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Dedication

To my parents, Fuan Huo and Kaiming Bi

Abstract

This dissertation includes three chapters.

The first two chapters are co-authored with Naoki Takayama. The first chapter presents a model of business cycles driven by shocks to agents' beliefs about economic fundamentals. Agents are hit both by common and idiosyncratic shocks. Common shocks act as confidence shocks, which cause economy-wide optimism or pessimism and consequently, aggregate fluctuations in real variables. Idiosyncratic shocks generate dispersed information, which prevents agents from perfectly inferring the state of the economy. Crucially, asymmetric information induces the infinite regress problem, that is, agents need to forecast the forecasts of others. We develop a method that can solve the infinite regress problem without approximation. Even though agents face a complicated learning problem, the equilibrium policy can be represented by a small number of state variables. Theoretically, we prove that the persistence of aggregate output is increasing in the degree of information frictions and strategic complementarity, and there is a hump-shaped relationship between the variance of output and the variance of the confidence shock. Quantitatively, our model with confidence shocks can match a number of the key business cycle moments.

The second chapter develops a general method of solving rational expectations models with higher order beliefs. Higher order beliefs are crucial in an environment with dispersed information and strategic complementarity, and the equilibrium policy depends on infinite higher order beliefs. It is generally believed that solving this type of equilibrium policy requires an infinite number of state variables (Townsend, 1983). This paper proves that the equilibrium policy rule can always be represented by a finite number of state variables if the signals observed by agents follow an ARMA process, in which case we obtain a general solution formula. We also prove that when the signals contain endogenous variables, a finite-state-variable representation of the equilibrium may not exist. For this case, we develop a tractable algorithm that can approximate the solution arbitrarily well. The key innovation in our method is to use the factorization identity and Wiener filter to solve signal extraction problems conditional on infinite

observables. This method can be used in a wide range of applications. We demonstrate its strong practicability by solving several classical models featuring higher order beliefs.

The third chapter is co-authored with José-Víctor Ríos-Rull. We build a variation of the neoclassical growth model in which both wealth shocks (in the sense of wealth destruction) and financial shocks to households generate recessions. The model features three mild departures from the standard model: (1) adjustment costs make it difficult to expand the tradable goods sector by reallocating factors of production from nontradables to tradables; (2) there is a mild form of labor market frictions (Nash bargaining wage setting with Mortensen-Pissarides labor markets); (3) goods markets for nontradables require active search from households wherein increases in consumption expenditures increase measured productivity. These departures provide a novel quantitative theory to explain recessions like those in southern Europe without relying on technology shocks.

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Chapter 1

Higher Order Beliefs, Confidence, and Business Cycles

1.1 Introduction

Motivated by the Great Recession, there has been an increased interest in business cycles driven by confidence shocks ([5, 4, 6]). A confidence shock can be understood as a shock to agents' beliefs about the economic activities that others are capable of. When this shock is correlated across agents, it induces economy-wide optimism or pessimism, and therefore, aggregate fluctuations in the main macro variables. Intuitively, confidence is promising as a source of business cycle fluctuations since it is well known that people's perceptions of business conditions vary dramatically. However, there have been substantial difficulties to incorporate confidence shocks into a rational expectations framework because of the *infinite regress problem* ([7]). Namely, with asymmetric information and interconnection between agents' economic activities, agents' payoffs depend on their beliefs about others' actions, and rationality requires agents to forecast the forecast of others. While it is necessary to allow for some persistence in shocks for empirical relevance, rational agents have to keep all the information learned from the past to forecast all higher order beliefs, which leads to an infinite-dimensional state space. The goal of this paper is to overcome this technical difficulty, and to explore whether the confidence shock could be an important factor in accounting for business cycles.

Our first contribution is to solve the infinite regress problem by applying our method developed in [8]. It is widely believed that if a rational expectations model involves higher order beliefs and persistent hidden states, the Kalman filter has to be applied to solve the signal extraction problem and to keep track of an infinite number of state variables in order to forecast all higher order beliefs. To short-circuit this problem, the existing literature typically assumes that the information become public after a certain number of periods, or imposes a heterogeneous prior formulation. Instead of modifying the original problem, we confront and solve the infinite regress problem directly. We prove that for any linear rational expectations model with an ARMA signal process, the equilibrium policy rule always allows a finite-state-variable representation.[1] We also provide a procedure to find these state variables and their laws of motion. By using a small set of state variables, agents can perform their best inference in equilibrium, and economists can calibrate or estimate the model as standard DSGE models with perfect information.

The idea is to find the true solution in the space spanned by the entire history of signals in the first place. In this infinite-dimensional state space, we use the Wiener filter to handle the signal extraction problem, as opposed to the standard Kalman filter. It turns out that if the signal process follows an ARMA process, the equilibrium policy will inherit this property and also be of the ARMA type. This implies that information can be summarized in a relatively compact way, and it allows us to find a finite-state-variable representation of the equilibrium policy rule. In addition, after we find this representation, the equilibrium is characterized by a simple linear system, and we no longer need to solve any inference problems when simulating the economy.

Our second contribution is to formalize the idea of confidence shocks in a rational expectations model and to apply our method to evaluate its quantitative importance. We first construct an illustrating model with decentralized trading and information frictions, which is based on the structure specified in [4]. The economy consists of a continuum of islands, and the islands differ in their productivity. At every period, each island is randomly matched with another island and trades with it. Households value both domestic and foreign goods, resulting in the local output increasing in their trading

[1] The linearity may be obtained by log-linearization, and the ARMA process assumption is compatible with the shock structure specified in most macroeconomic models.

partner's output. Information frictions prevent households from observing their trading partner's productivity, and households only receive a noisy signal of this productivity. With a positive (negative) noise, islands tend to overestimate (underestimate) their trading partners' productivity and output, and also to increase their own output due to strategic complementarity. If the noise shock is correlated across islands, then it will cause economy-wide output fluctuations. We label this shock a confidence shock.

When choosing the production level, agents need to infer their trading partner's productivity level, which is equivalent to inferring the confidence shock. However, this is not the end of the inference problem. Note that different islands receive different signals over time, and they will form different inferences about this confidence shock. As a result, agents also need to infer their trading partners' inference of the confidence shock, and all other higher order beliefs. If the confidence shock is persistent, the entire history of signals should be recorded since these signals contain information about the current state of the economy. Even though this is a fairly complicated learning problem, we manage to obtain a sharp analytic solution.

This model economy has two important properties. First, under the assumption that the confidence shock follows an AR(1) process, the aggregate output also follows an AR(1) process. Interestingly, the persistence of the aggregate output is increasing in the degree of strategic complementarity, the value of which is a function of the deep parameters related to preferences and technology. With a stronger interdependence, households respond more aggressively to signals, which magnifies the effects of the confidence shock. The persistence of the aggregate output is also increasing in the degree of information frictions, as it is more difficult to separate the confidence shock from a true productivity shock. Secondly, the unconditional variance of the aggregate output is not monotonically increasing in the variance of the confidence shock. On the one hand, if the variance of the confidence shock is small, the variance of aggregate output is also small since confidence shocks are the only exogenous disturbances. On the other hand, if the variance of the confidence shock is large, agents understand that signals become less useful for information extraction, and they optimally respond less to them. These two competing forces result in a hump-shaped relationship between the variance of output and the variance of the confidence shock. This nonlinearity is absent in standard DSGE models without information friction.

Another important property is that the forecast error is persistent. Supposing the forecast error is absent or there is no information friction, the equilibrium allocation is uniquely pinned down by economic fundamentals, leaving no room for the confidence shock. If we aim to generate persistent aggregate fluctuations, it is important to make sure that the forecast error is long-lasting. In our model, the forecast error is indeed persistent, and agents can never perfectly infer the underlying shocks. This is the result of our information structure, in which there are more shocks than signals, and agents do not have enough information to recover the true state of the economy. By contrast, in [9] and [10], the number of shocks equals the number of signals, and the forecast error disappears quickly. To ensure the persistent effects of the confidence shock, the information process has to be complicated enough to confuse agents for a relatively long time.

With these insights, we develop a quantitative business cycle model to examine the empirical relevance of the confidence shock. Our quantitative model has three key features: a rich information process, goods market frictions, and endogenous capital accumulation. (1) The rich information process provides the flexibility to pin down the degree of information frictions, which is the key factor in determining the performance of the model. The rational expectations framework allows us to link the signal extraction problem faced by agents in the model with the micro-level data. We set the variance and persistence of noise shocks to match the GDP forecast error in the Survey of Professional Forecasters. (2) Introducing goods market frictions a la [11] helps generate endogenous movements of the Solow residual. Goods market frictions create a wedge between potential and realized output. As consumers increase their demand, the utilization rate of potential output also increases, translating into a higher Solow residual. Without the endogenous Solow residual, employment becomes the only driving force of output in the short run, and it leads to the counter-factual prediction that the volatility of employment is much greater than that of output. (3) Capital accumulation brings additional endogenous persistence into the model economy. It also increases the complexity of the signal extraction problem substantially, which prevents us from obtaining an analytic solution. However, we can still represent the equilibrium policy rule by a small number of state variables.

In terms of quantitative performance, we find that the confidence shock alone accounts for much business cycle volatility and co-movement. For example, the standard deviation of output is close to 80% of its data counterpart. The persistence of main aggregate variables is endogenously determined, which represents about 50% of their data counterpart under our calibrations of information frictions. The persistence of aggregate variables hinges on the persistence of forecast errors, which are only modestly persistent in the data. This moment, the persistence of forecast errors, imposes an upper bound on the degree of information frictions, and it prevents generating large persistence of aggregate variables in our model with confidence shocks. Compared with a standard RBC model driven by TFP shocks, two differences stand out. First, our model driven by confidence shocks generates strong counter-cyclical labor wedges, a moment emphasized by [12]. Secondly, with confidence shocks, the standard deviation of employment is more than twice of that in the RBC mode, and it is much closer to the data.

Related literature From a methodological point of view, our paper is related to the literature that attempts to solve models with higher order beliefs. The most widely used method is truncating the relevant state by assuming all shocks become public information after a finite time or only a finite number of higher order beliefs matter for the equilibrium. With a finite number of state variables, the standard Kalman filter can be applied. This line of literature includes [7], [13], [5], [14], and [15] among others. Using these methods to solve our quantitative model with endogenous capital, the number of state variables needed is fairly large to achieve reasonable accuracy, and it is even more difficult to conduct calibration or estimation. The method we developed in [8] provides the true solution to the model, and it only requires a small number of state variables, which makes calibration or estimation possible. [9] and [10] also solve models with higher order beliefs without truncation, but in their environment, the number of signals is the same as the number of shocks, and the forecast error is not persistent. Our method allows us to use a general signal process when there are more shocks than signals, and the confidence shock has persistent effects.[2]

[2] In [16], the number of shocks is the same as the number of signals, but they assume that the underlying shock process is not invertible, which leads to persistent forecast error. We think it is more natural to introduce persistent forecast error by allowing more shocks than signals, a feature that is prevalent in signal extrication problems.

[17] assume agents have heterogeneous prior. This assumption avoids the difficult infinite regress problem, but as acknowledged by the authors, it also abstracts from agents' information extraction process. Under the common prior assumption, our method does not increase the computational difficulty, but allows us to link the model with micro-data and to pin down the degree of information frictions. The cross-sectional evidence on belief dispersion and forecast errors imposes an upper bound on the persistence and volatility of output that can be generated by confidence shocks.

Our quantitative application also complements the literature on aggregate fluctuations driven by shocks to agents' beliefs. In [5], [3], and [18], there is a shock to aggregate TFP, but agents only observe aggregate TFP contaminated by common noise. Even though this common noise can generate aggregate fluctuations, its effects are bounded above by the variance of the TFP shock. As the variance of the TFP shock approaches zero, agents will not respond to the noise shock. [4] introduce additional trading and communication frictions, and as a result, common noise can generate aggregate fluctuations with aggregate fundamentals being fixed. Our model environment is similar to [4], but we allow persistent common noise. Also, we highlight the role of higher order beliefs in shaping aggregate output. [6] propose another type of environment in which sentiments can generate aggregate fluctuations without resorting to trading or information frictions, and the variance of sentiment shocks is endogenously determined. Unlike our model, agents do not need to solve the infinite regress problem. In [19], confidence shocks affect agents' perceived uncertainty, while in our framework, confidence shocks change agents' mean beliefs. Our paper is related to the literature on news shocks and uncertainty shocks, such as [20], [21], [22], [23], and [24] among others.

The rest of the paper is organized as follows. Section 2.2 sets up a simple economy and describes how the infinite regress problem arises in this environment. We obtain an analytic solution, and discuss various properties of this economy. Section 1.3 considers the case when agents observe the signal which contains endogenous information. We compare the equilibrium outcome with and without endogenous information. Section 1.4 explores the quantitative performance of a full-blown model with confidence shocks. Section 2.8 concludes.

1.2 An Analytic Model with Higher Order Beliefs

In this section, we present a simple island model to introduce confidence shocks which trigger aggregate fluctuations. This model builds on [4], and we allow the signals to be persistent over time. This is a natural extension to make this model empirically relevant, but it induces the infinite regress problem which is difficult to solve. We apply the method developed in [8] to solve the model and obtain a sharp analytic solution.

1.2.1 Model Setup

The economy consists of a continuum of islands indexed by $i \in [0, 1]$. The total factor productivity on island i is a_i, which is drawn from a normal distribution $\mathcal{N}(0, \sigma_a^2)$ but fixed over time. Each island is populated by a continuum of identical households. In each household, there is a producer and a shopper. The producer decides how much to produce. The shopper then receives the output from the producer and makes transaction and consumption plans.

Every period, island i is randomly matched with another island. Households value both local and foreign goods, and they trade with the island they are matched with. There is no centralized market in the economy and all the trading is decentralized. Let $m(i, t)$ denote the index of island i's trading partner in period t. With a slight abuse of notation, sometimes we will use j to denote $m(i, t)$ as the index of island i's contemporary trading partner to simplify notation. It should be clear that island i is matched and trades with a different island j at each period.

We assume that the production plan has to be made at the beginning of a period without perfect knowledge of their trading partner's productivity level. The producers receive noisy signals about $a_{m(i,t)}$ (which will be specified below), and choose their output level conditional on these signals. After production, the two islands matched trade with each other.

The average productivity in the economy is fixed over time, but island i's specific trading partner changes every period. Even though households in each island understand that there is no aggregate change of fundamentals, they still face uncertainty due to the decentralized trading arrangement and the communication frictions. The need to infer their trading partner's output and the lack of perfect information leaves room for confidence shocks and also for higher order beliefs.

Timing and Information Each period has two stages: production and trade. At the beginning of the production stage, island i is randomly matched with another island. Once the match is drawn, producers on island i receive two signals. The first signal x_{it}^1 is on their trading partner's productivity, but is corrupted by a common noise ξ_t

$$x_{it}^1 = a_{m(i,t)} + \xi_t, \tag{1.1}$$

where $a_{m(i,t)} \sim \mathcal{N}(0, \sigma_a^2)$. Crucially, we assume that common noise ξ_t follows a persistent process

$$\xi_t = \rho \xi_t + \eta_t, \tag{1.2}$$

where $\rho \in (0,1)$ and $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$. A positive (negative) realization of ξ_t makes all agents in the economy overestimate (underestimate) their trading partner's productivity. Therefore, we label this common noise shock as a confidence shock.

The second signal x_{it}^2 provides private information on the confidence shock

$$x_{it}^2 = \xi_t + u_{it}, \tag{1.3}$$

where $u_{it} \sim \mathcal{N}(0, \sigma_u^2)$ is idiosyncratic noise. The variance of u_{it} determines the degree of information friction in the economy. If $\sigma_u^2 = 0$, then the producers observe ξ_t perfectly, and can figure out their trading partner's productivity using the first signal without error. The learning problem is trivial in this scenario. If $\sigma_u^2 > 0$ but $\rho = 0$, the producers face a static learning problem, because the information is independent of previous periods. If $\sigma_u^2 > 0$ and $\rho > 0$, the producers face a persistent learning problem, which is the focus of this paper.

The producers' information set on island i at time t includes all the signals received up to time t

$$\Omega_{it} = \left\{ a_i,\ x_{it}^1, x_{it-1}^1, x_{it-2}^1, \ldots,\ \ x_{it}^2, x_{it-1}^2, x_{it-2}^2, \ldots \right\}. \tag{1.4}$$

To fix notation, we use $\mathbb{E}_{it}[\cdot]$ to denote the expectation conditional on i's information up to period t, i.e., $\mathbb{E}_{it}[\cdot] = \mathbb{E}[\cdot|\Omega_{it}]$. Since trading histories and idiosyncratic noises differ across islands, producers on different islands share heterogeneous information sets. It follows that $\mathbb{E}_{it}[\cdot] \neq \mathbb{E}_{jt}[\cdot]$. After observing the signals, the producers decide the output level Y_{it}, which completes the first stage of a period.

The second stage is the trade stage. Shoppers on island i receive output from their producers and trade with shoppers from island $m(i,t)$ in a competitive goods market. In this stage, shoppers can observe the other island's output and productivity. To prevent information from being fully revealed, we assume that shoppers die after consumption and are replaced by new shoppers in the following period. Effectively, shoppers cannot communicate with producers after the transaction stage.

Remark The assumption that shoppers die after they trade and consume is only a means to implement the idea that the communication between producers and shoppers is not perfect. Supposing we allow imperfect communication between producers and shoppers, producers will receive another noisy signal on $a_{m(i,t)}$ or ξ_t, but this is equivalent to setting the variance of u_{it} to a smaller value. Therefore, what is really important is how much producers can learn, but not exactly how they learn.

Shoppers' Problem In the trade stage, goods markets are competitive and the prices for local goods and foreign goods are P_i and P_j respectively.[3] Shoppers receive the output Y_i produced in the first stage on their islands. The shoppers on island i solves the following static problem

$$\max_{C_{ii},C_{ij}} \left(\frac{C_{ii}}{\omega}\right)^{\omega} \left(\frac{C_{ij}}{1-\omega}\right)^{1-\omega}$$

subject to

$$P_i C_{ii} + P_j C_{ij} = P_i Y_i,$$

where C_{ii} is local consumption goods and C_{ij} is foreign consumption goods. We adopt a Cobb-Douglas preference structure and use ω to denote the degree of home bias. The first order condition for the shoppers' problem is

$$\frac{C_{ii}}{C_{ij}} = \frac{\omega}{1-\omega}\frac{P_j}{P_i},$$

[3] Because shoppers solve a static problem in the second stage, we use j to denote $m(i,t)$ to simplify the notation.

The goods market clearing condition in equilibrium is

$$C_{ii} + C_{ji} = Y_i,$$
$$C_{ij} + C_{jj} = Y_j.$$

Combining the equilibrium condition and the first order condition for both islands, we have

$$C^*_{ii} = \omega Y_i,$$
$$C^*_{ij} = (1-\omega) Y_j.$$

In equilibrium, local and foreign consumption are equal to a fixed fraction of local and foreign output, thanks to the Cobb-Douglas preference. The terms of trade is

$$\frac{P_i}{P_j} = \frac{Y_j}{Y_i}, \tag{1.5}$$

which as expected, is increasing in foreign output. In addition, for producers on island i, the utility value of 1 additional unit of local output is given by

$$\mathcal{U}_i = \left(\frac{C_{ij}}{C_{ii}} \frac{\omega}{1-\omega}\right)^{1-\omega} = \left(\frac{P_i}{P_j}\right)^{1-\omega}. \tag{1.6}$$

Note that $\mathcal{U}_i$ only depends on the terms of trade, and is independent of individual producer's output.

Producers' Problem Producers choose how much to produce. They understand that in the second period, the marginal value of their output is given by equation (1.6), which depends on their trading partners' output. If there is no information friction ($\sigma_u = 0$), the productivities on both islands become common knowledge, and the output level on both islands will only be a function of the fundamentals. When there are information frictions, the output level on island i is determined by the expected output level on island $m(i,t)$.

Because there is no capital, the producers' problem on island i is choosing output Y_{it} and labor N_{it} to maximize their expected utility in the current period. Since production is a static choice, the only intertemporal link in producers' problem is through information.

$$\max_{Y_{it}, N_{it}} \mathbb{E}_{it} \left[\left(\frac{P_{it}}{P_{m(i,t)t}} \right)^{1-\omega} Y_{it} - N_{it}^{1+\gamma} \right]$$

subject to

$$Y_{it} = \exp(a_i)\ N_{it}^{\theta}.$$

Here, γ is the inverse of Frisch elasticity, and θ determines the labor share. Producers' optimal choice is equating the marginal utility of local output for the shoppers with the marginal disutility of producing the output. When expected $Y_{m(i,t)t}$ increases, the terms of trade improves and the marginal utility of local output also increases, which encourages producers on island i to produce more output. In this sense, there is strategic complementarity between local and foreign output. The first order condition is [4]

$$Y_{it} = \left(\frac{\theta}{1+\gamma}\right)^{\frac{1}{\frac{1+\gamma}{\theta}-\omega}} \exp\left(\frac{1}{1-\frac{\theta}{1+\gamma}\omega}\ a_i\right) \mathbb{E}_{it}[Y_{m(i,t)t}^{1-\omega}]^{\frac{1}{\frac{1+\gamma}{\theta}-\omega}}. \tag{1.7}$$

Standard parametrization ensures that $\gamma > 0$, $\theta \in (0,1)$, and $\omega \in (0,1)$. This implies that $\frac{1}{\frac{1+\gamma}{\theta}-\omega}$, and that the local output is increasing in the expected output $Y_{m(i,t)t}$.

Log-Linearized Economy In this paper, we will work with log-linearzied model. Throughout, we use small letters to denote the log deviation from a variable's steady state value. The log-linearized version of the producers' decision rule (1.7) is

$$y_{it} = \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}[y_{m(i,t)t}], \tag{1.8}$$

where

$$\alpha_0 = \frac{1}{1-\frac{\theta}{1+\gamma}\omega},$$
$$\alpha_1 = \frac{1-\omega}{\frac{1+\gamma}{\theta}-\omega}.$$

As discussed before, α_1 is positive, and y_{it} is increasing in $\mathbb{E}_{it}[y_{m(i,t)t}]$. To guarantee a stable solution, we also restrict our parameter values such that $\alpha_1 < 1$. From now on, we will focus on equation (1.8). Note that the deep parameters related to preferences and technologies are all summarized by α_0 and α_1.

[4] In the first order condition, we have already used the equilibrium condition that the individual output choice coincides with the aggregate output level due to the representative agent assumption.

Perfect Information Benchmark Supposing the variance of the idiosyncratic noise u_{it} vanishes, then agents on island i can use the two signals to figure out $a_{m(i,t)}$ and ξ_t perfectly. In this case, there is no information friction. The optimal policy rule (1.8) becomes

$$y_{it} = \alpha_0 a_i + \alpha_1 y_{m(i,t)t}. \tag{1.9}$$

As expected, the output on island i is completely determined by the economic fundamentals

$$y_{it} = \frac{\alpha_0}{1-\alpha_1^2} a_i + \frac{\alpha_0 \alpha_1}{1-\alpha_1^2} a_{m(i,t)}. \tag{1.10}$$

By the law of large number, the aggregate output y_t stays at its steady state

$$y_t = \int y_{it} = 0. \tag{1.11}$$

The confidence shock ξ_t has no effect at all.

1.2.2 Infinite Regress Problem

When there are information frictions, agents have to infer their trading partners' productivity and output. Higher order beliefs become crucial in determining the production level. By equation (1.8), to infer the output on island $m(i,t)$, island i has to infer the productivity on island $m(i,t)$, which relies on i's prediction of the confidence shock ξ_t. But the same logic also applies to island $m(i,t)$. Therefore, island i needs to infer island $m(i,t)$'s prediction of ξ_t. But so does island $m(i,t)$. It turns out that island i has to predict $m(i,t)$'s prediction of i's prediction of ξ_t, and all other higher order beliefs eventually.

Proposition 1.2.1. *When $\alpha_1 \in (0,1)$, the optimal output rule is given by* [5]

$$y_{it} = \frac{\alpha_0}{1-\alpha_1^2} a_i + \frac{\alpha_0 \alpha_1}{1-\alpha_1^2} a_{m(i,t)} + \frac{\alpha_0}{1+\alpha_1} \sum_{k=1}^{\infty} \alpha_1^k (\xi_t - \mathbb{E}_{it}^k[\xi_t]) \tag{1.12}$$

[5] *Note that in equation (1.12), agents cannot observe $a_{m(i,t)}$ directly. If we sum up $\frac{\alpha_0\alpha_1}{1-\alpha_1^2} a_{m(i,t)}$ and $\frac{\alpha_0}{1+\alpha_1} \sum_{k=1}^{\infty} \alpha_1^k \xi_t$, it will give $\frac{\alpha_0 \alpha_1 x_{it}^1}{1-\alpha_1^2}$, which is a function of agents' first signal.*

where

$$\mathbb{E}^1_{it}[\xi_t] = \mathbb{E}_{it}[\xi_t]$$
$$\mathbb{E}^2_{it}[\xi_t] = \mathbb{E}_{it}\mathbb{E}_{m(i,t)t}[\xi_t]$$
$$\mathbb{E}^k_{it}[\xi_t] = \mathbb{E}_{it}\mathbb{E}_{m(i,t)t}\mathbb{E}^{k-2}_{it}[\xi_t], \ \textit{for } k = 3, 4, 5 \dots$$

Proof. See Appendix A.1.1 for the proof. □

Because islands differ in their information sets, the law of iterated expectation does not apply. Confidence shocks have real effects on the economy. More specifically, the effects of confidence shock on island i are captured by the last term of equation (1.12)

$$\frac{\alpha_0}{1+\alpha_1}\sum_{k=1}^{\infty}\alpha_1^k(\xi_t - \mathbb{E}^k_{it}[\xi_t]), \tag{1.13}$$

and the aggregate output is

$$y_t = \frac{\alpha_0}{1+\alpha_1}\sum_{k=1}^{\infty}\alpha_1^k\left(\xi_t - \int \mathbb{E}^k_{it}[\xi_t]\right). \tag{1.14}$$

Note that the higher order beliefs $\mathbb{E}^k_{it}[\xi_t]$ for $k = \{1, 2, \dots\}$ are different from ξ_t itself in general, which is the reason why the confidence shock can trigger aggregate fluctuation. If ξ_t is underestimated, then islands tend to overestimate their trading partners' productivities. By strategic complementarity, all the islands increase their own output because they expect a higher output from their trading partners, and a boom occurs.

The difficulty lies in computing the equilibrium policy rule of y_{it}. By Proposition 1.2.1, y_{it} depends on all the higher order beliefs $\mathbb{E}^k_{it}[\xi_t]$, but computing all the higher order beliefs is a fairly complicated task. The number of state variables needed to infer higher order beliefs is increasing in the order of the belief.

Proposition 1.2.2. *Given the signal process (1.1) to (1.3), the forecast of* $\mathbb{E}^k_{m(i,t)t}[\xi_t]$ *requires* $k+1$ *state variables.*

The state variables in this proposition are the priors of these higher order beliefs. To spell out all the higher order beliefs, island i needs to keep track of an infinite number of state variables, which is the infinite regress problem. In the next section, we define the equilibrium and use the method developed in [8] to solve the infinite regress problem.

It turns out that the geometric sum of all higher order beliefs follows a simple ARMA process, and a finite number of state variables is sufficient for agents to choose the optimal output y_{it}.

1.2.3 Equilibrium

The information set of producers on island i is $\Omega_{it} = (a_i, \{x^1_{it-\tau}\}^{\infty}_{\tau=0}, \{x^2_{it-\tau}\}^{\infty}_{\tau=0})$. Therefore, island i's policy rule belongs to the space spanned by square-summable linear combinations of current and past realizations of x^1_{it}, x^2_{it}, and also by the time independent local productivity a_i

$$y_{it} = h_a a_i + h_1(L)x^1_{it} + h_2(L)x^2_{it},$$

where $h_a \in \mathbb{R}$, $h_1(L)$ and $h_2(L)$ are lag polynomials

$$h_1(L) = \sum_{\tau=0}^{\infty} h_{1\tau} L^{\tau},$$
$$h_2(L) = \sum_{\tau=0}^{\infty} h_{2\tau} L^{\tau}.$$

The infinite sequences $\{h_{1\tau}\}^{\infty}_{\tau=0}$ and $\{h_{2\tau}\}^{\infty}_{\tau=0}$ belong to the square-summable space ℓ^2, which guarantees that y_{it} is a covariance-stationary process. The equilibrium is defined as follows

Definition 1.2.1. *Given the signal process (1.1) to (1.3), the equilibrium of model (1.8) is a policy rule $h = \{h_a, h_1, h_2\} \in \mathbb{R} \times \ell^2 \times \ell^2$, such that*

$$y_{it} = \alpha_0 a_i + \alpha_1 \ \mathbb{E}_{it}[y_{m(i,t)t}],$$

where

$$y_{it} = h_a a_i + h_1(L)x^1_{it} + h_2(L)x^2_{it}.$$

The equilibrium policy rule is given by the following theorem.

Theorem 1. *Assume that $\alpha_1 \in (0,1)$. Given the signal process (1.1) to (1.3), the equilibrium policy rule is given by*

$$h_a = \frac{\alpha_0}{1-\alpha_1^2\varphi_1}, \tag{1.15}$$

$$h_1(L) = \frac{h_a\alpha_1(\varphi_1 - \vartheta L)}{1-\vartheta L}, \tag{1.16}$$

$$h_2(L) = -\frac{h_a\alpha_1\varphi_2}{1-\vartheta L}, \tag{1.17}$$

where

$$\varphi_1 = \frac{\rho\tau_1 + \vartheta\tau_2}{\rho(\tau_1+\tau_2)}, \quad \varphi_2 = \frac{\tau_1(\rho-\vartheta)}{\rho(\tau_1+\tau_2)}, \tag{1.18}$$

$$\vartheta = \frac{1}{2}\left[\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha_1)(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right) - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha_1)(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right)^2 - 4}\right], \tag{1.19}$$

and

$$\tau_1 = \frac{\sigma_a^2}{\sigma_\eta^2}, \quad \tau_2 = \frac{\sigma_u^2}{\sigma_\eta^2}.$$

The aggregate output follows

$$y_t = \vartheta y_{t-1} + \frac{h_a\alpha_1\vartheta}{\rho}\,\eta_t. \tag{1.20}$$

Proof. See Appendix A.1.2 for proof. □

Even though agents face a fairly complicated learning problem, the equilibrium policy rule is simple. $h_1(L)$ is an ARMA(1,1) process and $h_2(L)$ is an AR(1) process. The aggregate output follows an AR(1) process. To understand the equilibrium policy rule, we discuss the following: the persistence of y_t, the unconditional variance of y_t, and the forecast error of y_t.

1.2.4 Characterization

Endogenous Persistence of y_t Crucially, the persistence of y_t is given by ϑ in equation (2.54), which also determines the persistence of the effects of the confidence shock. We have derived the following properties for ϑ.

Proposition 1.2.3. *Assume that* $\alpha_1 \in (0,1), \rho \in (0,1), \tau_1 > 0$ *and* $\tau_2 > 0$. *Then* ϑ *satisfies*

1. $0 < \lambda < \vartheta < \rho$, *where*

$$\lambda = \frac{1}{2}\left[\frac{\tau_1+\tau_2}{\rho\tau_1\tau_2} + \frac{1}{\rho} + \rho - \sqrt{\left(\frac{\tau_1+\tau_2}{\rho\tau_1\tau_2} + \frac{1}{\rho} + \rho\right)^2 - 4}\right]. \tag{1.21}$$

2. ϑ *is increasing in* α_1 *and*

$$\lim_{\alpha_1 \to 1} \vartheta = \rho$$
$$\lim_{\alpha_1 \to 0} \vartheta = \lambda$$

3. ϑ *is increasing in* τ_1, τ_2 *and* ρ.

Proposition 1.2.3 states that ϑ is bounded from above by the persistence of the confidence shock ρ. Intuitively, agents gradually learn ξ_t from the signals and once they can infer ξ_t relatively accurately, we return to the perfect information benchmark and the confidence shock will have little effect on output. Consequently, the persistence of output is always smaller than the confidence shock. At the same time, ϑ is also bounded from below by λ. Here, λ controls the persistence of the forecast of ξ_t, $\mathbb{E}_{it}[\xi_t]$. If we use the Kalman filter, it follows that

$$\mathbb{E}_{it}[\xi_t] = \lambda\mathbb{E}_{it-1}[\xi_{t-1}] + k_1 x_{it}^1 + k_2 x_{it}^1 \tag{1.22}$$

where k_1 and k_2 are the corresponding Kalman gains. To put it differently, λ determines the speed at which information is revealed, and it serves as the lower bound for the persistence of y_t.

Given the information related parameters ρ, σ_ϵ^2, σ_u^2, and σ_η^2, ϑ is increasing in α_1. As α_1 increases, there is stronger strategic complementarity. Agents respond more aggressively to possible good (bad) trading opportunities. As a result, the effects of confidence shocks last longer. In the extreme case, as α_1 approaches 1, the persistence of y_t approaches the persistence of ξ_t itself. Even though the information obtained by agents does not vary with α_1, the persistence of output chosen by individual agent varies with α_1 because of strategic complementarity.

It is not surprising that the persistence is increasing in τ_1 and τ_2, because the values of these two determine the degree of information frictions. Given the variance of innovation to the confidence shock σ_η^2, as σ_a^2 or σ_u^2 increases, it becomes more difficult to infer the confidence shock ξ_t, and the effects of the confidence shock last longer. Similarly, given the magnitude of idiosyncratic noise, the persistence of output decreases in σ_η^2.

Unconditional Variance of y_t The following proposition characterizes several properties of the variance of aggregate output:

Proposition 1.2.4. *Assume that $\alpha_1 \in (0,1), \rho \in (0,1), \tau_1 > 0$ and $\tau_2 > 0$. The unconditional variance of output y_t is given by*

$$\mathrm{Var}(y_t) = \frac{1}{1-\vartheta^2}\left(\frac{h_a \alpha_1 \vartheta}{\rho}\right)^2 \sigma_\eta^2, \tag{1.23}$$

and it has the following properties:

1. *There is a hump-shaped relationship between Var(y_t) and the variance of confidence innovation σ_η^2. Furthermore,*

$$\lim_{\sigma_\eta^2 \to 0} \mathrm{Var}(y_t) = 0$$

$$\lim_{\sigma_\eta^2 \to \infty} \mathrm{Var}(y_t) = 0$$

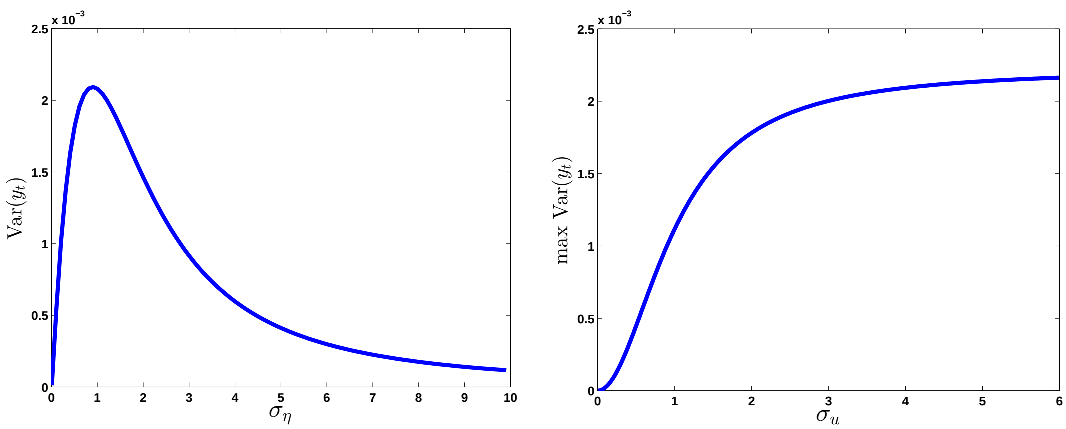
2. $\mathrm{Var}(y_t)$ *is increasing in α_1, σ_a^2, σ_u^2 and ρ.*

Note that in equation (1.23), h_a and ϑ are also functions of σ_η. As discussed in the introduction, there are two competing forces that determine the variance of output. The volatility of output tends to increase with σ_η^2 because there are stronger exogenous disturbances. At the same time, with a larger σ_η^2, agents attenuate their response to signals because they understand that signals are less useful for information extraction. We can also define the maximum amount of volatility that can be generated by confidence shocks given certain information frictions

$$\max \mathrm{Var}(y_t) = \max_{\sigma_\eta^2} \frac{1}{1-\vartheta^2}\left(\frac{h_a \alpha_1 \vartheta}{\rho}\right)^2 \sigma_\eta^2. \tag{1.24}$$

The left graph in Figure 1.1 shows an example of how the variance of output changes with the variance of η_t, which displays a hump-shaped relationship. The right graph in Figure 1.1 shows that the maximum of the variance of output is increasing in the variance of idiosyncratic noise, but it is also bounded from above. The upper bound is determined by the underlying productivity dispersion across islands. This graph clearly illustrates that there exists a limit for the effects of confidence shocks on the aggregate economy.

Figure 1.1: Illustration of Proposition 1.2.4



Proposition 1.2.4 has two implications for our quantitative exercise in the next section. First, given the degree of information frictions, there is an upper bound for the variance of aggregate output by varying the variance of the confidence shock. If the degree of information frictions is relatively low, we may not be able to generate enough volatility of output. Second, there are two different values of variance of the confidence shock which can generate the same volatility of output. These two choices of σ_η will imply different degrees of information frictions and consequently, different magnitudes of forecast errors. Both of these implications indicate that it is crucial to discipline the degree of information frictions in order to evaluate the quantitative importance of the confidence shock.

Persistent Forecasting Error An important feature of the learning problem in this model is that the forecast error is persistent. In [9] and [10] where the number of signals equals the number of shocks, the forecast error only exists in one period and agents can learn the true state fairly quickly. The reason is that there are enough signals for agents to figure out the true state of the economy. In our economy, there are more shocks than signals. Agents can never infer the state of the economy perfectly and the forecast error is long lasting. This is crucial in generating the persistent effects of the confidence shock, because once the forecast error disappears, the economy returns to the perfect information case and the confidence shock no longer plays a role.

We look in particular at differences between the aggregate output and the average predicted aggregate output, since this statistic is important in the calibration of the quantitative model. The inference of the aggregate output by producers on island i is given by

$$\mathbb{E}_{it}[y_t] = \frac{h_a \alpha_1 \vartheta \lambda}{\rho^2(1-\vartheta\lambda)} \frac{1-\rho L}{(1-\lambda L)(1-\vartheta L)} \left(\frac{1}{\tau_1} x_{it}^1 + \frac{1}{\tau_2} x_{it}^2 \right). \tag{1.25}$$

The mean forecast error is then

$$y_t - \int \mathbb{E}_{it}[y_t] = h_a \alpha_1 \vartheta \frac{1 - \frac{\lambda(\tau_1+\tau_2)}{\rho\tau_1\tau_2(1-\vartheta\lambda)} - \lambda L}{\rho(1-\lambda L)(1-\vartheta L)} \eta_t, \tag{1.26}$$

which follows an ARMA(2,1) process. Clearly, the forecast error is persistent over time.

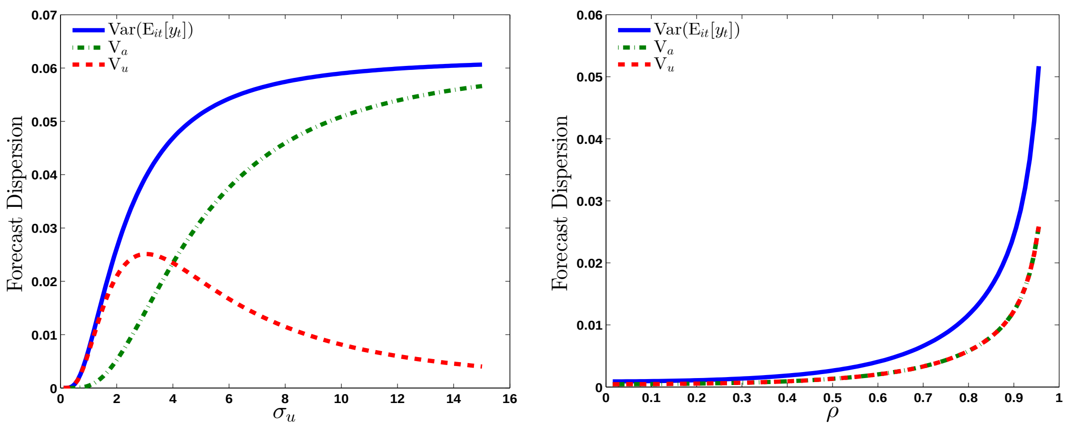
Forecast Dispersion Another interesting and relevant statistic to look at is the forecast dispersion. Based on equation (1.25), the forecast dispersion can be derived as

$$\begin{aligned}
&\mathrm{Var}(\mathbb{E}_{it}[y_t]) && (1.27)\\
=&\int \left(\mathbb{E}_{it}[y_t] - \int \mathbb{E}_{it}[y_t] \right)^2 \\
=&\left(\frac{h_a \alpha_1 \vartheta \lambda}{\rho^2(1-\vartheta\lambda)} \right)^2 \mathrm{Var}\left(\frac{1-\rho L}{(1-\lambda L)(1-\vartheta L)\tau_1} a_{m(i,t)} \right) && (1.28)\\
&+ \left(\frac{h_a \alpha_1 \vartheta \lambda}{\rho^2(1-\vartheta\lambda)} \right)^2 \mathrm{Var}\left(\frac{1-\rho L}{(1-\lambda L)(1-\vartheta L)\tau_2} u_{it} \right) \\
\equiv&\mathrm{V}_a + \mathrm{V}_u && (1.29)
\end{aligned}$$

As expected, the forecast dispersion can be decomposed into two components: the part related to the dispersion of productivity $a_{m(i,t)}$ and that related to the dispersion of idiosyncratic noise u_{it}. It should be clear that V_a and V_u depend on both the variances of the idiosyncratic shocks and the persistence of the confidence shock.

Figure 1.2 presents how $\mathrm{Var}(\mathbb{E}_{it}[y_t])$, V_a, and V_u vary with the variance of idiosyncratic noise and the persistence of the confidence shock. First, the forecast dispersion is monotonically increasing in σ_u^2. However, the part due to the variance of idiosyncratic noise V_u displays a hump-shaped relationship with σ_u^2. The reason is that as σ_u^2 increases, agents also optimally respond less to the second signal. Second, the forecast dispersion is also monotonically increasing in ρ. The change of ρ have similar effects on V_a and V_u, and therefore both components are monotonically increasing in ρ.

Figure 1.2: Forecast Dispersion



1.2.5 Example

In this section, we provide an example to show how the simple economy responds to a confidence shock. We choose parameters exogenously and they are summarized in Table 1.1. The impulse response to confidence shocks is shown in Figure 1.3. At the beginning, agents underestimate the confidence shock on average and consequently, they overestimate their trading partners' productivity and output. Due to strategic complementarity, their best response is to increase their own output, resulting in an increase in aggregate output. The confusion will not be resolved immediately. Agents gradually learn the true state of the economy, and during this process, the output remains above

Table 1.1: Parameters for the Simple Economy

Primitive	Description	Value
ω	Home bias	0.70
$\frac{1}{\gamma}$	Frisch elasticity	0.55
θ	Labor share	0.68
ρ	Persistence of confidence shock	0.95
σ_η	Std of confidence shock	1.00
σ_a	Std of productivity distribution	4.00
σ_u	Std of noise shock	4.00
Implied	**Description**	**Value**
α_0	Response to own productivity	1.20
α_1	Strategic complementarity	0.09
ϑ	Endogenous persistence of output	0.70

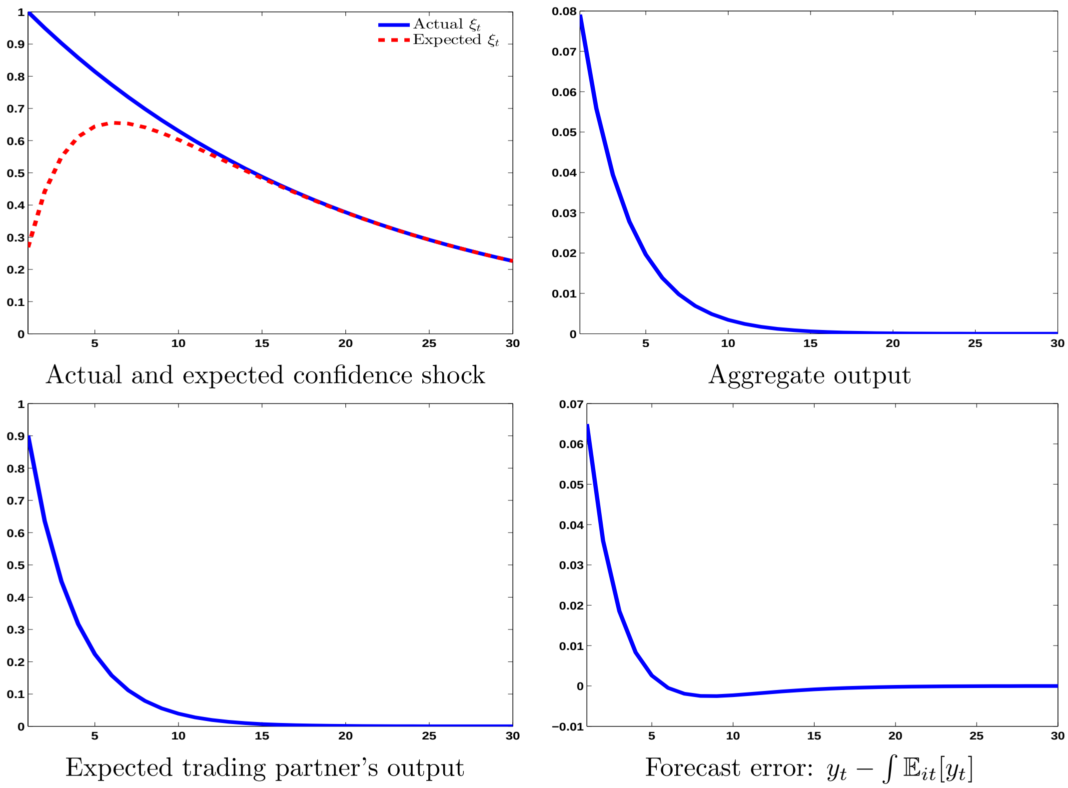
its steady state. Meanwhile, the aggregate output forecast error is persistent, and it resembles the pattern of the actual output.

Higher Order Beliefs By Proposition 1.2.1, the aggregate output can also be written in the form of higher order beliefs

$$y_t = \frac{\alpha_0}{1+\alpha_1} \sum_{k=1}^{\infty} \alpha_1^k \left(\xi_t - \int \mathbb{E}_{it}^k[\xi_t] \right). \tag{1.30}$$

The effects of the confidence shock depend on the difference between the confidence shock and the higher order beliefs about the confidence shock. Figure 1.4 plots the the impulse response of the higher order beliefs. Initially, all the higher order beliefs are smaller than the true ξ_t, which implies that $\xi_t - \int \mathbb{E}_{it}^k[\xi_t] > 0$ and the output y_t will be high in the short run. Gradually, all the higher order beliefs converge to ξ_t, and the output y_t returns to its steady state value. As the order of the beliefs increases, the difference between ξ_t and $\mathbb{E}_{it}^k[\xi_t]$ also becomes greater. However, the effects of these higher order beliefs decay at rate α_1, meaning that as k approaches infinity, the effects of $\mathbb{E}_{it}^k[\xi_t]$ become zero. This intuition is discussed extensively in [25].

Figure 1.3: Impulse Response to a Confidence Shock in the Simple Economy



Actual and expected confidence shock — Aggregate output — Expected trading partner's output — Forecast error: $y_t - \int \mathbb{E}_{it}[y_t]$

Heterogeneous Prior In [4] and [17], a heterogeneous-prior formulation is applied to avoid the infinite regress problem. The heterogeneous prior assumption works as follows. Assume that agents on island i observe both ξ_t and $a_{m(i,t)t}$ perfectly. However, they believe agents on island $m(i,t)$ observe a_i with bias ξ_t. If agent i's policy rule is

$$y_{it} = f_1 a_i + f_2 a_{m(i,t)} + f_3 \xi_t,$$

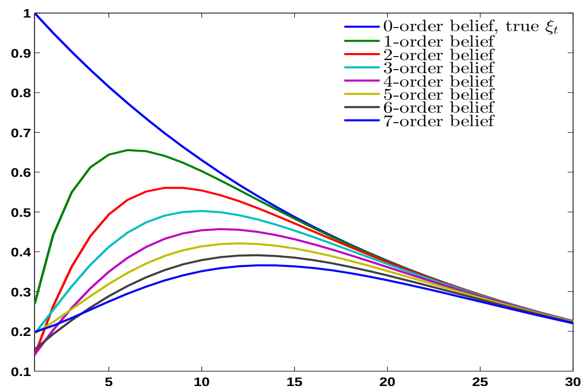
then agent i believes that her trading partner's output is

$$y_{m(i,t)t} = f_1 a_{m(i,t)} + f_2 (a_i + \xi_t) + f_3 \xi_t.$$

In equilibrium,

$$y_{it} = \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}[y_{m(i,t)t}],$$

Figure 1.4: Impulse Response of Higher Order Beliefs to the Confidence Shock



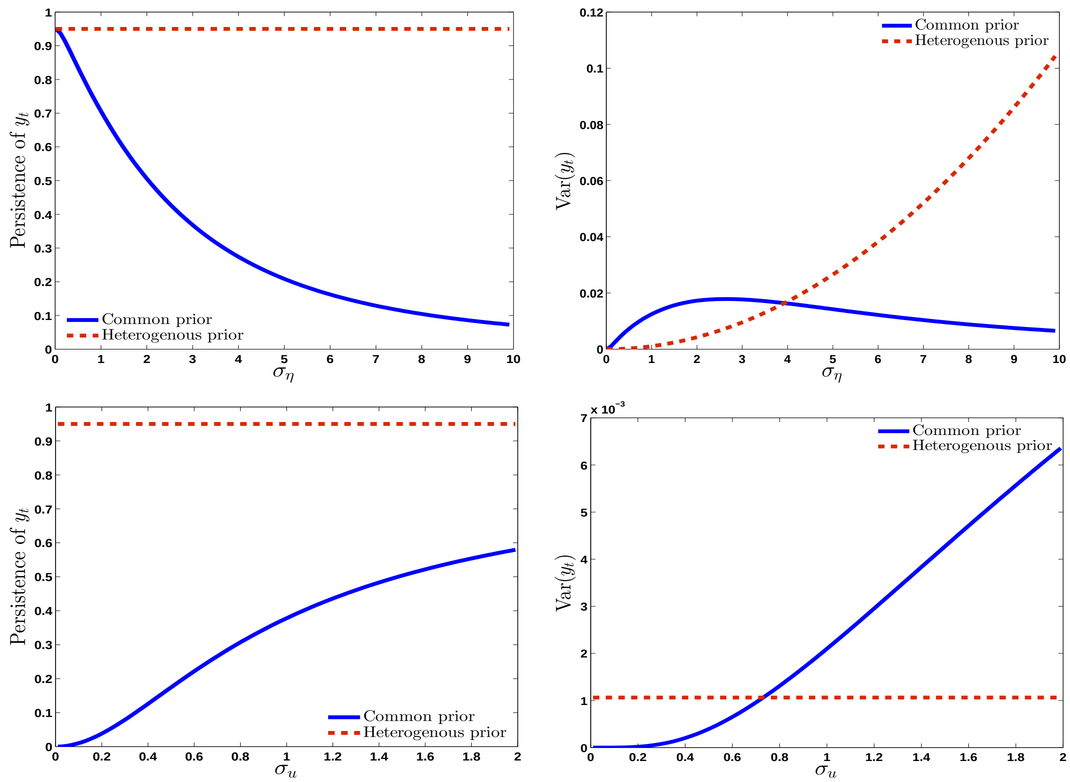
which leads to

$$y_{it} = \frac{1}{1-\alpha^2} a_i + \frac{\alpha}{1-\alpha^2} a_{m(i,t)} + \frac{\alpha_1^2}{(1-\alpha_1^2)(1-\alpha)} \xi_t \tag{1.31}$$

$$y_t = \frac{\alpha^2}{(1-\alpha^2)(1-\alpha)} \xi_t \tag{1.32}$$

By assuming heterogeneous prior beliefs, y_t is perfectly correlated with ξ_t, since the belief process is exogenously given. In Figure 1.5, we show how the persistence and variance of output vary with the variance of the confidence shock and the variance of idiosyncratic noise. With common prior, as we increase the variance of the confidence shock, τ_1 and τ_2 both decrease, and by Proposition 1.2.3, the persistence of output also decreases. By Proposition 1.2.4, there is a hump-shaped relationship between the variance of output and the variance of the confidence shock. In terms of information frictions, both of the persistence and variance of output are monotonically increasing in the variance of idiosyncratic noise σ_u^2. With heterogeneous prior, the persistence of output is independent of the variance of the confidence shock, and the variance of output is monotonically increasing with the variance of the confidence shock. In addition, both of these two statistics are independent of the degree of information frictions.

Figure 1.5: Common Prior v.s. Heterogeneous Prior



1.3 Endogenous Information

In the previous section, the signal process was exogenously determined and independent of agents' actions. An important theme in the literature on dispersed information and higher order beliefs is the role of an endogenous signal in coordinating beliefs and revealing information.[6] In this section, we allow agents to observe signals that contain a variable which is endogenously determined in equilibrium.

More specifically, we allow agents to observe two signals. The first signal is the same as before, which is their trading partner's productivity plus the confidence shock. The second signal is the aggregate output with an idiosyncratic noise. The aggregate output is endogenously determined by agents' output choice, but at the same time it serves as

[6] See [9], [26], and [16] for example.

a signal for agents to infer the state of the economy. Agents understand that η_t is the underlying shock that drives the confidence shock and the aggregate output. Hence, observing the noisy signal of aggregate output will help them predict η_t and in turn the confidence shock. Formally, the equilibrium with endogenous information is defined as follows.

Definition 1.3.1. *The equilibrium is an endogenous stochastic process Ω_{it}, a policy rule for individual agents $\phi = \{\phi_a, \phi_1, \phi_2, \phi_3\} \in \mathbb{R} \times \ell^2 \times \ell^2 \times \ell^2$ and the law of motion for aggregate output $\Phi \in \ell^2$, such that*

1. *Information process generating Ω_{it} is given*

$$\begin{aligned} x_{it}^1 &= a_{m(i,t)} + \xi_t, && (1.33) \\ x_{it}^2 &= y_t + u_{it}, && (1.34) \end{aligned}$$

where

$$\begin{aligned} \xi_t &= \frac{1}{1-\rho L}\eta_t, && (1.35) \\ y_t &= \Phi(L)\eta_t. && (1.36) \end{aligned}$$

2. *Individual rationality*

$$y_{it} = \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}[y_{m(i,t)}], \tag{1.37}$$

where

$$y_{it} = \phi_a a_i + \phi_1(L) a_{m(i,t)} + \phi_2(L) u_{it} + \phi_3(L)\eta_t. \tag{1.38}$$

3. *Aggregate consistency*

$$\Phi(L) = \phi_3(L). \tag{1.39}$$

The policy rule in this definition is in terms of the underlying shocks. As proved in [8], there is a one-to-one mapping between the policy defined in terms of signals and shocks. With endogenous information, it is more convenient to express the policy rule in terms of shocks, because it clearly separates the idiosyncratic components from the aggregate components. The equilibrium with endogenous information involves two fixed points. The first fixed point is individual rationality. Given the signal process, all islands choose the same policy rule ϕ that solves their optimization problem. Agents need to infer higher order beliefs, and the infinite regress problem still exists. The second fixed point is absent in the equilibrium with exogenous information. It requires that the perceived law of motion for aggregate output be the same as the law of motion for actual aggregate output. This can be viewed as the cross-equation restriction in the sense that agents perceptions are in line with the reality generated by their own actions.

Since there are more shocks than signals, agents cannot infer the shocks perfectly. The information role of output depends on the volatility of output. If the aggregate output is very volatile, then the second signal will be very informative about the confidence shock. However, once agents can learn quickly the state of the economy from aggregate output, the effects of the confidence shock will be very limited, which implies that the aggregate output can not respond to the confidence shock aggressively. Conversely, if there is little movement of aggregate output, then agents will pay little attention to the second signal and attribute a big portion of the confidence shock to their trading partner's productivity. Under this scenario, the confidence shock will generate large movements of aggregate output, which is a contradiction. The argument above provides the intuition for the existence of the equilibrium: there exists a point such that the volatility of aggregate output is neither too large nor too small.

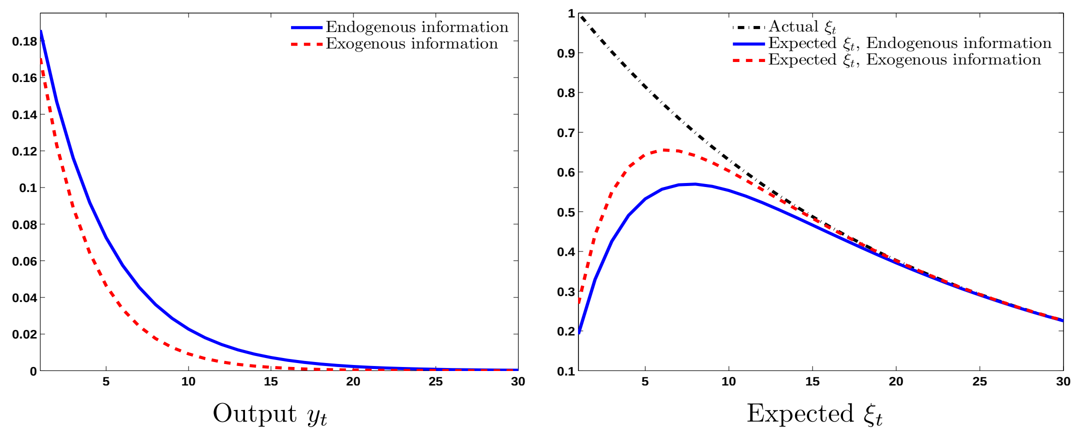
Theorem 2. *If $\alpha_1 \in (0,1)$, then there exists a unique equilibrium of the model in Definition 2.5.1.*

Proof. See Appendix A.2.15 for the proof. □

As shown in [8], even though there exists a unique equilibrium, aggregate output follows an infinite-order process. As a result, no analytic solution is possible any more. We use the method discussed in [8], and approximate the aggregate output by an ARMA (3,2) process. This approximation is close enough to the true solution.

Figure 1.6 compares the impulse response of the aggregate output to the confidence shock under endogenous information with the one under exogenous information. It can be seen that the output under endogenous information is more responsive to the confidence shock. To understand the results, we need to highlight the information role of the aggregate output. Since α_1 is small in our example, the force of strategic complementarity is weak, and hence the aggregate output is not very volatile. As a result, the

Figure 1.6: Endogenous Information versus Exogenous Information



endogenous signal $x_{it}^2 = y_t + u_{it}$ conveys less information about the confidence shock compared with the exogenous signal $x_{it}^2 = \xi_t + u_{it}$. Note that in Figure 1.6 the prediction of the confidence shock is indeed less accurate with endogenous information. Therefore, when α_1 is small, the effects of the confidence shock are greater under endogenous information than under exogenous information. Conversely, if we set α_1 to a large number, the aggregate output will be more volatile than ξ_t itself. It follows that the endogenous signal $x_{it}^2 = y_t + u_{it}$ will contain more information than the exogenous signal $x_{it}^2 = \xi_t + u_{it}$. Consequently, the effects of the confidence shock will be greater with exogenous information.

This example illustrates that whether agents observe exogenous signals or endogenous signals does not really matter. What matters is how much information agents can learn about the underlying state of the economy. At the end of the day, individual agents treat all signals as exogenously given, and we can change the size of the noise shocks to control the amount of information agents can extract. Based on this observation, in our quantitative model, we assume all information follows an exogenous process, but it should be noted that this assumption is not crucial to the purpose of evaluating the role of confidence shocks.

1.4 Quantitative Model

In this section, we present the full-blown business cycle model driven by confidence shocks. To evaluate its quantitative performance and confront the model with data, several issues need to be addressed. First, the confidence shock itself and the idiosyncratic noises cannot be observed, but we need to pin down the degree of information frictions. Second, because the confidence shock does not affect aggregate technology, the Solow residual remains constant. As a result, all the short-run fluctuations are driven by changes in labor, which is at odds with data. Third, aggregate investment is important in shaping business cycles, and agents constantly make inter-temporal decisions. Based on these considerations, we extend the simple model presented in Section 2.2 along three dimensions: (1)we adopt a more flexible matching process and information structure, which allows us to link the model with the survey data in order to discipline information frictions; (2)we introduce competitive search in the goods market à la [11]), which generates endogenous movements of the Solow residual; (3) we allow households to accumulate capital.

1.4.1 Model

Matching and Information In the simple model, we assume that the matching follows an i.i.d. process, that is, the quality of island i's trading partner in period t is completely independent of its trading partner in period $t-1$. This assumption is convenient for deriving analytic results, but it is far from being realistic. If we interpret an island as an establishment, a firm, or a region, the output or revenue of these entities is typically correlated over time. Meanwhile, the exact form of the matching process is also related to the degree of information frictions. Therefore, we allow the matching process to be persistent. Namely, if island i is matched with a good trading partner today, it is more likely that island i is also matched with a good trading partner tomorrow. Recall that we denote the index of island i's trading partner in period t as $m(i,t)$, and we now assume that the productivity of $m(i,t)$ follows an AR(1) process

$$a_{m(i,t)} = \rho_a a_{m(i,t-1)} + \epsilon_{it}, \tag{1.40}$$

where $\epsilon_{it} \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $\sigma_\epsilon^2 = (1-\rho_a^2)\sigma_a^2$. Note that the choice of σ_ϵ^2 guarantees that the unconditional variance of $a_{m(i,t)}$ is consistent with σ_a^2. If we set $\rho_a = 0$, it collapses to the original i.i.d. matching process. The following proposition proves the existence of the persistent matching process.

Proposition 1.4.1. *Let $m(i,t)$ be island i's trading partner at time t and $a_{m(i,t)}$ be its productivity. There exists a stochastic process such that, for all $i \in [0,1)$,*

$$
\begin{aligned}
a_{m(i,t)} &= \rho_a a_{m(i,t-1)} + \epsilon_{it}, \\
\epsilon_{it} &\sim \mathcal{N}(0, \sigma_\epsilon^2)
\end{aligned}
$$

where $\rho_a \in (0,1)$.

Proof. See Appendix A.1.4 for the proof. □

The signal process is almost the same as the simple model. At the beginning of each period, we still assume that producers receive two signals. The first signal concerns their trading partner's productivity, but it is contaminated by the common confidence shock

$$x_{it}^1 = a_{m(i,t)} + \xi_t. \tag{1.41}$$

The process of $a_{m(i,t)}$ is specified in equation (1.40). The confidence shock ξ_t follows the same AR(1) process as the simple model

$$\xi_t = \rho \xi_t + \eta_t, \tag{1.42}$$

where $\rho \in (0,1)$ and $\eta_t \sim \mathcal{N}(0, \sigma_\eta^2)$.

The second signal is the confidence shock plus an idiosyncratic noise.

$$x_{it}^2 = \xi_t + u_{it}, \tag{1.43}$$

The information set, up to time t, is

$$\Omega_{it} = \Big\{ a_i,\ x_{it}^1, x_{it-1}^1, x_{it-2}^1, \ldots,\ \ x_{it}^2, x_{it-1}^2, x_{it-2}^2, \ldots \Big\}.$$

Competitive Search and Shoppers' Problem In the simple model, the goods market between the two trading partners is frictionless. Shoppers from the two islands meet in a centralized market, and the prices P_i and P_j clear the goods market. Since the distribution of productivity is fixed, there is no change of aggregate TFP.

To introduce endogenous TFP movement, we assume there exist goods market frictions as in [11]. The basic idea is simple. Shoppers have to search for goods before they can consume them, and goods have to be found before they can be sold. A standard matching friction prevents standard market clearing. The probability that goods can be sold is determined by the amount of search effort exerted by shoppers. As a result, the search effort creates a wedge between potential output and actual output, which corresponds to the measured Solow residuals. Crucially, the amount of search effort exerted by shoppers depend on the level of production in the first period, which induces the Solow residuals move with business cycles.

Now we describe the implementation of goods market frictions. In the second stage, shoppers serve both as buyers and sellers. As sellers, each shopper is endowed with a unit measure of location and they can choose in which market to sell the goods inherited from their producers. As buyers, shoppers have to consume the goods produced by others but not by themselves, similarly to [27]. Goods market frictions require buyers to exert search effort to find the locations of others.

Different markets are indexed by their price and market tightness (P, Q), where market tightness is defined as the ratio of the measure of location to the measure of search effort. Exerting one unit of search effort in market (P, Q), a buyer expects to find a location with probability $\Psi^d(Q)$ at price P. At the same time, a seller in in market (P, Q) expects to sell her goods with probability $\Psi^f(Q)$ at price P. In equilibrium, not all markets are active. In fact, it is understood that there is an equilibrium-determined expected revenue per unit of good, $\zeta = P\ \Psi^f(Q)$, that active markets have to satisfy.

Because there are two different types of goods, local goods Y_i and foreign goods Y_j, there are two equilibrium-determined expected revenues ζ_i and ζ_j. Buyers on island i choose the local market (P_{ii}, Q_{ii}) and foreign markets (P_{ij}, Q_{ij}), while buyers on island j choose (P_{jj}, Q_{jj}) and (P_{ji}, Q_{ji}). In equilibrium, sellers have to be indifferent between allocating their locations to domestic customers and foreign customers, resulting in

$$P_{ii}\Psi^f(Q_{ii}) = P_{ji}\Psi^f(Q_{ji}) = \zeta_i, \tag{1.44}$$

$$P_{jj}\Psi^f(Q_{jj}) = P_{ij}\Psi^f(Q_{ij}) = \zeta_j. \tag{1.45}$$

It is important to note that not all goods can be sold and the produced goods Y_i and Y_j are only potential output. The realized output depends on the probability Ψ^f that goods are purchased, which is determined by the amount of search effort. This probability Ψ^f can be understood as the utilization rate, and we will show that it increases with the production level of Y_i and Y_j. When the production level changes, the amount of search effort and the utilization rate also change, generating endogenous movements of the measured Solow residual.

The shoppers' problem on island i can be written as

$$\max_{\substack{C_{ii},C_{ij},I_{ii},I_{ij},\\ Q_{ii},Q_{ij},D_{ii},D_{ij}}} \left(\frac{C_{ii}}{\omega}\right)^{\omega}\left(\frac{C_{ij}}{1-\omega}\right)^{1-\omega} - \chi_d D_i \tag{1.46}$$

subject to

$$P_{ii}(C_{ii} + I_{ii}) + P_{ij}(C_{ij} + I_{ij}) = \zeta_i Y_i, \tag{1.47}$$

$$C_{ii} + I_{ii} = D_{ii}\Psi^d(Q_{ii})Y_i, \tag{1.48}$$

$$C_{ij} + I_{ij} = D_{ij}\Psi^d(Q_{ij})Y_j, \tag{1.49}$$

$$P_{ii}\Psi^f(Q_{ii}) = \zeta_i, \tag{1.50}$$

$$P_{ij}\Psi^f(Q_{ij}) = \zeta_j, \tag{1.51}$$

$$I_i = \left(\frac{I_{ii}}{\omega}\right)^{\omega}\left(\frac{I_{ij}}{1-\omega}\right)^{1-\omega}, \tag{1.52}$$

$$D_i = D_{ii} + D_{ij}. \tag{1.53}$$

This calls for several comments. (1) Producers now determines both the level of production Y_i and the level of capital investment I_i in the first stage. As a result, they not only transfers the output Y_i to shoppers, but also require shoppers to purchase the investment good such that the composite of I_{ii} and I_{ij} satisfies producers' investment demand I_i. (2) The search effort D_i is the new element in the shoppers' problem, and the variation in D_i leads to changes in the utilization rate. (3) Related to the search effort, as discussed in [28], shoppers with different income levels choose markets with different prices and search intensities.

The equilibrium conditions include

$$Q_{ii} = \frac{T_{ii}}{D_{ii}}, \quad Q_{ij} = \frac{T_{ji}}{D_{ij}}, \quad Q_{ji} = \frac{T_{ij}}{D_{ji}}, \quad Q_{jj} = \frac{T_{jj}}{D_{jj}}, \tag{1.54}$$

$$T_{ii} + T_{ij} = 1, \quad T_{ji} + T_{jj} = 1, \tag{1.55}$$

$$\zeta_i = P_{ii}\Psi^f(Q_{ii}) = P_{ji}\Psi^f(Q_{ji}), \tag{1.56}$$

$$\zeta_j = P_{jj}\Psi^f(Q_{jj}) = P_{ij}\Psi^f(Q_{ij}). \tag{1.57}$$

Implicitly, shoppers also choose the allocation of their locations T_{ii} and T_{ij} to local and foreign markets, but they are indifferent since they will obtain the same expected revenue ζ_i.

We assume that the matching function in the goods market is of Cobb-Douglas form

$$\Psi^d(Q) = \nu Q^{1-\mu}, \tag{1.58}$$

$$\Psi^f(Q) = \nu Q^{-\mu}, \tag{1.59}$$

where μ is the matching elasticity and ν is a constant that determines the average matching probability. The equilibrium allocations satisfy

$$C_{ii}^* = \omega\nu \left(\frac{\mu\nu}{\chi d}\right)^{\frac{\mu}{1-\mu}} Y_i^{\frac{1-\mu+\mu\omega}{1-\mu}} Y_j^{\frac{\mu(1-\omega)}{1-\mu}} - \omega \left(\frac{Y_i}{Y_j}\right)^{1-\omega} I_i, \tag{1.60}$$

$$C_{ij}^* = (1-\omega)\nu \left(\frac{\mu\nu}{\chi d}\right)^{\frac{\mu}{1-\mu}} Y_i^{\frac{\mu\omega}{1-\mu}} Y_j^{\frac{1-\mu\omega}{1-\mu}} - (1-\omega) \left(\frac{Y_i}{Y_j}\right)^{-\omega} I_i, \tag{1.61}$$

$$D_i^* = \left(\frac{\mu\nu}{\chi d} Y_i^{\omega} Y_j^{1-\omega}\right)^{\frac{1}{1-\mu}}. \tag{1.62}$$

As the production level increases, shoppers purchase more consumption goods. At the same time, they also exert more search efforts. Because the total measure of locations is fixed, more search effort translates into a higher utilization rate and the matching elasticity μ determines the percentage increase of the utilization rate.

Similar to the simple model, we can derive the the utility value of 1 additional unit of local output

$$\mathcal{U}_i = \nu_g \left(\frac{\mu\nu_g}{\chi}\right)^{\frac{\mu}{1-\mu}} Y_i^{\frac{\mu-\eta}{1-\mu}} Y_j^{\frac{\eta}{1-\mu}}. \tag{1.63}$$

Note that $\mathcal{U}_i$ only depends on the aggregate output Y_i and Y_j, and individual producer takes it as given. It can also be shown that the utility value for shoppers by decreasing 1 unit of local investment demand is simply 1.

Producers' Problem Compared to the simple model, the complication of the producers' problem is the addition of investment choice. Instead of a static decision problem, the producers' problem becomes choosing a state contingent plan for Y_{it}, K_{it+1} and N_{it} to maximize their expected present value.

$$\max_{Y_{it},N_{it},K_{it+1},I_{it}} \mathbb{E}_{i0} \sum_{t=0}^{\infty} \beta^t \frac{[\mathcal{U}_{it} Y_{it} - I_{it} - \chi_n N_{it}^{1+\gamma}]^{1-\sigma}}{1-\sigma} \tag{1.64}$$

subject to

$$Y_{it} = \exp(a_i)\ K_{it}^{1-\theta} N_{it}^{\theta}, \tag{1.65}$$

$$K_{it+1} = (1-\delta) K_{it} + I_{it} - \Xi(I_{it}, K_{it}). \tag{1.66}$$

We assume that the investment is subject to a standard capital adjustment cost $\Xi(I_{it}, K_{it})$ with the following functional form

$$\Xi(I_{it}, K_{it}) = \frac{\varphi}{2} \left(\frac{I_{it}}{K_{it}} - \delta \right)^2 K_{it}. \tag{1.67}$$

To derive the first order conditions, we first substitute the production function into the objective function and define

$$\mathcal{V}(Y_{it}, a_i, K_{it}) = \chi_n \left(\frac{Y_{it}}{\exp(a_i)\ K_{it}^{1-\theta}} \right)^{\frac{1+\gamma}{\theta}}. \tag{1.68}$$

The first order condition with respect to Y_{it} is

$$\mathbb{E}_{it} \left[[\mathcal{U}_{it} Y_{it} - I_{it} - \chi_n N_{it}^{1+\gamma}]^{-\sigma} \left(\mathcal{U}_{it} - \mathcal{V}_{y_{it}} \right) \right] = 0 \tag{1.69}$$

The first order condition with respect to K_{it+1} is

$$\frac{\mathbb{E}_{it} \left[[\mathcal{U}_{it} Y_{it} - I_{it} - \chi_n N_{it}^{1+\gamma}]^{-\sigma} \right]}{1 - \Xi_i(K_{it}, I_{it})} = \beta \mathbb{E}_{it} \Bigg[[\mathcal{U}_{it+1} Y_{it+1} - I_{it+1} - \chi_n N_{it+1}^{1+\gamma}]^{-\sigma} \\ \left(\mathcal{U}_{it+1} (1-\theta) \exp(a_i) K_{it+1}^{-\theta} N_{it+1}^{\theta} + \frac{1 - \delta - \Xi_k(K_{it+1}, I_{it+1})}{1 - \Xi_i(K_{it+1}, I_{it+1})} \right) \Bigg] \tag{1.70}$$

These two first order conditions are quite similar to those in standard stochastic growth models, except marginal returns to production depend on producers' expectation of their trading partners' output level. As in a two-country business cycle model, the output and investment decisions both increase with their trading partners' output level.

Log-Linearized Economy Equation (1.69) and (1.70) summarize the producers' decisions. The log-linearized version of these two equations is:

$$\Gamma_1 a_i + \Gamma_2 y_{it} + \Gamma_3 k_{it} + \Gamma_4 \mathbb{E}_{it}[y_{m(i,t)t}] = 0, \tag{1.71}$$

$$\Upsilon_1 k_{it} + \Upsilon_2 k_{it+1} + \Upsilon_3 \mathbb{E}_{it}[y_{m(i,t)t}] + \Upsilon_4 \mathbb{E}_{it}[y_{it+1}] + \Upsilon_5 \mathbb{E}_{it}[k_{it+2}] + \Upsilon_6 \mathbb{E}_{it}[y_{m(i,t+1)t+1}] = 0, \tag{1.72}$$

where $\{\Gamma_1, \ldots, \Gamma_4\}$ and $\{\Upsilon_1, \ldots, \Upsilon_6\}$ are functions of the deep parameters. Similarly to the simple model, the equilibrium is defined as:

Definition 1.4.1. *Given the signal process (1.40) to (1.43), the equilibrium is policy rules* $h^y = \{h^y_a, h^y_1, h^y_2\} \in \mathbb{R} \times \ell^2 \times \ell^2$ *and* $h^k = \{h^k_a, h^k_1, h^k_2\} \in \mathbb{R} \times \ell^2 \times \ell^2$

$$y_{it} = h^y_a a_i + h^y_1(L) x^1_{it} + h^y_2(L) x^2_{it}, \tag{1.73}$$

$$k_{it+1} = h^k_a a_i + h^k_1(L) x^1_{it} + h^k_2(L) x^2_{it}, \tag{1.74}$$

such that equations (1.71) and (1.72) are satisfied.

To solve for the equilibrium, we apply the method developed in [8]. The details of the computation can be found in our online appendix.

1.4.2 Calibration and Estimation

The model period is a quarter. We separate the parameters into two groups: those in the first group (shown in Table 1.2) are determined exogenously, and those in the second group (shown in Table 1.3) are jointly determined by solving a large system: the equations require that the steady-state model statistics equal the targets, and the parameters are the unknowns. In addition, we also estimate the variance of the idiosyncratic noises and innovation to confidence shock to match the forecast error data.

Many parameters of preferences and technology are standard, and we choose them to reflect commonly used values. We set the discount rate β to 0.99, which implies that the rate of return is 4%. We set the risk aversion σ to 1. We choose the Frisch elasticity to be $\frac{1}{\gamma} = 0.55$, which lies between the micro and macro estimates. We choose the labor share $\theta = 0.68$, in line with [29]. The home bias parameter matters for the degree of strategic complementarity. We set $\omega = 0.7$ as our benchmark value.

Turning to the matching process. If we interpret each island as a firm, the persistence of the matching process directly translates into the persistence of the measured firms' profit or productivity even though their technology is unchanged. The empirical estimate of the persistence of the firms' productivity varies in the literature, ranging from 0.5 ([30]) to 0.8 ([31]) for the United States, and it varies even more when examining other countries ([32]). We set $\rho_a = 0.7$ and $\sigma_\epsilon^2 = 0.01$, which lie in the middle of various estimates. Note that σ_a^2 is determined residually by $\sigma_a^2 = \frac{\sigma_\epsilon^2}{1-\rho_a^2}$.

The matching elasticity is particularly important in shaping the endogenous Solow residual. The realized aggregate output is:

$$\overline{y} = \int \Psi^f(q_{ii}) + \int y_i = z + y$$

Here, we use $\overline{y}$ to denote the aggregate output, or realized sales, y to denote the potential output, or produced goods, and z to denote the measured Solow residual. Using equation (1.62), the measured Solow residual is proportional to the potential output

$$z = \int \Psi^f(q_{ii}) \quad \propto \quad \frac{\mu}{1-\mu}\int d_i \quad \propto \quad \frac{\mu}{1-\mu}y \quad = \mu\overline{y}$$

Therefore, the matching elasticity μ determines the portion of the output fluctuations which can be attributed to the Solow residual. We set $\mu = 0.4$, which imply that 40% of output fluctuations are due to Solow residual.

In terms of the endogenously determined parameters, we associate the parameters with the targets for which they are most directly responsible, even though these parameters are eventually determined simultaneously. We choose χ_n to target the average working time to be 0.4 which only serves as a normalization. We target the capital-output ratio to be 2, which pins downs the capital depreciation rate δ. Two parameters are related to goods market frictions: the units of search costs ξ_d and the matching efficiency ν. We choose the values for them so that the average occupation rate is 81%

Table 1.2: Exogenously Determined Parameters

Parameter	Description	Value
β	Discount rate	0.99
σ	Risk aversion	1.00
ω	Home bias	0.70
$\frac{1}{\gamma}$	Frisch elasticity	0.55
θ	Labor share	0.64
μ	Matching elasticity	0.40
ρ	Persistence of confidence shock	0.95
ρ_a	Persistence of matching quality	0.70
σ_a	Std of island specific productivity	0.14

and the average market tightness is 1. We set the capital adjustment cost ψ to match the relative volatility of investment to output.

An important part of the calibration is to discipline the confidence shock process (the persistence ρ and the standard deviation of confidence innovation σ_η), and the size of idiosyncratic noises, σ_u. We choose these parameters to match the mean forecast error of real output and also the average standard deviation of cross-sectional forecasts (forecast dispersion) in the Survey of Professional Forecasters (SPF). The mean forecast error and the forecast dispersion are jointly determined by these three parameters. We set ρ to match the forecast dispersion and estimate σ_η and σ_u using Bayesian method, to match the mean forecast error of real output growth rate (from 1969 Q1 to 2014 Q2). Table 1.4 shows the choice of prior distributions, the estimated posterior mode obtained by maximizing the log of the posterior distribution with respect to the parameters, the posterior mean, and also the 10 and 90 percentile of the posterior distribution of the parameters obtained through the Metropolis-Hastings sampling algorithm.

In a standard log-linearized DSGE model, the standard deviation of a shock is independent of policy rules. By contrast, in our model with information frictions, the relative volatility of various shocks, i.e., $\frac{\sigma_\epsilon^2}{\sigma_\eta^2}$ and $\frac{\sigma_u^2}{\sigma_\eta^2}$, does have direct effects on the policy

Table 1.3: Endogenously Determined Parameters

Parameter	Value	Target	Value	Model
χ_n	1.02	Average labor	0.40	0.40
χ_d	0.68	Average market tightness	1.00	1.00
ν	0.81	Average utilization rate	0.81	0.81
δ	0.02	Capital-to-output ratio	2.00	2.00
ψ	24.00	Ratio of Std of investment to output	4.00	3.92
ρ	0.91	Average Std of cross-sectional forecasts	0.00	3.92

Table 1.4: Estimated Parameters in the Baseline Model

	Prior			Posterior		
	Distribution	Mean	Std	Mode	Mean	90% HPD
σ_η	Inv Gamma	0.30	3.00	0.25	0.28	[0.20, 0.39]
σ_u	Inv Gamma	0.30	3.00	0.34	0.37	[0.31, 0.42]

rules, and σ_η have non-linear effects on aggregate variables. When we choose σ_η, it is not only a normalization. As shown in subsection 1.2.4, the volatility of output is not monotonically increasing in σ_η.

It should be noted that the survey participants have formal, advanced training in economic theory. These survey forecasts are generally better than forecasts generated by econometric models. Agents in the model are interpreted as normal households and firms, who have less information compared with the professional forecasters in the SPF in general.

1.4.3 Results

Baseline model with confidence shocks Figure 1.7 shows the impulse response of the main aggregate variables to the confidence shock.[7] At the beginning, agents underestimate the confidence shock and attribute a part of the confidence shock to a good realization of the matching process. As a result, producers believe that their trading partners' output is higher than average, and it will be so for a while due to the fact that the matching process is persistent. Because of strategic complementarity, believing that there is higher output on other islands leads to a higher output and investment level on their own islands, and thereby to a high aggregate output and investment. This belief is partially true, since the output on other islands is indeed higher than average. However, it is not because the productivity is higher, but because all the islands are optimistic. After a confidence shock, agents on average overestimate their trading partners' output and underestimate the aggregate output in the short run.

Table 1.5 compares the business cycle statistics from the data and our model driven by confidence shocks (Baseline model with ξ shock). The confidence shock model can produce reasonable aggregate volatility. From the demand side, the standard deviation of investment is approximately 4 times larger than that of output, similarly to the data. The volatility of consumption is smaller than the volatility of output, but it is less volatile compared with the data. From the supply side, the change in the output can be decomposed into the change in labor and the change in the measured Solow residual. The standard deviation of labor is close to 60% of its data counterpart, which we think is acceptable given that we choose a relatively low Frisch elasticity. Recall that there is no change in aggregate TFP, the changes in the measured Solow residual are entirely endogenous, driven by shoppers' searching activities. We have chosen the matching elasticity $\mu = 0.4$, which implies that when total output increases by 1%, the measured Solow residual increases by 0.4%.

The model cannot generate the same persistence of aggregate variables as in the data. The basic mechanism of the model is that the behavior output mirrors the behavior of forecast errors. In the data, the forecast errors are only modest persistent, which implies that the persistent of output in the model cannot be too high. Even though

[7] In Figure 1.7, we choose the size of the confidence innovation such that the initial response of output is 1.

capital introduces additional persistence, this effect is not strong enough to allow the model to achieve the same persistence as in the data. To match the autocorrelation in the data, it seems necessary to include other more persistent real shocks.

Comparison with RBC Model without goods market search Now we compare our baseline model driven by confidence shocks with the RBC model driven only by TFP shocks. The RBC model we use is the same as our quantitative model presented in Section 1.4.1 except for three differences: (1)there is no competitive search in the goods market and hence no endogenous Solow residual; (2)there are exogenous shocks to aggregate TFP; and (3) there is no information friction.

We assume that the aggregate TFP shock follows an AR(1) process

$$z_t = \rho_z z_{t-1} + \varsigma_t, \tag{1.75}$$

where $\varsigma_{it} \sim \mathcal{N}(0, \sigma_\varsigma^2)$. After subtracting a linear trend, we estimate process (1.77) and obtain $\rho_z = 0.96$ and $\sigma_\varsigma = 0.0078$. With the aggregate TFP shock, the productivity of an individual island's follows

$$z_{it} = a_i + z_t. \tag{1.76}$$

That is, the productivity in each island equals the sum of the island specific productivity and the aggregate TFP. Note that producers now can observe their trading partners' productivity perfectly.

We set the same exogenously determined parameters as before and calibrate the endogenously determined parameters to the same targets. As can be seen in Table 1.5, the two models have similar performances in matching the volatility of consumption and investment. The model with confidence shocks is more successful in accounting for the volatility of labor, a variable that the RBC model has difficulty matching. The RBC model with TFP shocks outperforms the model with confidence shocks in accounting for the Solow reisdual, but this is mainly due to the exogenously assumed TFP shock process.

As emphasized by [12], standard RBC models fail to capture the pattern of labor wedges. In our model with confidence shocks, the labor wedge is highly counter-cyclical. The reason is that agents increase or decrease their labor supply not because there is a real change in labor productivity, but because they believe the demand from other islands is high thanks to information frictions. The confidence shock creates a wedge between labor productivity and the marginal rate of substitution.

Baseline model with both confidence shock and TFP shock The baseline model driven only by confidence shocks does not generate enough persistence compared with data, and now we add exogenous aggregate TFP shock into the baseline model. The aggregate TFP shock process also follows

$$z_t = \rho_z z_{t-1} + \varsigma_t, \quad \varsigma_{it} \sim \mathcal{N}(0, \sigma_\varsigma^2), \tag{1.77}$$

but because of the existence of goods market search frictions, the measured Solow residual will not be the same as the exogenous aggregate TFP shock. We jointly estimate the TFP shock process and the confidence shock process using Bayesian method, to match the real output growth rate and the mean forecast error. The estimation results are shown in Table 1.6. Because goods market frictions generate endogenous movement of Solow residual, the estimated standard deviation of TFP innovation is smaller than that in the standard RBC model.

Table 1.5 compares the business cycle statistics of baseline model with and without TFP shocks. First, the model with both shocks improve the match of aggregate volatility. Second, the model with both shocks brings the persistence closer to the data, which can be viewed as a weighted average of the baseline model with only confidence shock and the RBC model with only TFP shock. Figure 1.8 displays the fraction of volatility that can be attributed to confidence shocks in a variance decomposition analysis. It can be seen that after adding TFP shocks, the confidence shock still plays an important role in accounting for aggregate fluctuations, especially in the short to medium term.

Comparison with the Heterogeneous-Prior Formulation To compare with the heterogeneous-prior formulation, we use the baseline model in [17].[8] In this formulation, the persistence and variance of output are independent of information frictions. With the same confidence shock process, the persistence of various aggregate variables is sympathetically higher than the one in our common-prior formulation. Unlike our common-prior model in which there is an upper bound for the variance of output, one

[8] The details of the model specification can be found on our online appendix.

can obtain any variance of output with heterogeneous-prior formulation. To capture the effects of information frictions, [17] choose a relatively low persistence of the confidence shock. Our paper implements this notion by solving the common-prior model and examining whether the forecast errors in the model match the micro data.

1.5 Conclusion

In this paper, we study a business cycle model in which aggregate fluctuations are driven by confidence shocks. Because of asymmetric information, higher order beliefs are crucial in shaping equilibrium outcomes, and the infinite regress problem arises. We use our method developed in [8] to solve the infinite regress problem without approximation. It turns out that the persistence aggregate output is increasing in the degree of information frictions and strategic complementarity. Also, there is an upper bound for the volatility of output that can be obtained by confidence shocks. In our quantitative model, we calibrate the parameters that determine information frictions to match micro-level data. We find that our model with confidence shocks can match a number of salient features of business cycles. However, the confidence shock itself does not generate enough persistence of aggregate variables. These results imply that confidence shocks or other non-fundamental shocks could play an important role in accounting for business cycles, but more persistent real shocks are also necessary. We believe the method and the insights discussed in this paper can also be applied to a broad class of models with higher order beliefs.

Figure 1.7: Impulse Response to the Confidence Shock

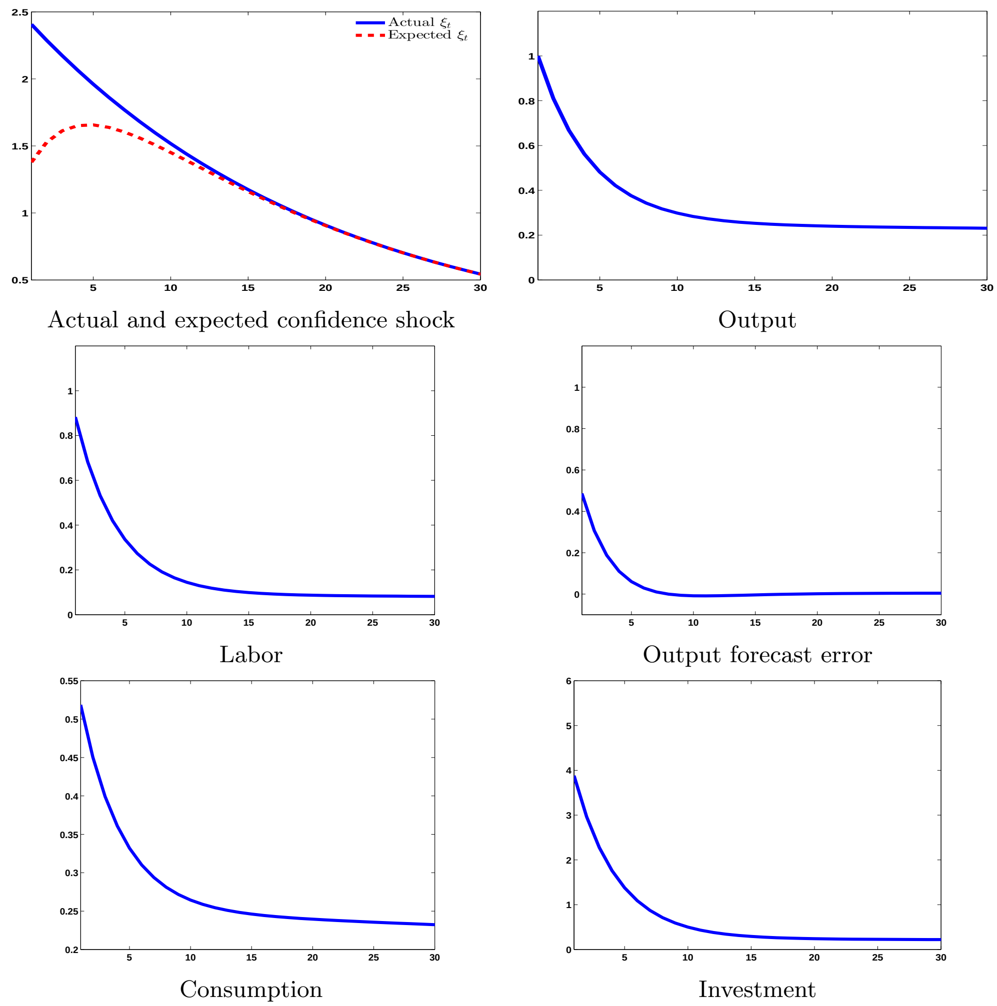


Table 1.5: Business Cycle Statistics

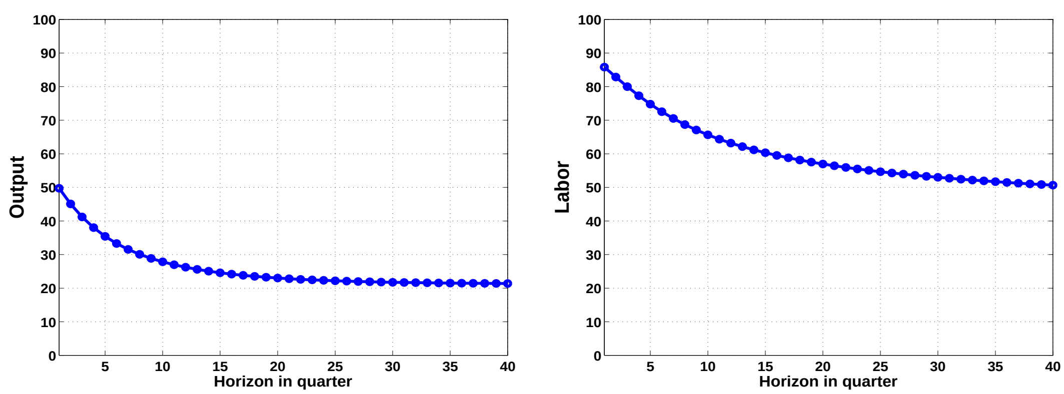
	Data	Baseline		RBC	Hetero-prior
		ξ shock	ξ, TFP shcok	TFP shock	ξ shock
Std. deviation					
Y	1.54	1.19	1.63	1.16	1.54
C	1.26	0.62	1.04	0.62	0.96
I	6.87	4.72	5.89	4.49	4.24
N	1.86	1.06	1.04	0.41	2.20
Z	1.24	0.47	1.03	0.88	—
LW_1	4.87	2.39	2.19	—	2.71
LW_2	3.96	1.79	1.59	—	1.76
Corr with Y					
Y	1.00	1.00	1.00	1.00	1.00
C	0.88	0.99	0.91	0.99	0.99
I	0.91	0.99	0.89	0.99	0.99
N	0.86	1.00	0.90	1.00	1.00
Z	0.77	1.00	0.95	1.00	—
LW_1	-0.84	-0.99	-0.93	—	-1.00
LW_2	-0.75	-0.99	-0.64	—	-1.00
Autocorrelation					
Y	0.87	0.42	0.64	0.74	0.70
C	0.88	0.45	0.70	0.75	0.71
I	0.83	0.42	0.56	0.73	0.69
N	0.92	0.42	0.56	0.74	0.69
Z	0.81	0.42	0.67	0.73	—
LW_1	0.92	0.41	0.58	—	0.70
LW_2	0.91	0.41	0.53	—	0.69

Note: All variables are HP-filtered logarithms of the original series. The standard deviations are multiplied by 100. LW_1 is the labor wage defined by the standard separable utility function $U(C,N) = \log C - \frac{N^{1+\gamma}}{1+\gamma}$, and $LW_1 = \log(\frac{Y}{N}) - \log(CN^{\gamma})$. LW_2 is the labor wage defined by the GHH utility function in this paper, and $LW_2 = \log(\frac{Y}{N}) - \log(N^{\gamma})$.

Table 1.6: Estimated Parameters in the Baseline Model with Confidence and TFP shock

	Prior			Posterior		
	Distribution	Mean	Std	Mode	Mean	90% HPD
σ_η	Inv Gamma	0.30	3.00	0.15	0.14	[0.12, 0.17]
σ_u	Inv Gamma	0.30	3.00	0.38	0.38	[0.32, 0.43]
ρ_z	Beta	0.50	0.20	0.25	0.93	[0.89, 0.96]
σ_ς	Inv Gamma	0.10	2.00	0.34	0.37	[0.31, 0.38]

Figure 1.8: Variance Decomposition: Fraction Due to Confidence Shocks



Chapter 2

Rational Expectations Models with Higher Order Beliefs

2.1 Introduction

In many economic models with information frictions, an agent's payoff depends on his own actions, the actions of others, and some unknown economic fundamentals. Rational behaviors not only depend on an agent's beliefs on economic fundamentals, but also depend on higher order beliefs, that is, agents' beliefs of others' beliefs, agents' beliefs of others' beliefs of others' beliefs, and so on. If the economic fundamentals are persistent over time and hence the past information is worth keeping track of, forecasting all the higher order beliefs would require an infinite number of priors of them, which would amount to an infinite number of state variables. This type of problem is known as the *infinite regress problem*, and has been explored by a large number of works.[1]

The difficulty of solving models with higher order beliefs lies in the fact that inferring others' action requires the functional form of the policy rule in the first place, but the policy rule is the solution to the inference problem. As argued in [7], if an agent assumes that other agents keep track of n state variables, he in turn needs to keep track of $n+1$ state variables (the prior of the economic fundamental and the n priors of others' state variables). Therefore, the equilibrium policy rule does not permit a finite-state

[1] A partial list of these works includes [33], [7], [34], [35], [9], [1], [5], [3], [13], [16], and so on.

representation. In terms of higher order beliefs, to predict $k-$th order belief requires at least k state variables, and to predict all the higher order beliefs requires infinite state variables. In light of these considerations, it is generally believed that an infinite number of state variables are needed to solve this type of model.

In this paper, we pursue the following question. With higher order beliefs, is it really impossible to find a small set of state variables that are sufficient statistics for agents to make the optimal inference? If possible, how do we find these state variables and what are the laws of motion for these variables? If it does require an infinite number of state variables, how do we approximate the true solution with a finite number of state variables?

Our first main result is that given a linear rational expectations model, when observed signals follow an ARMA process, the equilibrium policy rule always allows a finite-state representation. To make sure signals follow an ARMA process, we start from the case in which the information process is given exogenously. Like in standard problems with symmetric information, solving for the equilibrium requires finding the fixed point in the functional space. Unlike in standard models, when higher order beliefs are involved, it is difficult to figure out the sufficient state variables in the first place. Given this difficulty, we start from the state space that is spanned by the entire history of signals. This implies that solving for the equilibrium requires solving for a lag polynomial with an infinite number of coefficients. Our work is based on [36] and [9]. The idea is to transform the problem which solves for a lag polynomial into a simpler problem which solves for an analytical function, labelled as the frequency-domain method. When signals follow an ARMA process, we prove that the equilibrium policy rule, the lag polynomial, is also of the ARMA form. Therefore, we can find a finite-state representation for the equilibrium policy rule.

We extend the work of [9] and others in two important ways. First, we do not restrict the number of signals to being equal to the number of shocks. A necessary step in the inference problem with infinite sample is to find the Wold (fundamental) representation for the signal process. Previous works rely on the Blaschke matrices to find the fundamental representation, which require that the number of signals equals the number of shocks.[2] We adopt a different approach for finding the Wold representation. We show that one can first convert the signal process into its state-space, and then use the innovation representation and factorization identity to solve for the Wold representation conveniently. This procedure works for any information structure that follows an ARMA process: it is not restricted by the number of signals or the number of shocks. The restriction that there has to be the same number of signals as shocks is quite limited. In general signal extraction problems, there are more shocks than signals, as criticized in [25]. This restriction is indeed violated in many applications, such as [1], [3] and [4]. When this restriction is actually satisfied, agents often learn 'too much', in the sense that the prediction error is not long-lasting, because there are insufficient numbers of noisy shocks to really confuse them, unless assuming a confounding shock process in the first place.[3] In both [9] and [10], agents can learn the true state of the economy after one period. When there are more shocks than signals, agents never fully learn the true state of the economy and the prediction error is typically persistent. As a result, the model economy features more relevant and richer dynamics.

Secondly, we allow agents to solve a general signal extraction problem. The majority of existing literature that applies the frequency-domain technique only studies a pure forecasting problem. That is, only future values of signals are pay-off relevant. To forecast future signals, one can simply use the Hansen-Sargent formula. In the examples presented in this paper, agents need to solve a generic signal extraction problem conditional on infinite observables. The Hansen-Sargent formula does not apply in these environments. Instead, we apply the Wiener-Hopf prediction formula, which is well suited for these types of problems and includes Hansen-Sargent formula as a special case. Applying the Wiener-Hopf prediction formula in the univariate case has been discussed extensively in [40]. In this paper, we extend the application to multivariate case.

We illustrate our method in various applications. We first consider a two-player model in which asymmetric information and strategic complementarity make higher order beliefs relevant.[4] We discuss the case in which agents only receive private

[2] See [16], [37] and [10] for example. [38] and [39] solved a special signal process with more shocks than signals.

[3] In [16], they assume a non-invertible shock process.

[4] The two-player game should not be taken literally. The two players can be an individual agent

signals regarding the economic fundamental (similar to [1]), and the case in which agents also receive a public signal regarding the economic fundamental (similar to [3]). In both cases, we obtain a sharp analytic solution which can be characterized by finite state variables. The intuition for the finite-state representation is that agents do not directly care about each of the higher order beliefs, but they only care about a specific linear combination of all the higher order beliefs. The latter indeed follows an ARMA process. We also consider a model where agents are randomly matched, an extension of [4] with persistent shocks. In this case, an agent randomly interacts with a different agent every period, and needs to form higher order beliefs on each of them. Even though it complicates the inference problem, our method is general enough to solve these models as well.

The above first result is for the cases where agents solve their inference problem given an exogenous ARMA signal process. We label them as problems with exogenous information. We also explore cases when agents observe signals that contain information which is endogenously determined in the equilibrium. We label them as problems with endogenous information. The equilibrium with endogenous information imposes an additional cross-equation restriction, in the sense that the perceived law of motion has to be consistent with the realized law of motion. The endogenous variable that appears in the signal has an information role as well, similar to the concept of information equilibrium defined in [16].

Our second main result is that we prove that in our model with endogenous information, the equilibrium cannot be represented by finite state variables.[5] The endogenous variable that plays an information role follows an infinite order process. This result is somewhat surprising given that the exogenous driving force of the economy is very simple. It should be noted that it is not because of the infinite regress problem that agents have to keep track of infinite state variables. For each individual, they still take the signal process as exogenously given, even though the signals contain an equilibrium object. From our first main result, once the endogenous variable follows an ARMA process, the individual policy rule will also follow an ARMA process and permit a finite state variable representation. If the endogenous variable does not follow

and the whole economy.

[5] [33] proved a similar impossibility theory.

an ARMA process, the signal received by agents cannot follow an ARMA process. Note that in [9] and other papers where the number of signals is the same as the number of shocks, the equilibrium permits a finite-state representation even with endogenous information. When we allow for this more general information process, this result does not hold any more.

This finding is interesting from a theoretical point of view, but it also implies that finding the exact solution is no longer possible. To solve the problem with endogenous information, we approximate the law of motion of the endogenous variable that shows up in signals by an ARMA process. We can prove that as the order of the ARMA process increases, it can approximate the true solution arbitrarily well, and we also find that a relatively low order ARMA process can give accurate approximation. Note that this ARMA approximation method is different from [35] and others in an important way. Even though we approximate the law of motion of the endogenous variable, each individual still faces the infinite regress problem. The prediction problem still cannot be solved by the Kalman filter. Using our method, each individual's policy rule is solved exactly.

To demonstrate that our method can be applied in an empirically relevant environment, we solve a full-blown business cycle model in a companion paper ([41]). In this paper, the confidence shock is the sole driving force of business cycles, and agents face a complicated learning problem, i.e., they need to forecast the forecasts of others. Different from the applications solved in this paper, agents also make dynamic decisions (the investment decision), and the infinite regress problem becomes much more involved. We show that there is a hump-shaped relationship between the variance of output and the variance of the confidence shock, and under our favored calibration of information frictions, the model with confidence shocks can account for a number of salient features of business cycles.

Related literature Our paper is closely related to the literature that attempts to solve the infinite regress problem. Broadly speaking, there are two approaches to solving the infinite regress problem. The first approach is to short-circuit the infinite regress problem by modifying the original problems. For example, by assuming that information becomes public after certain periods, the relevant state space is finite and one can use the Kalman filter. A partial list of literature that employs this method includes [7], [13], [5], [14]. This assumption is unsatisfying from a modeling perspective, and it is proved by [38], [9] and [42] that the approximate solution can be very different from the true solution. Another type of approximation is developed by [15] and [25]. The idea is that only a finite order of higher order beliefs matter for the equilibrium, based on the observation that the effects of higher order beliefs diminish as the order increases. This method provides important insights into the nature of the higher order beliefs, but as shown in our examples, this method can be difficult to implement when the degree of strategic complementarity is strong, or when the model is complicated to express the policy rule in terms of higher order beliefs. [35] approximated the equilibrium via the ARMA process. The forecasting problem is transformed into fitting vector ARMA models, which is particularly useful when agents do not need to solve a pure forecasting problem.

The second approach is to solve the infinite regress problem exactly without approximation. [9] first uses the frequency-domain method to solve the [7] original problem and found that agents actually share the same belief and there is no infinite regress problem. [38], [16], and [37] apply the frequency-domain method to study various asset pricing models proposed by [43] and [34]. [10] applies this method to study the effects of noises on business cycles. These papers assume that the number of shocks equals the number of signals, a restriction that prevents this method from being applied in more general settings. Furthermore, in previous literature, agents solve a pure forecasting problem most of the time. This paper eliminates these restrictions and a much broader class of models can be solved by our method.

Our applications in this paper complement the literature on macroeconomics with higher order beliefs. We obtain analytical solutions for models closely related to [1], [3], and [4]. We believe our method is also useful in solving models similar to [5], [13], [44] and others. In our companion paper ([41]), we study a business cycle model driven by confidence shocks. We characterize how information frictions affect the persistence and variance of output, and show that the confidence shock could be an important factor in explaining business cycles.

The rest of the paper is organized as follows. Section 2.2 sets up a two-player model to introduce higher order beliefs and the infinite regress problem. Section 2.3 presents the main theorems. We show how to jointly use the Kalman filter and the Wiener-Hopf prediction formula to form the optimal expectation with infinite observables. We also show how to obtain a finite-state representation for a rational expectations model with higher order beliefs. Section 2.4 solves the two-player game with and without public signals. Section 2.5 explores the case in which the signals contain an endogenous variable. We prove that the equilibrium policy rule does not have a finite-state representation in this environment. Section 2.6 considers the case where an agent has to form higher order beliefs of many different agents. Section 2.7 discusses an application of the method in a quantitative business cycle model. Section 2.8 concludes.

2.2 A Two-Player Model with Infinite Regress Problem

In this section, we present a simple two-player model with the infinite regress problem. This model naturally assigns an important role to infinite higher order beliefs, and numerous variations of it have been used in the literature.

2.2.1 Model setup

Consider a game between two agents i and j. Time is discrete and lasts forever. In period t, agents' payoff depends on a common persistent economic fundamental ξ_t. The payoff also depends on the action of the other agent and we consider the case with strategic complementarity. However, information frictions prevent agents from perfectly observing ξ_t or the action of the other agent.

We assume that the best response of agent i, denoted by y_{it}, has to satisfy

$$y_{it} = \mathbb{E}[\xi_t|\Omega_{it}] + \alpha\mathbb{E}[y_{jt}|\Omega_{it}], [6] \tag{2.1}$$

where $\alpha \in (0,1)$ determines the strength of strategic complementarity and Ω_{it} denotes the information set of agent i at time t. Agent j follows the same strategy. Note that agents make a purely static decision every period, and the link across different periods is only through the information set. There are various micro-foundations that lead to this specification, such as [1] and [3]. For now we only focus on this abstract form and

[6] Here, $y_{it} \in \mathbb{R}$ and $y_{jt} \in \mathbb{R}$. The operator $\mathbb{E}$ denotes the linear projection on the information set.

discuss its general properties. The information structure of the model is specified as follows.

Signal process We assume that ξ_t follows a covariance stationary ARMA (p, q) process

$$\xi_t = \frac{\Pi_{k=1}^{q}(1+\theta_k L)}{\Pi_{k=1}^{p}(1-\rho_k L)}\,\eta_t, \tag{2.2}$$

where $\eta_t \sim N(0, \sigma_\eta)$. As opposed to observing ξ_t directly, agents receive two signals that are related to ξ_t. These two signals are simply the sum of ξ_t and some idiosyncratic noises.

$$x_{it}^1 = \xi_t + \epsilon_{it}, \tag{2.3}$$

$$x_{it}^2 = \xi_t + u_{it}, \tag{2.4}$$

where $\epsilon_{it} \sim N(0, \sigma_\epsilon^2)$ and $u_{it} \sim N(0, \sigma_u^2)$. Note that the idiosyncratic noises are indexed by i. More compactly, the signal process can be expressed as

$$x_{it} \equiv \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{\Pi_{k=1}^{q}(1+\theta_k L)}{\Pi_{k=1}^{p}(1-\rho_k L)} \\ 0 & 1 & \frac{\Pi_{k=1}^{q}(1+\theta_k L)}{\Pi_{k=1}^{p}(1-\rho_k L)} \end{bmatrix} \begin{bmatrix} \epsilon_{it} \\ u_{it} \\ \eta_t \end{bmatrix} \equiv M(L) s_{it}, \tag{2.5}$$

The information set of agent i at time t contains all the signals he has received up to time t

$$\Omega_{it} = \left\{ x_{it}^1,\ x_{it}^2,\ x_{it-1}^1,\ x_{it-1}^2,\ x_{it-2}^1,\ x_{it-2}^2,\ \ldots \right\}. \tag{2.6}$$

Agent j receives signals of ξ_t, but are corrupted by his idiosyncratic noises ϵ_{jt} and u_{jt}. As a result, these two agents do not share the same information set.

To simplify notation, we will use $\mathbb{E}_{it}[\,\cdot\,]$ to denote $\mathbb{E}[\,\cdot \mid \Omega_{it}]$ from now on.

Remark Several remarks about the model should be made here before we move on.

1. A wide range of models can be interpreted as the two-player model. If we assume that there are a continuum of agents in the economy, and each individual agent i interacts with the economy average $y_t = \int y_{jt}$, the model becomes

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}[y_t]. \tag{2.7}$$

As we show in Section 2.4, the solution to this model remains the same as the original model (2.1). What matters is whether to infer the action of a fixed agent (Section 2.4), or to infer the action of a random agent that changes over time (Section 2.6).

2. To introduce the infinite regress problem, it will be sufficient if agents only receive one of the two signals. The assumption that agents receive multiple signals is to demonstrate that our method can manage multivariate systems.

3. The information structure we have specified in equation (2.5) is a very special one. We can relax this assumption to allow any finite number of signals that follows any finite ARMA process. The structure we adopt here should not be taken in a narrow way. For example, we allow some of the signals to be shared by all agents (Section 2.4.2), and allow some of the signals to contain endogenous information (Section 2.5).

2.2.2 Higher order beliefs

The best response of agent i is given by equation (2.1), and the same rule applies to agent j,

$$y_{jt} = \mathbb{E}_{jt}[\xi_t] + \alpha \mathbb{E}_{jt}[y_{it}]. \tag{2.8}$$

We can repeatedly substitute equation (2.8) into equation (2.1), and vice versa, which leads to

$$\begin{aligned} y_{it} &= \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}[y_{jt}] \\ &= \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}\left[\mathbb{E}_{jt}[\xi_t] + \alpha \mathbb{E}_{jt}[y_{it}]\right] \\ &= \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}\mathbb{E}_{jt}[\xi_t] + \alpha^2 \mathbb{E}_{it}\mathbb{E}_{jt}[y_{it}] \\ &= \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}\mathbb{E}_{jt}[\xi_t] + \alpha^2 \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}[\xi_t] + \alpha^3 \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}[y_{jt}] \\ &\;\;\vdots \\ &= \sum_{k=0}^{\infty} \alpha^k \, \mathbb{E}_{it}^{k+1}[\xi_t], \end{aligned} \tag{2.9}$$

where $\mathbb{E}_{it}^k[\xi_t]$ stands for k-th order belief. These higher order beliefs are defined recursively as follows

$$\begin{aligned}
\mathbb{E}_{it}^1[\xi_t] &= \mathbb{E}_{it}[\xi_t] \\
\mathbb{E}_{it}^2[\xi_t] &= \mathbb{E}_{it}\mathbb{E}_{jt}[\xi_t] \\
\mathbb{E}_{it}^k[\xi_t] &= \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}^{k-2}[\xi_t], \text{ for } k = 3, 5, 7, \ldots \\
\mathbb{E}_{it}^k[\xi_t] &= \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}^{k-2}[\xi_t], \text{ for } k = 4, 6, 8, \ldots
\end{aligned}$$

Crucially, agents have heterogeneous information sets, and the law of iterated expectations does not apply. Hence, the optimal action y_{it} depends on all the higher order beliefs. Mathematically, the means of all these higher order beliefs can be calculated by the standard Kalman filter, but there are an infinite number of such objects to be calculated. One may think that if a certain pattern of these higher order beliefs is found, these beliefs may be summarized in a compact way. However, this approach does not work in general, due to a growing complexity with the order of beliefs.

Similarly, if we consider model (2.7), successive substitution leads to

$$y_{it} = \sum_{k=0}^{\infty} \alpha^k \, \mathbb{E}_{it}\overline{\mathbb{E}}_t^k[\xi_t]. \tag{2.10}$$

Here, as opposed to inferring agent j's beliefs, the higher order beliefs $\overline{\mathbb{E}}_t^k[\xi_t]$ are about the economy average expectations of ξ_t, defined recursively by

$$\begin{aligned}
\overline{\mathbb{E}}_t^0[\xi_t] &= \xi_t \\
\overline{\mathbb{E}}_t^1[\xi_t] &= \int \mathbb{E}_{jt}[\xi_t] \\
\overline{\mathbb{E}}_t^k[\xi_t] &= \int \mathbb{E}_{jt}\overline{\mathbb{E}}_t^{k-1}[\xi_t].
\end{aligned}$$

In both cases, it is apparent that agents' optimal response is related to infinite higher order beliefs. Forecasting all of these higher order beliefs requires an infinite number of priors of these beliefs, and these priors are functions of the entire history of agents' signals. As a result, it is generally believed that the policy rule has to include the entire history of signals as state variables.

2.2.3 Equilibrium

Recall that the information set of agent i is $\Omega_{it} = x_i^t$. The linear policy rule of agent i belongs to the space spanned by square-summable linear combinations of current and past realizations of x_{it}. We use $\mathcal{H}_t^x$ to denote this space. We assume that the policy rule takes the following form

$$y_{it} = \sum_{k=0}^{\infty} h_{1k} x_{it-k}^1 + \sum_{k=0}^{\infty} h_{2k} x_{it-k}^2, \tag{2.11}$$

and it is obvious that $y_{it} \in \mathcal{H}_t^x$. In standard models without higher order beliefs, the policy rule still depends on the entire history of signals, but a finite number of state variables can be easily found to effectively summarize the past information. In contrast, due to the infinite higher order beliefs, there is no way to figure out whether there exists a finite number of state variables in the first place (even though later on we prove that this is indeed the case), and we have to assume it is necessary to keep track of the entire history of signals.

More compactly, we use lag polynomials to denote the infinite sum

$$y_{it} = h_1(L) x_{it}^1 + h_2(L) x_{it}^2, \tag{2.12}$$

with $h_1(L) = \sum_{k=0}^{\infty} h_{1k} L^k$ and $h_2(L) = \sum_{k=0}^{\infty} h_{2k} L^k$.

To make sure that y_{it} is co-variance stationary, the infinite sequences $\{h_{1\tau}\}_{\tau=0}^{\infty}$ and $\{h_{2\tau}\}_{\tau=0}^{\infty}$ have to be in the square-summable space ℓ^2.[7] From now on, if an infinite sequence $\phi = \{\phi_k\}_{k=0}^{\infty} \in \ell^2$, then we denote $\phi(L) = \sum_{k=0}^{\infty} \phi_k L^k$ as its corresponding lag polynomial. The definition of the equilibrium is straightforward.

Definition 2.2.1 (Signal form)**.** *Given the signal process (2.5), the equilibrium of model (2.1) is a policy rule $h = \{h_1, h_2\} \in \ell^2 \times \ell^2$, such that*

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\, \mathbb{E}_{it}[y_{jt}],$$

[7] [45], [9] assume that the policy rule ϕ belong to β-summable space, i.e., $\sum_{k=0}^{\infty} \beta^k \phi_k^2 < \infty$. This is a less strict requirement than original ℓ^2 assumption, which arises naturally in linear-quadratic type models. However, it is less obvious whether this relaxation is valid or not in our model setting, and we will work with the original ℓ^2 space in this paper.

where

$$y_{it} = h_1(L)x^1_{it} + h_2(L)x^2_{it},$$
$$y_{jt} = h_1(L)x^1_{jt} + h_2(L)x^2_{jt}.$$

Since the signals $\{x_{it}\}$ are ultimately generated by the underlying shocks $\{s_{it}\}$, y_{it} also lies in the space spanned by the square-summable linear combinations of current and past shocks, denoted by $\mathcal{H}^s_t$. It should be clear that $\mathcal{H}^x_t \subset \mathcal{H}^s_t$. We say that the equilibrium is of signal form if the equilibrium policy is written in terms of signals, and the equilibrium is of innovation form if it is written in terms of the underlying shocks. The equilibrium in innovation form is defined as follows

Definition 2.2.2 (Innovation form)**.** *Given the signal process (2.5), the equilibrium of model (2.1) is a policy rule* $\phi = \{\phi_1, \phi_2, \phi_3\} \in \ell^2 \times \ell^2 \times \ell^2$*, such that*

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}[y_{jt}],$$

where

$$y_{it} = \phi_1(L)\epsilon_{it} + \phi_2(L)u_{it} + \phi_3(L)\eta_t,$$
$$y_{jt} = \phi_1(L)\epsilon_{jt} + \phi_2(L)u_{jt} + \phi_3(L)\eta_t.$$

In the literature, when solving the infinite regress problem in the frequency domain, the innovation form is exclusively used. The advantage of working with innovation form is that all the objects are expressed in terms of the underlying shocks and it is convenient to discuss its statistical properties. However, from an economic perspective, it is more natural to think of the policy rule in terms of signals, because agents do not observe those shocks directly.[8] In Theorem 6, we show that there is a one-to-one mapping between the equilibrium in signal form and in innovation form.

In terms of the existence and uniqueness of the equilibrium, we have the following result.

Proposition 2.2.1. *Assume that the signals follow a co-variance stationary process. If* $\alpha \in (0, 1)$*, then there exists a unique equilibrium of model (2.1).*

[8] [9] claims that the limited-information equilibrium does not exist in the space spanned by signals but only exists in the space spanned by the innovations. We find this conclusion questionable.

Proof. See Appendix A.2.1 for proof. □

The core of the proof is to show that the equilibrium is a fixed point of a contraction mapping. On one hand, to prove this proposition, we only require that the signals follow a co-variance stationary process, but not necessarily a finite ARMA process. On the other hand, this proposition does not imply whether the policy rule in equilibrium permits a finite-state representation or not. In principle, it could be that agents do need to keep track of the entire history of observables. Next theorem, however, shows that the equilibrium indeed has a finite-state representation when the signals follow a finite ARMA process.

2.2.4 Finite-state representation

Theorem 3. *Assume that (1) the exogenous variable ξ_t follows*

$$\xi_t = \frac{\Pi_{k=1}^{q}(1+\theta_k L)}{\Pi_{k=1}^{p}(1-\rho_k L)}\eta_t.$$

(2) The signals follow the following co-variance stationary process (2.5)

$$x_{it} = \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{\Pi_{k=1}^{q}(1+\theta_k L)}{\Pi_{k=1}^{p}(1-\rho_k L)} \\ & & \\ 0 & 1 & \frac{\Pi_{k=1}^{q}(1+\theta_k L)}{\Pi_{k=1}^{p}(1-\rho_k L)} \end{bmatrix} \begin{bmatrix} \epsilon_{it} \\ u_{it} \\ \eta_t \end{bmatrix}.$$

(3) The structural parameter $\alpha \in (0,1)$.

Then there exists a unique solution $y_{it} = h_1(L)x_{it}^1 + h_2(L)x_{it}^2$ satisfies model (2.1)

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\mathbb{E}_{it}[y_{jt}].$$

The equilibrium policy rule $h_1(L)$ and $h_2(L)$ have the following properties

1. *Both $h_1(L)$ and $h_2(L)$ have a finite ARMA representation*

$$y_{it} = \begin{bmatrix} h_1(L) & h_2(L) \end{bmatrix} \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} = \begin{bmatrix} \tau_1 \frac{\Pi_{k=1}^{n}(1+\zeta_k^1 L)}{\Pi_{k=1}^{m}(1-\vartheta_k L)} & \tau_2 \frac{\Pi_{k=1}^{n}(1+\zeta_k^2 L)}{\Pi_{k=1}^{m}(1-\vartheta_k L)} \end{bmatrix} \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \tag{2.13}$$

where the order of the ARMA process m and n, the coefficients τ_1, τ_2, $\{\vartheta_k\}_{k=1}^m$, $\{\zeta_k^1\}_{k=1}^n$, and $\{\zeta_k^2\}_{k=1}^n$ are all functions of the structural parameter α and the parameters that determine the signal process.

2. *Let* $r = max\ \{m, n\}$. *Given a particular signal realization* $\{x_{it}\}_{t=-\infty}^{-1}$, *there exists* r *state variables* $z_{it} = [z_{it}^1, z_{it}^2, \ldots, z_{it}^r]'$, *such that the policy rule in (2.13) has the following finite-state representation*

$$y_{it} = \Gamma_x\ x_{it} + \Gamma_z\ z_{it}, \tag{2.14}$$

with the law of motion of z_{it}

$$z_{it+1} = \Upsilon_x x_{it} + \Upsilon_z z_{it} \tag{2.15}$$

The initial state z_{i0} *is given by*

$$z_{i0} = (I_r - \Upsilon_z L)^{-1}\,\Upsilon_x x_{i-1} \tag{2.16}$$

The constant matrices $\Gamma_x, \Gamma_z, \Upsilon_x$, *and* Υ_z *are all functions of* τ_1, τ_2, $\{\vartheta_k\}_{k=1}^m$, *and* $\{\zeta_k^1\}_{k=1}^n$ *in equation (2.13).*

Proof. The proof of this theorem is a subset of the proof of Theorem 4, and the exact form of equation (2.13) can be derived by Theorem 5 in the next section. □

The first part of this theorem establishes that the equilibrium policy rule follows a finite ARMA process in terms of the signals. The second part of this theorem states that the policy rule has a finite-state representation, which is a natural result of the first part. Therefore, there indeed exists a small set of state variables that are sufficient for agents' inference problem. This theorem also implies that the infinite sum of higher order beliefs in equation (2.9) follows a finite ARMA process, even though $\mathbb{E}_{it}^k[\xi_t]$ follows an infinite ARMA process as k approaches to infinity.

To solve for the equilibrium policy rules $h_1(L)$ and $h_2(L)$, the difficulty lies in how to solve the inference problem

$$\mathbb{E}_{it}[y_{jt}] = \mathbb{E}_{it}[h_1(L)x_{jt}^1 + h_2(L)x_{jt}^2],$$

in which the variable to be predicted is with infinite states. The Kalman filter requires the predicted variable to have finite states, and therefore it is inapplicable for this type of the problem. In contrast, the Wiener filter can solve the inference problem that is conditional on infinite observables, and it allows the predicted variable to have infinite states (the details of these two filters are discussed in the next section). A key step to employ the Wiener is to find the Wold representation of the signal process, which is not provided by the Wiener filter itself but can be obtained by the Kalman filter. Therefore, a joint use of the Kalman filter and the Wiener filter solves this inference problem. The lack of an efficient way to find the Wold representation is exactly what prevents others from solving models with higher order beliefs, and we show that the Kalman filter can achieve this goal with ease. After solving $\mathbb{E}_{it}[y_{jt}]$, it turns out that $h_1(L)$ and $h_2(L)$ are of finite ARMA type, and it allows a finite-state representation.

The model we considered in this section is a very special one in the following sense: (1) there is only one choice variable y_{it}; (2) there is no endogenous state variables, such as capital; (3) there is no need to forecast variables in the future; (4) the signal process is very special. These limitations make model (2.1) only theoretically interesting, and far from empirically relevant. In the following section, we eliminate these restrictions, and extend Theorem 3 to a much more general statement.

2.3 Methodology: General Linear Rational Expectations Models

In this section, we develop the method that solves the general rational expectations models with higher order beliefs. We first lay out the structure of the model and the signal process, and state the main theorem that the equilibrium policies admit finite-state representation. We then show how to prove this theorem in steps. The key part is to use the Wold representation and the Wiener filter to solve the general signal extraction problem.

2.3.1 General rational expectations models

Now we move to the general form of the linear system. The input of the model includes two parts: the first part is the signal process; the second part is the linear system which corresponds to the equilibrium conditions that various kinds of variables need to satisfy. There are three kinds of variables involved here: choice variables, choice variables chosen by others, and exogenous variables.

Signal process Assume that the signals observed by agents follow a finite ARMA process,

$$x_t = \begin{bmatrix} x_t^1 \\ \vdots \\ x_t^n \end{bmatrix} = \begin{bmatrix} \frac{a_{11}(L)}{b_{11}(L)} & \dots & \frac{a_{1m}(L)}{b_{1m}(L)} \\ \vdots & \ddots & \vdots \\ \frac{a_{n1}(L)}{b_{n1}(L)} & \dots & \frac{a_{nm}(L)}{b_{nm}(L)} \end{bmatrix} \begin{bmatrix} s_{1t} \\ \vdots \\ s_{mt} \end{bmatrix} = M(L)s_t, \tag{2.17}$$

where the signal x_t is a stochastic $n \times 1$ vector and the shock s_t is a stochastic $m \times 1$ vector. We allow m to be different from n. We normalize the co-variance matrix of s_t to be an identity matrix. In each element of $M(L)$, $a_{ij}(L)$ and $b_{ij}(L)$ are finite order polynomials in the lag operator L. Particularly,

$$a_{ij}(L) = \sum_{k=0}^{q_{ij}} \alpha_{ijk} L^k,$$
$$b_{ij}(L) = \sum_{k=0}^{p_{ij}} \beta_{ijk} L^k,$$

and we normalize $\beta_{ij0} = 1$. The information set is $\Omega_t = x^t = \{x_t, x_{t-1}, x_{t-2}, \dots\}$.

Choice variable We assume there are d choice variables, which are functions of the signals:

$$y_t = \begin{bmatrix} y_{1t} \\ \vdots \\ y_{dt} \end{bmatrix} = h(L)x_t = \begin{bmatrix} h_{11}(L) & \dots & h_{1n}(L) \\ \vdots & \dots & \vdots \\ h_{d1}(L) & \dots & h_{dn}(L) \end{bmatrix} \begin{bmatrix} x_{1t} \\ \vdots \\ x_{nt} \end{bmatrix} = h(L)M(L)s_t. \tag{2.18}$$

$h(L)$ is the equilibrium policy rule we want to solve. We assume that each element in $h(L)$ has an infinite MA representation. We do not impose that $h(L)$ admits a finite ARMA representation in the first place (even though we prove this is indeed the case later). Because these choice variables only depend on signals up to t, $h_{ij}(L)$ cannot contain any negative powers in L. To write it more compactly for future derivation, define

$$\phi(L) \equiv \begin{bmatrix} h_{11}(L) & \dots & h_{1n}(L) & \dots & h_{d1}(L) & \dots & h_{dn}(L) \end{bmatrix}. \tag{2.19}$$

$\phi(L)$ effectively collapse all the lag polynomials to be solved into a vector, the dimension of which is denoted as $w \equiv dn$. Reversely, the elements of y_t can be expressed in terms of $\phi(L)$ as

$$
\begin{aligned}
y_{it} &= \phi(L)A_i x_t && (2.20)\\
&= \phi(L)A_i M(L)s_t && (2.21)
\end{aligned}
$$

where A_i is the constant matrix that selects $[h_{i1}\dots h_{in}]$ from $\phi(L)$. Later we will use $h(L)$ and $\phi(L)$ interchangeably.

Endogenous variables related to other agents' actions Crucially, the optimal policy may depend on other agents' actions or depend on some aggregate endogenous variables. These variables cannot be observed, but matter for agents' payoff. Assume there are d_f such endogenous variables, denoted by $f_t = [f_{it}, \dots, f_{d_f t}]'$ denote these endogenous variables. They are related to the policy rule $\phi(L)$ and the underlying shocks s_t in the following way

$$
f_{it} = \phi(L)f^i(L)s_t = \phi(L)\begin{bmatrix} f^i_{11}(L) & \dots & f^i_{1m}(L) \\ \vdots & \dots & \vdots \\ f^i_{w1}(L) & \dots & f^i_{wm}(L) \end{bmatrix}\begin{bmatrix} s_{1t} \\ \vdots \\ s_{mt} \end{bmatrix} \qquad (2.22)
$$

Here, each $f^i(L)$ is a $w \times m$ matrix in the lag operator L. We assume that all the elements in $f^i(L)$ are finite rational functions in L and do not contain negative powers of L in expansion (others' action cannot be a function of future shocks either).

Note that actions of others may also depend on shocks other than $\{s_{1t}, \dots, s_{mt}\}$. However, these shocks are uncorrelated with the shocks $\{s_{1t}, \dots, s_{mt}\}$ that drive $\{x_t\}$, and the best forecasts of those shocks conditional on $\{x_t\}$ are zero. As a result, what is relevant for agents are the parts that are correlated with $\{s_{1t}, \dots, s_{mt}\}$.

Exogenous variables Generally, the optimal policy depends on the evolution of some exogenous variables. We assume there are d_g such variables, denoted by $g_t = [g_{it}, \dots, g_{d_g t}]'$

$$
g_t = g(L)s_t = \begin{bmatrix} g_{11}(L) & \dots & g_{1m}(L) \\ \vdots & \dots & \vdots \\ g_{d_g 1}(L) & \dots & g_{d_g m}(L) \end{bmatrix}\begin{bmatrix} s_{1t} \\ \vdots \\ s_{mt} \end{bmatrix} \qquad (2.23)
$$

Note that these exogenous variables are independent of the equilibrium policy rule $\phi(L)$ Similarly, we assume that all the elements of $g(L)$ are rational functions in L.

General model Assume the policy rule needs to satisfy the following linear system in equilibrium

$$\sum_{j=0}^{p} C^{y,j} L^{j} y_t + \sum_{j=0}^{p} \mathbb{E}\left[C^{f,j} L^{j} f_t + C^{g,j} L^{j} g_t \middle| x^t \right] + \sum_{j=1}^{q} \mathbb{E}\left[C^{y,-j} L^{-j} y_t + C^{f,-j} L^{-j} f_t + C^{g,-j} L^{-j} g_t \middle| x^t \right] = 0 \quad (2.24)$$

For $j \in \{-q, \ldots, p\}$, $C^{y,j}$ is a constant $d \times d$ matrix, $C^{f,j}$ is a constant $d \times d_f$ matrix, and $C^{g,j}$ is a constant $d \times d_g$ matrix. These matrices are structural parameters that result from optimality conditions and resource constraints. This system of equations incorporates the possibilities that the choice variables y_t depend on the past, the current and the future values of the endogenous variables of others and the exogenous variables, and also y_t's own and future values. This specification includes the majority of applications that one may encounter.

Special cases The structure we have specified includes two special cases which are common in the literature.

1. Perfect information.

$$\sum_{j=0}^{p} C^{y,j} L^{j} y_t + \sum_{j=0}^{p} \mathbb{E}\left[C^{f,j} L^{j} f_t + C^{g,j} L^{j} g_t \middle| s^t \right] + \sum_{j=1}^{q} \mathbb{E}\left[C^{y,-j} L^{-j} y_t + C^{f,-j} L^{-j} f_t + C^{g,-j} L^{-j} g_t \middle| s^t \right] = 0$$

   In standard real business cycle models and New Keynesian models without information frictions, the underlying shocks $\{s_t\}$ are observed directly by agents. That is, the space spanned by shocks is the same as the space spanned by signals, $\mathcal{H}_t^s = H_t^x$. Also, because all the shocks are observed directly, the actions of other agents are also known perfectly. As a result, the expectations in model (2.24) can be calculated in a trivial way.

2. Imperfect information, but no roles of higher order beliefs [9]

$$\sum_{j=0}^{p} C^{y,j} L^j y_t + \sum_{j=0}^{p} \mathbb{E}\left[C^{g,j} L^j g_t \middle| x^t\right] + \sum_{j=1}^{q} \mathbb{E}\left[C^{y,-j} L^{-j} y_t + C^{g,-j} L^{-j} g_t \middle| x^t\right] = 0$$

   This is the case in which information frictions exist, i.e., $\mathcal{H}_t^x \subset H_t^s$, but there is no need to infer others' choices. Agents only need to infer the exogenous variables g_t, and standard Kalman filter will be sufficient in solving the problem.

The solution to model (2.24) defined as follows

Definition 2.3.1. *Given the signal process (2.17), a solution to model (2.24) (or an equilibrium) is a matrix of lag polynomials $h(L)$ or equivalently $\phi(L)$, such that*

1. *For all (i, j), $h_{ij}(L)$ has an infinite MA representation*

$$h_{ij}(L) = \sum_{k=0}^{\infty} h_{ijk} L^k,$$

   with $\sum_{k=0}^{\infty} h_{ijk} < \infty$.

2. *For all possible realizations of $\{x_t\}$,*

$$y_t = h(L) x_t$$

   satisfies equation (2.24).

Given the model, we are interested in the following questions:

1. Under what conditions does a unique solution to this problem exist?

2. Suppose there indeed exists a $h(L)$ that solves the problem, what its formula?

3. Does the solution admit a finite-state representation which allows agents to summarize the past information using a small set of statistics?

Theorem 5, which involves more technical details, answers the first two questions. The following theorem answers the third question.

[9] This case is also discussed in [46].

Theorem 4 (Finite-state representation)**.** *Suppose the signal process follows (2.17) and the model (2.24) has a solution $y_t = h(L)x_t$. Then $y_t = h(L)x_t$ has a finite ARMA representation*

$$y_t = h(L)x_t = \begin{bmatrix} \frac{c_{11}(L)}{d_{11}(L)} & \cdots & \frac{c_{1n}(L)}{d_{1n}(L)} \\ \vdots & \ddots & \vdots \\ \frac{c_{d1}(L)}{d_{d1}(L)} & \cdots & \frac{c_{dn}(L)}{d_{dn}(L)} \end{bmatrix} \begin{bmatrix} x_{1t} \\ \vdots \\ x_{nt} \end{bmatrix}, \tag{2.25}$$

where $c_{ij}(L)$ and $d_{ij}(L)$ are finite degree polynomials in the lag operator L.

Given a particular signal realization $\{x_t\}_{t=-\infty}^{-1}$, there exists a finite set of state variables z_t, such that

$$y_t = \Gamma_x\, x_t + \Gamma_z\, z_t, \tag{2.26}$$

with the law of motion of z_t

$$z_{t+1} = \Upsilon_x x_t + \Upsilon_z z_t. \tag{2.27}$$

The initial state z_0 is given by

$$z_0 = (I - \Upsilon_z L)^{-1}\, \Upsilon_x x_{-1} \tag{2.28}$$

Proof. See Appendix A.2.10 for proof. □

This theorem implies that higher order beliefs do not create infinite state variables. It is always possible to use a small set of variables to summarize the necessary information in the past, given that the signal process is of ARMA type. We present the proof of this theorem and the proof of Theorem 5 in steps in the following subsections.

2.3.2 Preview of the main steps

The proof of these theorems is quite lengthy and it involves a number of building blocks. The initial input includes the signal process (2.17) and the model (2.24). Here, we first sketch the main steps that lead to Theorem 5 and Theorem 4.

Step 1: Given the signal process (2.17), find its state-space representation.

Step 2: With the state-space of the signal process, use the innovation representation and factorization identity matrix to find the Wold representation of the signal process.

Step 3: With the Wold representation of the signal process, use Wiener filter to solve the inference problem in model (2.24).

Step 4: Applying the Riesz-Fisher Theorem, transform the infinite-dimension problem of solving the sequences of coefficients in the lag polynomials into the finite-dimension problem of solving a system of analytic functions.

Step 5: Use Cramer's rule to solve the system of analytic functions, which leads to the solution $h(L)$ with ARMA representation.

Step 6: Given the solution with ARMA representation, find its finite-state representation.

2.3.3 Mathematical background: z transformation

By the Riesz-Fisher Theorem, there is a one-to-one mapping between the space of square-summable sequences and the space of complex-valued functions. Given a two-sided lag polynomials

$$\psi(L) = \sum_{k=-\infty}^{\infty} \psi_k L^k,$$

with $\sum_{k=-\infty}^{\infty} |\psi_k|^2 < \infty$, we will use the complex-valued function $\psi(z)$ to denote its corresponding z transformation

$$\psi(z) = \sum_{k=0}^{\infty} \psi_k z^k,$$

where $\psi(z)$ is defined on the unit circle.

If $\psi(L)$ is a one-sided polynomial with $\sum_{k=0}^{\infty} |\psi_k|^2 < \infty$, then its z transformation is an analytic function on the open unit disk.

Particularly, assume there are two univariate co-variance stationary processes

$$x_t = M(L)s_t,$$
$$y_t = \psi(L)s_t.$$

The auto-covariance generating function for x_t is

$$\rho_{xx}(z) = M(z)M'(z^{-1}),$$

and the cross-covariance generating function between y_t and x_t is

$$\rho_{yx}(z) = \psi(z)M'(z^{-1}).$$

Most of the time, working with a complex function is much more convenient than working with a square-summable sequence.

2.3.4 State-space representation, Factorization Identity, and Wold representation

We need the Wold representation of the signal process for the following reason. All the prediction is conditional on the observed signals, but ultimately, the linear projection is on the space spanned by shocks. The original underlying shocks s^t contain more information than the signals, and the prediction conditional on s^t is different from the prediction conditional on x^t. The Wold representation provides a new sequence of shocks w^t. Different from the underlying shocks s^t, the space spanned by the signals x^t is equivalent to the space spanned by w^t, and we can conduct the linear projection on w^t. Given a finite ARMA signal process, in this subsection we present how to find its state-space representation and Wold representation using the factorization identity.

Lemma 2.3.1. *Assume that x_t follows a finite ARMA process and is co-variance stationary,*

$$x_t = \begin{bmatrix} x_t^1 \\ \vdots \\ x_t^n \end{bmatrix} = \begin{bmatrix} \frac{a_{11}(L)}{b_{11}(L)} & \cdots & \frac{a_{1m}(L)}{b_{1m}(L)} \\ \vdots & \ddots & \vdots \\ \frac{a_{n1}(L)}{b_{n1}(L)} & \cdots & \frac{a_{nm}(L)}{b_{nm}(L)} \end{bmatrix} \begin{bmatrix} s_{1t} \\ \vdots \\ s_{mt} \end{bmatrix} = M(L)s_t, \tag{2.29}$$

where x_t is a $n \times 1$ vector and s_t is a $m \times 1$ vector. The co-variance matrix of s_t is normalized to be an identity matrix. In each element of $M(L)$, $a_{ij}(L)$ and $b_{ij}(L)$ are finite degree polynomials in the lag operator L. Particularly,

$$a_{ij}(L) = \sum_{k=0}^{q_{ij}} \alpha_{ijk} L^k$$
$$b_{ij}(L) = \sum_{k=0}^{p_{ij}} \beta_{ijk} L^k$$

and we normalize $\beta_{ij0} = 1$. *The signal process admits at least one state-space representation.*

The state equation is

$$Z_t = FZ_{t-1} + Qs_t,$$

and the observation equation is

$$x_t = HZ_t,$$

where F, Q *and* H *are functions of* $\left\{p_{ij}, q_{ij}, \{\alpha_{ijk}\}_{k=1}^{q_{ij}}, \{\beta_{ijk}\}_{k=1}^{p_{ij}}\right\}$.

In addition, the eigenvalues of F *all lie inside the unit circle.*

Proof. See Appendix A.2.3 for proof. □

This lemma states that any finite ARMA process has a state-space representation. Note that there are many different state-state representations for the same ARMA process. Generally, we can write the state equation as

$$Z_t = FZ_{t-1} + Qs_t,$$

and the observation equation as

$$x_t = HZ_t + Rv_t,$$

where the covariance matrix of v_t is also an identity matrices. Lemma 2.3.1 only provides one of the state-space representation with the feature that there is no shock in the observation equation.

Finding the state-space representation is a necessary step to find the Wold representation of the signal process. Suppose that we have $B(L)$ and $\{w_t\}$ such that

$$x_t = M(L)s_t = B(L)w_t, \tag{2.30}$$

$B(L)$ is invertible,[10] and w_t is serially uncorrelated shocks with co-variance matrix V, then we say $x_t = B(L)w_t$ is a Wold representation of x^t. Since $B(L)$ is invertible, x^t

[10] This is equivalent to that the determinant of $B(z)$ does not contain any roots (zeros) within the unit circle.

contains the same information as w^t, i.e., $\mathcal{H}_t^x = \mathcal{H}_t^w$. Further, equation (2.30) implies that

$$\rho_{xx}(z) = M(z)M'(z^{-1}) = B(z)VB'(z^{-1}). \tag{2.31}$$

$B(z)$ and V is called a canonical factorization of $\rho_{xx}(z)$. Therefore, find the Wold representation is equivalent to find the canonical factorization. The following theorem provides the canonical factorization for the state-space representation of the signal process x_t, which uses the factorization identity.

Theorem (Canonical Factorization)**.** *Let F denote an $(r \times r)$ matrix whose eigenvalues are all inside the unit circle; let $Q'Q$ or $R'R$ be positive definite matrix of dimension $(r \times r)$ or $(n \times n)$; let H denote an arbitrary $(n \times r)$ matrix. Let P satisfy*

$$P = F[P - PH'(HPH' + R'R)^{-1}HP]F' + Q'Q$$

and K be defined as

$$K = PH'(HPH' + R'R)^{-1}$$

Then

1. *The eigenvalues of $(F - FKH)$ are all inside the unit circle.*

2. *The canonical factorization is*

$$\begin{aligned}\rho_{xx}(z) &= H[I_r - Fz]^{-1}Q'Q[I_r - Fz^{-1}]^{-1}H' + R'R \\ &= [I_n + H(I_r - Fz)^{-1}FKz][HPH' + R'R][I_n + K'F'(I_r - F'z^{-1})^{-1}H'z^{-1}] \\ &= B(z)VB'(z^{-1}).\end{aligned}$$

3. *$B(z)$ is*

$$B(z) = I_n + H[I_r - Fz]^{-1}FKz,$$

the inverse of $B(z)$ is

$$B(z)^{-1} = I_n - H[I_r - (F - FKH)z]^{-1}FKz,$$

and the co-variance matrix V is

$$V = HPH' + R'R$$

Proof. The proof is in Hamilton (1994). □

To prove this theorem, one essentially uses the Kalman filter. The requirement that all the eigenvalues of F lie inside the unit circle guarantees $(I_r - Fz)$ is invertible. The eigenvalues of $(F - FKH)$ are very important in understanding the prediction problem, which essentially determines the persistence of the forecasts.

2.3.5 Wiener-Hopf prediction formula

Now we turn to the inference problems incorporated in equation (2.24). The following theorem states the Wiener-Hopf prediction formula. Note that this prediction formula does not hinge on whether the signal follows a finite ARMA process or not.

Theorem (Wiener-Hopf)**.** *Suppose the multivariate co-variance stationary signal process follows*

$$x_t = M(L)s_t,$$

and y_t is a univariate co-variance stationary process

$$y_t = \psi(L)s_t.$$

Assume all the elements of $M(L)$ and $\psi(L)$ have an infinite MA representation. The canonical factorization of $\rho_{xx}(z)$ is given by

$$\rho_{xx}(z) = M(z)M'(z^{-1}) = B(z)VB'(z^{-1}).$$

Then the optimal linear prediction of y_t conditional on $\{x_t\}$ is

$$\mathbb{E}[y_t|x^t] = \left[\rho_{yx}(L)B'(L^{-1})^{-1}\right]_+ V^{-1}B(L)^{-1}. \tag{2.32}$$

Proof. See Appendix A.2.4 for proof. □

If we further assume that the signal follows a finite ARMA process, we can obtain a sharper and more specific prediction formula.

Lemma 2.3.2. *Assume the signal process follows equation (2.17). Then*

$$M'(z^{-1})B'(z^{-1})^{-1} = \frac{1}{\Pi_{k=1}^{u}(z-\lambda_k)}G(z) \tag{2.33}$$

where $B(z)$ is given by the Canonical Factorization Theorem , $G(z)$ is a polynomial matrix in z, and $\{\lambda_k\}_{k=1}^u$ are non-zero eigenvalues of $F - FKH$ which all lie inside the unit circle.

Proof. See Appendix A.2.5 for proof □

Proposition 2.3.1. *Given the signal process in equation (2.17), suppose there is a univariate random variable y_t follows*

$$y_t = \psi(L)s_t,$$

where the elements of $\phi(L)$ has an infinite MA representation.

Assume $\{\lambda_k\}_{k=1}^u$ in Lemma 2.3.2 are distinct, the prediction formula for current and past y_t is

$$\mathbb{E}\left[L^j y_t \mid x^t\right] = \psi(L)L^j M'(L^{-1})\rho_{xx}(L)^{-1}x_t - \sum_{k=1}^{u} \frac{\psi(\lambda_k)\lambda^k G(\lambda_k)V^{-1}B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}x_t \quad (2.34)$$

where $j = \{0, 1, 2 \ldots\}$.

The prediction formula for j-step ahead prediction is

$$\begin{aligned} & \mathbb{E}\left[L^{-j} y_t \mid x^t\right] & (2.35) \\ = \;& \psi(L)L^{-j} M'(L^{-1})\rho_{xx}(L)^{-1}x_t - \sum_{k=1}^{u} \frac{\psi(\lambda_k)G(\lambda_k)L^{-j}V^{-1}B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}x_t \\ - \;& \sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j}\left[\frac{\psi(L)G(L)}{\Pi_{k=1}^u(L-\lambda_k)} - \sum_{k=1}^{u}\frac{\psi(\lambda_k)G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} V^{-1}B(L)^{-1}x_t \end{aligned}$$

where $[\cdot]_0^{(\ell)}$ denote the ℓ-th order derivative evaluated at 0.

Proof. See Appendix A.2.6 for proof. □

The key in applying the Wiener-Hopf prediction formula is to find the Wold representation for x_t or the canonical factorization for $M(z)$. When the number of signals equals the number of shocks, $M(L)$ is a square matrix. Suppose $M(L)$ is invertible, then $M(L)$ itself is a Wold representation and the Wiener-Hopf prediction formula can be applied directly. This corresponds to the case when there is no information friction or the signals fully reveal the state of the economy. If $M(L)$ is a square but not an invertible matrix, then there exists at least one inside root of the determinant of $M(L)$. In this case, the Wold representation can be found by multiplying the Blaschke matrices $B_j(z)$ to flip the inside roots outside the unit circle

$$B(z) \quad = \quad M(z)\Pi_j(W_j B_j(z)).$$

The details of the Blaschke matrix can be found in Rozanov (1967). [9], [16], [37] and [10] all use this method to find the Wold representation.

If the number of shocks is larger than the number signals, $M(L)$ is a non-square matrix and is not invertible. To find the canonical factorization of $M(L)$ is more involved, but we just show this can be achieved by using the Canonical Factorization Theorem.

As criticized by [25], in most signal extraction problems, the number of shocks is larger than the number of signals. Existing literature restricts the number of signals to being the same as the number of shocks so that the Blaschke matrix is applicable in finding the Wold representation. However, this restriction often leaves some informative variables to be observed without noise. As a result, the true state of the economy is revealed too quickly. For example, [9], [35] and [42] all show that in [7], agents share the same belief about the common demand shock and there is no *forecast the forecasts of others* problem. Also, the forecast error only exists for one period, and agents figure out the demand shock fairly quickly. The one period delay is due to the fact that output is predetermined. Similarly, in [10], agents observe the last period's aggregate output perfectly, and effects of aggregate noise only last for one period because agents can infer the underlying shock accurately by observing aggregate output. [16] and [37] both have square observation matrix, and to prevent the price from fully revealing the information, they have to abandon the standard AR(1) process but assume that the underlying shock follows a confounding process.

More importantly, a lot of interesting models naturally require that there are more shocks than signals, such as [34], [1], [5], [3], [4] and so on. In this paper, we show that by using the factorization identity, the Wold representation is readily available for any finite ARMA process. Joint with the Wiener filter, we can easily solve the signal extraction problem.

2.3.6 System of analytic functions

After we apply the Wiener filter, solving for $h(L)$ or $\phi(L)$ in model (2.24) still requires solving sequences of infinite coefficients in the lag polynomials, which is an infinite dimension problem. By the Riesz-Fisher Theorem, instead of solving the sequences of infinite coefficients, we can solve for a finite number of analytic functions instead, as shown in the following proposition.

Proposition 2.3.2. *Given the signal process (2.17), there exists a solution $\phi(L)$ to model (2.24) if and only if there exists a vector analytic function $\phi(z)$ that solves*

$$T(z)\phi(z) = D\Bigg[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\Bigg] \tag{2.36}$$

where $T(z)$ is a $w \times w$ matrix given by

$$T(z) \equiv \begin{bmatrix} \sum_{j=-q}^{p} z^j \left[\sum_{i=1}^{r} C_{1,i}^{y,j} A_i + \sum_{i=1}^{v} C_{1,i}^{f,j} f_i(z) M'(z^{-1})\rho_{xx}(z)^{-1}\right]' \\ \vdots \\ \sum_{j=-q}^{p} z^j \left[\sum_{i=1}^{r} C_{r,i}^{y,j} A_i + \sum_{i=1}^{v} C_{r,i}^{f,j} f_i(z) M'(z^{-1})\rho_{xx}(z)^{-1}\right]' \end{bmatrix} \tag{2.37}$$

and $D\Bigg[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\Bigg]$ is a $w \times 1$ vector given by

$$D\Bigg[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\Bigg] = \tag{2.38}$$

$$\left[\begin{array}{c} \Bigg\{ -\sum_{j=-q}^{p} C_1^{g,j} g(z) z^j M'(z^{-1}) \rho_{xx}(z)^{-1} \\ + \sum_{k=1}^{u} \frac{\sum_{j=-q}^{p} \lambda_k^j \left[\sum_{i=1}^{d_f} C_{1,i}^{f,j} \phi(\lambda_k) f_i(\lambda_k) + \sum_{i=1}^{d_g} C_1^{g,j} g(\lambda_k)\right] G(\lambda_k) V^{-1} B(z)^{-1}}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \\ + \sum_{j=1}^{q} \sum_{\ell=0}^{j-1} \ell!\, z^{\ell-j} \Bigg(\left[\frac{\sum_{i=1}^{d} \phi(z) C_{1,i}^{y,-j} A_i M(z) G(z)}{\Pi_{k=1}^{u}(z-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{d} \phi(\lambda_k) C_{1,i}^{y,-j} A_i M(\lambda_k) G(\lambda_k)}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\ + \left[\frac{\sum_{i=1}^{d_f} \phi(z) C_{1,i}^{f,-j} f_i(z) G(z)}{\Pi_{k=1}^{u}(z-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{d_f} \phi(\lambda_k) C_{1,i}^{f,-j} f_i(\lambda_k) G(\lambda_k)}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\ + \left[\frac{C_{1,i}^{g,-j} g(z) G(z)}{\Pi_{k=1}^{u}(z-\lambda_k)} - \sum_{k=1}^{u} \frac{C_{1,i}^{g,-j} g(\lambda_k) G(\lambda_k)}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \Bigg) V^{-1} B(z)^{-1} \Bigg\}' \\ \vdots \\ \vdots \\ \Bigg\{ -\sum_{j=-q}^{p} C_d^{g,j} g(z) z^j M'(z^{-1}) \rho_{xx}(z)^{-1} \\ + \sum_{k=1}^{u} \frac{\sum_{j=-q}^{p} \lambda_k^j \left[\sum_{i=1}^{d_f} C_{d,i}^{f,j} \phi(\lambda_k) f_i(\lambda_k) + \sum_{i=1}^{d_g} C_d^{g,j} g(\lambda_k)\right] G(\lambda_k) V^{-1} B(z)^{-1}}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \\ + \sum_{j=1}^{q} \sum_{\ell=0}^{j-1} \ell!\, z^{\ell-j} \Bigg(\left[\frac{\sum_{i=1}^{d} \phi(z) C_{1,i}^{y,-j} A_i M(z) G(z)}{\Pi_{k=1}^{u}(z-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{d} \phi(\lambda_k) C_{r,i}^{y,-j} A_i M(\lambda_k) G(\lambda_k)}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\ + \left[\frac{\sum_{i=1}^{d_f} \phi(z) C_{d,i}^{f,-j} f_i(z) G(z)}{\Pi_{k=1}^{u}(z-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{d_f} \phi(\lambda_k) C_{d,i}^{f,-j} f_i(\lambda_k) G(\lambda_k)}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\ + \left[\frac{C_{d,i}^{g,-j} g(z) G(z)}{\Pi_{k=1}^{u}(z-\lambda_k)} - \sum_{k=1}^{u} \frac{C_{d,i}^{g,-j} g(\lambda_k) G(\lambda_k)}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \Bigg) V^{-1} B(z)^{-1} \Bigg\}' \end{array}\right]$$

Proof. See Appendix A.2.7 for proof. □

To solve for $\phi(z)$, one can use the Cramer's rule. However, one also needs to determine the following constants, $\{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}$, which are generated when applying the Wiener-Hopf prediction formula,. As discussed in [36], these constants can be set to remove the poles of $\phi(z)$ that are inside the unit circle. This makes sure that $\phi(z)$ is analytic. The following lemma shows that the number of free constants that can be used in eliminating the inside poles is not the same as the total number of $\{\phi(\lambda_k)\}_{k=1}^{u}$ and $\{\phi^{(j)}(0)\}_{j=0}^{q}$, because of some of them may be linearly dependent on each other.

Lemma 2.3.3. *There exists a $w \times N_1$ matrix $D_1(z)$, a $w \times 1$ vector $D_2(z)$, and a $N_1 \times 1$ constant vector ψ, such that*

$$D\left[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\right] \tag{2.39}$$

$$= \widehat{D}_1(z)\left[\phi(\lambda_1) \quad \dots \quad \phi(\lambda_u) \quad \phi^0(0) \quad \dots \quad \phi^q(0)\right]' + D_2(z)$$

$$= D_1(z)\psi + D_2(z) \tag{2.40}$$

where N_1 *is the column rank of* $\widehat{D}_1(z)$ *and* ψ *is a linear combination of the constant vector*

$$[\phi(\lambda_1) \;\; \ldots \;\; \phi(\lambda_u) \;\; \phi^0(0) \;\; \ldots \;\; \phi^q(0)]'.$$

Proof. See Appendix A.2.8 for proof. □

Here, N_1 is the actual number of free constants that can used to remove the inside poles of $\phi(z)$. Theorem 5 shows that possible inside poles of $\phi(z)$ are from the inside roots of the determinant of $T(z)$. It follows that whether there exists a solution to model (2.24) or not hinges on whether there are enough free constants to eliminate all the inside roots of $\det[T(z)]$. Furthermore, there exists a unique solution if there are exactly N_1 conditions to determine the N_1 free constants.

Theorem 5 (General solution formula)**.** *Assume the signal process follows (2.17) and the model is given by equation (2.24). Let* N_2 *denote the number of roots of* $\det[T(z)]$ *that are inside the unit circle and let* $\{\vartheta_1, \ldots, \vartheta_{N_2}\}$ *denote these inside roots. Assume these roots are distinct. Define*

$$\begin{aligned} & U_1\psi + U_2 \\ \equiv & \begin{bmatrix} \det\begin{bmatrix} D_1(\vartheta_1)\psi + D_2(\vartheta_1) & T_1(\vartheta_1) & \ldots & T_{\ell_1-1}(\vartheta_1) & T_{\ell_1+1}(\vartheta_1) & \ldots & T_w(\vartheta_1)\end{bmatrix} \\ \vdots \quad\quad \vdots \quad \ldots \quad \vdots \quad\quad \vdots \quad \ldots \quad \vdots \\ \det\begin{bmatrix} D_1(\vartheta_{N_2})\psi + D_2(\vartheta_{N_2}) & T_1(\vartheta_{N_{N_2}}) & \ldots & T_{\ell_{N_2}-1}(\vartheta_{N_2}) & T_{\ell_{N_2}+1}(\vartheta_{N_2}) & \ldots & T_w(\vartheta_{N_2})\end{bmatrix} \end{bmatrix} \end{aligned}$$

where $T_{\ell_i}(\vartheta_i)$ *is a linear combination of* $\left\{T_1(\vartheta_i), \ldots, T_{\ell_i-1}(\vartheta_i),\; T_{\ell_i+1}(\vartheta_i), \ldots, T_w(\vartheta_i)\right\}$.

1. *If* $N_1 < N_2$*, there is no solution.*

2. *If* $N_1 = N_2 = \text{rank}\ (U_1)$*, there exists a unique solution* $\phi(z)$*. For* $i \in \{1, \ldots, w\}$

$$\phi_i(z) = \frac{\det\begin{bmatrix} T_1(z) & \ldots & T_{i-1}(z) & D_1(z)\psi + D_2(z) & T_{i+1}(z)\ldots & \ldots & T_w(z)\end{bmatrix}}{\det\begin{bmatrix}T(z)\end{bmatrix}} \tag{2.41}$$

and

$$\psi = -U_1^{-1}U_2 \tag{2.42}$$

3. If $N_1 > N_2$ or $N_1 = N_2 > rank\ (U_1)$, there exists an infinite number of solutions.

Proof. See Appendix A.2.9 for proof. □

With this theorem, we can prove our finite-state-representation theorem (Theorem 4), which is the last step of our method.

2.3.7 Innovation form and signal form

The solution we discussed in Section 2.3.6 is in terms of signals

$$y_t = h(L)x_t. \tag{2.43}$$

This is the most natural way to represent the policy rule because agents' actions depends on what they observe. However, sometimes it is more convenient to work with the policy function in terms of the underlying shocks.

$$y_t = d(L)s_t. \tag{2.44}$$

We label the solution in terms of signals as *signal form* and the solution in terms of underlying shocks as *innovation form*. Similar to the procedure to solve for $h(L)$, which effectively solves a system of equations in terms of signals, one can also write down the system of equations in terms of the underlying shocks $\{s_t\}$. A detailed description of the problem in innovation form can be found in Appendix A.2.11.

From a practical point of view, the signal form is typically easier to solve, because the dimension of the problem in signal form is smaller than the dimension of the problem in innovation form. However, the innovation form often provides a sharper characterization of the equilibrium, for the reason that the statistical properties are easier to derive in terms of the underlying shocks. Therefore, it is useful to obtain the solution in both forms. One may be concerned about whether the solution in innovation form is the same as the solution in signal form, and the following theorem shows that one can indeed work with either of them.

Theorem 6. *Assume the signal process follows (2.17) and the model is given by equation (2.24). There exists a solution in signal form,*

$$y_t = h(L)x_t, \tag{2.45}$$

if and only if there exists a solution in innovation form

$$y_t = d(L)s_t, \tag{2.46}$$

where $h(L)$ *and* $d(L)$ *satisfy*

$$\begin{aligned} d(L) &= h(L)M(L) \\ h(L) &= d(L)M'(L^{-1})\rho_{xx}(L)^{-1} - \sum_{k=1}^{u} \frac{d(\lambda_k)\lambda^k G(\lambda_k)V^{-1}B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k - \lambda_\tau)} \end{aligned}$$

Proof. See Appendix A.2.12 for proof. □

If $M(L)$ is not invertible, the space spanned by signals is a subset of the space spanned by shocks. It should be clear that whether we use the innovation form or the signal form, $\{y_t\}$ always lies in the space spanned by current and past signals because agents can only condition their choice on their observables, that is, , $\{y_t\} \subset \mathcal{H}_t^x \subset H_t^s$.

2.4 Application I: Two-Player Model

In this section, we use the method developed in Section 2.3 to solve two particular two-player games. There are only private signals in the first case, but we allow agents to share a common public signal in the second case.

2.4.1 Private Signal: [1]

The model we use is akin to model (2.7) introduced in Section 2.3.

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\mathbb{E}_{it}[y_t]. \tag{2.47}$$

There is a continuum of agents, and each individual agent i's optimal choice satisfies equation (2.47). The aggregate action y_t is given by

$$y_t = \int y_{it} \tag{2.48}$$

We assume the economic fundamental ξ_t follows an AR(1) process

$$\xi_t = \rho\xi_{t-1} + \eta_t, \tag{2.49}$$

where $\eta_t \sim N(0,1)$ and we have normalized to the variance of η_t to be 1.

We assume that an agent i receives two private signals about ξ_t

$$x_{it}^1 = \xi_t + \epsilon_{it}, \tag{2.50}$$

$$x_{it}^2 = \xi_t + u_{it}, \tag{2.51}$$

where $\epsilon_{it} \sim N(0, \sigma_\epsilon^2)$ and $u_{it} \sim N(0, \sigma_u^2)$.

The equilibrium is defined as follows

Definition 2.4.1. *Given the signal process (2.49) to (2.51), the equilibrium of model (2.47) is a policy rule $h = \{h_1, h_2\} \in \ell^2 \times \ell^2$, such that*

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\, \mathbb{E}_{it}[y_t],$$

where

$$y_{it} = h_1(L)x_{it}^1 + h_2(L)x_{it}^2,$$

$$y_t = \int y_{it}.$$

The structure of this model is similar to [1]. In [1], y_{it} is the price chosen by an individual firm, y_t is the aggregate price level, and ξ_t can be interpreted as some aggregate demand shock. The focus of [1] is that higher order beliefs generate inertia of the aggregate price level in response to the demand shock ξ_t (hump-shaped response), which is shown numerically. The following proposition gives the analytic solution to model (2.47), and the underlying reason for the inertia becomes transparent.

Proposition 2.4.1. *Assume that $\alpha \in (0,1)$. Given the signal process (2.49) to (2.51), the equilibrium policy rule in model (2.47) is given by*

$$h_1(L) = \frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}\frac{1}{1-\vartheta L}, \tag{2.52}$$

$$h_2(L) = \frac{\vartheta}{\rho\sigma_u^2(1-\rho\vartheta)}\frac{1}{1-\vartheta L}, \tag{2.53}$$

where

$$\vartheta = \frac{1}{2}\left[\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)(\sigma_\epsilon^2 + \sigma_u^2)}{\rho\sigma_\epsilon^2\sigma_u^2}\right) - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)(\sigma_\epsilon^2 + \sigma_u^2)}{\rho\sigma_\epsilon^2\sigma_u^2}\right)^2 - 4}\,\right] \tag{2.54}$$

The finite-state representation is given by

$$y_{it} = \frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}x_{it}^1 + \frac{\vartheta}{\rho\sigma_u^2(1-\rho\vartheta)}x_{it}^2 + z_{it}, \tag{2.55}$$

where

$$z_{it+1} = \vartheta z_{it} + \frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}x_{it}^1 + \frac{\vartheta}{\rho\sigma_u^2(1-\rho\vartheta)}x_{it}^2. \tag{2.56}$$

The aggregate y_t is given by

$$y_t = \frac{\vartheta}{\rho(1-\rho\vartheta)}\frac{\sigma_\epsilon^2+\sigma_u^2}{\sigma_\epsilon^2\sigma_u^2}\frac{1}{(1-\vartheta L)(1-\rho L)}\eta_t \tag{2.57}$$

Proof. See Appendix A.2.13 for proof. □

The individual policy rule follows an AR(1) process, and the aggregate y_t follows an AR(2) process. The two signals only differ by the variance of their idiosyncratic noises. As expected, $h_1(L)$ and $h_2(L)$ are symmetric, but the weight on each signal is adjusted according to their informativeness.

Crucially, the persistences of $h_1(L), h_2(L)$, and the persistence of aggregate y_t are governed by ϑ. Given ρ, as ϑ increases, the peak of the impulse response of y_t to η_t shifts to the right, which makes it possible to have a hump-shaped response. If ϑ is small enough, then there may not be any hump-shaped response. The following proposition provides a sharp characterization of ϑ.

Proposition 2.4.2. *Assume that $\alpha_1 \in (0,1), \rho \in (0,1), \sigma_\epsilon > 0$, and $\sigma_u > 0$. Then ϑ satisfies*

1. *$0 < \lambda < \vartheta < \rho$, where λ is given by*

$$\lambda = \frac{1}{2}\left[\left(\frac{1}{\rho}+\rho+\frac{\sigma_\epsilon^2+\sigma_u^2}{\rho\sigma_\epsilon^2\sigma_u^2}\right) - \sqrt{\left(\frac{1}{\rho}+\rho+\frac{\sigma_\epsilon^2+\sigma_u^2}{\rho\sigma_\epsilon^2\sigma_u^2}\right)^2 - 4}\right] \tag{2.58}$$

2. *ϑ is increasing in α and*

$$\lim_{\alpha_1 \to 1} \vartheta = \rho$$

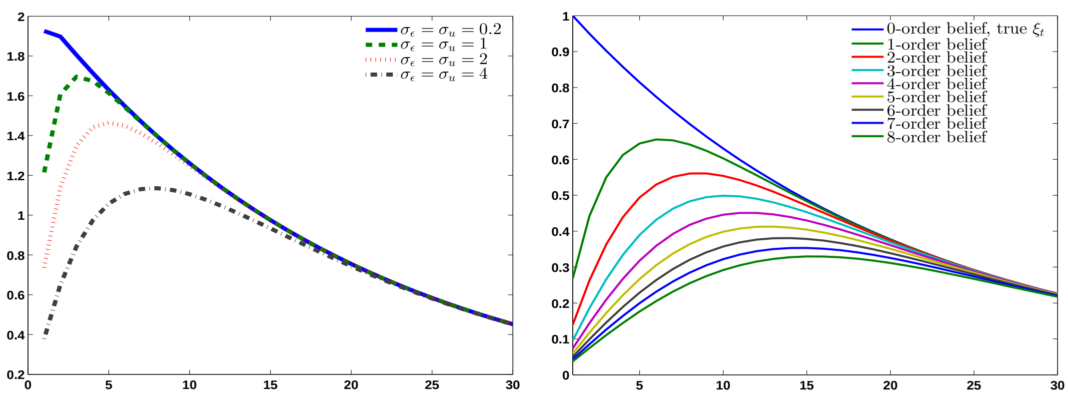
$$\lim_{\alpha_1 \to 0} \vartheta = \lambda$$

3. *ϑ is increasing in σ_ϵ, σ_u, and ρ.*

Here, ϑ is bounded from above by the persistence of ξ_t, and it is also bounded from below by λ, where $1-\lambda$ is the Kalman gain when using the Kalman filter to predict ξ_t. Note that ϑ is increasing in α, and with a large α, it is more likely for y_t to have a hump-shaped response . This is because with information frictions, higher order beliefs respond slowly to the shock. When the degree of strategic complementarity increases, higher order beliefs become more important in shaping the behavior of y_t, as shown in equation (2.10).

Example We use a numerical example to further illustrate the properties of the model economy. We set the degree of strategic complementarity $\alpha = 0.5$ and the persistence of ξ to be 0.95. As the variances of idiosyncratic shocks increase, the degree of information frictions also increases. As shown in Figure 2.1, the hump-shaped response of y_t

Figure 2.1: Impulse Response to η Shock in the Private-Signal Model



Response of y_t with various σ_ϵ and σ_u Higher order bliefs with $\sigma_\epsilon = \sigma_u = 4$

to η_t is more pronounced when there are larger information frictions. This is because ϑ is increasing in σ_ϵ and σ_u. When there is little information friction, ϑ is small and there is no hump-shaped response any more.

The higher order beliefs have the following feature: as the order increases, the higher order belief becomes less responsive, and the peak of its response shifts to the right. To predict ξ_t, agent i discounts his signals by the Kalman gain $1-\lambda$, which leads to that $\mathbb{E}_{it}[\xi_t]$ is less volatility than ξ_t. When agent i infers others' forecasts of ξ_t, he realizes others also discount their signals by $1-\lambda$. Agent i' best forecast of others signal is $\mathbb{E}_{it}[\xi_t]$, and his forecast of $\overline{\mathbb{E}}[\xi_t]$ in turn discounts the original ξ_t twice. This logic applies to all the higher order beliefs. Consider k-th order belief. As k increases, the forecasts of k-th order beliefs puts less weight on current signals, and more weight on the priors of beliefs with order lower than k, which makes the inertia increase in the order of beliefs.

2.4.2 Public Signal: [2], [3]

Now we introduce the following variation to the model discussed in the last section. The economic fundamental ξ_t still follows an AR(1) process

$$\xi_t = \rho\xi_{t-1} + \eta_t, \tag{2.59}$$

but we assume the first signal about the economic fundamental ξ_t is the same across all the agents

$$x_{it}^1 = \xi_t + \epsilon_t, \tag{2.60}$$

$$x_{it}^2 = \xi_t + u_{it}, \tag{2.61}$$

where $\epsilon_t \sim N(0,\sigma_\epsilon^2)$ and $u_{it} \sim N(0,\sigma_u^2)$. Note that now the noise in the first signal is not indexed by i. Effectively, the first signal now becomes a public signal. The model is the same as before

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\mathbb{E}_{it}[y_t]. \tag{2.62}$$

The structure of this model is similar to [2] and [3].[11] In [3], y_{it} is the output chosen by an individual firm i, y_t is the aggregate out, ξ_t is the aggregate TFP shock. The focus of their paper is to understand the effects of the common noise ϵ_t, which can be interpreted as animal spirits or sentiments. The question is whether this common noise can introduce aggregate output fluctuations. [3] use a guess-and-verify method and obtain a numerical solution. Here, we obtain an analytic solution.

[11] The original model in [3] is $y_{it} = \xi_t + u_{it} + \alpha\mathbb{E}_{it}[y_t]$, where u_{it} is firm specific technology shock. Here, we modify their original model to better contrast with our private-signal model, but the main dynamics remain the same.

Proposition 2.4.3. *Assume that $\alpha \in (0,1)$. Given the signal process (2.59) to (2.61), the equilibrium policy rule in model (2.62) is given by*

$$y_{it} = h_1(L)x_{it}^1 + h_2(L)x_{it}^2,$$

where

$$h_1(L) = \frac{1}{1-\alpha}\frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}\frac{1}{1-\vartheta L}, \tag{2.63}$$

$$h_2(L) = \frac{\vartheta}{\rho\sigma_u^2(1-\rho\vartheta)}\frac{1}{1-\vartheta L}, \tag{2.64}$$

and

$$\vartheta = \frac{1}{2}\left[\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)\sigma_\epsilon^2 + \sigma_u^2}{\rho\sigma_\epsilon^2\sigma_u^2}\right) - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)\sigma_\epsilon^2 + \sigma_u^2}{\rho\sigma_\epsilon^2\sigma_u^2}\right)^2 - 4}\right] \tag{2.65}$$

The finite-state representation is given by

$$y_{it} = \frac{1}{1-\alpha}\frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}x_{it}^1 + \frac{\vartheta}{\rho\sigma_u^2(1-\rho\vartheta)}x_{it}^2 + z_{it} \tag{2.66}$$

where

$$z_{it+1} = \vartheta z_{it} + \frac{1}{1-\alpha}\frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}x_{it}^1 + \frac{\vartheta}{\rho\sigma_u^2(1-\rho\vartheta)}x_{it}^2 \tag{2.67}$$

The aggregate y_t is given by

$$y_t = \frac{\vartheta}{\rho(1-\rho\vartheta)}\frac{(1-\alpha)\sigma_\epsilon^2 + \sigma_u^2}{(1-\alpha)\sigma_\epsilon^2\sigma_u^2}\frac{1}{(1-\vartheta L)(1-\rho L)}\eta_t + \frac{1}{1-\alpha}\frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}\frac{1}{1-\vartheta L}\epsilon_t \tag{2.68}$$

Proof. See Appendix A.2.14 for proof. □

We can see that the public-model is clearly different from the private-signal model. Because the common noise ϵ_t in the first signal now affects all agents in the economy, each individual agent will respond more strongly to the first signal, due to the strategic complementarity. As the strength of the strategic complementarity increases (α increases), the instantaneous response to the first signal, $\frac{1}{1-\alpha}\frac{\vartheta}{\rho\sigma_\epsilon^2(1-\rho\vartheta)}$, also becomes larger. In addition, σ_ϵ and σ_u are not symmetric in shaping the information frictions, reflected in how they affect the persistence ϑ in equation (2.65).

In terms of the aggregate y_t, it is now a function of both η shock and ϵ shock. However, the response to an η shock follows an AR(2) process, the same as the private-signal model, but the response to an ϵ shock follows an AR(2) process. Figure 2.2 plots the responses to these two shocks.[12]

2.5 Application II: Endogenous Information

So far we have only discussed the cases where the signal process is exogenously determined and independent of agents' actions. This section we consider the case where an observed signal contains endogenous information.

An important theme of the literature on dispersed information is the role of the endogenous signal in coordinating beliefs and revealing information. [9] and [42] show that by observing prices in other industries, agents share the same beliefs. [38] and [16] show that whether the price in the asset market reveals the state of the economy depends on whether the underlying shock follows a confounding process or not. However, most of the studies restrict their attention to the special case in which the number of signals equals the number of shocks and agents observe the endogenous variable without noise. In this section, we will analyse the role of endogenous information when there are more shocks than signals, and the endogenous variable cannot be observed perfectly.

2.5.1 Infinite state variables

The model we use is similar to the private-signal model in Section 2.4.1, but we assume a different information structure. Agents still receive two signals. The first signal is the same as before, but the second one is the aggregate y_t with an idiosyncratic noise

$$x_{it}^2 = y_t + u_{it} = \int y_{jt} + u_{it} \tag{2.69}$$

The aggregate y_t is endogenously determined by all the individual choices, while at the same time, it is served as a signal for agents to infer the state of the economy. In this case, we find it is more convenient to define the equilibrium with innovation form.

[12] We set the degree of strategic complementarity α to be 0.5 and the persistence of ξ to be 0.95. We also set the variance of the noise to be $\sigma_\epsilon = \sigma_u = 4$. The implied persistence $\vartheta = 0.77$.

Definition 2.5.1. *The equilibrium is an endogenous stochastic process Ω_{it}, a policy rule for an individual agent $\phi = \{\phi_1, \phi_2, \phi_3\} \in \ell^2 \times \ell^2 \times \ell^2$, and the law of motion for aggregate y_t, $\Phi \in \ell^2$, such that*

1. *Agent i's information set $\Omega_{it} = \left\{x^1_{it},\ x^2_{it},\ x^1_{it-1},\ x^2_{it-1},\ x^1_{it-2},\ x^2_{it-2},\ \ldots\right\}$ is determined by*

$$\begin{aligned} x^1_{it} &= \xi_t + \epsilon_{it}, && (2.70)\\ x^2_{it} &= y_t + u_{it}, && (2.71) \end{aligned}$$

   where

$$\begin{aligned} \xi_t &= \frac{\Pi_{k=1}^{n}(1+\kappa_k L)}{\Pi_{k=1}^{m}(1-\zeta_k L)}\eta_t, && (2.72)\\ y_t &= \Phi(L)\eta_t. && (2.73) \end{aligned}$$

2. *Individual rationality*

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\, \mathbb{E}_{it}[y_t], \tag{2.74}$$

   where

$$y_{it} = \phi_1(L)\epsilon_{it} + \phi_2(L)u_{it} + \phi_3(L)\eta_t. \tag{2.75}$$

3. *Aggregate consistency: $y_t = \int y_{it}$*

$$\Phi(L) = \phi_3(L) \tag{2.76}$$

To show the generality of our claim, we allow ξ_t to follow any finite ARMA process. The equilibrium with endogenous information involves two fixed points. The first fixed point is individual rationality. Given the policy rule of others and the signal process, agent i optimally chooses the same policy rule as others. The second fixed point is absent in the equilibrium with exogenous information. It requires that agents perceived law of motion of the aggregate y_t is the same as the actual law of motion of the aggregate y_t. This can be viewed as the cross-equation restriction in the sense that agents perception is in line with the reality generated by their own action.

Similar to Proposition 2.2.1, the following proposition guarantees that there exists a unique equilibrium with endogenous information.

Proposition 2.5.1. *If $\alpha \in (0, 1)$, then there exists a unique equilibrium of the model in Definition 2.5.1.*

Proof. See Appendix A.2.15 for proof. □

This proposition only proves the existence and uniqueness of the equilibrium, but it is silent on whether the agents need to keep track of infinite number of state variables or not. With exogenous information, we have shown that the equilibrium always permits a finite-state representation provided that the signals follow a finite ARMA process. In contrast, the following theorem shows that with endogenous information, even though there exists a unique equilibrium, the aggregate y_t does not follow a finite ARMA process. As a result, the equilibrium cannot have a finite-state representation.

Theorem 7. *If $\alpha \in (0, 1)$, the equilibrium of the model in Definition 2.5.1 does not have a finite-state representation.*

Proof. See Appendix A.2.16 for proof.[13] □

The proof of this theorem shows that if assuming the perceived aggregate y_t follows a finite ARMA process, the implied actual aggregate y_t cannot be the same as the perceived aggregate y_t. With exogenous information, Proposition 2.4.1 shows that if ξ_t follows an AR(1) process, the implied aggregate y_t follows an AR(2) process. With endogenous information, if we assume ξ_t follows an AR(1) process and the perceived y_t follows an AR(2) process, the implied actual y_t follows an ARMA $(4, 2)$ process. If we assume perceived y_t follows ARMA $(4, 2)$, the actual y_t will follow an ARMA $(6, 4)$ process. Iterating this process, the aggregate y_t follows an infinite ARMA process in the limit.

This is a somewhat surprising result. [9] and [42] show that in the [7] model, there is actually no infinite regress problem and the equilibrium permits a finite-state representation. Similarly, in [16] and [10], the equilibrium policy rule has a finite-state representation as well. [42] suspects that to resuscitate the infinite regress problem, there should be more shocks than signals. Theorem 7 proves that in our model with

[13] We thank Eduardo Faingold for pointing out a mistake in the original proof and suggestions on the current proof.

endogenous information, agents need to keep track of infinite state variables in equilibrium. [33] proved a similar impossibility theorem for a particular univariate case, and we prove this theorem in a multivariate system with an arbitrary ARMA process.

The reason for the infinite state variables, however, is not the infinite regress problem. When the signals follow an exogenous ARMA process, the infinite regress problem does exist but the equilibrium rule always has a finite-state representation. With endogenous information, each individual still treats the signal process as exogenous. If the perceived law of motion for y_t is a finite ARMA process, we return to the case covered by Theorem 4: each individual needs to solve the infinite regress problem, but the number of state variables is finite. With endogenous information, what complicates the issue is that the signal process itself cannot be represented as a finite ARMA process, but this is independent of the infinite regress problem faced by each individual.

Compared with the literature, the equilibrium policy rule in [9], [16] and [10] all follows an ARMA process, even though the signals contain endogenous information. The key difference is that they assume the number of signals equals the number shocks, i.e., the signals $x_t = M(L)s_t$ with $M(L)$ being a square matrix. In this case, one can use the Blaschke matrix to obtain the Wold representation without knowing the exact signal process. The cost of this assumption is that the signal process is not complicated enough to create interesting dynamics. In [10] or [9], the endogenous variable that has an information role is observed without noise, and the forecast error is transitory. In our model, because there are more shocks than signals, agents can never infer the shocks perfectly, and the forecast error is persistent.

2.5.2 Computation

The infinite-state result is theoretically interesting, but it excludes the possibility of obtaining the exact solution. Here we provide a tractable algorithm that can approximate the true solution arbitrarily well. The idea is to use a low order ARMA process to approximate aggregate y_t, which enables the Winer-Hopf prediction formula.

1. Assume that the perceived aggregate y_t follows an ARMA (p, q) process

$$y_t^p = \Phi(L)\eta_t = \sigma_y \frac{\Pi_{k=1}^q (1 + \theta_k L)}{\Pi_{k=1}^p (1 - \rho_k L)}.$$

2. Given the law of motion of the aggregate y_t, the signal process follows a finite ARMA process. Use the method in Section 2.3 to solve for the individual policy rule $\phi = \{\phi_1(L), \phi_2(L), \phi_3(L)\}$. The actual aggregate y_t follows

$$y_t^a = \phi_3(L)\eta_t.$$

3. To update $\Phi(L)$, expand $\phi_3(L)$ to obtain the infinite moving average representation. Choose the new σ_y, $\{\rho_k\}_{k=1}^p$ and $\{\theta_k\}_{k=1}^q$ to equate $\{\Phi_0, \Phi_1, \ldots, \Phi_{p+q}\}$ with $\{\phi_{30}, \phi_{31}, \ldots, \phi_{3p+q}\}$

4. Iterate 1 to 3 until the difference between $\{\Phi_0, \ldots, \Phi_{p+q}\}$ and $\{\phi_{30}, \ldots, \phi_{3p+q}\}$ is smaller than the tolerance level.

5. Compute $\|\Phi - \phi\|$ (one can simply use the norm of ℓ^2). If $\|\Phi - \phi\|$ is larger than the tolerance level, increase (p, q) and repeat 1 to 4.

Based on the proof of Proposition A.2.15, this algorithm is a contraction mapping that converges to the true solution as the order of the ARMA approximation increases. [35] also uses an ARMA process to approximate the signal process, but our method differs from his in an important way. In [35], only the forecasts of future signals are pay-off relevant. Once the law of motion of the signal is specified, agents do not need to solve the signal extraction problem and there is no need to forecast the forecasts of others. In our model, although the signal process is given, agents still face the infinite regress problem. Step 2 in the algorithm makes sure that each individual always performs their optimal prediction.

Compared with [25], our method has the following advantage. The first advantage is that our method requires fewer state variables. Nirmark's method needs to keep track of a large number of higher order beliefs to accurately approximate the policy rule. In principle, Nirmark's method is to use MA(∞) process to approximate the policy rule while our method uses an ARMA process for approximation, which is more efficient. In our numerical example, it requires to keep track of the higher order beliefs up to order 30 to achieve the same accuracy as our ARMA (4,2) approximation. The second advantage is that our method is easier to implement and is applicable in more general environments. Nirmark's method relies on the correct conjecture of the law of motion of the higher order beliefs. When the signal process is more complicated than an AR(1) process, it is not obvious what the correct conjecture should be. In addition, Nirmark's method also relies on that the equilibrium policy rule is a relatively simple function of higher order beliefs, but this may not be true in many empirical applications where the system is complicated (see the quantitative model in [41] for example). Instead, our method instead does not hinge on these assumptions.

Example To check whether our approximation method is accurate enough, we need to compare the perceived aggregate y_t with the implied aggregate y_t. We set $\alpha = 0.5$. We assume ξ_t follows an AR(1) process with persistence $\rho = 0.95$. We also set $\sigma_a = \sigma_u = 4$. As shown in Figure 2.3, if we use an AR(2) process to approximate the aggregate y_t, the difference between perceived and implied aggregate y_t is quite noticeable. If we use an ARMA (4,2) process to approximate y_t, the perceived and implied y_t are almost identical to each other. Given the existence of the equilibrium, this method can easily extend to other more complicated environments when there does not exist a finite-state representation.

2.6 Application III: Random-Matching Model, [4]

In this section we discuss another type of model, in which an agent meets a different player every period. [4] consider an interesting model environment with this feature, but they assume there is no persistent shock in their baseline model. This assumption does not affect their qualitative prediction, and it helps to avoid the infinite regress problem. However, this assumption prevents the model from exploring more relevant learning problems, and it makes the model unsuitable for empirical work. We extend [4] to allow persistent shocks and the infinite regress problem in the model.

Assume that there is a continuum of agents in the economy. An individual agent i is endowed with a productivity a_i, which is drawn from a normal distribution $N(0, \sigma_a^2)$. Note that both individual's productivity and the distribution is fixed over time, and there is no aggregate uncertainty with respect to the economic fundamentals. At the beginning of each period, an agent i is randomly matched with another agent $m(i,t)$ and trades goods with $m(i,t)$, where $m(i,t)$ is the index of agent i's trading partner in period t. Note that the production has to take place before trading, and agents have to infer others' output based on their signals. Due to strategic complementarity, agent i's optimal output choice y_{it} needs to satisfy

$$y_{it} = a_i + \alpha\, \mathbb{E}_{it}[y_{m(i,t)t}], \tag{2.77}$$

where $\alpha \in (0,1)$ controls the degree of strategic complementarity, and $y_{m(i,t)t}$ is the output choice of i's trading partner $m(i,t)$ at period t. Equation (2.77) says that agent i' output is increasing in his own productivity and the output of his trading partner, while the detailed micro-foundation that leads to this equation is not important for us to understand the infinite regress problem.

Agent i receives two signals

$$x^1_{it} = a_{m(i,t)} + \epsilon_{it}, \tag{2.78}$$

$$x^2_{it} = x^1_{m(i,t)t} + \xi_t + u_{it}, \tag{2.79}$$

where $\epsilon_{it} \sim N(0, \sigma^2_\epsilon)$ and $u_{it} \sim N(0, \sigma^2_u)$, both of which are idiosyncratic noise. The productivity of i's trading partner is $a_{m(i,t)}$, and from i's perspective, it is also an i.i.d shock that follows $N(0, \sigma^2_a)$. $x^1_{m(i,t)t}$ is the first signal received by agent i's trading partner $m(i,t)$. ξ_t is common for all agents, which follows an AR(1) process

$$\xi_t = \rho \xi_{t-1} + \eta_t, \tag{2.80}$$

where $\eta_t \sim N(0,1)$. In [4], ξ_t is an i.i.d shock, but we assume $\rho \in (0,1)$ here to introduce the infinite regress problem. The information set of agent i is[14]

$$\Omega_{it} = \left\{ a_i,\ x^1_{it},\ x^2_{it},\ x^1_{it-1},\ x^2_{it-1},\ x^1_{it-2},\ x^2_{it-2},\ \ldots \right\}. \tag{2.81}$$

Note that a_i needs to be included in the information set because agent i's action directly depends on a_i, and it also helps to predict i's trading partner's signal. The equilibrium is defined as

[14] There is an implicit assumption that agents do not observe their trading partner's output or productivity level after production. This assumption is only to implement the notion of imperfect communication between producers and transactors, but is not important for our purpose.

Definition 2.6.1. *Given the signal process (2.78) to (2.80), the equilibrium of model (2.77) is a policy rule* $h = \{h_a, h_1, h_2\} \in \mathbb{R} \times \ell^2 \times \ell^2$, *such that*

$$y_{it} = a_i + \alpha\, \mathbb{E}_{it}[y_{m(i,t)t}],$$

where

$$y_{it} = h_a a_i + h_1(L)x^1_{it} + h_2(L)x^2_{it}.$$

As emphasized by [4], agent i's estimate of his trading partners' productivity $a_{m(i,t)}$ is pinned downed by the i's first signal alone, and not affected by the second signal. However, agent i's estimate of $x^1_{m(i,t)t}$ is affected by the common noise ξ_t. With a positive realization of ξ_t, agent i attributes part of ξ_t to $x^1_{m(i,t)t}$, and believes that agent $m(i,t)$ will overestimate i's productivity a_i and produce more output. Therefore, agent i's also optimally produces more output due to strategic complementarity. In aggregate, ξ_t leads to fluctuations in total output by affecting all agents' higher order beliefs.

Different from the applications discussed in Section 2.4, agent i has to form higher order beliefs about a random player $m(i,t)$ every period. This change may prevent the use of the guess-and-verify method, but our method developed in Section 2.3 can still be applied to solve the model.

Proposition 2.6.1. *Assume that* $\alpha \in (0,1)$. *Given the signal process (2.78) to (2.80), the equilibrium policy rule in model (2.77) is given by*

$$h_a = 1 + \alpha\varphi - \frac{\alpha\vartheta\varphi(1-\rho)}{\rho(1-\vartheta)}\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_u^2} \tag{2.82}$$

$$h_1(L) = \varphi \tag{2.83}$$

$$h_2(L) = \frac{\alpha\vartheta\varphi}{\rho}\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_u^2}\frac{1-\rho L}{1-\vartheta L} \tag{2.84}$$

where

$$\vartheta = \frac{1}{2}\left[\frac{1}{\rho} + \rho + \frac{1-\alpha}{\rho(\sigma_\epsilon^2+\sigma_u^2)} - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{1-\alpha}{\rho(\sigma_\epsilon^2+\sigma_u^2)}\right)^2 - 4}\,\right], \tag{2.85}$$

$$\varphi = \frac{\alpha}{1-\alpha^2+\frac{\sigma_\epsilon^2}{\sigma_a^2}\left(1-\alpha^2\frac{\vartheta}{\rho}\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_u^2}\right)}. \tag{2.86}$$

The finite-state representation is given by

$$y_{it} = h_a a_i + \varphi x_{it}^1 + \frac{\alpha\vartheta\varphi}{\rho}\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_u^2} x_{it}^2 + z_{it} \tag{2.87}$$

where

$$z_{it+1} = \vartheta z_{it} + \frac{(1-\rho)\alpha\vartheta\varphi}{\rho}\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_u^2} x_{it}^2 \tag{2.88}$$

The aggregate y_t is given by

$$y_t = \frac{\alpha\vartheta\varphi}{\rho}\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_u^2}\frac{1}{1-\vartheta L}\eta_t \tag{2.89}$$

Proof. See Appendix A.2.17 for proof. □

Note that $h_1(L)$ is a constant, which implies that the policy rule does not depend on $\{x_{i\tau}^1\}_{\tau=-\infty}^{t-1}$. The reason is that the first signal is only useful in predicting the productivity of current trading partner, but it is independent of the persistent shock ξ_t. It turns out that $h_2(L)$ follows an ARMA(1,1) process, and the aggregate output y_t follows an AR(1) process.

Comparing with heterogeneous prior In the literature, a convenient device to avoid the infinite regress problem is to assume that agents have heterogeneous prior. The heterogeneous prior assumption works as follows. Assume that agent i observes both ξ_t and $a_{m(i,t)t}$ perfectly. However, agent i believes his trading partner $m(i,t)$ observes a_i with bias ξ_t. If agent i's policy rule is

$$y_{it} = f_1 a_i + f_2 a_{m(i,t)} + f_3 \xi_t,$$

then agent i believes that the output of his trading partner is

$$y_{m(i,t)t} = f_1 a_{m(i,t)} + f_2(a_i + \xi_t) + f_3 \xi_t.$$

In equilibrium,

$$y_{it} = \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}[y_{m(i,t)t}],$$

which leads to

$$y_{it} = \frac{1}{1-\alpha^2} a_i + \frac{\alpha}{1-\alpha^2} a_{m(i,t)} + \frac{\alpha_1^2}{(1-\alpha_1^2)(1-\alpha)}\xi_t \tag{2.90}$$

$$y_t = \frac{\alpha^2}{(1-\alpha^2)(1-\alpha)}\xi_t \tag{2.91}$$

Quantitatively, by assuming heterogeneous prior, y_t is perfectly correlated with ξ_t, while in our model with common prior, the persistence of aggregate output is endogenously determined by the structural parameter α and the information related parameters, and it is always different from the the persistence of ξ_t. A numerical example is shown in Figure 2.4.

Note that the both the persistence and instantaneous response of y_t under heterogeneous prior is very different from the solution under rational expectation. The solution under heterogeneous prior assumption is independent of the degree of information frictions, that is, the distribution of idiosyncratic productivity and the size of the idiosyncratic noise do not affect the behaviour of output. By assuming heterogeneous prior, one effectively assumes away the information frictions, which is the reason that higher order beliefs arise in the first place. The method we provide to solve the infinite regress problem retains the notion of rationality, and we can pin down the degree of information frictions by comparing the model results with data.

2.7 Application IV: a Quantitative Business Cycle Model

Application I to Application III can be thought of as various extensions of the basic model presented in Section 2.2. These applications are theoretically interesting, but have not fully taken advantage of our method. In the general model structure we outline in equation (2.24), we allow the model to include the past, the present, and the future values of the choice variables, the choices of others, and the exogenous variables.

In a companion paper ([41]), we apply our method and solve a full-blown quantitative model in which the confidence shock alone is sufficient to account for the main aggregates in business cycles. The idea is related to [4], but our model differs from theirs in several crucial ways. We maintain the strong notion of rationality and solve the infinite regress problem directly. Agents need to choose both labor and investment, and need to infer the output and capital of both their current and future trading partners. There are multiple persistent shocks in the signal process to match various micro and macro moments. Therefore, higher order beliefs affect agents' decisions in a fairly complex way. With our preferred calibration of information frictions, we find that the model with confidence shocks generate much of the volatility and co-movement of aggregate variables, but it has difficulty in matching the persistence of the aggregate variables.

2.8 Conclusion

In this paper, we have shown how to solve general rational expectations models with higher order beliefs. When the signal follows an ARMA process, we prove that the policy rule always admits a finite-state representation. It turns out the infinite regress problem does not require infinite state variables, because the total effects of the higher order beliefs can be summarized by a small set of variables. We provide a procedure that gives an explicit solution formula. The key of our method is to apply the Kalman filter to obtain the Wold representation of the signal process, and then use the Wiener filter to solve the inference problems. We also prove that when the signal process contains endogenous information, the equilibrium policy rule may not have a finite-state representation, which is in some sense the 'true' infinite regress problem. This is due to the fact that cross-equation restriction imposes an additional equilibrium condition that the perceive law of motion of an endogenous variable has to be the same as the law of motion that is generated by agents' actions. We provide a tractable algorithm that can approximate the true solution accurately with a small number of state variables. Various applications are easily solved by our method. We expect that the method we develop in this paper can be applied in a much broader class of models, especially in the areas of macroeconomics and financial economics with dispersed information. Preliminary findings in [41] show that this is a promising direction to pursue.

Figure 2.2: Impulse Response of y_t in the Public-Signal Model

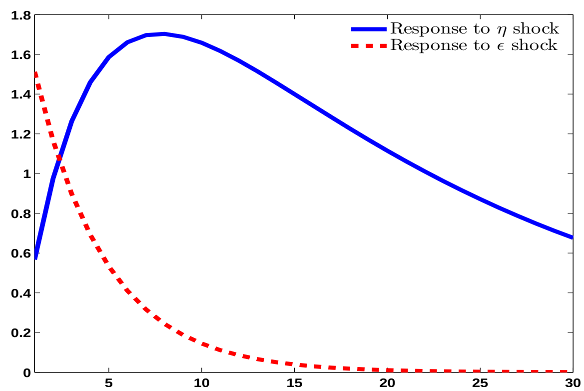
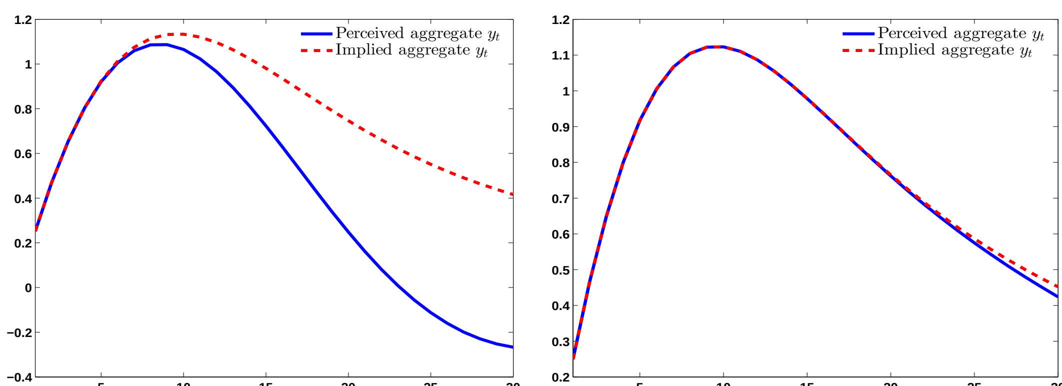
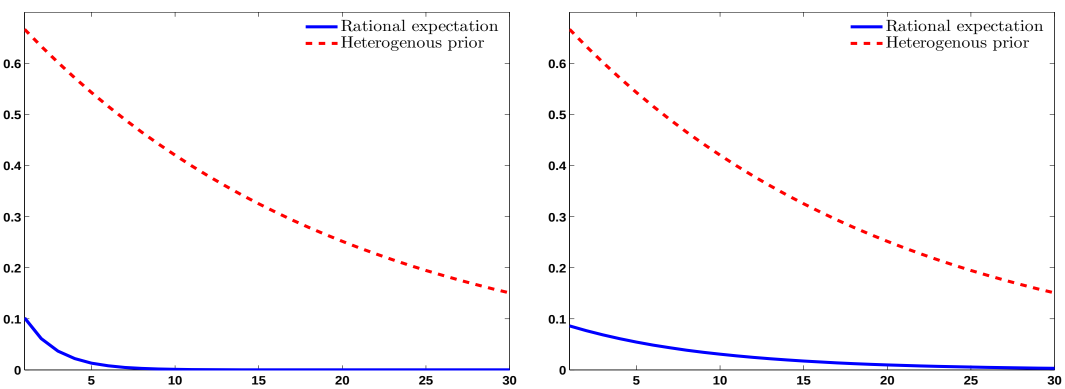


Figure 2.3: Comparing Approximation Accuracy for $\alpha = 0.8$



AR(2) approximation for y_t ARMA(4,2) approximation for y_t

Figure 2.4: Impulse Response of y_t to η Shock in Random-Player Model



Samll information friction. $\sigma_\epsilon = \sigma_u = 1$ Large information friction. $\sigma_\epsilon = \sigma_u = 4$

Chapter 3

Paradox of Thrift Recessions

3.1 Introduction

We develop a model in which recessions are triggered by households' desire to save more. Although mapped to a standard modern economy, our model features three ingredients that represent a mild departure from standard neoclassical growth theory:

1. Adjustment costs make it difficult to reallocate resources from the production of nontradable goods to the production of tradables, thereby preventing a rapid reallocation of production from consumption to investment or exportation.

2. The labor market is not competitive; instead, it is subject to search frictions à la Mortensen-Pissarides with Nash bargaining over the wage.

3. Goods markets for nontradables require active search from households. We extend [11] to an environment in which reductions in consumption generate reductions in productivity. This happens because households reduce consumption by reducing the number of consumption varieties as well as the quantity spent on each variety, and the reduction in the number of consumption varieties reduces the economy's capacity utilization rate.

We show that, contrary to standard growth models, households' desire to increase savings is a catalyst for a recession, not an expansion. Moreover, the onset of the recession reduces firms' value enough to reduce total household wealth despite households' increased savings. In this sense, our economy presents a *paradox of thrift*. Wealth recovers its initial value only after a few months. Although the novel mechanism that we model here—that households choose both the number of consumption varieties and the quantity of each variety that they consume—is not necessary for an increase in household savings or a negative wealth shock to spark a recession, its effects reduce by 2.3 times the size of the shocks needed for a given size of output contraction, which we deem to be large. Although our model economy does not include price rigidities, we document the extent to which such rigidities make recessions easier to obtain (via smaller shocks).

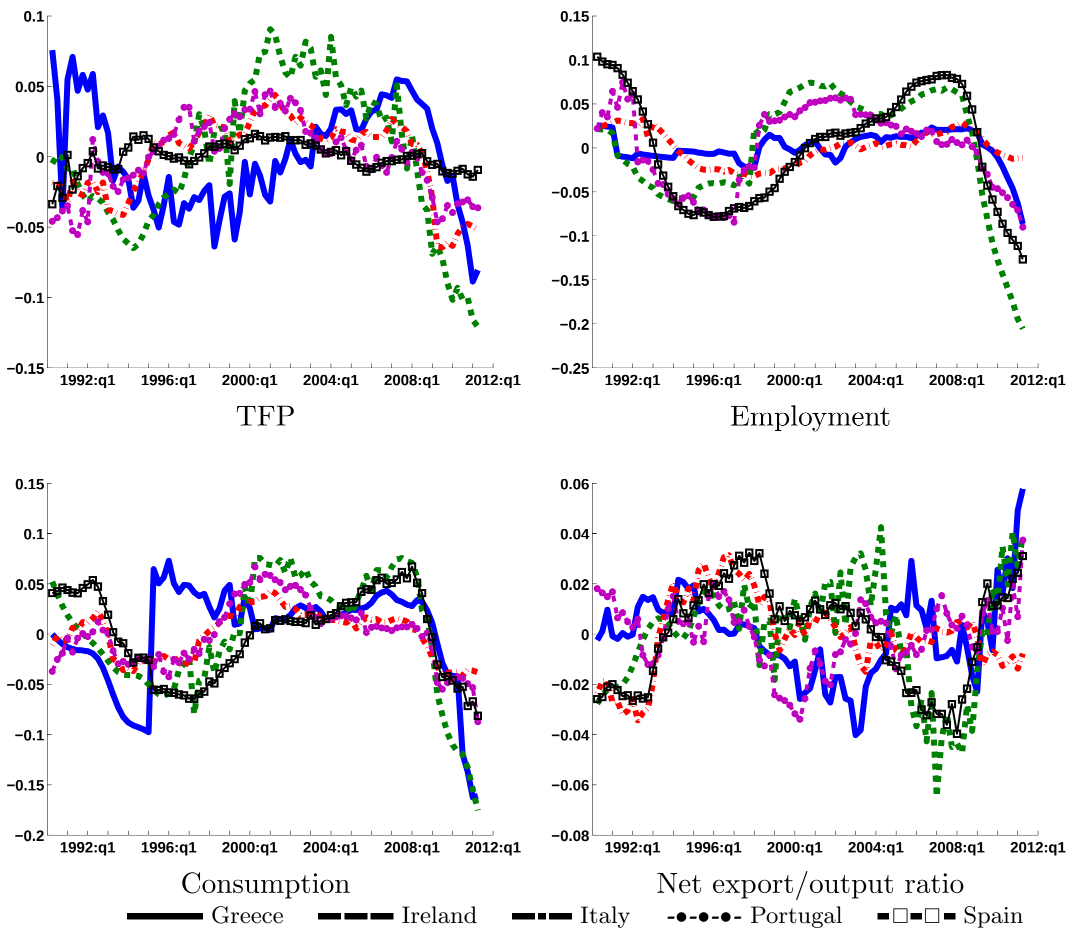
Our baseline economy uses shocks to patience to trigger households' increased desire to save for expository reasons.[1] We also study a recession that is generated by a sudden reduction in the wealth of households that triggers a reduction of consumption and hence a recession. Such a reduction in wealth could be linked to the experience of southern Europe (due perhaps to larger public debts than previously believed or to reductions in the generosity of their northern neighbors). We also provide a version of our model in which the recession is again generated via an increase in household's desire to save, only this time, instead of shocks to patience, shocks to financial intermediation—specifically, shocks to the costs to provide insurance to the unemployed—are responsible for sparking the recession. Our implementation of financial shocks has the advantage of being implementable within the representative agent framework.

Figure 3.1 displays the main aggregate variables in southern European countries. The current recession starting from 2008 features a big drop in measured total factor productivity (TFP), a fairly large decline in employment and consumption, and a rise in net exports. The predictions of our model are consistent with what is currently happening in southern Europe.

In order for a recession to be generated via households' increased desire to save, the environment has to be such that saving for the future through both investment and exports is difficult. In our economy, adjustment costs prevent a rapid reallocation of production from consumption to investment or exporting goods. [53] argue that without labor adjustment costs, too much shifting of resources into the tradable sector occurs, whereas [54] find that frictions in exports are necessary to match the gradual increase

[1] [47], [48], [49], [50], [51], and [52] all use shocks to the discount factor as the mechanism to trigger increases in savings. In these papers, insufficient demand triggers a recession because the economy is stacked at the zero lower bound on the nominal interest rate and there are rigid prices or wages.

Figure 3.1: Aggregate Economic Variables in Southern European Countries



in exports that follows a devaluation.[2]

Whatever reason that induces a household to save more—because its preferences have shifted toward the future, because it is poorer than before, or because a financial shock increases its desired wealth to earnings ratio—it would also make the household to want to work harder. The typical strategy to avoid this response is to prevent the labor market from clearing via some form of wage stickiness, so that labor demand will determine employment ([57], [55], and [58]). We follow a different approach, breaking down the static first-order condition of the household by posing standard labor market search frictions à la Mortensen–Pissarides. Clearly, wage rigidity makes recessions more likely, as we document later on, but even the mild deviation from competitive labor markets implied by the search friction is sufficient to generate recessions.

Our theoretical contribution is an extension to the work of [11], who model goods markets as having frictions where more intense search on the part of households translates into productivity gains as the economy operates at a higher capacity without more intense use of productive inputs. In their paper, search effort essentially behaves as a substitute for labor, and hence a desire to work harder or to save more would imply more search and increased productivity—hardly the trademark of recessions. In our paper, we provide a different channel through which search frictions affect productivity, ensuring that search and consumption are complements. Preferences are such that households have a taste for variety à la Dixit-Stiglitz, but each variety must be found, which requires search. In our model, when consumers want to increase their consumption, they do so by increasing the number of consumption varieties and consuming more of each variety. Hence, search effort is not a substitute for the resources spent when consuming but rather a complement to them. In this manner, an increased desire to save reduces productivity.

The predictions of our model for the number of varieties are consistent with the empirical findings of [59] and especially [60]: consumers increase consumption by increasing both the number varieties and the quantity of each variety. In fact, [60] shows that for the vast majorities of goods, both varieties and quantities are increasing in consumption expenditure, and that the Engel curve for varieties is upward sloping. The

[2] Extreme versions of this assumption can be found in [55], who assume that labor is not perfectly substitutable among different sectors, and the work of [56], [57], and [58], where tradable goods are given exogenously.

baseline model with a representative consumer displays no income dispersion, and the predictions of the model for are only for time series. However, in the extended version of the model that accommodates financial frictions, employed households members consume more varieties than unemployed members. The time-series and cross-section predictions of the model are a clear positive correlation between the number of varieties and households' income.

In the version of the model with financial frictions, employed and unemployed household members consume different amounts of varieties but also search at different intensities. The search friction implies that high-consumption agents (the employed) consume more varieties than low-consumption agents (the unemployed), which in general requires more search. Moreover, it also implies that the market splits locations into those that cater to the employed, which requires little search because of low market tightness, and those that cater to the unemployed, which necessitates more search. In this context, the unemployed substitute their own search for resources, finding cheaper prices. This behavior is documented in the United States for retirees and the unemployed by [61, 62] and for the unemployed by [63]. This extended model implements two features of the process of acquiring and enjoying consumption goods: finding out about goods and looking for cheaper prices for these goods. In the model, both activities involve more searching but have different effects. We think that our model captures the essence of the data showing that the poor search more per unit of consumption or per variety.

One crucial prediction in our model is that consumers reduce their search effort during recessions. The idea is that, because consumers search less, the probability that firms will sell their products decreases. This feature occurs at the same time that the employed search more than the unemployed. Consequently, we want to make a distinction between search effort and shopping time because we do not view these efforts as identical. In our model, we interpret search effort as the disutility associated with engaging in consumption, such as waiting for a restaurant table, searching for and booking movie tickets online, and driving to an out-of-town car dealership. We interpret shopping time, on the other hand, as the time spent looking for a lower price for a particular good or service, such as clipping newspaper coupons, searching for supermarket sales, and buying goods at shopping outlets far from home. During a recession, consumers cut their spending by eating at restaurants less often, watching fewer movies, and so on. At the same time, the associated search effort also decreases, which slows business for many firms. Shopping time, however, may actually increase as consumers spend more time looking for good deals, collecting coupons, and shopping at warehouse club stores in order to obtain lower prices for the same goods. Empirically, [64] document that the shopping time increased by around 7% during the last recession. Conversely, [61] show that unemployed workers and retirees spend more time shopping, but they spend eating at restaurants significantly less than employed workers do.

Related literature. A large and growing literature studies recessions generated by a disturbance to the discount factor. Recent key references include [47], [48], [49], [50], [51], and [52]. Although our paper shares the same view with this literature that a recession is the result of insufficient demand, it does not hinge on the economy being stacked at the zero lower bound on the nominal interest rate nor on the existence of rigid prices or wages. Instead, we provide a novel channel for increased savings generating a recession.

To provide a rationale for our theory that financial shocks to households are a catalyst for generating recessions, we turn to evidence provided by [65] and [66]. Using county-level data, they show that household demand is crucial in explaining aggregate economic performance and that it is also closely linked with households' financial conditions. In this context, [67] consider a shock to households' borrowing capacity in an Aiyagari-type model and show that this shock causes a decline in output. The shock does so, however, by reducing the work effort of the best-performing agents—hardly what characterizes the current Great Recession. Furthermore, if combined with nominal rigidities, the financial shock can potentially push the economy into a liquidity trap. [51] also study the effect of an exogenous reduction of the debt limit and highlight a Fisher deflation mechanism. [55] focus on the home equity borrowing issue and show that a drop in the leverage ratio reduces the liquidity of households and, correspondingly, their demand.

In terms of goods market frictions, [63] assume that unemployed workers spend more time shopping and that total shopping time increases in recessions mechanically as the unemployment rate rises. Similarly, in [68], households endogenously put more effort into shopping time during recessions because of the negative wealth effect. However, in both papers firms' capacity or the probability of selling their products is constant over the business cycle, a major departure from our paper. As mentioned, this paper is closely related to [11], who show how search frictions in the goods markets can make an economy with demand shocks look like an economy with productivity shocks and that estimating the model gives strong empirical support to this view of the cycle.

This paper is also related to the literature on sudden stops and business cycles in a small open economy. Most of the literature focuses on shocks that affect the production side directly, such as shocks to TFP, investment technology, interest rate premium, terms of trade, or firms' collateral constraints. We do not consider any of those shocks; instead, we consider shocks to the households' desire to spend, which endogenously change measured TFP. In [69], imported intermediate goods enter the production function and a reduction of imports leads to an endogenous decline in TFP. Our approach is quite different because we want to capture the idea that it is the internal demand of households that changes the production possibility frontier.

Section 3.2 explains how our new mechanism works in a simple two-period version of the model. The model that can be used for quantitative analysis is described in Section 3.3. Calibration details are found in Section 3.4, and the analysis of the baseline economy is in Section 3.5. Section 3.6 describes the quantitative importance of the new mechanism involving search frictions that we develop in this paper, and we deem this mechanism to be large. Section 3.7 explains that in versions of the growth model with flexible prices, both adjustment costs and labor market frictions are necessary ingredients for generating recessions via household increases in savings arising from shocks in patience. Section 3.8 describes what happens when the baseline economy becomes suddenly poorer (wealth destruction shocks). Section 3.9 analyzes how our findings vary as we change some particular targets. We look at various sizes of adjustment costs in the tradable sector, at alternative job finding and losing rates, and at different wage determination protocols (staggered wage contracts and constant labor share). We also explore the performance of the model economies with respect to some other margins (elasticity of substitution between tradables and nontradables, size of vacancy costs, labor matching elasticity, goods market elasticity, and the elasticity of substitution between varieties of nontradable consumption). Throughout our analysis, all versions of the economy have been recalibrated so that it is the targets that are constant and not the parameter values that implement them. Section 3.10 extends the model to accommodate financial shocks as the trigger to households' increased desire to save without the need to abandon the representative agent abstraction. Section 3.11 concludes. A technical appendix describes technical details and provides additional tables of interest.

3.2 A Simple Version of the Model

In our model, households choose both the number of consumption varieties and the quantity of each variety that they consume. To see how this mechanism works, consider a simple two-period version of our model. Households care about two sets of goods in the first period, which we call tradables and nontradables, and about the amount of tradable goods saved for the second period. Nontradables come in different varieties that have to be searched for and found before any purchase of that variety is made. Households choose how many varieties to consume because, even though they have a taste for variety, they incur a disutility when they search. Nontradable consumption varieties provide utility via a Dixit-Stiglitz aggregator, $\left(\int_0^{\mathrm{I}} c_{Ni}^{\frac{1}{\rho}}\, d_i\right)^{\rho}$. Under equal consumption of each variety, this aggregate collapses to $c_N\ \mathrm{I}^{\rho}$.[3] We can write the utility function of the household as $u(c_T, \mathrm{I}^{\rho} c_N, d) + \beta v(b')$, where d is search effort and the second-period terms have the standard interpretation of a discount rate and an indirect utility function of savings b'. Households have an endowment of one unit of the tradable good and they can borrow or save at a zero interest rate; they also own the nontradable-producing firms.

There is a continuum of measure one of consumption varieties. Households choose how many of those varieties to consume $\mathrm{I} < 1$ by means of exerting sufficient search effort, d, to overcome a matching friction. We denote by $\Psi^d(Q^g)$ the probability that a unit of search effort finds a variety, where Q^g is market tightness in the goods market.

[3] We deal explicitly with the determination of the price of each variety below (Section 3.3), where we explicitly account for the possibility of choosing different amounts for each variety.

We write the household problem as

$$\max_{c_T, \mathrm{I}, c_N, d, b'} \qquad u\left[c_T, \mathrm{I}^{\rho}\, c_N, d\right] + \beta \;\; v(b') \tag{3.1}$$

$$c_T + \mathrm{I}\, c_N\, p + b' = \pi_N + 1, \tag{3.2}$$

$$\mathrm{I} = d\;\Psi^d(Q^g), \tag{3.3}$$

where π_N are the profits from the firms in the nontradable sector. The solution to this problem yields demand functions that, using aggregate notation (capital letters denote aggregate quantities), are $C_T(p, Q^g, \pi_N; \beta)$, $C_N(p, Q^g, \pi_N; \beta)$, $\mathrm{I}(p, Q^g, \pi_N; \beta)$, $B'(p, Q^g, \pi_N; \beta)$, and $D(p, Q^g, \pi_N; \beta)$, where we are explicitly posing the dependence on the price of nontradables, on market tightness, and on profits, as well as on the households' discount rate which we can treat as a source of shocks.

There is a continuum of measure one of firms producing the nontradables, and each one of those firms has a measure one of locations. The probability that a location finds a household is $\Psi^f(Q^g) = \Psi^f(\frac{1}{D}) = M^g(D, 1)$, and the probability that a search unit, or shopper, finds a variety is $\Psi^d(Q^g) = \Psi^d(\frac{1}{D}) = \frac{M^g(D,1)}{D}$. In equilibrium $\Psi^f(Q^g) = \mathrm{I}$.

Firms and consumers are matched in the nontradable goods markets according to matching function $M^g(D, T)$, where D is the aggregate search effort of households and T is the measure of firms.

The equilibrium conditions are simple given that production is predetermined:

$$Q^g = \frac{1}{D(p, Q^g, \pi; \beta)}, \tag{3.4}$$

$$1 = C_T(p, Q^g, \pi; \beta) + B'(p, Q^g, \pi; \beta), \tag{3.5}$$

$$F^N = C_N(p, Q^g, \pi; \beta), \qquad \text{or} \qquad \pi_N = p\; F^N\; \Psi^f\left(\frac{1}{D}\right). \tag{3.6}$$

The first condition states that market tightness is the result of household search; the second, that tradable output is either consumed or saved; and the third, that the amount of nontradable consumption of every variety is what is available at each location. Walras' law allows us to choose between the last two equations.

To see what is special in this economy, note that in standard models, $Q^g = 1$ and the relative price of the two consumptions adjusts to clear the market. Since the interest rate is fixed, preferences determine savings. If both types of consumption are complements, when households want to save more, say, because of bigger β, a decrease in the price of the nontradables maintains market clearing, which in standard models occurs without any change in its quantity. This is not the case in our economy. Output of nontradables can decrease despite using all factors of production. With the preferences that we pose,[4] households reduce nontradable consumption by reducing the number of varieties as well as the amount consumed of each variety. In this simple economy, the amount consumed of each variety is predetermined so it cannot drop, but the number of varieties does drop, and hence so does total output because the economy is now operating at a lower capacity. In this example, profits decrease. If this mechanism were persistent, future profits would also decrease, which is why the paradox of thrift may show up.

This simplified version of our economy illustrates how an increase in savings generates a reduction in output via a reduction in measured TFP without either technology or the measured inputs changing. It is the search efforts of households that decrease. We next build these ideas into a growth model suitable for dynamic quantitative analysis.

3.3 The Baseline Economy

Our baseline economy poses a small open economy with the interest rate set by the rest of the world.[5] There is a representative household, or a family with a measure one of individual members, all of whom can work. The household fully insures all of its members.

Goods There are two types of goods: tradables, which can be imported and exported and used for consumption and investment, and nontradables, which can be used only for local consumption. Nontradables are subject to additional frictions that we now describe in detail.

There is a measure one of varieties of nontradables $i \in [0, 1]$, and each one is produced by a monopoly that posts prices and has to deliver the amount of goods demanded at that price. Each one of these firms or varieties has a measure one continuum of

[4] We have the type of preferences described in [70] (hereafter GHH preferences) between consumption and search effort, although many other types yield the same properties.

[5] To ensure that this section is self-contained, some repetition with respect to the previous section may occur.

locations, each with its own capital and labor and a standard constant returns to scale (CRS) technology, $F^N(k,n)$.

Each period, consumers have to search and find varieties, and they value both the number of varieties and the quantity consumed of each variety. To obtain varieties, consumers need to search for them, incurring a shopping disutility while doing so. Shoppers that find a variety are randomly allocated to one and only one of its locations. We denote the aggregate measure of shoppers or shopping effort as D. The total number of matches between shoppers and firms is determined by a CRS matching function $M^g(D,1)$. If we denote market tightness in the goods market by $Q^g = \frac{1}{D}$, the probability that a shopper finds a location becomes

$$\Psi^d(Q^g) = \frac{M^g(D,1)}{D}, \tag{3.7}$$

and the probability that a location in each firm finds a shopper is equal to the measure of locations of each variety that is filled and is given by

$$\Psi^f(Q^g) = \frac{M^g(D,1)}{1}. \tag{3.8}$$

Firms in the tradable goods sector operate in a standard competitive market, and we use tradables as the numeraire. Let the aggregate production function of tradables be given by $F^T(k,n)$.

Labor Market Work is indivisible, and all workers are either employed or unemployed. The labor market has a search friction à la Mortensen and Pissarides: firms have to post job vacancies, and unemployed workers are matched to those vacancies via a neoclassical matching function. There is a single labor market where all firms post vacancies, denoted as V_N by nontradable producers and V_T by tradable producers. The number of new matches is given by a CRS matching function $M^e(U,V)$, where U is the unemployment rate and $V = V_N + V_T$ is the total number of vacancies. The probability of finding a job for an unemployed worker is

$$\Phi^w(Q^e) = \frac{M^e(U,V)}{U}. \tag{3.9}$$

The probability of a job vacancy being filled is

$$\Phi^f(Q^e) = \frac{M^e(U,V)}{V}, \tag{3.10}$$

where $Q^e = \frac{V}{U}$ is labor market tightness. An employed worker faces a constant probability λ of job loss. Wage determination will be discussed in Section 3.3.

Preferences The representative household cares about a consumption aggregate c_A, shopping effort d, and the fraction of its members that work n. The aggregate consumption basket is valued via an Armington aggregator of tradables and nontradables, whereas nontradables themselves aggregate via a Dixit-Stiglitz formulation with a variable upper bound, yielding

$$c_A = \left(\omega \left[\int_0^{\mathrm{I}} c_{N,i}^{\frac{1}{\rho}} \, di \right]^{\frac{\rho(\eta-1)}{\eta}} + (1-\omega) c_T^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}}, \tag{3.11}$$

where $c_{N,i}$ is the amount of nontradable good of variety i, $\mathrm{I}_N \in [0,1]$ is the measure of varieties of nontradable goods that the household has acquired, $\rho > 1$ determines the substitutability among nontradable goods, and η controls the substitutability between nontradables and tradables. The period utility function is given by $u(c_A, d, n)$. Even though the search and matching features imply that workers are rationed, the disutility of working matters for wage determination. Households discount the future at rate β and are expected utility maximizers.

Asset Markets Households own the firms inside their own country that yield dividends $\pi_N + \pi_T$ and receive labor income. Households have access to (noncontingent) borrowing and lending from abroad at an internationally determined interest rate r. We denote the foreign asset position by b.

The state vector for a household, in addition to the aggregate state S to be specified later, is the pair (b, n), its assets and the fraction of its members with a job. Households take as given the prices of each variety p_i, the wage w, the probability of finding a variety Ψ^d, the probability of finding a job Φ^w, and the firms' dividends, all of which are equilibrium functions of the state.

Household's Problem We can write the recursive problem of the household as

$$V(S, b, n) = \max_{c_T, \mathrm{I}_N, c_{N,i}, d} u(c_A, d, n) + \beta \, \mathbb{E} \left\{ V(S', b', n') \mid \theta \right\}, \tag{3.12}$$

subject to the definition of the consumption aggregate (3.11) and

$$\int_0^{\mathrm{I}} p_i(S)\; c_{N,i}\; di + c_T + b' = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S), \tag{3.13}$$

$$\mathrm{I} = d\; \Psi^d[Q^g(S)], \tag{3.14}$$

$$n' = (1-\lambda)n + \Phi^w[Q^e(S)](1-n), \tag{3.15}$$

$$S' = G(S). \tag{3.16}$$

The household's budget constraint is (3.13). The requirement that varieties have to be found, which requires search effort d and depends on the goods market tightness, is given by (3.14). The evolution of the household's employment is (3.15), and condition (3.16) is the rational expectations requirement.

We define standard aggregates of nontradable consumption bundles and prices:

$$c_N = \left[\frac{1}{\mathrm{I}} \int_0^{\mathrm{I}} c_{N,i}^{\frac{1}{\rho}}\; di\right]^{\rho}, \tag{3.17}$$

$$p = \left[\frac{1}{\mathrm{I}} \int_0^{\mathrm{I}} p_i^{\frac{1}{1-\rho}}\; di\right]^{1-\rho}. \tag{3.18}$$

Note that p is not a function of I. We can derive the demand schedule for the goods from a particular variety (or firm) i, given c_N and p,

$$c_{N,i} = \left(\frac{p_i}{p}\right)^{\frac{\rho}{1-\rho}} c_N. \tag{3.19}$$

We can rewrite the consumption aggregate (3.11) and the budget constraint (3.14) as[6]

$$c_A = \left[\omega \left(c_N \mathrm{I}_N^{\rho}\right)^{-\eta} + (1-\omega)\; c_T^{-\eta}\right]^{-\frac{1}{\eta}}, \tag{3.20}$$

$$p(S)c_N\mathrm{I} + c_T + b' = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S). \tag{3.21}$$

The first-order conditions are

$$u_{c_N} = p(S)\mathrm{I}u_{c_T}, \tag{3.22}$$

$$u_{\mathrm{I}} = p(S)c_N u_{c_T} - \frac{u_d}{\Psi^d[Q^g(S)]}, \tag{3.23}$$

$$u_{c_T} = (1+r)\mathbb{E}\left\{\beta u'_{c_T} \mid \theta\right\}. \tag{3.24}$$

[6] See Appendix A.3.1 for a more detailed derivation.

Equation (3.22) shows the optimality condition between nontradable and tradable goods. Equation (3.23) determines the trade-off between the number of varieties and the quantity consumed of each variety: since $\rho > 1$, increasing I is more efficient than increasing c_N, but searching for different firms is costly. An implication of this equation is that in general, increases in consumption imply an increase of both the amount consumed of each variety and the number of varieties. Equation (3.24) is the standard Euler equation.

Firms in the Nontradable Goods Sector Firms post prices in each location. If a shopper shows up, it chooses how much of the good to buy according to the demand schedule derived earlier. We rewrite this demand schedule as a function that depends explicitly on both the aggregate state and goods prices:

$$C(p_i, S) = \left(\frac{p_i}{p(S)}\right)^{\frac{\rho}{1-\rho}} C_N(S). \tag{3.25}$$

To produce the goods, firms have a CRS production function that uses capital k and labor n. Recall that there is also a search friction in the labor market, so firms need to post vacancies at cost κ per unit in order to increase their labor the following period. Both investment and vacancies use tradable goods. The individual firm's state is (k, n), and its problem is

$$\Omega^N(S, k, n) = \max_{p_i, i, v} \Psi^f[Q^g(S)]p_i C(p_i, S) - w(S)n - i - v\kappa + \mathbb{E}\left\{\frac{\Omega^N(S', k', n')}{1+r} \mid \theta\right\}, \tag{3.26}$$

subject to

$$C(p_i^c, S) \leq F^N(k, n), \tag{3.27}$$

$$k' = (1-\delta)k + i - \phi^N(k, i), \tag{3.28}$$

$$n' = (1-\lambda)n + \Phi^f[Q^e(S)]v, \tag{3.29}$$

$$S' = G(S), \tag{3.30}$$

where $\phi^N(k, i)$ is a capital adjustment cost, which slows down the adaptation of firms to new conditions. Note that both capital and employment are predetermined, and therefore firms have to set the price such that demand does not exceed output. The first-order conditions are

$$\frac{(1+r)}{1-\phi_i^N} = \mathbb{E}\left\{\Psi^f[Q^g(S')]p_i'(F_k^N)'\frac{1}{\rho} + \frac{1-\delta-(\phi_k^N)'}{1-(\phi_i^N)'} \mid \theta\right\}, \tag{3.31}$$

$$\frac{\kappa}{\Phi^f[Q^e(S)]} = \frac{1}{1+r}\mathbb{E}\left\{\Psi^f[Q^g(S')](p_i^c)'(F_n^N)'\frac{1}{\rho} - w(S') - \frac{(1-\lambda)\kappa}{\Phi^f[Q^e(S')]} \mid \theta\right\} \tag{3.32}$$

Equations (3.31) and (3.32) equate the marginal benefits and marginal costs of increasing investment and vacancies. All firms choose the same price in equilibrium, i.e., $p_i = p(S)$ for all $i \in [0, 1]$.

Firms in the Tradable Goods Sector Unlike firms in the nontradable goods sector, firms in the tradable goods sector operate in a frictionless, perfectly competitive environment. To accommodate the possibility of decreasing returns to scale, we pose that in addition to capital and labor, firms also need to use another factor, land, available in fixed supply, as an input of production. Without loss of generality, we assume that there is a firm that operates each unit of land. There are also adjustment costs to expand capital and employment, given by functions $\phi^{T,k}(k, i)$ and $\phi^{T,n}(n', n)$, which makes it difficult for this sector to expand quickly. The problem of the firms in the tradable goods sector is

$$\Omega^T(S, k, n) = \max_{i,v} F^T(k, n) - w(S)n - i - v\kappa - \phi^{T,n}(n', n) + \mathbb{E}\left\{\frac{\Omega^T(S', k', n')}{1+r} \mid \theta\right\}, \tag{3.33}$$

subject to

$$k' = (1-\delta)k + i - \phi^{T,k}(k, i), \tag{3.34}$$

$$n' = (1-\lambda)n + \Phi^f[Q^e(S)]v, \tag{3.35}$$

$$S' = G(S). \tag{3.36}$$

The first-order conditions are

$$\frac{1+r}{1-\phi_i^{T,k}} = \mathbb{E}\left\{\left(F_k^T\right)' + \frac{1-\delta-(\phi_k^{T,k})'}{1-(\phi_i^{T,k})'} \mid \theta\right\}, \tag{3.37}$$

$$\frac{\kappa}{\Phi^f[Q^e(S)]} + \phi_{n'}^{T,n} = \frac{\mathbb{E}\left\{(F_n^T)' - w(S') - (\phi_n^{T,n})' + (1-\lambda)\frac{\kappa}{\Phi^f[Q^e(S')]} \mid \theta\right\}}{1+r}. \tag{3.38}$$

Equations (3.37) and (3.38) are similar to the optimality condition for nontradable firms. When necessary, we use the subindex T to refer to tradables.

Wage Determination The wage rate is determined via Nash bargaining. Unlike in [71] and [72], where agents internalize the effect of additional saving on their bargaining position, here we assume that individual workers and firms take the wage as given and act as though a worker-firm pair like themselves bargain over the wage rate.[7] The value of an additional employed worker for the household with wage w is

$$\widetilde{V}_n(w,S) = wu_{c_T}(S) - \varsigma + \beta\big(1 - \lambda - \Phi^w[Q^e(S)]\big)\mathbb{E}\{V_n(S') \mid \theta\}, \tag{3.39}$$

where $V_n(S) = \widetilde{V}_n(w(S),S)$ and $u_{c_T}(S)$ is the marginal utility for the representative household. The value of an additional worker for a firm in the nontradable goods sector with wage w is

$$\widetilde{\Omega}_n^N(w,S) = \Psi^f[Q^g(S)]p(S)F_n^N(S)\frac{1}{\rho} - w + \frac{(1-\lambda)}{1+r}\mathbb{E}\{\Omega_n^N(S') \mid \theta\} \tag{3.40}$$

and for a firm in the tradable goods sector is

$$\widetilde{\Omega}_n^T(w,S) = F_n^T(S) - w - \phi_n^{T,n}(S) + \frac{(1-\lambda)}{1+r}\mathbb{E}\{\Omega_n^T(S') \mid \theta\}, \tag{3.41}$$

where $\Omega_n^N(S) = \widetilde{\Omega}_n^N(w(S),S)$ and $\Omega_n^T(S) = \widetilde{\Omega}_n^T(w(S),S)$. Firms may not value workers equally, that is, $\widetilde{\Omega}_n^T$ may not be the same as $\widetilde{\Omega}_n^N$. We assume that the wage that is set in the market is the outcome from a bargaining process between a representative worker and a weighted value of the valuation of the worker by firms, with weights given by the employment share of each sector. With these elements, the Nash bargaining problem becomes

$$w(S) = \max_w \left[\widetilde{V}_n(w,S)\right]^{\varphi}\left[\chi(S)\widetilde{\Omega}_n^N(w,S) + (1-\chi(S))\widetilde{\Omega}_n^T(w,S)\right]^{1-\varphi}, \tag{3.42}$$

where φ is the bargaining power of households and $\chi(S) = \frac{n_N}{n_N+n_T}$ is the employment share of the nontradable goods sector. Taking the derivative with respect to w yields

[7] If instead, for example, we allow an individual household to bargain directly with firms for their workers, the household will have an incentive to accumulate additional assets to improve its outside option and increase the wage rate when bargaining. As shown in both [71] and [72], however, the effect of additional savings on the wage rate is small when the household's wealth is not close to zero, as is the case with representative households. This issue is also discussed in [73].

the first-order condition

$$\varphi u_{c_T}(S)\left[\chi(S)\widetilde{\Omega}_n^N(w,S)+(1-\chi(S))\widetilde{\Omega}_n^T(w,S)\right]=(1-\varphi)\widetilde{V}_n(w,S). \tag{3.43}$$

In steady state, the wage rate is given by

$$w=\varphi\left[\chi\left(\Psi^f(Q^g)pF_n^N\frac{1}{\rho}\right)+(1-\chi)F_n^T+Q^e\kappa\right]+(1-\varphi)\frac{\varsigma}{u_{c_T}}. \tag{3.44}$$

We can think of the wage rate as a weighted average of the marginal product of labor and the savings on vacancy postings on the one hand, and of the worker's forfeited leisure on the other.[8]

We will also explore staggered wage environments in which wages are set through Nash bargaining, but the workers and firms can only renegotiate contracts with a certain probability. In Section 3.9, we investigate how wage rigidity affects the model's performance.

Aggregate State The aggregate state of the economy consists of the shocks, θ, the production capacity of the economy (capital and labor in each sector), and the net foreign asset position, $S=\{\theta,K_N,N_N,K_T,N_T,B\}$.

Equilibrium Equilibrium is a set of decision rules and values for the household: $\{c_N,c_T,d,\mathrm{I},b',V\}$ as functions of its state (S,b,n), nontradable and tradable firms' decision rules and values: $\{i_N,v_N,k'_N,p_i,\Omega^N\}$, and $\{i_T,v_T,k'_T,\Omega^T\}$ as functions of their states (S,k_N,n_N) and (S,k_T,n_T), and aggregate variables for nontradable goods C_N and tradable goods C_T, total employment N, total vacancies V, total shopping effort D, labor market tightness Q^e, goods market tightness Q^g, total bond holdings B, aggregate capital $\{K_N,K_T\}$, employment $\{N_N,N_T\}$, investment $\{I_N,I_T\}$, vacancies $\{V_N,V_T\}$, and profits $\{\pi_N,\pi_T\}$ in both sectors, the aggregate price index p, and the wage rate w as functions of aggregate state $S=(\theta,K_N,N_N,K_T,N_T,B)$, such that

1. Policy and value functions solve the corresponding problems.

2. Individual decisions are consistent with aggregate variables.

[8] A minor departure from the standard labor search model is that the wage rate has a dynamic component under uncertainty. The reason is that firms discount future profits using the world interest rate r instead of the households' stochastic discount factor.

3. The wage rate w is determined via the Nash bargaining process (3.42).

4. Tradables and nontradables markets clear.

Note that in equilibrium, $\mathrm{I} = \Psi^f(Q^g)$ (i.e., consumers' demand directly translates into firms' capacity). Also note that this economy may have multiple steady states with varying foreign asset positions.[9] In fact, any unexpected temporary change in any parameter will result in the economy being in a long-run position that is different from the one in which it started.

3.4 Calibration

We start by discussing some details of national accounting, describing how the variables in the model correspond to those measured in the national income and product accounts (NIPA) (Section 3.4.1). We then discuss the functional forms used and the parameters involved (Section 3.4.2), and finally we set the targets that the model economy has to satisfy (Section 3.4.3).

3.4.1 NIPA and Variable Definitions Issues

Real output is given by

$$Y = p^* \Psi^f(Q^g) F^N(K_N, N_N) + F^T(K_T, N_T), \tag{3.45}$$

where p^* is the steady-state price of nontradables. This amounts to measuring output using base year prices instead of current prices. Let $Y_N = p^* \Psi^f(Q^g) F^N(K_N, N_N)$ denote nontradable output and $Y_T = F^T(K_T, N_T)$ tradable output. Total consumption is $C = p^* \mathrm{I} C_N + C_T$. Total employment is $N = N_N + N_T$. Total capital is $K = K_N + K_T$. Total investment is $I = I + I_T$. Let υ denote the labor share in steady state. Total factor productivity or the measured Solow residual, Z, is defined as

$$Z = \frac{Y}{K^{1-\upsilon} N^{\upsilon}}. \tag{3.46}$$

[9] A stationary recursive equilibrium for the stochastic version requires $1 + r < \beta^{-1}$ because of precautionary savings. Given the small quantitative nature of these issues, we ignore them in the discussion that follows.

3.4.2 Functional Forms and Parameters

Preferences We adopt GHH preferences between consumption and shopping effort, which suffices to yield that consumption per variety and the number of varieties move together, making measured TFP procyclical. Other specifications do not have this property (see Appendix A.3.2 for a more detailed discussion). The working disutility enters as an additively separable term (any consideration of Frisch elasticities is irrelevant because the work disutility matters only for wage determination). The period utility function is then given by

$$u(c_A, d, n) = \frac{1}{1-\sigma} (c_A - \xi d)^{1-\sigma} - \varsigma n. \tag{3.47}$$

The units for search effort do not matter. We write ξ only because we have a steady-state target for d.

The preference parameters are the discount factor β, the risk aversion parameter of sorts, σ, the parameter that determines average shopping effort ξ, and the working disutility ς. As discussed before, c_A, the aggregator of consumption, is

$$c_A = \left[\omega \left(c_N \mathrm{I}_N^{\rho} \right)^{\frac{\eta-1}{\eta}} + (1-\omega) c_T^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}, \tag{3.48}$$

where η is the elasticity of substitution between nontradable and tradable goods, ρ is the elasticity of substitution among nontradables, and ω is the nontradable home bias or home bias parameter.

Technology The production function of nontradables is

$$F^N(k, n) = z_N \, k^{\theta^N} \, n^{1-\theta^N}, \tag{3.49}$$

where z_N is a parameter determining units. The production function of tradables is

$$F^T(k, n) = z_T \, k^{\theta_k^T} n^{\theta_n^T} \, L^{1-\theta_k^T-\theta_n^T} = z_T k^{\theta_k^T} n^{\theta_n^T}. \tag{3.50}$$

Land is limited, $L = 1$, hence there are decreasing returns to scale (DRS) in capital and labor.

Adjustment Costs The capital adjustment cost in the nontradable goods sector is given by

$$\phi^N(k,i) = \frac{\epsilon^N}{2}\left(\frac{i}{k} - \delta\right)^2 k, \tag{3.51}$$

where δ is the capital depreciation rate and ϵ^N determines the size of the adjustment cost. Similarly, the capital adjustment cost in the tradable goods sector is

$$\phi^{T,k}(k,i) = \frac{\epsilon^{T,k}}{2}\left(\frac{i}{k} - \delta\right)^2 k. \tag{3.52}$$

In addition to the capital adjustment cost, producing for tradable goods also involves adjustment costs in employment,

$$\phi^{T,n}(n',n) = \frac{\epsilon^{T,n}}{2}\left(\frac{n'}{n} - 1\right)^2 n. \tag{3.53}$$

Nash Bargaining Workers' bargaining power is φ.

Matching The matching technologies in the labor and nontradable goods markets are

$$M^e(U,V) = \nu^e U^\mu V^{1-\mu}, \tag{3.54}$$

$$M^g(D,T) = \nu^g D^\alpha T^{1-\alpha}, \tag{3.55}$$

where μ and α determine the elasticity of the matching probability with respect to market tightness.

Wealth This economy has a continuum of steady states differing in the net foreign asset position. Here, we look at the steady state with a zero net foreign asset position.

3.4.3 Targets and Values

We choose a period to be six weeks so that the unemployment duration can be short. A first group of 5 parameters can be determined exogenously (i.e., they imply targets that are independent of the equilibrium allocation). Table 3.1 summarizes the targets and the implied parameter values. We set risk aversion to 2 and the rate of return to 4% annually. We choose the elasticity of substitution between tradable and nontradable goods, η, to be 0.83, the benchmark value used in [74], which is also similar to the value estimated by [75]. We set the elasticity of the labor matching rate with respect to labor market tightness, μ, to 0.5, which lies in the middle of existing empirical estimates.[10] The price markup ρ reflects the substitutability among the nontradable goods as well as the price markup that the monopolistic firms will set. The literature provides no solid evidence on how large this parameter should be. [79], using micro reasoning, claim that the implied markup is not significantly greater than 1 (1.03), whereas [80] estimate the price markup using macro data and obtain a value ranging from 1.01 to 1.85. Here, we have set $\rho = 1.05$.

Table 3.1: Exogenously Determined Parameters of the Baseline Economy

Parameter	Value
Risk aversion, σ	2.0
Annual rate of return, β	$\frac{1}{\beta^8} - 1 = 4\%$
Labor matching elasticity, μ	0.50
Elasticity of substitution between tradables and nontradables, η	0.83
Price markup ρ	1.05

The second group of parameters is not the direct implication of any single target, but can be determined by steady-state conditions, which requires the specification of sufficient steady-state moments. There are 14 such parameters: 3 preference parameters, $\{\omega, \xi, \varsigma\}$, 6 production parameters $\{z_N, z_T, \theta_N, \theta_T^k, \theta_T^n, \delta\}$, 2 search friction parameters $\{\nu^e, \nu^g\}$, and 3 labor market parameters $\{\varphi, \lambda, \kappa\}$. Table 3.2 lists the steady-state targets and associated parameters for the baseline economy.[11] Although many of the parameters in Table 3.2 have economic meaning, others are just the determinants of units. Accordingly, the table displays the unit parameters separately.

The targets of the job flows are standard: an employment rate of 93% to accommodate movements in labor force participation and a monthly job finding rate of 45%. We target a capacity utilization or occupancy rate of 81%, which is the average of the

[10] [76] considers the elasticity to be 0.4, [77] 0.72, and [78] 0.24.

[11] The term "associated" refers to the attempt to link targets and moments according to some intuitive link between them. Mathematically, they are all interdependent.

official data series ([81]), and a labor share of 60% in both the nontradable sector and tradable goods sector. We target the tradable goods to output ratio (share of tradables) to be 30%. Following the literature, the tradable goods sector typically includes agriculture, mining, and manufacturing industries. We choose a contribution of land to output of tradables to be a size equal to that of capital, which determines the size of the decreasing returns of the sector. We target a vacancy posting cost to output ratio of 0.0374. The literature has few direct estimates of this vacancy cost. [82] report the flow vacancy costs to be 4.3% of the quarterly wage and the training costs to be 55% of the quarterly wage. We consider the vacancy costs as the sum of all of these recruitment-related costs.[12] [83] and [84] have a smaller vacancy cost because they take only the flow vacancy cost into account. [77] sets the workers' bargaining power equal to 0.72 solely to satisfy the Hosios condition, whereas [83] use a much smaller number: 0.05. We target the value of the unemployment (or leisure) $\frac{\varsigma}{u_{c_T}}$ to wage ratio of 0.35, and it turns out that the bargaining power $\varphi = 0.42$, which is in the middle of those two polar cases. We also target an annual capital-output ratio of 2.75.

We normalize output, the relative price of nontradables, and market tightness in both labor and goods markets to 1. The parameters more closely related to these unit targets are the definition of units in the production function z_T and z_N as well as the value of leisure ς, and the parameter that transforms search units into utils, ξ.

The last group of parameters has no steady-state implications, and we set these parameters according to their dynamic implications (see Table 3.3). We choose the capital adjustment cost in the nontradable goods sector ϵ^N such that the immediate response of nontradable investment i_N is four times as large as the response of nontradable output Y_N at its lowest point. That is, we want a 1% increase (decrease) in nontradable output in our exercises to be associated with a 4% decrease in investment in nontradables. We want output in the tradable sector to expand by 5% when total real output Y drops by 1% (which may even be too large a target), and we want adjustments in labor and capital of tradables to be symmetric. A higher α implies a larger volatility of capacity in the goods market, as well as a larger role played by consumers' demand in shaping TFP. We choose α such that when total output declines by 1%, the employment rate

[12] In the robustness check, we will show that by targeting a smaller value of the vacancy costs, the model results are improved. Therefore, the target we use here should be considered as a conservative benchmark.

Table 3.2: Steady-State Targets and Associated Parameters of the Baseline Economy

Target	Value	Parameter	Value
Share of tradables, $\frac{F_T^*}{Y^*}$	0.3	ω	0.91
Unemployment rate, U^*	7%	λ	0.05
Monthly job finding rate	45%	ν^e	0.67
Occupancy rate, $\frac{C_N^*}{F_N^*}$	0.81	ν^g	0.81
Capital to output ratio, $\frac{K^*}{Y^*}$	2.75	δ	0.007
Labor share in nontradables	0.6	θ_N	0.67
Labor share in tradables	0.6	θ_T^N	0.64
Equal role of capital and land in tradables,	$2\theta_T^K + \theta_T^N = 1$	θ_T^K	0.18
Vacancy posting to output ratio	0.037	κ	0.53
Value of leisure to wage ratio	0.35	φ	0.42
Units Parameters			
Output, Y^*	1	z_N	0.45
Relative price of nontradables, p^*	1	z_T	0.52
Market tightness in labor markets, $\frac{U^*}{V^*}$	1	ς	0.54
Market tightness in goods markets, D^*	1	ξ	0.02

decreases by 0.5%.

3.5 A Recession Induced by a Shock to the Discount Factor

We are now ready to explore the properties of recessions induced by households' attempt to save more. We use relatively permanent shocks to the discount factor as a proxy for financial shocks, but in Section 3.10 we extend the model so as to accommodate explicit financial shocks that make consumption smoothing difficult.

A household that suffers a shock to its patience wants to work harder and save more by reducing its consumption of both tradables and nontradables. Its willingness to work more translates to a wage drop but not in more work unless firms pose more job vacancies. Less tradable consumption translates directly into more net exports. Given our assumptions on preferences, households implement a reduction of nontradable consumption by reducing both the number of consumption varieties and the quantity of

Table 3.3: Dynamically Calibrated Parameters of the Baseline Economy

Target	Value	Parameter	Value
Response of nontradable investment	$\frac{\Delta I^N}{\Delta Y^N} = 4$	ϵ^N	14.17
Response of tradable output	$\frac{\Delta Y^T}{\Delta Y} = -5$	$\epsilon^{T,n}$	7.70
Symmetry of tradable adjustment costs	$\epsilon^{T,k} = \epsilon^{T,n}$	$\epsilon^{T,k}$	7.70
Response of labor to output	$\frac{\Delta N}{\Delta Y} = .5$	α	0.22

each variety. This in turn reduces productivity (fewer locations are occupied) and the prices of nontradables and, consequently, the output and profits of nontradables for a few periods. The tradable sector expands because of the reduction in wages, but only in a limited way because of the decreasing returns to scale of this sector and to the adjustment costs that slow its expansion.

Specifically, consider the following AR(1) stochastic process: $\log \tau_t = \rho^\tau \log \tau_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}(0, \sigma^\tau)$, with persistence $\rho^\tau = 0.95$. Now consider the following version of the utility function:

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \tau_t \beta^t u(c_t, d_t, n_t)\right\}. \tag{3.56}$$

Our strategy is to look for an innovation ε_t capable of reducing real output by 1%. Clearly, the lower the required value of ε_t, the more vulnerable the economy is to recessions.

Performance of the Baseline Economy The first row of Table 3.4 displays the size and the sign of the innovation of the shock required to produce a drop in output of 1%, as well as the implied change of employment, of the measured Solow residual, and of total consumption. The size of the temporary increase in the discount rate is a little less than 1%. By itself, this statistic does not tell us much, but it is useful for comparisons. Recall that the economy was calibrated to generate a drop in employment of 0.5%. We see that there is a reduction in measured TFP of 0.69% and that consumption drops by 3.8%. The reduction of nontradable consumption is responsible for the reduction in measured TFP.

Figure 3.2 displays the impulse responses of the main macroeconomic variables to

Table 3.4: Statistics for a 1% Drop in Output

Model economy	Pref Shock	Labor	TFP	Consumption
Baseline economy	**0.88**	-0.50	**-0.69**	-3.86
Baseline without goods market friction	**2.00**	-1.22	**-0.16**	-7.50
Baseline with very low adjustment costs	1.29	0.12	-1.80	-8.39
Frictionless markets	**-0.48**	-1.77	0.00	4.18
Frictionless labor with goods market friction	**-0.53**	-1.96	0.10	4.50
Baseline + high adjustment cost	0.66	-0.80	-0.47	-2.49
Baseline + lower job finding rate	0.90	-0.38	-0.72	-4.03
Baseline + lower job finding rate + new parameters	0.95	-0.50	-0.65	-4.00
Baseline + staggered wage	0.55	-0.78	-0.50	-2.67
Baseline + staggered wage + high adj. costs	0.45	-0.94	-0.40	-2.03
Baseline + constant labor share	0.85	-0.51	-0.67	-3.75
Baseline w/o goods market friction and high adj. costs	1.10	-1.36	-0.09	-3.40
Baseline w/o goods market friction and staggered wages	0.90	-1.37	-0.13	-3.69
Baseline w/o goods market friction and fixed labor share	1.88	-1.22	-0.16	-7.11

the shock in the baseline economy (blue dots). Here are eight interesting features of the ensuing recession beyond those that we imposed (i.e., the 1% drop in output and the 0.5% drop in employment):

1. The Solow residual drop of 0.69% lingers for a while and does not recover its original value for at least five years.

2. Employment recovers quite fast, within a year.

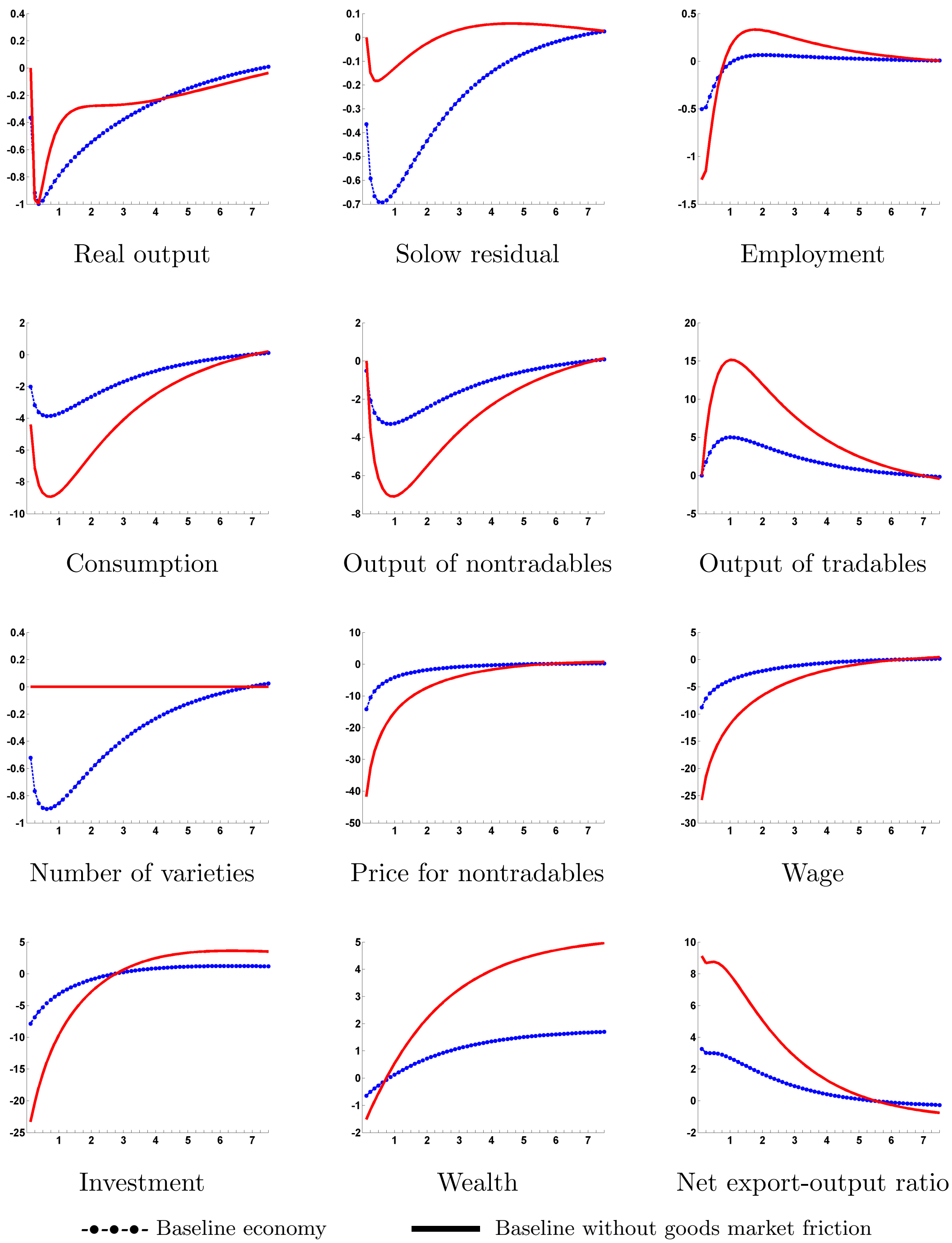
3. Consumption drops about 4% and recovers slowly. The drop is much higher for tradables than for nontradables: the price of the latter drops quite dramatically, about 15%.

4. The large increase in the output of tradables is due to an increase in net exports, which jumps to 3.5% of GDP as investment suffers quite a large reduction, almost 8%.

5. The drop in nontradable consumption is due to both the number of consumption varieties and the quantity consumed of each variety, albeit more of the latter.

6. Wages measured in tradables goods drop quite dramatically, almost 10%.

Figure 3.2: IRF: Baseline Economy and Economy w/o Goods Markets Frictions



7. A paradox of thrift arises. Despite households' attempt to increase savings, the value of wealth is reduced for a few periods, as measured by the sum of the foreign bonds and the present discounted sum of profits, $W_t = (1+r)b_t + \sum_{k=t}^{\infty} \frac{\pi_{N,k}+\pi_{T,k}}{(1+r)^{k-t}}$. It takes roughly a year for wealth to recover its initial level. Eventually, wealth increases by 1.6%.

8. A massive increase in net exports of about 3.5% occurs. In the long run, the economy has a current account deficit because of its long-run positive net foreign asset position.

To summarize, in the baseline economy, an increase in savings generates a long-lasting recession with loss of both employment and productivity, reductions in consumption and investment, and an increase in net exports. As stated before, all of these features are consistent with the experience in southern Europe (see Figure 3.1).

3.6 The Role of Frictions in the Goods Market

To consider the quantitative importance of the mechanism that is novel in this paper, we pose an economy like the baseline except that there are no search frictions in the goods market, and therefore consumers use all consumption varieties and the economy works at full capacity. [13]

The second row in Table 3.4 shows that to get a 1% recession, the size of the shock required in an economy without the goods market friction is 2.00%, 2.3 times larger than in the baseline and a very large number. Moreover, such a recession is made up of a 1.22% reduction in employment and a 0.16% reduction in TFP (which comes from the decreasing returns to scale of the tradable goods sector).

The solid red lines in Figure 3.2 show the dynamic paths of this economy. In the absence of the shopping friction, the requirements for the recession are dramatic: a reduction of consumption of 7.5% rather than 4%, a reduction of investment of 22% rather than 8%, and also enormous reductions in the wage rate (over 20%) and in the price of nontradables (over 40%). We conclude that the contribution of consumers to the

[13] Appendix A.3.3 includes a more detailed description of this model economy and the economies without labor market frictions.

determination of productivity is a major ingredient for the understanding of business cycles.

3.7 The Role of Adjustment Costs and Frictions in the Labor Market

Adjustment costs and labor market frictions are crucial to generate savings induced recessions.

Adjustment Costs Nontrivial adjustment costs is a required ingredient of the recession. To see this, it suffices to look at the third row of Table 3.4 which displays an economy like the baseline except that adjustment costs are almost zero, $\epsilon^{T,k} = \epsilon^{T,n} = 0.01$ (recall that the value in the baseline is 7.7). Now a shock that is 50% larger than that in the baseline generates a 1% drop in output but a small increase in employment. The recession comes about only because of the lower productivity implied by consumption in nontradables and lower shopping. The implied drop in consumption is much larger, more than two times larger than in the baseline.

Frictions in the Labor Market The labor market friction prevents the household from choosing how much to work. In our economy, a shock to patience induces households to be willing to reduce consumption and increase labor today relative to tomorrow. If households are able to choose how much to work, the economy will yield an increase in labor, thereby generating an expansion, not a recession. This occurs because the households are willing to accept a lower wage to delay gratification.[14] Therefore, in our economy with competitive markets, it is a reduction of patience that generates a recession with lower work and higher wages. Figure 3.3 and the fourth and fifth rows of Table 3.4 show the performance of two versions of the baseline economy with no labor market frictions, one with and one without goods markets frictions, and we compare them with the performance of the baseline economy. Except for output and employment, all the other aggregate variables move in a direction opposite from the

[14] The traditional way to avoid this problem is to assume wage or price stickiness as in most New Keynesian models. Recent practice includes [55], [67], [57], and [85].

baseline: the tradable sector shrinks, the nontradable sector expands, consumption increases and net exports decrease. The shock needed to induce a 1% decrease in output is larger when the goods market friction exists. This is because the larger demand for nontradable goods also increases the Solow residual, which is countercyclical in this context.

3.8 Shocks to Wealth

We now engineer a recession by a sudden reduction in the net foreign asset position. We set the size of the reduction to reduce output in the long run by 1% starting from a steady state with a zero net foreign asset position. This exercise explores the implication that when households become impoverished, they lower their consumption, which brings about a permanent drop in output and the Solow residual. We think that such a wealth shock is a good description of the mechanism that triggered the recession in southern Europe—the size of the public debt was larger and the banking system was in worse shape than previously thought, and the generosity of the northern neighbors became greatly reduced.

We model this shock as an unexpected onetime shock ϵ^w to the households' budget constraint:

$$p(S)c_N \mathrm{I} + c_T + b' = (1+r)(b - \epsilon^w) + w(S)n + \pi_N(S) + \pi_T(S). \tag{3.57}$$

In the first row of Table 3.5, we list the size of the shock as a percentage of initial total wealth. This is a sizable shock, since the total value of wealth in this economy is about five times the yearly output. As shown in the second row the size of the shock in a version of the economy without the shopping friction is twice as large as in the baseline.

Table 3.5: Statistics for Wealth Shock to Induce 1% Output Drop

Model Economy	Wealth Shock	Employment	TFP	Consumption
Baseline	9.51	-0.21	-0.37	-3.07
Baseline, no shopping	18.74	-0.39	-0.05	-4.57

Figure 3.3: IRF: Baseline Economy and the Economies without Labor Market Frictions

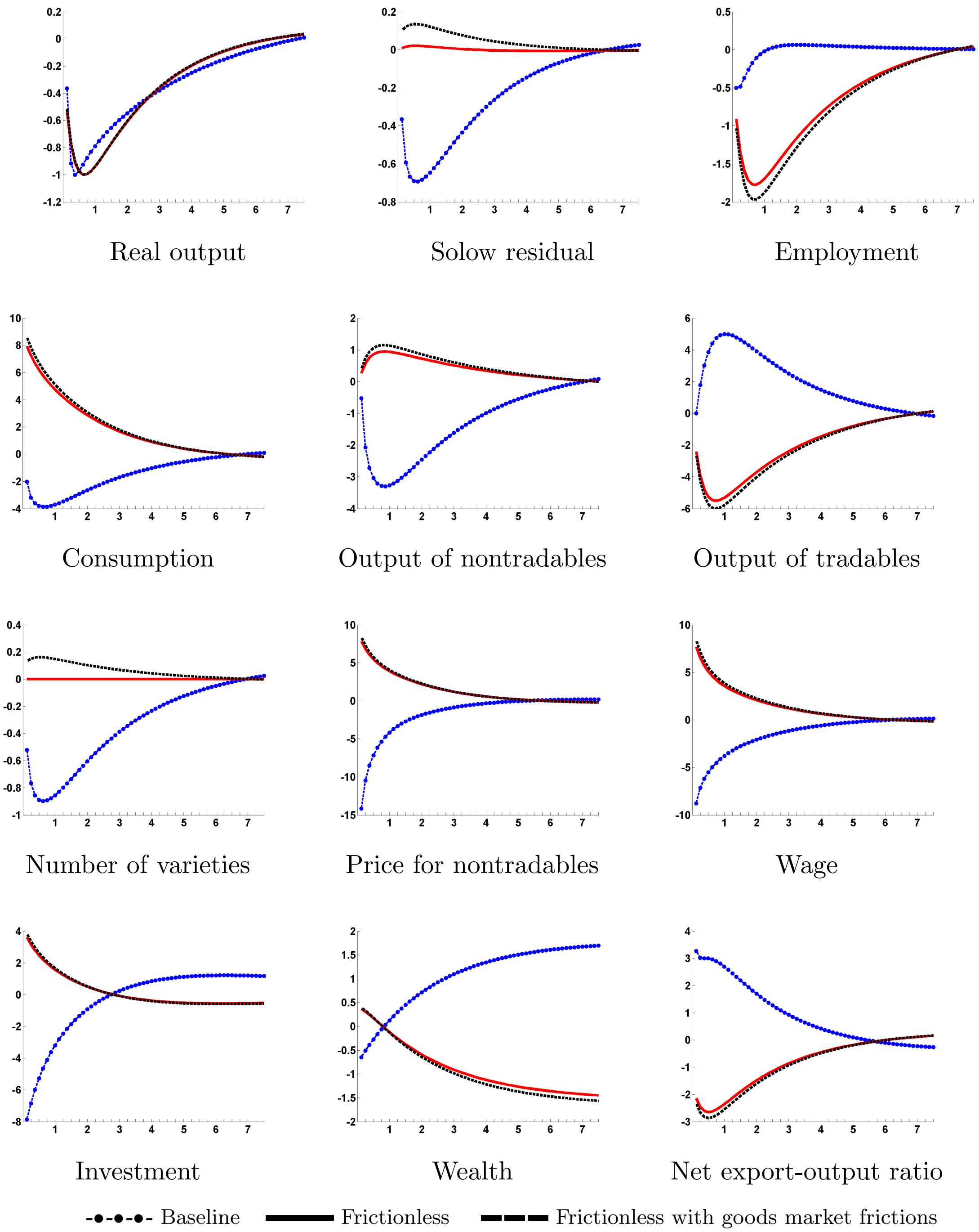


Figure 3.4 covers the first 10 years after the shock. The changes are now permanent. The impoverishment requires the economy to reallocate resources into the tradable goods sector, resulting in a permanent expansion of tradable goods production and net exports. There is also a permanent decline in wages, which will encourage a permanent increase in employment, but only after a decline in the short run arising from the adjustment costs.

Shocks to the discount factor β and to wealth both induce a recession in the short run. However, the paths of recovery are quite different. After a discount factor shock, consumption rebounds fairly quickly, and it inherits the statistical properties of the shock. Other aggregate variables follow a similar pattern. With a shock to wealth, output, the Solow residual, and consumption transit to a quite different and lower steady state.

Southern European economies have stagnated for a relatively long time. From this point of view, a shock to wealth looks more like a plausible trigger than a shock that just increases the desire to save (e.g., a shock to the discount factor) because some aggregate variables such as output and consumption do not recover. However, under the baseline calibration, neither the β shock nor the wealth shock produces a slow recovery in employment. If we add staggered wage contracts to the baseline model, employment takes about two years to return to its original level, as shown in Table 3.6. Further, if we combine both staggered wage contracts and high adjustment costs in the tradable sector, employment takes more than three years to fully recover when the recession is triggered by a wealth shock. Figure 3.5 compares the impulse responses in economies with β shocks and wealth shocks, both of which adopt staggered wage contracts and high adjustment costs in the tradable sector.

3.9 Various Other Alternatives to the Baseline Economy

We now explore variations of the baseline economy that sharpen the characterization of how adjustment costs, wage setting mechanisms, the shopping friction, and labor market turnover affect recessions. Appendix A.3.3 includes a detailed study of the various other alternatives to the baseline economy and also the robustness analysis when using different calibration targets.

Figure 3.4: IRF: Shocks to Wealth

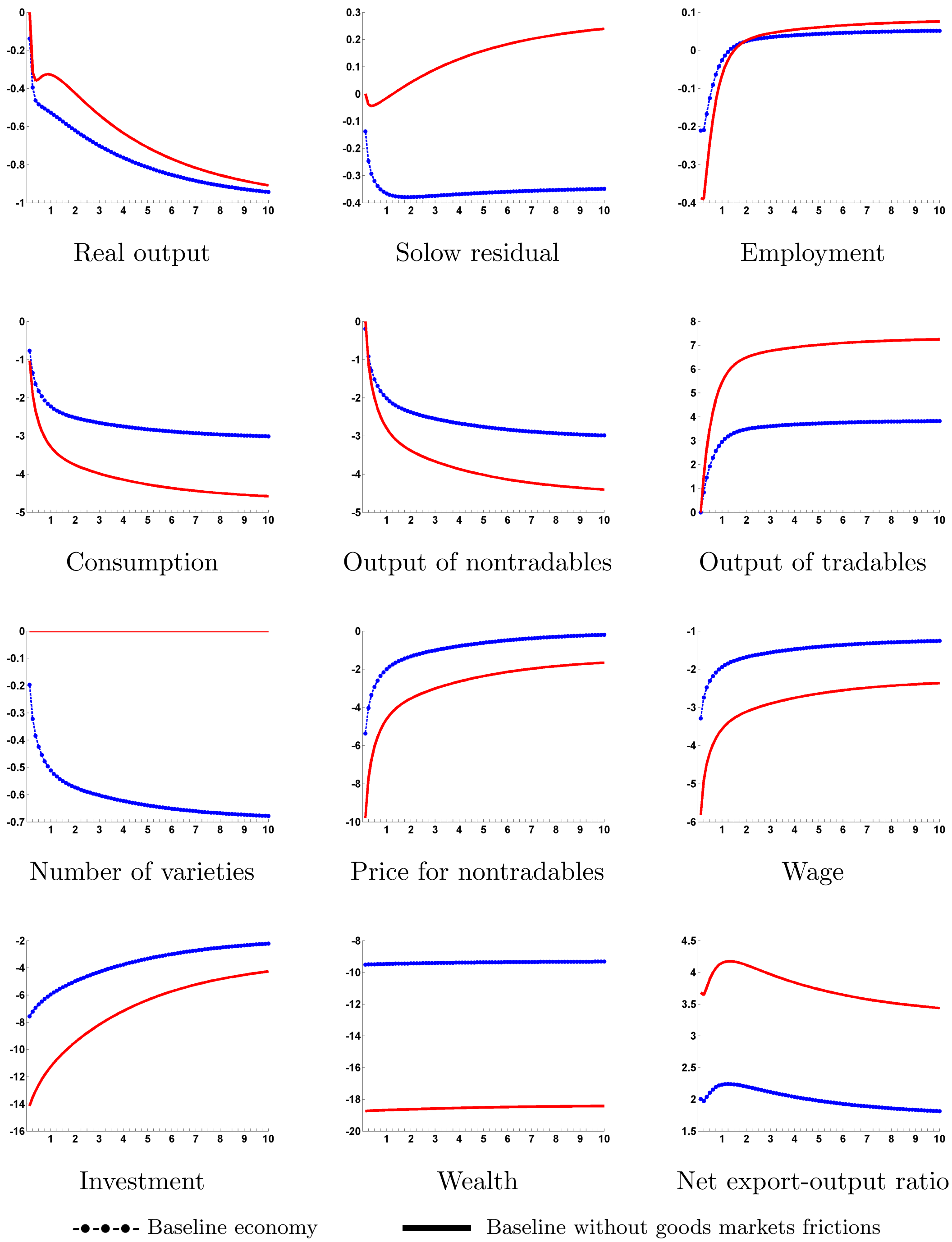


Figure 3.5: IRF: Shocks to β and Wealth with Staggered Wage and High Adj Costs

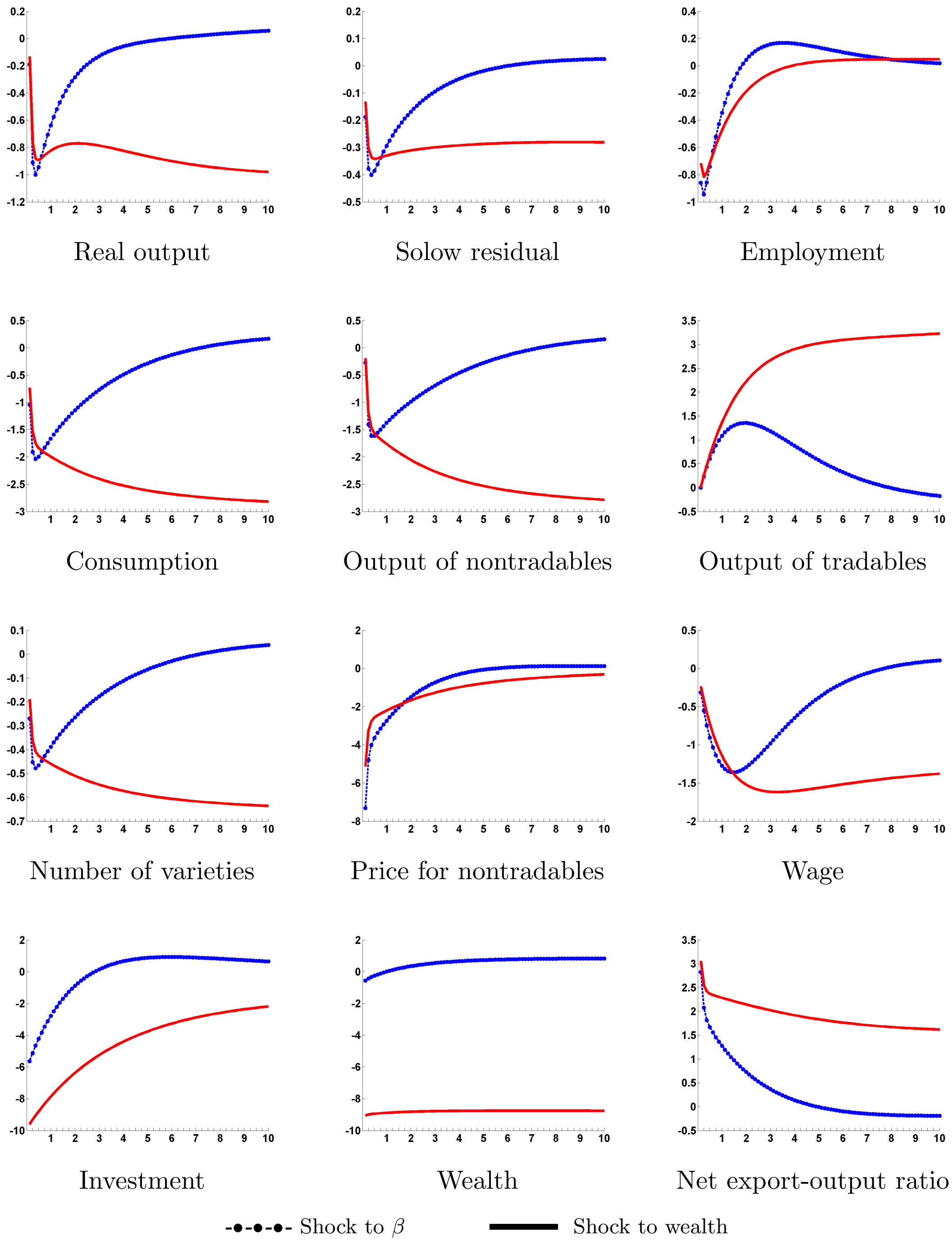


Table 3.6: Statistics for β Shock and Wealth Shock to Induce 1% Output Drop

Model Economy	Labor	TFP	Consumption	Recovery
β shock	-0.50	-0.72	-4.50	4
β shock + staggered wage	-0.85	-0.47	-2.72	7
β shock + staggered wage + high cost	-1.02	-0.36	-2.03	9
Wealth shock	-0.21	-0.37	-3.44	2
Wealth shock + staggered wage	-0.46	-0.43	-3.44	9
Wealth shock + staggered wage + high cost	-0.82	-0.46	-3.25	14

Note: Recovery time is the number of quarters it takes employment to recover its initial steady-state level.

High Adjustment costs in the Tradable Goods Sector We increase the adjustment costs for labor and capital in equal magnitude ($\epsilon^{T,n} = \epsilon^{T,k}$) to reduce the expansion of the tradable sector to 2% instead of 5%. The sixth row of Table 3.4 shows that the size of the shock needed to generate a 1% reduction in output is about 75% of that in the baseline economy, but now the drop in employment is larger (0.80%) and that of TFP smaller (0.47%) because of the lower employment creation in tradables. The dynamic analysis (shown in Appendix A.3.3 in Figure A.1) shows a smaller (2.5%) reduction in consumption and a larger (9%) reduction in investment relative to the baseline economy, whereas the drop in the wage is smaller. As in the baseline economy, a paradox of thrift also occurs, but this time the final increase in wealth is about half of that in the baseline. Not only is there a larger drop in employment compared with the baseline model, but it also takes longer for employment to recover. In an economy with no adjustment costs, total employment will not decrease at all; instead, there would be an export-based expansion. We take this result as evidence that the tradable sector has to have sizable adjustment costs.

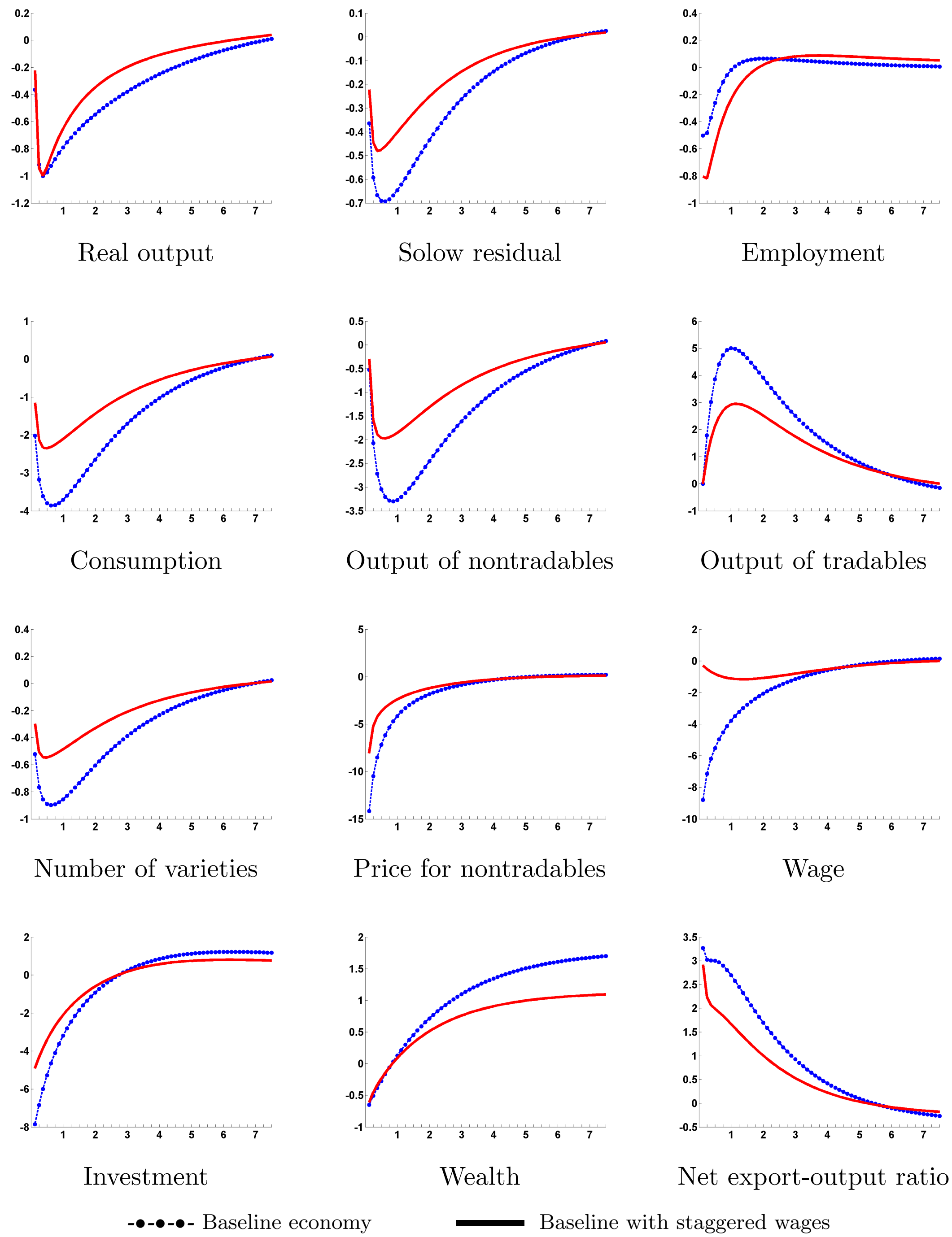
Low Labor Market Turnover Southern European economies are characterized by having a much less dynamic labor market than that of the United States or northern Europe. To explore the implications of smaller flows in and out of employment, we pose two economies with a lower job finding rate. The seventh row of Table 3.4 displays the size of the required shock for a 1% output reduction and its associated decomposition into employment and measured productivity of an economy calibrated to have the same steady state as the baseline except for having a monthly job finding rate of 22% (one-half of that of the baseline) and an unemployment rate of 10% (7% in the baseline). The dynamic parameters have not changed. We see that the size of the shock is very similar to that of the baseline and that the reduction in output is due more to a reduction in productivity than in the baseline. The eighth row of Table 3.4 displays the results for a low labor market turnover economy in which the adjustment costs have been increased to have the same employment response as in the baseline (0.5%). The required shock is a bit higher but not by too much. We conclude that our main findings are robust to the amount of labor market turnover.

Staggered Wage Contracts à la Calvo In the baseline economy, despite the holdup problem implied by Nash bargaining with labor search frictions, there is a large drop in wages. An extensive literature (see [78], for example) documents that adding wage stickiness can help Mortensen-Pissarides type models to account for employment volatility. In this section, we examine the role of wage stickiness in a Calvo-style wage contracting environment, similar to [86]. We assume that, every period, a fraction θ_w of employed workers have the chance to renegotiate their wages with firms.[15] We set $\theta_w = 12.5\%$, so that the average duration of a wage contract is one year. The ninth row of Table 3.4 displays the required shock size and the decomposition of the fall in output into labor and productivity. As we expected, the required size of the shock is much smaller than in the baseline. Moreover, most of the fall in output is due to a fall in employment. The drop in consumption is also much smaller.

Figure 3.6 compares the impulse responses to the patience shock in the baseline economy and the staggered wage economy. Comparing the two sets of impulse responses, we see large differences among them. In the staggered wages economy, there is a larger initial drop in employment that reverses after three years. There are also slower drops in consumption and nontradables but a smaller increase in tradables. After the initial drop in wealth that also displays the paradox of thrift, the eventual increase in wealth is lower than in the baseline economy.

[15] Again, Appendix A.3.3 gives the details of the specification of this economy.

Figure 3.6: IRF: Baseline and Staggered Wage Economies



Both Staggered Wages and High Adjustment Costs With these two frictions together, the required size of the shock is about 50% of that of the baseline (tenth row of Table 3.4). In this case, the drop in labor is almost as large as that of output.

Constant Labor Share To get a sense of the role of different forms of wage setting, the eleventh row of Table 3.4 displays an economy with constant labor share. Its performance is quite similar to the baseline with Nash bargaining wage setting: the size of the shock required to generate a 1% reduction in output is slightly smaller than in the baseline (0.85 versus 0.88).

The Role of Shopping in Alternative Economies In the last three rows of Table 3.4 we report nonshopping versions of economies with high adjustment costs, staggered wages, and constant labor share. As before, the size of the shock is also much larger than in the shopping counterpart.

3.10 Economies with Financial Frictions

Shocks to patience are not what we have in mind as a trigger for increased savings. We now extend our model to allow a limited form of heterogeneity within the household that is capable of accommodating shocks to the financial system that trigger changes in savings. We assume financial costs to providing unemployment insurance, implying that employed and unemployed workers may have different consumption levels. These costs are lower when wealth is higher because the transfers of the employed to the unemployed become smaller. Let ψ be a financial cost per unit of transfer to unemployed workers. A relatively permanent increase of ψ makes it more expensive to insure unemployed workers, which encourages the household to increase savings. Shocks to ψ have similar effects than shocks to the discount factor.

In this economy employed and unemployed agents search in different goods markets with different prices, different market tightness, and different amounts consumed. In other words, goods markets can be segmented.[16] In this economy the unemployed

[16] Appendix A.3.4 describes a version of this economy with nonsegmented goods markets. We think of these two versions of the model as extreme cases of a general version in which there are different goods markets for the two types of agents, but some noise sends them to the wrong market.

care relatively more than the employed for lower prices relative to search intensity, which generates a rationale for firms to price discriminate. Firms will gear some of their locations to cater to unemployed workers and the rest to employed workers. The former will face higher market tightness and lower prices, but, as it turns out, not lower quantities of each good. This result is consistent with the evidence provided by [64] and [63]. The wage determination mechanism of the financial friction economy maintains a constant labor share. The firm's problem is

$$\Omega^N(S,k,n) = \max_{\substack{i,v,x^e,p_j^e \\ x^u,p_j^u}} x^e p_j^e \Psi^f(Q^{g,e}) C^e(p_j^e, S) + x^u p_j^u \Psi^f(Q^{g,u}) C^u(p_j^u, S) - w\ell - i - \kappa v + \frac{\Omega^j(k',n')}{1+r^*} \quad (3.58)$$

subject to

$$F^N(k,n) \geq x^e C^e(p_i^e, S) + x^u C^u(p_i^u, S), \quad (3.59)$$

$$x^e + x^u \leq 1, \quad (3.60)$$

$$k' = (1-\delta)k + i - \phi^N(k,i), \quad (3.61)$$

$$n' = (1-\lambda)n + \Phi^f[Q^e(S)]v, \quad (3.62)$$

$$S' = G(S). \quad (3.63)$$

To have an equilibrium where firms enter both markets, the following conditions have to be satisfied:

$$p^e \Psi^f(Q^{g,e}) = p^u \Psi^f(Q^{g,u}) = \zeta, \quad (3.64)$$

$$c_N^e = c_N^u = F^N(K_N, N_N). \quad (3.65)$$

The market tightness in equilibrium equals

$$Q^{g,e} = \frac{X^e}{nD^e}, \qquad Q^{g,u} = \frac{X^u}{(1-n)D^u}. \quad (3.66)$$

Satisfying (3.66) requires $p^e > p^u$ and $\Psi^f(Q^{g,e}) < \Psi^f(Q^{g,u})$. The employed shop at locations with smaller tightness, but they pay a higher price; the unemployed pay a lower price but search harder. The problem of the household is

$$V(S,b,n) = \max_{\substack{c_T^e, \mathrm{I}^e, p^e \\ c_T^u, \mathrm{I}^u, p^u}} nu(c_A^e, d^e, 1) + (1-n)u(c_A^u, d^u, 0) + \beta\, \mathbb{E}\left\{V(S', b', n') \mid \theta\right\} \qquad \text{s.t.} \tag{3.67}$$

$$n[p^e(S)\mathrm{I}^e c_N^e + c_T^e] + (1-n)[p^u \mathrm{I}^u c_N^u + c_T^u] = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S) - \psi(1-n)\mathrm{T}_r - b'. \tag{3.68}$$

$$\mathrm{T}_r = p^u \mathrm{I}^u c_N^u + c_T^u - [(1+r)b + \pi_N(S) + \pi_T(S)], \tag{3.69}$$

$$\mathrm{I}^s = \Psi^d(Q^{g,s})\, d^s, \qquad \text{for } \ s \in \{e,u\} \tag{3.70}$$

$$n' = (1-\lambda)n + \Phi^w[Q^e(S)](1-n), \tag{3.71}$$

$$\zeta = p^e \Psi^f(Q^{g,e}) = p^u \Psi^f(Q^{g,u}), \tag{3.72}$$

$$c_N^e = c_N^u = F^N(K_N, N_N), \tag{3.73}$$

$$S' = G(S). \tag{3.74}$$

The first-order conditions for the multiple market environment can be summarized as

$$u_I^s = \frac{1}{1-\alpha} p^s c_N^s u_{c_T^s}, \qquad \text{for } \ s \in \{e,u\} \tag{3.75}$$

$$u_{c_T^e} = u_{c_T^u}(1+\psi), \tag{3.76}$$

$$u_{\mathrm{I}^s} = p^s c_N^s u_{c_T^s} - \frac{u_{d^s}}{\Psi^d(Q^{g,s})}, \qquad \text{for } \ s \in \{e,u\} \tag{3.77}$$

$$u_{c_T^e} = (1+r)\mathbb{E}\left\{\beta u'_{c_T^e}[1+\psi'(1-n')] \mid \theta\right\}. \tag{3.78}$$

We calibrate ψ such that the steady-state financial cost $\psi(1-n)\mathrm{T}_r$ is 1% of output, although what matters the size of the shocks. We assume that ψ follows an AR(1) process with persistence of 0.95. The realization the shock results in a 1% output drop.

To see how a shock to ψ is related to a shock to β, we can log-linearize the Euler equation (3.78) as

$$\widehat{u}_{c_T^e,t} = \widehat{u}_{c_T^e,t+1} + \frac{\psi^*(1-n^*)}{1+\psi^*(1-n^*)}\widehat{\psi}_{t+1} - \frac{\psi^* n^*}{1+\psi^*(1-n^*)}\widehat{n}_{t+1}, \tag{3.79}$$

where ψ^* and n^* are their steady-state values. Define $\widehat{\beta}$ as the wedge between the intertemporal marginal utilities of consumption today and tomorrow, $\widehat{\beta} = \frac{\psi^*(1-n^*)}{1+\psi^*(1-n^*)}\widehat{\psi}_{t+1} -$

Table 3.7: Calibration of the Financial Frictions Economy

Target	Value	Parameter	Value
Share of tradables, $\frac{F_T^*}{Y^*}$	0.3	ω	0.93
Unemployment rate, U^*	7%	λ	0.05
Monthly job finding rate	45%	ν^e	0.67
Occupancy rate, $\frac{C_N^*}{F_N^*}$	0.81	ν^g	0.81
Capital to output ratio, $\frac{K^*}{Y^*}$	2.75	δ	0.007
Labor share in tradables	0.6	θ_T^N	0.64
Equal role of capital and land in tradables,	$2\theta_T^K + \theta_T^N = 1$	θ_T^K	0.18
Vacancy posting to output ratio	0.037	θ_N	0.67
Financial cost to output ratio	0.01	ψ	0.28
Units Parameters			
Output, Y^*	1	z_N	0.45
Relative price of nontradables, p^*	1	z_T	0.52
Market tightness in labor markets, $\frac{U^*}{V^*}$	1	κ	0.53
Market tightness in goods markets, D^*	1	ξ	0.02

$\frac{\psi^* n^*}{1+\psi^*(1-n^*)}\widehat{n}_{t+1}$. Recall that in the baseline economy with the discount factor shock, the log-linearized Euler equation is

$$\widehat{u}_{c_T,t} = \widehat{u}_{c_T,t+1} + \tau_t, \tag{3.80}$$

with $\widehat{\beta} = \tau$. Note how the effect of shock to ψ is similar to a shock to β. We then compare the implied $\widehat{\beta}$ with the shock to β in the baseline economy. We adopt a wage determination protocol that permits a constant labor share.[17]

After a shock to ψ, agents reduce both the amount of consumption of each variety and the number of varieties. The unemployed reduce their shopping effort less than the employed, but their number of varieties decreases more because of a lower probability of finding a location.

The economy (it has the same exogenous parameters as the baseline and its ts calibration is described in Table 3.7) displays similar behavior to a version of the baseline

[17] We do not use Nash bargaining because after a ψ shock, one additional employed worker becomes much more valuable to the household, which greatly weakens the household's bargaining power and leads to an implausible decrease of wages.

Table 3.8: Baseline and Financial Friction Economies with Constant Labor Share

Model Economy	$\widehat{\beta}$	Labor	TFP	Consumption	Cost/Output
Baseline + constant labor share	0.85	-0.51	-0.67	-3.75	—
Multiple nontradable goods markets	1.15	-0.53	-0.67	-3.72	1.36

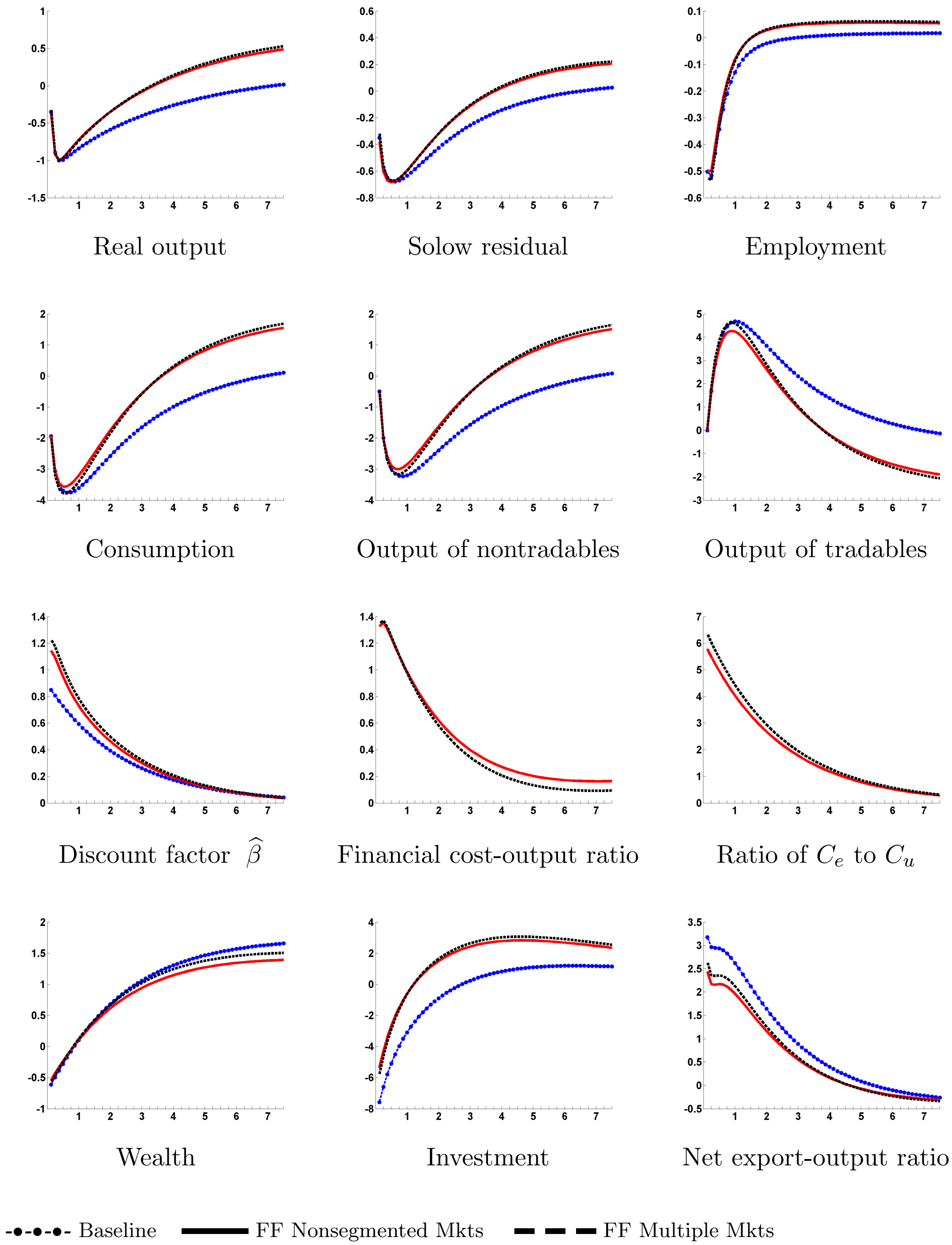
economy where the wage determination mechanism implies constant labor share as can be seen in Figure 3.7 (which also shows the behavior of the economy with financial frictions and nonsegmented markets) and Table 3.8. The implied discount rate, $\widehat{\beta}$, in the financial frictions economy is 1.15% larger than β which compares to an increase of 0.85% in the corresponding baseline economy. The shock induces an increase in the financial cost to output ratio of 34%.

As we can see in Figure 3.7 the main difference between the financial frictions economy and the corresponding version of the baseline is that the former recovers faster and has a bit less reallocation of resources than the latter. As such, net exports go up to 2.5% of GDP, not 3%, and the increase in total wealth is about 1.3% and slowly disappears as the financial frictions return to normal. In other respects the financial friction economy behaves like the baseline. We also see that both financial friction economies (with and without segmented markets) behave almost identically.

3.11 Conclusion

In this paper, we generated demand-induced recessions in an otherwise standard neoclassical growth model. The two necessary ingredients are (1) adjustment costs that make it difficult for the economy to expand the tradable sector by reallocating factors of production from nontradables to tradables, and (2) some form of noncompetitive labor markets (Mortensen-Pissarides labor search frictions and wage setting via Nash bargaining being is enough). In addition, our model poses frictions in the goods markets, where increases in consumers' search efforts enable the economy to operate at a higher capacity (an extension of [11]). Consequently, reductions in household consumption

Figure 3.7: IRF: Baseline and Financial Friction Economies



reduce measured TFP. This feature is quantitatively important: its presence amplifies by two and a half times the effects of shocks. The recessions that we induce display the paradox of thrift in the sense that increases in household savings reduce wealth at the start of the recession, and it takes a few quarters before it recovers its initial level.

Finally, an extension of our model features financial frictions that, when subject to shocks, generate fluctuations like those derived from shocks to patience, even in the context of a representative agent model. We think that in many ways, the type of recession we have posed resembles what is currently happening in southern Europe.

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Appendix A

A.1 Proof of Theorems and Propositions in Chapter 1

A.1.1 Proof of Proposition 1.2.1

Proof. Let j denote $m(i,t)$. With the optimal output rule (1.8), successive iteration leads to

$$\begin{aligned}
y_{it} &= \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}\left[y_{jt}\right] \\
&= \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}\left[\alpha_0 a_j + \alpha_1 \mathbb{E}_{jt}\left[y_{it}\right]\right] \\
&= \alpha_0 a_i + \alpha_0\alpha_1 \mathbb{E}_{it}\left[a_j\right] + \alpha_1^2 \mathbb{E}_{it}\mathbb{E}_{jt}[y_{it}] \\
&= \alpha_0 a_i + \alpha_0\alpha_1 \mathbb{E}_{it}\left[a_j\right] + \alpha_1^2 \mathbb{E}_{it}\mathbb{E}_{jt}[\alpha_0 a_i + \alpha_1 \mathbb{E}_{it}\left[y_{jt}\right]] \\
&= \alpha_0 a_i + \alpha_0\alpha_1^2 \mathbb{E}_{it}\mathbb{E}_{jt}[a_i] + \alpha_0\alpha_1 \mathbb{E}_{it}\left[a_j\right] + \alpha_1^3 \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}[y_{jt}] \\
&= \alpha_0 a_i + \alpha_0\alpha_1^2 \mathbb{E}_{it}\mathbb{E}_{jt}[a_i] + \alpha_0\alpha_1 \mathbb{E}_{it}\left[a_j\right] + \alpha_0\alpha_1^3 \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}[a_j] + \alpha_1^4 \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}\mathbb{E}_{jt}[y_{it}] \\
&\;\;\vdots \\
&= \alpha_0 \sum_{k=0}^{\infty} \alpha_1^{2k} \mathbb{E}_{it}^{2k}[a_i] + \alpha_0 \sum_{k=0}^{\infty} \alpha_1^{2k+1} \mathbb{E}_{it}^{2k+1}[a_j].
\end{aligned}$$

Given that $\alpha_1 \in (0,1)$ and the modulus of the expectation is bounded from above, the summation in the last line is well defined. The expectation operator $\mathbb{E}_{it}^k$ stands for higher order beliefs and is given by

$$\begin{aligned}
\mathbb{E}_{it}^0[a_i] &= a_i \\
\mathbb{E}_{it}^1[a_j] &= \mathbb{E}_{it}[a_j] \\
\mathbb{E}_{it}^k[a_i] &= \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}^{k-2}[a_i], \text{ for } k = 2, 4, 6 \ldots \\
\mathbb{E}_{it}^k[a_j] &= \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}^{k-2}[a_j], \text{ for } k = 3, 5, 7 \ldots
\end{aligned}$$

We can derive $\mathbb{E}^k_{it}[a_i]$ or $\mathbb{E}^k_{it}[a_j]$ in the following way recursively

$$\begin{aligned}
&\mathbb{E}_{it}[a_j] = x^1_{it} - \mathbb{E}_{it}[\xi_t] \\
&\mathbb{E}^2_{jt}[a_i] = \mathbb{E}_{it}[x^1_{jt} - E_{jt}[\xi_t]] = a_i + \mathbb{E}_{it}[\xi_t] - \mathbb{E}_{it}\mathbb{E}_{jt}[\xi_t] \\
&\mathbb{E}^3_{it}[a_j] = \mathbb{E}_{it}[a_j + \mathbb{E}_{jt}[\xi_t] - \mathbb{E}_{jt}\mathbb{E}_{it}[\xi_t]] = \mathbb{E}_{it}[a_j] + \mathbb{E}_{it}\mathbb{E}_{jt}[\xi_t] - \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}[\xi_t] \\
&\mathbb{E}^4_{it}[a_i] = \mathbb{E}_{it}[\mathbb{E}_{jt}[a_i] + \mathbb{E}_{jt}\mathbb{E}_{it}[\xi_t] - \mathbb{E}_{jt}\mathbb{E}_{it}\mathbb{E}_{jt}[\xi_t]] = \mathbb{E}_{it}\mathbb{E}_{jt}[a_i] + \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}[\xi_t] - \mathbb{E}_{it}\mathbb{E}_{jt}\mathbb{E}_{it}\mathbb{E}_{jt}[\xi_t]
\end{aligned}$$

More compactly,

$$\mathbb{E}^k_{it}[a_i] = a_i - \sum_{n=1}^{k}(-1)^n\mathbb{E}^n_{it}[\xi_t], \text{ for } k = 0, 2, 4, 6 \ldots$$

$$\mathbb{E}^k_{it}[a_j] = x^1_{it} + \sum_{n=1}^{k}(-1)^n\mathbb{E}^n_{it}[\xi_t], \text{ for } k = 1, 3, 5, 7 \ldots$$

The the output in island i is

$$\begin{aligned}
y_{it} &= \alpha_0 \sum_{k=0}^{\infty} \alpha_1^{2k}\mathbb{E}^{2k}_{it}[a_i] + \alpha_0 \sum_{k=0}^{\infty} \alpha_1^{2k+1}\mathbb{E}^{2k+1}_{it}[a_j] \\
&= \frac{\alpha_0}{1-\alpha_1^2}a_i + \frac{\alpha_0\alpha_1}{1-\alpha_1^2}x^1_{it} - \frac{\alpha_0}{1+\alpha_1}\sum_{k=1}^{\infty}\alpha_1^k\mathbb{E}^k_{it}[\xi_t] \\
&= \frac{\alpha_0}{1-\alpha_1^2}a_i + \frac{\alpha_0\alpha_1}{1-\alpha_1^2}a_j + \frac{\alpha_0\alpha_1}{1-\alpha_1^2}\xi_t - \frac{\alpha_0}{1+\alpha_1}\sum_{k=1}^{\infty}\alpha_1^k\mathbb{E}^k_{it}[\xi_t] \\
&= \frac{\alpha_0}{1-\alpha_1^2}a_i + \frac{\alpha_0\alpha_1}{1-\alpha_1^2}a_j + \frac{\alpha_0}{1+\alpha_1}\sum_{k=1}^{\infty}\alpha_1^k(\xi_t - \mathbb{E}^k_{it}[\xi_t])
\end{aligned}$$

□

A.1.2 Proof of Theorem 1

Proof. The signal process in our simple economy can be written as

$$x_{it} = \begin{bmatrix} x^1_{it} \\ x^2_{it} \end{bmatrix} = \begin{bmatrix} \sigma_a & 0 & \frac{1}{1-\rho L} \\ 0 & \sigma_u & \frac{1}{1-\rho L} \end{bmatrix} \begin{bmatrix} \widehat{a}_{m(i,t)} \\ \widehat{u}_{it} \\ \widehat{\eta}_t \end{bmatrix} = \widehat{M}(L)\widehat{s}_{it},$$

where we have normalized the shock process to be with unit variance. By the Canonical Factorization Theorem discussed in [8], the matrices for the fundamental representation are

$$B(z) \quad = \quad \frac{1}{1-\rho z}\begin{bmatrix} 1 - \frac{\tau_1\rho+\lambda\tau_2}{\tau_1+\tau_2}z & \frac{\tau_1\rho-\lambda\tau_1}{\tau_1+\tau_2}z \\ \frac{\tau_2\rho-\lambda\tau_2}{\tau_1+\tau_2}z & 1 - \frac{\tau_2\rho+\lambda\tau_1}{\tau_1+\tau_2}z \end{bmatrix},$$

$$V^{-1} = \frac{1}{\rho(\tau_1+\tau_2)} \begin{bmatrix} \frac{\tau_1\rho+\lambda\tau_2}{\tau_1} & \lambda-\rho \\ \lambda-\rho & \frac{\tau_2\rho+\lambda\tau_1}{\tau_2} \end{bmatrix},$$

where $\tau_1 = \frac{\sigma_a^2}{\sigma_\eta^2}$ and $\tau_2 = \frac{\sigma_u^2}{\sigma_\eta^2}$. τ_1 and τ_2 are the relative variance of idiosyncratic shocks to the confidence shock.[1] λ is given by

$$\lambda = \frac{1}{2}\left[\frac{\tau_1+\tau_2}{\rho\tau_1\tau_2} + \frac{1}{\rho} + \rho - \sqrt{\left(\frac{\tau_1+\tau_2}{\rho\tau_1\tau_2} + \frac{1}{\rho} + \rho\right)^2 - 4}\right].$$

In equilibrium

$$y_{it} = \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}[y_{m(i,t)t}].$$

We are looking for policy rule

$$y_{it} = h_a a_i + h_1(L)x_{it}^1 + h_2(L)x_{it}^2$$

such that the equilibrium condition is satisfied. To predict $y_{m(i,t)t}$, it is equivalent to forecast

$$y_{m(i,t)t} = h_a a_{m(i,t)} + h_1(L)\left(a_{m(m(i,t),t)} + \frac{1}{1-\rho L}\eta_t\right) + h_2(L)\left(u_{m(i,t)t} + \frac{1}{1-\rho L}\eta_t\right).$$

Note that $\mathbb{E}_{it}[a_{m(m(i,t),\tau)}] = a_i$ for $\tau = t$ and $\mathbb{E}_{it}[a_{m(m(i,t),\tau)}] = 0$ for $\tau \neq t$. Also, $\mathbb{E}_{it}[u_{m(i,t)\tau}] = 0$ for all τ. The Wiener-Hopf prediction formula gives

$$\mathbb{E}_{it}[a_{m(i,t)}] = \frac{1}{1-\lambda L}\begin{bmatrix} \frac{\tau_1\rho+\tau_2\lambda}{\rho(\tau_1+\tau_2)} - \lambda L \\ \frac{\tau_1(\lambda-\rho)}{\rho(\tau_1+\tau_2)} \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix},$$

$$\begin{aligned} & \mathbb{E}_{it}\left[\frac{h_1(L)+h_2(L)}{1-\rho L}\eta_t\right] \\ = \; & \frac{1}{1-\lambda L}\begin{bmatrix} \frac{\lambda}{\rho\tau_1(L-\lambda)}\left(L[h_1(L)+h_2(L)] - \lambda[h_1(\lambda)+h_2(\lambda)]\frac{1-\rho L}{1-\rho\lambda}\right) \\ \frac{\lambda}{\rho\tau_2(L-\lambda)}\left(L[h_1(L)+h_2(L)] - \lambda[h_1(\lambda)+h_2(\lambda)]\frac{1-\rho L}{1-\rho\lambda}\right) \end{bmatrix} \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix}. \end{aligned}$$

[1] Since we assume $\sigma_\eta = 1$, it follows that $\tau_1 = \sigma_a^2$ and $\tau_2 = \sigma_u^2$.

Using the equilibrium condition, the following system has to be true

$$\begin{aligned}
& h_a a_i + h_1(L)x_{it}^1 + h_2(L)x_{it}^2 \\
= \; & \alpha_0 a_i \\
+ \; & \alpha_1 h_a \begin{bmatrix} \frac{\frac{\tau_1\rho+\tau_2\lambda}{\rho(\tau_1+\tau_2)}-\lambda L}{1-\lambda L} \\ \frac{\tau_1(\lambda-\rho)}{\rho(\tau_1+\tau_2)}\frac{1}{1-\lambda L} \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} + \alpha_1 h_1(0) a_i \\
+ \; & \alpha_1 \begin{bmatrix} \frac{\lambda}{\rho\tau_1}\frac{L}{(1-\lambda L)(L-\lambda)}h_1(L) - \frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_1(\lambda) \\ \frac{\lambda}{\rho\tau_2}\frac{L}{(1-\lambda L)(L-\lambda)}h_1(L) - \frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_1(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \\
+ \; & \alpha_1 \begin{bmatrix} \frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda L)(L-\lambda)}h_2(L) - \frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho z}{(1-\lambda L)(L-\lambda)}h_2(\lambda) \\ \frac{\lambda}{\rho\tau_2}\frac{L}{(1-\lambda L)(L-\lambda)}h_2(L) - \frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_2(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix}
\end{aligned}$$

By the Reise-Fisher Theorem, the following system in the analytic function space has to be true

$$C(z)\begin{bmatrix} h_1(z) \\ h_2(z) \end{bmatrix} = d[z, h_1(\lambda) + h_2(\lambda)]$$

where

$$C(z) = \begin{bmatrix} 1 - \alpha_1 \frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} & -\alpha_1 \frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} \\ -\alpha_1 \frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} & 1 - \alpha_1 \frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix}$$

$$d(z) = \begin{bmatrix} h_a\alpha_1 \frac{\frac{\tau_1\rho+\tau_2\lambda}{\rho(\tau_1+\tau_2)}-\lambda z}{1-\lambda z} - \alpha_1\frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho z}{(1-\lambda z)(z-\lambda)}[h_1(\lambda)+h_2(\lambda)] \\ h_a\alpha_1 \frac{\tau_1(\lambda-\rho)}{\rho(\tau_1+\tau_2)}\frac{1}{1-\lambda z} - \alpha_1\frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho z}{(1-\lambda z)(z-\lambda)}[h_1(\lambda)+h_2(\lambda)] \end{bmatrix}$$

To solve for $h_1(z)$ and $h_2(z)$, we use Cramer's rule, which requires the determinant of $C(z)$.

$$\begin{aligned}
\det C(z) &= 1 - \alpha_1\left[\frac{\lambda(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)}\right] \\
&= \frac{\rho\tau_1\tau_2(1-\lambda z)(z-\lambda) - \alpha_1\lambda(\tau_1+\tau_2)z}{\rho\tau_1\tau_2(1-\lambda z)(z-\lambda)} \\
&= \frac{-\lambda\left[z^2 - \left(\frac{1}{\lambda} + \lambda - \frac{\alpha_1(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right)z + 1\right]}{(1-\lambda z)(z-\lambda)}.
\end{aligned}$$

The determinant of $C(z)$ has two roots which are reciprocal for each other. The inside root is

$$\vartheta = \frac{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha_1)(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right) - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha_1)(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right)^2 - 4}}{2}$$

Therefore

$$\det C(z) = \frac{\frac{\lambda}{\vartheta}(z-\vartheta)(1-\vartheta z)}{(1-\lambda z)(z-\lambda)}$$

Using Cramer's rule,

$$h_1(z) = \frac{\det \begin{bmatrix} d_1(z) & -\alpha_1 \frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} \\ d_2(z) & 1-\alpha_1 \frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix}}{\det C(z)}$$

To make sure $h_1(z)$ does not have poles in the unit circle, we need to choose $h_1(\lambda)+h_2(\lambda)$ to remove the pole at ϑ, which requires

$$\det \begin{bmatrix} d_1(\vartheta) & -\alpha_1 \frac{\lambda}{\rho\tau_1}\frac{\vartheta}{(1-\lambda\vartheta)(\vartheta-\lambda)} \\ d_2(\vartheta) & 1-\alpha_1 \frac{\lambda}{\rho\tau_2}\frac{\vartheta}{(1-\lambda\vartheta)(\vartheta-\lambda)} \end{bmatrix} = 0$$

Note that evaluating z at ϑ, we have

$$d_1(\vartheta)+d_2(\vartheta)=0.$$

We can then solve for $h_1(\lambda)+h_2(\lambda)$ as a function of h_a.

$$h_1(\lambda)+h_2(\lambda) = \frac{h_a(\vartheta-\lambda)\left(\frac{\lambda}{\rho}-\lambda\vartheta\right)}{\frac{\lambda^2}{\rho}\frac{1}{1-\rho\lambda}(1-\rho\vartheta)(\frac{1}{\tau_1}+\frac{1}{\tau_2})} = \frac{h_a(\vartheta-\lambda)(1-\rho\lambda)\tau_1\tau_2}{\lambda(\tau_1+\tau_2)}$$

Using this result, it follows that

$$\det \begin{bmatrix} d_1(z) & -\alpha_1 \frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} \\ d_2(z) & 1-\alpha_1 \frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix}$$
$$= \frac{1}{(1-\lambda z)(z-\lambda)}\alpha_1 h_a^y(-\lambda)(z-\vartheta)\left(z-\frac{\rho\tau_1+\vartheta\tau_2}{(\tau_1+\tau_2)\vartheta\rho}\right).$$

Therefore,

$$h_1(z) = \frac{\alpha_1 h_a \vartheta\left(\frac{\rho\tau_1+\vartheta\tau_2}{(\tau_1+\tau_2)\vartheta\rho}-z\right)}{1-\vartheta z}.$$

Similarly, we can solve for $h_2(z)$ as

$$h_2(z) = -\frac{\alpha_1 h_a \frac{\tau_1(\rho-\theta)}{\rho(\tau_1+\tau_2)}}{1-\vartheta z}.$$

Finally, h_a can be obtained by solving the following linear equation

$$h_a = \alpha_0 + \alpha_1 h_1(0) = \alpha_0 + \alpha_1^2 h_a \frac{\rho\tau_1 + \vartheta\tau_2}{(\tau_1+\tau_2)\rho} = \frac{\alpha_0}{1-\alpha_1^2 \frac{\rho\tau_1+\vartheta\tau_2}{(\tau_1+\tau_2)\rho}}.$$

□

A.1.3 Proof of Theorem 2.5.1

Proof. Let $\phi = \{\phi_a, \phi_1, \phi_2, \phi_3\} \in \mathbb{R} \times \ell^2 \times \ell^2 \times \ell^2$. The norm of ϕ can de defined as

$$\|\phi\| = \sqrt{\sigma_a^2\phi_a^2 + \sigma_a^2 \sum_{i=0}^{\infty} \phi_{1i}^2 + \sigma_u^2 \sum_{i=0}^{\infty} \phi_{2i}^2 + \sigma_\eta^2 \sum_{i=0}^{\infty} \phi_{3i}^2}.$$

Given an arbitrary ϕ, let

$$\Phi(L) = \phi_3(L)$$

Then the signal process is well defined.

The corresponding individual policy rule is

$$y_{it}^{\phi} = \phi_a a_i + \phi_1(L) a_{m(i,t)} + \phi_2(L) u_{it} + \phi_3(L)\eta_t,$$

and the optimal linear forecast is given by

$$\mathbb{E}_{it}[y_{m(i,t)t}^{\phi}] = \widehat{\phi}_a a_i + \widehat{\phi}_1(L) a_{m(i,t)} + \widehat{\phi}_2(L) u_{it} + \widehat{\phi}_3(L)\eta_t.$$

If $y_{it}^{\phi} = \alpha_0 a_i + \alpha_1 \mathbb{E}_{it}[y_{m(i,t)t}^{\phi}]$, then ϕ and Φ consist an equilibrium.

Define the operator $\mathcal{T} : \mathbb{R} \times \ell^2 \times \ell^2 \times \ell^2 \to \mathbb{R} \times \ell^2 \times \ell^2 \times \ell^2$ as

$$\mathcal{T}(\phi) = \mathcal{T}(\{\phi_a, \phi_1, \phi_2, \phi_3\}) = (\{\alpha_0 + \alpha_1\widehat{\phi}_a, \alpha_1\widehat{\phi}_1, \alpha_1\widehat{\phi}_2, \alpha_1\widehat{\phi}_3\}).$$

The equilibrium is a fixed point of the operator $\mathcal{T}$. If we can show that $\mathcal{T}$ is a contraction mapping, it is sufficient to prove the theorem.

Let $\phi \in \mathbb{R} \times \ell^2 \times \ell^2 \times \ell^2$ and $\psi \in \mathbb{R} \times \ell^2 \times \ell^2 \times \ell^2$. The distance between ϕ and ψ is

$$\|\phi - \psi\| = \sqrt{\sigma_a^2(\phi_a-\psi_a)^2 + \sigma_a^2 \sum_{i=0}^{\infty}(\phi_{1i}-\psi_{1i})^2 + \sigma_u^2 \sum_{i=0}^{\infty}(\phi_{2i}-\psi_{2i})^2 + \sigma_\eta^2 \sum_{i=0}^{\infty}(\phi_{3i}-\psi_{3i})^2}.$$

The distance between $\mathcal{T}(\phi)$ and $\mathcal{T}(\psi)$ is

$$\|\mathcal{T}(\phi) - \mathcal{T}(\psi)\| =$$
$$|\alpha_1| \sqrt{\sigma_a^2(\widehat{\phi}_a-\widehat{\psi}_a)^2 + \sigma_a^2 \sum_{i=0}^{\infty}(\widehat{\phi}_{1i}-\widehat{\psi}_{1i})^2 + \sigma_u^2 \sum_{i=0}^{\infty}(\widehat{\phi}_{2i}-\widehat{\psi}_{2i})^2 + \sigma_\eta^2 \sum_{i=0}^{\infty}(\widehat{\phi}_{3i}-\widehat{\psi}_{3i})^2}.$$

Note that the variance of a variable is always larger than the variance of its predictor

$$
\begin{aligned}
&\mathrm{Var}[y^{\phi-\psi}_{m(i,t)t}] \\
=&\mathrm{Var}[(\phi_a-\psi_a)a_{m(i,t)}+(\phi_1(L)-\psi_1(L))a_{m(m(i,t),t)}] \\
&+\mathrm{Var}[(\phi_2(L)-\psi_2(L))u_{m(i,t)t}+(\phi_3(L)-\psi_3(L))\eta_t] \\
=&\sigma_a^2(\phi_a-\psi_a)^2+\sigma_a^2\sum_{i=0}^{\infty}(\phi_{1i}-\psi_{1i})^2+\sigma_u^2\sum_{i=0}^{\infty}(\phi_{2i}-\psi_{2i})^2+\sigma_\eta^2\sum_{i=0}^{\infty}(\phi_{3i}-\psi_{3i})^2 \\
=&\|\phi-\psi\|^2
\end{aligned}
$$

$$
\begin{aligned}
\geq&\mathrm{Var}[\mathbb{E}_{it}[y^{\phi-\psi}_{m(i,t)}]] \\
=&\mathrm{Var}[(\widehat{\phi}_a-\widehat{\psi}_a)a_i+(\widehat{\phi}_1(L)-\widehat{\psi}_1(L))a_{m(i,t)}+(\widehat{\phi}_2(L)-\widehat{\psi}_2(L))u_{it}+(\widehat{\phi}_3(L)-\widehat{\psi}_3(L))\eta_t] \\
=&\sigma_a^2(\widehat{\phi}_a-\widehat{\psi}_a)^2+\sigma_a^2\sum_{i=0}^{\infty}(\widehat{\phi}_{1i}-\widehat{\psi}_{1i})^2+\sigma_u^2\sum_{i=0}^{\infty}(\widehat{\phi}_{2i}-\widehat{\psi}_{2i})^2+\sigma_\eta^2\sum_{i=0}^{\infty}(\widehat{\phi}_{3i}-\widehat{\psi}_{3i})^2 \\
=&\|\mathcal{T}(\phi)-\mathcal{T}(\psi)\|^2\frac{1}{|\alpha_1|^2}.
\end{aligned}
$$

Therefore, $\|\mathcal{T}(\phi)-\mathcal{T}(\psi)\|\leq\alpha_1\|\phi-\psi\|$ when $\alpha_1\in(0,1)$. The operator $\mathcal{T}$ is a contraction mapping. There exists a unique fixed point. □

A.1.4 Proof of Proposition 1.4.1

Proof. Let $m(i,t)$ be island i's partner at time t and $a_{m(i,t)}$ be its productivity. We want to guarantee that there exists stochastic process such that, for all $i\in[0,1)$,

$$
\begin{aligned}
a_{m(i,t)} &= \rho a_{m(i,t-1)}+\epsilon_t, \\
\epsilon_t &\sim \mathcal{N}(0,\sigma^2)
\end{aligned}
$$

where $\rho\in(0,1)$.

Without loss of generality, we can assume that at some t every island $x\in[0,\frac{1}{2})$ meets an island $m(x,t)=x+\frac{1}{2}$ and vice versa. Define a shift operator as

$$
a\oplus \mathrm{b} \equiv a-\frac{1}{2}+b-\frac{1}{2}\left\lfloor 2(a-\frac{1}{2}+b)\right\rfloor,
$$

where $\lfloor c\rfloor$ is the largest integer not exceeding c. Then, for all $n\in Z_+$, for all $x\in[0,\frac{1}{2})$, let

$$
m(x,t+n+1) = m(x,t+n)\oplus\Delta,
$$

where $\Delta\in R$, and $\Delta\notin Q$. As for $x\in[\frac{1}{2},1)$, vice versa. In a discrete analog with countably infinite islands, the next partner island is obvious e.g. its neighbor to the left or right. Here, however, there is no naturally next number to x, and hence we need to guarantee that there exists a step size Δ such that, for all $x \in [\frac{1}{2}, 1)$,

$$a_{x\oplus\Delta} - \rho a_x \sim \mathcal{N}(0, \sigma^2),$$

and similarly for $x \in [0, \frac{1}{2})$. This is not an obvious task.

Now, there exists an Ornstein-Uhlenbeck process $\{Z_x\}$ obeying

$$\begin{aligned} dZ_x &= -\hat{\rho} Z_x + \hat{\sigma} dW_x, \\ Cov[Z_y, Z_x] &= \frac{\hat{\sigma}^2}{2\hat{\rho}} \exp(-\hat{\rho}|y-x|), \end{aligned}$$

where $\{W_x\}$ is the Wiener process and its discrete analog (an AR(1) process) is written as

$$\begin{aligned} z_n &= \kappa_N z_{n-1} + \sqrt{1-\kappa_N^2}\hat{\epsilon}_n, \\ \kappa_N &= \exp(-\frac{\hat{\rho} X}{N}), \\ \hat{\epsilon}_n &\sim \mathcal{N}(0, \frac{\hat{\sigma}^2}{2\hat{\rho}}), \end{aligned}$$

where $n = 1, ..., N$ and N is a large number[2] . Then, let

$$\begin{aligned} X &= \frac{1}{2} \\ \Delta &= \frac{X}{N}, \\ \rho &= \kappa_N, \\ \sigma &= \hat{\sigma}\sqrt{\frac{\Delta(\rho^2-1)}{2\log\rho}}. \end{aligned}$$

It follows that

$$\begin{aligned} z_n &= \rho z_{n-1} + \sqrt{1-\rho^2}\hat{\epsilon}_n, \\ &= \rho x_{n-1} + \epsilon_n, \end{aligned}$$

and this can be interpreted as a discrete analog of a_x. The corresponding Ornstein-Uhlenbeck process is rewritten as

$$dZ_x = \frac{\log\rho}{\Delta} Z_x + \sigma\sqrt{\frac{2\log\rho}{\Delta(\rho^2-1)}} dW_x,$$

and hence

$$Cov[Z_{x+\Delta}, Z_x] = \frac{\rho\sigma^2}{1-\rho^2}.$$

[2] Finch, Steven (2004) “Ornstein-Uhlenbeck Process, ” mimeo.

Note this is identical to the first auto-correlation of the discrete analog and W_x is normally distributed so is the sum of innovation of Z_x between $x + \Delta$ and x. Therefore, if we assume $a_x = Z_{x-\frac{1}{2}}$ for $x \in [\frac{1}{2}, 1)$ and similarly for $x \in [0, \frac{1}{2})$ (with another identical stochastic process), the step size we want is Δ, given no wrap-around happens at $x = 1$, and the wrap-around can be ignored when $\Delta \to 0$. $\square$

A.2 Proof of Theorems and Propositions in Chapter 2

A.2.1 Proof of Proposition 2.2.1

Proof. We consider the equilibrium in the innovation form. Let $\phi = \{\phi_1, \phi_2, \phi_3\} \in \ell^2 \times \ell^2 \times \ell^2$. The norm of ϕ can de defined as

$$\|\phi\| = \sqrt{\sigma_\epsilon^2 \sum_{k=0}^{\infty} \phi_{1k}^2 + \sigma_u^2 \sum_{k=0}^{\infty} \phi_{2k}^2 + \sigma_\eta^2 \sum_{k=0}^{\infty} \phi_{3k}^2}.$$

Given ϕ, let

$$y_{it} = \phi_1(L)\epsilon_{it} + \phi_2(L)u_{it} + \phi_3(L)\eta_t,$$

and let

$$\mathbb{E}_{it}[\phi_1(L)\epsilon_{jt} + \phi_2(L)u_{jt} + \phi_3(L)\eta_t] \equiv \widehat{\phi}_1(L)\epsilon_{it} + \widehat{\phi}_2(L)u_{it} + \widehat{\phi}_3(L)\eta_t$$

The inference of ξ_t is independent of ϕ and is given by

$$\mathbb{E}_{it}[\xi_t] \equiv g_1(L)\epsilon_{it} + g_2(L)u_{it} + g_3(L)\eta_t.$$

If $y_{it} = g_1(L)\epsilon_{it} + g_2(L)u_{it} + g_3(L)\eta_t + \alpha\left(\widehat{\phi}_1(L)\epsilon_{it} + \widehat{\phi}_2(L)u_{it} + \widehat{\phi}_3(L)\eta_t\right)$, then ϕ is an equilibrium.

Define the operator $\mathcal{T} : \ell^2 \times \ell^2 \times \ell^2 \to \ell^2 \times \ell^2 \times \ell^2$ as

$$\mathcal{T}(\phi) = \mathcal{T}(\{\phi_1, \phi_2, \phi_3\}) = \{g_1 + \alpha\widehat{\phi}_1, g_2 + \alpha\widehat{\phi}_2, g_3 + \alpha\widehat{\phi}_3\}$$

The equilibrium is a fixed point of the operator $\mathcal{T}$. If we can show that $\mathcal{T}$ is a contraction mapping, it is sufficient to prove the theorem.

Let $\phi \in \ell^2 \times \ell^2 \times \ell^2$ and $\psi \in \ell^2 \times \ell^2 \times \ell^2$. The distance between ϕ and ψ is

$$\|\phi - \psi\| = \sqrt{\sigma_\epsilon^2 \sum_{k=0}^{\infty} (\phi_{1k} - \psi_{1k})^2 + \sigma_u^2 \sum_{k=0}^{\infty} (\phi_{2k} - \psi_{2k})^2 + \sigma_\eta^2 \sum_{k=0}^{\infty} (\phi_{3k} - \psi_{3k})^2}.$$

The distance between $\mathcal{T}(\phi)$ and $\mathcal{T}(\psi)$ is

$$\|\mathcal{T}(\phi) - \mathcal{T}(\psi)\| = |\alpha| \sqrt{\sigma_\epsilon^2 \sum_{i=0}^{\infty} (\widehat{\phi}_{1i} - \widehat{\psi}_{1i})^2 + \sigma_u^2 \sum_{i=0}^{\infty} (\widehat{\phi}_{2i} - \widehat{\psi}_{2i})^2 + \sigma_\eta^2 \sum_{i=0}^{\infty} (\widehat{\phi}_{3i} - \widehat{\psi}_{3i})^2}$$

Note that the variance of a variable is always larger than the variance of its predictor

$$\begin{aligned} &\mathrm{Var}\Big[[\phi_1(L) - \psi_1(L)]\epsilon_{jt} + [\phi_2(L) - \psi_2(L)]u_{jt} + [\phi_3(L) - \psi_3(L)]\eta_t\Big] \\ \geq &\mathrm{Var}\Big[\mathbb{E}_{it}\Big[[\phi_1(L) - \psi_1(L)]\epsilon_{jt} + [\phi_2(L) - \psi_2(L)]u_{jt} + [\phi_3(L) - \psi_3(L)]\eta_t\Big]\Big] \end{aligned}$$

We have

$$
\begin{aligned}
&\operatorname{Var}\Bigg[[\phi_1(L)-\psi_1(L)]\epsilon_{jt}+[\phi_2(L)-\psi_2(L)]u_{jt}+[\phi_3(L)-\psi_3(L)]\eta_t\Bigg]\\
=&\sigma_\epsilon^2\sum_{k=0}^{\infty}(\phi_{1k}-\psi_{1k})^2+\sigma_u^2\sum_{k=0}^{\infty}(\phi_{2k}-\psi_{2k})^2+\sigma_\eta^2\sum_{k=0}^{\infty}(\phi_{3k}-\psi_{3k})^2\\
=&\|\phi-\psi\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Var}\Bigg[\mathbb{E}_{it}\Bigg[[\phi_1(L)-\psi_1(L)]\epsilon_{jt}+[\phi_2(L)-\psi_2(L)]u_{jt}+[\phi_3(L)-\psi_3(L)]\eta_t\Bigg]\Bigg]\\
=&\operatorname{Var}\Bigg[[\widehat{\phi}_1(L)-\widehat{\psi}_1(L)]\epsilon_{it}+[\widehat{\phi}_2(L)-\widehat{\psi}_2(L)]u_{it}+[\widehat{\phi}_3(L)-\widehat{\psi}_3(L)]\eta_t\Bigg]\\
=&\sigma_\epsilon^2\sum_{i=0}^{\infty}(\widehat{\phi}_{1i}-\widehat{\psi}_{1i})^2+\sigma_u^2\sum_{i=0}^{\infty}(\widehat{\phi}_{2i}-\widehat{\psi}_{2i})^2+\sigma_\eta^2\sum_{i=0}^{\infty}(\widehat{\phi}_{3i}-\widehat{\psi}_{3i})^2\\
=&\|\mathcal{T}(\phi)-\mathcal{T}(\psi)\|^2\left(\frac{1}{\alpha}\right)^2.
\end{aligned}
$$

Therefore, $\|\mathcal{T}(\phi)-\mathcal{T}(\psi)\|\leq\alpha\|\phi-\psi\|$. When $\alpha\in(0,1)$, the operator $\mathcal{T}$ is a contraction mapping, and there exists a unique fixed point. □

A.2.2 Riesz-Fisher Theorem

Theorem (Riesz-Fisher)**.** *Let $\{c_\tau\}$ be a square-summable sequence of complex numbers for which $\sum_{\tau=-\infty}^{\infty}|c_\tau|^2<\infty$. Then there exists a complex-valued function $g(z)$, defined at least on the unit circle in the complex plane such that*

$$g(z)=\sum_{\tau=-\infty}^{\infty}c_\tau z^\tau,$$

where the infinite series converges in the mean square sense that

$$\lim_{n\to\infty}\oint\left|\sum_{\tau=-n}^{n}c_\tau z^\tau-g(z)\right|^2\frac{dz}{z}=0$$

where the integral is a contour integral on the unit circle. The function $g(z)$ is square-integrable

$$\left|\frac{1}{2\pi i}\oint|g(z)|^2\frac{dz}{z}\right|<\infty$$

The function $g(z)$ is called the z transform of the sequence $\{c_\tau\}$.

Conversely, given a square-integrable $g(z)$, there exists a square- summable sequence $\{c_\tau\}$ where

$$c_\tau=\frac{1}{2\pi i}\oint g(z)z^{-\tau-1}dz.$$

Furthermore, suppose $\{c_\tau\}$ *be a one-side square-summable sequence for which* $\sum_{\tau=0}^{\infty} |c_\tau|^2 < \infty$. *Then there exists an analytic function* $g(z)$ *on the open unit disk such that*

$$g(z) = \sum_{\tau=0}^{\infty} c_\tau z^\tau.$$

Conversely, given an analytic function on the unit disk, there exists a one-side square-summable sequence $\{c_\tau\}$ *where*

$$c_\tau = \frac{1}{2\pi i} \oint g(z) z^{-\tau-1} dz.$$

Proof. The proof of this theorem is referred to [40] and [9]. □

A.2.3 Proof of Lemma 2.3.1

Proof. There can be many different state-space representations and we only give one of them here, which is sufficient to prove the claim. [87] shows how to represent a univariate ARMA process in state space, and what we construct below is a natural extension to the multivariate case.

Let $r_{ij} = \max\{p_{ij}, q_{ij} + 1\}$, and let $\alpha_{ijk} = 0$ if $k > q_{ij}$ and $\beta_{ijk} = 0$ if $k > q_{ij}$. Let $r = \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij}$. F is a $r \times r$ matrix with the following form

$$F = \begin{bmatrix} F_{11} & \mathbf{0} & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} \\ \mathbf{0} & F_{12} & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \dots & \dots & \dots & \vdots \\ \mathbf{0} & \dots & \dots & F_{1m} & \dots & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & F_{nm-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & F_{nm} \end{bmatrix}. \tag{A.1}$$

The element F_{ij} in F is a $r_{ij} \times r_{ij}$ matrix

$$F_{ij} = \begin{bmatrix} \alpha_{ij1} & \alpha_{ij2} & \dots & \alpha_{ijr_{ij}-1} & \alpha_{ijr_{ij}} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Q is a $r \times m$ matrix with the following form

$$Q = \begin{bmatrix} Q_{11} \\ Q_{12} \\ \vdots \\ Q_{1m} \\ \vdots \\ Q_{nm-1} \\ Q_{nm} \end{bmatrix}. \tag{A.2}$$

The element D_{ij} in D is a $r_{ij} \times m$ matrix

$$Q_{ij} = \begin{bmatrix} 0 & \dots & \alpha_{ij0} & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix}, \tag{A.3}$$

where α_{ij0} is at the jth column.

H is a $n \times r$ matrix with the following form

$$H = \begin{bmatrix} H_{11} & \dots & H_{1m} & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & H_{21} & \dots & H_{1m} & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \ddots & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & H_{n1} & \dots & H_{nm} \end{bmatrix} \tag{A.4}$$

The element H_{ij} in H is a $1 \times r_{ij}$ matrix

$$H_{ij} = \begin{bmatrix} 1 & \beta_{ij1} & \beta_{ij2} & \dots & \beta_{ijr_{ij}} \end{bmatrix}.$$

Let Z_t follows

$$Z_t = FZ_{t-1} + Qs_t.$$

We have

$$x_t = M(L)s_t = HZ_t$$

To show that the eigenvalues of F lie inside the unit circle, we can iterate the Z_t to obtain

$$Z_t = \sum_{j=0}^{\infty} F^j L^j Q s_t = (I - FL)^{-1} Q s_t$$

If the eigenvalues of F lies outside the unit circle, it follows that Z_t is not co-variance stationary, which contradicts the assumption that x_t is co-variance stationary. □

A.2.4 Proof of Theorem 2.3.5

Proof. A formal proof can be found in Whittle (1983). Here we provide a sketch of the proof.

Suppose the prediction is based on all the realization of the signals x^∞ instead of x^t. The optimal linear prediction of y_t is

$$E[y_t|x^\infty] = \rho_{yx}(L)\rho_{xx}(L)^{-1}x_t.$$

This formula resembles the familiar formula in OLS regression. ρ_{yx} measures the correlation between y and x, adjusted by ρ_{xx}. Given the fundamental representation

$$x_t = B(L)w_t,$$

the prediction is equivalent to the prediction conditional on w^∞ and the prediction formula can be written as

$$\begin{aligned}
E[y_t|x^\infty] =&E[y_t|w^\infty] \\
=&\rho_{yx}(L)\rho_{xx}(L)^{-1}x_t, \\
=&\rho_{yx}(L)B'(L^{-1})^{-1}V^{-1}B(L)^{-1}B(L)w_t, \\
=&\rho_{yx}(L)B'(L^{-1})^{-1}V^{-1}w_t.
\end{aligned}$$

Now imagine the prediction is conditional on only current and past signals x^t, which is equivalent to conditional on w^t. Since w_t is serially uncorrelated, the best forecast of w_i for $i > t$ is zero. Note that $\rho_{yx}(L)B'(L^{-1})^{-1}$ contains negative powers of L and the best forecast of w_i for $i > t$ is zero, the optimal prediction for y_t is simply

$$\begin{aligned}
E[y_t|x^t] =&E[y_t|w^t] \\
=&[\rho_{yx}(L)B'(L^{-1})^{-1}]_+V^{-1}w_t, \\
=&[\rho_{yx}(L)B'(L^{-1})^{-1}]_+V^{-1}B(L)^{-1}x_t, \\
=&[\rho_{yx}(L)B'(L^{-1})^{-1}]_+V^{-1}B(L)^{-1}M(L)s_t.
\end{aligned}$$

Recall that $B(L)$ is invertible, so $B(L)^{-1}$ contains only positive powers of L. □

A.2.5 Proof of Lemma 2.3.2

Proof. By the Canonical Factorization Theorem, it follows that the inverse of $B(z)$ is given by

$$\begin{aligned}
B(z)^{-1} =&I_n - H[I_r - (F - FKH)z]^{-1}FKz \\
=&\frac{I_n \det[I_r - (F - FKH)z] - H\text{Adj}[I_r - (F - FKH)z]FKz}{\det[I_r - (F - FKH)z]} \\
=&\frac{\widehat{B}(z)}{\Pi_{k=1}^{u}(1 - \lambda_k z)}
\end{aligned} \tag{A.5}$$

where $\widehat{B}(z)$ is a matrix and the elements are all polynomials in z with finite degree, u is the degree of $\det[I_r - (F - FKH)z]$, and $\{\lambda_k\}_{k=1}^{u}$ are non-zero eigenvalues of $F - FKH$. To see why this is true, note that if λ_k is the eigenvalue of $F - FKH$, it satisfies

$$\det[\lambda_k I_r - (F - FKH)] = 0$$

which implies

$$\det\left[I_r - (F - FKH)\frac{1}{\lambda_k}\right] = 0$$

That is, $\frac{1}{\lambda_i}$ is the root of the determinant of $I_r - (F - FKH)z$. Reversely, the roots of $I_r - (F - FKH)z$ must be the reciprocals of the non-zero eigenvalues of $F - FKH$. In addition, Theorem 2.3.4 guarantees all of these eigenvalues of $F - FKH$ lie inside the unit circle.

Meanwhile, we have

$$B(z) = I_n + H[I_r - Fz]^{-1}FKz,$$

and

$$\begin{aligned} B(z)^{-1} =& \left[I_n + H[I_r - Fz]^{-1}FKz\right]^{-1} \\ =& \left[\frac{I_n \det[I_r - Fz] - H\text{Adj}[I_r - Fz]FKz}{\det[I_r - Fz]}\right]^{-1} \\ =& \det[I_r - Fz]\left[I_n \det[I_r - Fz] - H\text{Adj}[I_r - Fz]FKz\right]^{-1} \end{aligned}$$

Note that equation (A.5) has to be satisfied at the same time. As a result, there exists a matrix $\widehat{C}(z)$ such that the elements of it are all finite polynomials in z, and

$$B(z)^{-1} = \det[I_r - Fz]\frac{\widehat{C}(z)}{\Pi_{k=1}^{u}(1 - \lambda_k z)}. \tag{A.6}$$

The roots of $\det[I_r - Fz]$, which are the inverse of the eigenvalues of F, are different from $\{\lambda_k\}_{k=1}^{u}$, which are the inverse of the eigenvalues of $F - FKH$. By construction, the degree of $\Pi_{k=1}^{u}(1 - \lambda_k z)$ is larger than the degree of $\det[I_r - Fz]\widehat{C}(z)$.

By Lemma 2.3.1,

$$x_t = M(L)s_t = HZ_t = H(I_r - FL)^{-1}s_t \tag{A.7}$$

Combining equation (A.6) and (A.7) leads to

$$\begin{aligned} B(z)^{-1}M(z) =& \det[I_r - Fz]\frac{\widehat{C}(z)}{\Pi_{k=1}^{u}(1 - \lambda_k z)}\ H(I_r - Fz)^{-1} \\ =& \det[I_r - Fz]\frac{\widehat{C}(z)}{\Pi_{k=1}^{u}(1 - \lambda_k z)}\ H\ \frac{\text{Adj}[I_r - Fz]}{\det[I_r - Fz]} \\ =& \frac{\widehat{C}(z)H\text{Adj}[I_r - Fz]}{\Pi_{k=1}^{u}(1 - \lambda_k z)}. \end{aligned}$$

Here, the numerator of $B(z)^{-1}M(z)$ is a finite polynomial in z, and the degree of $\det[I_r - Fz]$ is larger than the degree of $\text{Adj}[I_r - Fz]$. Therefore,

$$\begin{aligned}
M'(z^{-1})B'(z^{-1})^{-1} =& \frac{\left(\widehat{C}(z^{-1})H\text{Adj}[I_r - Fz^{-1}]\right)'}{\Pi_{k=1}^{u}(1-\lambda_k z^{-1})} = \frac{\left(z^u\widehat{C}(z^{-1})H\text{Adj}[I_r - Fz^{-1}]\right)'}{\Pi_{k=1}^{u}(z-\lambda_k)} \\
=& \frac{G(z)}{\Pi_{k=1}^{u}(z-\lambda_k)},
\end{aligned}$$

where $G(z)$ is a polynomial in z because the degree of $\widehat{C}(z)H\text{Adj}[I_r - Fz]$ is less than u. □

A.2.6 Proof of Proposition 2.3.1

Proof. By the Wiener-Hopf Theorem, the prediction formula is

$$\mathbb{E}\left[y_t \mid x^t\right] = \left[\psi(L)M'(L^{-1})B'(L^{-1})^{-1}\right]_+ V^{-1}B(L)^{-1}x_t$$

We need to obtain the formula for

$$\left[\psi(L)M'(L^{-1})B'(L^{-1})^{-1}\right]_+ = \sum_{i=1}^{m}\begin{bmatrix} \frac{1}{\Pi_{k=1}^{u}(L-\lambda_k)}\psi_i(L)G_{i1}(L) \\ \vdots \\ \frac{1}{\Pi_{k=1}^{u}(L-\lambda_k)}\psi_i(L)G_{in}(L)\end{bmatrix}'_+ \tag{A.8}$$

Suppose $g(z)$ is a rational function of z that does not contains negative powers of z in expansion, then

$$\left[\frac{g(z)}{(z-\lambda_1)\cdots(z-\lambda_u)}\right]_+ = \frac{g(z)}{(z-\lambda_1)\cdots(z-\lambda_u)} - \sum_{k=1}^{u}\frac{g(\lambda_k)}{(z-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}$$

It follows that

$$\begin{aligned}
\left[\psi(L)M'(L^{-1})B'(L^{-1})^{-1}\right]_+ &= \sum_{i=1}^{m}\begin{bmatrix} \frac{1}{\Pi_{k=1}^{u}(L-\lambda_k)}\psi_i(L)G_{i1}(L) - \sum_{k=1}^{u}\frac{\psi_i(\lambda_k)G_{i1}(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \\ \vdots \\ \frac{1}{\Pi_{k=1}^{u}(L-\lambda_k)}\psi_i(L)G_{in}(L) - \sum_{k=1}^{u}\frac{\psi_i(\lambda_k)G_{in}(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\end{bmatrix}' \\
= \quad \psi(L)M'(L^{-1})B'(L^{-1})^{-1} - \sum_{i=1}^{m}&\begin{bmatrix}\sum_{k=1}^{u}\frac{\psi_i(\lambda_k)G_{i1}(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \\ \vdots \\ \sum_{k=1}^{u}\frac{\psi_i(\lambda_k)G_{in}(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\end{bmatrix}'
\end{aligned}$$

Also note that if $g(z) = [f(z)]_+$, then for $j = \{1, 2, \ldots\}$

$$\begin{aligned}
&[z^{-j} f(z)]_+ \\
=&\left[z^{-j}[g(z) + f(z) - g(z)]\right]_+ \\
=&[z^{-j} g(z)]_+ + \left[z^{-j}[f(z) - g(z)]\right]_+ \\
=&z^{-j}\left(g(z) - \sum_{p=0}^{j-1} p!\, z^p [g(z)]_0^{(p)}\right)
\end{aligned}$$

where $[g(z)]_0^{(p)}$ denotes p-th order derivative evaluated at 0. Applying this formula, we have the desired formula. □

A.2.7 Proof of Proposition 2.3.2

Proof. By Proposition 2.3.1, the system (2.24) can be written as

$$
\begin{aligned}
& \begin{bmatrix} \phi(L)\left(\sum_{j=-q}^{p}\sum_{i=1}^{r} C_{1,i}^{y,j} A_i L^j\right) x_t \\ \vdots \\ \phi(L)\left(\sum_{j=-q}^{p}\sum_{i=1}^{r} C_{r,i}^{y,j} A_i L^j\right) x_t \end{bmatrix} + \begin{bmatrix} \phi(L)\left(\sum_{j=-q}^{p}\sum_{i=1}^{v} C_{1,i}^{f,j} f_i(L) L^j M'(L^{-1})\rho_{xx}(L)^{-1}\right) x_t \\ \vdots \\ \phi(L)\left(\sum_{j=-q}^{p}\sum_{i=1}^{v} C_{r,i}^{f,j} f_i(L) L^j M'(L^{-1})\rho_{xx}(L)^{-1}\right) x_t \end{bmatrix} \\
+ & \begin{bmatrix} \sum_{j=-q}^{p} C_1^{g,j} g(L) L^j M'(L^{-1})\rho_{xx}(L)^{-1} x_t \\ \vdots \\ \sum_{j=-q}^{p} C_r^{g,j} g(L) L^j M'(L^{-1})\rho_{xx}(L)^{-1} x_t \end{bmatrix} \\
= & \begin{bmatrix} \sum_{k=1}^{u} \frac{\phi(\lambda_k)\left(\sum_{j=-q}^{p}\sum_{i=1}^{v} \lambda_k^j C_{1,i}^{f,j} f_i(\lambda_k) G(\lambda_k) V^{-1} B(L)^{-1}\right)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} x_t \\ \vdots \\ \sum_{k=1}^{u} \frac{\phi(\lambda_k)\left(\sum_{j=-q}^{p}\sum_{i=1}^{v} \lambda_k^j C_{r,i}^{f,j} f_i(\lambda_k) G(\lambda_k) V^{-1} B(L)^{-1}\right)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} x_t \end{bmatrix} + \begin{bmatrix} \sum_{k=1}^{u} \frac{\sum_{j=-q}^{p} \lambda_k^j C_1^{g,j} g(\lambda_k) G(\lambda_k) V^{-1} B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} x_t \\ \vdots \\ \sum_{k=1}^{u} \frac{\sum_{j=-q}^{p} \lambda_k^j C_r^{g,j} g(\lambda_k) G(\lambda_k) V^{-1} B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} x_t \end{bmatrix} \\
+ & \begin{bmatrix} \sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j} \left[\frac{\sum_{i=1}^{r} \phi(L) C_{1,i}^{y,-j} A_i M(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r} \phi(\lambda_k) C_{1,i}^{y,-j} A_i M(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \right]_0^{(\ell)} V^{-1} B(L)^{-1} x_t \\ \vdots \\ \sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j} \left[\frac{\sum_{i=1}^{r} \phi(L) C_{r,i}^{y,-j} A_i M(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r} \phi(\lambda_k) C_{r,i}^{y,-j} A_i M(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \right]_0^{(\ell)} V^{-1} B(L)^{-1} x_t \end{bmatrix} \\
+ & \begin{bmatrix} \sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j} \left[\frac{\sum_{i=1}^{v} \phi(L) C_{1,i}^{f,-j} f_i(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r} \phi(\lambda_k) C_{1,i}^{y,-j} f_i(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \right]_0^{(\ell)} V^{-1} B(L)^{-1} x_t \\ \vdots \\ \sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j} \left[\frac{\sum_{i=1}^{v} \phi(L) C_{r,i}^{f,-j} f_i(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r} \phi(\lambda_k) C_{r,i}^{f,-j} f_i(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \right]_0^{(\ell)} V^{-1} B(L)^{-1} x_t \end{bmatrix} \\
+ & \begin{bmatrix} \sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j} \left[\frac{C_{1,i}^{g,-j} g(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{C_{1,i}^{g,-j} g(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \right]_0^{(\ell)} V^{-1} B(L)^{-1} x_t \\ \vdots \\ \sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j} \left[\frac{C_{r,i}^{g,-j} g(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{C_{r,i}^{g,-j} g(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} \right]_0^{(\ell)} V^{-1} B(L)^{-1} x_t \end{bmatrix}
\end{aligned}
$$

Rearranging the system of equations above to isolate $\phi(L)$ leads to the following more compact way

$$
\begin{bmatrix}
\phi(L)\sum_{j=-q}^{p} L^j \left[\sum_{i=1}^{r} C_{1,i}^{y,j} A_i + \sum_{i=1}^{v} C_{1,i}^{f,j} f_i(L) M'(L^{-1})\rho_{xx}(L)^{-1}\right] x_t \\
\vdots \\
\phi(L)\sum_{j=-q}^{p} L^j \left[\sum_{i=1}^{r} C_{r,i}^{y,j} A_i + \sum_{i=1}^{v} C_{r,i}^{f,j} f_i(L) M'(L^{-1})\rho_{xx}(L)^{-1}\right] x_t
\end{bmatrix}
$$

$$
\begin{aligned}
= \quad & - \begin{bmatrix}
\sum_{j=-q}^{p} C_1^{g,j} g(L) L^j M'(L^{-1})\rho_{xx}(L)^{-1} x_t \\
\vdots \\
\sum_{j=-q}^{p} C_r^{g,j} g(L) L^j M'(L^{-1})\rho_{xx}(L)^{-1} x_t
\end{bmatrix} \\
& + \begin{bmatrix}
\sum_{k=1}^{u} \frac{\sum_{j=-q}^{p} \lambda_k^j \left[\sum_{i=1}^{v} C_{1,i}^{f,j}\phi(\lambda_k) f_i(\lambda_k) + \sum_{i=1}^{v2} C_1^{g,j} g(\lambda_k)\right] G(\lambda_k) V^{-1} B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} x_t \\
\vdots \\
\sum_{k=1}^{u} \frac{\sum_{j=-q}^{p} \lambda_k^j \left[\sum_{i=1}^{v} C_{r,i}^{f,j}\phi(\lambda_k) f_i(\lambda_k) + \sum_{i=1}^{v2} C_r^{g,j} g(\lambda_k)\right] G(\lambda_k) V^{-1} B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)} x_t
\end{bmatrix} \\
& + \begin{bmatrix}
\sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j}\Bigg(\left[\frac{\sum_{i=1}^{r}\phi(L) C_{1,i}^{y,-j} A_i M(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r}\phi(\lambda_k) C_{1,i}^{y,-j} A_i M(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\
\quad + \left[\frac{\sum_{i=1}^{v}\phi(L) C_{1,i}^{f,-j} f_i(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r}\phi(\lambda_k) C_{1,i}^{y,-j} f_i(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\
\quad + \left[\frac{C_{1,i}^{g,-j} g(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{C_{1,i}^{g,-j} g(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \Bigg) V^{-1} B(L)^{-1} x_t \\
\vdots \\
\sum_{j=1}^{q}\sum_{\ell=0}^{j-1} \ell!\, L^{\ell-j}\Bigg(\left[\frac{\sum_{i=1}^{r}\phi(L) C_{r,i}^{y,-j} A_i M(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r}\phi(\lambda_k) C_{r,i}^{y,-j} A_i M(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\
\quad + \left[\frac{\sum_{i=1}^{v}\phi(L) C_{r,i}^{f,-j} f_i(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{\sum_{i=1}^{r}\phi(\lambda_k) C_{r,i}^{y,-j} f_i(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \\
\quad + \left[\frac{C_{r,i}^{g,-j} g(L) G(L)}{\Pi_{k=1}^{u}(L-\lambda_k)} - \sum_{k=1}^{u} \frac{C_{r,i}^{g,-j} g(\lambda_k) G(\lambda_k)}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k-\lambda_\tau)}\right]_0^{(\ell)} \Bigg) V^{-1} B(L)^{-1} x_t
\end{bmatrix}
\end{aligned}
$$

This has to be true for all the possible realizations of $\{x_t\}$. By Riesz-Fischer Theorem, it is equivalent to the following system of functional equations

$$
T(z)\phi(z) = D\left[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\right]
$$

where $T(z)$ and $D\left[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\right]$ are defined in equation (2.37) and (2.38), respectively.

By the Riesz-Fisher Theorem, there exists $\phi(L)$ that solves model (2.24) if and only if there exists a vector analytic function $\phi(z)$ that solves equations (2.36). □

A.2.8 Proof of Lemma 2.3.3

Proof. Note $D\Big[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\Big]$ is linear in constants $\{\{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\}$. As a result, we can arrange $D\Big[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\Big]$ to obtain

$$D\Big[z, \{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\}_{j=0}^{q}\Big] = \widehat{D}_1(z)\begin{bmatrix}\phi(\lambda_1) & \dots & \phi(\lambda_u) & \phi^0(0) & \dots & \phi^q(0)\end{bmatrix}' + D_2(z)$$

Let $N_c = wu+w(q+1)$ denote the length of the vector $\begin{bmatrix}\phi(\lambda_1) & \dots & \phi(\lambda_u) & \phi^0(0) & \dots & \phi^q(0)\end{bmatrix}'$. Therefore, $\widehat{D}_1(z)$ is a $w \times N_c$ matrix. Let N_1 denote the column rank of $\widehat{D}_1(z)$. It follows that there exists N_1 vectors from $\widehat{D}_1(z)$ that consists a basis of $\widehat{D}_1(z)$. Denote these N_1 vectors as $D_1(z)$. Therefore, there exists a constant matrix Λ of dimension $N_1 \times N_c$, such that

$$\widehat{D}_1(z) = D_1(z)\Lambda\begin{bmatrix}\phi(\lambda_1) & \dots & \phi(\lambda_u) & \phi^0(0) & \dots & \phi^q(0)\end{bmatrix}'$$

Let $\psi \equiv \Lambda\begin{bmatrix}\phi(\lambda_1) & \dots & \phi(\lambda_u) & \phi^0(0) & \dots & \phi^q(0)\end{bmatrix}'$ completes the proof. □

A.2.9 Proof of Theorem 5

Proof. By Cramer's rule, the i-th element of $\phi(z)$ that solves equation (2.36) is given by

$$\phi_i(z) = \frac{\det\Big[T_1(z) \ \dots \ T_{i-1}(z) \ \ D_1(z)\psi + D_2(z) \ \ T_{i+1}(z)\dots \ \dots \ T_w(z)\Big]}{\det\Big[T(z)\Big]}$$

By Proposition 2.3.2, Proposition 2.3.1, and the assumption on model (2.24), the functions in $T(z)$, $D_1(z)$, and $D_2(z)$ are all rational functions with finite degree. As a result, whether $\phi_i(z)$ is an analytic function or not is equivalent to whether $\phi_i(z)$ has poles within the unit circle or not.

In principle, the poles of $\phi_i(z)$ are either the roots of $\det[T(z)]$, i.e., $\{\vartheta_i, \dots, \vartheta_{N_2}\}$, or the poles of

$$\widehat{\phi}_i(z) \equiv \Big[T_1(z) \ \dots \ T_{i-1}(z) \ \ D_1(z)\psi + D_2(z) \ \ T_{i+1}(z)\dots \ \dots \ T_w(z)\Big]. \tag{A.9}$$

By construction, the only poles of $\widehat{\phi}_i(z)$ are $\{\lambda_k\}_{k=1}^{u}$ and 0. However, $\{\lambda_k\}_{k=1}^{u}$ and 0 cannot be poles of $\phi_i(z)$ because these poles are generated from forming expectations using the Wiener-Hopf prediction formula, and by Proposition 2.3.1, these poles are already eliminated by $\Big\{\{\phi(\lambda_k)\}_{k=1}^{u}, \{\phi^{(j)}(0)\Big\}$.

Consider the inside roots of $\det[T(z)]$. For any ϑ_i, it is always possible to find ℓ_i such that $T_{\ell_i}(\vartheta_i)$ is a linear combination of $\left\{T_1(\vartheta_i), \ldots, T_{\ell_i-1}(\vartheta_i),\ T_{\ell_i+1}(\vartheta_i), \ldots, T_w(\vartheta_i)\right\}$. That is

$$T_{\ell_i}(\vartheta_i) = \sum_{k \neq \ell_i} \varphi_k^i T_k(\vartheta_i) \tag{A.10}$$

Suppose

$$\det\begin{bmatrix} D_1(\vartheta_1)\psi + D_2(\vartheta_1) & T_1(\vartheta_i) & \ldots & T_{\ell_i-1}(\vartheta_i) & T_{\ell_i+1}(\vartheta_i) & \ldots & T_w(\vartheta_i)\end{bmatrix} = 0 \tag{A.11}$$

Then for any $j \in \{1, \ldots, \ell_i - 1, \ell_i + 1, \ldots, w\}$, we have

$$\begin{aligned} & \det\begin{bmatrix} T_1(\vartheta_i) & \ldots & \overbrace{D_1(\vartheta_i)\psi + D_2(\vartheta_i)}^{j-th\ column} & \ldots & \overbrace{T_{\ell_i}(\vartheta_i)}^{\ell_i-th\ column} & \ldots & T_w(\vartheta_i)\end{bmatrix} \\ = & \sum_{k \neq \ell_i} \det\begin{bmatrix} T_1(\vartheta_i) & \ldots & \overbrace{D_1(\vartheta_i)\psi + D_2(\vartheta_i)}^{j-th\ column} & \ldots & \overbrace{\varphi_k^i T_k(\vartheta_i)}^{\ell_i-th\ column} & \ldots & T_w(\vartheta_i)\end{bmatrix} \\ = & \ 0. \end{aligned}$$

This implies that if equation (A.11) holds, for $j \in \{1, \ldots, w\}$, ϑ_i is the root of the determinant

$$\det\begin{bmatrix} T_1(\vartheta_i) & \ldots & D_1(\vartheta_1)\overset{j}{\psi} + D_2(\vartheta_1) & \ldots & T_w(\vartheta_i)\end{bmatrix}$$

Consequently, ϑ_i cannot be a pole of $\phi(z)$. Now consider the following problem,

$$\begin{aligned} & U_1\psi + U_2 \equiv \\ & \begin{bmatrix} \det\begin{bmatrix} D_1(\vartheta_1)\psi + D_2(\vartheta_1) & T_1(\vartheta_1) & \ldots & T_{\ell_1-1}(\vartheta_1) & T_{\ell_1+1}(\vartheta_1) & \ldots & T_w(\vartheta_1)\end{bmatrix} \\ \vdots \qquad \vdots \qquad \ldots \qquad \vdots \qquad \vdots \qquad \ldots \qquad \vdots \\ \det\begin{bmatrix} D_1(\vartheta_{N_2})\psi + D_2(\vartheta_{N_2}) & T_1(\vartheta_{N_{N_2}}) & \ldots & T_{\ell_{N_2}-1}(\vartheta_{N_2}) & T_{\ell_{N_2}+1}(\vartheta_{N_2}) & \ldots & T_w(\vartheta_{N_2})\end{bmatrix} \end{bmatrix} \end{aligned}$$

If there exists ψ such that

$$U_1\psi + U_2 = 0 \tag{A.12}$$

Then $\{\vartheta_i\}_{i=1}^{N_2}$ are not poles of $\phi(z)$.

1. If $N_1 < N_2$, then there are more equations than unknowns. There does not exist ψ such that equation (A.2.9) holds. As a result, there is no solution to (2.36).
2. If $N_1 = N_2 = \text{rank}(U_2)$, then there exists a unique ψ that solves (A.2.9). Therefore, $\{\vartheta_i\}_{i=1}^{N_2}$ are not poles of $\phi(z)$.
3. If $N_1 > N_2$ or $N_1 = N_2 > \text{rank}(U_2)$, there are infinite solutions to (A.2.9). As a result, there are infinite number of analytic functions $\phi(z)$ that solves (2.36).

□

A.2.10 Proof of Theorem 4

Proof. By Theorem 5, if there exists a solution to (2.24), for $i \in \{1, \ldots, w\}$, $\phi_i(z)$ is a rational function with finite degree. Therefore, $y_t = h(L)x_t$ can be written as (2.25). By Lemma 2.3.1, there exists a state space representation of $y_t = h(L)x_t$, which is given by

$$z_{t+1} = Fz_t + Qx_t \tag{A.13}$$

$$y_t = HQx_t + HFz_t \tag{A.14}$$

where F, Q and H are given by (A.1), (A.3), and (A.4) respectively. Define

$$\Gamma_x = HQ \tag{A.15}$$

$$\Gamma_z = HF \tag{A.16}$$

$$\Upsilon_x = Q \tag{A.17}$$

$$\Upsilon_z = F, \tag{A.18}$$

and we obtain the finite-state representation. Note that the eigenvalues of Γ_z all lie inside the unit circle. The law of motion of z_t

$$z_{t+1} = \Upsilon_x x_t + \Upsilon_z z_t \tag{A.19}$$

can be written as

$$z_{t+1} = (I - \Upsilon_z L)^{-1} \Upsilon_x x_t \tag{A.20}$$

Therefore, given $\{x_t\}_{t=-\infty}^{-1}$,

$$z_0 = (I - \Upsilon_z L)^{-1} \Upsilon_x x_{-1} \tag{A.21}$$

□

A.2.11 More on solution in innovation form

We follow the same procedure as the signal form to define the solution in innovation form. Here, we use similar notations as the signal form to make them comparable to each other, but it should be clear that they may stand for different objects.

Choice variable The policy rule we want to solve is $y_t = [y_{it}, \ldots, y_{rt}]'$, where

$$y_t = \begin{bmatrix} y_{1t} \\ \vdots \\ y_{rt} \end{bmatrix} = \begin{bmatrix} d_{11}(L) & \ldots & h_{1m}(L) \\ \vdots & \ldots & \vdots \\ h_{r1}(L) & \ldots & h_{rm}(L) \end{bmatrix} \begin{bmatrix} s_{1t} \\ \vdots \\ s_{mt} \end{bmatrix} = d(L)s_t. \tag{A.22}$$

We assume that each element in $d(L)$ has an infinite MA representation. More compactly, define

$$\phi(L) \equiv \begin{bmatrix} d_{11}(L) & \ldots & d_{1m}(L) & \ldots & d_{r1}(L) & \ldots & d_{rm}(L) \end{bmatrix}. \tag{A.23}$$

Endogenous variables related to other agents' actions Let $f_t = [f_{it}, \ldots, f_{vt}]'$ denote the endogenous variables chosen by other agents. They are related to the policy rule $\phi(L)$ and the driving shocks s_t in the following way

$$f_{it} = \phi(L) f^i(L) s_t = \phi(L) \begin{bmatrix} f^i_{11}(L) & \ldots & f^i_{1m}(L) \\ \vdots & \ldots & \vdots \\ f^i_{w1}(L) & \ldots & f^i_{wm}(L) \end{bmatrix} \begin{bmatrix} s_{1t} \\ \vdots \\ s_{mt} \end{bmatrix} \tag{A.24}$$

Here, each $f_i(L)$ is a $w \times m$ matrix in the lag operator L. We assume that all the elements in $f_i(L)$ are finite rational functions in L and do not contain negative powers of L in expansion.

Exogenous variables This part is the same as the signal form exposition.

General model This part is the same as signal form.

Definition A.2.1. *A solution to model (2.24) (or an equilibrium) in innovation form is a vector of lag polynomials $\phi(L)$ such that*

1. *For each $i \in \{1, \ldots, w\}$, $\phi_i(L)$ has an infinite MA representation*

$$\phi_i(L) = \sum_{k=0}^{\infty} \phi_{ik} L^k,$$

with $\sum_{k=0}^{\infty} \phi_{ik} < \infty$.

2. *For all possible realizations of $\{s_t\}$,*

$$y_t = \phi(L) \begin{bmatrix} A_1 & \ldots & A_r \end{bmatrix}' x_t = d(L) s_t$$

satisfies equation (2.24).

A.2.12 Proof of Theorem 6

Proof. Suppose there exists a solution in signal form

$$y_t = h(L) x_t$$

By the definition of the signal process (2.17), it follows that

$$y_t = h(L) M(L) s_t.$$

Because $y_t = h(L) x_t$ satisfies model (2.24), $y_t = h(L) M(L) s_t$ also satisfies model (2.24). Reversely, suppose there exists a solution in innovation form

$$y_t = d(L) s_t.$$

We can rearrange model (2.24) such that

$$y_t = -\left(\sum_{j=0}^{p} C^{y,j} L^j\right)^{-1}$$
$$\left(\sum_{j=0}^{p} \mathbb{E}\left[C^{f,j} L^j f_t + C^{g,j} L^j g_t \middle| x^t\right] + \sum_{j=1}^{q} \mathbb{E}\left[C^{y,-j} L^{-j} y_t + C^{f,-j} L^{-j} f_t + C^{g,-j} L^{-j} g_t \middle| x^t\right]\right) \tag{A.25}$$

Note that $\left(\sum_{j=0}^{p} C^{y,j} L^j\right)$ has to be invertible. Otherwise, y_t is not co-variance stationary, which contradicts to the assumption that $y_t = d(L)s_t$ is a solution to the model. Therefore, $\{y_t\} \subset \mathcal{H}_t^x$ and $\{d(L)s_t\} \subset \mathcal{H}_t^x$. By Proposition 2.3.1, it follows that

$$y_t = d(L)s_t = \mathbb{E}[d(L)s_t|x^t] = \left(d(L)M'(L^{-1})\rho_{xx}(L)^{-1} - \sum_{k=1}^{u} \frac{d(\lambda_k)\lambda^k G(\lambda_k)V^{-1}B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k - \lambda_\tau)}\right) x_t$$

Defining

$$h(L) = d(L)M'(L^{-1})\rho_{xx}(L)^{-1} - \sum_{k=1}^{u} \frac{d(\lambda_k)\lambda^k G(\lambda_k)V^{-1}B(L)^{-1}}{(L-\lambda_k)\Pi_{\tau\neq k}(\lambda_k - \lambda_\tau)}$$

gives us the signal form solution. □

A.2.13 Proof of Proposition 2.4.1

Proof. Consider the state-space representation of the signal process. The state equation is

$$\xi_t = \rho \xi_{t-1} + \eta_t$$

The observation equation is

$$x_{it} = \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_t + \begin{bmatrix} \epsilon_{it} \\ u_{it} \end{bmatrix}.$$

By the Canonical Factorization Theorem, the Wold representation is

$$B(z)^{-1} = \frac{1}{1-\lambda z} \begin{bmatrix} 1 - \frac{\tau_2\rho + \lambda\tau_1}{\tau_1+\tau_2} z & \frac{\tau_1(\lambda-\rho)}{\tau_1+\tau_2} z \\ \frac{\tau_2(\lambda-\rho)}{\tau_1+\tau_2} z & 1 - \frac{\tau_1\rho+\lambda\tau_2}{\tau_1+\tau_2} z \end{bmatrix},$$

$$V^{-1} = \frac{1}{\rho(\tau_1+\tau_2)} \begin{bmatrix} \frac{\tau_1\rho+\lambda\tau_2}{\tau_1} & \lambda - \rho \\ \lambda - \rho & \frac{\tau_2\rho+\lambda\tau_1}{\tau_2} \end{bmatrix},$$

where $\tau_1 = \sigma_\epsilon^2$ and $\tau_2 = \sigma_u^2$, and

$$\lambda = \frac{1}{2}\left[\frac{\tau_1+\tau_2}{\rho\tau_1\tau_2} + \frac{1}{\rho} + \rho - \sqrt{\left(\frac{\tau_1+\tau_2}{\rho\tau_1\tau_2} + \frac{1}{\rho} + \rho\right)^2 - 4}\right].$$

Assuming $y_{it} = h_1(L)x_{it}^1 + h_2(L)x_{it}^2$, it follows that

$$y_t = h_1(L)\xi_t + h_2(L)\xi_t.$$

By Proposition 2.3.1, we have

$$\mathbb{E}_{it}[\xi_t] = \begin{bmatrix} \frac{1}{1-\lambda L}\frac{\lambda}{(1-\rho\lambda)\rho\tau_1} \\ \frac{1}{1-\lambda L}\frac{\lambda}{(1-\rho\lambda)\rho\tau_2} \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix},$$

and

$$\begin{aligned} \mathbb{E}_{it}[y_t] &= \begin{bmatrix} \frac{\lambda}{\rho\tau_1}\frac{L}{(1-\lambda L)(L-\lambda)}h_1(L) - \frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_1(\lambda) \\ \frac{\lambda}{\rho\tau_2}\frac{L}{(1-\lambda L)(L-\lambda)}h_1(L) - \frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_1(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \\ &+ \begin{bmatrix} \frac{\lambda}{\rho\tau_1}\frac{L}{(1-\lambda L)(L-\lambda)}h_2(L) - \frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_2(\lambda) \\ \frac{\lambda}{\rho\tau_2}\frac{L}{(1-\lambda L)(L-\lambda)}h_2(L) - \frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_2(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix}. \end{aligned}$$

The model requires that

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\mathbb{E}_{it}[y_t],$$

which leads to

$$\begin{aligned} & h_1(L)x_{it}^1 + h_2(L)x_{it}^2 \\ = & \begin{bmatrix} \frac{1}{1-\lambda L}\frac{\lambda}{(1-\rho\lambda)\rho\tau_1} \\ \frac{1}{1-\lambda L}\frac{\lambda}{(1-\rho\lambda)\rho\tau_2} \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \\ + & \alpha \begin{bmatrix} \frac{\lambda}{\rho\tau_1}\frac{L}{(1-\lambda L)(L-\lambda)}h_1(L) - \frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_1(\lambda) \\ \frac{\lambda}{\rho\tau_2}\frac{L}{(1-\lambda L)(L-\lambda)}h_1(L) - \frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_1(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \\ + & \alpha \begin{bmatrix} \frac{\lambda}{\rho\tau_1}\frac{L}{(1-\lambda L)(L-\lambda)}h_2(L) - \frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_2(\lambda) \\ \frac{\lambda}{\rho\tau_2}\frac{L}{(1-\lambda L)(L-\lambda)}h_2(L) - \frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho L}{(1-\lambda L)(L-\lambda)}h_2(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \end{aligned}$$

By the Riesz-Fisher Theorem, we can transform it into the following problem

$$C(z)\begin{bmatrix} h_1(z) \\ h_2(z) \end{bmatrix} = d(z, h(\lambda))$$

where $h(\lambda) = h_1(\lambda) + h_2(\lambda)$, and

$$\begin{aligned} C(z) &= \begin{bmatrix} 1 - \alpha\frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} & -\alpha\frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} \\ \\ -\alpha\frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} & 1 - \alpha\frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix}, \\ d(z, h(\lambda)) &= \begin{bmatrix} d_1(z, h(\lambda)) \\ d_2(z, h(\lambda)) \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\lambda z}\frac{\lambda}{(1-\rho\lambda)\rho\tau_1} - \alpha\frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}\frac{1-\rho z}{(1-\lambda z)(z-\lambda)}h(\lambda) \\ \frac{1}{1-\lambda z}\frac{\lambda}{(1-\rho\lambda)\rho\tau_2} - \alpha\frac{\lambda^2}{\rho\tau_2}\frac{1}{1-\rho\lambda}\frac{1-\rho z}{(1-\lambda z)(z-\lambda)}h(\lambda) \end{bmatrix}. \end{aligned}$$

Note that

$$\det C(z) = \frac{-\lambda\left[z^2 - \left(\frac{1}{\lambda} + \lambda - \frac{\alpha(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right)z + 1\right]}{(1-\lambda z)(z-\lambda)} = \frac{\frac{\lambda}{\vartheta}(z-\vartheta)(1-\vartheta z)}{(1-\lambda z)(z-\lambda)}$$

The inside root of the determinant of $C(z)$ is

$$\vartheta = \frac{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right) - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)(\tau_1+\tau_2)}{\rho\tau_1\tau_2}\right)^2 - 4}}{2}$$

Using Cramer's rule,

$$h_1(z) = \frac{\det\begin{bmatrix} d_1(z) & -\alpha\frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} \\ \\ d_2(z) & 1-\alpha\frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix}}{\det C(z)}.$$

The numerator is

$$\det\begin{bmatrix} d_1(z) & -\alpha\frac{\lambda}{\rho\tau_1}\frac{z}{(1-\lambda z)(z-\lambda)} \\ \\ d_2(z) & 1-\alpha\frac{\lambda}{\rho\tau_2}\frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix}$$
$$= \frac{1}{(1-\lambda z)(z-\lambda)}\left\{\frac{\lambda(z-\lambda)}{(1-\rho\lambda)\rho\tau_1} - \alpha\frac{\lambda^2}{\rho\tau_1}\frac{1}{1-\rho\lambda}(1-\rho z)h(\lambda)\right\}.$$

To make sure $h_1(z)$ does not have poles in the unit circle, we need to choose $h(\lambda)$ to remove the pole at ϑ, which requires

$$h(\lambda) = \frac{\vartheta - \lambda}{\alpha\lambda(1-\rho\vartheta)}.$$

Therefore,

$$h_1(z) = \frac{\vartheta}{\rho\tau_1(1-\rho\vartheta)}\frac{1}{1-\vartheta z},$$

and similarly,

$$h_2(z) = \frac{\vartheta}{\rho\tau_2(1-\rho\vartheta)}\frac{1}{1-\vartheta z}$$

□

A.2.14 Proof of Proposition 2.4.3

Proof. The signal process and the Wold representation is the same as the proof A.2.13. The difference is when assuming $y_{it} = h_1(L)x_{it}^1 + h_2(L)x_{it}^2$, the aggregate y_t becomes

$$y_t = (h_1(L) + h_2(L))\xi_t + h_1(L)\epsilon_t.$$

By Proposition 2.3.1, we have

$$\mathbb{E}_{it}[\xi_t] = \begin{bmatrix} \frac{1}{1-\lambda L} \frac{\lambda}{(1-\rho\lambda)\rho\tau_1} \\ \frac{1}{1-\lambda L} \frac{\lambda}{(1-\rho\lambda)\rho\tau_2} \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix},$$

and

$$\begin{aligned}
\mathbb{E}_{it}[y_t] &= \begin{bmatrix} \frac{\lambda}{\rho\tau_1} \frac{L}{(1-\lambda L)(L-\lambda)} h_1(L) - \frac{\lambda^2}{\rho\tau_1} \frac{1}{1-\rho\lambda} \frac{1-\rho L}{(1-\lambda L)(L-\lambda)} h_1(\lambda) \\ \frac{\lambda}{\rho\tau_2} \frac{L}{(1-\lambda L)(L-\lambda)} h_1(L) - \frac{\lambda^2}{\rho\tau_2} \frac{1}{1-\rho\lambda} \frac{1-\rho L}{(1-\lambda L)(L-\lambda)} h_1(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \\
&+ \begin{bmatrix} \frac{\lambda}{\rho\tau_1} \frac{L}{(1-\lambda L)(L-\lambda)} h_2(L) - \frac{\lambda^2}{\rho\tau_1} \frac{1}{1-\rho\lambda} \frac{1-\rho L}{(1-\lambda L)(L-\lambda)} h_2(\lambda) \\ \frac{\lambda}{\rho\tau_2} \frac{L}{(1-\lambda L)(L-\lambda)} h_2(L) - \frac{\lambda^2}{\rho\tau_2} \frac{1}{1-\rho\lambda} \frac{1-\rho L}{(1-\lambda L)(L-\lambda)} h_2(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix} \\
&+ \begin{bmatrix} \frac{\tau_1}{\tau_1+\tau_2} h_1(L) + \frac{\tau_2 \frac{\lambda}{\rho}(L-\rho)(1-\rho L)}{(\tau_1+\tau_2)(1-\lambda L)(L-\lambda)} h_1(L) - \frac{\tau_2 \frac{\lambda}{\rho}(\lambda-\rho)(1-\rho L)}{(\tau_1+\tau_2)(1-\lambda L)(L-\lambda)} h_1(\lambda) \\ -\frac{\tau_1}{\tau_1+\tau_2} h_1(L) + \frac{\tau_1 \frac{\lambda}{\rho}(L-\rho)(1-\rho L)}{(\tau_1+\tau_2)(1-\lambda L)(L-\lambda)} h_1(L) - \frac{\tau_1 \frac{\lambda}{\rho}(\lambda-\rho)(1-\rho L)}{(\tau_1+\tau_2)(1-\lambda L)(L-\lambda)} h_1(\lambda) \end{bmatrix}' \begin{bmatrix} x_{it}^1 \\ x_{it}^2 \end{bmatrix}
\end{aligned}$$

The model requires that

$$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha \mathbb{E}_{it}[y_t],$$

which leads to the following system of analytic functions

$$C(z) \begin{bmatrix} h_1(z) \\ h_2(z) \end{bmatrix} = d(z, h(\lambda))$$

where $h(\lambda) = h_2(\lambda)$, and

$$C(z) = \begin{bmatrix} 1 - \alpha \frac{\lambda}{\rho\tau_1} \frac{z}{(1-\lambda z)(z-\lambda)} - \alpha \left(\frac{\tau_1}{\tau_1+\tau_2} + \frac{\tau_2 \frac{\lambda}{\rho}(z-\rho)(1-\rho z)}{(\tau_1+\tau_2)(1-\lambda z)(z-\lambda)} \right) & -\alpha \frac{\lambda}{\rho\tau_1} \frac{z}{(1-\lambda z)(z-\lambda)} \\ -\alpha \frac{\lambda}{\rho\tau_2} \frac{z}{(1-\lambda z)(z-\lambda)} - \alpha \left(-\frac{\tau_1}{\tau_1+\tau_2} + \frac{\tau_1 \frac{\lambda}{\rho}(z-\rho)(1-\rho z)}{(\tau_1+\tau_2)(1-\lambda z)(z-\lambda)} \right) & 1 - \alpha \frac{\lambda}{\rho\tau_2} \frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix},$$

$$d(z, h(\lambda)) = \begin{bmatrix} d_1(z, h(\lambda)) \\ d_2(z, h(\lambda)) \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\lambda z} \frac{\lambda}{(1-\rho\lambda)\rho\tau_1} - \alpha \frac{\lambda^2}{\rho\tau_1} \frac{1}{1-\rho\lambda} \frac{1-\rho z}{(1-\lambda z)(z-\lambda)} h(\lambda) \\ \frac{1}{1-\lambda z} \frac{\lambda}{(1-\rho\lambda)\rho\tau_2} - \alpha \frac{\lambda^2}{\rho\tau_2} \frac{1}{1-\rho\lambda} \frac{1-\rho z}{(1-\lambda z)(z-\lambda)} h(\lambda) \end{bmatrix}.$$

Note that

$$\det C(z) = \frac{(1-\alpha)\lambda(z-\vartheta)(1-\vartheta z)}{\vartheta(1-\lambda z)(z-\lambda)}$$

The inside root of the determinant of $C(z)$ is

$$\vartheta = \frac{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)\tau_1+\tau_2}{\rho\tau_1\tau_2}\right) - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{(1-\alpha)\tau_1+\tau_2}{\rho\tau_1\tau_2}\right)^2 - 4}}{2}$$

Using Cramer's rule,

$$h_1(z) \quad = \quad \frac{\det \begin{bmatrix} d_1(z) & -\alpha \frac{\lambda}{\rho\tau_1} \frac{z}{(1-\lambda z)(z-\lambda)} \\ & \\ d_2(z) & 1-\alpha \frac{\lambda}{\rho\tau_2} \frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix}}{\det C(z)}.$$

The numerator is

$$\begin{aligned} & \det \begin{bmatrix} d_1(z) & -\alpha \frac{\lambda}{\rho\tau_1} \frac{z}{(1-\lambda z)(z-\lambda)} \\ & \\ d_2(z) & 1-\alpha \frac{\lambda}{\rho\tau_2} \frac{z}{(1-\lambda z)(z-\lambda)} \end{bmatrix} \\ = \quad & \frac{1}{(1-\lambda z)(z-\lambda)} \left\{ \frac{\lambda(z-\lambda)}{(1-\rho\lambda)\rho\tau_1} - \alpha \frac{\lambda^2}{\rho\tau_1} \frac{1}{1-\rho\lambda}(1-\rho z)h(\lambda) \right\}. \end{aligned}$$

To make sure $h_1(z)$ does not have poles in the unit circle, we need to choose $h(\lambda)$ to remove the pole at ϑ, which requires

$$h(\lambda) = \frac{\vartheta - \lambda}{\alpha\lambda(1-\rho\vartheta)}.$$

Therefore,

$$h_1(z) = \frac{\vartheta}{\rho\tau_1(1-\alpha)(1-\rho\vartheta)} \frac{1}{1-\vartheta z},$$

and similarly,

$$h_2(z) = \frac{\vartheta}{\rho\tau_1(1-\rho\vartheta)} \frac{1}{1-\vartheta z}$$

□

A.2.15 Proof of Proposition 2.5.1

Proof. Let $\phi = \{\phi_1, \phi_2, \phi_3\} \in \ell^2 \times \ell^2 \times \ell^2$. The norm of ϕ can de defined as

$$\|\phi\| = \sqrt{\sigma_\epsilon^2 \sum_{k=0}^{\infty} \phi_{1k}^2 + \sigma_u^2 \sum_{k=0}^{\infty} \phi_{2k}^2 + \sigma_\eta^2 \sum_{k=0}^{\infty} \phi_{3k}^2}.$$

Given ϕ, the signal process is well defined

$$\begin{aligned} x_{it}^1 &= \xi_t + \epsilon_{it}, \\ x_{it}^2 &= \phi_3(L)\eta_t + u_{it}. \end{aligned}$$

The individual action is then given by

$$y_{it} = \phi_1(L)\epsilon_{it} + \phi_2(L)u_{it} + \phi_3(L)\eta_t,$$

and the optimal linear forecast is given by

$$\mathbb{E}_{it}[y_t] = \widehat{\phi}_1(L)\epsilon_{it} + \widehat{\phi}_2(L)u_{it} + \widehat{\phi}_3(L)\eta_t.$$

If $y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\mathbb{E}_{it}[y_t]$, then ϕ and Φ consist an equilibrium.

Define the operator $\mathcal{T} : \ell^2 \times \ell^2 \times \ell^2 \to \ell^2 \times \ell^2 \times \ell^2$ as

$$\mathcal{T}(\phi) = \mathcal{T}(\{\phi_1, \phi_2, \phi_3\}) = \{\alpha\widehat{\phi}_1, \alpha\widehat{\phi}_2, \alpha\widehat{\phi}_3\}$$

The equilibrium is a fixed point of the operator $\mathcal{T}$. The proof of the contraction mapping is the same as the proof of Proposition 2.2.1. The modification is that the expectation will be conditional on the signal process that depends on ϕ. □

A.2.16 Proof of Theorem 7

Proof. Here, we only layout the structure of the proof, and the details can be found in the online appendix.

1. Assume the law of aggregate y_t has a finite ARMA representation in condition 1 of definition 2.5.1.

   $$\Phi(L) = \sigma_y \frac{\Pi_{k=1}^{q}(1 + \theta_k L)}{\Pi_{k=1}^{p}(1 - \rho_k L)}, \tag{A.26}$$

   where σ_y is a constant.

2. Solve agents optimal policy $\phi = \{\phi_1, \phi_2, \phi_3\}$ in a partial equilibrium. The partial equilibrium consists of two conditions

   - Each individual conduct inference conditional on the following signal process

     $$\begin{aligned} x_{it}^1 &= \xi_t + \epsilon_{it} \\ x_{it}^2 &= y_t + u_{it} \end{aligned}$$

     where

     $$\begin{aligned} \xi_t &= \frac{\Pi_{k=1}^{n}(1 + \kappa_k L)}{\Pi_{k=1}^{m}(1 - \zeta_k L)}\eta_t \\ y_t &= \Phi(L)\eta_t \end{aligned}$$

   - The policy rule ϕ satisfies that

     $$y_{it} = \mathbb{E}_{it}[\xi_t] + \alpha\, \mathbb{E}_{it}\left[\int y_{jt}\right] = \mathbb{E}_{it}[\xi_t] + \alpha\, \mathbb{E}_{it}\left[\phi_3(L)\eta_t\right],$$

     where

     $$y_{it} = \phi_1(L)\epsilon_{it} + \phi_2(L)u_{it} + \phi_3(L)\eta_t.$$

Note that in this partial equilibrium, agents reply on exogenous information, but their optimal policy rule does depend on others' action. Also note that we do not require $\int y_{jt} = \phi_3(L)\eta_t = \Phi(L)\eta_t$. Solving this partial equilibrium is similar to the problem in Section 2.3.

3. Show $\Phi(L)$ cannot be the same as $\phi_3(L)$. That is, condition 3 of definition 2.5.1 cannot be satisfied.

□

A.2.17 Proof of Proposition 2.6.1

Proof. Note that $x^1_{m(i,t)t} = a_i + \epsilon_{m(i,t)t}$, the signal process can be rewritten as

$$\begin{aligned} x^1_{it} &= a_{m(i,t)} + \epsilon_{it} \\ \widehat{x}^2_{it} &= x^2_{m(i,t)t} - a_i = \xi_t + \epsilon_{m(i,t)t} + u_{it}, \\ \xi_t &= \rho\xi_{t-1} + \eta_t. \end{aligned}$$

The two signals are independent of each other, and we can find the Wold representation for each of them separately. The canonical representation for $\widehat{x}^2_{it}$ is

$$\begin{aligned} B(z) &= \frac{1-\lambda z}{1-\rho z}, \\ V^{-1} &= v = \frac{\lambda}{\rho(\sigma^2_\epsilon + \sigma^2_u)}, \end{aligned}$$

where

$$\lambda = \frac{1}{2}\left[\frac{1}{\rho} + \rho + \frac{1}{\rho(\sigma^2_\epsilon + \sigma^2_u)} - \sqrt{\left(\frac{1}{\rho} + \rho + \frac{1}{\rho(\sigma^2_\epsilon + \sigma^2_u)}\right)^2 - 4}\,\right].$$

The prediction of $y_{m(i,t)t}$ is

$$\mathbb{E}_{it}[y_{m(i,t)t}] = \mathbb{E}_{it}[h_a a_{m(i,t)} + h_1(L)(a_{m(m(i,t),t)} + \epsilon_{m(i,t)t}) + h_2(L)(u_{(m(i,t)t)} + \epsilon_{m(m(i,t),t)} + \xi_t)],$$

where

$$\begin{aligned} \mathbb{E}_{it}[a_{m(i,t)}] &= \frac{\sigma^2_a}{\sigma^2_a + \sigma^2_\epsilon} x^1_{it} \\ \mathbb{E}_{it}[a_{m(m(i,\tau),\tau)}] &= a_i \quad \text{if } \tau = t, \text{ otherwise } 0 \\ \mathbb{E}_{it}[\epsilon_{m(i,\tau)\tau}] &= \frac{\sigma^2_\epsilon v(1-\rho L)}{1-\lambda L}\widehat{x}^2_{it} \quad \text{if } \tau = t, \text{ otherwise } 0 \\ \mathbb{E}_{it}[u_{m(i,t)t}] &= 0 \\ \mathbb{E}_{it}[\epsilon_{m(m(i,\tau),\tau)}] &= \frac{\sigma^2_\epsilon}{\sigma^2_a + \sigma^2_\epsilon} x^1_{it} \quad \text{if } \tau = t, \text{ otherwise } 0 \\ \mathbb{E}_{it}[h_2(L)\xi_t] &= \left(\frac{vLh_2(L)}{(L-\lambda)(1-\lambda L)} - \frac{v\lambda(1-\rho L)h_2(\lambda))}{(1-\rho\lambda)(L-\lambda)(1-\lambda L)}\right)\widehat{x}^2_{it}. \end{aligned}$$

The system is

$$\begin{aligned}
&h_a a_i + h_1(L)x_{it}^1 + h_2(L)\widehat{x}_{it}^2 \\
=&a_i + \alpha\Bigg[h_a \frac{\sigma_a^2}{\sigma_a^2+\sigma_\epsilon^2}x_{it}^1 + h_1(0)a_i + h_1(0)\frac{\sigma_\epsilon^2 v(1-\rho L)}{1-\lambda L}\widehat{x}_{it}^2 \\
&+\left(\frac{vLh_2(L)}{(L-\lambda)(1-\lambda L)} - \frac{v\lambda(1-\rho L)h_2(\lambda))}{(1-\rho\lambda)(L-\lambda)(1-\lambda L)}\right)\widehat{x}_{it}^2 + h_2(0)\frac{\sigma_\epsilon^2}{\sigma_a^2+\sigma_\epsilon^2}x_{it}^1\Bigg],
\end{aligned}$$

which leads to

$$\begin{aligned}
h_a &= 1 + \alpha h_1(0) \\
h_1(0) &= \alpha h_a \frac{\sigma_a^2}{\sigma_a^2+\sigma_\epsilon^2} + \alpha_1 h_2(0)\frac{\sigma_1^2}{\sigma_a^2+\sigma_1^2} \\
h_2(z) &= \alpha h_1(0)\frac{\sigma_\epsilon^2 v(1-\rho z)}{1-\lambda z} + \alpha\left(\frac{vzh_2(z)}{(z-\lambda)(1-\lambda z)} - \frac{v\lambda(1-\rho z)h_2(\lambda)}{(1-\rho\lambda)(z-\lambda)(1-\lambda z)}\right).
\end{aligned}$$

The third equation can be written as

$$-\lambda(z-\vartheta)\left(z-\frac{1}{\vartheta}\right)h_2(z) = \alpha_1 h_1\sigma_1^2 v(1-\rho z)(z-\lambda) - \alpha\frac{\sigma_\eta^2 v\lambda(1-\rho z)h_2(\lambda)}{(1-\rho\lambda)}$$

where

$$\vartheta = \frac{1}{2}\left[\frac{1}{\rho} + \rho + \frac{1-\alpha}{\rho(\sigma_\epsilon^2+\sigma_u^2)} - \sqrt{\left(\frac{1}{\rho}+\rho+\frac{1-\alpha}{\rho(\sigma_\epsilon^2+\sigma_u^2)}\right)^2 - 4}\right]. \tag{A.27}$$

Use $h_2(\lambda)$ to removes the inside root ϑ, we have

$$\begin{aligned}
h_1(z) = h_1(0) &= \frac{\alpha}{1-\alpha^2+\frac{\sigma_\epsilon^2}{\sigma_a^2}\left(1-\alpha^2\frac{\vartheta}{\rho}\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_u^2}\right)} \\
h_a &= 1+\alpha h_1(0) \\
h_2(z) &= \frac{\alpha\vartheta h_1(0)\sigma_\epsilon^2}{\rho(\sigma_\epsilon^2+\sigma_u^2)}\frac{1-\rho z}{1-\vartheta z}
\end{aligned}$$

□

A.3 Additional Materials for Chapter 3

A.3.1 Simplification of the Household's Problem

The original problem for households is

$$V(S, b, n) = \max_{c_{N,i}, c_T, \mathrm{I}_N, d} u(c_A, d, n) + \beta \, \mathbb{E} \left\{ V(S', b', n') \mid \theta \right\}, \tag{A.28}$$

subject to

$$\int_0^{\mathrm{I}} p_i(S) \; c_{N,i} \; di + c_T + b', = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S), \tag{A.29}$$

$$\mathrm{I} = \Psi^d[Q^g(S)] \; d, \tag{A.30}$$

$$n' = (1-\lambda)n + \Phi^w[Q^e(S)](1-n), \tag{A.31}$$

$$S' = G(S). \tag{A.32}$$

This problem involves choosing how much to consume of each variety, $c_{N,i}$. Instead, we can solve a two-stage problem. In the first stage, we choose the number of varieties, the expenditures in nontradable consumption, and the expenditures in tradable consumption. In the second stage, we solve how much $c_{N.i}$ to purchase of each variety i given the number of varieties I and the total expenditure Z of nontradable consumption. We can rewrite the second stage as written as

$$\max_{c_i} \left[\int_0^{\mathrm{I}} c_i^{\frac{1}{\rho}} \right]^{\rho}, \tag{A.33}$$

subject to

$$\int_0^{\mathrm{I}} p_i c_{N,i} \leq Z. \tag{A.34}$$

The first-order condition gives

$$c_{N,i} = c_{N,j} \left(\frac{p_i}{p_j} \right)^{\frac{\rho}{1-\rho}}. \tag{A.35}$$

Define the consumption bundle c_N and the price index p as

$$c_N = \left[\frac{1}{\mathrm{I}} \int_0^{\mathrm{I}} c_{N,i}^{\frac{1}{\rho}} \right]^{\rho}, \tag{A.36}$$

$$p = \left[\frac{1}{\mathrm{I}} \int_0^{\mathrm{I}} p_i^{\frac{1}{1-\rho}} \right]^{1-\rho}. \tag{A.37}$$

Substituting equation (A.35) into the budget constraint gives

$$c_{N.i} = \left(\frac{p_i}{p} \right)^{\frac{\rho}{1-\rho}} \frac{Z}{p\mathrm{I}}. \tag{A.38}$$

Combining equation (A.38) and the definition of c_N leads to

$$\int_0^{\mathrm{I}} p_i c_{N,i} = p\mathrm{I}c_N. \tag{A.39}$$

It is then straightforward to derive the demand schedule for each variety:

$$c_{N,i} = \left(\frac{p_i}{p}\right)^{\frac{\rho}{1-\rho}} c_N', \tag{A.40}$$

and we only need to keep track of c_N and I in the utility function:

$$\left[\int_0^{\mathrm{I}} c_{N,i}^{\frac{1}{\rho}}\right]^{\rho} = c_N \mathrm{I}^{\rho}. \tag{A.41}$$

Note that under the assumption that search in the goods market is undirected, the price index p is independent of the number of varieties I. All the derivations above do not rely on the assumption that all prices for nontradables are equal, even though this is indeed the case in equilibrium. In the end, we can rewrite the household's problem as

$$V(S, b, n) = \max_{c_N, c_T, \mathrm{I}_N, d} u(c_A, d, n) + \beta\, \mathbb{E}\left\{V(S', b', n') \mid \theta\right\}, \tag{A.42}$$

subject to

$$c_A = \left[\omega \left(c_N \mathrm{I}_N^{\rho}\right)^{-\eta} + (1-\omega)\, c_T^{-\eta}\right]^{-\frac{1}{\eta}}, \tag{A.43}$$

$$p(S)c_N\mathrm{I} + c_T + b' = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S), \tag{A.44}$$

$$\mathrm{I} = \Psi^d[Q^g(S)]\, d, \tag{A.45}$$

$$n' = (1-\lambda)n + \Phi^w[Q^e(S)](1-n), \tag{A.46}$$

$$S' = G(S). \tag{A.47}$$

A.3.2 Discussion of GHH Preferences

We choose GHH preferences between consumption and the shopping disutility to allow the number of varieties of nontradable goods to be a normal good. Consider the following simplified static problem without tradable goods:

$$\max_{c,\mathrm{I},d} \quad \frac{1}{1-\sigma}\left(c\mathrm{I}^{\rho} - d\right)^{1-\sigma} \tag{A.48}$$

$$\text{subject to} \quad c\mathrm{I} = E, \tag{A.49}$$

$$\mathrm{I} = d\Psi^d(Q), \tag{A.50}$$

where E is total income and the price is normalized to 1. After substituting the constraints into the objective function and defining $A = (\Psi^d(Q))^{-1}$, the original problem can be rewritten as

$$\max_{\mathrm{I}} \; E\mathrm{I}^{\rho-1} - A\mathrm{I}. \tag{A.51}$$

The first-order condition gives

$$(\rho - 1)E\mathrm{I}^{\rho-2} = A. \tag{A.52}$$

The solution of the problem is

$$\mathrm{I}^* = E^{\frac{1}{2-\rho}}(\rho - 1)^{\frac{1}{2-\rho}} A^{\frac{-1}{2-\rho}}. \tag{A.53}$$

Note that I^* is increasing in E if $2 - \rho > 0$. Since typical estimates of ρ are between 1 and 1.5, this condition is not restrictive. The number of varieties is a normal good.

A.3.3 Alternatives to the Baseline Economy

Baseline Economy Minus Goods Market Frictions The first alternative model economy that we consider has frictions in the labor market but not in the goods market. Households cannot choose their labor. The period utility function for the household is $u(c, n) = \frac{c^{1-\sigma}}{1-\sigma} - \varsigma n$. Here, the consumers have neither a shopping choice nor a labor choice (no need to shop, and they work as much as they can). The problem of the household is

$$V(S, b, n) \quad = \quad \max_{c_N, c_T, b} u(c, n) + \mathbb{E}\{\beta V(S', b', n')|\theta\}, \tag{A.54}$$

$$\text{subject to} \qquad p(S)c_N + c_T + b' = (1 + r)b + w(S)n + \pi_N(S) + \pi_T(S), \tag{A.55}$$

$$n' = (1 - \lambda)n + \Phi^w[Q^e(S)](1 - n), \tag{A.56}$$

$$S' = G(S), \tag{A.57}$$

where $\left[\omega\,(c_N)^{\frac{\eta-1}{\eta}} + (1-\omega)c_T^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$. Although the problem of the firms in the tradable goods sector is the same as in the baseline economy, firms in the nontradable goods sector solve

$$\Omega^N(S, k, n) \quad = \quad \max_{p_i, i, v} p_i C(p_i, S) - w(S)n - i - v\kappa + \mathbb{E}\left\{\frac{\Omega^N(S', k', n')}{1 + r} \mid \theta\right\}, \tag{A.58}$$

$$\text{subject to} \qquad C(p_i^c, S) \leq F^N(k, n), \tag{A.59}$$

$$k' = (1 - \delta)k + i - \phi^N(k, i), \tag{A.60}$$

$$n' = (1 - \lambda)n + \Phi^f[Q^e(S)]v, \tag{A.61}$$

$$S' = G(S). \tag{A.62}$$

Frictionless Economy The frictionless economy we considered is a two-sector small open economy without frictions in either the labor market or the goods market. Households still value varieties but do not need to search to find them. Therefore, I = 1 and aggregated nontradable consumption is $c_N = \left[\int_0^1 c_{N,i}^{\frac{1}{\rho}} di\right]^{\rho}$, with the price index defined as $p = \left[\int_0^1 p_i^{\frac{1}{1-\rho}} di\right]^{1-\rho}$. Households choose how much labor to supply, firms are able to immediately adjust labor input and the wage rate clears the market. We assume a standard additively separable utility function $u(c,n) = \frac{c^{1-\sigma}}{1-\sigma} - \xi_n \frac{n^{1+\gamma}}{1+\gamma}$. The problem of households is

$$V(S,b) \quad = \quad \max_{c_N, c_T, n, b'} u\left(c,n\right) + \beta \mathbb{E}\{V(S',b') \mid \theta\}, \tag{A.63}$$

subject to

$$p(S)c_N + c_T + b' = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S), \tag{A.64}$$

$$S' = G(S), \tag{A.65}$$

where $\left[\omega\left(c_N\right)^{\frac{\eta-1}{\eta}} + (1-\omega)c_T^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$. Unlike in the baseline economy where n evolves exogenously, here n is a choice variable for households.

Firms in the nontradable goods sector solve the following problem:

$$\Omega^N(S,k) \quad = \quad \max_{p_i, i, n} p_i C(p_i, S) - w(S)n - i + \mathbb{E}\left\{\frac{\Omega^N(S',k')}{1+r} \mid \theta\right\}, \tag{A.66}$$

subject to

$$C(p_i^c, S) \leq F^N(k,n), \tag{A.67}$$

$$k' = (1-\delta)k + i - \phi^N(k,i), \tag{A.68}$$

$$S' = G(S). \tag{A.69}$$

Firms in the tradable goods sector solve the following problem:

$$\Omega^T(S,k,n^-) \quad = \quad \max_{i,n} F^T(k,n) - w(S)n_T - i_T - \phi^{T,n}(n,n^-) + \mathbb{E}\left\{\frac{\Omega^T(S',k',n)}{1+r}\right\}, \tag{A.70}$$

subject to

$$k' = (1-\delta)k + i - \phi^{T,k}(k,i), \tag{A.71}$$

$$S' = G(S). \tag{A.72}$$

Frictionless Economy Plus Goods Market Friction This model economy has a competitive labor market and frictions in the goods market; that is, households need to search for varieties, but they can choose how much to work. The period utility function for the household is $u(c,d,n) = \frac{1}{1-\sigma}\left(c - \xi_d d\right)^{1-\sigma} - \xi_n \frac{n^{1+\gamma}}{1+\gamma}$. The problem of households is

$$V(S,b,n) \quad = \quad \max_{c_N, c_T, \mathrm{I}_N, d} u(c,d,n) + \mathbb{E}\{\beta V(S',b',n')|\theta\}, \tag{A.73}$$

$$\text{subject to} \qquad p(S)c_N\mathrm{I} + c_T + b' = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S), \tag{A.74}$$

$$\mathrm{I}_N = d\Psi^d[Q^g(S)], \tag{A.75}$$

$$n' = (1-\lambda)n + \Phi^w[Q^e(S)](1-n), \tag{A.76}$$

$$S' = G(S). \tag{A.77}$$

where $\left[\omega\left(c_N \mathrm{I}^\rho\right)^{\frac{\eta-1}{\eta}} + (1-\omega)c_T^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$.

The problem of the firms in the nontradable goods sector is

$$\Omega^N(S,k) \quad = \quad \max_{p_i,i,n} \Psi^f[Q^g(s)]p_i C(p_i,S) - w(S)n - i + \mathbb{E}\left\{\frac{\Omega^N(S',k')}{1+r} \mid \theta\right\}, \tag{A.78}$$

$$\text{subject to} \qquad C(p_i^c, S) \leq F^N(k,n), \tag{A.79}$$

$$k' = (1-\delta)k + i - \phi^N(k,i), \tag{A.80}$$

$$S' = G(S). \tag{A.81}$$

The problem of the firms in the tradable goods sector is the same as in the frictionless economy.

Economy with Staggered Wage Contracts Assume that, every period, a fraction θ_w of employed workers have the chance to renegotiate their wages with firms and denote the economy-wide average wage rate by $w(S)$ and the newly negotiated wage rate by $\widetilde{w}(S)$. The evolution of the average wage rate is as follows:

$$w(S) = (1-\theta_w)w(S^-) + \theta_w \widetilde{w}(S), \tag{A.82}$$

where $w(S^-)$ denotes the average wage rate last period. Note that equation (A.82) implies that those who just became employed negotiate their wage with probability θ_w. Otherwise, they receive last period's average wage rate $w(S^-)$.

For households and firms, only the average wage rate $w(S)$ matters, and therefore, their problems are unchanged. The difference lies in the process of Nash bargaining. Under wage stickiness, the marginal value of a worker for the household with an average wage differs from the value of a worker with a newly set wage. Let $\widetilde{V}(w,S)$ be the value of a worker for the household if they just bargain the wage rate at w:

$$\widetilde{V}(w,S) = wu_{c_T}(S) - \varsigma + \big(1-\lambda\big)\mathbb{E}\left\{\beta(1-\theta_w)\widetilde{V}(w,S') + \beta\theta_w\widetilde{V}(\widetilde{w}(S'),S') \mid \theta\right\}$$
$$- \Phi^w[Q^e(S)]\mathbb{E}\{\beta V_n(S') \mid \theta\}. \tag{A.83}$$

Notice that the wage rate may be the same next period with probability θ_w or may become the newly set wage $\widetilde{w}(S')$. The value of a worker with a newly set wage rate w for firms in the nontradable sector is

$$\widetilde{J}^N(w,S) = \Psi^f[Q^g(S)]p(S)F_n^N(S)\frac{1}{\rho} - w + \frac{(1-\lambda)}{1+r}\mathbb{E}\{(1-\theta_w)\widetilde{J}^N(w,S') + \theta_w \widetilde{J}^N(\widetilde{w}(S'),S') \mid \theta\}, \tag{A.84}$$

and for the firms in the tradable sector it is

$$\widetilde{J}^T(w,S) = F_n^T(S) - w - \phi_n^{T,n}(S) + \frac{(1-\lambda)}{1+r}\mathbb{E}\{(1-\theta_w)\widetilde{J}^T(w,S') + \theta_w \widetilde{J}^T(\widetilde{w}(S'),S') \mid \theta\}. \tag{A.85}$$

As in the baseline economy, we maintain the assumption that the value of a worker for firms is a weighted value of the evaluation of the worker by the firms with weights given by the employment share of each sector. Recall that $\chi(S) = \frac{n_N}{n_N+n_T}$ is the employment share of the nontradable sector. Then, the Nash bargaining problem is

$$\widetilde{w}(S) = \max_w \left[\widetilde{V}(w,S)\right]^{\varphi} \left[\chi(S)\widetilde{J}^N(w,S) + (1-\chi(S))\widetilde{J}_n^T(w,S)\right]^{1-\varphi}. \tag{A.86}$$

Taking the derivative with respect to w yields the first-order condition

$$\varphi \widetilde{V}_w(w,S)\left[\chi(S)\widetilde{J}^N(w,S) + (1-\chi(S))\widetilde{J}^T(w,S)\right] = (1-\varphi)\frac{1+r}{r+\lambda+\theta_w-\lambda\theta_w}\widetilde{V}(w,S), \tag{A.87}$$

where $V_w(w,S)$ is the discounted sum of marginal utility by increasing the wage rate by one unit:

$$\widetilde{V}_w(w,S) = u_{c_T}(S) + \mathbb{E}\{\beta(1-\theta_w)(1-\lambda)\widetilde{V}_w(w,S') \mid \theta\}. \tag{A.88}$$

Calibration, Tables, and Figures of Alternative Model Economies

Table A.1: Exogenously Determined Parameters
in the Economy with Labor Frictions but without Goods Markets Frictions

Parameter	Value
Risk aversion, σ	2.0
Annual rate of return, β	$\frac{1}{\beta^8} - 1 = 4\%$
Labor matching elasticity, μ	0.50
Elasticity of substitution bw tradables and nontradables, η	0.83
Price markup, ρ	1.05

Table A.2: Steady-State Targets and Associated Parameters
in the Economy with Labor Frictions but without Goods Markets Frictions

Target	Value	Parameter	Value
Share of tradables, $\frac{F_T^*}{Y^*}$	0.3	ω	0.91
Unemployment rate, U^*	7%	λ	0.05
Monthly job finding rate	45%	ν^e	0.67
Capital to output ratio, $\frac{K^*}{Y^*}$	2.75	δ	0.007
Labor share in nontradables	0.6	θ_N	0.67
Labor share in tradables	0.6	θ_T^N	0.64
Equal role of capital and land in tradables,	$2\theta_T^K + \theta_T^N = 1$	θ_T^K	0.18
Vacancy posting to output ratio	0.037	κ	0.53
Value of leisure to wage ratio	0.35	φ	0.42
Units Parameters			
Output, Y^*	1	z_N	0.36
Relative price of nontradables, p^*	1	z_T	0.52
Market tightness in labor markets, $\frac{U^*}{V^*}$	1	ς	0.53

Table A.3: Exogenously Determined Parameters in the Frictionless Economy

Parameter	Value
Risk aversion, σ	2.0
Annual rate of return, β	$\frac{1}{\beta^8} - 1 = 4\%$
Elasticity of substitution bw tradables and nontradables, η	0.83
Working Frisch elasticity, γ	1.50
Price markup, ρ	1.05

Table A.4: Targets and Associated Parameters in Frictionless Economy

Target	Value	Parameter	Value
Share of tradables, $\frac{F_T^*}{Y^*}$	0.3	ω	0.90
Fraction of time working, n^*	0.3	ξ_n	26.66
Capital to output ratio, $\frac{K^*}{Y^*}$	2.75	δ	0.009
Labor share in nontradables	0.6	θ_N	0.63
Labor share in tradables	0.6	θ_T^N	0.60
Equal role of capital and land in tradables,	$2\theta_T^K + \theta_T^N = 1$	θ_T^K	0.20
Units Parameters			
Output, Y^*	1	z_N	0.64
Relative price of nontradables, p^*	1	z_T	0.95

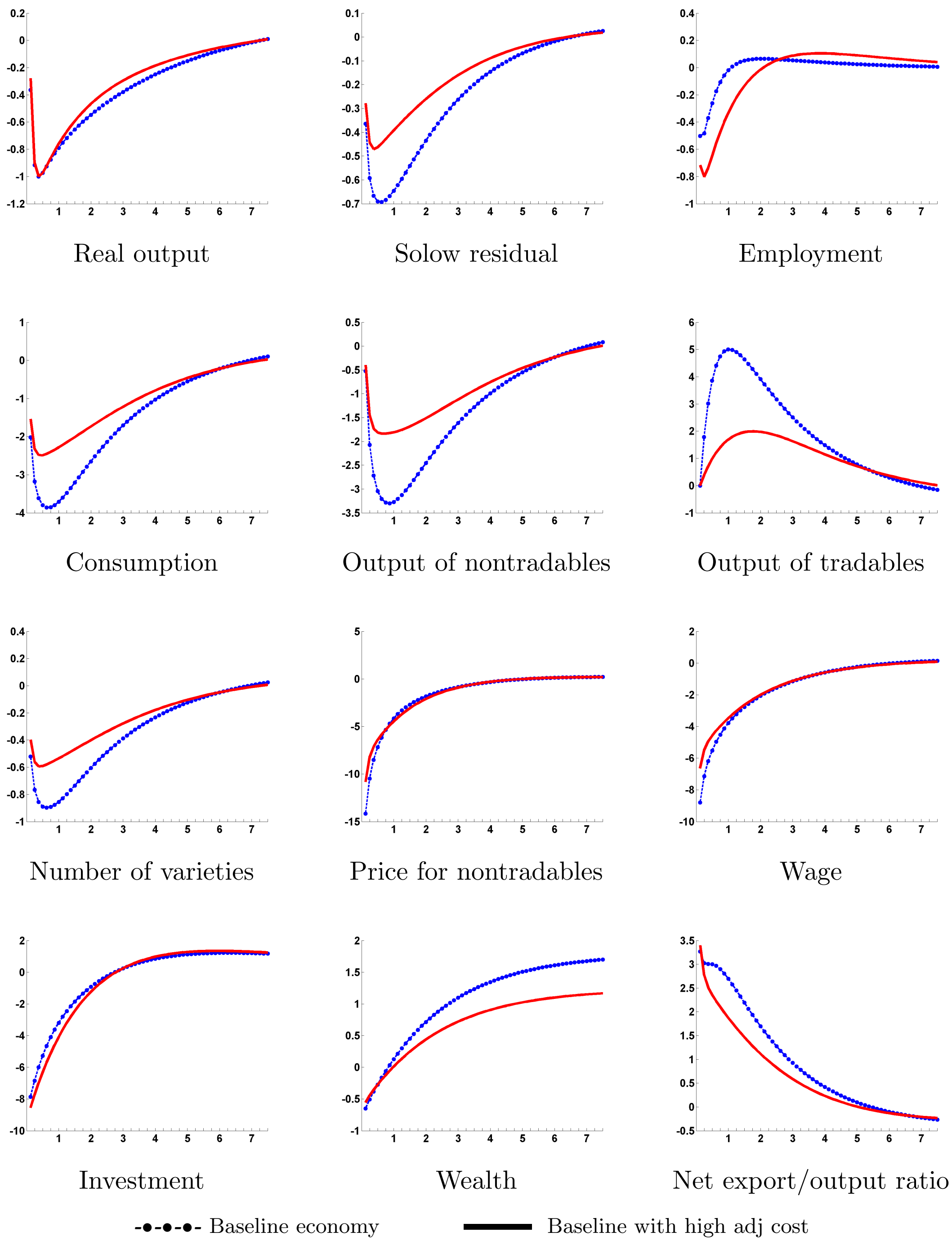
Table A.5: Exogenously Determined Parameters
in the Economy with Goods Frictions but without Labor Frictions

Parameter	Value
Risk aversion, σ	2.0
Annual rate of return, β	$\frac{1}{\beta^8} - 1 = 4\%$
Elasticity of substitution bw tradables and nontradables, η	0.83
Working Frisch elasticity, γ	1.50
Price markup, ρ	1.05

Table A.6: Steady-State Targets and Associated Parameters
in the Economy with Goods Frictions but without Labor Frictions

Target	Value	Parameter	Value
Share of tradables, $\frac{F_T^*}{Y^*}$	0.3	ω	0.91
Fraction of time working, n^*	0.3	ξ_n	27.89
Occupancy rate, $\frac{C_N^*}{F_N^*}$	.81	ν^g	0.81
Capital to output ratio, $\frac{K^*}{Y^*}$	2.75	δ	0.009
Labor share in nontradables	0.6	θ_N	0.63
Labor share in tradables	0.6	θ_T^N	0.60
Equal role of capital and land in tradables,	$2\theta_T^K + \theta_T^N = 1$	θ_T^K	0.20
Units Parameters			
Output, Y^*	1	z_N	0.80
Relative price of nontradables, p^*	1	z_T	0.95
Market tightness in goods markets, D^*	1	ξ_d	0.02

Figure A.1: IRF: Baseline and High Adjustment Cost Economies



Other Alternative Calibration Targets

Table A.7 displays the main properties of the recession in various alternative economies. A higher elasticity of substitution between nontradables and tradables allows households to greatly reduce their consumption of tradables without reducing much of their nontradables, which makes recessions harder to create. The required shock increases by 5% when the elasticity of substitution increases about 40%. We conclude that the differences are small.

For the same reduction in employment as in the baseline economy, a higher labor market matching elasticity μ, makes hiring a new worker cheaper. The total reallocation costs are also cheaper. As a result, to get the same size recession, a 20% larger initial shock is needed.

The role of goods market matching elasticity α is straightforward. For the same decrease in aggregate search effort, D, a higher matching elasticity α leads to a larger decline in the probability of meeting customers. Therefore, the Solow residual and aggregate output decrease further. The size of the shock needed to obtain a 1% recession is 20% smaller when α changes from 0.22 to 0.30.

A lower elasticity of substitution between nontradable goods ($\frac{\rho}{\rho-1}$) increases the elasticity of the number of varieties I with respect to consumption per variety, c_N. In other words, with the same reduction in c_N, an economy with higher ρ will have a larger drop in I. As can be seen from Table A.7, the elasticity of I with respect to c_N is largest when $\rho = 1.08$, corresponding to a larger drop in the Solow residual and a smaller required shock to patience.

As we decrease vacancy costs, the required shock is smaller and the wage rate drop less. The main reason for this change is that workers' bargaining power increases as vacancy costs decrease in order to calibrate to a constant labor share. The issue of how to model the bargaining process between workers and firms is still in debate in the labor market search literature.[3] Most studies focus on the effect of productivity shocks on labor market volatility. Higher bargaining power for workers implies large wage volatility and low employment volatility. However, this argument cannot be carried over into the current environment. In this paper, the recession originates from changes in the willingness to enjoy consumption and leisure today versus tomorrow. These changes result in increased volatility of employment when workers' weight is larger, which in turn implies that the required size of the shock is much lower.

A.3.4 Financial Frictions with Nonsegmented Goods Markets

In the financial frictions economy with nonsegmented goods markets, financial frictions occur between shopping and consuming, meaning that the prices faced and the number of consumption varieties used by both employed and unemployed workers are the same. In other words, shoppers

[3] See [77] and [83].

Table A.7: Alternative Calibration Targets

Elasticity of substitution between nontradable and tradable

Parameter	Pref Shock	Employment	TFP	Tradable Consumption
$\eta = 0.83$	0.88	-0.50	-0.69	-12.33
$\eta = 1.20$	0.93	-0.49	-0.70	-17.17
$\eta = 0.60$	0.84	-0.50	-0.68	-9.17

Labor market matching elasticity

Parameter	Pref Shock	Employment	TFP	Elast of N_N wrt N_T
$\mu = 0.50$	0.88	-0.50	-0.69	-68.95
$\mu = 0.70$	1.05	-0.31	-0.80	-56.27
$\mu = 0.40$	0.79	-0.58	-0.64	-76.97

Goods market matching elasticity

Parameter	Pref Shock	Employment	TFP	Number of varieties
$\alpha = 0.22$	0.88	-0.50	-0.69	-0.89
$\alpha = 0.30$	0.70	-0.38	-0.77	-1.03
$\alpha = 0.10$	1.32	-0.78	-0.48	-0.55

Elasticity of substitution between nontradable goods

Parameter	Pref Shock	Employment	TFP	Elast of I wrt c_N
$\rho = 1.05$	0.88	-0.50	-0.69	58.55
$\rho = 1.08$	0.83	-0.46	-0.72	62.49
$\rho = 1.02$	0.93	-0.54	-0.65	53.88

Vacancy costs

Vacancy cost	Pref Shock	Employment	TFP	Wage rate
3.74%	0.88	-0.50	-0.69	-8.79
2.00%	0.72	-0.66	-0.58	-7.40
1.00%	0.54	-0.84	-0.47	-5.66

first buy a certain amount of goods at each firm and then distribute the goods to the two groups of workers. The amount of goods bought at each location is simply $nc_N^e + (1-n)c_N^u$, where e and u stand for employed and unemployed, respectively. In the absence of financial frictions, the household would equate consumption between employed and unemployed workers. Financial frictions induce the household to provide different amounts of consumption. The problem of the household is now

$$V(S,b,n) = \max_{c_N^e, c_T^e, c_N^u, c_T^u, \mathrm{I}, d} nu(c_A^e, d, 1) + (1-n)u(c_A^u, d, 0) + \beta\,\mathbb{E}\left\{V(S', b', n') \mid \theta\right\}, \tag{A.89}$$

subject to

$$n[p(S)\mathrm{I}c_N^e + c_T^e] + \\ (1-n)[p(S)\mathrm{I}c_N^u + c_T^u] = (1+r)b + w(S)n + \pi_N(S) + \pi_T(S) - \psi(1-n)\mathrm{T}_r - b', \tag{A.90}$$

$$\mathrm{T}_r = p(S)\mathrm{I}c_N^u + c_T^u - [(1+r)b + \pi_N(S) + \pi_T(S)], \tag{A.91}$$

$$\mathrm{I} = \Psi^d[Q^g(S)]\, d, \tag{A.92}$$

$$n' = (1-\lambda)n + \Phi^w[Q^e(S)](1-n), \tag{A.93}$$

$$S' = G(S). \tag{A.94}$$

The total consumption expenditures of each unemployed worker are $p(S)\mathrm{I}c_N^u + c_T^u$, and the financial assets available to each worker are bond holdings plus the profits from firms $(1+r)b + \pi_N(S) + \pi_T(S)$. In the budget constraint, the transfer to an unemployed worker is T_r, the difference between consumption and per agent financial assets, and the financial costs of this transfer are $\psi(1-n)\mathrm{T}_r$. When the household accumulates more savings, the financial costs to achieve the same consumption for the unemployed are smaller. The first-order conditions are

$$u_{c_N^e} = p(S)\mathrm{I}u_{c_T^e}, \tag{A.95}$$

$$u_{c_N^u} = p(S)\mathrm{I}u_{c_T^u}, \tag{A.96}$$

$$u_{c_T^e} = u_{c_T^u}(1+\psi), \tag{A.97}$$

$$n\left[u_{\mathrm{I}}^e - p(S)c_N^e u_{c_T^e} + \frac{u_d^e}{\Psi^d[Q^g(S)]}\right] = -(1-n)\left[u_{\mathrm{I}}^u - p(S)c_N^u u_{c_T^u} + \frac{u_d^u}{\Psi^d[Q^g(S)]}\right], \tag{A.98}$$

$$u_{c_T^e} = (1+r)\mathbb{E}\left\{\beta u'_{c_T^e}[1+\psi'(1-n')] \mid \theta\right\}. \tag{A.99}$$

Equations (A.95) and (A.96) describe the optimality condition between tradables and nontradables. Equation (A.97) implies that unless financial costs are zero (i.e, $\psi = 0$), the consumption level of the employed will be higher than the unemployed. The inequality is increasing in ψ. In the baseline economy, the optimal choice of I equalizes the benefits of one variety and the cost of its associated shopping disutility. With two groups sharing the same number of varieties, the optimal I equalizes a weighted average of costs and benefits, with the weights given by the employment rate. Equation (A.99) is the Euler equation, which we write in terms of consumption of the employed. The problems of the firms are the same as in the baseline economy, except that aggregate demand is now $C_N(S) = n\, C_N^e(S) + (1-n)\, C_N^u(S)$.

We still assume that ψ follows an AR(1) process with persistence of 0.95 and the size of the shock is chosen to get a 1% real output drop. In the first two rows of Table 3.8, we compare the size of the shocks in terms of the explicit or implied proportional change in the discount factor. We use a version of the baseline economy with constant factor shares as well as the economies with financial shocks and constant factor shares. The value goes from 0.85% to 1.14%. The financial cost to output ratio goes from 1% in steady state to 1.33% after the shock to ψ. In terms of employment and the Solow residual, the financial friction economy is very similar to the baseline economy with shocks to the patience.

The exogenously determined parameters are the same as for the baseline economy, and we do not report them here.

Table A.8: Steady-State Targets and Associated Parameters
in the Financial Shocks Economy with Nonsegmented Goods Markets

Target	Value	Parameter	Value
Share of tradables, $\frac{F_T^*}{Y^*}$	0.3	ω	0.91
Unemployment rate, U^*	7%	λ	0.05
Monthly job finding rate	45%	ν^e	0.67
Occupancy rate, $\frac{C_N^*}{F_N^*}$	0.81	ν^g	0.81
Capital to output ratio, $\frac{K^*}{Y^*}$	2.75	δ	0.007
Labor share in tradables	0.6	θ_T^N	0.64
Equal role of capital and land in tradables,	$2\theta_T^K + \theta_T^N = 1$	θ_T^K	0.18
Vacancy posting to output ratio	0.037	θ_N	0.67
Financial cost to output ratio	0.01	ψ	0.28
Units Parameters			
Output, Y^*	1	z_N	0.45
Relative price of nontradables, p^*	1	z_T	0.52
Market tightness in labor markets, $\frac{U^*}{V^*}$	1	κ	0.53
Market tightness in goods markets, D^*	1	ξ	0.02