X_2^* and V^*, (c) from the Cauchy-Schwartz inequality, and (d) from the fact that entropy can only be reduced by conditioning.

Definition 5.3.1. Define the rate region $\mathcal{R}_{out}^{\alpha,\beta_1,\beta_2}$ to be the convex hull of all rate triplets (R_0, R_1, R_2) satisfying

$$\begin{aligned}
&R_0 \leq \tfrac{1}{2} \log\left(1 + \tfrac{\beta_1 P_1 + b^2(1-\alpha)P_2 + 2b\sqrt{(\beta_2(1-\alpha)P_1 P_2)}}{(1+b^2\alpha P_2)}\right), \\
&R_1 \leq \tfrac{1}{2} \log\left(2\pi e \left(1 + \left(1 - \beta_2(1-\alpha)\right) P_1\right)\right), \\
&R_0 + R_1 \leq \tfrac{1}{2} \log\left(1 + \tfrac{P_1 + b^2(1-\alpha)P_2 + 2b\sqrt{(\beta_2(1-\alpha)P_1 P_2)}}{(1+b^2\alpha P_2)}\right), \\
&R_2 \leq \tfrac{1}{2} \log(\alpha P_2 + 1), \\
&R_1 \geq \mu R_0,
\end{aligned} \tag{5.11}$$

for some $\alpha \in [0,1]$, $\beta_1 \in [0,1]$, and $\beta_2 \in [0, \beta_1]$.

Definition 5.3.2. Define the rate region $\mathcal{R}_{out}$ to be convex hull of the union of rate region $\mathcal{R}_{out}^{\alpha,\beta}$:

$$\mathcal{R}_{out} \triangleq \overline{\bigcup_{0\le\alpha,\beta_1\le 1, 0\le\beta_2\le\beta_1} \mathcal{R}_{out}^{\alpha,\beta_1,\beta_2}}. \tag{5.12}$$

We denote $\mathcal{C}$ to be the capacity region of the Gaussian weak partially cognitive radio channel. An outer bound for $\mathcal{C}$ is obtained as follows.

Theorem 5.3.6. *$\mathcal{R}_{out}$ is an outer bound of the capacity region for the Gaussian weak partially cognitive radio channel:*

$$\mathcal{C} \subset \mathcal{R}_{out}.$$

Proof: We start from the rate region in Theorem 5.3.1.

$$\begin{aligned} R_0 &\le I(U, X_1; Y_1|V) = h(Y_1|V) - h(Y_1|V, U, X_1) \\ &= h(Y_1|V) - h(Y_1|U, X_1), \end{aligned} \tag{5.13}$$

$$R_1 \le I(X_1; Y_1|X_2) = h(Y_1|X_2) - h(N_1), \tag{5.14}$$

$$R_0 + R_1 \le I(U, X_1; Y_1) = h(Y_1) - h(Y_1|U, X_1), \tag{5.15}$$

$$R_2 \le I(X_2; Y_2|U, X_1) = h(Y_2|U, X_1) - h(N_2). \tag{5.16}$$

(5.13) follows from the Markov chain, $V \Rightarrow (U, X_1) \Rightarrow Y_1$. First, we set

$$h(Y_2|U, X_1) = \frac{1}{2}\log(2\pi e(1 + \alpha P_2)), \tag{5.17}$$

without loss of generality for some $\alpha \in [0, 1]$. Note that

$$\begin{aligned} Y_1 &= b(X_2 + Z_1) + X_1 + Z', \\ h(Y_1|U, X_1) &= h(b(X_2 + Z_1) + Z'|U, X_1), \end{aligned} \tag{5.18}$$

where $b < 1$ because legitimate receiver faces a weak interference, and Z' is a Gaussian distributed random variable with variance $1 - b^2$. By entropy power inequality (EPI) [67], we have,

$$\begin{aligned} 2^{2h(Y_1|U,X_1)} &\geq 2^{2h(bY_2|U,X_1)} + 2^{2h(Z')}. \\ &= b^2 2^{2h(Y_2|U,X_1)} + 2\pi e(1 - b^2) \\ &= 2\pi e(1 + b^2 \alpha P_2), \end{aligned}$$

which yields

$$h(Y_1|U, X_1) \geq \frac{1}{2} \log(2\pi e(1 + b^2 \alpha P_2)). \tag{5.19}$$

Next, we need to bound $h(Y_1)$, $h(Y_1|V)$, and $h(Y_1|X_2)$. Note that, by setting $h(Y_2|U, X_1) = \frac{1}{2} \log(2\pi e(1 + \alpha P_2))$, we have the following result.

$$\begin{aligned} h(Y_2|U, X_1) &\leq h(X_2 + Z_2|X_1) \\ &\leq h(X_2^* + Z_2|X_1^*) \\ &= \frac{1}{2} \log(2\pi e(1 + (X_2^*|X_1^*))), \end{aligned} \tag{5.20}$$

where $(\cdot|\cdot)$ denotes the conditional covariance. Combining (5.17) with (5.20), we obtain the bound

$$(X_2^*|X_1^*) \geq \alpha P_2. \tag{5.21}$$

Also,

$$(X_2^*|X_1^*) = \mathbb{E}[(X_2^*)^2] - \mathbb{E}[(\mathbb{E}[X_2^*|X_1^*])^2]. \tag{5.22}$$

From (5.21) and (5.22), we obtain,

$$\mathbb{E}[(\mathbb{E}[X_2^*|X_1^*])^2] \leq (1-\alpha)P_2. \tag{5.23}$$

Note that

$$\mathbb{E}[(X_1^*)^2|V^*] \leq P_1, \tag{5.24}$$

since conditioning only reduces the entropy. Again, we set $\mathbb{E}[(X_1^*)^2|V^*] = \beta_1 P_1$ for some $\beta_1 \in [0,1]$ without loss of generality. Now combining Lemma 5.3.4, Lemma 5.3.5, and the above result, (5.23),

$$\mathbb{E}[X_1X_2] \leq \sqrt{\beta_1 P_1}\sqrt{(1-\alpha)P_2}. \tag{5.25}$$

We can set

$$\mathbb{E}[X_1X_2] = \sqrt{\beta_2 P_1}\sqrt{(1-\alpha)P_2}, \tag{5.26}$$

where $\beta_2 \in [0, \beta_1]$. Therefore, we obtain the bound for $h(Y_1)$ as

$$\begin{aligned} h(Y_1) &\leq \frac{1}{2}\log\left(2\pi e\begin{pmatrix} 1+(X_1)+b^2(X_2) \\ +2b\mathbb{E}[X_1X_2] \end{pmatrix}\right) \\ &= \frac{1}{2}\log\left(2\pi e\begin{pmatrix} 1+P_1+b^2P_2 \\ +2b\sqrt{\beta_2(1-\alpha)P_1P_2} \end{pmatrix}\right). \end{aligned} \tag{5.27}$$

For $h(Y_1|V)$, note that (Y_1^*, V^*) has the same covariance matrix as (Y_1, V) if $Y_1 = X_1^* + bX_2^*$. Also, Y_1 is a mean zero Gaussian distributed random variable. Thus,

$$\begin{aligned} h(Y_1|V) \leq& h(Y_1^*|V^*) \\ =& h(X_1^* + bX_2^* + Z_1|V^*) \\ =& \frac{1}{2}\log\left(2\pi e\begin{pmatrix} 1+(X_1^*|V^*) \\ +b^2(X_2^*|V^*) \\ +2b\mathbb{E}[X_1^*X_2^*|V^*] \end{pmatrix}\right) \\ \leq& \frac{1}{2}\log\left(2\pi e\begin{pmatrix} 1+\beta_1P_1+b^2P_2 \\ +2b\sqrt{(\beta_2(1-\alpha)P_1P_2)} \end{pmatrix}\right). \end{aligned} \tag{5.28}$$

For $h(Y_1|X_2)$,

$$\begin{aligned} h(Y_1|X_2) =& h(X_1 + bX_2 + Z_1|X_2) \\ =& h(X_1 + Z_1|X_2) \\ \leq& h(X_1^* + Z_1|X_2^*) \\ =& \frac{1}{2}\log\left(2\pi e\left(1 + (X_1^*|X_2^*)\right)\right) && (5.29) \\ =& \frac{1}{2}\log\left(2\pi e\left(1 + P_1 - \frac{\mathbb{E}[X_1^* X_2^*]^2}{P_2}\right)\right) && (5.30) \\ =& \frac{1}{2}\log\left(2\pi e\left(1 + \left(1 - \beta_2(1-\alpha)\right) P_1\right)\right), && (5.31) \end{aligned}$$

which gives the desired outer bound for the capacity region. Rate region $\mathcal{R}_{out}^{\alpha,\beta_1,\beta_2}$ shows that outer bound can be obtained by having β_1 set to 1.

5.4 Achievable Region for the Gaussian Channel

In this section, we describe an achievable region for the Gaussian channel model described in (5.1). In deriving the achievable region, we combine superposition coding, dirty paper coding [22], and Han and Kobayashi coding [4]. The reason for using this combination is to cope with the channel status with different μ's. We have more strict restriction on how much data can be shared between cognitive and legitimate transmitters with large μ. Thus, as μ increases, the channel becomes more close to an interference channel. Han and Kobayahi coding is known to have best achievable rate region to date for the general interference channel. Also, as μ decreases, the channel becomes more similar to cognitive radio with full knowledge of legitimate transmitters message sets. In such a case, dirty paper coding is known

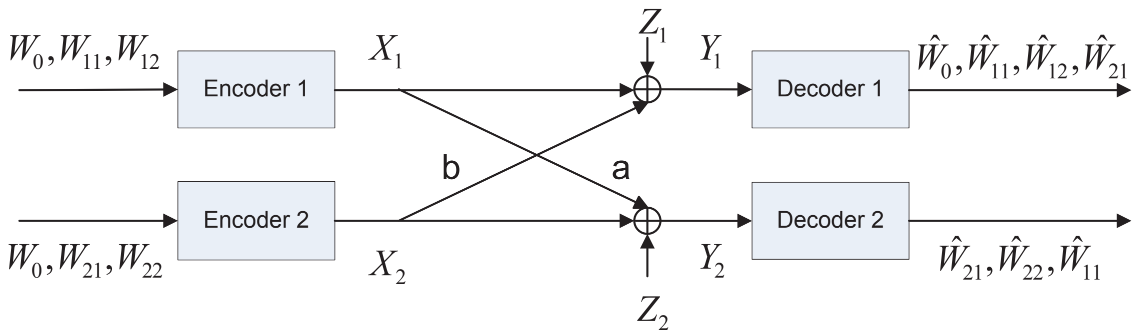


Figure 5.3: The Gaussian partial cognitive radio channel

to be optimal [41] [42]. By combining superpositon coding, dirty paper coding, and Han and Kobayashi coding, achievable scheme can cope with the best possible strategy in two extremes. Figure 5.3 shows the messages sets that encoded and decoded at each transmitter and receiver.

The legitimate transmitter encodes messages W_0, W_{11}, and W_{12} using Gaussian codebooks and superimposes them to form its final codeword. W_0 is the message set that is shared between legitimate and cognitive transmitters. W_{11} and W_{12} correspond to the individual message set for legitimate message, which is W_1 in Fig. 1 and Fig. 2. W_{12} is a public message set which is intended to be decoded in both legitimate and cognitive receivers. W_{11} is a private message set which is decoded only in the legitimate receiver. The cognitive transmitter allocates a portion of the power in communicating message W_0 to the legitimate receiver. The remaining power is used in encoding its own message W_2. Again, W_2 is divided into a public message set, W_{21}, and a private message set, W_{22}. The cognitive transmitter encodes message W_{22} using dirty paper coding treating the codewords from W_0 as non-causally known interference.

Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 > 0$ such that

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \qquad \beta_1 + \beta_2 + \beta_3 = 1.$$

We define function $L : R^+ \to R^+$ as $L(x) = \frac{1}{2}\log(1 + x)$. Let $Q = \left(1 + \sqrt{\frac{\beta_1 P_2}{\alpha_1 P_1}}\right)^2 \alpha_1 P_1$ and $S = \left(a + \sqrt{\frac{\beta_1 P_2}{\alpha_1 P_1}}\right)^2 \alpha_1 P_1$.

We define the constants $r_0, r_1, r_2, \ldots r_{17}$ as follows:

$$\begin{aligned}
r_0 &= L\left(\tfrac{Q}{1+\beta_3 P_2}\right) & r_1 &= L\left(\tfrac{\alpha_2 P_1}{1+\beta_3 P_2}\right) \\
r_2 &= L\left(\tfrac{\alpha_3 P_1}{1+beta_3 P_2}\right) & r_3 &= L\left(\tfrac{b^2\beta_2 P_2}{1+\beta_3 P_2}\right) \\
r_4 &= L\left(\tfrac{Q+\alpha_2 P_1}{1+\beta_3 P_2}\right) & r_5 &= L\left(\tfrac{Q+\alpha_3 P_1}{1+\beta_3 P_2}\right) \\
r_6 &= L\left(\tfrac{Q+b^2\beta_2 P_2}{1+\beta_3 P_2}\right) & r_7 &= L\left(\tfrac{(\alpha_2+\alpha_3)P_1}{1+\beta_3 P_2}\right) \\
r_8 &= L\left(\tfrac{\alpha_2 P_1+b^2\beta_2 P_2}{1+\beta_3 P_2}\right) & r_9 &= L\left(\tfrac{\alpha_3 P_1+b^2\beta_2 P_2}{1+\beta_3 P_2}\right) \\
r_{10} &= L\left(\tfrac{Q+(\alpha_2+\alpha_3)P_1}{1+\beta_3 P_2}\right) & r_{11} &= L\left(\tfrac{Q+\alpha_2 P_1+b^2\beta_2 P_2}{1+\beta_3 P_2}\right) \\
r_{12} &= L\left(\tfrac{Q+\alpha_3 P_1+b^2\beta_2 P_2}{1+\beta_3 P_2}\right) & r_{13} &= L\left(\tfrac{(\alpha_2+\alpha_3)P_1+b^2\beta_2 P_2}{1+\beta_3 P_2}\right) \\
r_{14} &= L\left(\tfrac{Q+(\alpha_2+\alpha_3)P_1+b^2\beta_2 P_2}{1+\beta_3 P_2}\right) & r_{15} &= L\left(\tfrac{a^2\alpha_3 P_1}{1+S+a^2\alpha_2 P_1+beta_3 P_2}\right) \\
r_{16} &= L\left(\tfrac{\beta_2 P_2}{1+S+a^2\alpha_2 P_1+\beta_3 P_2}\right) & r_{17} &= L\left(\tfrac{a^2\alpha_3 P_1+\beta_2 P_2}{1+S+a^2\alpha_2 P_1+\beta_3 P_2}\right) \\
r_{18} &= L\left(\tfrac{\beta_3 P_2}{1+a^2\alpha_2 P_1}\right)
\end{aligned} \tag{5.32}$$

Define the rate region $\mathcal{R}_i^{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3}$ to be the convex hull of all rate triplets (R_0, R_1, R_2) satisfying

$$
\begin{aligned}
R_0 &\leq r_0 \\
R_1 &\leq \min(r_7, r_1 + r_{15}) \\
R_2 &\leq \min(r_3 + r_{18}, r_{16} + r_{18}) \\
R_0 + R_1 &\leq \min(r_{10}, r_4 + r_{15}) \\
R_0 + R_2 &\leq r_6 + r_{18} \\
R_1 + R_2 &\leq \min(r_{13} + r_{18}, r_8 + r_{15} + r_{18}, r_+ r_{17} + r_{18}) \\
R_0 + R_1 + R_2 &\leq \min(r_{14} + r_{18}, r_{11} + r_{15} + r_{18}, r_4 + r_{17} + r_{18}) \\
2R_0 + R_1 &\leq r_4 + r_5 \\
R_1 + 2R_2 &\leq \min(r_8 + r_9 + 2r_{18}, r_8 + r_{17} + 2r_{18}) \\
2R_0 + R_1 + R_2 &\leq \min(r_5 + r_{11} + r_{18}, r_4 + r_{12} + r_{18}) \\
R_0 + R_1 + 2R_2 &\leq \min(r_9 + r_{11} + 2r_{18}, r_8 + r_{!2} + 2r_{18}, r_{11} + r_{17} + 2r_{18}) \\
2R_0 + R_1 + 2R_2 &\leq r_{11} + r_{12} + 2r_{18}
\end{aligned}
\tag{5.33}
$$

Define the rate region $\mathcal{R}_i$ to be convex hull of the union of rate region $\mathcal{R}_i^{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3}$:

$$
\mathcal{R}_i \triangleq \overline{\bigcup_{\substack{\alpha_1+\alpha_2+\alpha_3=1 \\ \beta_1+\beta_2+\beta_3=1}} \mathcal{R}_i^{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3}}. \tag{5.34}
$$

Theorem 5.4.1. *For the Gaussian channel with partially cognitive radio as described in (5.1), the region described by*

$$
\mathcal{R}_{in} = \{(R_0, R_1, R_2) \in \mathcal{R}_i : R_1 \geq \mu R_0\} \tag{5.35}
$$

is achievable.

In proving the theorem, we use an encoding strategy that combines superposition coding, dirty paper coding, and Han and Kobayashi coding. We first describe the encoding strategy at the two transmitters. We fix $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\beta_1 + \beta_2 + \beta_3 = 1$.

Encoding Strategy at legitimate transmitter: For every message $W_0 \in \{1, \ldots, M_0\}$, the legitimate transmitter generates a codeword $X_{10}^n(W_0)$ from the distribution $p(X_{10}^n) = \Pi_{i=1}^n p(X_{10}(i))$, where $X_{10}(i) \sim \mathcal{N}(0, \alpha_1 P_1)$. For every message $W_{11} \in \{1, \ldots, M_{11}\}$, the legitimate transmitter generates a codeword $X_{11}^n(W_1)$ from the distribution $p(X_{11}^n) = \Pi_{i=1}^n p(X_{11}(i))$, where $X_{11}(i) \sim \mathcal{N}(0, \alpha_2 P_1)$. For every message $W_{12} \in \{1, \ldots, M_{12}\}$, the legitimate transmitter generates a codeword $X_{12}^n(W_2)$ from the distribution $p(X_{12}^n) = \Pi_{i=1}^n p(X_{12}(i))$, where $X_{12}(i) \sim \mathcal{N}(0, \alpha_3 P_1)$. The legitimate transmitter then superimposes these codewords to form the net codeword X_1^n as

$$X_1^n = X_{10}^n + X_{11}^n + X_{12}^n.$$

Encoding strategy at cognitive transmitter: The cognitive transmitter allocates a portion of its power in communicating the message W_0 to the legitimate receiver. For message W_0, the cognitive transmitter generates a codeword $X_{20}^n(W_0)$ as follows:

$$X_{20}^n(W_0) = \sqrt{\frac{\beta_1 P_2}{\alpha_1 P_1}} X_{10}^n(W_0).$$

That is, the cognitive transmitter uses the same codeword for encoding message W_0 as used by the legitimate transmitter except that it is scaled to power $\beta_1 P_2$. Next, the cognitive transmitter encodes message W_{21} to codeword X_{21}^n. The cognitive transmitter generates a codeword $X_{21}^n(W_1)$ from the distribution $p(X_{21}^n) = \Pi_{i=1}^n p(X_{21}(i))$, where $X_{21}(i) \sim \mathcal{N}(0, \beta_2 P_2)$. Then, the cognitive transmitter encodes message W_{22} to codeword X_{22}^n using dirty paper coding treating $aX_{10}^n + X_{20}^n$

as non-causally known interference. X_{22}^n is independent of the interference, $aX_{10}^n + X_{20}^n$, and is distributed as $p(X_{22}^n) = \Pi_{i=1}^n p(X_{22}(i))$ and $X_{22}(i) \sim \mathcal{N}(0, \beta_3 P_2)$. The cognitive transmitter superimposes the three codewords X_{20}^n, X_{21}^n, and X_{22}^n to form its net codeword X_2^n. That is,

$$X_2^n = X_{20}^n + X_{21}^n + X_{22}^n.$$

Next, we describe the decoding strategy and the rate constraints associated at the two receivers.

Decoding strategy at legitimate receiver: The legitimate receiver obtains the signal

$$Y_1^n = X_{10}^n + X_{11}^n + X_{12}^n + bX_{20}^n + bX_{21}^n + bX_{22}^n + Z_1^n.$$

The licensed receiver decodes the messages $W_0, W_{11}, W_{12}, W_{21}$ jointly treating X_{22}^n as noise. The decoding is successful if the rates satisfy the constrainst given by

$$\begin{aligned} R_0 &\leq r_0 & R_{11} &\leq r_1 \\ R_{12} &\leq r_2 & R_{21} &\leq r_3 \\ R_0 + R_{11} &\leq r_4 & R_0 + R_{12} &\leq r_5 \\ R_0 + R_{21} &\leq r_6 & R_1 &\leq r_7 \\ R_{11} + R_{21} &\leq r_8 & R_{12} + R_{21} &\leq r_9 \\ R_0 + R_1 &\leq r_{10} & R_0 + R_{11} + R_{21} &\leq r_{11} \\ R_0 + R_{12} + R_{21} &\leq r_{12} & R_1 + R_{21} &\leq r_{13} \\ R_0 + R_1 + R_{21} &\leq r_{14}. \end{aligned} \tag{5.36}$$

Decoding strategy at cognitive receiver: The cognitive receiver obtains the signal

$$Y_2^n = aX_{10}^n + aX_{11}^n + aX_{12}^n + X_{20}^n + X_{21}^n + X_{22}^n + Z_2^n.$$

The cognitive receiver decodes messages W_{12} and W_{21} jointly treating X_{10}^n, X_{20}^n, X_{11}^n and X_{22}^n as Gaussian noise. The receiver can decode message W_{12} and W_{21} successfully if

$$\begin{aligned} R_{12} &\leq r_{15} \\ R_{21} &\leq r_{16} \\ R_{12} + R_{21} &\leq r_{17} \end{aligned} \tag{5.37}$$

Finally, the cognitive receiver decodes W_{22} using Costa's dirty paper decoding. In decoding W_{22}, X_{10}^n and X_{20}^n do not appear as noise as they were canceled out at the encoder side using Costa's dirty paper coding. The decoding is successful if

$$R_{22} \leq r_{18}. \tag{5.38}$$

Using Fourier-Motzkin elimination, we can easily show that the region given by $\mathcal{R}_i^{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3}$ is achievable. By taking the closure of the convex hull over the set of α's and β's, we show that the region given by $\mathcal{R}_i$ is achievable,. This completes the achievability proof.

Remark 5.4.1. As μ grows to infinity, transmission of the shared message sets are not allowed, which means that channel becomes more close to the interference channel with no cognitive message sets. Our achievable scheme enforces β_1 and α_1 to be fixed at 0, and use regular Han and Kobayashi coding. In the other extreme, the channel becomes cognitive radio channels with full message sets of the legitimate user. In this case, α_2, α_3 are fixed to zero, and cognitive user make the dirty paper coding with the transmission support to legitimate user, which is optimal.

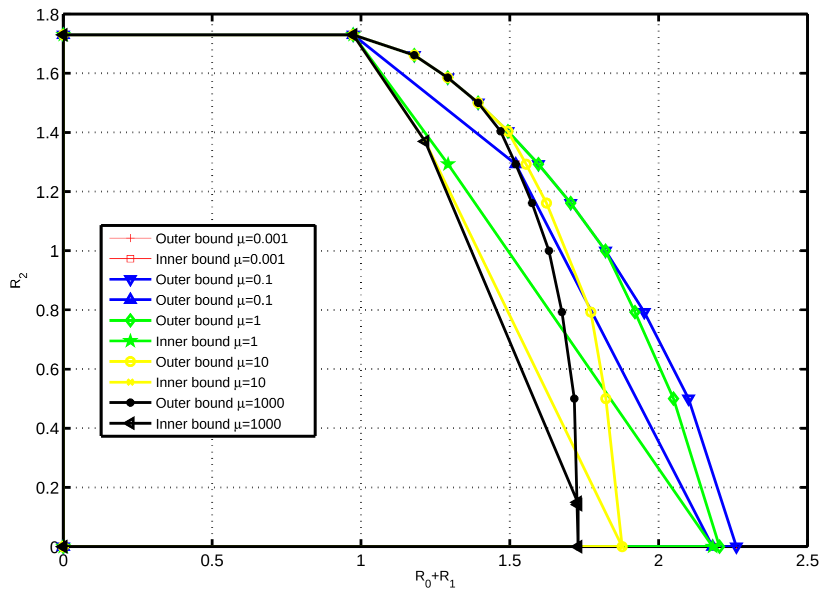


Figure 5.4: Achievable region and Outer bound

Achievable region and outer bound are compared in Fig. 5.4. Both transmit powers, P_1 and P_2, are set to 10, and interference gain a and b are fixed to 2 and 0.5 respectively. For the licensed user, we use the total rate $R_0 + R_1$.

Notice that as the value μ grows, achievable region asymptotically approaches the outer bound.

5.5 Conclusion

In this chapter, we investigated the capacity region of interference channel with partially cognitive transmitter. For the general discrete memoryless IFC setting, we obtained the outer bound for the capacity region when the legitimate receiver observes the weak interference. We also derived an outer bound and achievable region for the Gaussian partial cognitive-radio channel.

Chapter 6

K User Gaussian Interference Channel

In this chapter, we deviate from cognitive radio channel models and study the K user interference channel with K transmitter-receiver pairs. The goal of this chapter is to understand the capacity behavior of such large networks and to determine if the capacity scales with the number of users in the network and to derive transmission strategies that help in understanding capacity behavior at all power levels. We use lattice coding as an interference alignment transmission strategy and derive capacity results for the K user Gaussian interference channel.

6.1 Introduction

Determining the capacity region of large Gaussian interference network has been a long standing open problem. Several capacity results have been derived for the two user interference channel [17–21]. Recently, it has been shown in [13] that the gap between the Han-Kobayashi achievable region [4] and a genie aided outer bound for the two user Gaussian IC is at most one bit per channel use. In [14]– [16], the sum capacity of the two user Gaussian IC has been determined for a range of "very weak" or "noisy" interference cases where treating interference as noise is optimal. While the results of [14]– [16] are generalizable to more than two users, other capacity results such as [13, 18, 19] do not extend to interference channels with more than two transmitter-receiver pairs.

For interference networks with more than two transmitter-receiver pairs, degrees of freedom characterization (capacity approximations within $o(\log(\text{SNR}))$) have been found for a class of time or frequency varying channels in [68]– [72]. These results do not apply to interference networks with constant channels, i.e., channels that are not time or frequency varying. In [73], the authors compute the approximate capacity of constant many-to-one Gaussian interference channels (where only one receiver sees interference from the other transmitters, and the other receivers see no interference) by building and using an approximate deterministic model for the channel. In [74], the generalized degrees of freedom (GDOF) of the symmetric K user Gaussian interference channels are derived. However, this result holds only in the high SNR regime for channels where the channel gains scale with power. In [7], some examples of K user interference channels are presented which come close to achieving the outer limit of $K/2$ degrees of freedom.

Very recently, it has been shown that for the interference channel with real and rational coefficients, total degrees of freedom is bounded away from $K/2$ [75]. In the same work, authors present an achievable scheme for a class of interference channel with a mix of rational and algebraic irrational channel gains channel gains to achieve $K/2$ degrees of freedom. For the case of complex channel gains, [83] show that at least $6/5$ total degrees of freedom are achievable for almost all values of channel coefficients.

Note that the main emphasis of a majority of previous work on this topic has been on the degrees of freedom characterization for a K user interference network. The primary difference between prior work and the work in this thesis is that our aim is to determine new achievable regions for the fully connected, symmetric K user interference channel at any SNR. To this end, we utilize structured transmission schemes based on lattice codes. Note that the use of lattice codes to effect an interference alignment can also be found in [73] where it is applied to the many-to-one Gaussian interference channel.

Lattice coding has also been used as an effective transmission strategy in achieving the capacity of several other channels. It is used (along with lattice decoding) on an AWGN channel in [84]– [86] to achieve a rate equal to $\frac{1}{2}\log(\text{ SNR})$. In [87, 88], lattice coding, along with simplified maximum likelihood decoding, is shown to achieve the capacity of the AWGN channel. Lattice coding has also been used to determine the approximate capacity of two-way relay channels in [89, 90]. Some other relevant results on lattice coding include [91]– [97].

In this chapter, we study a class of K user Gaussian interference channels (see Figure 6.1) from a capacity and a degree of freedom perspective. The primary tool we use in deriving achievable rates is lattice coding. Lattice coding helps in aligning the interference at each receiver and enables us to decode the *total interference* without decoding each individual interference signal or each message. Note that, for two user Gaussian interference channels, decoding the net interference is equivalent to decoding each interfering signal/message (as there is only one interferer), but there is a clear distinction between "total interference" and "each interfering transmitter's signal" for channels with more than two users. First, we derive a “very strong” interference regime for symmetric K user Gaussian interference channels and extend the result to a class of non-symmetric channels. A “very strong” interference regime is one in which the capacity region of the interference channel is the same as the capacity region of the interference channel with no interference. That is, the interference can be completely canceled out first by each receiver without incurring a rate penalty. This extends the work in [17] where the “very strong” interference regime is derived for two user interference channels. Note that there is a fundamental difference between the “very strong” interference channels in [17] and those in this chapter. In [17], each receiver decodes all the messages from all the transmitters. In our work, each receiver decodes only its message and a *function* of the other signals. Second, we use this “very strong” interference result to propose a layered lattice coding scheme for a class of K user Gaussian interference channel beyond the very strong interference regime. We use this layered lattice coding strategy to show that we can achieve more than one degree of freedom for a large range of channel parameters. In particular, we also show that there exist channels which achieve degrees of freedom arbitrarily close to $K/2$. We also numerically compare the layered lattice coding strategy with a coding/decoding scheme that resembles Han-Kobayashi scheme in [4] with codebooks that are generated i.i.d Gaussian, to show that significant rate benefits can be achieved by decoding the interference instead of part (or whole) of undesired messages from the interfering transmitters. The main contributions of this work are summarized below:

- We derive a “very strong” interference regime for a class of K user Gaussian interference channels,

- We propose a layered lattice coding strategy for any SNR. This coding scheme is also shown to achieve more than one degree of freedom for a large range of channel parameters (in the class of interference channels considered),

- We show numerically that significant rate benefits can be achieved by the layered lattice coding strategy when compared with the extension of the Han-Kobayashi style strategy with Gaussian codebooks.

It is to be noted that the results presented in this chapter are joint work with Amin Jafarian, a Ph.D. student at the Department of Electrical and Computer Engineering. Amin Jafarian derived the “very strong” interference regime for symmetric K user interference channels. This dissertation applies the very strong interference result to develop a layered lattice alignment scheme for interference networks and analyzes the degree of freedom of such networks. The layered lattice alignment scheme also presents a very effective transmission strategy that works at any signal power levels. For the sake of completeness of the chapter, we present the “very strong” interference result for interference networks in Section 6.4.

The rest of the chapter is organized as follows: In Section 6.2, we present the system model. We describe notations and present some lattice preliminaries in Section 6.3. In Section 6.4, we summarize the “very strong” interference conditions for the two user Gaussian interference channel and state and prove our results on “very strong” interference regime for the K user Gaussian interference channel. In Section 6.5, we present the layered lattice coding strategy for symmetric

K user Gaussian interference channels and analyze the total degrees of freedom achieved by that strategy. In Section 6.6, we extend the layered lattice coding approach to a class of non-symmetric channels. In Section 6.7, we present numerical results comparing our layered lattice coding approach with the extension of Han-Kobayashi coding strategy with i.i.d. Gaussian inputs for the symmetric three user Gaussian interference channels. We conclude with Section 6.8.

6.2 System Model

A K user Gaussian interference channel consists of K transmitter-receiver pairs and K independent messages such that message m_k originates at Transmitter k and is intended for Receiver k for all $k \in \{1, 2, \ldots, K\}$. The system model is described in Figure 6.1 and the channel equations are described by

$$Y_j(i) = X_j(i) + \sum_{k=1, k\neq j}^{K} h_{jk} X_k(i) + Z_j(i),\ j \in \{1, 2, \ldots, K\} \tag{6.1}$$

where $Y_j(i)$ is the received signal at the j^{th} receiver at the i^{th} channel use, $X_k(i)$ is the transmitted signal at the k^{th} transmitter at the i^{th} channel use, and h_{jk} denotes the channel gain from the k^{th} transmitter to the j^{th} receiver. In Equation (6.1), all the direct channel gains have been normalized to unity. $Z_j(i)$ is the zero mean, unit variance additive white Gaussian noise at receiver j at time i. The Gaussian noise at each receiver is i.i.d. across time, but the noise at one receiver maybe correlated with noise at any other receiver, and this correlation does not affect the capacity region of the system. In this setup, it is assumed that the channel gains are constant and are known at all the transmitters and receivers. We also restrict the system

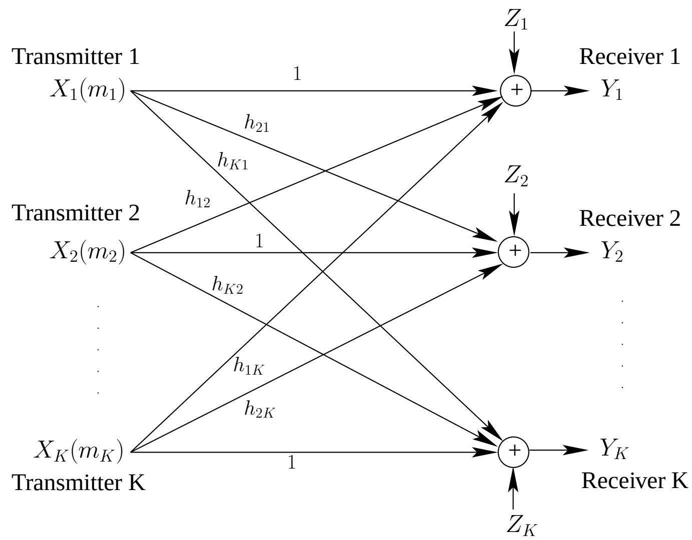


Figure 6.1: System Model for K User Gaussian Interference Channel

model to real channel inputs and channel outputs. The channel inputs are subject to the following average power constraints:

$$\frac{1}{n}\sum_{i=1}^{n} X_k(i)^2 \leq P_k,\ \forall\, k \in \{1, 2, \ldots, K\}. \tag{6.2}$$

Let H denote the $K \times K$ matrix of channel gains

$$H = \begin{pmatrix} 1 & h_{12} & \cdots & h_{1K} \\ h_{21} & 1 & \cdots & h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ h_{K1} & h_{K2} & \cdots & 1 \end{pmatrix}.$$

Let $\mathcal{H}_0, \mathcal{H}_2$ denote the following classes of channel matrices:

$$\mathcal{H}_0 = \left\{ H \in \mathbb{R}^{K \times K} : h_{ii} = 1 \right\}$$
$$\mathcal{H}_2 = \left\{ H \in \mathbb{Q}^{K \times K} : h_{ii} = 1 \right\},$$

where $\mathbb{Q}$ is the set of all rational numbers.

The class of channels to which our coding strategy applies is given by $\mathcal{H}_1$, where $\mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0$. For example, in three user case, $\mathcal{H}_1^3$ is:

$$\mathcal{H}_1^3 = \left\{ H \in \mathbb{Q}^{3\times 3} : \frac{h_{12}}{h_{21}} \times \frac{h_{23}}{h_{32}} \times \frac{h_{31}}{h_{13}} \in \mathbb{Q} \right\}. \tag{6.3}$$

Note that this is a (fairly) non-trivial class of channels which includes the symmetric interference channel and interference channel with rational gains as special cases. In a symmetric interference channel,

$$h_{ij} = \begin{cases} 1 & \text{if } i = j \\ a & \text{if } i \neq j \end{cases}.$$

That is, all the cross channel gains are equal. Moreover, in a K user symmetric interference channel, all the power constraints are equal, i.e., $P_j = P \; \forall \; j \in \{1, 2 \ldots, K\}$.

We represent the interference to noise ratio of user j caused by transmitter k as $\text{INR}_{j,k}$. That is,

$$\text{INR}_{j,k} = h_{j,k}^2 P_k.$$

A $(2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_K}, n, \lambda)$ code for the K user Gaussian interference channel consists of K message sets

$$M_k = \{1, 2, \ldots, 2^{nR_k}\}, \; \forall \; k \in \{1, 2, \ldots, K\},$$

K encoding functions

$$F_k : M_k \to \mathcal{X}_k^n, \; \forall \; k \in \{1, 2, \ldots, K\},$$

and K decoding functions

$$G_k : \mathcal{Y}_k^n \to M_k, \; \forall\, k \in \{1, 2, \ldots, K\}$$

such that the average probability of decoding error is less than or equal to λ. A rate tuple $(R_1, R_2, \ldots, R_K)$ is said to be achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_K}, n, \lambda^{(n)})$ codes such that $\lambda^{(n)} \to 0$ as $n \to \infty$. The capacity region of the channel is the set of all achievable rate tuples and is denoted by $\mathcal{C}_{ap}$.

The degrees of freedom region of the K user Gaussian interference channel is defined as follows:

$$\mathcal{D} = \Bigg\{ (d_1, \ldots, d_K) \in \mathbb{R}_+^K : \forall (\mu_1, \ldots, \mu_K) \in \mathbb{R}_+^K, \\ \mu_1 d_1 + \ldots \mu_K d_K \leq \limsup_{P_1+\ldots+P_K \to \infty} \sup_{(R_1,\ldots,R_K) \in \mathcal{C}} \frac{\mu_1 R_1 + \ldots + \mu_K R_K}{\frac{1}{2}\log(P_1 \ldots + P_K)} \Bigg\} \quad (6.4)$$

The total degrees of freedom of the three user Gaussian IC is denoted by $\mathcal{D}_{sum}$ and is defined as

$$\mathcal{D}_{sum} \triangleq \limsup_{P_1+\ldots+P_K \to \infty} \max_{(R_1,\ldots,R_K) \in \mathcal{C}_{ap}} \frac{R_1 + \ldots + R_K}{\frac{1}{2}\log(P_1 + \ldots + P_K)}. \quad (6.5)$$

The total degrees of freedom represents the rate of growth of sum capacity in terms of $\log(\text{SNR})$ and thus corresponds to the number of non-interfering links in the channel. We desire to determine an achievable region for this channel that simultaneously has a good performance in terms of degrees of freedom.

In the next section, we provide a brief introduction to lattice coding and also summarize some known results on lattice coding for a point to point AWGN channel.

6.3 Lattice Coding Preliminaries

A lattice Λ of dimension n is a discrete subset of $\mathbb{R}^n$ described by

$$\Lambda = \{\lambda = Gx : x \in \mathbb{Z}^n\},$$

where G is the generator matrix that describes the lattice. Let Ω_Λ denote the fundamental Voronoi region (as defined in [86]) of the lattice Λ and V_Λ denote the volume of Ω_Λ. In this chapter, we use lattices generated using a mechanism known as Construction-A [86], which we describe below.

For any positive prime integer p, let $\mathbb{Z}_p$ denote the set of integers modulo p. Let $g : \mathbb{Z}^n \to \mathbb{Z}_p^n$ denote the component wise modulo p operation over integer vectors. Let C denote a linear (n, k) code over $\mathbb{Z}_p$. Then the lattice Λ_C given by

$$\Lambda_C = \{v \in \mathbb{Z}^n : g(v) \in C\} \quad (6.6)$$

is said to be generated using Construction-A with respect to the linear code C. In this work, we consider scaled versions of lattices generated in this construction, that is, lattices of the form $\gamma\Lambda_C$ for some $\gamma \in \mathbb{R}$. The fundamental volume of $\gamma\Lambda_C$ is equal to $\gamma^n p^{n-k}$.

A set $\mathcal{B}$ of linear codes over $\mathbb{Z}_p$ is said to be balanced if every nonzero element of $\mathbb{Z}_p^n$ is contained in the same number of codes in $\mathcal{B}$. An example of a balanced linear code is given in [87, Section VII]. Let $\mathcal{L}_B$ be the set of lattices denoted by

$$\mathcal{L}_B = \{\Lambda_C : C \in \mathcal{B}\}. \quad (6.7)$$

We now state here the Minkowski-Hlawka Theorem (as established in [86]) with some minor modifications.

Lemma 6.3.1 (Minkowski-Hlawka Theorem)**.** *Let f be a Riemann integrable function $\mathbb{R}^n \to \mathbb{R}$ of bounded support. For any integer k, $0 < k < n$ and any fixed V, let $\mathcal{B}$ be any balanced set of linear (n, k) codes over $\mathbb{Z}_p$. As $p \to \infty$, $\gamma \to 0$ such that $\gamma^n p^{n-k} = V$, at least three-fourths of the lattices in the set $\mathcal{L}_B$ satisfy the following relationship*

$$\sum_{v \in \gamma\Lambda_C : v \neq 0} f(v) \leq \frac{4}{V} \int_{\mathbb{R}^n} f(v) dv. \tag{6.8}$$

The proof of this lemma is similar to the proof of [86, Theorem 1] with few elementary changes, and is therefore omitted.

Next, we consider a point to point additive noise channel

$$Y = X + Z, \tag{6.9}$$

where X is the transmitted signal, Y the received signal and Z is the additive noise of zero mean and variance equal to σ^2 that corrupts the transmitted signal at the receiver. If the transmitted word over time is a lattice point, then it can be shown that a suitable lattice and a decoding strategy exists such that the probability of decoding error can be made arbitrarily small as the number of dimensions of the lattice increases. This result is stated formally.

Lemma 6.3.2 ([86])**.** *Consider a single user point to point additive noise channel described in (6.9). Let $\mathcal{B}$ be a balanced set of linear (n, k) codes over $\mathbb{Z}_p$. Averaged* *over all lattices from the set $\mathcal{L}_B$ given in (6.7), each with a fundamental volume V, we have that for any $\delta > 0$, the average probability of decoding error is bounded by*

$$\overline{P_e} < (1+\delta)\frac{2^{n\frac{1}{2}\log(2\pi e\sigma^2)}}{V}. \tag{6.10}$$

for sufficiently large p and small γ such that $\gamma^n p^{n-k} = V$. Hence, the probability of decoding error for at least three fourths of the lattices in $\mathcal{L}_B$ satisfies

$$P_e < 4(1+\delta)\frac{2^{n\frac{1}{2}\log(2\pi e\sigma^2)}}{V}. \tag{6.11}$$

The proof of Lemma 6.3.2 is also described in [86] and is therefore omitted. In essence, Lemma 6.3.2 describes the existence of a lattice code with sufficient codewords. The next lemma summarizes the main result of [88].

Lemma 6.3.3. *Consider a point to point additive noise channel in (6.9) where the noise is AWGN with zero mean and variance equal to σ^2. Let Λ be any lattice generated from Construction A that satisfies (6.11). Then, we can choose the fundamental volume of the lattice V, shift s and a shaping region S such that the lattice code $(\Lambda + s) \cap S$ achieves a rate R with arbitrarily small average probability of error if*

$$R \leq \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right).$$

The proof of Lemma 6.3.3 is provided in [88]. It is important to note that Lemma 6.3.3 requires that the additive noise be i.i.d Gaussian distributed. The three lemmas introduced in this section is used to derive a "very strong" interference regime for the K user Gaussian interference channel.

An important point to note is that these three lemmas which originate from [86] and [88] assume the noise added in the point to point channel is statistically independent of the transmitted codeword and independent of the structure of the codeword. However, we are often presented with scenarios in this chapter where this may not be the case, and the noise may in fact depend on the structure of the codeword being transmitted. The following lemma considers a channel where no assumption is made about the independence of noise and the structure of the codebook (or of the codeword being transmitted).

Lemma 6.3.4. *Consider a single user point to point additive noise channel in (6.9) where the noise Z is zero mean and the n- dimensional noise vector $\mathbf{Z}$ satisfies $||\mathbf{Z}||^2 \leq n\sigma^2$. We assume that the noise is statistically independent of the transmitted signal (it may be dependent on the structure of the transmitted signal). Let $\mathcal{B}$ be a balanced set of linear (n, k) codes over $\mathbb{Z}_p$. Averaged over all lattices from the set $\mathcal{L}_B$ given in (6.7), each with a fundamental volume V, we have that for any $\delta > 0$, the average probability of decoding error is bounded by*

$$\overline{P_e} < (1+\delta)\frac{2^{n\frac{1}{2}\log(2\pi e\sigma^2)}}{V}. \tag{6.12}$$

for sufficiently large p and small γ such that $\gamma^n p^{n-k} = V$. Hence, the probability of lattice decoding error for at least three fourths of the lattices in $\mathcal{L}_B$ satisfies

$$P_e < 4(1+\delta)\frac{2^{n\frac{1}{2}\log(2\pi e\sigma^2)}}{V}. \tag{6.13}$$

Proof. : The proof is a minor modification of the proof of Lemma 6.3.2 as described in [86]. Let $\mathcal{E}$ denote the typical set of noise vectors. Let $\overline{\mathcal{E}}$ denote the sphere of radius $\sqrt{n\sigma^2}$. We assume that the transmitted signal X is an element of lattice Λ. If the noise $Z \in \mathcal{E}$, an error may occur in decoding if Z can be expressed as $Z = Z' + X^*$ where $X^* \in \Lambda^*$ and $Z' \in \mathcal{E}$. Then the probability of error P_e can be upper bounded by

$$P_e \leq P_{amb|\mathcal{E}} + Pr(Z \notin \mathcal{E}).$$

In proving the lemma, we work with the set $\overline{\mathcal{E}}$ instead of the typical set of noise vectors $\mathcal{E}$ as in [86]. We can also upper bound the probability of error by

$$Pe \leq P_{amb|\overline{\mathcal{E}}} + Pr(Z \notin \mathcal{E}).$$

We can show that averaged over all lattices from the set $\mathcal{L}_B$ given in (6.7), each with a fundamental volume V, we have that for any $\delta > 0$, we can upper bound $\overline{P_{amb|\overline{\mathcal{E}}}}$ by

$$\overline{P_{amb|\overline{\mathcal{E}}}} \leq (1+\delta)\frac{V(\overline{\mathcal{E}})}{V_f}.$$

The remainder of this proof now proceeds along the same lines as [86, Theorems 4,5] and the details are therefore omitted. □

6.4 “Very Strong” Interference Regime

An interference channel is said to be in the “very strong” interference regime if the capacity region of the channel is the same as the capacity region of the channel obtained by removing all the interfering links. That is, in the “very strong” interference regime, user j can achieve a rate of $\frac{1}{2}\log(1+\text{SNR}_j)$ for $j \in \{1, \ldots, K\}$, where $\text{SNR}_j = \frac{P_j}{1} = P_j$. Note that this is the maximum rate user j can achieve given its resource constraints. The essential strategy in the very strong interference regime is to decode the net interference first and then decode the desired message. In this regime, the interference is so strong that the rate constraints due to decoding the interference are not binding on the capacity region. In the next subsection, we briefly summarize the “very strong” interference regime for a two user Gaussian interference channel and provide a generalization of the result to the K user channel. In Section 6.4.3, we state the main results on “very strong” interference for the K user symmetric Gaussian interference channel and for a class of non-symmetric channels. In Section 6.4.4, we provide the proofs for the results in Section 6.4.3.

6.4.1 Two User Gaussian Interference Channel - Very Strong Interference Regime

In this section, we describe the “very strong” interference regime for the two user Gaussian interference channel as shown in Figure 6.2. Carleial [17] showed

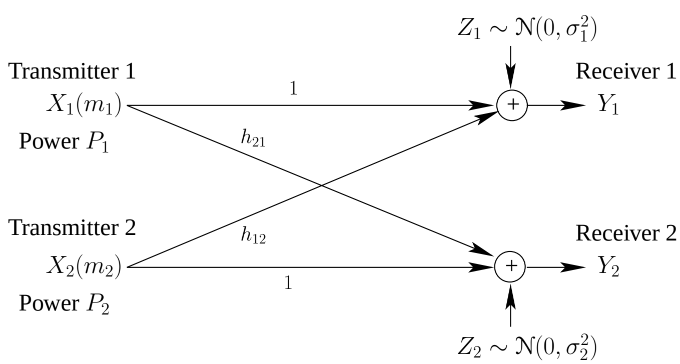


Figure 6.2: Two User Gaussian Interference Channel

that for the channel in Figure 6.2, interference does not degrade capacity when it is very strong, because the interfering signal can be decoded without any rate penalty for either the desired or the interfering user's message. This result is stated formally in the next lemma.

Lemma 6.4.1. *For the two user Gaussian interference channel shown in Figure 6.2, if the channel parameters satisfy*

$$h_{12}^2 \geq \frac{P_1 + \sigma_1^2}{\sigma_2^2}, \quad h_{21}^2 \geq \frac{P_2 + \sigma_2^2}{\sigma_1^2}, \tag{6.14}$$

then the capacity region of the channel is given by

$$\mathcal{C}_{ap} = \left\{ \begin{array}{l} (R_1, R_2) \in \mathbb{R}_+^2 : \\ R_1 \leq \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_1^2}\right), \quad R_2 \leq \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_2^2}\right) \end{array} \right\}.$$

The proof of this lemma is described in [17]. The essential idea is that the receivers decode the interfering message first before decoding their message. If the channel parameters satisfy (6.14), then we can see that the rate constraints due to decoding the interfering message at receiver 1 (or 2) is non-binding, and the constraint resulting from decoding the desired message at each receiver is the primary rate limiting factor. We now provide a direct extension of the above result for the K user symmetric Gaussian interference channel as shown in Figure 6.3. A generalization to the non-symmetric case is similar, but is fairly unwieldy to express due to the number of parameters and factors involved.

Lemma 6.4.2. *Consider a K user symmetric Gaussian interference channel as shown in Figure 6.3. If the channel parameters satisfy*

$$a^2 \geq \left(\left(1 + \frac{P}{\sigma^2}\right)^{K-1} - 1\right)\frac{P + \sigma^2}{(K-1)P}, \tag{6.15}$$

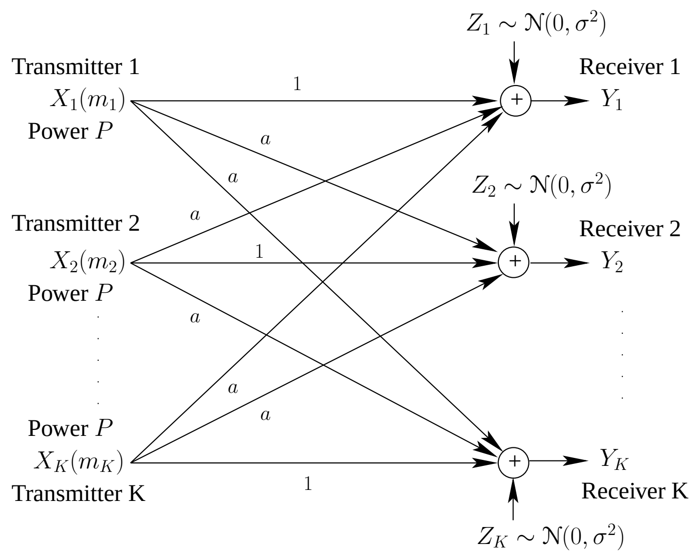


Figure 6.3: K User Gaussian Interference Channel

then the capacity region of the channel is given by

$$\mathcal{C}_{ap} = \left\{ \begin{array}{l} (R_1, \ldots, R_K) \in \mathbb{R}_+^K : \\ R_i \leq \frac{1}{2} \log\left(1 + \frac{P}{\sigma^2}\right), \quad i \in \{1, \ldots, K\} \end{array} \right\}.$$

Proof. : The proof of this lemma is similar to the proof of Lemma 6.4.1 and is described next. Each transmitter encodes it messages by choosing codewords from a suitable i.i.d. Gaussian distribution. Each receiver first decodes all the interfering messages by treating its own codeword as noise. After canceling the effect of all interference, the receiver then decodes its own message. We now analyze the rate constraints imposed by this encoding/decoding strategy at Receiver 1. Due to the symmetry of the channel, the constraints imposed on other receivers are similar.

Receiver 1 observes

$$Y_1 = X_1 + \sum_{k=2}^{K} aX_k + Z_1.$$

As Receiver 1 first decodes all the interfering messages, it treats its own codeword X_1 as noise and hence sees a total noise power of $P+\sigma^2$. The receiver can decode the interfering messages $m_2, \ldots, m_K$ successfully if the rate tuple $(R_2, \ldots, R_K)$ satisfies

$$\sum_{j \in \mathcal{S}} R_j \leq \frac{1}{2} \log\left(1 + \frac{|S| a^2 P}{P + \sigma^2}\right), \quad \forall\, \mathcal{S} \subseteq \{2, \ldots, K\}. \tag{6.16}$$

After decoding all the interfering messages, receiver 1 can decode its message m_1 successfully if

$$R_1 \leq \frac{1}{2} \log\left(1 + \frac{P}{\sigma^2}\right).$$

Hence, we can describe the achievable rate region $\mathcal{R}_1$ as follows:

$$\mathcal{R}_1 = \left\{ \begin{array}{l} (R_1, R_2, \ldots, R_K) \in \mathbb{R}_+^K \\ R_j \leq \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right) \\ \sum_{k \in \mathcal{S}_1} R_k \leq \frac{1}{2}\log\left(1 + \frac{|S_1| a^2 P}{P+\sigma^2}\right), \quad \forall\, \mathcal{S}_1 \subseteq \{2, \ldots, K\} \\ \sum_{k \in \mathcal{S}_2} R_k \leq \frac{1}{2}\log\left(1 + \frac{|S_2| a^2 P}{P+\sigma^2}\right), \quad \forall\, \mathcal{S}_2 \subseteq \{1, 3, \ldots, K\} \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \sum_{k \in \mathcal{S}_K} R_k \leq \frac{1}{2}\log\left(1 + \frac{|S_K| a^2 P}{P+\sigma^2}\right), \quad \forall\, \mathcal{S}_K \subseteq \{1, \ldots, K-1\} \end{array} \right\}. \tag{6.17}$$

We can now see that if the channel parameter satisfy (6.15), then the only constraints in $\mathcal{R}_1$ that are binding are

$$R_i \leq \frac{1}{2} \log\left(1 + \frac{P}{\sigma^2}\right), \quad i \in \{1, \ldots, K\},$$

and this is the capacity region of the channel as this is the maximum possible rates that each user can achieve even in the absence of all interference. □

Remark 6.4.1. From the above lemma, the lower bound on a^2 for very strong interference increases exponentially with K.

In the next section, we investigate the very strong interference regime for a deterministic K user symmetric interference channel. We show that the very strong interference condition remains the same for all K.

6.4.2 Very Strong Interference Regime for K User Interference Channel: A Deterministic Model

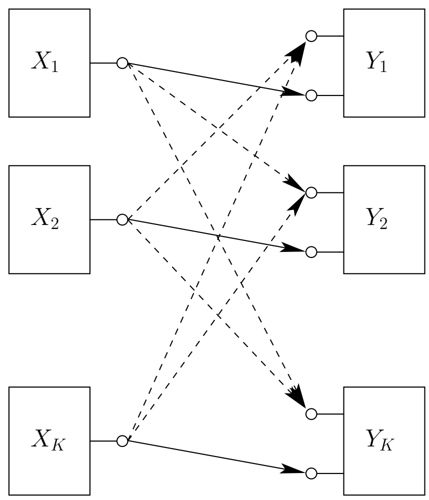


Figure 6.4: A Deterministic K User Gaussian Interference Channel

In Figure 6.4, we describe an example of a deterministic channel model of K user fully connected Gaussian interference channel (as proposed by [73]). In this example, each user achieves a rate equal to the capacity that he would achieve in the absence of all interference. Note that with all K users transmitting at capacity, a receiver is able to decode the desired message but cannot decode any of the other interfering messages (as they all add up in the first terminal of each receiver). However, each receiver is able to decode the *sum* of the codewords sent by the interfering users. For example, Receiver 1 cannot decode the messages $m_2, \ldots, m_K$, but it can decode the sum of the interfering codewords $X_2 + \ldots + X_K$. In the terminology of generalized degrees of freedom [73] the "very strong interference" condition for this symmetric deterministic channel can be stated as:

$$\frac{\log(\text{INR})}{\log(\text{SNR})} \geq 2 \tag{6.18}$$

or in our notation, the very strong interference condition can be stated as $a^2 \geq P$. This shows that the very strong interference condition for a deterministic K user interference channel is independent of K. In the next section, we show that the even for the K user fully connected Gaussian interference channel, the very strong interference condition is independent of K. As in [73], we use the deterministic channel model to help us devise a good transmission strategy for the Gaussian channel. As the receivers decode only the sum of the interference (and not each interfering message) in the deterministic model, we apply the same principle to the Gaussian model. Through lattice codes, we "align" the interference at each receiver so that to cancel out the interference, the receivers do not have to decode all the interfering messages, but can directly decode the sum of all the interference.

6.4.3 Very Strong Interference Regime for K User Interference Channel - Main Results

In this section, we derive a "very strong" interference regime for symmetric K user Gaussian interference channels and then extend the result to a class of non-symmetric channels. We use lattice codes to align interference at each receiver in such a way that the sum of the interfering codewords can be decoded, without requiring the decodability of the messages carried by the interfering signals. Relaxing the message decodability constraint produces a much tighter "very strong" interference condition for the K user symmetric interference channel. Lattice codes have previously been used in [73] for interference alignment on the *many-to-one* interference channels, leading to capacity characterizations within a fixed number of bits per channel use for these channels. However, since we are interested in fully connected interference networks, several key aspects of the lattice code constructions in this section are unique to our setup. The next theorem presents a "very strong" interference region for the symmetric K user Gaussian interference channel.

Theorem 6.4.3. *[99] [101] Consider a K user symmetric Gaussian interference channel in Figure 6.3 where a represents the cross channel gain and P is the power constraint at each transmitter. If the channel gain a satisfies*

$$a^2 \geq \frac{(P+1)^2}{P}, \tag{6.19}$$

then the capacity region of the channel, denoted by $\mathcal{C}_{ap}$ is given

$$\mathcal{C}_{ap} = \left\{ \begin{array}{c} (R_1, \ldots, R_k) : \\ \quad R_k \leq \frac{1}{2}\log(1+P) \;\; \forall\, k \in \mathcal{K} \end{array} \right\}. \tag{6.20}$$

The region described by (6.20) is an outer bound on the capacity region for a K user interference channel for any value of a. This is because $\frac{1}{2}\log(1+P)$ is the maximum rate achieved by any user when there is no interference. To show that the region described by (6.20) is achievable under "very strong" interference given by (6.19), we show that the symmetric rate point $\left(\frac{1}{2}\log(1+P), \ldots, \frac{1}{2}\log(1+P)\right)$ is achievable when (6.19) is satisfied. The transmitters use lattice coding to encode their messages, while the receivers first decode the total interference and then decode their message after canceling all the interference. We use the results of the Lemmas in Section 6.3 proved in [86] and [88] in proving Theorem 6.4.3. The proof is presented in Section 6.4.4.

Note that the "very strong" interference condition for the K user symmetric Gaussian interference channel is different from the condition for the two user case given by $a^2 \geq P + 1$. In fact, we have the following approximate capacity result for $a^2 \geq P + 1$ for the K user symmetric Gaussian interference channel.

Theorem 6.4.4. *[99, 101]*

For a K user symmetric Gaussian interference channel with cross channel gain a and power constraint P, if the channel gain a satisfies $a^2 \geq P+1$, then each user can achieve a rate of $\frac{1}{2}\log(P)$. Hence, for $a^2 \geq P + 1$, each user achieves within half a bit per channel use of his maximum possible rate.

The proof of the theorem is very similar to the proof of Theorem 6.4.3. In the proof of Theorem 6.4.3, we use the Loeliger framework in decoding the interference and the Urbanke-Rimoldi framework in decoding the message at each receiver. However, in proving Theorem 6.4.4, we use the Loeliger framework for decoding both the interference and the message at each receiver. In this chapter, we do not prove Theorem 6.4.4 completely as the proof is similar to the proof of Theorem 6.4.3. However, we provide the essential details of the proof in Section 6.4.4.

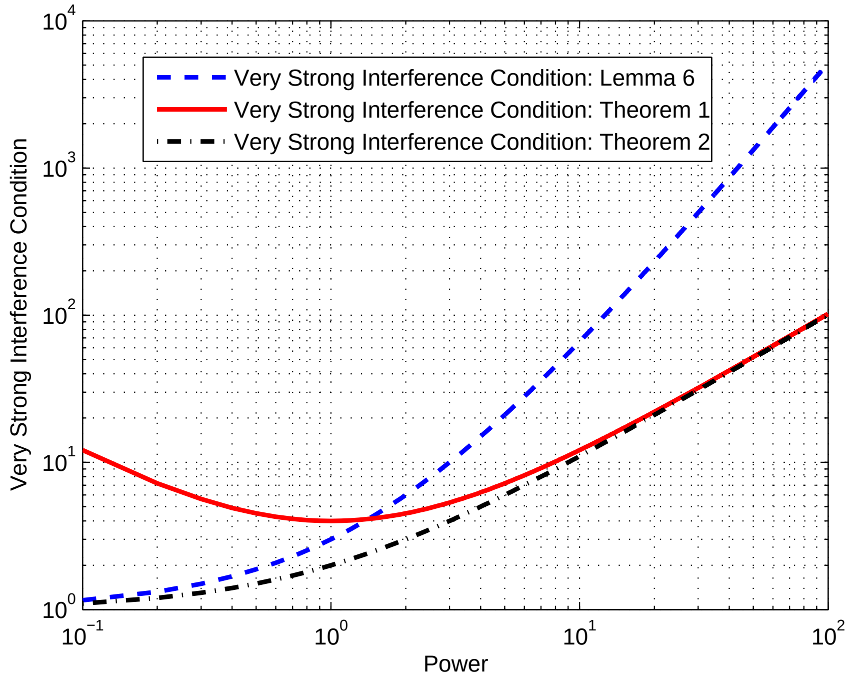


Figure 6.5: Comparing Very Strong Interference Conditions of Lemma 6.4.2, theorems 6.4.3 and 6.4.4

In Figure 6.5, we plot the very strong interference condition of Lemma 6.4.2, Theorems 6.4.3 and 6.4.4 for a three user symmetric Gaussian interference channel. We can see that the very strong interference condition of Lemma 6.4.2 beats the very strong interference condition of Theorem 6.4.3 for low values of power P. This is due to mixing the Urbanke-Rimoldi and Loeliger approach of decoding. By using only the Loeliger approach for decoding at the receivers, we get the very strong interference condition of Theorem 6.4.4. But, we get only an approximate capacity result. If we can use the Urbanke-Rimoldi framework at the receivers for decoding the interference and the message, then we can get a very strong interference condition of $a^2 \geq P + 1$ and still get capacity. However, we have not been able to use the Urbanke-Rimoldi framework for decoding the interference. This is because when decoding the interference, the receiver observes a non-Gaussian noise. As the Urbanke-Rimoldi decoding approach works only in the presence of AWGN noise, we cannot use this approach. But, for moderate and high values of power P, the very strong interference condition of Theorem 6.4.3 clearly outperforms the very strong interference condition of Lemma 6.4.2.

We now generalize the "very strong" interference result to a class of non-symmetric channels. For simplicity, we restrict ourselves to three user interference channels as shown in Fig 6.6. However, the results can be generalized to any K user interference channels satisfying similar channel conditions. In this section, we

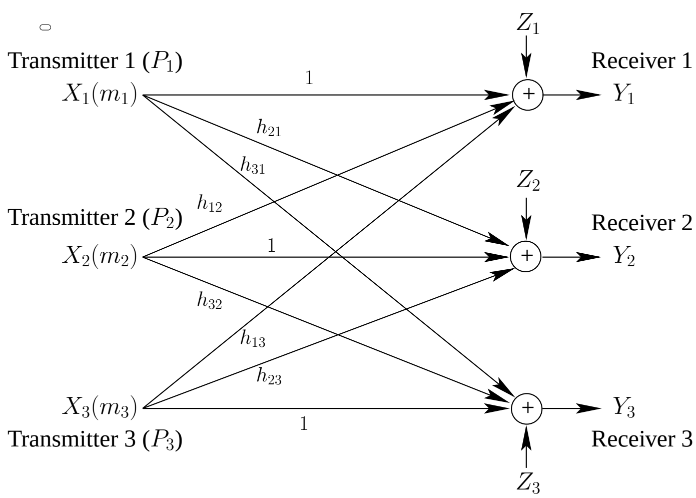


Figure 6.6: Three User Non Symmetric Gaussian Interference Channel

consider three user Gaussian interference channels whose channel matrix $H \in \mathbb{R}^{3\times 3}$

is an element of $\mathcal{H}_1^3$ as described in Equation (6.3). That is, we have

$$\frac{h_{12}}{h_{21}} \times \frac{h_{23}}{h_{32}} \times \frac{h_{31}}{h_{13}} \in \mathbb{Q}.$$

Without loss of generality, we assume that

$$\frac{h_{12}}{h_{21}} \times \frac{h_{23}}{h_{32}} \times \frac{h_{31}}{h_{13}} = \frac{p}{q} \tag{6.21}$$

where p and q are co-prime integers. Then, Theorem 6.4.5 describes "very strong" interference conditions for such a class of interference channels. This theorem is the generalization of Theorem 6.4.3 to the class of non-symmetric channels being considered. The proof of the channel is described in Section 6.4.4

Theorem 6.4.5. *[101] Consider a three user Gaussian IC, whose channel matrix $H \in \mathcal{H}_1$ and whose channel gains satisfy (6.21). We assume that the power constraints at the transmitters are P_1, P_2, P_3 and the noise variances at the receivers are σ_1^2, σ_2^2 and σ_3^2. If the channel gains satisfy one of the following three conditions*

$$\begin{gathered}
\exists N_i \in \mathbb{R}, \quad N_i \geq \sigma_i^2 \quad \textit{for } i \in \{1,2,3\}: \\
h_{12}^2 N_2 = p^2 h_{13}^2 N_3, \quad h_{21}^2 N_1 = q^2 h_{23}^2 N_3, \quad h_{31}^2 N_1 = h_{32}^2 N_2, \\
h_{12}^2 \geq p^2 \frac{P_1 + N_1}{N_2}, \quad h_{13}^2 \geq \frac{P_1 + N_1}{N_3} \\
h_{21}^2 \geq q^2 \frac{P_2 + N_2}{N_1}, \quad h_{23}^2 \geq \frac{P_2 + N_2}{N_3} \\
h_{31}^2 \geq \frac{P_3 + N_3}{N_1}, \quad h_{32}^2 \geq \frac{P_3 + N_3}{N_2}
\end{gathered} \tag{6.22}$$

(or)

$$
\begin{gathered}
\exists N_i \in \mathbb{R}, \quad N_i \geq \sigma_i^2 \quad \textit{for } i \in \{1,2,3\}: \\
h_{12}^2 N_2 = h_{13}^2 N_3, \quad p^2 h_{21}^2 N_1 = h_{23}^2 N_3, \quad q^2 h_{31}^2 N_1 = h_{32}^2 N_2, \\
h_{12}^2 \geq \frac{P_1 + N_1}{N_2}, \quad h_{13}^2 \geq \frac{P_1 + N_1}{N_3} \\
h_{21}^2 \geq \frac{P_2 + N_2}{N_1}, \quad h_{23}^2 \geq p^2 \frac{P_2 + N_2}{N_3} \\
h_{31}^2 \geq \frac{P_3 + N_3}{N_1}, \quad h_{32}^2 \geq q^2 \frac{P_3 + N_3}{N_2}
\end{gathered} \tag{6.23}
$$

(or)

$$
\begin{gathered}
\exists N_i \in \mathbb{R}, \quad N_i \geq \sigma_i^2 \quad \textit{for } i \in \{1,2,3\}: \\
q^2 h_{12}^2 N_2 = h_{13}^2 N_3, \quad h_{21}^2 N_1 = h_{23}^2 N_3, \quad h_{31}^2 N_1 = p^2 h_{32}^2 N_2, \\
h_{12}^2 \geq \frac{P_1 + N_1}{N_2}, \quad h_{13}^2 \geq q^2 \frac{P_1 + N_1}{N_3} \\
h_{21}^2 \geq \frac{P_2 + N_2}{N_1}, \quad h_{23}^2 \geq \frac{P_2 + N_2}{N_3} \\
h_{31}^2 \geq p^2 \frac{P_3 + N_3}{N_1}, \quad h_{32}^2 \geq \frac{P_3 + N_3}{N_2}
\end{gathered} \tag{6.24}
$$

then, the users can achieve rates given by

$$
R_i \leq \frac{1}{2} \log \left(\frac{P_i}{N_i} \right), \quad i \in \{1,2,3\}. \tag{6.25}
$$

This theorem is the generalization of Theorem 6.4.4 to the class of non-symmetric channels considered. The proof of the theorem is described in Section 6.4.4.

6.4.4 Very Strong Interference Regime for K User Interference Channel - Proofs

In this section, we give the proofs for Theorems 6.4.3 and 6.4.4. In proving Theorem 6.4.3, we prove only the achievability portion, as the converse part can be proved in a straightforward manner by removing all the interference from the receivers.

Achievability Proof of Theorem 6.4.3: In this proof, we show that in a K user symmetric Gaussian interference channel (with cross channel gain a and power constraint P), each user can achieve a symmetric rate $R < \frac{1}{2}\log(1+P)$ under very strong interference condition given by (6.19). We first describe the encoding strategy at the transmitters.

Encoding Strategy: The transmitters employ lattice coding as a transmission strategy. That is their codewords are elements of a shifted lattice within a shaping region. Due to the symmetry of the channel, we use the lattice Λ at each transmitter. We denote the Voronoi region of the lattice Λ by Ω and the volume of the Voronoi region by V. The transmitters use codebooks of the form $\mathcal{C} = (\Lambda + s) \cap \mathcal{S}$, where s is a shift, and $\mathcal{S}$ is a shaping region (to satisfy the power constraint). The shaping region is taken to be an n- dimensional sphere of radius $\sqrt{nP}$. Note that the shift s is there just to ensure a sufficient number of codewords inside the shaping region $\mathcal{S}$. Let $V_{\mathcal{S}}$ denote the volume of the shaping region $\mathcal{S}$. for $j \in \{1, 2, \ldots, K\}$, transmitter j communicates message $m_j \in M = \{1, \ldots, 2^{nR}\}$ to receiver j. For each $m_j \in M$, transmitter j assigns a codeword $X_i(m) \in \mathcal{C}$. We choose R', P' such that

$$R < R' < \frac{1}{2}\log(1+P') < \frac{1}{2}\log(1+P).$$

We denote the interference seen by receiver j as I_j and is given by

$$I_j = \sum_{i=1, i \neq j}^{K} aX_i. \tag{6.26}$$

Next, we describe the decoding strategy at the receivers.

Decoding Strategy: For $j \in \{1, 2, \ldots, K\}$, receiver j first decodes its total interference I_j and then decodes its message m_j. In decoding the interference I_j, receiver j treats its own codeword X_j as noise. Hence, the total noise power seen by receiver j when decoding interference I_j is upper bounded by $P + 1$. It is important to note here that, due to the symmetric nature of the channel, the interference I_j at receiver j is an element of lattice $a\Lambda$. We describe the decoding strategy for receiver j. The analysis is similar for other receivers and the details are omitted here. We first describe the choice of lattice Λ and the shift s. The lattice Λ is chosen such that:

- Condition (6.8) (Minkowski-Hlawka condition) is satisfied.
- The volume of the Voronoi region $V = 2^{-nR'} V_{\mathcal{S}}$.
- In decoding the interference, the probability of error is upper bounded by (6.11) with $\sigma^2 = 1 + P$.

We choose a shift s such that the codebook $|\mathcal{C}| \geq 2^{nR}$. The existence of such a shift is guaranteed by [88] for large n.

Decoding Strategy for Receiver j: Receiver j first cancels the sum of the interference caused by other transmitters and then decodes the message intended for it. The received output Y_j is given by

$$Y_j = X_j + a \sum_{i=1, i \neq j}^{K} X_i + Z_j.$$

As each transmitter uses the same lattice Λ, the interference caused by the interfering transmitters at receiver j is aligned and is an element of $a\Lambda$. Here, we use the fact that the receiver knows the shift s used by the interfering transmitter and cancels them out. We use the Loeliger framework in [86] in decoding the total interference. The volume of the Voronoi region of the interference lattice is given by $a^n V$. The total noise seen in decoding the interference is given by

$$I_j = X_j + Z_j.$$

The noise power is limited in power by $1 + P$ and the noise is independent of the interference I_j. With the choice of our lattice, the probability of decoding error denoted by $P_{e,I}$ is upper bounded by

$$P_{e,I} < 4(1+\delta) \frac{2^{n \frac{1}{2} \log(2\pi e(1+P))}}{a^n V} \tag{6.27}$$

Hence, the probability of error decays if

$$\frac{1}{2} \log \left(\frac{2\pi e(1+P)}{a^2} \right) - \frac{1}{n} \log V < 0. \tag{6.28}$$

Lemma 6.3.2 guarantees the choice of lattice Λ such that (6.27) is satisfied. After decoding the total interference I_j, receiver j decodes its message from the resulting point to point AWGN channel. In decoding its own message, receiver j uses the nearest neighbor decoding approach as described in [88]. As the lattice Λ

satisfies (6.8), we can use the Urbanke - Rimoldi approach to decode the intended message at the receiver.

Then, from [88], it follows that the average probability of decoding error decays with n. Hence, receiver j can decode its message successfully if

$$R' < \frac{1}{2}\log(1+P) \tag{6.29}$$

Also by choosing sufficiently large n, the condition for decoding the interference with decaying probability of error as given in (6.28) reduces to

$$R' < \frac{1}{2}\log\left(\frac{a^2 P}{1+P}\right). \tag{6.30}$$

The very strong interference condition comes when the rate constraints imposed by decoding the interference is less binding than the constraint imposed by decoding their respective messages at the receivers. Hence, the very strong interference condition is given when the constraint on R' due to (6.30) is less binding than that due to (6.29), or when

$$a^2 \geq \frac{(P+1)^2}{P}. \tag{6.31}$$

By choosing R' and P' appropriately, we can show that user j can achieve a rate arbitrarily close to $\frac{1}{2}\log(1+P)$ under very strong interference condition. The decoding strategy for other receivers is identical, and lead to identical constraints on rates. Hence, each user can achieve a rate arbitrarily close to $\frac{1}{2}\log(1+P)$ when the interference is very strong. This completes the proof of Theorem 6.4.3.

In Theorem 6.4.3, we derived a "very strong" interference regime for a K user symmetric Gaussian interference channel. The "very strong" interference condition we derived is weaker than the "very strong" interference condition for the two user symmetric Gaussian interference channel. In Theorem 6.4.4, we show that we can have the same "very strong" interference condition for K user symmetric Gaussian interference channel by compromising on the rate achieved by each user. The proof of Theorem 6.4.4 is very similar to the proof of Theorem 6.4.3. Hence, we just present the main steps of the proof here.

Proof of Theorem 6.4.4: We show that if the cross channel gain a satisfies

$$a^2 \geq P + 1,$$

then each user can achieve a rate given by

$$R_i \leq \frac{1}{2} \log(P),\ i \in \{1, \ldots, K\}.$$

The encoding strategy is similar to the one we described in Theorem 6.4.3. Each transmitter encodes its message using lattice coding by choosing the same lattice Λ. The shaping region used is an n dimensional sphere of radius $\sqrt{nP}$. The codebook used by each transmitter is of the form $\mathcal{C} = (\Lambda \cap \mathcal{S}) + s$ where s is the shift used.

The decoding strategy used is also similar to the one used in Theorem thm : very strong interference symmetric channel. Each receiver first decodes the total interference seen treating its own signal as noise. After canceling all the interference, the receiver decodes its own message. The only difference is that, while in Theorem 6.4.3, we used the Loeliger framework for decoding the interference and the Urbanke-Rimoldi framework for decoding the message, in Theorem 6.4.4, we use the Loeliger framework for decoding the interference and the message. We first describe the choice of lattice Λ and the shift s. The lattice Λ is chosen such that:

- Condition (6.8) (Minkowski-Hlawka condition) is satisfied.
- The volume of the Voronoi region $V = 2^{-nR}V_{\mathcal{S}}$.
- In decoding the interference, the probability of error is upper bounded by (6.11) with $\sigma^2 = 1 + P$.

We choose a shift s such that the codebook $|\mathcal{C}| \geq 2^{nR}$. The existence of such a shift is guaranteed by [86] for large n.

We describe the decoding strategy at receiver j and the associated rate constraints involved. The strategy for other receivers and the rate constraints involved are identical. Receiver j first cancels the total interference caused by other transmitters and then decodes the message intended for it. As the receiver uses the same Loeliger strategy for decoding the interference, the constraints involved are the same as in Theorem 6.4.3. Hence, receiver j can decode the total interference if

$$R \leq \frac{1}{2}\log\left(\frac{a^2P}{1+P}\right).$$

After decoding the total interference caused by other transmitters, receiver j decodes the message intended for it. In this theorem, we use the Loeliger strategy at receiver j for decoding message m_j. From [86], we can show that receiver j can decode message m_j with vanishingly small probability of error if

$$R \leq \frac{1}{2}\log(P).$$

The very strong interference condition comes when the rate constraints imposed by decoding the interference is less binding than the constraint imposed by decoding their respective messages at the receivers. Hence if the channel gain a satisfies

$$a^2 \geq 1 + P,$$

then each user can achieve a rate given by

$$R \leq \frac{1}{2}\log(P).$$

This completes the proof of Theorem 6.4.4. We showed that we can achieve the same "very strong" interference condition for the K user symmetric Gaussian interference channel as for the two user symmetric Gaussian interference channel if we allow for a $\frac{1}{2}$ bit per channel use rate penalty for each user.

Next, we prove Theorems 6.4.5 which is generalization of Theorem 6.4.4 for a class of three user non symmetric Gaussian interference channels. Note that equivalence generalization for Theorem 6.4.3 can be stated in a similar fashion.

Main Steps in Proof of Theorem 6.4.5: We show that if the channel gains satisfy (6.22), then each user can achieve the stated rate. Let Λ_1 be a lattice obtained from Construction A, that the volume of its Voronoi region is equal to N_1. Define $\Lambda_2 = \frac{h_{31}}{h_{32}}\Lambda_1$ and $\Lambda_3 = \frac{h_{21}}{qh_{23}}\Lambda_1$. Note that this assignment and conditions given by Equations 6.22 enforce the volume of the Voronoi regions of Λ_2 and Λ_3 to be N_2 and N_3, respectively.

For $j \in \{1, 2, 3\}$, transmitter j encodes its message by lattice coding using lattice Λ_j. The shaping region used by transmitter j is an n dimensional spherical region of radius $\sqrt{nP_j}$. The codebook used by transmitter j is of the form $\mathcal{C}_j = (\Lambda_j \cap \mathcal{S}_j) + s_j$, where s_j is the shift used by transmitter j.

The decoding strategy is following: each receiver first decodes the total interference it sees from all the interfering transmitters and then decodes its own message. Note that here, similar to that in Theorem 6.4.4, we use the Loeliger framework for decoding both the interference and the relevant message. We describe the rate constraints involved in the decoding process at receiver 1.

Rate Constraints at Receiver 1: The interference seen by receiver 1 is given by $h_{12}\Lambda_2 + h_{13}\Lambda_3$. From the choice of lattices, we can see that the interference is an element of the lattice $h_{13}\Lambda_3$. Hence the interference can be decoded successfully if

$$R_2 \leq \frac{1}{2}\log\left(\frac{h_{12}^2 P_2}{p^2(P_1 + \sigma_1^2)}\right) \tag{6.32}$$

$$R_3 \leq \frac{1}{2}\log\left(\frac{h_{13}^2 P_3}{P_1 + \sigma_1^2}\right). \tag{6.33}$$

One can check that the above inequalities hold using the fact that $N_i > \sigma_i^2$.

After decoding the interference, receiver 1 decodes its message m_1 using the Loeliger framework from the remnant point to point AWGN channel. The message m_1 can be decoded successfully if

$$R_1 \leq \frac{1}{2}\log\left(\frac{P_1}{N_1}\right). \tag{6.34}$$

The rate constraints involved at receivers 2 and 3 can be similarly derived. From the rate constraints, we can see that if (6.22) is satisfied, then each user can achieve a rate within half a bit per channel use of its maximum possible individual capacity. Similarly, we can prove Theorem 4 when (6.23) or (6.24) is satisfied.

6.5 Layered Lattice Coding for K User Symmetric Gaussian Interference Channels

In this section, we use the "very strong" interference result we derived in Section IV to derive a layered lattice coding approach to K user symmetric Gaussian interference channels. We show that the layered lattice coding scheme can achieve more than one degree of freedom for a large range of channel parameters. We also show that significant rate improvements can be obtained using the layered lattice coding scheme over the extension of the Han-Kobayashi coding scheme to K user interference channels. The main results of this section are described in the next theorem.

Theorem 6.5.1. *[100, 101] Consider a K user symmetric Gaussian interference channel with channel parameter a and noise variance 1 at each receiver. The total degrees of freedom of the channel satisfies*

$$\mathcal{D}_{sum} \geq \begin{cases} \max\left(1, K \times \frac{\log(a^2-1)}{\log((K-1)a^4-(K-2)a^2)}\right), & a^2 \geq 2 \\ 1, & \frac{1}{K} \leq a^2 \leq 2 \\ \max\left(1, 3 \times \frac{\log\left(\frac{1-a^2}{(K-1)a^2}\right)}{\log\left(\frac{1+(K-2)a^2}{(K-1)a^4}\right)}\right), & a^2 \leq \frac{1}{K} \end{cases} \tag{6.35}$$

Proof. : The proof of the Theorem for $\frac{1}{3} \leq a^2 \leq 2$ is obvious, because a simple time sharing scheme achieves one degree of freedom for any a. Hence, we focus on the other two cases.

First, we consider the case $a^2 \geq 2$. As the channel is symmetric and we are analyzing the total degrees of freedom, we look at only symmetric rate points. For $j \in \{1, 2, \ldots, K\}$, transmitter j communicates message $m_j \in \{1, \ldots, 2^{nR}\}$

to receiver j. Transmitter j splits its message m_j into N parts $m_{j1}, \ldots, m_{jN}$, such that a rate R_i is associated with the i^{th} sub-message of each message. For $i \in \{1, \ldots, N\}$, the i^{th} sub-message is encoded to codeword X_{ji}^n by the j^{th} transmitter, which transmits $X_j^n = \sum_{i=1}^N X_{ji}^n$. Also, each transmitter assigns a power P_i for encoding its i^{th} sub-message. Note that the subscript in rate and power does not indicate user, but the sub-messages. The power P_i is chosen as

$$P_i = (a^2 - 1)((K-1)a^4 - (K-2)a^2)^{N-i}, \;\; i \in \{1, 2, \ldots, N\}. \tag{6.36}$$

We explain the encoding and decoding strategy below in detail.

Encoding Strategy: Each transmitter encodes all its sub-messages using lattice coding, and chooses lattices $\Lambda_1, \ldots, \Lambda_N$, shifts $s_1, \ldots, s_N$ and spherical shaping regions $S_1, \ldots, S_N$. The codebook for i^{th} sub-message at each transmitters is denoted by $\mathcal{C}_i = (\Lambda_i + s_1) \cap S_i$.

Decoding Strategy: The received signal at receiver j is

$$Y_j^n = \sum_{i=1}^N X_{ji}^n + \sum_{k=1, k\neq j}^K \sum_{i=1}^N aX_{ki}^n + Z_j^n.$$

We denote the interference at receiver j due to the i^{th} sub-message from the other transmitters by I_{ji}^n given by

$$I_{ji}^n = \sum_{k=1, k\neq j}^K aX_{ki}^n. \tag{6.37}$$

The decoding process at receiver j proceeds through N stages. At stage i, receiver j first decodes interference I_{ji}^n and then decodes message m_{ji}. In decoding the interference I_{ji}^n, receiver j sees interference plus noise of

$$\sum_{k=i}^{N} X_{jk}^n + \sum_{l=1, l\neq j}^{K} \sum_{k=i+1}^{N} aX_{lk}^n + Z_j^n$$

with an interference plus noise power $\leq P_i + \sum_{k=i+1}^{N}((K-1)a^2+1)P_k + 1$. In decoding message m_{ji}, receiver j sees an interference plus noise

$$\sum_{k=i+1}^{N} X_{jk}^n + \sum_{l=1, l\neq j}^{3} \sum_{k=i+1}^{N} aX_{lk}^n + Z_j^n$$

with an interference plus noise power $\leq \sum_{k=i+1}^{N}((K-1)a^2+1)P_k + 1$. Next, we describe the choice of lattices, shifts and spherical regions, before proceeding to probability of error analysis and rate constraints at the receivers.

Choice of Lattices, Shifts and Shaping Regions: For $i \in \{1, \ldots, N\}$, each transmitter chooses shaping region S_i to be an n dimensional sphere of radius $\sqrt{nP_i}$. The volume of the shaping region S_i is denoted by V_{S_i}. Lattice Λ_i is generated using construction A such that

- the volume of the Voronoi region $V_i = 2^{-nR_i} V_{S_i}$,
- in decoding interference I_{ji}^n at receiver j, the probability of error is upper bounded by (6.11) with $\sigma^2 = P_i + \sum_{k=i+1}^{N}((K-1)a^2+1)P_k + 1$, and
- in decoding message m_{ji} at receiver j, the probability of error is upper bounded by (6.11) with $\sigma^2 = \sum_{k=i+1}^{N}((K-1)a^2+1)P_k + 1$.

Finally, shift s_i is chosen such that the cardinality of the codebook $\mathcal{C}_i$ satisfies $|\mathcal{C}_i| = |(\Lambda_i + s_i) \cap S_i| \geq 2^{nR_i}$. Next, we describe the probability of error analysis and rate constraints at receiver 1. The analysis and the rate constraints at other receivers are the same.

Receiver 1 first decodes interference I_{11}^n and message m_{11}. The interference plus noise power when decoding I_{11}^n is given by $P_1 + \sum_{k=2}^N ((K-1)a^2+1)P_k + 1$. With the choice of lattice Λ_1, the probability of decoding error is upper bounded by

$$P_{e1}^{int} \leq 4(1+\delta) \frac{2^{n\frac{1}{2}\log(2\pi e(P_1 + \sum_{k=2}^N ((K-1)a^2+1)P_k+1))}}{a^n V_1}, \tag{6.38}$$

where $a^n V_1$ is the volume of the Voronoi region of the lattice $a\Lambda_1$ (the interference lattice of message m_{21} and m_{31}). Hence, the probability of error decays with n if

$$R_1 \leq \frac{1}{2} \log \left(\frac{a^2 P_1}{P_1 + \sum_{k=2}^N ((K-1)a^2+1)P_k + 1} \right). \tag{6.39}$$

Similarly, in decoding the message m_{11}, the interference plus noise power seen by receiver 1 is equal to $\sum_{k=2}^N ((K-1)a^2+1)P_k + 1$. The probability of decoding error is upper bounded by

$$P_{e1}^{message} \leq 4(1+\delta) \frac{2^{n\frac{1}{2}\log(2\pi e(\sum_{k=2}^N ((K-1)a^2+1)P_k+1))}}{V_1}. \tag{6.40}$$

Hence, the probability of error decays with n if

$$R_1 \leq \frac{1}{2} \log \left(\frac{P_1}{\sum_{k=2}^N ((K-1)a^2+1)P_k + 1} \right). \tag{6.41}$$

Proceeding along similar lines, at stage i, interference I_{1i}^n and message m_{1i} can be decoded successfully if

$$R_i \leq \frac{1}{2} \log \left(\frac{a^2 P_i}{P_i + \sum_{k=i+1}^N ((K-1)a^2+1)P_k + 1} \right), \tag{6.42}$$

$$R_i \leq \frac{1}{2} \log \left(\frac{P_i}{\sum_{k=i+1}^N ((K-1)a^2+1)P_k + 1} \right). \tag{6.43}$$

The power values have been chosen so that the "very strong" interference condition is satisfied at each stage. The noise plus interference power seen at stage i in decoding interference I_{1i}^n and message m_{1i} is equal to $\sum_{k=i+1}^{N}((K-1)a^2+1)P_k+1$. From the power assignments in (6.36), we can see that

$$a^2 = \frac{P_i}{\sum_{k=i+1}^{N}(2a^2+1)P_k+1} + 1.$$

With the choice of power values as in (6.36), the rate at each stage is given by

$$R_i = \frac{1}{2}\log(a^2-1). \tag{6.44}$$

For R_i to be positive, we need $a^2 \geq 2$. Hence, the total rate achieved by each user is given by

$$R = \frac{1}{2}\log(a^2-1)^N. \tag{6.45}$$

Also, the total power used by each transmitter is given by

$$\begin{aligned} P &= P_1 + \ldots + P_N \\ &\leq ((K-1)a^4 - (K-2)a^2)^N. \end{aligned} \tag{6.46}$$

Taking N to ∞, we get the desired result. That is,

$$\lim_{N,P\to\infty} \frac{KR}{\frac{1}{2}\log(P)} \geq K \times \frac{\log(a^2-1)}{\log((K-1)a^4-(K-2)a^2)}. \tag{6.47}$$

Next, we consider the case $a^2 \leq \frac{1}{3}$. The proof for this case is very similar to that of $a^2 \geq 2$ with very few modifications. We again focus only on symmetric rates. For $j \in \{1,2,3\}$, transmitter j splits its message $m \in \{1,\ldots,2^{nR}\}$ into N sub-parts

$m_{j1}, \ldots, m_{jN}$ such that rate R_i and power P_i is associated with the i^{th} sub-message. Power P_i is chosen as

$$P_i = \frac{1-a^2}{(K-1)a^4} \left(\frac{1+(K-2)a^2}{(K-1)a^4} \right)^{N-i}. \tag{6.48}$$

The encoding strategy is similar to the one described for the case $a^2 \geq 2$ in that each transmitter uses lattice coding to encode all its sub-messages. However, the decoding strategy is slightly different. The decoding process again proceeds through N stages. In stage i, receiver j first decodes message m_{ji} and then decodes interference I_{ji}^n. This is because decoding interference first leads to rate constraints that are more binding than the constraints due to decoding the message.

Choice of Lattices, Shifts and Shaping Regions: For $i \in \{1, \ldots, N\}$, each transmitter chooses shaping region S_i to be a n dimensional sphere of radius $\sqrt{nP_i}$. Lattice Λ_i is generated using construction A such that

- the volume of the Voronoi region $V_i = 2^{-nR_i} V_{S_i}$,
- in decoding interference I_{ji}^n at receiver j, the probability of error is upper bounded by (6.11) with $\sigma^2 = \sum_{k=i+1}^{N}((K-1)a^2+1)P_k + 1$, and
- in decoding message m_{ji} at receiver j, the probability of error is upper bounded by (6.11) with $\sigma^2 = (K-1)a^2 P_i + \sum_{k=i+1}^{N}((K-1)a^2+1)P_k + 1$.

Finally, shift s_i is chosen such that the cardinality of the codebook $\mathcal{C}_i$ satisfies $|\mathcal{C}_i| = |(\Lambda_i + s_i) \cap S_i| \geq 2^{nR_i}$. The details of the probability of error analysis are similar to the case $a^2 \geq 2$ and are omitted here. Using the power assignments in (6.48), we see that each user achieves a rate R_i for its i^{th} sub message given by

$$R_i = \frac{1}{2}\log\left(\frac{1-a^2}{(K-1)a^2}\right). \tag{6.49}$$

For R_i to be positive, we must have $a^2 \leq \frac{1}{K}$. Hence, each user achieves a total rate R given by

$$R = \frac{1}{2}\log\left(\frac{1-a^2}{(K-1)a^2}\right)^N. \tag{6.50}$$

The total power expended by each transmitter is given by

$$\begin{aligned} P &= P_1 + \ldots + P_N \\ &\leq \left(\frac{1+(K-2)a^2}{(K-1)a^4}\right)^N. \end{aligned} \tag{6.51}$$

Taking N to ∞, we get the desired result. This completes the proof of Theorem 6.5.1. □

Remark 6.5.1. From (6.35), we can see that the achievable total degrees of freedom tends to $K/2$ as the channel gain $a \to \infty$, and when $a \to 0$. We should also see that when the channel gain $a = 0$, then we can actually achieve K degrees of freedom.

In Figure 6.7, we plot the degrees of freedom that we achieve for a symmetric three user Gaussian IC using the layered lattice coding approach.

In this section, we proposed a layered lattice coding scheme for the symmetric K user Gaussian interference channel and we analyzed the degrees of freedom that we can achieve using this approach. We showed that we can achieve more than one degree of freedom for a large range of channel parameters and showed that the

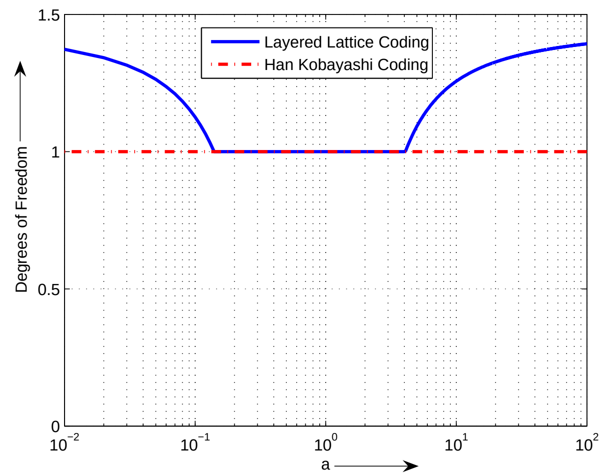


Figure 6.7: Plot of Achievable Degrees of Freedom versus a^2

total degrees of freedom achievable tends to $K/2$ when the cross channel gain tends to ∞.

6.6 Layered Lattice Coding Scheme for Non-Symmetric Interference Channels

In this section, we briefly analyze the degrees of freedom of three user non-symmetric Gaussian ICs. We use the same layered lattice coding scheme that we used for the symmetric case. To present the main ideas and for analytical tractability, we restrict ourselves to the following class of three user Gaussian IC with channel matrix given by

$$H = \begin{pmatrix} 1 & a_1 & a_1 \\ a_2 & 1 & a_2 \\ a_3 & a_3 & 1 \end{pmatrix}, \qquad (6.52)$$

where $a_1^2, a_2^2, a_3^2 \geq 2$. Without loss of generality we assume $a_1 \leq a_2 \leq a_3$. The analysis for other channel matrices in $H \in \mathcal{H}_1$ are similar to the one presented and is omitted here. We describe the encoding and decoding strategy below:

For $j \in \{1, 2, 3\}$, transmitter j communicates message $m_j \in \{1, \ldots, 2^{nR_j}\}$ to receiver j. Transmitter j splits its message into N parts - $m_{j_1}, m_{j_2}, \ldots, m_{jN}$ such that rate R_{ji} is associated with the i^{th} sub-message. For $i \in \{1, \ldots, N\}$, transmitter j encodes message m_{ji} into codeword X_{ji}^n and transmits $X_j^n = \sum_{i=1}^N X_{ji}^N$. Also, transmitter j assigns power P_{ji} to encode its i^{th} sub-message.

Encoding Strategy: Each transmitter encodes all its sub-messages using lattice coding, and chooses lattices $\Lambda_1, \ldots, \Lambda_N$. Transmitter j chooses shifts $s_{j1}, \ldots, s_{jN}$ and spherical shaping regions $S_{j1}, \ldots, S_{jN}$. The codebook for the i^{th} sub-message at transmitter j is denoted by $\mathcal{C}_{ji}$ and is given by $\mathcal{C}_{ji} = (\Lambda_i + s_{ji}) \cap S_{ji}$.

Decoding Strategy: The received signal at receiver j is given by

$$Y_j^n = \sum_{i=1}^{N} X_{ji}^n + \sum_{l=1, l \neq j}^{3} \sum_{i=1}^{N} a_j X_{li}^n + Z_j^n.$$

We denote the interference at receiver j due to the i^{th} sub-message from the other transmitter by I_{ji}^n and is given by

$$I_{ji}^n = \sum_{l=1, l \neq j}^{3} a_j X_{li}^n.$$

The decoding process at receiver j proceeds through N stages. At stage i, receiver j first decodes interference I_{ji}^n and then decodes its sub-message m_{ji}. In decoding interference I_{ji}^n, receiver j sees an effective noise power of

$$\sigma_{ji}^2 = 1 + P_{ji} + \sum_{l=i+1}^{N} P_{jl} + \sum_{l=1,l\neq j}^{3} \sum_{k=i+1}^{N} a_j^2 P_{lk}. \tag{6.53}$$

In decoding message m_{ji}, receiver j sees an interference plus noise power of

$$\sigma_{m_{ji}}^2 = \sum_{l=i+1}^{N} P_{jl} + \sum_{l=1,l\neq j}^{3} \sum_{k=i}^{N} a_j^2 P_{lk}. \tag{6.54}$$

The choice of lattices Λ_i, shifts s_{ji} and shaping regions S_{ji} are similar to those described in Theorem 6.4.3 and the details are omitted here. We choose the powers P_{ji} such that the "very strong" interference condition is satisfied at every decoding stage. That is, at stage i, the rate constraints on R_{ji} due to decoding interference I_{li}^n at receiver l is less binding than the constraint imposed due to decoding message m_{ji} at receiver j. Hence, we choose powers such that

$$\begin{aligned} P_{1i} &= \min(a_1^2\sigma_{m_{2i}}^2 - \sigma_{m_{1i}}^2, a_1^2\sigma_{m_{3i}}^2 - \sigma_{m_{1i}}^2) \\ P_{2i} &= \min(a_2^2\sigma_{m_{1i}}^2 - \sigma_{m_{2i}}^2, a_2^2\sigma_{m_{3i}}^2 - \sigma_{m_{2i}}^2) \\ P_{3i} &= \min(a_3^2\sigma_{m_{1i}}^2 - \sigma_{m_{3i}}^2, a_3^2\sigma_{m_{2i}}^2 - \sigma_{m_{3i}}^2). \end{aligned} \tag{6.55}$$

The rate achieved by user j at stage i is given by

$$R_{ji} = \frac{1}{2}\log\left(\frac{P_{ji}}{\sigma_{m_{ji}}^2}\right). \tag{6.56}$$

The total power used by transmitter j is given by

$$P_j = P_{j1} + P_{j2} + \ldots P_{jN}.$$

The total degrees of freedom then satisfies

$$\mathcal{D}_{sum} \geq \limsup_{P_1+P_2+P_3\to\infty} \frac{\sum_{j=1}^{3}\sum_{i=1}^{N} R_{ji}}{\frac{1}{2}\log(P_1+P_2+P_3)}.$$

However, unlike the symmetric channel case in Theorem 6.4.3, we have not been able to derive closed form expressions for the total degrees of freedom achievable for non-symmetric channels. We illustrate the total degrees of freedom achieved for an example channel (derived numerically) in Figure 6.8. The degree of freedom

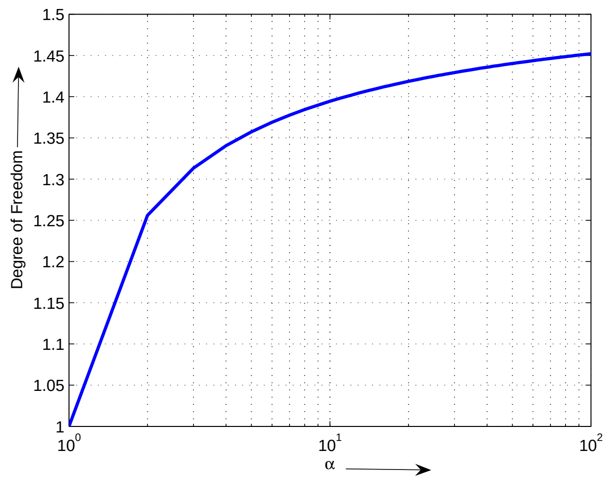


Figure 6.8: Achievable Degrees of Freedom for an example channel: $a_1 = 2\alpha, a_2 = 3\alpha, a_3 = 4\alpha$

analysis for other non symmetric three user Gaussian interference channels with channel matrix $H \in \mathcal{H}_1$ follows along the same lines as the analysis for the channel given by (6.52).

6.7 Comparing Lattice Coding with an extension of Han Kobayashi with i.i.d Gaussian Coding

In this section, we compare our layered lattice coding approach with a coding scheme that resembles of Han-Kobayashi scheme, extended to the case of a three user symmetric Gaussian IC. In the Han-Kobayashi coding scheme for the two user IC [4], each transmitter splits its message into two parts, a private part and a common part. For decoding, each receiver decodes its message and the common message transmitted by the interfering transmitter. In our extension of this scheme to the three user IC, each transmitter splits its message into four parts - one private part and three common parts. For instance, transmitter 1 splits its message m_1 into four parts - 1) m_{11}, the private part, 2) m_{12}, the common part which is also decoded by receiver 2, 3) m_{13}, the common part which is also decoded by receiver 3 and 4) m_{123}, the common part which is decoded by receivers 2 and 3. In this extended version of the Han-Kobayashi scheme, we restrict ourselves to Gaussian codebooks. Finally, as the channel is symmetric, we restrict our comparison to the maximum symmetric rate that can be achieved using the two approaches[1]. In the next Lemma, we derive a symmetric rate point that can be achieved using the layered lattice coding approach for the three user symmetric Gaussian IC with cross channel gain a and power constraint P. We define P_a^N as follows

$$P_a^N \triangleq \begin{cases} (a^2-1)\frac{(2a^4-a^2)^N-1}{2a^4-a^2-1}, \text{ if } a^2 \geq 2 \\ \frac{1-a^2}{2a^4}\frac{\left(\frac{1+a^2}{2a^4}\right)^{N_2}-1}{\left(\frac{1+a^2}{2a^4}\right)-1}, \text{ if } a^2 \leq \frac{1}{3} \end{cases} . \tag{6.57}$$

Let $\mathcal{R}_{HK}(P, \sigma^2, a)$ denote the maximum symmetric rate that can be obtained by Han-Kobayashi coding scheme in a three user symmetric Gaussian IC with power constraint P, noise at the receiver σ^2 and the cross channel gain equal to a.

[1]Note that, for asymmetric points on this channel's achievable region, our comparison does not hold and the extended Han-Kobayashi style coding may be better in performance.

Lemma 6.7.1. *Consider a symmetric three user Gaussian IC with cross channel gain a and power constraint P at each transmitter. Then*

a) if $a^2 \geq 2$ and $P \leq a^2 - 1$, each user can achieve a symmetric rate given by

$$R_{sym} = \frac{1}{2}\log(P). \tag{6.58}$$

b) If $a^2 \geq 2$, and there exists integer $N_1 > 0$ such that

$$P_a^{N_1} < P < P_a^{N_1+1},$$

then each user can achieve a symmetric rate given by

$$R_{sym} = \frac{N_1}{2}\log(a^2 - 1) + \frac{1}{2}\log\left(1 + \frac{(2a^2+1)(P - P_a^{N_1})}{1 + (2a^2+1)P_a^{N_1}}\right). \tag{6.59}$$

c) If $a^2 \geq 2$, and there exists integer $N_1 > 0$ such that $P_a^{N_1} = P$, then each user can achieve a symmetric rate given by

$$R_{sym} = \frac{N_1}{2}\log(a^2 - 1). \tag{6.60}$$

d) If $a^2 \leq \frac{1}{3}$ and $P \leq \frac{1-a^2}{2a^4}$, each user can achieve a symmetric rate of

$$R_{sym} = \mathcal{R}_{HK}(P, a, 1). \tag{6.61}$$

e) If $a^2 \leq \frac{1}{3}$ and there exists integer $N_2 > 0$ such that

$$P_a^{N_2} < P < P_a^{N_2+1}$$

then each user can achieve a symmetric rate of

$$\begin{aligned} R_{sym} = \max\nolimits_{i=N_2-1:N_2} \tfrac{i}{2}\log\left(\tfrac{1-a^2}{2a^2}\right) + \\ \mathcal{R}_{hk}(P - P_a^i, (2a^2+1)P_a^i, a) \end{aligned} \tag{6.62}$$

f) If $a^2 \leq \frac{1}{3}$ and there exists integer $N_2 > 0$ such that $P = P_a^{N_2}$, then each user can achieve a symmetric rate given by

$$R_{sym} = \frac{N_2}{2} \log \left(\frac{1 - a^2}{2a^2} \right). \tag{6.63}$$

The proof of the above Lemma is very similar to the proof of Theorem 6.5.1 and is therefore omitted. It should be noted that for cases (b) and (e) in the lemma, we use Han-Kobayashi style encoding and decoding for the first layer of the codebook. This is because, the power allocated to this level is not sufficient enough to benefit from lattice coding. Figure 6.9 compares the symmetric rate point achievable using the layered lattice coding approach with the maximum symmetric rate that can be achieved using Han-Kobayashi scheme for $a = 2.5$ and $a = \frac{1}{3}$. Note that in our layered lattice coding approach, we restrict ourselves to identical power splitting approach by all the transmitters. This can be generalized to different power splitting schemes and can lead to a higher rate achieved by the lattice coding scheme. However, it is interesting to note that even a possibly suboptimal lattice coding scheme significantly outperforms our extended version of the Han-Kobayashi style scheme with i.i.d. Gaussian codebooks.

This shows that while the Han-Kobayashi coding scheme (message splitting and random coding) with Gaussian codebooks is optimal to within one bit for a two user Gaussian IC [13], a natural extension of this scheme optimal even in terms of degrees of freedom for larger ICs with more than two transmitter-receiver pairs. Lattice coding, while allowing the interference to be decoded without decoding the interfering messages places fewer constraints on the rates of the interfering users.

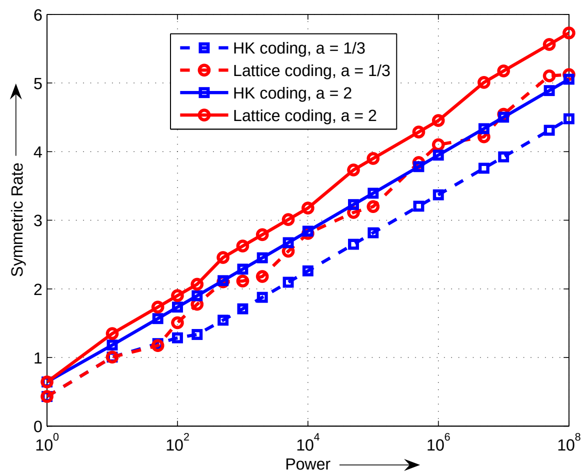


Figure 6.9: Comparing Han-Kobayashi and Layered Lattice Coding for $a = 2.5$

In particular, it eliminates the MAC type constraints that arise when decoding the interfering messages separately.

6.8 Conclusion

In this chapter, we study the impact of using structured codes on a $K > 2$ user interference channel. We find that it benefits both the characterization of the achievable rate, and enables us to characterize the channel's capacity for a class of very strong interference channels. Lattices enable us to align interference signals, and thus allow for achievable rate characterizations for a large class of Gaussian interference channels. Note that extending this work to arbitrary (irrational channel gains) asymmetric Gaussian interference channels may not be straightforward. However, there is recent work on determining the DoF of such channels [75, 98].

Chapter 7

Conclusions and Future Work

This dissertation has focused on analyzing the capacity region of a two broad classes of interference networks - cognitive networks and K user interference channel. The capacity region of interference networks has been an open problem for several decades. In this dissertation, we have taken significant steps in understanding the capacity behavior of several cognitive radio models. We have also analyzed the K user interference channel and devised a lattice based interference alignment scheme to derive significant rate benefits over other traditional transmission strategies. We summarize the main conclusions of the dissertation and discuss possible future work.

7.1 Cognitive Radio Networksl

In Chapter 2, we studied the MIMO cognitive radio channel and derived an achievable region and outer bound on the capacity region. The achievable region is based on lattice coding and is quite similar to the single antenna model. The outer bound was derived through a series of channel transformations and is significantly different than the bounds derived for single antenna case. We also derive possible channel conditions in which the achievable region might meet the outer bound.

In Chapter 3, we extended the cognitive radio channel to multiple access networks. We considered a three transmitter, two receiver systems with two licensed transmitters transmitting to a common licensed receiver and a cognitive transmitter transmitting to a cognitive receiver. We derived outer bounds and achievable region for the discrete memoryless channel. We also showed that for the Gaussian channel model, Gaussian signalling at the transmitters is optimal when the cross channel gain from the cognitive transmitter to the licensed receiver is weak (≤ 1).

In chapter 4, we analyzed the capacity region of cognitive relay networks. In this channel model, we essentially have a two user interference channel with a cognitive relay which has access to the messages of the transmitter. We derive an achievable region based on Han-Kobayashi message splitting and dirty paper coding for both the messages. We also derive an outer bound on the capacity region of such networks. The outer bounds are derived by permitting transmitter and receiver co-operation. We also derived the degree of freedom region of such networks.

In all the above cognitive radio models, it is assumed that the cognitive node has access to the messages transmitted by the other nodes. In chapter 5, we study a more practical model of cognitive radio in which the cognitive transmitter has access to only a portion of the message of the licensed transmitter. We analyze the capacity region of partial cognitive radio channel and derive outer bounds and achievable region for the discrete memoryless and the Gaussian channel model.

7.1.1 Future Work

While a lot of research has been done on cognitive radios in the last decade, there are still a lot of open problems. The capacity region of the cognitive radio channel with single antennas has been well understood. However, under multiple antenna setting, optimal strategies are still unknown. In this dissertation, we extended the cognitive radio to a two user multiple access network. Extensions to larger MAC and other network configurations are still possible. A lot of work remains to be done on the field of cognitive radios with partial cognition. Such a channel model is very practical and and needs to be understood in greater detail.

7.2 K User Interference Channel

In chapter 6, we analyzed the capacity region of a K user interference channel with K transmitter-receiver pairs. use lattice coding as an interference alignment transmission strategy to study the channel. We derived a very strong interference regime for the K user symmetric Gaussian channel and extended it to a class of non-symmetric channels. We used the very strong interference regime to derive a layered lattice coding scheme. We use the layered lattice coding scheme to analyze the degrees of freedom of the channel and also to derive a clever transmission strategy for all power levels. We show that significant rate benefits can be obtained over other traditional transmission strategies.

7.2.1 Future Work

Several prominent researchers are currently working on solving the capacity region of the K user interference channel. Significant advances have been made in this effort over the last two or three years. It has been shown recently that $K/2$ degrees of freedom can be achieved for almost all channel parameters. Future work will revolve around characterizing the capacity region of the K user interference channel to within a finite number of bits. Extending the lattice coding scheme to any general interference network will also be a challenging problem.

Bibliography

[1] C.E. Shannon, "Two-way Communication channels," In *Proc. 4th Berkely Symp. on Mathematical Statistics and Probability*, vol. 1, pp. 611–644, Berkely, CA : Univ. California Press, 1961.

[2] R. Ahlswede, "The capacity region of a channel with two senders and two receivers," *Annals Probability*, vol. 2, no. 7, pp. 805–814, 1974.

[3] A.B. Carleial, "Interference Channels," *IEEE Trans. Inform. Theory*, vol. 24, no. 1, pp. 60–70, Jan. 1978.

[4] T.S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. Inform. Theory*, vol. 27, no. 1, pp. 49–60, Jan. 1981.

[5] V.R. Cadambe and S.A. Jafar, "Interference Alignment and the Degrees of Freedom for the K User Interference Channel," *in Preprint available through http://newport.eecs.uci.edu/ syed/papers/intK.pdf*, July 2007.

[6] V.R.Cadambe abd S.A. Jafar, "Degrees of Freedom for Wireless X Networks," *in Preprint available through http://newport.eecs.uci.edu/ syed/papers/dofxnet.pdf*, 2007.

[7] V.R. Cadambe, S.A. Jafar and S. Shamai (Shitz), "Interference Alignment on the Deterministic Channel and Application to Fully Connected AWGN Inter-

ference Networks," *in Preprint available through http://newport.eecs.uci.edu/ syed/papers/dofconst.pdf*, 2007.

[8] V. R. Cadambe and S. A. Jafar, "Reflections on Interference Alignment and the Degrees of Freedom of the K User Interference Channel," *IEEE Information Theory Society Newsletter*, Vol. 59, No. 4, December 2009, Pages 5-9.

[9] A.B. Carleial, "Outer bounds on the capacity of interference channels," *IEEE Trans. Inform. Theory*, vol. 29, no. 4, pp. 602–606, July 1983.

[10] H. Sato, "On degraded Gaussian two-way channels," *IEEE Trans. Inform. Theory*, vol. 24, no. 4, pp. 638–640, Sept. 1978.

[11] M. Costa, "On the Gaussian interference channel," *IEEE Trans. Inform. Theory*, vol. 31, no. 5, pp. 607–615, Sep 1985.

[12] G. Kramer, "Outer Bounds on the capacity region of Gaussian interference channels," *IEEE Trans. Inform. Theory*, vol. 50, bo. 3, pp. 581–586, March 2004.

[13] R.H. Etkin, D.N.C. Tse and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. Inform. Theory*, vol. 54, no. 12, pp. 5534 5562, Dec. 2008.

[14] X. Shang, G. Kramer and B. Chen, "A new outer bound and noisy-interference sum-rate capacity for the Gaussian interference channels," *IEEE Trans. Inform. Theory*, vol. 55, no. 2, pp. 689-699, Feb. 2009.

[15] V.S. Annapureddy and V. Veeravalli, “Gaussian Interference Networks: Sum Capacity in the Low Interference Regime and New Outer Bounds on the Capacity Region,” *IEEE Trans. Inform. Theory*, July 2009.

[16] A.S. Motahari and A.K. Khandani, “Capacity bounds for the Gaussian interference channel,” *IEEE Trans. Inform. Theory*, Vol. 55, No. 2, pp. 620-643, Feb. 2009.

[17] A.B. Carleial, “A case where interference does not reduce capacity,”*IEEE Trans. Inform. Theory*, vol. 21, no. 5, pp. 569–570, Sept. 1975.

[18] H. Sato, “The capacity of the Gaussian interference channel under strong interference,” *IEEE Trans. Inform. Theory*, vol. 27, no. 6, pp. 786–788, Nov. 1981.

[19] M. Costa and A.A.E. Gamal, “The capacity region of the discrete memoryless interference channel with strong interference,” *IEEE Trans. Inform. Theory*, vol. 33, no. 5, pp. 710–711, Sept. 1987.

[20] R. Benzel, “The capacity region of a class of discrete additive degraded interference channels,” *IEEE Trans. Inform. Theory*, vol. 25, no. 2, pp. 228–231, March 1979.

[21] A.A.E. Gamal and M. Costa, “The capacity region of a class of determinstic interference channels,” *IEEE Trans. Inform. Theory*, vol. 28, no. 2, pp. 343–346, March 1982.

[22] M. Costa, “Writing on dirty paper (corresp.),” *IEEE Trans. Inform. Theory*, vol. 29, no. 3, pp. 439–441, May 1983.

[23] *Federal Communications Commision, Cognitive Radio Technologies Proceeding (CRTP), ET Docket No. 03-108.* http://www.fcc.gov/oet/cognitiveradio/.

[24] J. Mitola, “Cognitive Radio: An Integrated Agent Architecture for Software Defined Radio”, *Ph.D. Dissertation*, KTH, Stockholm, Sweeden, December 2000.

[25] A. Ghasemi and E. Sousa, “Collaborative spectrum sensing for opportunistic access in fading environments,” in *Proc. First IEEE International Symp. on New Frontiers in Dynamic Spectrum Access Networks (DySPAN)*, 8-11 Nov. 2005, pp. 131–136.

[26] S. Haykin, “Cognitive radio: brain-empowered wireless communications,” *IEEE Journal on selected areas in Communications*, vol. 23, no. 2, pp. 201–220, Feb 2005.

[27] S. Srinivasa, S. Jafar and N. Jindal, “On the capacity of the cognitive tracking channel,” in *IEEE International Symposium on Information Theory*, July 2006, pp. 2077–2081.

[28] S. A. Jafar and S. Srinivasa, “Capacity limits of cognitive radio with distributed and dynamic spectral activity,” *IEEE Journal on selected Areas in Communications*, vol. 25, no. 3, pp. 529–537, April 2007.

[29] S. Srinivasa and S. A. Jafar, "The throughput potential of cognitive radio: A theoretical perspective," *IEEE Communications Magazine*, vol. 45, no. 5, 2007.

[30] M. Gastpar, "On capacity under receive and spatial spectrum-sharing constraints," *IEEE Trans. Inform. Theory*, vol. 53, pp. 471-487, Feb. 2007.

[31] H. Suraweera and J. Gao and P. Smith and M. Shafi and M. Faulkner, "Channel capacity limits of cognitive radio in asymmetric fading environments," in *Proc. IEEE Int. Conf. Comm.*, May 2008.

[32] W. Zhang and U. Mitra, "A spectrum sharing perspective on cognitive radio: Uncoded primary transmission case," in *Proc. IEEE ISIT*, July 2008.

[33] P. Popovski, H. Yomo, K. Nishimori, R. Di Taranto and R. Prasad, "Opportunistic interference cancelation in cognitive radio systems," in *Proc. IEEE Int. Symp. New Frontiers Dyn. Spectrum Access Networks*, April 2007.

[34] W. Wang, T. Peng and W. Wang, "Optimal power control under interference temperature constraints in cognitive radio network," in *Proc. IEEE Wireless Comm. Netw. Conf.*, March 2007.

[35] Y. Xing and R. Chandramouli, "QoS constrained secondary spectrum sharing," in *Proc. IEEE Int. Symp. New Frontiers Dyn Spectrum Access Networks*, Nov. 2005

[36] J. O. Neel, J. H. Reed and R. P. Giles, "Convergence of cognitive radio networks,"in *Proc. IEEE Wireless Comm. Netw. Conf.*, 2004.

[37] D. Niyato and E. Hossain, “A game theoretic approach to competetive spectrum sharing in cognitive radio networks,” in *Proc. IEEE Wireless Comm. Netw. Conf.*, March 2007.

[38] X. Wang and Q. Zhu, “Power control for cognitive radio based on game theory,” in *Proc. Int. Conf. Wireless Comm. Netw. Mobile Comput.*”, Sep. 2007.

[39] N. Devroye, P. Mitran and V. Tarokh, “Achievable rates in cognitive radio channels,” *IEEE Trans. Inform. Theory*, vol. 52, no. 5, pp. 1813–1827, May 2006

[40] I. Maric, R. Yates and G. Kramer, “On the Capacity of Interference Channels with One Cooperating Transmitter,” *European Trans. on Telecomm.*, 2008.

[41] A. Jovicic and P. Viswanath, “Cognitive radio: An information-theoretic perspective,” to appear in *IEEE Trans. Inform. Theory*, Preprint available at http://www.ifp.uiuc.edu/ pramodv/pubs.html.

[42] W. Wu, S. Vishwanath and A. Arapostathis, “Capacity of a class of cognitive radio channels: Interference channels with degraded message sets,” *IEEE Trans. Inform. theory*, vol. 53, no. 11, pp. 4391–4399, Nov. 2007.

[43] S. Rini, D. Tuninetti and N. Devroye, “The Capacity Region of the Gaussian Cognitive Radio Channels at High SNR,’ *Proc. of ITW*, Taormina, October 2009.

[44] S. Rini, D. Tuninetti and N. Devroye, “The Capacity Region of Gaussian Cognitive Radio Channels to within 1.87 bits,” in *Proc. of ITW*, Cairo, Januray 2010.

[45] S. Rini, D. Tuninetti and N. Devroye, “State of the cognitive interference channel: a new unified inner bound, and capacity to within 1.87 bits,” to be presented in *International Zurich Seminar on Communications*, Zurich, March 2010.

[46] H. Weingarten, Y. Steinberg, and S. Shamai (Shitz), “The capacity region of the Gaussian multiple-input multiple-output broadcast channel,” *IEEE Trans. Inform. Theory*, vol. 52, no. 9, pp. 3936–3964, Sept. 2006.

[47] M. Mohseni and J. M. Cioffi, “A proof of the converse for the capacity of Gaussian MIMO broadcast channels,” in *Proc. IEEE Intl. Symp. on Inform. Theory, Seattle, 2006*, July 2006, pp. 881–885.

[48] H. Weingarten, T. Liu, S. S. (Shitz), Y. Steinberg, and P. Viswanath, “The capacity region of the degraded multiple input multiple output compound broadcast channel,’ *IEEE Trans. Inform. Theory*, vol. 55, no. 11, pp. 5011-5023, November 2009.

[49] O. Sahin and E. Erkip, “On achievable rates for interference relay channel with interference cancelation,” in *Proc. of Forty First Annual Asilomar Conf. on Signals, Systems and Computers*, Pacific Grove, California, Nov. 2007.

[50] O. Sahin and E. Erkip, “Cognitive relaying with one-sided interference,” in *Proc. of Forty Second Annual Asilomar Conference on Signals, Systems and Computers, Pacific Grove, California*, Oct. 2008.

[51] O. Sahin and E. Erkip, “Achievable rates for the Gaussian interference relay channel,” in *Proc. of 2007 GLOBECOM Communication Theory Symposium, Washington D.C.*, Nov. 2007.

[52] O. Sahin, O. Simeone and E. Erkip, “Interference channel aided by an infrastructure relay,” in *Proc. IEEE ISIT*, June 2009.

[53] A. S. Baruch, S. Sridharan, S. Vishwanath, S. Verdu and S. Shamai (Shitz), “On the capacity of Cognitive Radios in Multiple access networks, in *Proc. of Asilomar*, Pacic Grove, CA, Oct. 2008.

[54] A. S. Baruch, S. Sridharan, S. Vishwanath, S. Verdu and S. Shamai (Shitz), “On the capacity of Cognitive Radios in Multiple access networks, to be submitted to *IEEE Trans. Inform. Theory*.

[55] A. Somekh-Baruch, S. Shamai (Shitz), and S. Verdú, “Cognitive interference channels with state information,” in *Proc. of International Symp. on Inform. Theory*, July 2008.

[56] S. Sridharan and S. Vishwanath, “On the Capacity of a Class of MIMO Cognitive Radios,” *Proc. of Information Theory Workshop*, Lake Tahoe, California, Sept. 2007.

[57] S. Sridharan and S. Vishwanath, "On the Capacity of a Class of MIMO Cognitive Radios," in *Journal of Selected Topics in Signal Proc.*, March 2008.

[58] H. Weingarten, Y. Steinberg, and S. Shamai (Shitz), "The capacity region of the Gaussian multiple-input multiple-output broadcast channel," *IEEE Trans. Inform. Theory*, vol. 52, no. 9, pp. 3936–3964, Sept. 2006.

[59] Y. Steinberg and S. Shamai (Shitz), "Achievable rates for the broadcast channel with states known at the transmitter," in *Proc. of Intl. Symp. of Inform. Theory, Sep. 2005*, 2005.

[60] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley Series in Telecommunications, July 2006.

[61] W. Yu and J. Cioffi, "Sum capacity of Gaussian vector broadcast channels," *IEEE Trans. Inform. Theory*, vol. 50, no. 9, pp. 1875–1892, Sept. 2004.

[62] S. Vishwanath, G. Kramer, S. Shamai (Shitz), S. Jafar, and A. Goldsmith, "Capacity bounds for Gaussian vector broadcast channels," in *DIMACS Workshop on Signal Processing for Wireless Transmission, Rutgers, NJ*, October 2002.

[63] R. M. Dudley, *Real analysis and probability*. Cambridge University Press, 2002.

[64] K. Fan, "Minimax theorems," in *Proc. Nat. Acad. Sci, 39*, 1953, pp. 42–47.

[65] S. Sridharan, S. Vishwanath, S.A. Jafar and S. Shamai (Shitz), "On the capacity of cognitive relay assisted Gaussian interference channel," *Proc. of IEEE ISIT*, 2008.

[66] I. Marić, R. D. Yates, and G. Kramer, "Capacity of interference channels with partial transmitter cooperation," *IEEE Trans. Inform. Theory*, vol. 53, no. 10, pp. 3536–3548, Oct. 2007.

[67] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley Series in Telecommunications, July 2006.

[68] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom for the K user interference channel," *IEEE Trans. Inform. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[69] ——, "Degrees of freedom for wireless x networks," *IEEE Trans. Inform. Theory*, Vol. 55, No. 9, Sep. 2009,Pages: 3893-3908.

[70] ——, "Capacity of wireless networks within o(log(snr)) - the impact of relays, feedback, cooperation and full-duplex operation," Feb. 2008, arXiv: 0802.0534, e-print.

[71] M. Shen, A. Host-Madsen, and J. Vidal, "An improved interference alignment scheme for frequency selective channels," in *Proc. of ISIT*, July 2008.

[72] C. Suh and D. N. C. Tse, "Interference alignment for cellular networks," to appear in *Proc. of the 46th Allerton Conference*, Sept. 2008.

[73] G. Bresler, A. Parekh, and D. N. C. Tse, "The approximate capacity of the many-to-one and one-to-many Gaussian interference channels," in *Proc. of 45th Allerton Conference*, Sept. 2007.

[74] S. A. Jafar and S. Vishwanath, “Generalized degrees of freedom of the k user symmetric Gaussian interference channel,” April 2008, arXiv:0804.4489, e-print.

[75] R. Etkin and E. Ordentlich, “On the degrees-of-freedom of the k-user Gaussian interference channel,” *submitted to IEEE Trans. Inform. Theory*, June 2008.

[76] I. Maric, A. Goldsmith, G. Kramer and S. Shamai (Shitz), “On the capacity of interference channel with a partially cognitive transmitter,” *Proc. of IEEE ISIT*, Nice, France, June 2007.

[77] H. Bagheri, A. S. Motahari and A. K. Khandani, “On the sum capacity of symmetric Gaussian interference channel with partial unidirectional transmitter cooperation,” preprint available at *arXiv:0909.2777*

[78] H. Bagheri, A. S. Motahari, and A. K. Khandani, Generalized degrees of freedom of the symmetric Gaussian interference channel with partial unidirectional transmitter cooperation, *Proc. IEEE 43rd Conference of Information Sciences and Systems* Johns Hopkins University, MD, USA, March 2009.

[79] G. Chung, S. Sridharan, S. Vishwanath and C. S. Hwang, “On the capacity of partially cognitive radios,” *proc. of IEEE ISIT*, Seoul, Korea, June 2009.

[80] G. Chung, S. Sridharan, S. Vishwanath and C. S. Hwang, “On the capacity of partially cognitive radios,” submitted to *IEEE Trans. Inform. Theory*.

[81] I. Maric and R. Yates, "The strong interference channel with common information," in *Proc. of Allerton Conference on Communications*, Sept. 2005.

[82] J. A. Thomas, "Feedback can at most double Gaussian multiple access channel capacity," *IEEE Trans. Inform. Theory*, vol. 52, pp. 3936-3964, Sep. 2006.

[83] C. W. V. R. Cadambe, S. A. Jafar, "Interference alignment with asymmetric complex signaling - settling the host-madsen-nosratinia conjecture," *can be found at http://arxiv.org/abs/0904.0274*, 2009.

[84] R. de Buda, "The upper error bound of a new near-optimal code," *IEEE Trans. Inform. Theory*, vol. 21, no. 4, pp. 441–445, July 1975.

[85] G. Poltyrev, "On coding without restrictions for the AWGN channel," *IEEE Trans. Inform. Theory*, vol. 40, no. 2, pp. 409–417, March 1994.

[86] H. A. Loeliger, "Averaging bounds for lattices and linear codes," *IEEE Trans. Inform. theory*, vol. 43, no. 6, pp. 1767–1773, Nov. 1997.

[87] U. Erez and R. Zamir, "Achieving 1/2 log(1 + SNR) on the AWGN channel with lattice encoding and decoding," *IEEE Trans. Inform. Theory*, vol. 50, no. 10, pp. 2293–2314, Oct. 2004.

[88] R. Urbanke and B. Rimoldi, "Lattice codes can achieve capacity on the AWGN channel," *IEEE Trans. Inform. Theory*, vol. 44, no. 1, pp. 273–278, Jan. 1998.

[89] M. P. Wilson, K. Narayanan, H. Pfister, and A. Sprintson, "Joint physical layer coding and network coding for bi-directional relaying," April 2008, arXiv:0805.0012, e-print.

[90] W. Nam, S. Y. Chung, and Y. H. Lee, “Capacity bounds for two-way relay channels,” in *Proc. of IEEE Intl. Zurich Seminar on Commn.*, March 2008, pp. 144–147.

[91] U. Erez, S. Litsyn, and R. Zamir, “Lattices which are good for (almost)everything,” *IEEE Trans. Inform Theory*, vol. 51, no. 10, pp. 3401–3416, Oct. 2005.

[92] R. Zamir, S. Shamai (Shitz), and U. Erez, “Nested linear/lattice codes for structured multiterminal binning,” *IEEE Trans. Inform. Theory*, vol. 48, no. 6, pp. 1250–1276, June 2002.

[93] U. Erez, S. Shamai (Shitz), and R. Zamir, “Capacity and lattice strategies for canceling known interference,” *IEEE Trans. Inform. Theory*, vol. 51, no. 11, pp. 3820–3822, Nov. 2005.

[94] Y. H. Nam and H. E. Gamal, “Joint lattice decoding achieves the optimal diversity multiplexing tradeoff for multiple access channels,” submitted to IEEE Trans. Inform. Theory, April 2007.

[95] D. Krithivasan and S. S. Pradhan, “Lattices for distributed source coding: Jointly Gaussian sources and reconstructions of a linear function,” *IEEE Trans. Inform. Theory*, Dec. 2009.

[96] B. Nazer and M. Gastpar, “Lattice coding increases multicast rates for Gaussian multiple-access channels,” in *Proc. of 45th Allerton Conference*, Sept. 2007.

[97] ——, "The case for structured random codes in network capacity theorems," *European Trans. on Comm, Special Issue on New Directions in Inform. Theory*, vol. 19, no. 4, pp. 455–474, June 2008.

[98] S. O. G. A. Motahari and A. K. Khandani, "On the degrees of freedom of the 3–user Gaussian interference channel: the symmetric case," *Proc. IEEE ISIT*, 2009.

[99] S. Sridharan, A. Jafarian, S. Vishwanath, S. A. Jafar, S. Shamai (Shitz), "Capacity of Symmetric K-User Gaussian Very Strong Interference Channels," in *Proc. of Globecom*, New Orleans, Dec. 2008.

[100] S. Sridharan and A. Jafarian and S. Vishwanath and S. A. Jafar and S. Shamai (Shitz), "A Layered Lattice Coding Scheme for a Class of Three User Gaussian Interference Channels, in *Proc. of Allerton*, Sep. 2008.

[101] S. Sridharan, A. Jafarian, S. Vishwanath, S. A. Jafar and S. Shamai (Shitz), "Lattice Coding for K User Gaussian interference channels," submitted to *IEEE Trans. Inform. Theory*.

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