Essays on Universal Portfolios

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Dedication

I dedicate this thesis to myself — I enjoyed working on this material, and I hope that I can put some of this expertise to practical and profitable use in the future. Hopefully this will amount to some kind of preparation or value-added for my new job teaching financial engineering at Northern Illinois University.

Speaking of the subject matter, I find it far more interesting to envision one's self as a player of the game, rather than a passive observer of equilibria. People play and study games because they want to win them. The individual-sequence approach to decision-making pursued in these pages is simple, elegant, and completely general in the sense that it requires no prior knowledge of the nature and extent of the opponents' motives and [ir]rationality. It leads at once back to the classical techniques of Zermelo, Von Neumann, Wald, and Blackwell, and to combinatorial and computational amusements of a very fundamental character.

The point of capitalism is to accumulate as much capital as possible in the long run. The excitement comes from the fact that optimal capital accumulation requires one to take a very precise level of risk: too little risk leads to practically no growth at all, but too much risk leads to wild fluctuations of the bankroll that, apart from their intrinsic undesirability, serve to destroy the asymptotic growth rate. Which is to say, missed opportunities and false steps can both have equally disastrous consequences at infinity.

In practical life, one is almost always ignorant of the *actual* extent of the risks he has taken — and he is often ignorant of the possible rewards as well. In the manner of Donald Rumsfeld and his *unknown unknowns*, there is just as much risk from model uncertainty as there is inherent in the random outcomes generated by the unknown model. As far as I can see, then, hardly anthing could be more relevant than Cover's austere and brilliant theory of asymptotic capital accumulation. Evidently he was not exaggerating when he called it “universal.”

Abstract

This thesis has three chapters.

Chapter 1 concentrates on a family of sequential portfolio selection algorithms called *multilinear trading strategies*. A multilinear strategy is characterized by the fact that its final wealth is linear separately in each period's gross-return vector for the stock market. These strategies are simple, intuitive, and general enough for many purposes — and yet they retain a basic level of analytic and computational tractability. Thus, instead of the usual method of specifying his portfolio vector each period as a function of the return history, a trader can proceed differently. Rather, he selects a desired final wealth function (which, however, must be *feasible*) and works backward to recover the implied trading strategy.

I show that the class of multilinear strategies is general enough for superhedging derivatives in discrete time. A *superhedge* for a derivative D is a self-financing trading strategy that guarantees to generate cash flows greater than or equal to those of the derivative in any outcome. In dominating D by a multilinear final wealth function, one is able to put upper bounds on the no-arbitrage price of D. This is relevant to realistic trading environments, which are hampered by transaction costs and the impossibility of continuous-time trading. Superhedging is a possible solution: the cost of the cheapest superhedge for D amounts to the greatest possible (model-independent) rational price for the derivative.

Multilinear super-hedging amounts to interpolating D with a multilinear payoff, and then dynamically replicating the interpolating form. If D is a convex function separately of each period's return vector, then there is a multilinear superhedge that is cheaper than any other (multilinear or not). For this reason, I give a detailed guide to the practical computation of multilinear strategies. The key requirement for tractibility is that the form (or derivative) be *symmetric* in the sense that its final wealth depend only on the numerical magnitudes of the return vectors x_t, and not their order. For example, if the daily returns of the U.S. stock market before, during, and after the crash of 1929 were re-ordered in some way, the final wealth of a symmetric multilinear strategy would not have been affected.

Chapter 1 concludes with an extensive study of the high-water mark of Cover's theory of "universal portfolios." Universal portfolios are best understood as superhedges (of varying efficiency) of a specific fictitious "lookback" derivative. The idea is this: a trader imagines a derivative D whose payoff represents the final wealth of a *non-causal* trading strategy, e.g. a trading strategy whose activities at t are in some way a function of the future path of stock prices. In the manner of Biff's sports almanac, the payoff D has been rigged to "beat the market" by a significant margin. Obviously, the trader himself cannot use such a strategy: his behavior can be conditioned on the past, but not the future. However, what he *can* do is try to superhedge D. Cover found (1986, 1991, 1996, 1998) that D could be chosen so as to generate superhedges that (under some tacit restrictions on market behavior) *de facto* "beat the market asymptotically." Any reasonably efficient superhedging strategy for this derivative will enjoy the asymptotic optimality property, and it turns out that there is a large collection of such strategies. The chapter then turns its attention to the question of just how long it takes to reach the asymptote, and what the practical consequences are of increasing the trading frequency.

Chapter 2 studies a family of superhedging and trading strategies that are optimal from the standpoint of sequential minimax. The concept is that, given a path dependent-derivative, a multilinear superhedge (even the cheapest one) that was conceived at $t = 0$ will not necessarily make credible choices for all variations of market behavior. As the path of stock prices is slowly revealed to the trader, it (in everyday cases) becomes apparent that actual cost of superhedging will ultimately prove to be much lower than originally thought. This phenomenon is the result of the fact that superhedging ultimately hinges upon planning for a set of worst-case scenarios, albeit ones that will rarely occur in practice. When these worst cases fail to actually materialize, it has irrevocable consequences for the final payoff of the path-dependent derivative. A sophisticated superhedging strategy will exploit this to dynamically reduce the hedging cost.

Instead of approximating D by a multilinear form and then hedging the approximation, I explicitly calculate a backward induction solution from the end of the investment horizon. The superhedging strategies so-derived are the sharpest possible in all variations. Universal portfolios are the major impetus for the technique, the point being to dynamically reduce the time needed to beat the market asymptotically. In addition to their greater robustness, the sequential minimax trading strategies derived in the chapter are easier to calculate and implement than multilinear superhedges. This being done, I extend the trading model to account for leverage and *a priori* linear restrictions on the daily return vector in the stock market. In deriving a strategy that is robust to a smaller, more reasonable set of outcomes, the trader is able to use leverage in a reliable and perspicacious manner. In the sharpened model, the linear restrictions serve to narrow the set of nature's choices, while simulateneously allowing the trader the privilege of a richer set of (leveraged) strategies. To be specific, nature is required to choose the stock market's return vector from a given cone, and the trader is allowed to pick any admissible (non-bankruptable) portfolio from the dual cone. *a fortiori*, this dynamic is guaranteed to increase the superhedging efficiency, sometimes substantially. This point is illustrated with many numerical examples. Again, the chapter studies the extent to which this trick reduces the time needed to beat the market.

Chapter 2 concludes with a sequential minimax version of Cover's (1996) universal portfolio with side information. In this environment, a discrete-valued signal (the "side information") is available to the trader prior to each period's trading session. The trader starts the game in total ignorance of the meaning of the signal, and he strives to interpret it in the most robust way possible. I provide a universal portfolio under "adversarial" signals whose performance guarantees are a significant refinement to those in Cover (1996). The idea is that a trader, making use of side information, should come to fear the possibility that nature chooses the signal maliciously, intending to create dynamic confusion vis-a-vis the exact meaning of the signal. This meaning is only ever revealed in hindsight, and the trader comes to regret the fact that he was ignorant of the most profitable interpretation of the signal. The trader plays to minimize this regret in the worst case. On account of the complicated environment, the implied optimum trading strategy is only practically computable for horizons on the order of 10-20 periods, and thus is suitable chiefly as, say, an annual trading model.

Chapter 3 is a comprehensive study of universal sequential betting schemes, where the bets are placed on the outcomes of discrete events (colloquially called "horse races"). The Kelly horse race markets studied in the chapter get at the essential features that drive both the multilinear and sequential minimax universal portfolios. The chapter discusses the manner in which these two strategies particularize to one and the same thing under the Kelly horse race. In this connection, the two strategies just amount to the universal source code of Shtarkov (1987), suitably reinterpreted. The sharp performance of the minimax strategy is then compared to the horizon-free strategies that result from particularizing the "Dirichlet-weighted" (1996) universal portfolios and the "Empirical Bayes" (1986) portfolio. Careful attention is given to on-line computation of the universal bets, and several numerical visualizations and simulations are provided. The chapter ends with a sequential minimax refinement to the empirical Bayes stock portfolio. Whereas Cover (1986) is a direct instantiation of Blackwell's (1956) geometric method for approaching a set of vector payoffs, the sequential minimax approach studied here is, on a fixed horizon, the most robust possible strategy for approaching the set.

For convenient reference, a glossary of concepts and notation is given at the end of the thesis.

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Chapter 1

Multilinear trading strategies

There are three related ways to envision the decision problem of a trader in the stock market. In all three, stock prices (or equivalently, returns) are considered to be chosen by "nature." Firstly, we can consider a repeated game whereby in each period t, nature picks the returns of all stocks while the trader simultaneously picks a portfolio vector. This is the point of view adopted by Blackwell's approachability theory and Cover's first (1986) universal portfolio. Second, we can consider a one-shot game whereby the trader picks an entire portfolio selection *algorithm*, and nature simultaneously draws the future price chart for all stocks. This is the point of view adopted in Cover's more evolved later work (1991, 1996, 1998) on universal portfolios. Finally, we can consider a sequential interaction whereby the trader and nature choose their "moves" one at a time. Each period, nature waits for the trader to pick his portfolio and then, having observed this choice, nature picks the returns of all stocks. This back-and-forth happens repeatedly until the end of the trader's investment horizon. For instance, the binomial option pricing model takes just this view. In that model, the trader is able to rig his strategy so that his final wealth exactly replicates the payoff of some derivative, in every possible variation of countermove.

1.1 Definitions and notation

Suppose there are m underlying assets (henceforth called "stocks") that are traded in T discrete sessions, called $t \in \{1, ..., T\}$. Let S_{tj} be the price of stock j at the close of session t, and let x_{tj} be the gross return on a \$1 investment in stock j in session t. That is, $x_{tj} = \frac{S_{tj}}{S_{t-1,j}}$, assuming that no dividends were paid in trading session t. More generally, if each share of stock j receives a dividend of δ_{tj} at the end of session t, then $x_{tj} = \frac{S_{tj}+\delta_{tj}}{S_{t-1,j}}$.

We assume ourselves to be small in relation to the market, so that, at the start of session t, it is possible to buy any number of shares of stock j at the opening price $S(t-1, j)$ (so long as we can afford it), and thereby participate fully in the gross return x_{tj}. The portfolio that we buy at the open of session t (= the close of session $t-1$) must be held until the open of session $t+1$, at which time our holdings in the various stocks will be adjusted. It is presumed that this buying activity is too insignificant to sway the behavior of the wider market. There are no taxes or transaction costs. The gross-return vector in session t is denoted $x_t = (x_{t1}, ..., x_{tj}, ..., x_{tm})$. The return history after session t is denoted $x^t = (x_1, ..., x_t)$. The $T \times m$ matrix $x^T = [x_{tj}]$ is denoted simply X. In accordance with limited liability, we have all positive prices and gross returns, e.g. $S_{tj}, x_t \geq 0$

We take up the most general derivative security, which pays off an amount $D(x_1, ..., x_T)$ at the close of session T. This allows for path-dependence, e.g. the possibility that $D(\cdot)$ is not expressible as a function solely of the final stock prices $S_{T1}, ..., S_{Tm}$. The derivative is written (created and sold) by a primary-dealer at $t = 0$. In each trading session thereafter, D is traded on the secondary market alongside the m stocks. Note that in each session we have a continuum of possible outcomes $x_t \in \mathcal{R}^m_+$, so that D not generally a redundant asset.

Example 1. $D_j = \underset{1 \leq s \leq t \leq T}{Max} \; S_{sj} - S_{tj}$

This is the maximum drawdown suffered by an investor who owns one share of stock j for the duration $1 \leq t \leq T$. Somebody who particularly hates drawdowns, for instance, might be willing to pay a significant upfront fee in exchange for $\$D_j$ at T.

Example 2. $D_j = \underset{1 \leq s \leq t \leq T}{Max} \; S_{tj} - S_{sj}$

This is the profit that an investor could have realized from the single shrewdest trade over $1 \leq t \leq T$. That is, someone looks back at the chart of stock j, wishing he had bought in s^* and sold in t^*.

Example 3. $D = \underset{1 \le s \le t \le T}{Max} \; \sum_{j=1}^{m} n_j(S_{sj} - S_{tj})$

Here, D represents the maximum drawdown suffered by an investor who bought an initial portfolio with n_j shares of each stock j, and held it throughout $1 \le t \le T$. If $n_j < 0$, it means that asset j was sold short at the start of session 1, and covered at the end of T.

Example 4. $D_j = \underset{1 \le j \le m}{Max} \; S_{Tj}$

D_j gives the gross return on \$1 invested in the best performing stock over $1 \le t \le T$. Unlike the foregoing examples, this D is path-independent.

In all these examples, the full expression for $D(x_1, ..., x_T)$ is gotten by substituting $S_{tj} = S_{0j}x_{1j}x_{2j} \cdots x_{tj}$. In the sequel it will indeed prove convenient to deal with returns instead of prices, chiefly because a greedy trader or gambler only ever cares about returns, e.g. the factor by which he has multiplied his money on any given bet.

1.2 Self-financing trading strategies

We consider self-financing trading strategies, generally called $\theta(\cdot)$. Literally, the strategy finances its asset purchases internally, via the sales of other assets. A self-financing strategy is not subject to any deposits or withdrawals, except for the initial deposit of money into the strategy.

Thus, a trader deposits \$1 into $\theta(\cdot)$ at $t = 0$ and just "lets it ride." Let θ_{tj} be the fraction of wealth that the trader puts into stock j at the start of session t, where $\sum_{j=1}^{m} \theta_{tj} = 1$. Thus $\theta_{tj} = \theta_{tj}(x_1, ..., x_{t-1}) = \theta_{tj}(x^{t-1})$. The trader's portfolio vector in session t is denoted $\theta_t = (\theta_{t1}, ..., \theta_{tm})$.

For the time being we require that $\theta_{tj} \ge 0$, but this assumption will be relaxed in Chapter 2. For simplicity, we will merely write $\theta(x_1, ..., x_t) \in \Delta$ for the trader's portfolio in session $t+1$. Formally, then, $\theta : \cup_{t=1}^{\infty} (\mathcal{R}_{+}^{m})^t \to \Delta$ is a mapping of return histories into the portfolio simplex.

In the first trading session, when there is no data to speak of, a strategy must specify some initial portfolio, which I denote by $\theta(h^0)$, where h^0 is the empty history. For example, a common choice is $\theta(h^0) = (1/m, ..., 1/m)$.

In session t, $\theta(\cdot)$ multiplies the trader's wealth by the factor $\sum_{j=1}^{m} \theta_{tj}(x^{t-1})x_{tj} = \langle \theta_t(x^{t-1}), x_t \rangle$, the dot product of the portfolio vector and the return vector. After t sessions, the trader's initial dollar has grown into

$$W_\theta(x_1, ..., x_t) = \langle \theta(h^0), x_1 \rangle \langle \theta(x_1), x_2 \rangle \cdots \langle \theta(x_1, ..., x_{T-1}), x_T \rangle. \tag{1.1}$$

This equation formalizes the fact that $\theta(\cdot)$ is self-financing. Notice how every self-financing trading strategy θ induces a derivative $D_\theta(x_1, ..., x_T) = W_\theta(x_1, ..., x_T)$.

We can consider the dynamic programming state after t periods to be $\xi_t = (x_1, ..., x_t)$, with concatenation as the transition law: $\xi_{t+1} = (\xi_t, x_{t+1})$. Thus, both the trader's action and the next state are functions of the current state. The strategies considered in this thesis thus have *infinite memory.* In general, one can choose an arbitrary transition law $F(\cdot)$ and state-dependent portfolio vector $\theta(\cdot)$:

$$\xi_{t+1} = F(\xi_t, x_{t+1}) \tag{1.2}$$

$$\theta_{t+1} = \theta(\xi_t) \tag{1.3}$$

Example 5. *Helmbold's (1998) "exponentiated gradient" strategy only ever cares to remember yesterday's return x_t, and yesterday's portfolio vector θ_t, the latter incorporating all the information about the return history that the strategy deems necessary. The state is therefore $\xi_t = (\theta_t, x_t)$ with new portfolio vector $\theta_{t+1} = \theta(\xi_t)$ and transition law $F(\xi_t, x_{t+1}) = (\theta(\xi_t), x_{t+1})$. Helmbold's formula for the new portfolio vector is*

$$\theta_{t+1,j} = \frac{\theta_{tj}\, exp\left(\eta \frac{x_{tj}}{\langle \theta_t, x_t \rangle}\right)}{\sum_{k=1}^{m} \theta_{tk}\, exp\left(\eta \frac{x_{tk}}{\langle \theta_t, x_t \rangle}\right)}, \tag{1.4}$$

where η is a numerical parameter chosen by the practitioner. Here $\langle \theta_t, x_t \rangle = \sum_{j=1}^{m} \theta_{tj} x_{tj}$ is the scalar product. The strategy is self-financing on account of the fact that $\sum_{j=1}^{m} \theta_{t+1,j} = 1$.

Definition 1. *A derivative $D(x_1, ..., x_T)$ is said to be perfectly hedgeable (or replicable) iff there is a self-financing trading strategy $\theta(\cdot)$ and an initial deposit p such that $D = p \cdot W_\theta$. This (necessarily unique) θ is called the hedging (or replicating) strategy corresponding to $D(\cdot)$. The (unique) initial deposit p is called the hedging cost.*

Example 6. *The general price-weighted index (viz. the Dow) has value* $D(x_1, ..., x_t) = \lambda(S_{t1} + S_{t2} + \cdots + S_{tm})$

Here, λ is some scale factor rigged to make the index a "nice" number. The performance of the index is uniquely replicated by the strategy $\theta(S_{t1}, ..., S_{tm}) = \frac{1}{\sum_{j=1}^{m} S_{tj}}(S_{t1}, ..., S_{tm})$. In words, all you do is buy λ shares of each stock, and never trade again. The hedging cost is the initial value of the index.

Example 7. *The general market capitalization-weighted index (e.g. S&P, NASDAQ Composite) has value* $\lambda(n_1 S_{t1} + n_2 S_{t2} + \cdots + n_m S_{tm})$*, where* n_j *is the number of shares firm* j *has outstanding, and* λ *is a scale factor.*

The cap-weighted index is replicated by $\theta(S_{t1}, ..., S_{tm}) = \frac{1}{\sum_{j=1}^{m} n_j S_{tj}}(n_1 S_{t1}, ..., n_m S_{tm})$. In words, all you do is buy a "market portfolio" that holds each asset in proportion to its market capitalization, and never trade again. Notice how $\langle \theta(S_{t1}, ..., S_{tm}), x_{t+1} \rangle = \frac{\sum_{j=1}^{m} n_j S(t+1,j)}{\sum_{j=1}^{m} n_j S(t,j)}$, the ratio of the index value at the close of $t+1$ to the index value at the close of t.

Assuming that $D = p \cdot W_\theta$ can be hedged perfectly, the unique replicating strategy is derived as follows. Start with

$$\langle \theta(x^t), x_{t+1} \rangle = \frac{W_\theta(x^{t+1})}{W_\theta(x^t)}, \tag{1.5}$$

and substitute $x_{t+1} = e_j$, where e_j is the j^{th} unit basis vector. This gives $\theta_j(x^t) = \frac{W_\theta(x^t, e_j)}{W_\theta(x^t)}$. Summing over j, we get $W_\theta(x^t) = \sum_{j=1}^{m} W_\theta(x^t, e_j)$. Applying this last formula repeatedly, one gets the formulas

$$W_\theta(x^t) = \sum_{(j_{t+1},...,j_T) \in \{1,...,m\}^{T-t}} W_\theta(x^t, e_{j_{t+1}}, ..., e_{j_T}) \tag{1.6}$$

$$\theta_k(x^t) = \frac{\sum_{(j_{t+2},...,j_T) \in \{1,...,m\}^{T-t-1}} D(x^t, e_k, e_{j_{t+2}}, ..., e_{j_T})}{\sum_{(j_{t+1},...,j_T) \in \{1,...,m\}^{T-t}} D(x^t, e_{j_{t+1}}, ..., e_{j_T})} \tag{1.7}$$

Here $\theta_k(x^t)$ denotes the k^{th} coordinate of $\theta(x^t)$.

In general, the above formula for $\theta(\cdot)$ in terms of $D(\cdot)$ may well be an extraneous solution of the functional equation $D = W_\theta$. Of course, one must substitute the strategy θ so obtained back into the equation $D = W_\theta$, and verify that it is a solution. This phenomenon is illustrated below.

Example 8. *Consider a fictitious investor who has advance knowledge of all stock prices. Each trading session, he puts all his money into the stock that will happen have the greatest percentage increase in price, achieving a growth factor of* $\underset{1\leq j\leq m}{Max}\ x_{tj}$. *His final wealth is then* $D(x_1, ..., x_T) = \prod_{t=1}^{T} \left\{ \underset{1\leq j\leq m}{Max}\ x_{tj} \right\}$. *Obviously, this derivative should not be perfectly replicable — and in fact, substituting it into the above yields the extraneous solution* $\theta_k(x^t) \equiv 1/m$. *Substituting into* $D = W_\theta$ *then leads to the false statement*

$$m^{-T} \prod_{t=1}^{T} \left\{ \sum_{j=1}^{m} x_{tj} \right\} \equiv \prod_{t=1}^{T} \left\{ \underset{1\leq j\leq m}{Max}\ x_{tj} \right\} \tag{1.8}$$

The upper and lower envelopes of the market are illustrated below for 100 days in 2017, with $m = 2$ stocks (Amazon, Netflix). The value of the upper envelope at t represents the wealth of somebody who, in each period $1 \leq s \leq t$, was lucky enough to have all of his wealth in the best-performing stock that period. That is, the upper envelope after t periods is $\prod_{s=1}^{t} \left\{ \underset{1\leq j\leq m}{Max}\ x_{sj} \right\}$. Similarly, the value of the lower envelope, $\prod_{s=1}^{t} \left\{ \underset{1\leq j\leq m}{Min}\ x_{sj} \right\}$ represents the wealth of somebody who, in each period $1 \leq s \leq t$, was unlucky enough to have all his wealth in the worst-performing stock that period.

The wealth path of any trading strategy must lie between these two bounds. It is not necessarily the case, however, that every path that lies between these two bounds could have been generated as the wealth series of some trading strategy.

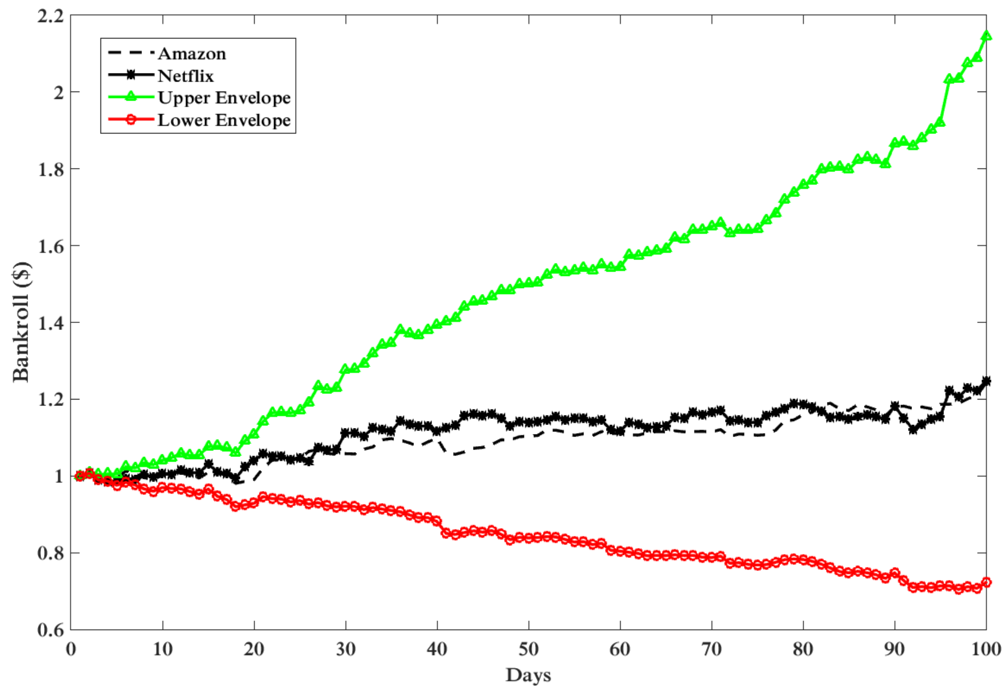

Figure 1.1: **The envelope of feasible wealth paths for 100 days in 2017, trading Amazon and Netflix**

However, this entire process has given an interesting argument in favor of the equal weight index $\theta(x^t) = (1/m, ..., 1/m)$. In any given trading session t, having no idea what the best performing stock j will be, we spread our money evenly among the m stocks, thereby guaranteeing that at least $1/m$ of our money will achieve the growth factor $||x_t||_\infty$.

Theorem 1. *A derivative $D(x_1, ..., x_T)$ can be exactly dynamically replicated if and only if it satisfies the following functional equation, identically for all $x_1, ..., x_T$:*

$$\prod_{t=1}^{T} \left(\frac{\sum\limits_{(j_t,...,j_T)\in\{1,...,m\}^{T-t+1}} D(x^{t-1}, e_{j_t}, ..., e_{j_T}) x_{tj_t}}{\sum\limits_{(j_t,...,j_T)\in\{1,...,m\}^{T-t+1}} D(x^{t-1}, e_{j_t}, ..., e_{j_T})} \right) \equiv \frac{D(x_1, ..., x_T)}{\sum\limits_{(j_1,...,j_T)\in\{1,...,m\}^{T}} D(e_{j_1}, ..., e_{j_T})} \tag{1.9}$$

Corollary 1. *If $D(\cdot)$ can be hedged exactly, then the hedging cost is* $p = \sum\limits_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$.

Corollary 2. *If* $D(x_1, ..., x_T) \geq 0$ *is a multilinear form, e.g. it is linear separately in each vector argument* x_t*, then* D *can be replicated exactly.*

To see this, we can write $D(x^{t-1}, e_{j_t}, ..., e_{j_T})x_{tj_t} = D(x^{t-1}, x_{tj_t}e_{j_t}, ..., e_{j_T})$ on account of the fact that D is homogeneous separately in each argument. We then sum this equation over the indices $j_t = 1, 2, ..., m$, and get $D(x^t, e_{j_{t+1}}, ..., e_{j_T})$ on account of the fact that D is additive separately in each vector argument. The product on the left-hand side of the functional equation is now seen to be telescopic; it collapses exactly to the ratio given on the right-hand side of the functional equation.

1.3 Multilinear derivatives

There are many examples of derivatives (or final wealth functions) that are linear separately in each of the vectors $x_1, ..., x_T$. In fact, any buy-and-hold strategy yields a multilinear final wealth function: start with \$1 and make some initial distribution $c = (c_1, ..., c_m) \in \Delta$ of wealth into the m stocks, and never trade again, yielding

$$W = \sum_{j=1}^{m} c_j x_{1j} x_{2j} \cdots x_{Tj} \tag{1.10}$$

In particular, the familiar price- and capitalization-weighted indexes are multilinear derivatives. Equal weight indexes are also multilinear, e.g.

$$W = m^{-T} \langle \mathbf{1}, x_1 \rangle \langle \mathbf{1}, x_2 \rangle \cdots \langle \mathbf{1}, x_T \rangle, \tag{1.11}$$

where $\mathbf{1}$ is a vector of ones.

We take up the problem of hedging multilinear derivatives $D(x_1, ..., x_T) \geq 0$. Let θ be the corresponding hedging strategy. On account of the fact that $W_\theta = \frac{D}{p}$, W_θ is itself a positive multilinear form. We thus have the expansion

$$W_\theta(x_1, ..., x_T) = \sum_{j_1, ..., j_T} W_\theta(e_{j_1}, ..., e_{j_T}) x_{1j_1} x_{2j_2} \cdots x_{Tj_T}, \tag{1.12}$$

where the form's coefficients $W_\theta(e_{j_1}, ..., e_{j_T})$ are all positive and sum to 1. After some careful substitutions and manipulations, and remembering that $W_\theta(\cdot)$ is supposed to be multilinear, we recover the replicating strategy

$$\theta_k(x^t) = \frac{\underset{(j_1,...,j_t,j_{t+2},...,j_T)\in\{1,...,m\}^{T-1}}{\Sigma} D(e_{j_1},...,e_{j_t},e_k,e_{j_{t+2}},...,e_{j_T})x_{1j_1}x_{2j_2}\cdots x_{tj_t}}{\underset{(j_1,...,j_T)\in\{1,...,m\}^T}{\Sigma} D(e_{j_1},...,e_{j_T})x_{1j_1}x_{2j_2}\cdots x_{tj_t}} \tag{1.13}$$

The denominator (independent of k) is just the sum of the numerators for $k = 1, ..., m$. The numerators are multilinear functions of the return data as it is known after t trading sessions. In the k^{th} numerator, the coefficient of the product $x_{1j_1}\cdots x_{tj_t}$ is given by

$$\alpha(j_1,...,j_t) = \underset{j_{t+2},...,j_T}{\Sigma} D(e_{j_1},...,e_{j_t},e_k,e_{j_{t+2}},...,e_{j_T}). \tag{1.14}$$

For illustrative purposes, I give some additional discussion of the fact that this is not an extraneous solution. The key thing to recognize is that, in the general derivation of θ above, we took the equation

$$\langle\theta(x^t), x_{t+1}\rangle = \frac{W_\theta(x^t, x_{t+1})}{W_\theta(x^t)}, \tag{1.15}$$

and made the m substitutions $x_{t+1} = e_j$. For general derivatives this transformation is not reversible, but for multilinear $D(\cdot)$ it is. Given the equations $\theta_j(x^t) = \frac{1}{W_\theta(x^t)}W_\theta(x^t, e_j)$, multiply both sides by $x_{t+1,j}$ and sum over j. The multilinearity of $W(\cdot)$ then gives the desired result. A brute force verification can also be given as follows: in the above formula given for $\theta_k(x^t)$, one multiplies both sides by $x_{t+1,k}$, sums over k, and then forms the product of the resulting equations for $t = 0, 1, ..., T-1$. This last product is telescopic, and evaluates to

$$\frac{D(x_1,...,x_T)}{\underset{j_1,...,j_T}{\Sigma} D(e_{j_1},...,e_{j_T})} \tag{1.16}$$

Definition 2. *A self-financing trading strategy θ is called multilinear iff its induced final wealth function $W_\theta(x_1,...,x_T) \geq 0$ is multilinear over the domain $(\mathcal{R}^m_+)^T$.*

Thus, rather than specify θ directly, one can simply give (m^T) numerical coefficients $W(e_{j_1},...,e_{j_T})$ that are nonnegative and sum to 1. The set of multilinear strategies can be identified with the unit simplex in $\mathcal{R}^{m^T}$.

1.3.1 Interpretation of multilinear final wealth as a convex combination of the final wealths of extremal strategies

The general (normalized) multilinear strategy is characterized by its final wealth function $W(x_1, ..., x_T) = \sum_{(j_1,...,j_T)\in\{1,...,m\}^T} \alpha(j_1, ..., j_T) x_{1j_1} x_{2j_2} \cdots x_{Tj_T}$, where the coefficients $\alpha(j_1, ..., j_T)$ are positive numbers that sum to 1. This has a very simple, intuitive explanation in terms of *extremal trading strategies*. Consider the trading strategy that, in period 1, puts all its wealth into stock j_1. Then in period 2, it puts all of its wealth into j_2, all into j_3 in period 3, and so on. After t periods, a \$1 investment in this strategy grows into $x_{1j_1} x_{2j_2} \cdots x_{tj_t}$ dollars. Now, imagine somebody who starts with a dollar at $t = 0$, and distributes this money in some way among the m^T extremal strategies. Say, he puts an amount $\alpha(j_1, ..., j_T)$ into the $j^T = (j_1, ..., j_T)^{th}$ strategy. Then his final wealth is precisely $\sum_{j^T\in\{1,...,m\}^T} \alpha(j^T) x_{1j_1} x_{2j_2} \cdots x_{Tj_T}$. The trick is that if m or T are large, accounting for the performance of so many (m^T) strategies becomes intractable. After t trading sessions, the $(j^T)^{th}$ strategy has accumulated $\alpha(j^T) x_{1j_1} x_{2j_2} \cdots x_{tj_t}$ dollars, and the trader's aggregate wealth is $\sum_{j_1,...,j_T} \alpha(j^T) x_{1j_1} \cdots x_{tj_t}$. Note that the coefficient of $x_{1j_1} \cdots x_{tj_t}$ is $\sum_{j_{t+1},...,j_T} \alpha(j_1, ..., j_T)$. To find the proportion of wealth to bet on stock k after partial history $x_1, ..., x_t$, we consider the extremal strategies $(j_1, ..., j_t, k, j_{t+2}, ..., j_T)$ that have their money in stock k in period $t + 1$. Thus, in aggregate, $\sum_{(j_1,...,j_t,j_{t+2},...,j_T)\in\{1,...,m\}^{T-1}} \alpha(j_1, ..., j_t, k, j_{t+2}, ..., j_T) x_{1j_1} \cdots x_{tj_t}$ dollars are bet on stock k in period $t + 1$. The final portfolio weight is thus

$$\theta_k(x_1, ..., x_t) = \frac{\sum_{(j_1,...,j_t,j_{t+2},...,j_T)\in\{1,...,m\}^{T-1}} \alpha(j_1, ..., j_t, k, j_{t+2}, ..., j_T) x_{1j_1} \cdots x_{tj_t}}{\sum_{j_1,...,j_T} \alpha(j^T) x_{1j_1} \cdots x_{tj_t}} \quad (1.17)$$

Put differently, we can consider extremal strategies to be the recommendations of a collection of m^T "experts," whereby Mr. j^T is thinks that j^t will be the best performing stock in session t. Notice that the final wealth of one such expert j^T will be the upper envelope $||x_1||_\infty \cdots ||x_T||_\infty$ of the market. Correspondingly, one of the experts will also fail miserably, achieving the lower envelope $\prod_{t=1}^T \underset{1\leq j\leq m}{Min}\, x_{tj}$ of the market. A multilinear trading strategy amounts to a prior distribution of wealth among these experts. However, the number of experts is far to large to actually carry this out in practice: instead we just do the accounting and calculate the portfolio vectors $\theta_t(x^{t-1})$

implied by the prior distribution of wealth $\alpha(j^T)$.

Illustration

To illustrate the concept of an extremal strategy, I plot the daily price histories of Amazon and Netflix, together with the bankrolls of a pair of extremal strategies $j^T = (j_1, ..., j_T)$ and $k^T = (k_1, ..., k_T)$ drawn *a priori* at random from the set $\{1, 2\}^T$. In general, the designer of a multilinear trading strategy specifies in advance a (possibly large) set of extremal strategies, along with an initial distribution of wealth among these extremals.

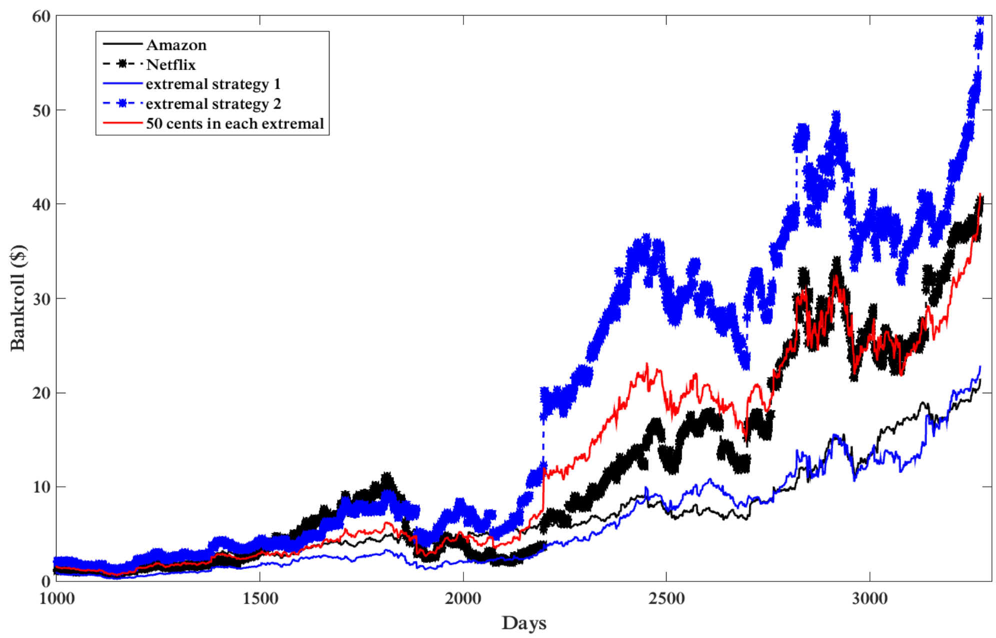

Figure 1.2: **Making an initial deposit of 50 cents into a pair of extremal strategies, and letting it ride**

1.4 Symmetric multilinear strategies and their computation

It becomes computationally onerous to carry out the accounting for m^T experts when m or T is large — some simplification must be made to the prior distribution $\alpha(j_1, ..., j_T)$.

For example, a buy-and-hold strategy only distributes wealth among m experts, e.g. $\alpha(j^T)$ is only positive for $j^T = (k, k, ..., k)$ for some stock k. That is, buy-and-hold strategies only consider experts Mr. k that recommend the same stock k each period. For the sake of tractability, we will assume that the probability mass function $\alpha(j_1, ..., j_T)$ is symmetric in the indices $j_1, ..., j_T$. Thus, the only relevant information about Mr. j^T is the fact that he makes n_1 total recommendations of stock 1, n_2 total recommendations of stock 2, ... , and stock n_m total recommendations of stock m. The numbers

$$n(j^T) = (n_1(j^T), ..., n_m(j^T)) \tag{1.18}$$

constitute the *type* of j^T. Two experts j^T and k^T are considered equivalent ($j^T \sim k^T$) iff they have the same type. The possible experts (types) now correspond to solutions of the equation $n_1 + \cdots + n_m = T$ in nonnegative integers. This *de facto* reduces the number of experts to $\binom{m+T-1}{m-1} = \mathcal{O}(T^{m-1})$. The type class corresponding to $(n_1, ..., n_m)$ is an amalgamation of $\binom{T}{n_1\, n_2 \cdots n_m} = \frac{T}{n_1! n_2! \cdots n_m!}$ experts. For a pairs trading strategy with 2 stocks, this amounts to just $T+1$ experts to keep track of. The assumed symmetry of the prior monetary distribution $\alpha(\cdot)$ expresses an agnosticism with respect to time: the trader has no *a priori* knowledge of the how the future will play out, and will learn on the fly from the observed performance of the experts in managing money. If experts j^T and k^T recommend Amazon and Netflix with the same sample frequencies, the designer of a symmetric multilinear strategy has no prior basis on which to assume that one will outperform the other. For instance, if somebody believes that the return vectors x_t will be independent and identically distributed across time (but he is ignorant of the CDF $F(\cdot)$ from which these returns will be drawn), then it would be preposterous to use a multilinear trading strategy that is not symmetric.

We will abuse notation and write $\alpha(n_1, ..., n_m)$ for the initial amount of money distributed to any given trader of type n. The type-n experts are thus given $\binom{T}{n_1, ..., n_m}\alpha(n)$ dollars in aggregate to manage. $\alpha(n)$ must then satisfy $\sum_{n_1+\cdots+n_m=T} \binom{T}{n_1, ..., n_m}\alpha(n) = 1$. It is a good idea to choose this function itself to be symmetric in the variables $n_1, ..., n_m$, so that no stock is treated *a priori* in a way that is different from any other.

Example 9. *$\alpha(n) = \lambda n_1^{n_1} n_2^{n_2} \cdots n_m^{n_m}$, where $\lambda = \left\{ \sum_{n_1+\ldots+n_m=T} \binom{T}{n_1, ..., n_m} n_1^{n_1} \cdots n_m^{n_m} \right\}^{-1}$. This is the prior distribution in Cover's (1998) universal portfolio.*

Example 10. *$\alpha(n) = \lambda n_1! n_2! \cdots n_m!$, where $\lambda = \left\{ \underset{n_1+\ldots+n_m=T}{\Sigma} \binom{T}{n_1,\ldots,n_m} n_1! \cdots n_m! \right\}^{-1} = \left\{ \binom{m+T-1}{m-1} T! \right\}^{-1}$. Thus $\alpha(n) = \left\{ \binom{m+T-1}{m-1} \binom{T}{n_1,\ldots,n_m} \right\}^{-1}$. This is the prior distribution in Cover's (1991) universal portfolio. This strategy is thus characterized by the fact that it distributes an equal amount of money into each type class.*

1.4.1 Simplification of $\theta_k(x^t)$

For a symmetric multilinear trading strategy, the numerator of $\theta_k(x^t)$ can be simplified as follows. Let $\alpha(j^t, k) = \underset{j_{t+1},\ldots,j_T}{\Sigma} \alpha(j^t, k, j_{t+1}, ..., j_T)$ be the marginal pmf obtained from α by summing over the coordinates $j_{t+1}, ..., j_T$. This number depends only on k and the type $(N_1, ..., N_m)$ of j^t, where $N_1 + \cdots + N_m = t$. In fact, if j^t has type N, then $\alpha(j^t, k)$ is equal to

$$\underset{n_1+\cdots+n_m=T-t-1}{\Sigma} \binom{T-t-1}{n_1, ..., n_m} \alpha(N_1 + n_1, ..., N_k + n_k + 1, , ..., N_m + n_m). \tag{1.19}$$

Denote this number by $\alpha_{tk}(N_1, ..., N_m)$. We then have

$$\underset{j^t}{\Sigma}\ \alpha(j^t, k) x_{1j_1} \cdots x_{tj_t} = \underset{N_1+\cdots+N_m=t}{\Sigma}\ \alpha_{tk}(N) \underset{j^t \, has\, type\, N}{\Sigma} x_{1j_1} \cdots x_{tj_t}. \tag{1.20}$$

Let $\sigma(N_1, ..., N_m; x^t)$ denote the number $\underset{j^t \, has\, type\, N}{\Sigma} x_{1j_1} \cdots x_{tj_t}$. Effective calculation of the numerator of $\theta_k(x^t)$ thus can be broken into three parts:

1. Calculate $\sigma(N_1, ..., N_m; x^t)$ by a recursive method

2. Calculate $\alpha_{tk}(N)$ by a recursive method (if $\alpha(\cdot)$ allows) or else by direct summation

3. Explicitly add all the terms in

$$\underset{N_1+\cdots+N_m=t}{\Sigma}\ \alpha_{tk}(N) \sigma(N; x^t) \tag{1.21}$$

A recurrence for $\sigma(N; x^t)$ is derived as follows. $\sigma(N; x^t) = \sum_{k=1}^m \left\{ \underset{j^{t-1}\, has\, type\, (N_1,\ldots,N_k-1,\ldots,N_m)}{\Sigma} x_{1j_1} \cdots x_{t-1,j_{t-1}} \right\} x_{tk} = \Sigma_{k=1}^m \sigma(N_1, ..., N_k - 1, ..., N_m) x_{tk}$

$$= \Sigma_{k=1}^m \sigma(N_1, ..., N_k - 1, ..., N_m; x^{t-1}) x_{tk}. \tag{1.22}$$

The recursion gradually reduces the numbers $N_1, ..., N_m$ until one of them (say, the k^{th}) is 1 and the rest are 0. The boundary conditions are then

$$\sigma(0, ..., \underset{k}{1}, ..., 0; x_1) = x_{1k}. \tag{1.23}$$

Thus, calculating $\sigma(N; x^t)$ requires m recursive calls, and the recursion tree is $t-1$ levels deep. One is required to calculate *all* the numbers $\sigma(r_1, ..., r_N; x^t)$ for which $r_k \geq 0$ and $1 \leq r_1 + \cdots + r_m \leq t-1$. This amounts to calculating and storing $\sum_{s=1}^{t-1} \binom{s+m-1}{m-1} = \mathcal{O}(t^m)$ numbers, which is possible for small values of m. A direct recursive implementation should not be attempted, as the recursion tree will involve enormous duplication. Rather, the numbers $\sigma(r_1, ..., r_N; x^t)$ should be tabulated according to the "bottom up" approach. At step s, we tabulate all the numbers $\sigma(r_1, ..., r_N; x^t)$ for which $r_1 + \cdots + r_N = s$, making use of all the numbers tabulated in step $s-1$. Once step s is completed, the numbers tabulated in step $s-1$ no longer need to be stored.

1.5 Cover's (1991, 1996) horizon-free universal strategies

Cover's horizon-free universal portfolio is a leading example of a symmetric multilinear trading strategy. In addition to its being characterized as that symmetric multilinear strategy that distributes its initial capital uniformly among the type classes of the extremal strategies, it can also be understood as a distribution of wealth among a class of experts whose recommendations are horizon-free in the sense that they are not tied to a particular time or investment horizon. In Cover's words, the strategy tries to locate the "financial center of gravity" of the stockmarket. The portfolio in session $t+1$ is given as a performance-weighted average of a set of simpler strategies, namely the constant rebalancing rules.

Definition 3. *A rebalancing rule is a constant trading strategy* $\theta(x^t) \equiv c = (c_1, ..., c_m) \in \Delta$.

This means that, at the start of each trading session t, the trader puts a (fixed) proportion $c_j \geq 0$ of his wealth into asset j, where $\sum_{j=1}^m c_j = 1$. At the close of trading session t (*after* x_t is realized, but *before* the portfolio is rebalanced), the trader no longer has exactly c_j of his wealth in asset j — rather, he has the fraction

$$\frac{c_j x_{tj}}{\sum_{k=1}^m c_k x_{tk}} \tag{1.24}$$

in asset j. In order to rebalance his portfolio back to the fixed fractions $c = (c_1, ..., c_m)$, the trader will have to sell some of his holdings in certain stocks j, and use the proceeds to increase his holdings in some of the other stocks. He will have to buy additional shares of stock j if the inequality

$$(1 - c_j)x_{tj} < \sum_{k \neq j} c_k x_{tk} \tag{1.25}$$

is true; otherwise he will have to sell some shares of stock j. Let n_{tj} denote the number of shares of stock j the gambler holds at the end of session t. Then c-rebalancing dictates that he buy

$$\frac{c_j}{S_{tj}} \sum_{k=1}^{m} n_{tk} S_{tk} - n_{tj} \tag{1.26}$$

additional shares of stock j.

We have already met the equal weight index $\theta(x^t) = 1/m$. Generally speaking, rebalancing rules provide mechanical procedures whereby one buys low and sells high. Just as soon as some stock j starts to outperform, the trader lightens his load, rerouting the profits into the (relatively cheaper) stocks $-j$. Empirical backtests are fairly definitive on the value of rebalancing: most buy-and-hold investors stand to gain from annual, semi-annual, or quarterly rebalancing. Most of the time, one finds that periodic rebalancing leads to some combination of higher rates of return, lower risk, lower max drawdowns, greater alpha, or higher Sharpe and Sortino ratios. The effect is most pronounced when the portfolio consists of several volatile, uncorrelated stocks.

Notice that *a fortiori*, the final wealth $\langle c, x_1 \rangle \cdots \langle c, x_T \rangle$ from the c-rebalancing rule is a symmetric multilinear form. After expansion, we have, using the terminology of the prequel,

$$W(x^t) = \sum_{n_1 + \cdots + n_m = T} c_1^{n_1} \cdots c_m^{n_m} \sigma(n_1, ..., n_m; x^t) \tag{1.27}$$

The rebalancing rules can be thought to constitute a continuum of experts $c \in \Delta$, with Mr. c recommending the portfolio $\theta_t = c$ each day. After T periods, some one of these experts $c^*(x_1, ..., x_T)$ will wind up achieving a growth factor that is greater than any other. In hindsight, we will regret not having used $c^*(x^T)$ to begin with. The key insight here is that someone should make an initial distribution of his dollar among the continuum of rebalancing rules $c \in \Delta$ and let it ride, in a vein similar to the extremal strategies. For the sake of what follows, a "rebalancing rule" will mean a point of the set $\mathcal{C} = \{(c_1, ..., c_{m-1}) \geq 0 : c_1 + \cdots + c_{m-1} \leq 1\}$, which is to say, the volume below the set $\Delta = \{(c_1, ..., c_m) \geq 0 : c_1 + \cdots + c_m = 1\}$ in the positive orthant.

Let $F(\cdot)$ be the CDF of this prior distribution of wealth. If F has a corresponding density $f(\cdot)$, this means that the rebalancing rules in the vicinity of c have been distributed money at a rate of $f(c)$ dollars per unit of volume; the total amount of money distributed to this vicinity is $f(c)dc_1 \cdots dc_{m-1}$ dollars. Assuming that f is positive and continuous, the neighbordhood of $c^*(x^T)$ is guaranteed to receive a positive amount of money to manage. Of course, this is in spite of the fact that the location of $c^*(x^T)$ is not known in advance. Over time, the various rebalancing rules will grow their bankrolls at different per-period geometric rates $\rho^*(c)$. The aggregate wealth of all the experts (read, the trader's wealth) will come to be dominated by $c^*(x^T)$.

For example, suppose that $F(\cdot)$ distributes the initial dollar among a finite number of rebalancing rules, call them $c^1, ..., c^r, ..., c^R$. Suppose that Mr. r grows his money at the rate ρ_r, so that the growth factor he achieves after t periods is $e^{\rho_r t}$. Let $f(c_r) > 0$ be the initial amount of money given to Mr. r to manage. Let r^* be the index of the rule with the highest geometric growth rate. Then the aggregate wealth after t periods is

$$\sum_{r=1}^{R} f(c_r) e^{\rho_r t} = e^{\rho_{r^*} t} \left[f(c_{r^*}) - \sum_{r \neq r^*} f(c_r) e^{(\rho_r - \rho_{r^*})t} \right], \tag{1.28}$$

which grows at an asymptotic rate of ρ_{r^*}. Notice that this holds good for any distribution of money at all, so long as each rebalancing rule r gets a positive amount of money.

We see that there is no need (at least theoretically) to be satisfied with distributing the initial dollar among a finite number of rebalancing rules. After t periods, the rules in the vicinity of c have

$$\langle c, x_1 \rangle \langle c, x_2 \rangle \cdots \langle c, x_t \rangle f(c) dc_1 \cdots dc_{m-1} \tag{1.29}$$

dollars, so that the trader's total wealth after t plays is

$$\int_{c \in \mathcal{C}} \langle c, x_1 \rangle \langle c, x_2 \rangle \cdots \langle c, x_t \rangle f(c) dc_1 \cdots dc_{m-1} \tag{1.30}$$

Of course, we cannot in reality keep tabs on a continuum of rebalancing rules: we must do the accounting and calculate the implied portfolio vectors $\theta(x_1, ..., x_t)$. What proportion of wealth with the trader (in aggregate) have in stock k in trading session

$t+1$? Since Mr. c keeps the fixed fraction c_k of his wealth in stock k at all times, evidently the trader has, in total

$$\int_{c\in\mathcal{C}} c_k\langle c,x_1\rangle\langle c,x_2\rangle\cdots\langle c,x_t\rangle f(c)dc_1\cdots dc_{m-1} \tag{1.31}$$

dollars invested in stock k. At last, we have the following compact formula for the trader's portfolio vector in period $t+1$:

$$\theta(x^t)=\frac{\int_{c\in\mathcal{C}} c\langle c,x_1\rangle\langle c,x_2\rangle\cdots\langle c,x_t\rangle f(c)dc_1\cdots dc_{m-1}}{\int_{c\in\mathcal{C}}\langle c,x_1\rangle\langle c,x_2\rangle\cdots\langle c,x_t\rangle f(c)dc_1\cdots dc_{m-1}}. \tag{1.32}$$

The trader's initial portfolio (when there is no return data as yet) is $\int_{c\in\mathcal{C}} cf(c)dc_1\cdots dc_{m-1}$, e.g. the the mean vector of the distribution $F(\cdot)$. Note that the volume of $\mathcal{C}$ is $\frac{1}{(m-1)!}$, so that the uniform density over $\mathcal{C}$ is $f(c)=(m-1)!$. This constant matters for direct calculation of the trader's wealth after t plays, but is irrelevant to the formula for $\theta(x^t)$ as it cancels from the numerator and denominator. Under a uniform prior, the initial portfolio is the centroid $(1/m,...,1/m)$.

Notice that (by direct inspection) the final wealth is linear in each period's return vector x_t, and obviously permuting the order of $x_1,...,x_T$ does not affect the final wealth. Thus, every prior distribution $F(\cdot)$ over $\mathcal{C}$ leads to a symmetric multilinear trading strategy. These strategies have the elegant feature that their behavior depends only on the return data $x_1,...,x_t$ as it is known after t plays, without depending in any specific way on a specific investment horizon T. More explicitly, $W(x^T)$ is equal to

$$\sum_{n_1+\cdots+n_m=T}\left(\int_{c\in\mathcal{C}} c_1^{n_1}\cdots c_{m-1}^{n_{m-1}}(1-c_1-...-c_{m-1})^{n_m}f(c)dc_1\cdots dc_{m-1}\right)\sigma(n_1,...,n_m;x^T) \tag{1.33}$$

Recall that monomial functions over $\mathcal{C}$ can be integrated exactly: the number

$$c_1^{\alpha_1-1}\cdots c_{m-1}^{\alpha_{m-1}-1}\left(1-\sum_{j=1}^{m-1}c_j\right)^{\alpha_m-1} \tag{1.34}$$

is the (unnormalized) multivariate Beta (or Dirichlet) density with parameter vector $\alpha=(\alpha_1,...,\alpha_m)$, and its integral is the multivariate Beta function,

$$B(\alpha)=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\cdots+\alpha_m)}. \tag{1.35}$$

In the above formula for $W(x^T)$ we have $\alpha_j = n_j + 1$, so that the coefficient of $\sigma(n_1, ..., n_m; x^T)$ is $(m-1)!\frac{n_1!\cdots n_m!}{(m+T-1)!}$, which is indeed equal to $\left\{\binom{m+T-1}{m-1}\binom{T}{n_1,...,n_m}\right\}^{-1}$, as promised above.

As usual, let $N = (N_1, ..., N_m)$ be the type of j^t. Similarly, let $n = (n_1, ..., n_m)$ be the type of $(j_{t+2}, ..., j_T)$. Then, making use of the notation established above, we have, for the uniform prior over $\mathcal{C}$,

$$\alpha_{tk}(N) = \frac{\sum_{N_1+\cdots+N_m=T-t-1} \binom{T-t-1}{N_1,...,N_m}(n_1+N_1)!\cdots\ (n_k+N_k+1)!\ \cdots\ (n_m+N_m)!}{\binom{T+m-1}{m-1}T!} \tag{1.36}$$

1.6 Multilinear superhedging

Examples 1-4 and Example 8 involved derivatives $D(x_1, ..., x_T)$ that, while not multilinear, were *multiconvex*, that is, convex separately as a function of each period's gross-return vector x_t. For instance, $D(x_1, ..., x_T) = ||x_1|| \cdots ||x_t||$ has this property for any choice of norm. In general, when exact replication is not possible, we can turn to super- and sub-hedging:

Definition 4. *A super-hedge (or super-replicating strategy) for $D(\cdot)$ is a self-financing trading strategy $\theta(\cdot)$ and an initial deposit p, such that $p \cdot W_\theta(x_1, ..., x_T) \geq D(x_1, ..., x_T)$ for all $x_1, ..., x_T$. When the sense of the inequality is reversed, then (p, θ) is called a sub-hedge for D.*

In words, a super-hedge is a trading strategy that generates cash flows greater than or equal to the derivative in any outcome. If (p, θ) is a super-hedge, then the price of the derivative at h^0 cannot exceed p: for otherwise, somebody could short the derivative for, say, $p + \epsilon$, pocket the ϵ, and then buy a super-hedge for p. This guarantees the trader arbitrage profits of at least ϵ.

Definition 5. *The superhedging cost (superhedging price) of a derivative at $t = 0$ is the infimum of all p such that (p, θ) is a super-hedge for some θ. If no superhedge exists, the superhedging price is therefore $+\infty$.*

If D is convex separately in each x_t, then, one derives the bound

$$D(x_1, ..., x_T) \leq \sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T}) x_{1j_1} \cdots x_{tj_t}. \tag{1.37}$$

This means that one can make an initial distribution of $D(e_{j_1}, ..., e_{j_T})$ dollars to Mr. j^T, and let it ride, and guarantee to have a final wealth at least as large as the derivative liability D. The inequality is sharp: it holds with equality on sample paths $x^T = (e_{j_1}, ..., e_{j_T})$. The bound is proved simply by expanding each $x_t = \sum_{j=1}^m x_{tj} e_j$ and writing $D(x_t, x_{-t}) \leq \sum_{j_t=1}^m D(e_{j_t}, x_{-t}) x_{tj_t}$. The general result follows by induction. Thus, we have already found a good super-hedge for D, one that requires an initial capital of $p = \sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$. Thus, the super-hedging price, being the infimum over all possible strategies, is no greater than this number. It is no less either: that is, if D is multiconvex, then a minimum-cost super-hedge can always be found among the multilinear trading strategies (generally super-hedges will not be unique).

Theorem 2. *For every derivative D, the superhedging cost of D is at least $\sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$. If D is convex separately in each of its vector arguments, then the bound holds with equality, and in fact there is a multilinear strategy (among others) that achieves the bound. This strategy amounts to exact replication of the unique multilinear form that interpolates D at the m^T points $(e_{j_1}, ..., e_{j_T}) \in \mathcal{R}^{Tm}$.*

Proof. Start by writing $p \cdot \langle \theta(x^{T-1}), x_T \rangle W_\theta(x^{T-1}) \geq D(x^{T-1}, x_T)$. Now substitute $x_T = e_j$, and sum these inequalities over j. On account of the fact that $\sum_{j=1}^m \theta_j(x^{T-1}) = 1$, we have

$$p \cdot W_\theta(x^{T-1}) \geq \sum_{j=1}^m D(x^{T-1}, e_j). \tag{1.38}$$

Again, we may substitute $x_{T-1} = e_k$, and then sum over k. Then $x_{T-2} = e_l$, sum over l, and so forth. After T iterations we are left with $p \geq \sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$. Now, if $D(\cdot)$ is multiconvex, then the multilinear strategy corresponding to the final wealth function $\sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T}) x_{1j_1} \cdots x_{Tj_T}$ is a superhedge that costs $\sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$, thereby achieving the bound. □

Theorem 3. *If D is any derivative at all, then its subhedging price at $t = 0$ is at most $\sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$. Thus, absent any prior knowledge of the joint distribution of $(x_1, ..., x_T)$, $\sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$ is a rational price for the derivative at $t = 0$.*

The proof that the subhedging price is at most $\sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$ is the same as that of Theorem 2, but with the inequalities reversed.

1.7 Specialization to Cover (1998)

We take up the derivative defined by

$$D(x_1, ..., x_T) = \underset{c \in \Delta}{Max} \ \langle c, x_1 \rangle \langle c, x_2 \rangle \cdots \langle c, x_T \rangle. \tag{1.39}$$

This was already mentioned above in the discussion of horizon-free universal portfolios. Not knowing what the maximizer $c^*(x_1, ..., x_T)$ will be in advance, one basically just dumps a bunch of money throughout the simplex so as to guarantee that c^* (wherever he may be) has some money to manage. In spite of its simplicity, this function is of extraordinary importance. The maximand amounts to the growth factor achieved by the rebalancing rule $\theta(x^t) \equiv c$. Looking back over the price history, $D(\cdot)$ represents the growth factor achieved by the most profitable rebalancing rule in hindsight. In passing, we note that D is indeed convex separately in each x_t, being that it is a pointwise max of linear functions.

A fortiori, the best rebalancing rule $c^*(x_1, ..., x_T)$ in hindsight achieves a growth factor greater than that of any individual stock. That is to say, the suboptimal choice $c = e_j$ in the above optimization problem would have yielded the growth factor $x_{1j} x_{2j} \cdots x_{Tj}$ achieved from buying and holding stock j. Put another way, the continuum of experts $c \in \Delta$ includes a Mr. c whose recommendation is to maintain 100% of wealth in stock j. The hindsight optimized rule $c^*(x_1, ..., x_T)$ therefore achieves more final wealth than even the best performing stock in the market. It thus beats every buy-and-hold portfolio, whose final growth factor is a convex combination of the growth factors of the m individual stocks. Of course, our class of primitive experts $c \in \Delta$ does not literally include experts that advocate buy and hold strategies, other than the m vertices e_j. However, any buy-and-hold strategy can be synthesized by making the appropriate initial distribution of wealth among the m experts $e_1, ..., e_m$, and letting it ride. It would thus be redundant to explicitly include the buy-and-hold strategies explicitly in some expanded set $\mathcal{E}$ of experts. Thus, the target rebalancing rule $c^*(x_1, ..., x_T)$ beats all the familiar price- and capitalization- weighted indices. By definition, it also beats the equal weight index $c = (1/m, ..., 1/m) \in \Delta$. Naturally, then, the corresponding super-hedging strategies θ are of great interest.

1.7.1 Interpretation of $c^*(x_1, ..., x_T)$ as an estimator of the Kelly rule

Assume for a moment that the x_t are drawn *iid* from the CDF $F : \mathcal{R}^m \to [0, 1]$. We can write $\langle c, x_t \rangle = e^{\rho(c)}$, where ρ is the rate of continuous compounding over the time interval $[t, t+1]$. The trader's per-period expected growth rate is then $E[log \langle c, x_t \rangle]$. Naturally, somebody who knew F might seek to maximize the expected growth rate:

$$c^*(F) = \underset{c \in \Delta}{argmax}\ E[log \langle c, x_t \rangle] \tag{1.40}$$

$c^*(F)$ is called the *Kelly rule*. The maximum possible expected growth rate is called the *Kelly growth rate*, which I denote $\rho^*(F)$. Somebody who uses the Kelly rule is called a *Kelly gambler*. The *realized* growth rate of the Kelly gambler is

$$\frac{\sum_{t=1}^{T} log \langle c^*(F), x_t \rangle}{T} \to^{a.s.} \rho^*(F) \tag{1.41}$$

by the law of large numbers. Suppose now that someone uses a non-Kelly rebalancing rule, say $c \notin c^*(F)$. For *iid* returns x_t distributed over the orthant according to an arbitrary CDF $F(\cdot)$, the per-period growth rate realized by c converges a.s. to the expected per-period growth rate, by the Law of Large Numbers. By the definition of c, this growth rate is less than $\rho^*(F)$. Thus, there is a T so large that for all $t \geq T$, the Kelly gambler has stricly more wealth than Mr. c. The ratio of Mr. c's wealth to that of the Kelly bettor will converge to 0. More or less, then, the main excuse for someone not using $c^*(F)$ is not knowing what F is. Notice how, by definition, $c^*(x_1, ..., x_T)$ achieves more wealth than $c^*(F)$ in any finite sample. That is to say, from the standpoint of rebalancing rules, it is slightly better to know the realized price paths than it is to know the distribution from which the returns are drawn.

$c^*(x_1, ..., x_t)$ is a natural estimator for $c^*(F)$. Given data $x_1, ..., x_t$, the sample analog of $E[log \langle c, x \rangle]$ is $\frac{\sum_{t=1}^{T} log \langle c, x_t \rangle}{T}$, and maximizing this quantity is equivalent to maximizing $\langle c, x_1 \rangle \cdots \langle c, x_t \rangle$. An immediate, natural idea is to use the trading strategy $\theta(x^t) = c^*(x^t)$. In the sequel, we will see that super-hedging $D(\cdot)$ is deeper still: even more than discovering what the Kelly rule is, the name of the game is to guarantee to compound one's money at the same asymptotic rate as the Kelly gambler. The strategy $c^*(x_1, ..., x_t)$ may perform badly if the *iid* assumption is violated (anybody who tries it on real return data can confirm this). Super-hedging D, on the other hand, is model-independent, and has strong pointwise optimality properties.

1.7.2 Superhedging cost

The superhedging cost can be evaluated exactly, on account of the fact that $D(e_{j_1}, ..., e_{j_T})$ turns out to have a closed form. If, as usual, $(n_1, ..., n_m)$ is the type of $(j_1, ..., j_T)$, then we have

$$D(n_1, ..., n_m) = \underset{c \in \Delta}{Max}\ c_1^{n_1} c_2^{n_2} \cdots c_m^{n_m} \tag{1.42}$$

This is a standard Cobb-Douglas optimization problem, the solution being $c_j^* = \frac{n_j}{T}$, and hence $D(n_1, ..., n_m) = (\frac{n_1}{T})^{n_1} \cdots (\frac{n_m}{T})^{n_m}$. The tacit convention here is that "$0^0 = 1$". Let us call the superhedging price $p(T, m)$. Then

$$p(T, m) = \sum_{n_1 + \cdots + n_m = T} \binom{T}{n_1\ n_2\ \cdots\ n_m} \left(\frac{n_1}{T}\right)^{n_1} \cdots \left(\frac{n_m}{T}\right)^{n_m} \tag{1.43}$$

Example 11. *For $m = 2$ assets, the superhedging cost is $p(T, 2) =$
$\sum_{j=0}^{T} \binom{T}{j} \left(\frac{j}{T}\right)^j \left(\frac{T-j}{T}\right)^{T-j} =$
$2 \sum_{j=0}^{\lceil \frac{T}{2} \rceil - 1} \binom{T}{j} \left(\frac{j}{T}\right)^j \left(\frac{T-j}{T}\right)^{T-j} + \boldsymbol{1}_{\{T\, is\, even\}} \binom{T}{\frac{T}{2}} 2^{-T}$.*

Taking logs of the defining inequality for super-hedging, we therefore have

$$\frac{log\, D(x_1, ..., x_T) - log\, W_\theta(x_1, ..., x_T)}{T} \leq \frac{log\, p(T, m)}{T}, \tag{1.44}$$

where θ is such that $(p(T, m), \theta)$ is a super-hedge for D. Let $\rho_\theta(x_1, ..., x_T)$ be the continuously compounded growth rate achieved by θ, and let $\rho^*(x_1, ..., x_T)$ be the growth rate achieved by the best rebalancing rule in hindsight. Then

$$\rho^*(x^T) - \rho_\theta(x^T) \leq \frac{log\, p(T, m)}{T} \tag{1.45}$$

holds for *all* $x_1, ..., x_T$.

Example 12. *Considered as a pairs trading strategy that rebalances annually, on a horizon of $T = 30$ years one can guarantee to achieve within 6.7% of the compound* *annual growth rate of the hindsight-optimized rebalancing rule for the relevant pair of stocks. This guarantee is the best possible.*

Example 13. *Over the 10-year period 5/1/2007 to 5/1/2017, Amazon (AMZN) grew at a continuously-compounded annual rate of 24%, and Netflix (NFLX) grew at 33.6%. The best annual rebalancing rule in hindsight was to put (16%,84%) into (AMZN, NFLX). The hindsight-optimized rule grew its wealth at a continuous annual rate of 38%. The pairs strategy was guaranteed a priori to achieve within 15.4% of this rate, e.g. it would have returned at least 22.6% per year, compounded continuously.*

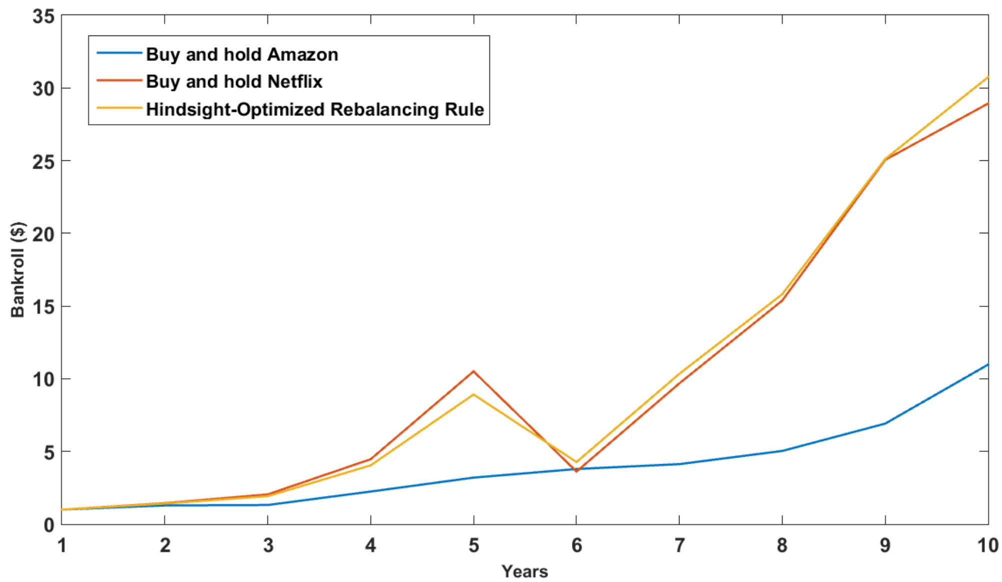

Figure 1.3: **The best rebalancing rule in hindsight: annual pairs trading (Amazon, Netflix) 2007-2017**

Cover's insight is that $\lim_{T\to\infty} p(T,m) = 0$. This is a beautiful result: it means that the super-replicating strategy θ compounds its money at the same asymptotic rate as the best rebalancing rule in hindsight. This convergence is *uniform*, on account of the fact that the above bound is independent of the return path $x_1, ..., x_T$. The actual process from which $\omega = (x_1, ..., x_T)$ is drawn is rendered completely irrelevant. Now, if it so happens that the x_t are indeed drawn *iid* from $F(\cdot)$, then on account of the fact that $\rho^*(F) \leq \rho^*(x_1, ..., x_T)$, the Kelly gambler's excess growth rate (over and above that of a trader who uses θ) converges uniformly to 0.

On account of the fixed horizon T, the term *uniform* demands explanation. To be quite correct, for any tolerance ϵ (say, 0.0001%, or whatever), there is a horizon T_ϵ so large that

$$\rho^*(x_1, ..., x_T) - \rho_\theta(x_1, ..., x_T) \leq \epsilon \tag{1.46}$$

for all $x_1, ..., x_T \in \mathcal{R}^m_+$.

It is edifying to express this result in Blackwell's approachability terminology. Consider the set of all (F, ρ) such that $\rho \geq E_F[log \langle c^*(F), x_t \rangle]$, where $c^*(F)$ is the Kelly rule corresponding to F. This set, called the *Bayes envelope of the stockmarket*, is a convex set of vector payoffs, on account of it being the epigraph of the convex function $F \mapsto E_F[log \langle c^*(F), x_t \rangle]$. In repeated play against the stock market, the trader can force the empirical average $(\overline{\rho}, \overline{F})$ to converge to the Bayes envelope, where $\overline{\rho}$ is the trader's realized per-period growth rate of capital, and $\overline{F}$ is the empirical CDF of the currently known return history. Whereas the Blackwell theory concentrates on minimizing the Euclidean distance to this envelope, the superhedging strategies seek to minimize the vertical slackness $E_F[log \langle c^*(F), x_t \rangle] - \rho$.

To give a simple proof of the fact that $p(T, m) \to 0$, we note first the inequality $\binom{T}{n_1\, n_2 \cdots n_m}(\frac{n_1}{T})^{n_1} \cdots (\frac{n_m}{T})^{n_m} \leq 1$. Granted this, we get that $p(T, m) \leq \sum_{n_1+\cdots+n_m=T} 1 = \binom{T+m-1}{m-1} = \mathcal{O}(T^{m-1})$, whence $\frac{log\, p(T,m)}{T} \to 0$. The bound $\mathcal{O}(T^{m-1})$ is not the best possible. In fact, $p(T, m) = \mathcal{O}(T^{\frac{m-1}{2}})$, e.g. for $m = 2$ stocks, the superhedging price grows like $\sqrt{T}$.

For a given number of stocks m, and a tolerance ϵ, a practitioner is advised to select the smallest horizon T_ϵ such that $\frac{log\, p(T,m)}{T} \leq \epsilon$. Of course, he will only have to wait T periods in a fictional worst case that amounts to wild market behavior, namely gross return vectors x_t that are unit basis vectors (or nearly so).

1.7.3 Efficient tabulation of $p(T, m)$

For the sake of exact solving the inequality $\frac{log\, p(T,m)}{T} \leq \epsilon$, I provide a simple recurrence for $p(T, m)$:

$$p(T, m) = 1 + \sum_{n=0}^{T-1} \binom{T}{n} \frac{n^n (T-n)^{T-n}}{T^T} p(T-n, m-1) \tag{1.47}$$

We have the boundary conditions $p(1, m) = m$ and $p(T, 1) = 1$. This recurrence allows one to rapidly tabulate $p(T, m)$. To avoid numerical overflow on the machine, the large powers and factorials should be calculated in log-space, and then exponentiated: $exp\Big(log\binom{T}{n} + n\,log\,n + (T-n)log\,(T-n) - T\,log\,T\Big)$. For large T and m this will start to require the calculation of an enormous number of logarithms. Instead, the logs $L_n = log\,n$ should be precomputed and stored for $1 \leq n \leq T$, along with the log-factorials $LF_n = log\,n + LF_{n-1}$. The numbers $log\binom{T}{n}$ are then calculated easily by $LF_T - LF_n - LF_{T-n}$. These procedures were used to generate the figure below:

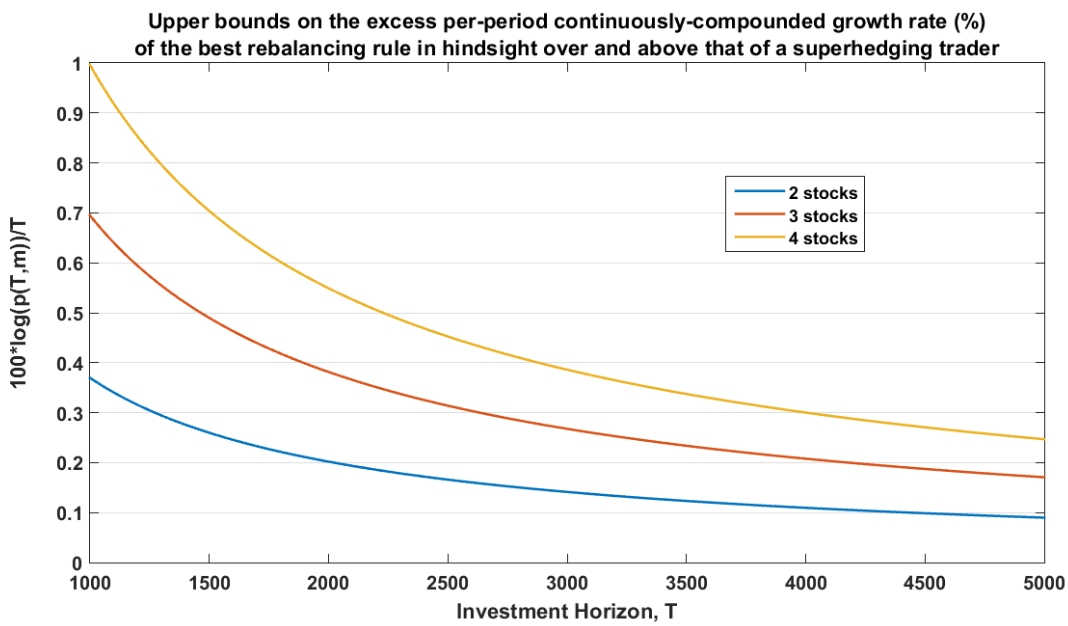

Figure 1.4: **Guaranteed upper bounds on the excess per-period growth rate of the best rebalancing rule in hindsight**

Naturally, we have the fact that $p(T, m) > p(T, m-1)$. Reason: $p(T, m) = \Sigma$(terms for which $n_m = 0$)+Σ(terms for which $n_m > 0$)= $p(T, m-1) + \Sigma$(terms for which $n_m > 0$). It is also true that $p(T, m) \leq p(T+1, m)$, e.g. the superhedging cost is increasing in the horizon. To prove this, we need the fact that $D(x_1, ..., x_t, \mathbf{1}) = D(x_1, ..., x_t)$, where $\mathbf{1}$ is a vector of ones. We will also require the fact that D is *sub-additive separately in each vector argument*, e.g. $D(x_t + y_t, x_{-t}) \leq D(x_t, x_{-t}) + D(y_t, x_{-t})$. This is true, because D is a pointwise max of additive functions. With these two facts in hand, we get $p(T,m) = \sum_{j_1,\dots,j_T} D(e_{j_1},...,e_{j_T},\mathbf{1}) \leq \sum_{j_1,\dots,j_{T+1}} D(e_{j_1},...,e_{j_{T+1}}) = p(T+1,m)$, where we have decomposed $\mathbf{1} = e_1 + \cdots + e_m$ and invoked the subadditivity.

For extremely large values of T and m (or for very small ϵ), direct calculation of $p(T,m)$ becomes unwieldy, even with the aid of the foregoing recurrences. Fortunately, we can use Shtarkov's (1987) bound, which is both very accurate and simple to calculate:

$$p(T,m) \leq \sqrt{\pi} \sum_{j=1}^{m} \binom{m}{j} \frac{(\frac{T}{2})^{\frac{j-1}{2}}}{\Gamma(\frac{j}{2})} = \sum_{j=1}^{m} a_j T^{\frac{j-1}{2}} = \mathcal{O}(T^{\frac{m-1}{2}}), \qquad (1.48)$$

where

$$a_j = \frac{\sqrt{\pi}\binom{m}{j}}{\Gamma(\frac{j}{2}) \cdot 2^{\frac{j-1}{2}}} \qquad (1.49)$$

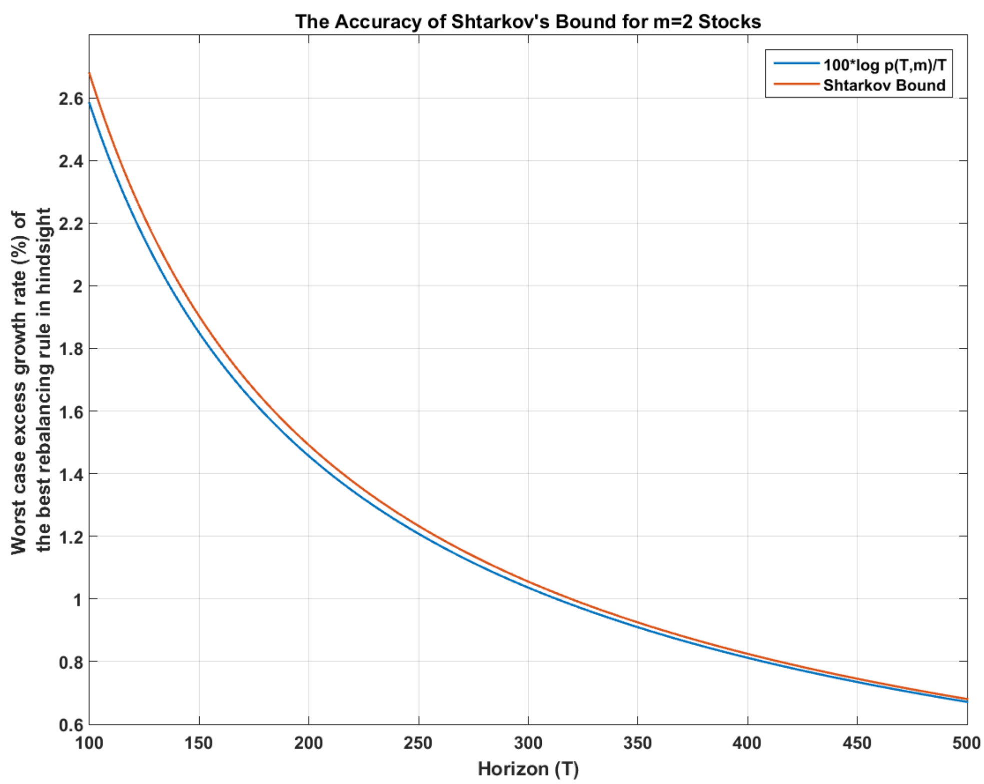

Figure 1.5: **The accuracy of Shtarkov's bound for $m = 2$ stocks**

For example, with $m = 2$ stocks, a superhedging strategy with horizon $T = 350$ periods will guarantee to compound its money at a per-period rate that is within 0.9% of the rate achieved by the best rebalancing rule in hindsight. As far as a practical method for obtaining fast, accurate solutions of the inequality $\frac{log\, p(T,m)}{T} < \epsilon$, one merely solves the following fixed-point iteration in one variable:

$$T = g(T) = \frac{1}{\epsilon}\; log\left(\sum_{j=1}^{m} a_j T^{\frac{j-1}{2}}\right) \tag{1.50}$$

For large T, since $log\, p(T,m)$ grows at an asymptotically negligible rate, we will need to roughly double the horizon in order to cut ϵ in half. It is thus tempting to think that one can "cheat" the situation by trading at a far higher frequency. But, realize that getting within 1% of the compound annual growth rate of the best rebalancing rule in hindsight under *annual trading* ($T = 320$ years if there are 2 stocks) is far different from getting within 1% of the hindsight-optimized growth rate under, say, *daily trading* ($T = 320$ days). No, the proper equivalence is that, under daily trading, one must get within $\frac{1}{252}$% of the daily growth rate (there are 252 trading days in a year). Daily trading multiplies the horizon by 252, it's true — but it also divides the required epsilon by 252. Intuitively, it should be *harder* to get within $\frac{1}{252}$% of the optimal daily growth rate, on account of the fact that the hindsight-optimized rebalancing rule will perform better than the hindsight-optimized rule under annual trading. In fact, for 2 stocks, this will require 156,500 days, or 621 years. Now, a given trading day (8:30 am to 3 pm central time) lasts 6.5 hours, making for 13 half-hour sessions. Assuming we make a trade every 30 minutes, we must get within $\frac{1}{252 \cdot 30}$% of the hindsight-optimized per-session growth rate. This will require nearly 2.5 million half-hour periods, or 759 years. In trading at a higher frequency, the gambler is accepting weaker guarantees about the veracity of the growth-rate approximation, in exchange for approximating something that hopefully has a higher CAGR. Of course, figures like 621 years represent fictional worst cases that (in all probability) will not actually come to pass.

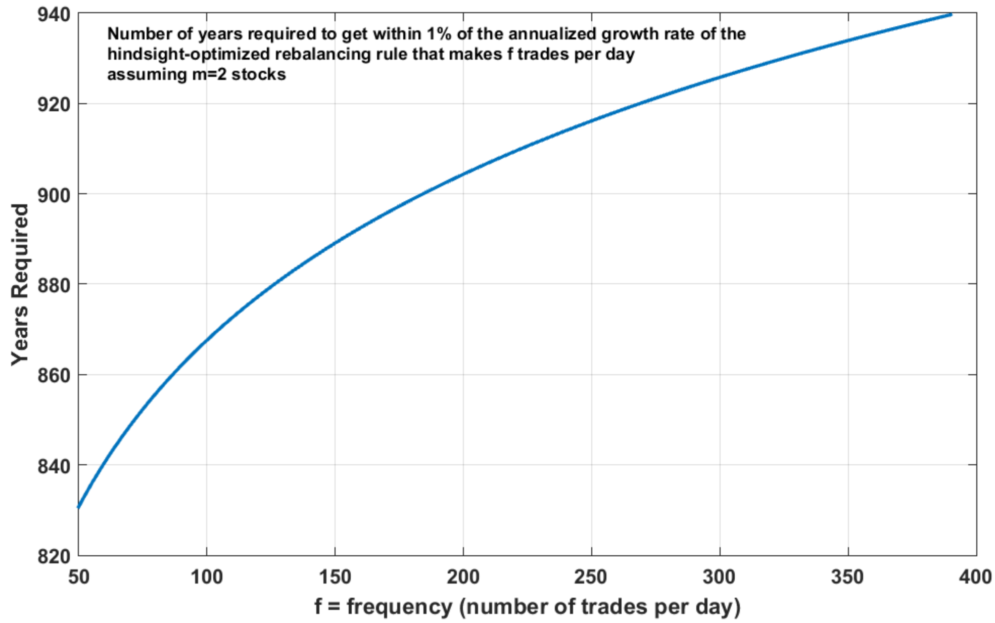

Figure 1.6: **Longest waiting time needed to get within 1% of the annualized growth rate of the hindsight-optimized rebalancing rule that makes f trades per day**

Does the number of years required in this chart settle upon a finite limit as $f \to \infty$? The answer is no.

Theorem 4. *Let f be the frequency (in trades per year) with respect to which we perform the hindsight optimization over rebalancing rules. Let ϵ be the desired annual tolerance, e.g. if $\epsilon = 0.01$, we want to get within 1% of the annualized growth rate of the best rebalancing rule in hindsight that makes trades at a rate of f per year. Then the number of years of trading (at a rate of f trades per year) required to make this guarantee is $\frac{1}{f}\, Min \left\{ T \in \mathbb{N} : \frac{log\, p(T,m)}{T} \le \frac{\epsilon}{f} \right\}$, which tends to ∞ as $f \to \infty$.*

To prove this, I bound the number $Min \left\{ T \in \mathbb{N} : \frac{log\, p(T,m)}{T} \le \frac{\epsilon}{f} \right\}$ from below, as follows. On account of its asymptotic expansion, there is a constant A such that $P(T,m) \ge A \cdot T^{\frac{m-1}{2}}$, so that $\frac{log\, p(T,m)}{T} \ge \frac{log\, A + \frac{m-1}{2} log\, T}{T}$. Thus, $\left\{ T \in \mathbb{N} : \frac{log\, p(T,m)}{T} \le \frac{\epsilon}{f} \right\} \subset \left\{ T \ge 1 : \frac{log\, A + \frac{m-1}{2} log\, T}{T} \le \frac{\epsilon}{f} \right\}$. Let $T^*(f)$ denote the min of this latter set, so that

$$\frac{1}{f}\,Min\left\{T \in \mathbb{N} : \frac{log\,p(T,m)}{T} \leq \frac{\epsilon}{f}\right\} \geq \frac{T^*(f)}{f}, \tag{1.51}$$

where $\frac{log\,A+\frac{m-1}{2}log\,T^*(f)}{T^*(f)} = \frac{\epsilon}{f}$. Thus, $\frac{T^*(f)}{f} = \frac{log\,A+\frac{m-1}{2}log\,T^*(f)}{\epsilon}$, which tends to ∞ as $f \to \infty$, since $T^*(f) \to \infty$. This gives the desired result.

1.7.4 Beating the market asymptotically

A trading strategy θ that achieves the superhedging price can be said to "beat the market asymptotically," in the following sense. Let us assume that the return path $x^{T_\epsilon} = (x_1, ..., x_{T_\epsilon})$ is such that the best rebalancing rule in hindsight is able to grow its money at a compound rate that is at least ϵ greater than a given index, such as the S&P 500. Then, since θ achieves a growth rate within ϵ of the best rebalancing rule, θ also beats the index. Let $\mathcal{S}(x_1, ..., x_T)$ denote the growth factor achieved by, say, the S&P 500 index. Then

$$\frac{log\,W_\theta - log\,\mathcal{S}}{T} = \frac{log\,W_\theta - log\,D + log\,D - log\,\mathcal{S}}{T} > -\epsilon + \epsilon = 0 \tag{1.52}$$

Thus, under this restriction on the return path, we have $W_\theta(x_1, ..., x_T) > \mathcal{S}(x_1, ..., x_T)$. This seems to be a fairly reasonable assumption: witness the extent to which Amazon or Netflix have crushed the S&P in the past decade. And D is guaranteed to outperform every individual stock in the index. On the paths x^T where W_θ underperforms $\mathcal{S}(\cdot)$, the worst that can happen is that θ compounds its money at a rate ϵ lower than the index.

Of course, it is not being claimed here that the S&P 500 (or any index) has been dominated vis-a-vis final wealth. Any strategy θ that aspires to beat some benchmark must of necessity expose itself to underperformance on certain paths x^T, so that it may win on the others.

Chapter 2

Sequential minimax trading strategies

2.1 Introduction

The foregoing chapter considered self-financing trading strategies θ, together with an initial deposit p, that could guarantee (at $t = 0$) to generate cash flows at T greater than or equal to some derivative $D(x_1, ..., x_T)$, regardless of the outcome $(x_1, ..., x_T)$. However, it is frequently the case that, on some partial history $x^t = (x_1, ..., x_t)$, it becomes apparent that less money than p was needed to construct a super-hedge. Equivalently, after observing x^t, the trader may be able to dynamically guarantee that $p \cdot W_\theta(x^t, x_{t+1}, ..., x_T) \geq D(x_1, ..., x_T) + \epsilon$ for some $\epsilon > 0$. This is on account of the fact that payoff of the general derivative $D(\cdot)$ is path-dependent.

Let $p(x_1, ..., x_t)$ denote the cost of the cheapest super-hedge, conditional on having observed the partial history x^t. The interpretation is that someone shorts the derivative after x^t, collects some amount of money that is at least as large as the conditional super-hedging cost, and uses some of the proceeds to buy a conditional super-hedge. Any left-over money gets immediately pocketed as an arbitrage profit. In the foregoing chapter we showed that if (p, θ) is a super hedge at $t = 0$, then $p = p(h^0) \geq \sum_{j_1,...,j_T} D(e_{j_1}, ..., e_{j_T})$. The proof of Theorem 2 actually showed more: if $(p(x^t), \theta)$ is a conditional super hedge, then

$$p(x^t) \geq \underset{j_{t+1},\ldots,j_T}{\Sigma} D(x_1, ..., x_t, e_{j_{t+1}}, ..., e_{j_T}) \tag{2.1}$$

In the foregoing chapter, I showed that if $D(x_t, x_{-t})$ is convex separately as a function of each x_t, then there is always a (multilinear) super-hedge that achieves this bound. This same logic already applies to the general situation.

After partial history x^t, we consider the derivative $\delta(x_{t+1}, ..., x_T) = D(x_1, ..., x_t, x_{t+1}, ..., x_T)$, which is just the restriction of $D(\cdot)$ to the set of all return paths descended from x^t. Then, as before, we simply use the multilinear bound

$$\delta(x_{t+1}, ..., x_T) \leq \underset{j_{t+1},\cdots,j_T}{\Sigma} \delta(e_{j_{t+1}}, ..., e_{j_T}) x_{t+1,j_{t+1}} \cdots x_{T,j_T} \tag{2.2}$$

Based on the goings-on in Chapter 1, this already proves that the conditional super-hedging price is $\underset{j_{t+1},\cdots,j_T}{\Sigma} D(x_1, ..., x_t, e_{j_{t+1}}, ..., e_{j_T})$. The original superhedging strategy at, say, $t = 0$ is not credible in all subgames x^t: when we reach x^t, we find occasion to use a strategy that achieves the conditional superhedging price instead. In the language of distributing money among extremal strategies, one makes an initial distribution of wealth among the extremals j^T, fully expecting to just "let it ride" so long as nature plays all unit basis vectors. If x_t was not a unit basis vector, then the trader starts afresh. In the continuation game after x^t, the extremal strategies now amount to all possible sequences $(j_{t+1}, ..., j_T) \in \{1, ..., m\}^{T-t}$. Given nature's sub-optimal play so far, an entirely new distribution of current wealth $\alpha(j_{t+1}, ..., j_T)$ among the extremals is called for. Moving forward, the trader will "let it ride" so long as nature gets back on the wagon and starts to play unit basis vectors. If x_{t+1} is not a unit basis vector, then this process repeats itself.

2.2 Sequential minimax interpretation

To fully pursue this logic, we consider a full-blown sequential trading game against nature. The trader is player 1, and he starts with \$1. At the start of each trading session t, nature (player 2) waits for the trader to make his "move" $\theta_t = \theta(x_1, ..., x_{t-1})$. Nature then responds by picking the gross return vector $x_t \in \mathcal{R}_+^m$. The situation repeats itself in periods $t+1, t+2, ..., T$, at which point the game is terminated. The position of the game after t periods is denoted $h^t = (\theta_1, x_1, \theta_2, x_2, ..., \theta_t, x_t)$, with transition law

$h^{t+1} = (h^t, \theta_{t+1}, x_{t+1})$. The nodes where nature is on the move are denoted (h^t, θ_{t+1}). Final nodes are denoted h^T, and the root of the game tree is called h^0. The trader starts with \$1, and his wealth after t periods is denoted $W(h^t) = \langle\theta_1, x_1\rangle\langle\theta_2, x_2\rangle \cdots \langle\theta_t, x_t\rangle$, with transition law $W(h^{t+1}) = W(h^t)\langle\theta_{t+1}, x_{t+1}\rangle$. The trader's final payoff at h^T is given by

$$\Pi(h^T) = \frac{W(h^T)}{D(x_1, ..., x_T)}. \tag{2.3}$$

Nature's payoff is $-\Pi(h^T)$. $\Pi(h^T)$ is called the "competitive ratio." The interpretation is that the trader seeks, in the worst case, to guarantee that his \$1 investment hedges the greatest possible proportion of the derivative payoff at a final node.

Let $V(h^t)$ denote the greatest possible final payoff that the trader can guarantee at final nodes $h^T \supset h^t$. At the same time, this number will represent the lowest possible final payoff that nature can guarantee, conditional on having reached h^t. Similarly for the values $V(h^t, \theta_{t+1})$ where nature is on the move. At final nodes, the value of the game is just $V(h^T) = \Pi(h^T)$. We have the recurrences

$$V(h^t) = \underset{\theta_{t+1}\in\Delta}{Max}\ V(h^t, \theta_{t+1}) \tag{2.4}$$

$$V(h^t, \theta_{t+1}) = \underset{x_{t+1}\in\mathcal{R}^m_+-\{0\}}{Min}\ V(h^t, \theta_{t+1}, x_{t+1}) \tag{2.5}$$

We will assume that $D(x_1, ..., x_T)$ is positively homogeneous of degree 1 separately in each argument x_t, meaning that $D(\lambda x_t, x_{-t}) = \lambda D(x_t, x_{-t})$ for $\lambda \geq 0$. As a consequence of this assumption, the final payoff $\Pi(h^T)$ is homogeneous of degree 0 separately in each return vector x_t. On account of this fact, we can assume without loss of generality that $||x_t||_1 = 1$ for all t, e.g. that $x_t \in \Delta$.

The compact action sets, together with Berge's max theorem, guarantee the existence of the value function, and of the subgame-perfect equilibrium strategies $\theta^*_{t+1}(h^t)$ and $x^*_{t+1}(h^t, \theta_{t+1})$. By its very definition, $\theta^*_{t+1}(h^t)$ is the sharpest possible superhedging strategy for $D(\cdot)$, and its behavior is credible in all subgames. If we assume that $D = D(x_t, x_{-t})$ is convex and positively homogeneous as a function of each x_t separately, then it becomes possible to directly calculate the value function and the sequential minimax strategies by backward induction. I illustrate the first few steps below.

At penultimate nodes (h^{T-1}, θ_T), we have

$$V(h^{T-1}, \theta_T) = W(h^{T-1}) \underset{x_T \in \Delta}{Min} \frac{\langle \theta_T, x_T \rangle}{D(x_1, ..., x_T)}. \tag{2.6}$$

Now, when viewed as a function of x_T alone, the minimand is quasi-concave, it being the ratio of a positive linear function to a positive convex function. Accordingly, its minimum must be achieved at one of the extreme points $e_1, ..., e_m$ of the simplex, namely, the unit basis vectors. Thus, we get

$$V(h^{T-1}) = W(h^{T-1}) \underset{\theta_T \in \Delta}{Max} \underset{1 \le j \le m}{Min} \frac{\theta_{Tj}}{D(x_1, ..., x_{T-1}, e_j)} \tag{2.7}$$

We now have to maximize a piecewise linear function over the simplex, and the unique solution is characterized by the fact that it equalizes all the numbers $\frac{\theta_{Tj}}{D(x_1,...,x_{T-1},e_j)}$, for $1 \le j \le m$. Say, that $\theta_{Tj} = B \cdot D(x_1, ..., x_{T-1}, e_j)$ for all j. Then, summing over j, we get

$$V(h^{T-1}) = \frac{W(h^{T-1})}{\sum_{j=1}^{m} D(x_1, ..., x_{T-1}, e_j)}. \tag{2.8}$$

Notice that $V(h^{T-1})$ is itself quasi-concave when viewed as a function of x_{T-1} alone. The same calculation repeats itself, and one gets

$$V(h^{T-2}) = \frac{W(h^{T-2})}{\sum_{k=1}^{m} \sum_{j=1}^{m} D(x_1, ..., x_{T-2}, e_k, e_j)}. \tag{2.9}$$

At each step we pick up an index of summation; in general, the denominator of $V(h^t)$ will feature a sum with $T - t$ indices of summation.

Theorem 5. *For multiconvex derivatives $D(\cdot)$, the general formula for the value of the hedging game is*

$$V(h^t) = \frac{W(h^t)}{\sum\limits_{(j_{t+1},...,j_T) \in \{1,...,m\}^{T-t}} D(x_1, ..., x_t, e_{j_{t+1}}, ..., e_{j_T})}. \tag{2.10}$$

We immediately recognize the denominator as the conditional superhedging cost $p(x^t)$, so that $V(h^t) = \frac{W(h^t)}{p(x^t)}$ is the number of conditional superhedges the trader can afford, given his current wealth $W(h^t)$.

Theorem 6. *The SPNE trading strategy* $\theta^*_{t+1}(h^t)$ *is characterized by the fact that it equalizes the continuation values* $V(h^t, \theta^*_{t+1}(h^t), e_j)$ *for* $1 \leq j \leq m$*:*

$$\theta^*_{t+1,k}(h^t) = \frac{\sum\limits_{(j_{t+2},...,j_T)\in\{1,...,m\}^{T-t-1}} D(x^t, e_k, e_{j_{t+2}}, ..., e_{j_T})}{\sum\limits_{(j_{t+1},...,j_T)\in\{1,...,m\}^{T-t}} D(x^t, e_{j_{t+1}}, ..., e_{j_T})}. \tag{2.11}$$

This formula gives the proportion of wealth to put into stock k as a function of the observed return data x^t. Notice how it is homogeneous of degree 0 separately in the variables $x_1, ..., x_t$. If D happens to be a *symmetric* derivative in the sense that $D(x_1, ..., x_T) = D(x_{\sigma(1)}, ..., x_{\sigma(T)})$ for any permutation σ of the indices $\{1, ..., T\}$, then the above portfolio weights will depend only on the numerical values of $x_1, ..., x_t$ and not their order.

Theorem 7. *Nature's SPNE gross-return policy is given by selecting a unit basis vector* e_{k^*} *that minimizes the continuation value:* $x^*_{t+1}(h^t, \theta_{t+1}) = e_{k^*(h^t,\theta_{t+1})}$*, where*

$$k^*(h^t, \theta_{t+1}) = \underset{1\leq k\leq m}{argmin} \frac{\theta_{t+1,k}}{\sum\limits_{j_{t+1},...,j_T} D(x_1, ..., x_t, e_k, e_{j_{t+2}}, ..., e_{j_T})}. \tag{2.12}$$

If $\theta_{t+1} = \theta^*(h^t)$ *is the correct (SPNE) portfolio vector given* h^t*, then* $k^*(h^t, \theta_{t+1}) = \{1, ..., m\}$*.*

2.3 Illustration

I illustrate the difference between the sequential minimax and multilinear universal portfolios for the case of $T = 2$ investment periods and $m = 2$ stocks. Both strategies use the same initial portfolio vector $(1/2, 1/2)$ in trading session 1. The superhedging cost is

$\sum_{j=0}^{2} \binom{2}{j} \left(\frac{j}{2}\right)^j \left(\frac{2-j}{2}\right)^{2-j} = 1 + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 = \frac{5}{2}$. The final wealth of a \$1 deposit into the multilinear superhedging strategy is

$$W_\theta\Big((x_{11}, x_{12}), (x_{21}, x_{22})\Big) = \frac{2}{5}(x_{11}x_{21} + \frac{1}{4}x_{11}x_{22} + \frac{1}{4}x_{12}x_{21} + x_{12}x_{22}) \tag{2.13}$$

According to the formulas derived in the multilinear chapter, we have

$$\theta_1(x_1) = \frac{W(x_1, e_1)}{W(x_1, e_1) + W(x_2, e_2)} = \frac{\frac{2}{5}x_{11} + \frac{1}{10}x_{12}}{\frac{1}{2}x_{11} + \frac{1}{2}x_{12}} = \frac{4}{5}\left(\frac{x_{11}}{x_{11} + x_{12}}\right) + \frac{1}{5}\left(\frac{x_{12}}{x_{11} + x_{12}}\right) \tag{2.14}$$

$$\theta_2(x_1) = 1 - \theta_1(x_1) = \frac{1}{5}\left(\frac{x_{11}}{x_{11}+x_{12}}\right) + \frac{4}{5}\left(\frac{x_{12}}{x_{11}+x_{12}}\right) \tag{2.15}$$

Now assume that stock 1 outperforms stock 2 in period 1, say, by a wide margin. As the relative performance $\frac{x_{11}}{x_{12}}$ becomes large, $\theta(x_1)$ approaches the vector $(\frac{4}{5}, \frac{1}{5})$. Thus, no matter how well stock 1 performs in period 1, the multilinear universal portfolio will bet no more than 4/5 of its wealth on stock 1 in period 2.

Consider a Shannon's-Demon type situation whereby $x_1 = (2, \frac{1}{2})$, so that stock 1 has doubled in price and stock 2 has been cut in half. Both strategies now have wealth $W(h^1) = \frac{5}{4}$. Then the multilinear strategy plays the vector $(\frac{16}{25}, \frac{9}{25}) = (0.64, 0.36)$ in trading session 2. By contrast, the subgame-perfect strategy uses the weights

$$\theta_1(x_1) = \frac{\underset{0\le\lambda\le1}{Max}\,(\lambda x_{11} + (1-\lambda)x_{12})\lambda}{\underset{0\le\lambda\le1}{Max}\,(\lambda x_{11} + (1-\lambda)x_{12})\lambda + \underset{0\le\lambda\le1}{Max}\,(\lambda x_{11} + (1-\lambda)x_{12})(1-\lambda)} \tag{2.16}$$

$$\theta_2(x_1) = \frac{\underset{0\le\lambda\le1}{Max}\,(\lambda x_{11} + (1-\lambda)x_{12})(1-\lambda)}{\underset{0\le\lambda\le1}{Max}\,(\lambda x_{11} + (1-\lambda)x_{12})\lambda + \underset{0\le\lambda\le1}{Max}\,(\lambda x_{11} + (1-\lambda)x_{12})(1-\lambda)} \tag{2.17}$$

Calculating these weights simply amounts to finding the vertices of two parabolas, using corner solutions where appropriate. Having observed $x_1 = (2, 1/2)$, the sequential minimax strategy uses the portfolio $(0.75, 0.25)$. Notice that at the beginning of the game, both strategies were guaranteed to achieve at least 40% of the final wealth of the best rebalancing rule in hindsight. Now, conditional on observing nature's play $x_1 = (2, 1/2)$, the sequential minimax strategy is able to guarantee to achieve the (greater) proportion

$$\frac{W(h^1)}{D(x_1, e_1) + D(x_1, e_2)} = \frac{5/4}{2 + 2/3} = 47\%. \tag{2.18}$$

Assume now that nature plays the same vector in session 2, e.g. $x_2 = x_1 = (2, 0.5)$. Then the final wealth of the best rebalancing rule in hindsight is \$4, the final wealth of the multilinear strategy is \$1.82, and the final wealth of the sequential minimax strategy is \$2.03. The multilinear strategy achieved 45.5% of the target final wealth, which is greater than the promised 40%. The sequential minimax strategy achieved 50.8% of the target wealth, which is greater than the initially promised (at $t = 0$) 40%, and the revised promise of 47% at $t = 1$. Of course, the point of the illustration was not to claim that the sequential minimax strategy will always make more money than the multilinear strategy. The point was that the sequential minimax strategy is always revising its promises upward whenever nature fails to play a unit basis vector. The revised promises are always the best possible.

Let us assume, then, that the shoe is on the other foot: $x_2 = (0.5, 2)$. Then the hindsight-optimized wealth is \$1.56. The sequential minimax trader's final wealth is \$1.09, or 70% of the hindsight-optimized wealth, which again is greater than the promised 40%.

2.4 Dynamic horizon adjustment

This example has illustrated the fact that the sequential minimax universal portfolio can dynamically announce ever lower final growth rate spreads. However, one can proceed differently. Rather than achieve a final growth-rate spread that is (significantly) better than the originally promised ϵ, the strategy can dynamically reduce its horizon, achieving the promise ϵ in a shorter amount of time than was originally thought.

Theorem 8. *Given current history h^t and tolerance ϵ for the excess per-period growth rate of the best rebalancing rule in hindsight, let $T_\epsilon(h^t)$ be the earliest date at which the trader can guarantee that the spread is $< \epsilon$. Then $T_\epsilon(h^t)$ is the smallest solution $T \in \mathbb{N}$ of the inequality*

$$\frac{log \sum_{j_{t+1},\dots,j_T} D(x_1, ..., x_t, e_{j_{t+1}}, ..., e_{j_T}) - log\, W(h^t)}{T} < \epsilon. \tag{2.19}$$

This generalizes the fact, noted earlier, that $T_\epsilon(h^0)$ is the smallest solution T of the inequality

$$\frac{log\, p(T, m)}{T} < \epsilon. \tag{2.20}$$

Thus, the most perspicacious possible trading strategy proceeds as follows. After t periods, the trader takes stock of his current wealth $W(h^t)$ and the observed return history $x_1, ..., x_t$. He updates his termination date to $T_\epsilon(h^t) \le T_\epsilon(h^{t-1})$. The trader then carries out a fresh backward induction from the new, smaller horizon, and uses the sequential minimax portfolio vector $\theta(x_1, ..., x_t)$.

The meaning of this trading strategy is best understood in relation to the terminology of announced mates in Chess. The analogy is pursued in the table below.

Chess	**Trading**
White	Player 1 (the trader)
Black	Player 2 (nature)
Ply	θ_t
Re-ply	x_t
Black is checkmated on the t^{th} move	$log\, D(x_1, ..., x_t) - log\, W(h^t) < \epsilon \cdot t$
Scoresheet after t moves	h^t
Board position after t moves	Payoff-relevant state $\xi_t = \left(x^t, W(h^t)\right)$
White announces "Mate in s" prior to the $t+1^{st}$ move	$\underset{j_{t+1},...,j_{t+s}}{\Sigma} D(x^t, e_{j_{t+1}}, ..., e_{j_{t+s}}) < W(h^t)e^{\epsilon(t+s)}$
Quickest "Mate in s" that can be announced	$s = T_\epsilon(h^t) - t$
Stiffest resistance by Black	Unit basis vectors
Principal variation	$h^{t+1} = \Big(h^t, \theta^*(h^t), e_{k^*(h^t,\theta^*(h^t)))}\Big)$

Table 2.1: **Analogy between announced mates in Chess and the sequential minimax trading strategy with dynamically adjusting horizon**

The performance of the strategy with dynamically adjusting horizon is illustrated below for monthly pairs trading of Amazon and Netflix shares. A tolerance of $\epsilon = 2\%$ was used. Note how the strategy starts to outperform the passive index as the mate gets nearer. A 46% reduction in horizon was achieved from the original declared time-to-mate of 136 months.

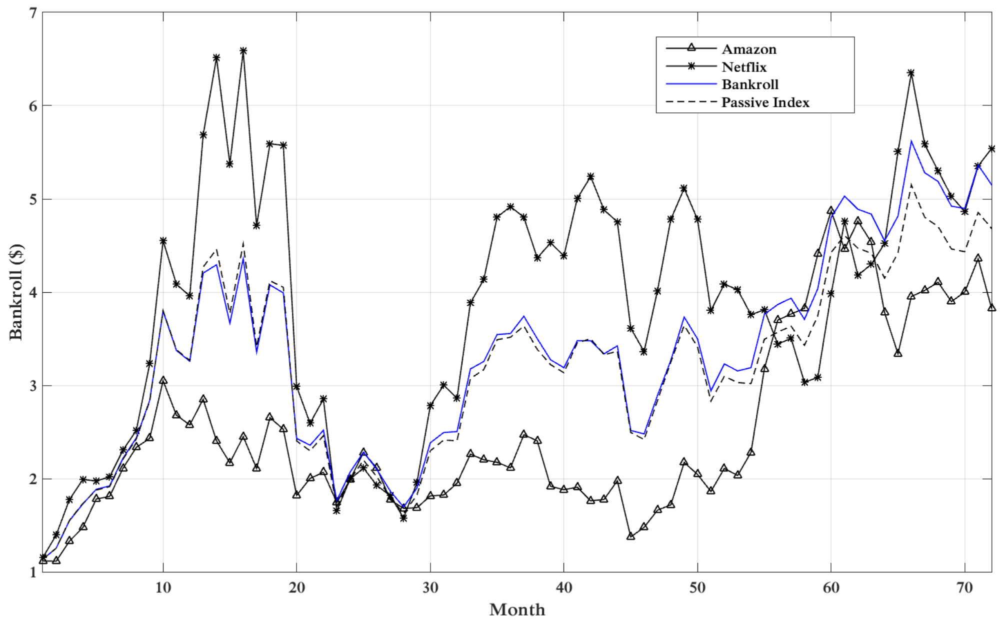

Figure 2.1: **Checkmate after 73 moves: monthly pairs trading for Amazon and Netflix with dynamically adjusting horizon and $\epsilon = 2\%$.**

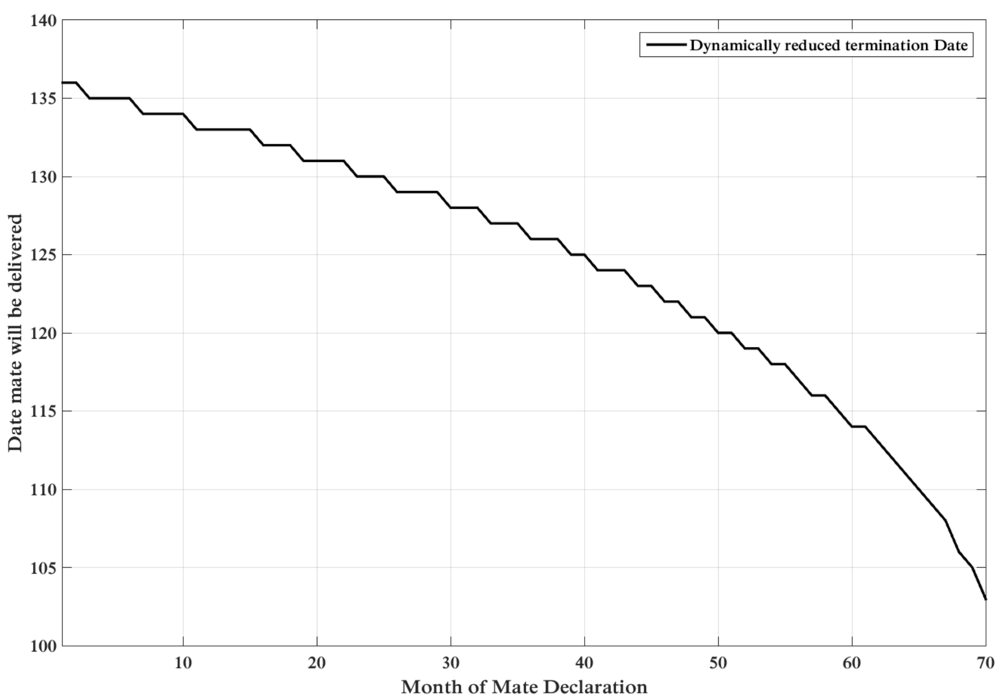

Figure 2.2: **Evolution of $T_\epsilon(h^t)$ in subgame perfect, horizon-adjusted monthly pairs trading**

2.5 Efficient calculation of the hindsight-optimized wealth

It is clear from the foregoing illustration that effective on-line computation of the sequential minimax universal strategy will hinge upon how efficiently one computes the hindsight-optimized wealth $D(x^T) = \underset{c\in\Delta}{Max}\ \langle c, x_1\rangle \cdots \langle c, x_T\rangle$. This amounts to a log-concave program over the simplex:

$$\underset{c\in\Delta}{Max}\ \phi(c) = \sum_{t=1}^{T} log\,\langle c, x_t\rangle. \tag{2.21}$$

The strategy calls for the summation of all possible values $D(x^t, e_k, e_{j_{t+1}}, ..., e_{j_T}) = D(x^t; n_1, ..., n_m)$, where $(n_1, ..., n_m)$ is the type of the sequence $(k, j_{t+1}, ..., j_T)$. The numerator of the portfolio weight on stock k after partial history x^t is then

$$\underset{n_1+\cdots+n_m=T-t-1}{\Sigma} \binom{T-t-1}{n_1\ \cdots\ n_m} D(x^t; n_1, ..., n_{k-1}, n_k+1, n_{k+1}, ..., n_m) \tag{2.22}$$

Notice that if $n_k > 0$, then of necessity $c_j^* > 0$, for otherwise the rebalancing rule c^* yields zero final wealth. Thus, the great majority of terms $(n_1, ..., n_m)$ in the sum will be correspond to interior solutions, regardless of the value of x^t. After eliminating Lagrange multipliers one has the first order conditions, for each i:

$$\frac{\partial\phi}{\partial c_i}(c) = \sum_{t=1}^{T} \frac{x_{ti}}{\langle c, x_t\rangle} = T. \tag{2.23}$$

A solution of these FOCs will automatically satisfy the condition $\sum_{i=1}^{m} c_i = 1$. Let $H(c)$ be the Hessian, e.g.

$$H_{ij}(c) = \frac{\partial^2\phi}{\partial c_i \partial c_j}(c) = -\sum_{t=1}^{T} \frac{x_{ti}x_{tj}}{\langle c, x_t\rangle^2} \tag{2.24}$$

If $g(c) = \left(\frac{\partial\phi}{\partial c_1}(c), ..., \frac{\partial\phi}{\partial c_m}(c)\right)'$ denotes the gradient, then the updates in Newton's method are given by $c \leftarrow c + H^{-1}(c)(T\cdot\mathbf{1} - g(c))$. Here $\mathbf{1} = (1, ..., 1)'$. The convergence to c^* will be very rapid, especially on account of the fact that the solver will be provided with analytic formulas for the gradient and Hessian.

Example 14. *For $m = 2$ stocks, denote $c = (c_1, c_2)$ by $(\lambda, 1-\lambda)$. After observing partial history x^t, and assuming a continuation path $(e_{j_{t+1}}, ..., e_{j_T})$ whereby e_1 appears n_1 times and e_2 appears n_2 times, the hindsight-optimized wealth given by*

$$D(x^t; n_1, ...n_m) = \underset{0\leq\lambda\leq 1}{Max}\left\{\prod_{s=1}^{t}(\lambda x_{s1} + (1-\lambda)x_{s2})\right\}\lambda^{n_1}(1-\lambda)^{n_2} \tag{2.25}$$

is found by the Newton iteration

$$\lambda \leftarrow \lambda + \frac{\sum_{s=1}^{t}\frac{x_{s1}-x_{s2}}{x_{s2}+\lambda(x_{s1}-x_{s2})} + \frac{n_1}{\lambda} - \frac{n_2}{1-\lambda}}{\sum_{s=1}^{t}\left[\frac{x_{s1}-x_{s2}}{x_{s2}+\lambda(x_{s1}-x_{s2})}\right]^2 + \frac{n_1}{\lambda^2} + \frac{n_2}{(1-\lambda)^2}}. \tag{2.26}$$

The iterates converge to the (unconstrained) maximizer λ^*, which must be projected back to the interval $[0, 1]$ by taking $Max\Big(0, Min(1, \lambda^*)\Big)$.

2.5.1 A simple method for computing the terms that may correspond to corner solutions

When dealing with terms $(n_1, ..., n_m)$ for which some of the values n_k are zero, we no longer have *a priori* knowledge that c^* has full support, e.g. that the hindsight-optimized rebalancing rule is active in all the stocks. However, the maximizer can still be found in a simple, practical way. Let $j_{best}(c)$ be the stock with the greatest marginal utility at the current iterate c, e.g.

$$j_{best}(c) \in \underset{1\leq j\leq m}{argmax}\ \frac{\partial\phi}{\partial c_j}(c). \tag{2.27}$$

Similarly, let j_{worst} be the stock in the current portfolio that has the lowest marginal utility. That is,

$$j_{worst}(c) \in \underset{j\in supp(c)}{argmin}\ \frac{\partial\phi}{\partial c_j}(c) \tag{2.28}$$

We will transfer some amount of mass λ from stock j_{worst} to the stock j_{best}, where $0 \leq \lambda \leq j_{worst}$. Define $f(\lambda) = \phi(c + \lambda d)$, where $d = (d_1, ..., d_m)$ is the direction of the variation: $d_{j_{best}} = 1$, $d_{j_{worst}} = -1$, and $d_k = 0$ otherwise. The line search in direction d is now easily resolved, again by Newton's method. We have

$$f'(\lambda) = \sum_{t=1}^{T}\frac{\langle d, x_t\rangle}{\langle c + \lambda d, x_t\rangle} = 0 \tag{2.29}$$

$$f''(\lambda) = -\sum_{t=1}^{T} \frac{\langle d, x_t \rangle^2}{\langle c + \lambda d, x_t \rangle^2} \tag{2.30}$$

$$\lambda \leftarrow \lambda - \frac{f'(\lambda)}{f''(\lambda)} \tag{2.31}$$

In this line search, a corner solution corresponds to $f'(c_{j_{worst}}) \geq 0$. This means that all of the mass currently on stock j_{worst} is transferred to stock j_{best}. This technique only changes c two coordinates at a time, but it makes the greediest possible transfer from the worst stock to the best one, at least vis-a-vis the current portfolio c. The convergence is extremely rapid in practice.

2.5.2 Log-barrier method

For the sake of reference and completeness, I provide the necessary formulas for successful implementation of an alternative (interior-point) method, namely the log-barrier method. This is a simple method whose iterates converge to c^* through the interior of the simplex. The idea is very straightforward. We choose a parameter $\eta > 0$, and augment the objective function with a penalty term, as follows

$$\phi(c) + \eta \sum_{j=1}^{m} log\, c_j. \tag{2.32}$$

This maintains the concavity, and on account of Inada conditions, forces an interior solution c^*_η. As $\eta \to 0$, $c^*(\eta) \to c^*(0) = c^*$. After eliminating the multiplier on the constraint $\sum_{j=1}^{m} c_j = 1$, we get the FOCs

$$\frac{\partial \phi}{\partial c_i} - \sum_{k=1}^{m} c_k \frac{\partial \phi}{\partial c_k} - m\eta + \frac{\eta}{c_i} = 0 \tag{2.33}$$

For the sake of Newton's method, the Hessian is now

$$H_{ij} = \frac{\partial^2 \phi}{\partial c_i \partial c_j} - \sum_{k=1}^{m} c_k \frac{\partial^2 \phi}{\partial c_k \partial c_j} - \frac{\partial \phi}{\partial c_j} - \frac{\eta}{c_j^2} \mathbf{1}_{\{j=i\}} \tag{2.34}$$

One can now find c^*_η by iterating $c \leftarrow c - H^{-1}(c) g(c)$.

2.6 Leverage and polytope restrictions

So far, the core reason for allowing nature to choose any realization $x_t \in \mathcal{R}_+^m - \{0\}$ has been analytic tractability. In relaxing nature's set of choices as much as possible, we arrive at a simple situation where nature's behavior is determined by a set of free generators of the orthant, e.g. the unit basis vectors after normalization. As a bonus, however, we obtain a trading strategy that is robust to all possible market behavior under the sun. Note also that, if the x_{tj} are lognormally distributed (as they would be under geometric Brownian motion), then the support of x_t is the entire orthant anyhow.

In reality, there is a large, but finite number of possible realizations $x_t \in \mathcal{X}$, where $\mathcal{X}$ is the support of x_t. For one thing, transaction prices are all rational numbers that are only quoted to a finite degree of precision. For another, assuming that transactions are settled in U.S. dollars, no transaction price can exceed the maximum amount of dollars in any bank or brokerage account.

Given the *a priori* belief that every point of the orthant is a possible realization, the trader cannot safely use leverage to any degree. A short sale of any amount of stock j can potentially bankrupt the trader, just as soon as nature chooses a value x_{tj} that is large enough. From the standpoint of getting within ϵ of the hindsight-optimized growth rate as quickly as possible, the ideal scenario is to reduce the set $\mathcal{X}$ of nature's choices, while at the same time allowing the trader to use leverage in so far as he can now guarantee that he will never go bankrupt. The value of the game would then increase on both fronts. The right generalization is to restrict $\mathcal{X} \subset \mathcal{R}_+^m$ to be a convex cone, or, after normalization, a polytope with m vertices that is contained in the unit simplex.

Accordingly, define $\mathcal{A} = co(A_1, ..., A_m)$, where $A_i = \left[A_{i1}, ..., A_{im}\right]$ are the generators of the convex cone. The $m \times m$ matrix of A_{ij} is called simply A. We will assume that the A_i are linearly independent (e.g. A is invertible), and their conic hull will be assumed to contain the vector $\mathbf{1}' = \left[1, ..., 1\right]$. In the case of the positive orthant, we formerly had $A_i = e_i$ and $A = I$, the identity matrix, and $\mathcal{A}$ being equal to the unit simplex.

Dually, we will allow the trader to use any non-bankruptable portfolio $\theta_t \in \mathcal{P}$, where

$$\mathcal{P} = \left\{ b \in \mathcal{R}^m : \langle b, A_i \rangle \geq 0 \, for \, all \, i, \, and \sum_{j=1}^{m} b_j = 1 \right\}. \tag{2.35}$$

These are precisely the (potentially leveraged) portfolios that make positive inner product with all possible realizations $x_t \in \mathcal{A}$. In the former case we had $\mathcal{P} = \Delta$. As usual, we will require that the derivative $D(x_1, ..., x_T)$ be subadditive and homogeneous separately in each x_t. When $D(\cdot)$ is homogeneous in each vector argument, we may assume without loss that each $A_i \in \Delta$, or equivalently, that A is a row-stochastic matrix.

The coordinate vector of $\left[1, ..., 1\right]$ with respect to the basis $A_1, ..., A_m$ will be denoted $\rho = (\rho_1, ..., \rho_m) \geq 0$, e.g. $\rho = \mathbf{1}' A^{-1}$. In the unrestricted case, we had $\rho = \left[1, ..., 1\right]$. The intuition behind requiring $\mathbf{1}'$ to be in the conic hull of the A_i is that we leave nature the option to have all assets perform equally well. In non-trivial situations, $\mathbf{1}'$ will be an interior point of the convex cone generated by the A_i, and nature will therefore have the option in any given period to have any asset j outperform all the others.

Under these assumptions, $\mathcal{P}$ will be a compact convex set. In fact, $\mathcal{P}$ is the convex hull of the vectors

$$\left\{ \frac{A^{-1}e_1}{\rho_1}, \frac{A^{-1}e_2}{\rho_2}, ..., \frac{A^{-1}e_m}{\rho_m} \right\}. \tag{2.36}$$

That is, the extreme points of $\mathcal{P}$, are just the columns of A^{-1}, but normalized so that their coordinates sum to 1.

Example 15. *Consider the daily gross return vectors for (Visa, Mastercard) over the last five years. After normalization to the simplex, the most extreme realizations on record are* $A_1 = (0.5136, 0.4864)$ *and* $A_2 = (0.4766, 0.5234)$. *We have* $\rho = (1.2636, 0.7364)$, *and the extreme points of* $\mathcal{P}$ *are* $(11.1992, -10.1992)'$ *and* $(-17.858, 18.858)'$. *That is to say, for every dollar of wealth, one can potentially short* \$10.20 *worth of Mastercard if he puts all the proceeds into Visa; he can potentially short* \$17.86 *worth of Visa if he puts all the proceeds into Mastercard. This is illustrated in the figure below. So long as the daily gross return vectors lie in the conic hull of* A_1 *and* A_2, *no convex combination of these (extreme) leveraged portfolios can go bankrupt. On account of the small size of the empirical support* $\mathcal{X}$, *leverage ratios as high as* $36:1$ *have become admissible.*

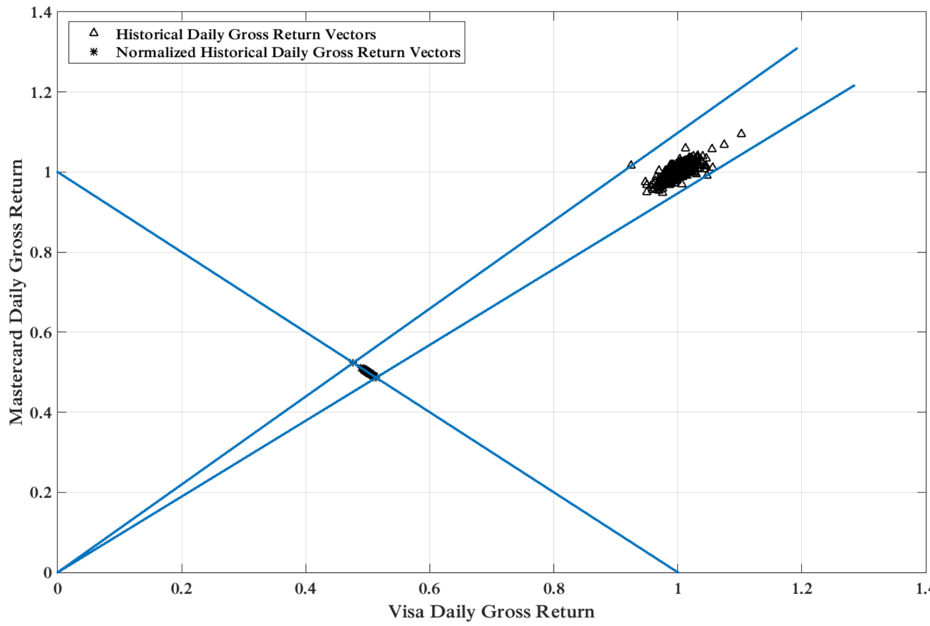

Figure 2.3: **Less to worry about: the conic hull of the past five years of daily gross return vectors for (Visa, Mastercard)**

The general Bellman equations now become

$$V(h^t) = \underset{\theta_{t+1} \in \mathcal{P}}{Max}\ V(h^t, \theta_{t+1}) \tag{2.37}$$

$$V(h^t, \theta_{t+1}) = \underset{x_{t+1} \in co(A_1, \ldots, A_m)}{Min}\ V(h^t, \theta_{t+1}, x_{t+1}), \tag{2.38}$$

with the same boundary conditions as before. *Mutatis mutandis*, the explicit calculations for the backward induction repeat themselves. Once again, in all subgames nature's best move is to play some extreme point A_j. The unique SPNE trading strategy $\theta^*(h^t)$ is characterized by the fact that it equalizes the continuation values $V(h^t, \theta^*(h^t), A_j)$ for $1 \leq j \leq m$. Should nature fail to play an extreme point (which it often will in practice), the continuation value of the game increases, to the advantage of the trader. It is also quite obvious that the restricted game with action sets $\mathcal{P}, \mathcal{A}$ has a higher value than the unrestricted one. *A fortiori*, at each step of the sequential minimax, the trader is maximizing over a larger set than before, and nature is minimizing over a smaller one. Should nature fail to play an extreme point at any time, he should get back on the wagon immediately: after the partial return history $x^t = (x_1, ..., x_t)$, the only final nodes that are relevant to the continuation game (in equilibrium) are $(x^t, A_{j_{t+1}}, ..., A_{j_T})$.

Theorem 9. *The general formula for the value of the hedging game is*

$$V(h^t) = \frac{W(h^t)}{\sum\limits_{j_{t+1},...,j_T} \rho_{j_{t+1}} \rho_{j_{t+2}} \cdots \rho_{j_T} D(x^t, A_{j_{t+1}}, ..., A_{j_T})}, \tag{2.39}$$

and the SPNE portfolio vector for trading session $t+1$ after observing history h^t is specified by

$$\langle \theta^*(h^t), A_k \rangle = \frac{\sum\limits_{j_{t+2},...,j_T} \rho_{j_{t+2}} \cdots \rho_{j_T} D(x^t, A_k, A_{j_{t+2}}, ..., A_{j_T})}{\sum\limits_{j_{t+1},...,j_T} \rho_{j_{t+1}} \cdots \rho_{j_T} D(x^t, A_{j_{t+1}}, ..., A_{j_T})} \tag{2.40}$$

This is a system of m linear equations in the m unknowns $\theta_k^*(h^t)$, $1 \leq k \leq m$. It has a unique solution on account of the fact that A is supposed to be invertible. *A fortiori* we see that this $\theta^*(h^t)$ is admissible: by its very definition, it makes positive inner product with the extreme points of $\mathcal{A}$, on account of the fact that $\rho \geq 0$ and $D(\cdot) > 0$. We also observe that

$$1 = \sum_{k=1}^{m} \rho_k \langle \theta^*(h^t), A_k \rangle = \langle \theta^*(h^t), \sum_{k=1}^{m} \rho_k A_k \rangle = \langle \theta^*(h^t), \mathbf{1} \rangle, \tag{2.41}$$

so that the coordinates of the portfolio vector do indeed sum to 1. Following the general logic of the prequel, the dynamic superhedging price of the derivative after x^t is

$$p_A(x^t) = \sum_{j_{t+1},...,j_T} \rho_{j_{t+1}} \cdots \rho_{j_T} D(x^t, A_{j_{t+1}}, ..., A_{j_T}). \tag{2.42}$$

Example 16. *Assuming that $D(x^T)$ is the hindsight-optimized wealth (vis-a-vis unlevered rebalancing rules c), the superhedging price is*

$$\sum_{n_1+\cdots+n_m=T} \binom{T}{n_1 \cdots n_m} \left\{ \rho_1^{n_1} \cdots \rho_m^{n_m} \cdot \underset{c \in \Delta}{Max} \langle c, A_1 \rangle^{n_1} \cdots \langle c, A_m \rangle^{n_m} \right\}. \tag{2.43}$$

For $m = 2$ stocks this reduces to

$$\sum_{j=0}^{m} \binom{T}{j} \lambda^*(j)^j \Big(1 - \lambda^*(j)\Big)^{T-j}, \tag{2.44}$$

where

$$\lambda^*(j) = Min\left(Max\left(\frac{j}{T}, \rho_1 Min(a_{11}, 1-a_{11})\right), \rho_1 Max(a_{11}, 1-a_{11})\right), \qquad (2.45)$$

and $\rho_1 = \frac{2a_{22}-1}{a_{11}+a_{22}-1}$. $\lambda^*(j)$ *is the point of the interval* $[\rho_1\, Min(a_{11}, 1-a_{11}), \rho_1\, Max(a_{11}, 1-a_{11})]$ *that is nearest to* $\frac{j}{T}$.

Example 17. *Assume that* $A_1 = (0.4, 0.6)$ *and* $A_2 = (0.6, 0.4)$*. Then* $0.4 \leq x_{tj} \leq 0.6$ *for all* t, j*. No stock can ever account for more than* 60% *of the gross return in any one trading session. This will be the case for the daily returns of practically all the stocks in the S&P 500, for instance. Then, as is illustrated below, cognizance of this restriction (plus the possibility of using leverage in certain variations of play) leads to significantly sharper guarantees on the final growth rate spread, especially for shorter horizons.*

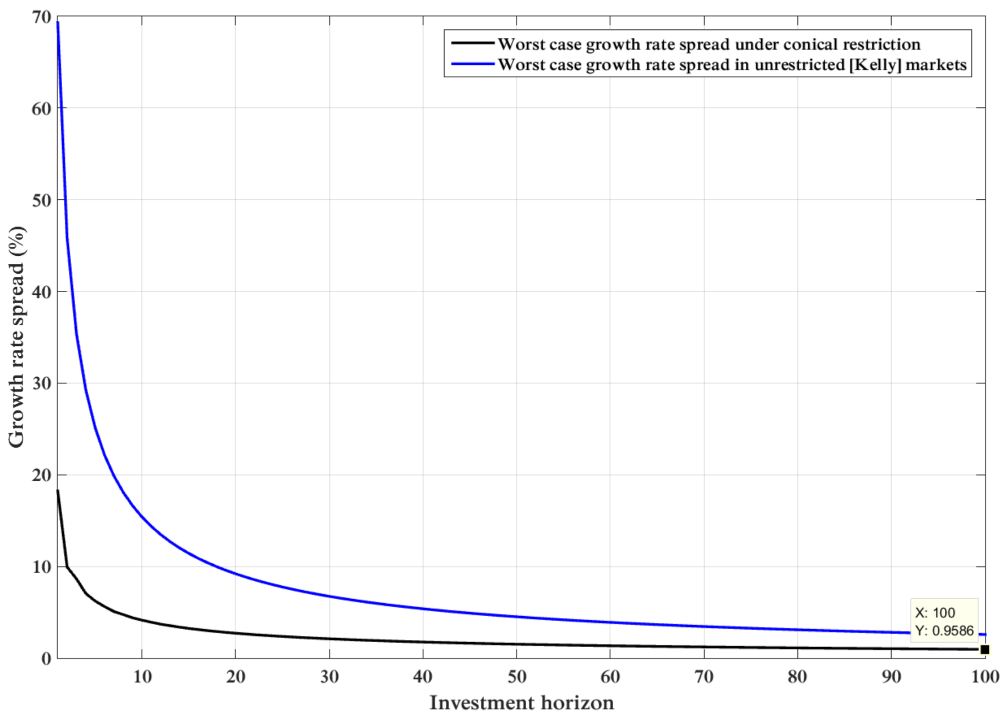

Figure 2.4: **Guaranteed final growth-rate spreads for levered and unlevered versions of the subgame-perfect universal portfolio**

The performance of the subgame-perfect universal portfolios is illustrated below for the period (1/1/1987 to 1/1/2017), for the stocks (Apple, Microsoft) with annual rebalancing. The generators of nature's action set were taken to be $A_1 = (0.75, 0.25)$ and

$A_2 = (0.25, 0.75)$. Obviously, the fact that the vanilla SPUP outperforms the polytope SPUP in this particular case is not a black mark against the theory. The vanilla strategy winds up beating both stocks outright, while the (theoretically much safer) polytope version beats a performance-weighted index that makes an initial $(0.5, 0.5)$ distribution of wealth into the two stocks and lets it ride.

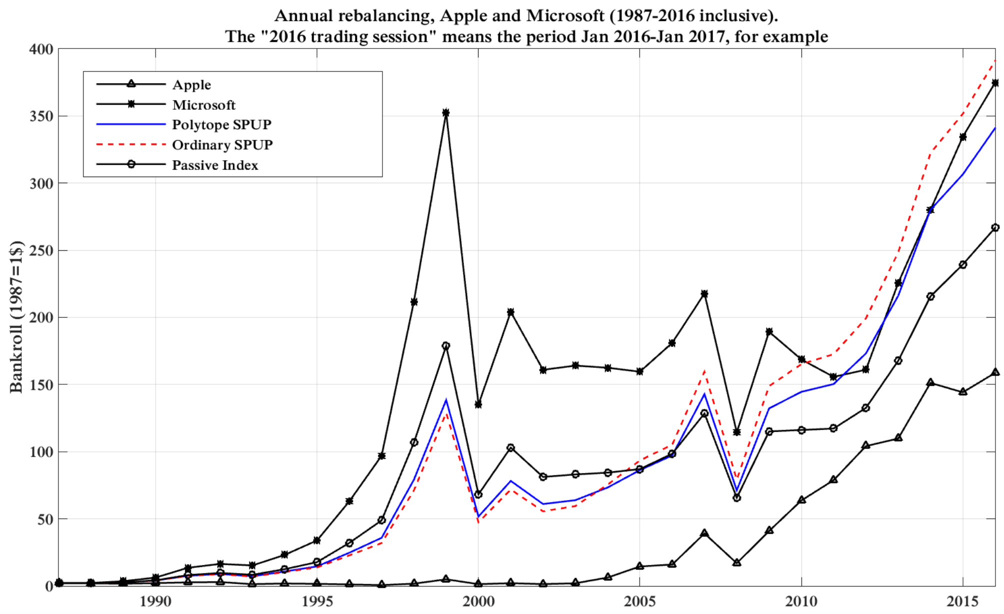

Figure 2.5: **Performance of the subgame-perfect universal strategies over the 30-year period Jan 1987-Jan 2017. (Pairs trading of Apple and Microsoft shares, with annual rebalancing)**

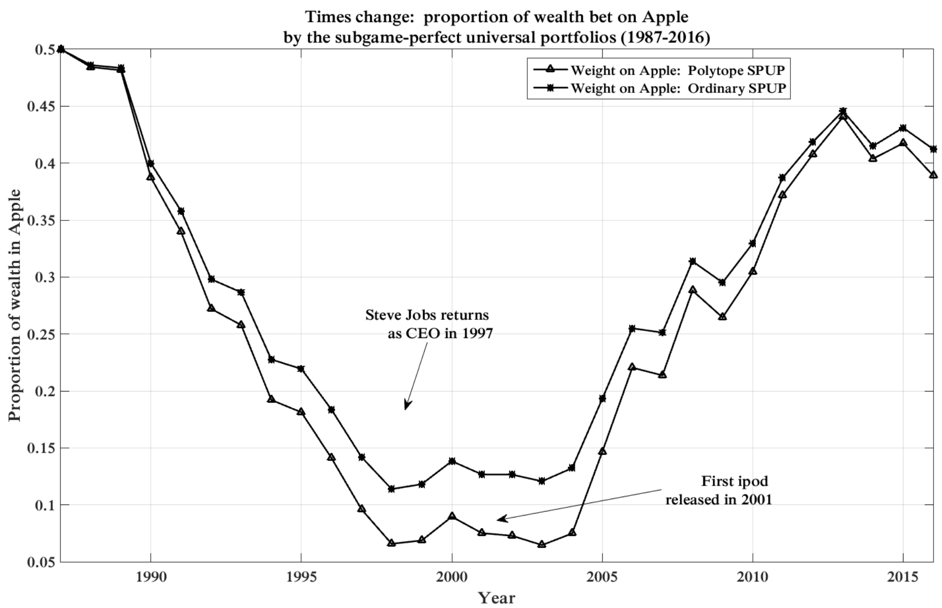

Figure 2.6: **Shrewd: the SPUPs buy back into Apple at the turn of the millenium**

The optimized worst-case bounds on the excess compound annual growth rate of the hindsight-optimized rebalancing rule are illustrated below. Notice how the polytope-restricted version is capable of making very sharp guarantees at the outset: within 4% of the hindsight-optimized CAGR without having seen any data. However, the levered/restricted bound falls more slowly than the vanilla bound. In this particular example the vanilla SPUP winds up outperforming the more perspicacious version. Of course, this in no way contradicts the fact that the levered SPUP has better worst-case performance. In fact, we see at once that the polytope version is extraordinarily conservative: how many of us would be willing to accept a CAGR spread that is more than 2.5 percentage points higher in the worst case?

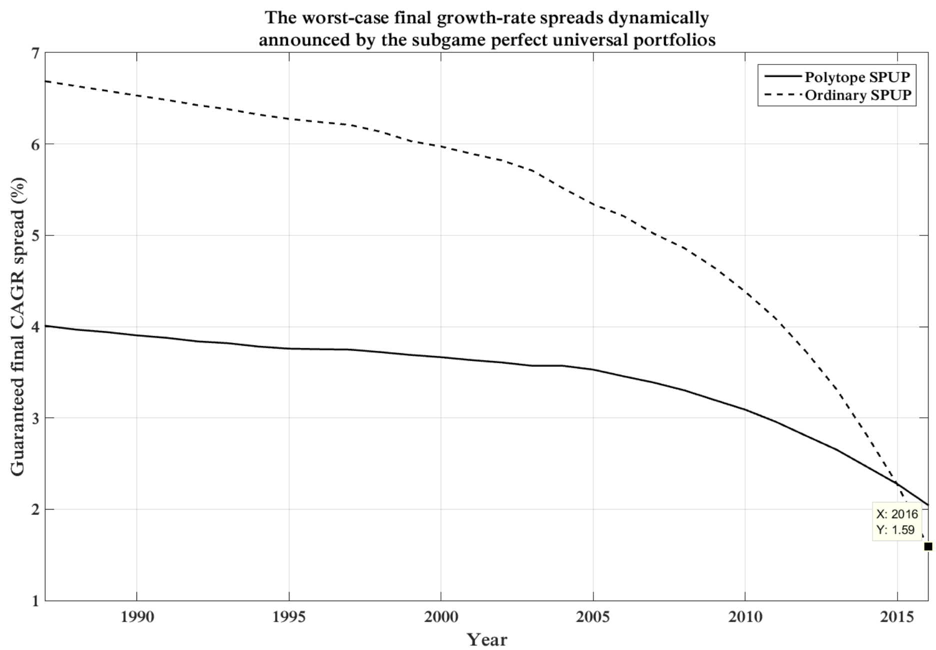

Figure 2.7: **Dynamically revised worst-case growth rate spreads announced by the subgame-perfect universal portfolios (Apple and Microsoft 1987-2016)**

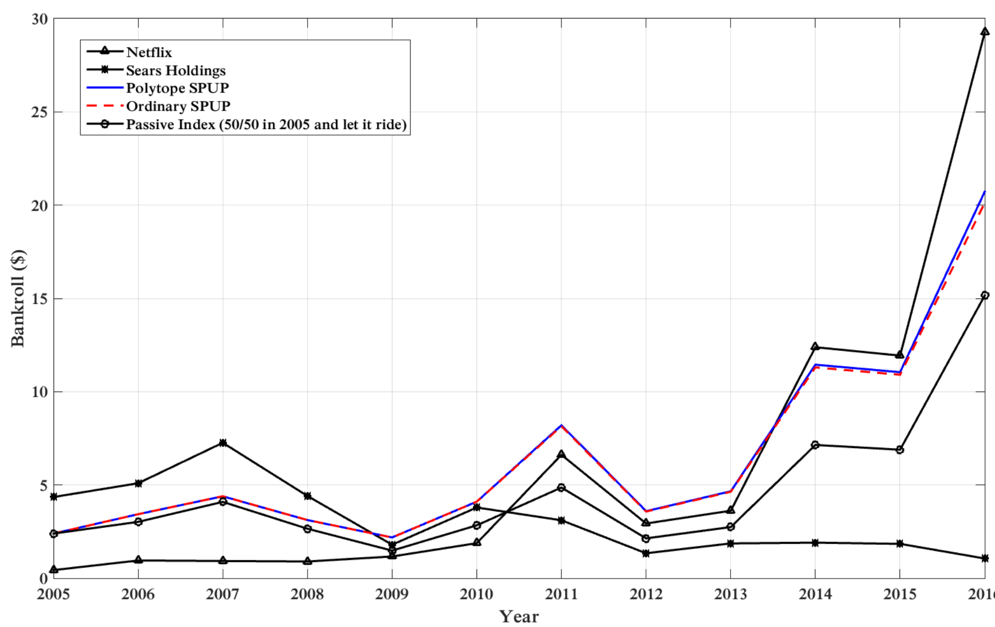

Figure 2.8: **Game history for annual pairs trading of Netflix and Sears (1/1/2005 to 1/1/2017), with** $A_1 = (0.9, 0.1)$ **and** $A_2 = (0.1, 0.9)$

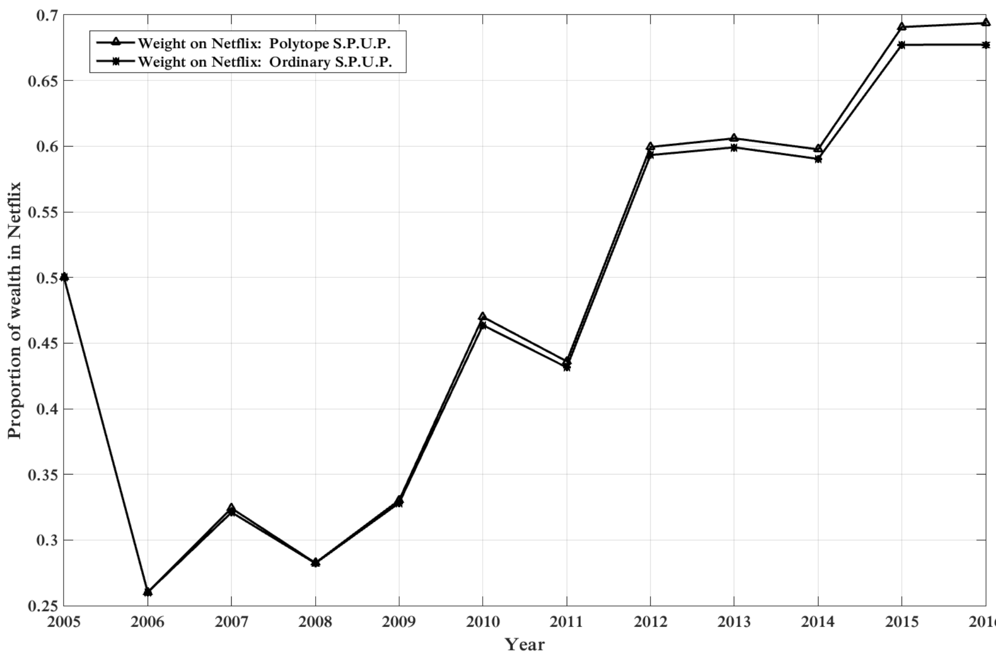

Figure 2.9: **The proportion of wealth bet on Netflix (paired with Sears), annual pairs trading (1/1/2005 to 1/1/2017)**

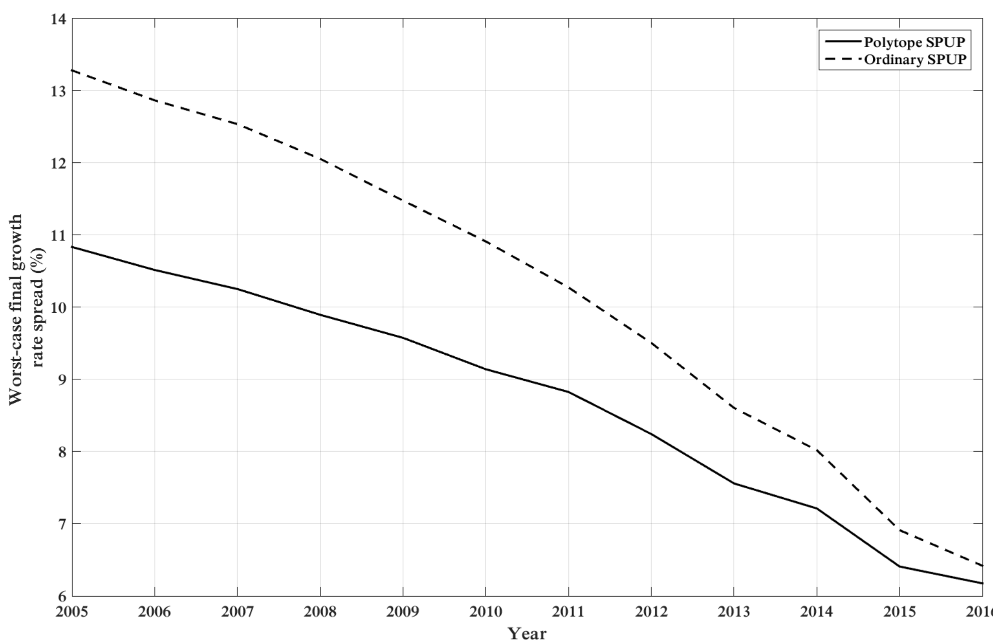

Figure 2.10: **The dynamically best possible guaranteed final excess growth rate of the best rebalancing rule in hindsight (Netflix and Sears, 1/1/2005 to 1/1/2017)**

2.7 Targeting the performance of the best *levered* rebalancing rule in hindsight

Until now we have allowed the trader himself to use leverage, but the hindsight optimization was restricted to ordinary rebalancing rules $c \in \Delta$. Empirically speaking, from the foregoing examples, the polytope S.P.U.P. gaves better dynamic guarantees in the middle of the game, but in practice this gap falls precipitously. Rather than use leverage so as to better track the performance of the best unlevered rebalancing rule in hindsight, the natural step is to allow the hindsight-optimization to use leverage as well. This amounts to the derivative

$$D(x_1, ..., x_T) = \underset{c \in \mathcal{P}}{Max} \ \langle c, x_1 \rangle \cdots \langle c, x_T \rangle \tag{2.46}$$

All of the usual properties of $D(\cdot)$ (convex and homogeneous in each x_t separately) obtain. On account of the fact that both the hindsight-optimizer and the trader are equally free to use leverage, the superhedging price remains what it was for the ordinary universal portfolio. To calculate

$$D(n_1, ..., n_m) = \underset{c \in \mathcal{P}}{Max} \ \langle c, A_1 \rangle^{n_1} \cdots \langle c, A_m \rangle^{n_m}, \tag{2.47}$$

we just make the substitution $b_j = \rho_j \langle c, A_j \rangle$ for $1 \leq j \leq m$. On account of the fact that this linear change of variable

$$(b_1, ..., b_m)' = diag(\rho_1, ..., \rho_m) A (c_1, ..., c_m)' \tag{2.48}$$

is a bijection of $\mathcal{P}$ onto the unit simplex, we get the same program

$$\underset{b \in \Delta}{Max} \ b_1^{n_1} \cdots b_m^{n_m} \tag{2.49}$$

as before, with optimized value $(\frac{n_1}{T})^{n_1} \cdots (\frac{n_m}{T})^{n_m}$. It follows at once that we can compound our money at the same asymptotic rate as the best *levered* rebalancing rule in hindsight, on the exact same worst-case timeframe as the ordinary universal portfolio. As we have seen, we can expect the subgame perfection to provide a significant dynamic improvement to the initial superhedging cost of $\underset{n_1+\cdots+n_m=T}{\Sigma}\binom{T}{n_1\cdots n_m}(\frac{n_1}{T})^{n_1}\cdots(\frac{n_m}{T})^{n_m}$.

Computation of the final wealth of the best levered rebalancing rule in hindsight

No additional techniques are required to calculate $D(x_1,...,x_T)$. On account of the fact that

$$(c_1,...,c_m)' = A^{-1}diag\left(\frac{1}{\rho_1},...,\frac{1}{\rho_m}\right)(b_1,...,b_m)', \tag{2.50}$$

we have $\langle c,x_t\rangle = \langle A^{-1}D^{-1}b,x_t\rangle = \langle b,x_tA^{-1}D^{-1}\rangle$, where $D = diag(\rho_1,...,\rho_m)$. Thus, all we must do is linearly transform the row vector $x_t = (x_{t1},...,x_{tm})$ and solve an ordinary (unlevered) hindsight optimization problem for the new data set $y_t = x_tA^{-1}D^{-1}$:

$$D(x_1,...,x_T) = \underset{b\in\Delta}{Max}\ \langle b,y_1\rangle\cdots\langle b,y_T\rangle. \tag{2.51}$$

All the numerical techniques discussed above therefore apply. Note that the matrix $A^{-1}D^{-1}$ is just the result of normalizing the columns of A^{-1} so that they sum to 1. The extreme points A_i (row vectors) get transformed to $A_iA^{-1}D^{-1} = \frac{e_i}{\rho_i}$. In calculating the numerator of the portfolio $\theta_k(x_1,...,x_t)$, one must calculate the number

$$\underset{b\in\Delta}{Max}\ \langle c,y_1\rangle\cdots\langle c,y_t\rangle b_1^{n_1}\cdots b_m^{n_m}. \tag{2.52}$$

Thus, the full numerator is just

$$\underset{n_1+\cdots n_m=T-t-1}{\Sigma}\binom{T-t-1}{n_1\ \cdots\ n_m}D(y_1,...,y_t;n_1,...,n_m). \tag{2.53}$$

Thus, the game with data $(x_t)_{t=1}^T$ whereby *both* the trader and the hindsight-optimizer lever is equivalent to the game where neither levers and the data is $(y_t)_{t=1}^T$.

Example 18. *If*

$$A = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}, \tag{2.54}$$

then

$$[y_{t1},...,y_{tm}] = [x_{t1},...,x_{tm}]\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \tag{2.55}$$

transforms the returns of the levered-levered game to the returns of the equivalent unlevered-unlevered game. For instance, the vector [1.2 0.9] *gets transformed into the (exxagerated) figures* [1.8 0.3]. *If* λ *is the proportion of wealth bet on stock* 1, *then* $-2 \leq \lambda \leq 3$.

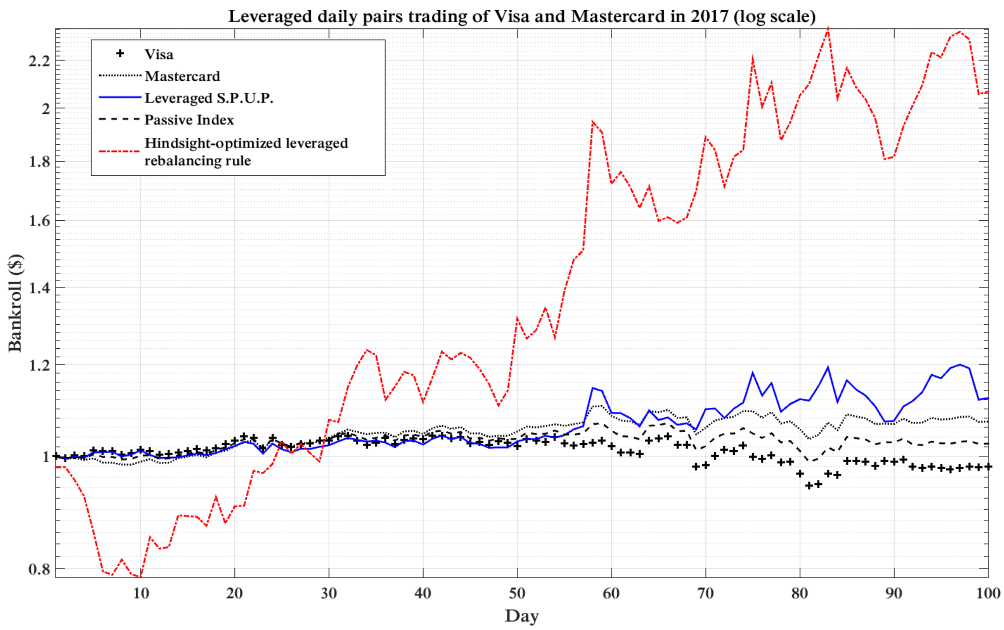

Figure 2.11: **Juiced: leveraged daily pairs trading of Visa and Mastercard in 2017. Target wealth is that of the best leveraged rebalancing rule in hindsight.**

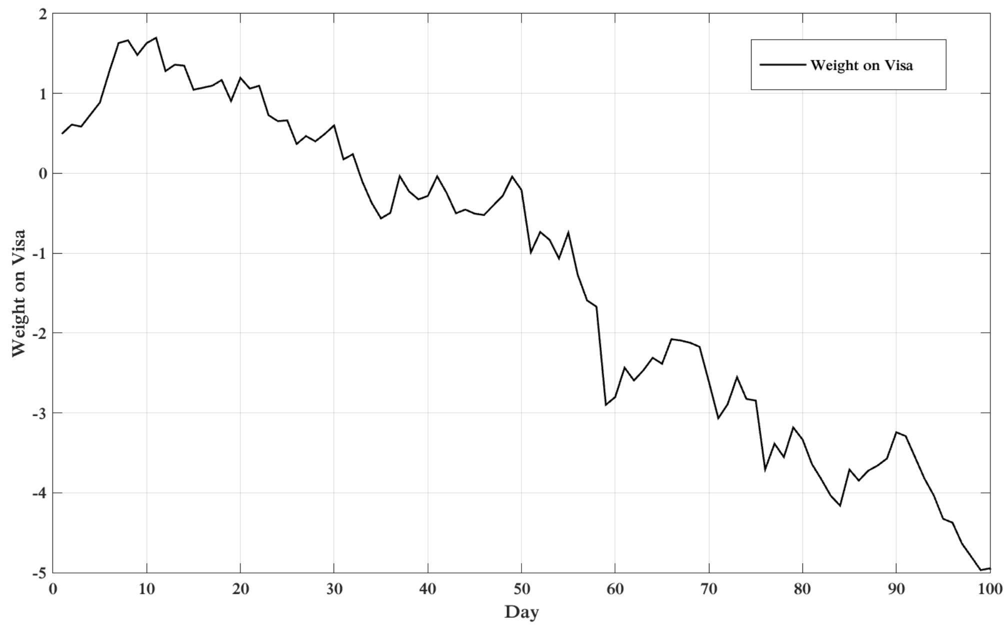

Figure 2.12: **Out of whack: the subgame-perfect universal strategy shorts Visa and puts the proceeds into Mastercard over a period of 100 days in 2017. Very high leverage ratios are possible on account of the high correlation (viz. small support) of the two stocks' daily returns**

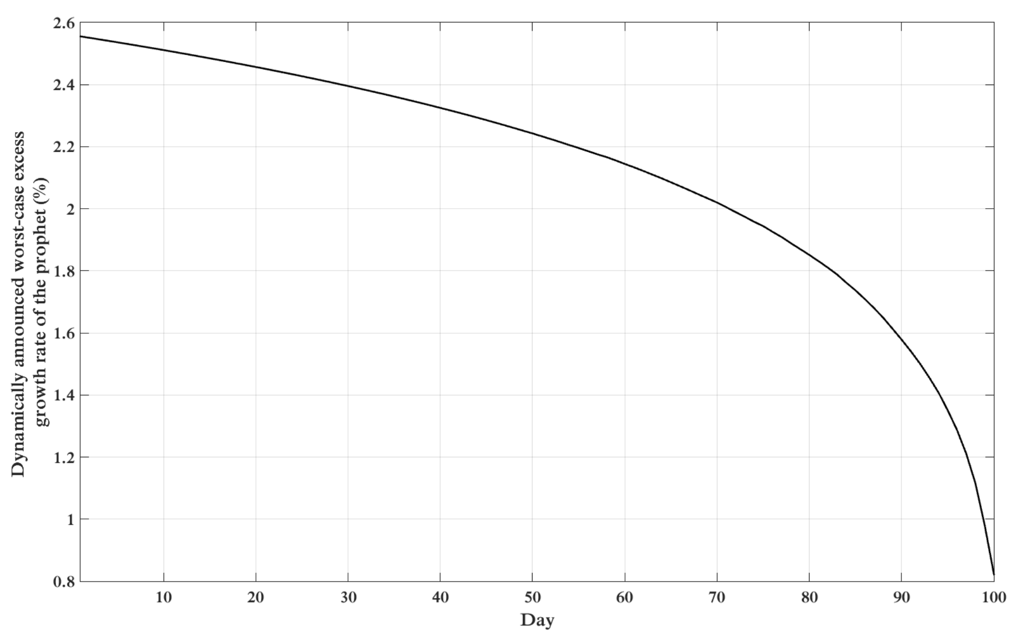

Figure 2.13: **Promises kept: the evolution of the worst-case final regret as nature fails to play extreme points of the (Visa, Mastercard) return support**

2.8 Multi-convex derivatives on a single stock

The superhedging framework we have developed is very flexible and general, and finds use in a wide array of situations. For one thing, we can simply assume that asset 1 is cash, say, with a constant gross return $x_{t1} = 1 + r$. We will assume that the single stock (asset 2) has compact support $x_{t2} \in [d, u]$, where d is the lowest possible gross return on down days, and u is the greatest possible gross return on up days. This contradicts the possibility of x_{t2} being, say, lognormally distributed, but nobody can deny that in reality the support $\mathcal{X}$ of x_t has a finite, if large, number of outcomes. Thorp said it himself: the utility of continuous return distributions is that they make for good quality, tractable approximations of this discrete reality. Notice that we have simply filled in the usual binomial lattice model of price dynamics, which assumes $x_{t2} \in \{d, u\}$. But at the end of the day everything will hinge for us on the extreme points anyhow. We have

$$A = \begin{bmatrix} 1+r & d \\ 1+r & u \end{bmatrix},\ A^{-1} = \frac{1}{(1+r)(u-d)} \begin{bmatrix} u & -d \\ -(1+r) & 1+r \end{bmatrix}, \tag{2.56}$$

$$\rho = \mathbf{1}' A^{-1} = \frac{(u-(1+r), (1+r)-d]}{(1+r)(u-d)} \tag{2.57}$$

$$A^{-1} D^{-1} = \begin{bmatrix} \frac{u}{u-(1+r)} & \frac{-d}{1+r-d} \\ \frac{-(1+r)u}{u-(1+r)} & \frac{1+r}{1+r-d} \end{bmatrix} \tag{2.58}$$

Under the usual assumption that $d < 1 + r < u$, the vector $(1, ..., 1)$ will then lie in the conic hull of the rows of A, as required. Naturally, leverage will be used for the sake of efficient superhedging. The admissible portfolios can have a weight on the stock no greater than $\frac{1+r}{1+r-d}$, and a weight no less than $\frac{-d}{1+r}$. Any purchase of shares on margin, or any short sale, that violates these bounds has the potential to go bust. The dynamic super-hedging price of a symmetric, multiconvex (but not necessarily homogeneous) derivative $D(x_1, ..., x_t)$ can be read off from our general formulas: it is

$$\frac{1}{(1+r)^{T-t}} \sum_{j=0}^{T-t} \binom{T-t}{j} p^j q^{T-t-j} D(x^t; j, T-j), \tag{2.59}$$

where $p = \frac{u-(1+r)}{u-d}$ and $q = 1 - p$. These are just the risk-neutral probabilities implied the usual binomial lattice $\{u, d\}$.

Theorem 10. *For a symmetric multiconvex derivative over a single stock in discrete time, assuming compact return support $x_{t2} \in [d, u]$, the dynamic superhedging price is exactly the price on the binomial lattice $\{d, u\}$.*

For example, if S_0 is the initial price of the stock, and a European call is written that expires at T for a striking price of K, the dynamic superhedging price in this environment is

$$p_t(S_t) = \frac{1}{(1+r)^{T-t}} \sum_{j=0}^{T-t} \binom{T-t}{j} p^j q^{T-tj} Max(S_t u^j d^{T-t-j} - K, 0), \tag{2.60}$$

where $S_t = S_0 \prod_{s=1}^{t} x_{s2}$. Since this price is a convex function separately of each period's gross return vector x_t, the greatest possible superhedging price after t periods would have obtained for extreme realizations $x_{t2} \in \{d, u\}$. Any realizations strictly between these two bounds constitutues off-path play by nature, which will have the effect of reducing the superheding price.

2.8.1 Subgame-perfect universal portfolios over cash and one fund

Even simpler than trading pairs, one can simply construct a desired fund (say, the S&P 500 ETF) to make leveraged bets on. Ideally this fund will be selected to have a high Sharpe or Sortino ratio, and the problem then becomes one of dynamically adjusting the leverage ratio. In this connection, shorting cash amounts to buying additional shares of the fund on margin. For example, in recent years there has been a bonanza of leveraged and inverse ETFs that maintain a constant (2x,3x, etc) leverage ratio. Asymptotically these ETFs will be beaten by levered subgame-perfect universal schemes. Note that leveraged rebalancing rules have the reverse intuition and mechanics than do unlevered ones. In the case of 50-50 rebalancing in Shannon's Demon, say, some shares are sold every time the stock goes up, and some shares are bought every time it goes down. In this new situation, suppose someone has a dollar of collateral and takes out a margin loan of $\lambda - 1$ dollars for the sake of achieving a leverage ratio of $\lambda : 1$. Equivalently, the trader aims for a debt to assets ratio of $1 - \frac{1}{\lambda}$. If the stock now goes up, then the trader must buy additional shares to preserve this ratio. The trader's assets have increased in value, although his cash debt to his broker has not changed. On account of the fact that his $\frac{debt}{assets}$ ratio is now lower, he borrows more. The reverse happens when the stock goes down: the debt to assets ratio has increased, and the trader must sell some shares to pay down some of the margin debt. If the 50-50 rebalancer buys low and sells high, then, someone with a leveraged rebalancing rule $(1 - \lambda, \lambda)$ has set himself a mechanical plan to buy high and sell low. The saving grace of such a strategy is that a good quality asset like the S&P 500 will have a very strong drift relative to its volatility.

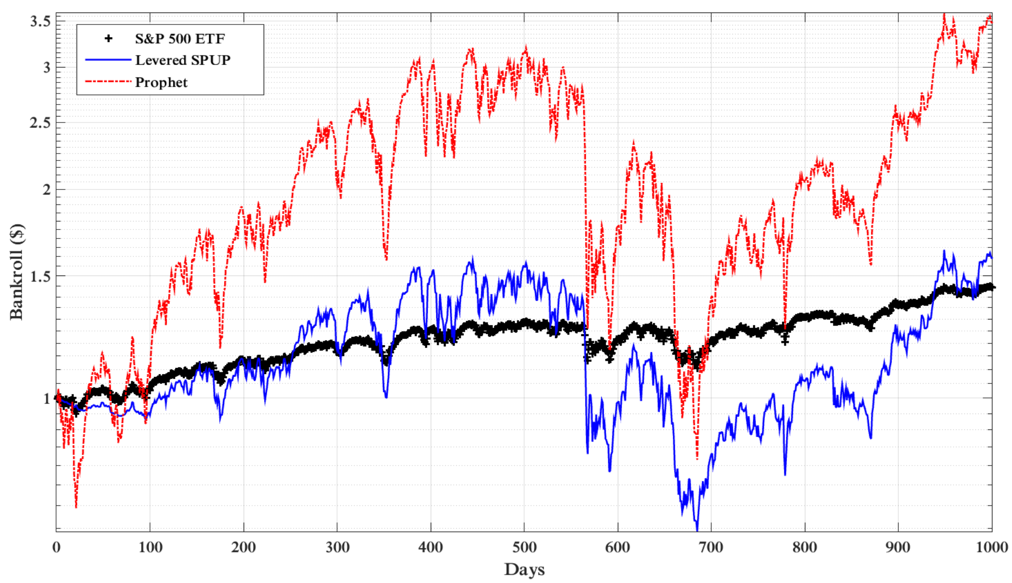

Figure 2.14: **Leveraged bets on the S&P 500 index over the last 1000 days (log scale on the vertical axis)**

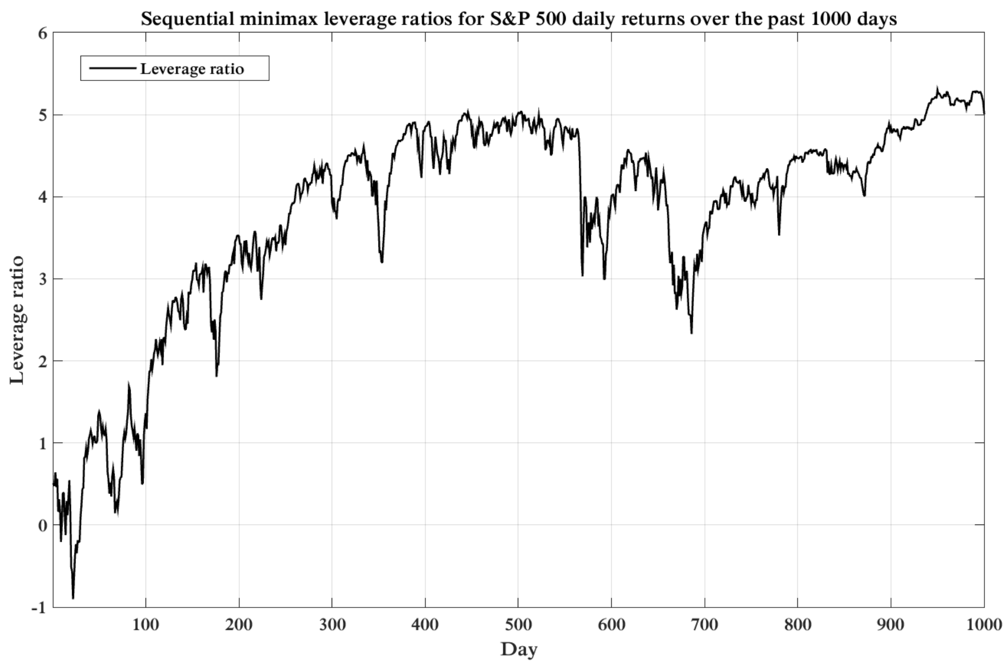

Figure 2.15: **Evolution of the subgame perfect leverage ratio for S&P 500 daily returns, assuming that cash earns zero interest**

2.9 Extensive form with side information

Assume that each period, before the gross return vector x_t is realized, the gambler is able to observe a signal $s_t \in \{1, ..., S\}$. A *state constant rebalancing rule* is a mapping $c : \{1, ..., S\} \to \Delta$. $c(s) = c_s \in \Delta$ will denote the rebalancing rule that is to be used in state s, and c_{sj} will denote the proportion of wealth to bet on stock j in state s. The signal history up to period t is denoted $s^t = (s_1, ..., s_t)$. The most profitable state-constant rebalancing rule in hindsight is given by

$$D(x^T, s_1, ..., s_T) = \underset{c_1,...,c_S \in \Delta}{Max} \prod_{s=1}^{S} \prod_{t:\, s_t=s} \langle c_s, x_t \rangle = \prod_{s=1}^{S} D\big((x_t)_{s_t=s}\big), \tag{2.61}$$

where $D\big((x_t)_{s_t=s}\big)$ denotes the optimized final growth factor of wealth achieved in state s.

On account of this factorization, $D(\cdot)$ is still positive, subadditive, homogeneous, (and therefore convex) in each x_t, and putting $x_t = \mathbf{1}'$ still deletes x_t from the history.

Among the times t for which signal s is observed, the x_t can still be permuted; let n_{sj} be the number of times horse j won in state s. Then, by abuse of notation,

$$D([n_{sj}]) = \prod_{s=1}^{S} D(n_{s1}, ..., n_{sm}). \tag{2.62}$$

Computing $D(X, s_1, ..., s_T)$ for general data is not much harder than before: we now solve S log-concave programs, but the individual objective functions are cheaper to compute.

We take up the general situation studied in the prequel, with vertices $A_1, ..., A_m$. Before period t's trading session, nature picks a public signal $s_t \in \{1, ..., S\}$. The histories are $h^t = (s_1, \theta_1, x_1, ..., s_t, \theta_t, x_t)$, and the final competitive ratio is

$$\Pi(h^T) = \frac{W(h^T)}{D(x^T, s^T)}. \tag{2.63}$$

Backward induction takes place according to the Bellman equations

$$V(h^t) = \underset{s_t \in \{1,...,S\}}{Min} V(h^t, s_{t+1}) \tag{2.64}$$

$$V(h^t, s_{t+1}) = \underset{\theta_{t+1} \in \mathcal{P}}{Max} V(h^t, s_{t+1}, \theta_{t+1}) \tag{2.65}$$

$$V(h^t, s_{t+1}, \theta_{t+1}) = \underset{x_{t+1} \in co(A_1,...,A_m)}{Min} V(h^t, s_{t+1}, \theta_{t+1}, x_{t+1}) \tag{2.66}$$

On account of the general properties that $D(\cdot)$ has retained, the backward induction can still be carried out "explicitly," although this "solution" is far less tractable than it was before.

Theorem 11. *In the extensive-form with adversarial signals, nature should always play a vertex $x_t = A_i$. After h^t, the unique θ^*_{t+1} is characterized by making nature indifferent among the vertices. The best guaranteed payoff after h^t, $V(h^t)$, is given by*

$$\frac{W(h^t)}{\underset{s_{t+1}}{Max} \sum_{j_{t+1}} \rho_{j_{t+1}} \cdots \underset{s_{T-1}}{Max} \sum_{j_{T-1}} \rho_{j_{T-1}} \underset{s_T}{Max} \sum_{j_T} \rho_{j_T} D(x^t, A_{j_{t+1}}, ..., A_{j_T}; s_1, ..., s_T)}. \tag{2.67}$$

The trader's policy is characterized by the numbers $<\theta^*(h^t, s_{t+1}), A_k>$ *for* $k = 1, ..., m$, *which are given by*

$$\frac{\underset{s_{t+2}}{Max} \cdots \underset{s_{T-1}}{Max} \sum_{j_{T-1}} \rho_{j_{T-1}} \underset{s_T}{Max} \sum_{j_T} \rho_{j_T} D(x^t, A_k, A_{j_{t+2}}, ..., A_{j_T}; s_1, ..., s_T)}{\sum_{j_{t+1}} \rho_{j_{t+1}} \underset{s_{t+2}}{Max} \cdots \underset{s_{T-1}}{Max} \sum_{j_{T-1}} \rho_{j_{T-1}} \underset{s_T}{Max} \sum_{j_T} \rho_{j_T} D(x^t, A_{j_{t+1}}, ..., A_{j_T}; s_1, ..., s_T)}. \tag{2.68}$$

The vertex policy is

$$j^*(h^t, s_{t+1}, \theta_{t+1}) = \underset{1 \le k \le m}{argmin}\ V(h^t, s_{t+1}, \theta_{t+1}, A_k)\langle \theta_{t+1}, A_k \rangle \tag{2.69}$$

The signal policy solves the auxiliary dynamic program

$$\delta(x^t; s_1, ..., s_t) = \underset{s_{t+1}}{Max} \sum_{k=1}^{m} \rho_k \delta(x^t, A_k; s_1, ..., s_{t+1}), \tag{2.70}$$

where $\delta(x^T; s_1, ..., s_T) = D(x^T; s_1, ..., s_T)$

For the sake of the asymptotic result, assume for the moment that there are no restrictions on nature's moves, e.g. $A_k = e_k$. Let $s_1^*, ..., s_T^*$ achieve the respective maxima in the denominator of $V(h^0)$. I will simply factor $V(h^0)^{-1}$ into a product of S sums, all of which are $\le p(T, m)$, the superhedging price in the absence of signals or leverage.

Let τ_s be the set of times at which signal s was realized. Sum s will have $|\tau_s|$ indices of summation, denoted $J^s = (J_1^s, ..., J_{|\tau_s|}^s) \in \{1, ..., m\}^{|\tau_s|}$. We have

$$V(h^0)^{-1} = \prod_{s=1}^{S} \sum_{J^s} D(J^s) = \prod_{s=1}^{S} p(|\tau_s|, m) \le p(T, m)^S. \tag{2.71}$$

Here, the symbol $D(J^s)$ denotes the number $D(e_{J_1^s}, ..., e_{J_{|\tau_s|}^s})$. Thus, the excess growth rate of the best (state-constant) rebalancing rule over and above the sequential minimax trader is at most $\frac{S \cdot log\, p(T,m)}{T} \to 0$. The first few values of the exact finite-sample improvement in worst-case growth spread are plotted below.

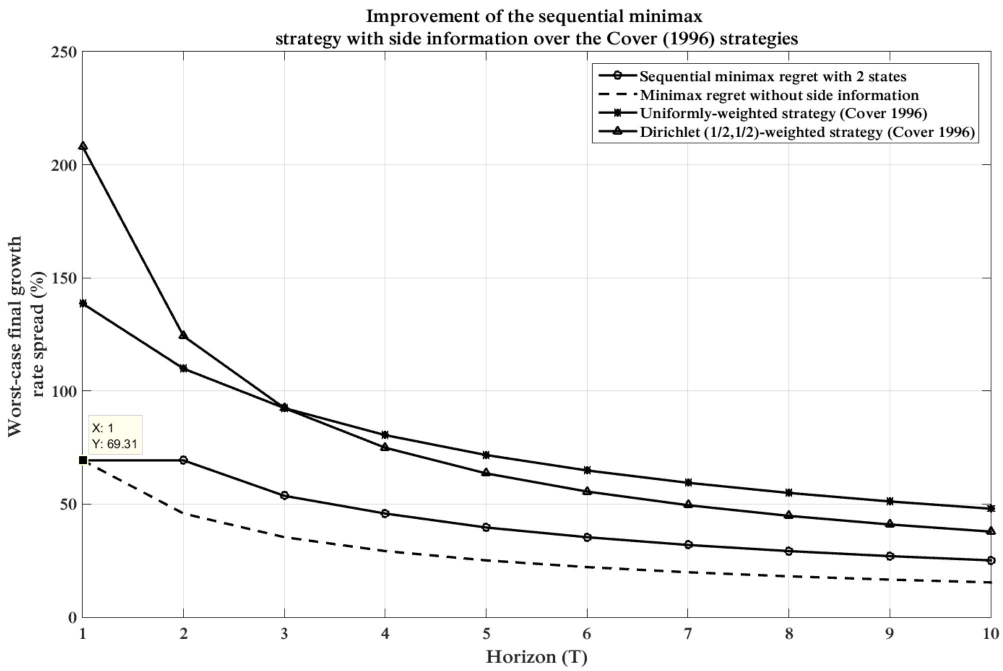

Figure 2.16: **The sequential minimax improvement to the Cover (1996) regret bounds under $S = 2$ states of side information**

Chapter 3

Universal horse race betting

3.1 Fundamental role of the unit basis vectors

Unit basis vectors have played a decisive role in the trading strategies disussed in this thesis. Mathematically, this owes itself to the fact that the conic hull of the unit basis vectors is the positive orthant. In the other interpretation, where the orthant has been normalized to the unit simplex, the unit basis vectors comprise the extreme points on which nature's behavior is determined. Why have we allowed nature to choose any return vector x_t in the positive orthant, save the origin? The answer is that, in relaxing nature's decision problem as much as possible, we are able to obtain analytic solutions for the value function and the SPNE trading strategy. Of course, it would be absurd to think that unit basis vectors would ever be encountered in practice. The trader's insight is that, arithmetically, no realization can damage the competitive ratio as much as the least favorable unit basis vector. In the case of universal portfolios, this method of being robust to a larger set of outcomes than necessary has led to a very favorable result. One is able to compound one's money at the same asymptotic rate as the best rebalancing rule in hindsight, under all possible types of market behavior, extreme or not. In fact, the value-added of the sequential minimax universal portfolio is that it can adapt to the fact that unit basis vectors have not actually been occuring, in so much as this constrains the possible derivative payoffs $D(x^t, e_{j_{t+1}}, ..., e_{j_T})$ at final nodes descended from the current position.

However, it can be a little irritating to try to interpret the (normalized) gross return vector e_j literally: it means that stock j provided 100% of the gross return in trading session t. The other stocks must have collapsed in price to 0. The best way to think about it is that stock j went through the roof relative to the others, which, say, collapsed in price to ϵ. In this situation $\frac{x_t}{||x_t||_1}$ will be very nearly a unit basis vector and hence, by continuity, its effect on the competitive ratio will be nearly that of a unit basis vector. This represents the most oscillatory and chaotic possible stockmarket.

3.2 Horse race interpretation

Assume that m horses run T races sequentially, $1 \leq t \leq T$. The horses are called $j \in \{1, ..., m\}$. A gambler arrives with no prior knowledge of the probability p_j that horse j wins any given race. In fact, this probability may change over time, in a manner that is unknown to the gambler as well.

Before each race t, the bookie posts the *odds* $\mathcal{O}_t = (\mathcal{O}_{t1}, ..., \mathcal{O}_{tm})$. This means that a \$1 bet on horse j pays off a gross return of $\mathcal{O}_{tj}$ if j wins the race, and \$0 otherwise. Implicitly, the bookie therefore believes that $\frac{1}{\mathcal{O}_{tj}}$ is the probability that horse j wins race t, so that the expected gross return on a \$1 bet is \$1. We will assume that the posted odds are "fair" in the sense that $\sum_{j=1}^{m} \frac{1}{\mathcal{O}_{tj}} = 1$. Of course, "fairness" here does not mean that the bookie's beliefs are necessarily correct.

Before each race, the gambler may distribute his wealth among a "portfolio" of bets on the various horses, where θ_{tj} is the proportion of his wealth he bets on horse j in race t. Note that if the gambler uses the portfolio $\theta_{tj} = \frac{1}{\mathcal{O}_{tj}}$, he has *de facto* just stored his money for a period. For simplicity, then, we can assume that the gambler distributes *all* of his wealth among the m horses, without leaving any in cash. In order for the gambler to grow his money, his portfolio θ_t must diverge in some way from the bookie's estimates $\left(\frac{1}{\mathcal{O}_{t1}}, ..., \frac{1}{\mathcal{O}_{tm}}\right)$.

The win history (data) from the first $t-1$ races is denoted $j^{t-1} = (j_1, .., j_{t-1})$, where j_s was the winner of race s. A betting strategy is denoted $\theta_t = \theta(j_1, ..., j_{t-1})$. The gross-return vector for race t is $\mathcal{O}_{tj_t} e_{j_t}$, where j_t is the winner of race t. After normalization to the simplex, this is just e_{j_t}. The worst-case behavior in the stock market thus corresponds to the payoff of a Kelly horse race. In this connection a rebalancing rule $c = (c_1, ..., c_m)$ corresponds to a fixed-fraction betting scheme whereby a gambler bets the proportion c_j of his wealth on horse j every period. The hindsight-optimized rebalancing rule is calculated by

$$D(j_1, ..., j_T) = \underset{c \in \Delta}{Max} \langle c, e_{j_1} \rangle \langle c, e_{j_2} \rangle \cdots \langle c, e_{j_T} \rangle = \underset{c \in \Delta}{Max} \ c_1^{n_1} c_2^{n_2} \cdots c_m^{n_m}, \tag{3.1}$$

where n_k is the number of times horse k won on the path $j^T = (j_1, ..., j_T)$. As discussed in Chapter 1, the hindsight rule is given by $c_j^* = \frac{n_j}{T}$, and the derivative payoff is $(\frac{n_1}{T})^{n_1} \cdots (\frac{n_m}{T})^{n_m}$. These quantities depend only the empirical distribution of wins for the various horses, and not on the order in which the wins occur. Thus, a gambler who adopts the SPNE trading strategy will compound his money at the same asymptotic rate as the best fixed fraction betting scheme in hindsight. If it so happens that the j_t are drawn *iid* according to probabilities $p_k = P(j_t = k)$, then the Kelly rule bets these same proportions $c_k = p_k$, regardless of the posted odds. Let us assume now that the odds do not vary with time, e.g. $\mathcal{O}_{tj} \equiv \mathcal{O}_j$. In any finite sample, *a fortiori*, the hindsight optimized rebalancing rule makes more money than the Kelly gambler. In fact, the hindsight-optimized wealth is

$$\mathcal{O}_1^{n_1} \cdots \mathcal{O}_m^{n_m} \left(\frac{n_1}{T} \right)^{n_1} \cdots \left(\frac{n_m}{T} \right)^{n_m}, \tag{3.2}$$

whereas the Kelly gambler's wealth after T races is

$$\mathcal{O}_1^{n_1} \cdots \mathcal{O}_m^{n_m} p_1^{n_1} \cdots p_m^{n_m}, \tag{3.3}$$

However, they both have the same limiting continuously compounded per-period growth rate of capital, namely

$$\sum_{j=1}^{m} p_j log \left(\frac{p_j}{1/\mathcal{O}_j} \right) \tag{3.4}$$

Let $\mathcal{B} = (\mathcal{B}_1, ..., \mathcal{B}_m)$ denote the bookie's beliefs, where $\mathcal{B}_j = \frac{1}{\mathcal{O}_j}$. Then the Kelly growth rate is simply the Kullback-Leibler divergence $D(p||\mathcal{B})$ of the true distribution from the bookie's beliefs. Although the posted odds are irrelevant to the Kelly (and universal) gambler's strategy, they do determine the highest asymptotic growth rate the gambler is able to achieve.

Example 19. *Suppose there are* $m = 2$ *horses, with the true win probabilities being* $p = (0.4, 0.6)$*, e.g. horse 1 has a* 40% *chance of winning any given race. Suppose* *the bookie thinks the chances are* $\mathcal{B} = (0.5, 0.5)$*, and posts odds* $\mathcal{O} = (2, 2)$*. Then the asymptotic growth rate achieved by the Kelly gambler (as well as the universal and hindsight-optimized gamblers) is* $D(0.4, 0.6||0.5, 0.5) = 0.4\,log\,\frac{0.4}{0.5} + 0.6\,log\,\frac{0.6}{0.5} = 2\%$ *per period. The gambler will roughly double his wealth every 35 races.*

320 such races have been simulated below, under which the per-race growth rate spread of the best rebalancing rule in hindsight over and above the universal gambler is guaranteed to be less than 1% at the end of the horizon. The actual final growth rate spread is 0.98%. Notice how the growth rate spread does not decrease to 0 monotonically. The hindsight-optimized fixed fraction betting scheme achieves slightly higher final wealth than a Kelly gambler, although his wealth can be lower at intermediate times $1 < t < T$. All three of these time series are converging to the same limit.

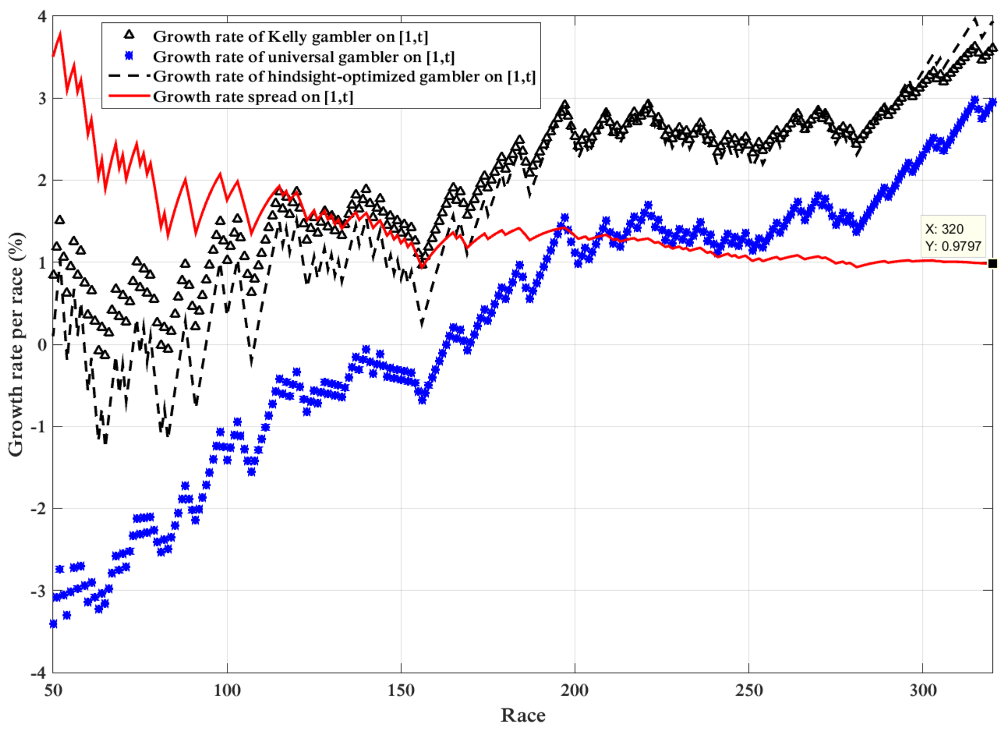

Figure 3.1: **Simulation of** $T = 320$ **races with** $m = 2$ **horses, with win probabilities** $p = (0.4, 0.6)$ **and odds** $\mathcal{O} = (2, 2)$

We have shown in the horse race example that the excess growth rate of the hindsight optimized rule over and above the Kelly gambler converges to 0 almost surely. With slightly more effort, this can be proved for the general stockmarket. Let $D(x_1, ..., x_T)$ be the hindsight optimized wealth, and let $c^*(F)$ be the Kelly rule against *iid* returns drawn from the CDF $F(\cdot)$. We must show that the growth rate spread

$$\mathcal{S}(x_1, ..., x_T) = \frac{log\, D(x_1, ..., x_T) - \sum_{t=1}^{T} log\, \langle c^*(F), x_t \rangle}{T} \to 0. \tag{3.5}$$

As usual, let $W_\theta(x_1, ..., x_T)$ be the wealth of a (horizon-free) universal strategy, e.g. that of Cover (1991). We have

$$\frac{log\, D(x_1, ..., x_T) - log\, W_\theta(x_1, ..., x_T)}{T} + \frac{log\, W_\theta(x_1, ..., x_T) - \sum_{t=1}^{T} log\, \langle c^*(F), x_t \rangle}{T} \tag{3.6}$$

By the universality of $\theta(\cdot)$, the term on the left converges to 0 (everywhere). According to *Breiman's theorem*, no non-anticipating trading strategy θ can asymptotically dominate the growth rate of the Kelly rule. That is, the lim sup of the term on the right is ≤ 0. Thus, $\underset{T\to\infty}{limsup}\, \mathcal{S}(x_1, ..., x_T) = 0$. On account of the fact that $\mathcal{S}(x_1, ..., x_T) \geq 0$, we get $\underset{T\to\infty}{liminf}\, \mathcal{S}(x_1, ..., x_T) \geq 0$ This proves that $\mathcal{S}(x^T) \to 0$ almost surely.

3.3 Variational approach to universal horse race gambling

Every universal portfolio algorithm, including the sequential minimax universal portfolio of this chapter, gives a universal gambling scheme for horse races when we force nature to choose axis vectors $x_t = \mathcal{O}_{tj} e_j$, or e_j after normalization. The empirical Bayes portfolio (Cover 1986) and the horizon-free portfolios (Cover 1991, 1996) specialize to different gambling schemes. Cover's (1998) Max-Min portfolio and the sequential minimax portfolio specialize to the same gambling scheme, which is the sharpest possible for a fixed number of races T. The sequential minimax strategy only rears its head when we step outside of the horse race markets.

In the concrete gambling situation a direct variational method can be substituted for the backward induction. The method amounts to nothing more than Shtarkov's (1987) universal source code, re-interpreted for this new context. It was (presumably) just this method that led cover to formulate his (1998) Max-Min portfolio in terms of extremal strategies.

Let $\theta_k(j^{t-1})$ be the proportion of one's wealth to bet on horse k in race t, assuming the current win history is $j^{t-1} = (j_1, ..., j_{t-1})$. As usual, let $W_\theta(j_1, ..., j_T)$ be the induced final wealth function. According to principles discussed above, it is convenient to normalize $\mathcal{O}_{tj} \equiv 1$. These are sub-fair odds: a \$1 bet on the winning horse gets you your dollar back. Any money bet on the other horses is lost. Again, this normalization is possible because we are concerned with optimizing relative, not absolute performance.

We are already aware of the general fact that θ is completely characterized by W_θ. One has, for all j^T:

$$\theta_{j_1}(j^0)\theta_{j_2}(j^1)\cdots\theta_{j_T}(j^{T-1}) = W_\theta(j^T), \tag{3.7}$$

where $j^1 \subset j^2 \subset \cdots \subset j^{T-1}$ are sub-histories of j^T. From this we recover

$$\theta_k(j^t) = \frac{\sum\limits_{j_{t+2},\ldots,j_T} W_\theta(j^t, k, j_{t+2}, ..., j_T)}{\sum\limits_{j_{t+1},\ldots,j_T} W_\theta(j^t, j_{t+1}, ..., j_T)}. \tag{3.8}$$

Of course this is old hat at this stage. Thus, we confine ourself to selecting final wealth functions that satisfy $\sum\limits_{j_1,\ldots,j_T} W(j^T) = 1$. In this context, a gambling strategy is merely vector of numbers $(W(j^T))_{j^T\in\{1,\ldots,m\}^T}$ in the m^T-dimensional simplex.

We solve the problem

$$\underset{W(\cdot)\in\Delta}{Max}\ \underset{j^T}{Min}\ \frac{W(j^T)}{D(j^T)} \tag{3.9}$$

The issue of maximizing this piecewise linear function over the simplex is easily resolved. Looking past the notation and high number of dimensions, it is just a Leontiev (perfect complements) demand problem. Accordingly, we pick the vector W so as to equalize all the numbers:

$$\frac{W(j^T)}{D(j^T)} = C \tag{3.10}$$

Multiplying through by $D(j^T)$ and summing over all j^T, we get $C = \frac{1}{\sum\limits_{j^T} D(j^T)}$ and $W(j^T) = \frac{D(j^T)}{\sum\limits_{j^T} D(j^T)}$.

The universal gambling scheme is therefore

$$\theta_k(j^t) = \frac{\sum\limits_{j_{t+2},\ldots,j_T} D(j^t, k, j_{t+1}, ..., j_T)}{\sum\limits_{j_{t+1},\ldots,j_T} D(j^t, j_{t+1}, ..., j_T)}, \tag{3.11}$$

where $D(j^T)$ is the final wealth of the hindsight-optimized rebalancing rule corresponding to the win history j^T. I give simplified formulas for the (unnormalized) k^{th} numerator $\theta_k(j^T)$. Then one just normalizes the vector of numerators to the simplex.

Let $(N_1, ..., N_m)$ be the type of j^t, e.g. horse 1 has won N_1 races so far, horse 2 has won N_2 races, and so on. Let $(n_1, ..., n_m)$ denote the type of the continuation history $(j_{t+1}, ..., j_T)$, e.g. horse k wins n_k races total in periods $t+1, t+2, ..., T$. Then the numerator of the portfolio weight on horse k after j^t is $\sum_{n_1+\cdots n_m=T-t-1} \binom{T-t-1}{n_1 \cdots n_m} \times \left(\frac{N_1+n_1}{T}\right)^{N_1+n_1} \cdots \left(\frac{N_k+n_k+1}{T}\right)^{N_k+n_k+1} \cdots \left(\frac{N_m+n_m}{T}\right)^{N_m+n_m}$ Just to avoid confusion, the long product on the right is equal to

$$\left(\frac{N_k+n_k+1}{T}\right)^{N_k+n_k+1} \prod_{j\neq k} \left(\frac{N_j+n_j}{T}\right)^{N_j+n_j} \tag{3.12}$$

This accounts for the special role of index k in calculating the k^{th} numerator.

Example 20. *For $m = 2$ stocks, suppose that win history j^t has been observed after t races. Suppose that horse 1 has won N_1 races, and horse 2 has won N_2 races so far. Then the numerator of the portfolio weight on stock 1 is*

$$\sum_{j=0}^{T-t-1} \binom{T-t-1}{j} \left(\frac{N_1+j+1}{T}\right)^{N_1+j+1} \left(\frac{T-N_1-j-1}{T}\right)^{T-N_1-j-1} \tag{3.13}$$

The numerator of the portfolio weight on stock 2 is

$$\sum_{j=0}^{T-t-1} \binom{T-t-1}{j} \left(\frac{N_1+j}{T}\right)^{N_1+j} \left(\frac{T-N_1-j}{T}\right)^{T-N_1-j} \tag{3.14}$$

The portfolio of bets to use is then $\frac{(Numerator_1, Numerator_2)}{Numerator_1+Numerator_2}$.

As an application of these formulas, I calculate a few portfolio weights under the assumption that $t = 10$ and $T = 100$. The universal gambler is systematically more conservative than somebody who just bets the empirical distribution of wins as known after 10 races. However, the universal scheme gets more aggressive as it accumulates data.

N_1	N_2	**Proportion of wealth to bet on horse 1** ($T = 100$)
1	9	13.5%
2	8	22.7%
3	7	31.8%
4	6	40.9%
5	5	50%
2	18	11.9%
4	16	21.4%
6	14	30.9%
8	12	40.4%
10	10	50%

Table 3.1: **Fortune favors the paranoid: how to bet after observing N_1 wins for horse 1 and N_2 wins for horse 2**

3.4 Orderly computation of the universal bets

In general, we must compute values of the function

$$f(N_1, ..., N_m; H) = \sum_{n_1+\cdots+n_m=H} \binom{H}{n_1 \cdots n_m} \left(\frac{N_1+n_1}{T}\right)^{N_1+n_1} \cdots \left(\frac{N_m+n_m}{T}\right)^{N_m+n_m}, \tag{3.15}$$

where H stands for "horizon." The formula for the (unnormalized) numerator of the portfolio weight on stock k is now expressed as $f(N_1, ..., N_{k-1}, N_k+1, N_{k+1}, ..., N_m; T-t-1)$. We can use the recurrence

$$f(N_1, ..., N_m; H) = \sum_{j=0}^{H} \binom{H}{j} \left(\frac{N_m+j}{T}\right)^{N_m+j} f(N_1, ..., N_{m-1}; H-j). \tag{3.16}$$

At each recursive step, the number of horses decreases by 1. We have the boundary conditions $f(N_1; H) = \left(\frac{N_1+H}{T}\right)^{N_1+H}$ and $f(N_1, ..., N_m; 0) = \left(\frac{N_1}{T}\right)^{N_1} \cdots \left(\frac{N_m}{T}\right)^{N_m}$. One should also exploit the fact that $f(N_1, ..., N_m; H)$ is a symmetric function of the variables $N_1, ..., N_m$. In the recursive function calls, every time a value $f(N_1, ..., N_m; H)$ is calculated it should be stored for future reference. This number only needs to be calculated once; every other request for $f(N_1, ..., N_m; H)$ in the function call tree should just look up the answer from memory. By symmetry, we need only ever calculate the function for values $N_1, ..., N_m$ for which $N_1 \geq N_2 \geq \cdots \geq N_m$. Whenever $f(\cdot)$ is called recursively, the input values should be re-ordered so that this condition is satisfied.

Example 21. *Assume* $m = 4$ *horses and* $T = 100$ *races. Assume that after* $t = 10$ *races the win profile is* $(N_1, N_2, N_3, N_4) = (5, 2, 2, 1)$*. Then the optimal (sequential minimax) portfolio of bets for race 11 is* $(45.9\%, 20.8\%, 20.8\%, 12.5\%)$*. After 20 races, if the win profile is* $(10, 4, 4, 2) = 2 \cdot (5, 2, 2, 1)$*, then the optimal portfolio is* $(47.7\%, 20.5\%, 20.5\%, 11.3\%)$*. Thus, having twice the data, the universal gambler's portfolio has moved closer into line with the empirical win distribution as known after 20 races.*

3.5 Comparison of the sequential minimax betting scheme with horizon-free schemes

3.5.1 Dirichlet-Weighted Schemes

Having observed win history $j^t = (j_1, ..., j_t)$ of type $(n_1, ...n_m)$, the uniformly weighted horizon-free universal portfolio has the numerator of $\theta_k(j^t)$ equal to $\int_{c_1+\cdots+c_{m-1}\leq 1} c_1^{n_1} \cdots c_{k-1}^{n_{k-1}} c_k^{n_k+1} c_{k+1}^{n_{k+1}} \cdots \left(1 - \sum_{j=1}^{m-1} c_j\right)^{n_m} dc_1 \cdots dc_{m-1} = \frac{n_1! \cdots n_{k-1}!(n_k+1)!n_{k+1}! \cdots n_m!}{(t+m)!}$. After simplification, one has

$$\theta_k(n_1, ..., n_m) = \frac{n_k + 1}{t + m} \tag{3.17}$$

Similarly, the $(1/2, ..., 1/2)$-Dirichlet weighted horizon-free universal portfolio makes the bets

$$\theta_k(n_1, ..., n_m) = \frac{n_k + \frac{1}{2}}{t + \frac{m}{2}}, \tag{3.18}$$

In general, the $(\alpha_1, ..., \alpha_m)$-Dirichlet weighted portfolio uses

$$\theta_k(n_1, ..., n_m) = \frac{n_k + \alpha_k}{t + \sum_{l=1}^{m} \alpha_l} \tag{3.19}$$

where t is the number of races completed. This is Laplace's generalized rule of succession. The practical performance of these strategies is illustrated below. Remember, of course, that although the horizon-free strategies may outperform the sequential minimax strategy in "practical" examples, their worst-case performance is inferior to that of the sequential minimax strategy.

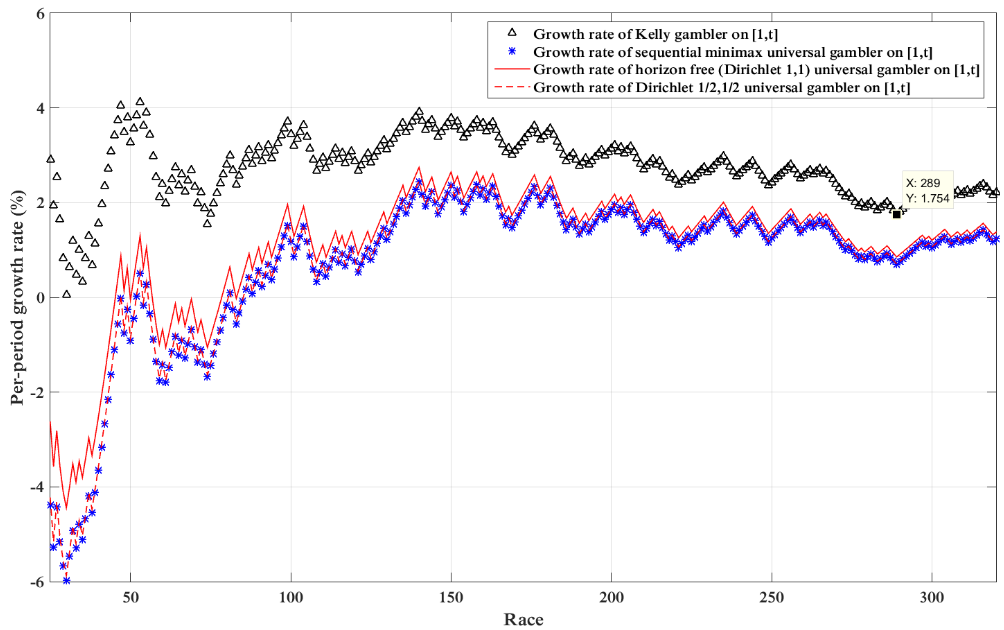

Figure 3.2: **Performance of the horizon-free strategies under 320 simulated iid horse races with** $\mathcal{O} = (2, 2)$ **and probabilities** $(0.4, 0.6)$**.**

The slight outperformance of the uniformly weighted (Dirichlet 1,1) scheme is typical. The Dirichlet 1/2,1/2 scheme tracks the fixed-horizon strategy rather closely, and has better worst case performance than the uniformly weighted betting system.

Example 22. *Suppose the gambler has an a priori belief that rebalancing rules near the centroid of the simplex will perform better. He decides to distribute his dollar among the rebalancing rules* $\mathcal{C}$ *according to a density*

$$d(c_1, ..., c_{m-1}) = \frac{Min\left(\underset{1 \le j \le m-1}{Min} c_j, 1 - \sum_{j=1}^{m-1} c_j\right)}{\int_{b_1 + \cdots + b_{m-1} \le 1} Min\left(\underset{1 \le j \le m-1}{Min} b_j, 1 - \sum_{j=1}^{m-1} b_j\right)} \tag{3.20}$$

dollars per unit of volume. He then "lets it ride," e.g. rebalancing rule c gets to manage *this money forever. In the case of two stocks this amounts to*

$$d(\lambda) = \frac{Min(\lambda, 1-\lambda)}{\int_0^1 Min(\lambda, 1-\lambda)d\lambda} = 4 \cdot Min(\lambda, 1-\lambda). \tag{3.21}$$

The on-line proportion of wealth to bet on stock 1 *after observing win counts* (n_1, n_2) *is*

$$\frac{\int_0^1 \lambda^{n_1+1}(1-\lambda)^{n_2} Min(\lambda, 1-\lambda)d\lambda}{\int_0^1 \lambda^{n_1}(1-\lambda)^{n_2} Min(\lambda, 1-\lambda)d\lambda} \tag{3.22}$$

Empirical Bayes Scheme

Cover (1986) gives an "empirical Bayes stock portfolio" that is universal for stock markets with finite support $x_t \in \mathcal{X}$. We can specialize this model to the Kelly horse race by assuming that $\mathcal{X} = \{\mathcal{O}_1 e_1, ..., \mathcal{O}_m e_m\}$, where $\mathcal{O}_j$ are the odds on horse j. The algorithm keeps track of the empirical win distribution $p_t = (\frac{n_{t1}}{T}, ..., \frac{n_{tm}}{T})$ and the sample average of the per race growth rates $\overline{G}_t = \frac{\sum_{s=1}^{t} log(\mathcal{O}_{j_s}\theta_{j_s}(j_1,...,j_{s-1}))}{t}$. Here n_{tj} is the number of races horse j has won after t races have been completed. Given the current empirical win distribution p, the best available average growth rate in hindsight over $1 \leq s \leq t$ is given by $B(p) = \sum_{j=1}^{m} p_j log(p_j \mathcal{O}_j)$, which would have been the result of betting the empirical frequencies $p_j = \frac{n_j}{t}$. The typical situation for 2 horses is illustrated in the figure below. The curve $\{(p, B(p)) : p \in \Delta\}$ is called the *Bayes envelope*. Following *Blackwell*, the objective of the algorithm is to force, in repeated plays, the convergence of the empirical payoff vector $(p_t, \overline{G}_t)$ to the Bayes envelope. This is achieved by finding the point $(q^*, B(q^*))$ on the Bayes envelope that is nearest to $(p_t, \overline{G}_t)$ in the Euclidean sense. On the next play of the game, the algorithm uses the portfolio of bets $\theta(p_t, \overline{G}_t) = q^*$, where

$$q^* = \underset{q\in\Delta}{argmin}\ ||q-p||^2 + (B(q) - \overline{G}_t)^2 \tag{3.23}$$

$$= \underset{q\in\Delta}{argmin} \sum_{j=1}^{m} \left(q_j - \frac{n_{tj}}{t}\right)^2 + \left(\sum_{j=1}^{m} q_j log(q_j \mathcal{O}_j) - \overline{G}_t\right)^2 \tag{3.24}$$

Intuitively, the purpose of this convex program is to provide a robust adjustment q^* to the currently known vector of empirical win frequencies p_t.

Example 23. *Given fixed odds* $\mathcal{O} = (2, 2)$*, if after* t *plays, the observed empirical frequency of wins for horse 1 is* 80%*, and assuming that the gambler has achieved an* *average per-race growth rate of* 0.1%*, the empirical Bayes gambler bets about* 76% *of his wealth on horse 1 in race* $t+1$*. Thus, there is a certain "disbelief" of the empirical win frequency, a disbelief that weakens as the gambler's performance approaches the Bayes envelope.*

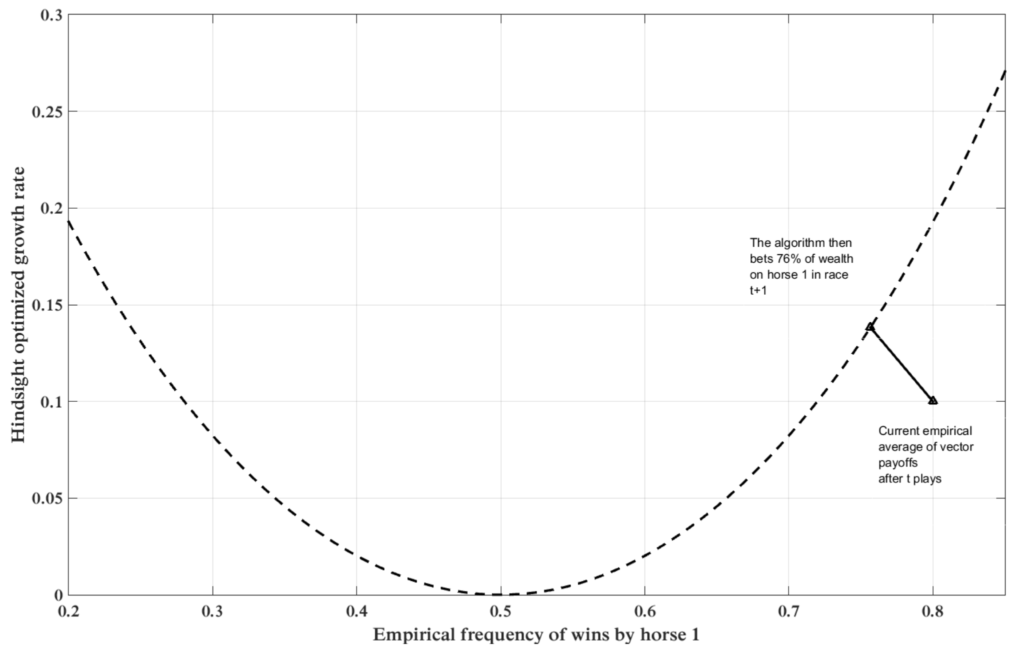

Figure 3.3: **Situation faced by an Empirical Bayes gambler who has measured an empirical win frequency of** 80% **for horse 1 and has grown his wealth at a rate of** 0.1% **per race**

The manner in which the empirical Bayes gambler is able to force his way into the Bayes envelope is illustrated below for 1000 iid races drawn from $p = (0.4, 0.6)$ and $\mathcal{O} = (2, 2)$.

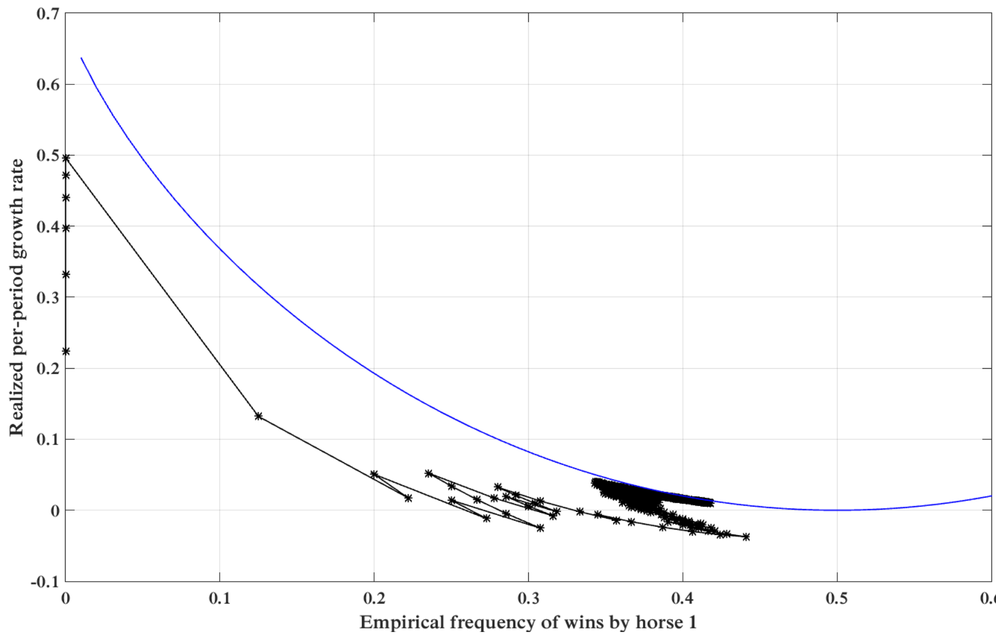

Figure 3.4: **Illustration of Blackwell's approachability theory applied to the Kelly horse race. The gambler is able to force convergence to the point** $(40\%, 2\%)$ **on the Bayes envelope.**

Example 24. *Consider the situation with 3 horses running iid races with true, unknown win probabilities* $(0.5, 0.3, 0.2)$ *and fixed odds* $\mathcal{O} = (3, 3, 3)$*. The Kelly growth rate is* 6.9%*. A gambler using the empirical Bayes scheme is able to force the sample average* $(\frac{n_1}{T}, \frac{n_2}{T}, \overline{G}_t)$ *to converge to the point* $(0.5, 0.3, 0.069)$ *on the Bayes envelope almost surely. A sample path for 1000 such races is illustrated below.*

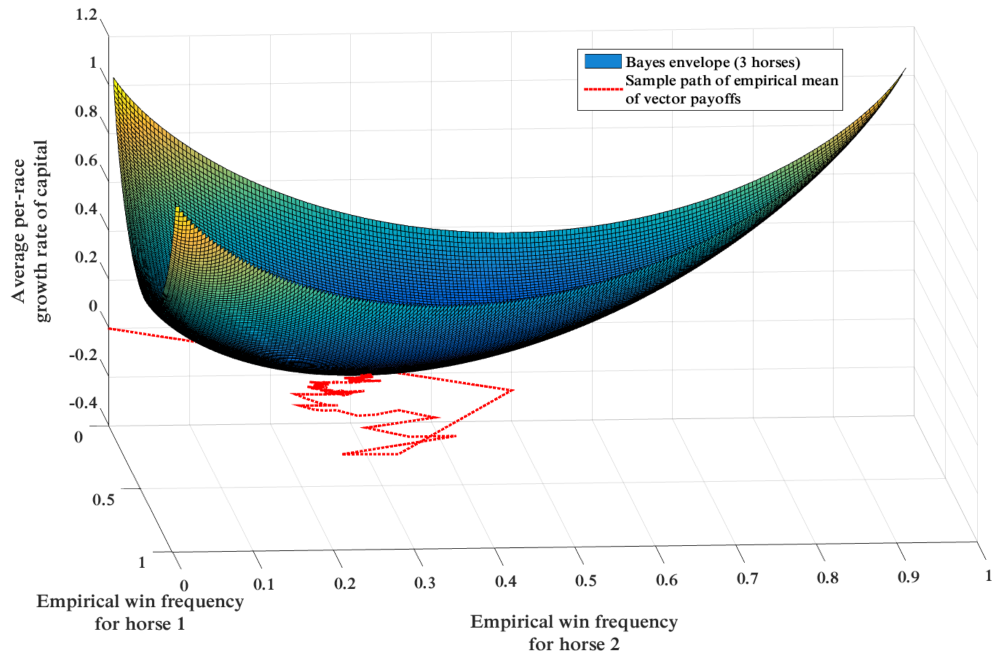

Figure 3.5: **Approaching the Bayes envelope in a sample path of 1000 races of 3 horses**

3.6 A sequential minimax improvement to the Cover/Blackwell method

The Blackwell/Cover method of projecting the current sample average $(p_t, \overline{G}_t)$ onto the Bayes envelope, and betting the "corrected" beliefs p^* is simple and elegant, but it is not strictly the best possible step that can be made toward the Bayes envelope $\mathcal{E}$. One naturally is led to consider the question: given the position $(p_t, \overline{G}_t)$ after t plays, what is the smallest numerical value of $d\big(\mathcal{E}, (p_{t+1}, \overline{G}_t)\big)$ we can guarantee to achieve after the next play? The most perspicacious strategy would take into account the time (after all it is the denominator of the sample average) and also the fact this problem will repeat itself several times, e.g. it is important not only to make progress now, but to set ourselves up well to make good progress in the future. Given an ultimate horizon of T plays, what is the smallest distance to the Bayes envelope that we can guarantee? Let $V_t(p, \overline{G})$ denote the lowest final distance to $\mathcal{E}$ we can guarantee to achieve after T

plays, given that the position after t plays is $(p, \overline{G})$. Note that p can only take one of a finite number of values after t plays, namely $(\frac{n_1}{t}, ..., \frac{n_m}{t})$, where the n_k are nonnegative integers summing to t. Let $b = (b_1, ..., b_m)$ be the portfolio of bets chosen by the gambler, and let j denote the index of the horse that wins the $t+1^{st}$ race. We then have the Bellman equation

$$V_t(p, \overline{G}) = \underset{b \in \Delta}{Min} \underset{1 \leq j \leq m}{Max} V_{t+1}\left(\frac{tp + e_j}{t+1}, \frac{t\overline{G} + log(b_j \mathcal{O}_j)}{t+1}\right) \tag{3.25}$$

and the boundary condition

$$V_T(p, \overline{G}) = d(\mathcal{E}, (p, \overline{G})) = \underset{q \in \Delta}{Min} \sqrt{||q - p||_2^2 + \left(\sum_{j=1}^{m} q_j log(q_j \mathcal{O}_j) - \overline{G}\right)^2}. \tag{3.26}$$

Appendix A

Glossary of concepts and notation

- **Bayes envelope of a Kelly horse race:** Given that a gambler has observed (after some indeterminate number of races) *empirical* win frequencies $p = (p_1, ..., p_m)$, where p_j is the proportion of races won so far by horse j, the highest per-period continuously compounded growth rate available to fixed-fraction betting schemes in hindsight is $B(p) = \sum_{j=1}^{m} p_j log(p_j \mathcal{O}_j)$, where $\mathcal{O}_j$ are the (gross) odds paid for a \$1 bet on horse j. The (convex) epigraph $\{(p, G) : G \geq B(p)\}$ is called the Bayes envelope. The gambler's objective is to force convergence of the empirical average payoff $(p, \overline{G})$ to the envelope, where $\overline{G}$ is the gambler's realized average per-period growth rate.

- **Best rebalancing rule in hindsight**: at T, having observed the realized return vectors $x_1, ..., x_T$, the rebalancing rule $c^*(x_1, ..., x_T)$ that would have yielded the greatest final wealth $W^*(x_1, ..., x_T) = \underset{c \in \Delta}{Max} \langle c, x_1 \rangle \cdots \langle c, x_T \rangle$. $W^*(\cdot)$ is a convex function (separately) of each vector argument x_t. Note that *a fortiori*, the best rebalancing rule in hindsight makes more money in any finite sample than the Kelly rule $c^*(F)$.

- **Capitalization-weighted index**: an index (like the S&P 500) that corresponds to the (time-varying) portfolio weights $c_j = \frac{CAP_{tj}}{\sum_{k=1}^{m} CAP_{tk}}$, where CAP_{tj} is the aggregate market value of firm j at the close of session t. In spite of the fact that these weights are always changing, the performance of a capitalization-weighted index is replicated by purchasing an initial "market portfolio," and "letting it ride" forever. The numerical value of the index after t sessions is $\lambda(CAP_{t1}+\cdots+CAP_{tm})$, where λ is just a constant chosen to keep the index level within a "nice" interval of numbers.

- **Conditional super-hedge**: conditional on the fact that the realizations $x_1, ..., x_t$ are already "baked in" to the return history, a conditional super-hedge for $D(\cdot)$ is a trading strategy that constitutes a super hedge for the restricted derivative $\delta(x_{t+1}, ..., x_T) = D(x^t, x_{t+1}, ..., x_T)$ in the continuation game that occurs for periods $t+1, t+2, ..., T$.

- **Conditional superhedging price**: the cost $p(x_1, ..., x_t)$ of the cheapest conditional superhedge for D, given that $x_1, ..., x_t$ have already occured. The corresponding superhedging strategy $\theta(\cdot)$ is initiated in period $t+1$, with initial deposit $p(x^t)$.

- **Equal weight index**: the final wealth that accrues from the using the rebalancing rule $\theta(x^t) \equiv (1/m, ..., 1/m)$. An example of this is the Guggenheim/Rydex Equal Weight S&P 500 ETF (ticker RSP) which is rebalanced quarterly. Empirically speaking, over long periods, equal weight rebalancing achieves higher growth rates and higher Sharpe and Sortino ratios than their "let it ride" counterparts.

- **Extremal strategy**: a trading strategy that puts all its wealth into some stock j_1 in period 1, then puts all the proceeds into stock j_2 in period 2, *et cetera*, then puts all the proceeds into stock j_T in period T. The general extremal strategy is characterized by a tuple $(j_1, ..., j_T) \in \{1, ..., m\}^T$. If x_{tj} is the gross return of stock j in period t, then the gross return on a \$1 investment in the $(j_1, ..., j_T)$-extremal strategy is $x_{1j_1}x_{2j_2}\cdots x_{Tj_T}$. For example, the extremal strategy $(1, 2, 1, 2, ..., 1, 2)$ puts all its wealth into stock 1 in odd periods, and bets it all on stock 2 in even periods. By contrast, the extremal strategy $(1, ..., 1)$ corresponds to buying and holding stock 1. The final wealth of every multilinear trading strategy is some convex combination of the payoffs of the m^T extremal strategies.

- **Final wealth function**: the function $W_\theta(x_1, ..., x_T)$ that gives the final wealth achieved by the self-financing strategy θ on the path $x_1, ..., x_T$. If η and θ are trading strategies such that $W_\eta = W_\theta$, then $\eta = \theta$.

- **Fixed-horizon (or horizon T) trading strategy**: a strategy $\theta(\cdot)$ that depends in an essential way on a definite investment horizon T. Cover (1998) is a horizon-T strategy, for example, but Cover (1991) is not. Fixed-horizon strategies have the advantage of (possibly) being optimized for a particular horizon T, but have the disadvantage of forcing the practitioner to commit to a course of action for a large, fixed number of periods into the future. Somebody who is concerned about his ability to stick to such a long range plan should use a horizon-free strategy.

- **Horizon-free trading strategy**: a strategy $\theta(\cdot)$ that is calculated purely from the available return data $x_1, ..., x_t$, without reference to any particular investment horizon T. A good horizon-free strategy will not be optimal for any specific horizon T, but will perform well for all horizons.

- **Kelly growth rate**: the greatest possible asymptotic growth rate $\rho^*(F)$ achievable against return vectors x_t drawn *iid* from cdf $F(\cdot)$. This achievement happens, simply enough, by acting each period so as to maximize the expected log of one's capital.

- **Kelly horse race**: there are m horses that run T races sequentially, $1 \leq t \leq T$. Before race t starts, a bookie posts odds $\mathcal{O}_t = (\mathcal{O}_{t1}, ..., \mathcal{O}_{tm})$, where $\mathcal{O}_{tj}$ are the odds on horse j. This means that a \$1 bet on horse j in race t will have a gross payoff of $\mathcal{O}_{jt}$ dollars. Let j_t denote the winner of race t. Then the gross-return vector of this "horse race market" in period t is $\mathcal{O}_{tj_t} e_{j_t}$, where e_{j_t} is the unit basis vector with a 1 in the j_t^{th} position. The stock market analog of this realization has the following interpretation: all the stocks except j_t collapsed in price to ϵ, and the price of stock j_t was unchanged.

- **Kelly rule (or log-optimal portfolio)**: if the x_t are distributed *iid* with cdf F, then the log-optimal portfolio $c^*(F)$ is that which maximizes the expected, continuously compounded per-period growth rate of capital: $\underset{c \in \Delta}{argmax}\ E_F[log\,\langle c, x_t \rangle]$.

- **Kullback-Leibler divergence** $D(p||\mathcal{B})$: in the Kelly horse race, let $p = (p_1, ..., p_m)$

be the probabilities that the various horses win a given race. The bookie, unaware of p, has his own beliefs $\mathcal{B} = (\mathcal{B}_1, ..., \mathcal{B}_m) \in \Delta$, which are implicit in the posted odds. Then $D(p||\mathcal{B})$ is the maximum possible asymptotic per-period capital growth rate that is achievable by a gambler who knows p (or uses a universal betting strategy). We have $D(p||\mathcal{B}) = \sum_{j=1}^{m} p_j \, log \, \frac{p_j}{\mathcal{B}_j}$. In general $D(p||\mathcal{B}) \geq 0$, and $= 0$ if and only if $p = \mathcal{B}$.

- **"Let it ride"**: the act of depositing some proportion of one's wealth into a strategy, asset, or portfolio, and never rebalancing. For example, if one puts half of his wealth into stocks and half in bonds, and lets it ride, then 30 years later he will in all probability have much more than half of his wealth invested in stocks.

- **"Multi-convex" derivative**: A derivative $D(x_1, ..., x_T)$ that is convex separately in each x_t. That is, $D(\lambda x_t + (1-\lambda)y_t, x_{-t}) \leq \lambda D(x_t, x_{-t}) + (1-\lambda)D(y_t, x_{-t})$. If D is also homogeneous separately in each x_t, then D can be majorized by a multilinear derivative:

$$D(x_1, ..., x_T) \leq \sum_{j_1, ..., j_T} D(e_{j_1}, ..., e_{j_T}) x_{1j_1} \cdots x_{Tj_T} \tag{A.1}$$

  If D is (jointly) convex in the variable $\xi = (x_1, ..., x_T)$ then it is multi-convex, but not conversely.

- **Multilinear hedging**: the act of interpolating a derivative $D(x_1, ..., x_T)$ by the unique multilinear form that passes through the m^T points $(e_{j_1}, ..., e_{j_T})$, where the e_{j_t} are unit basis vectors. A trader then uses the unique replicating strategy corresponding to the interpolating form.

- **Multilinear trading strategy**: a self-financing trading strategy $\theta(\cdot)$, and an initial deposit p, that perfectly replicates the final payoff of a multilinear derivative $D(x_1, ..., x_T)$. Every positive multilinear form corresponds to a unique replicating strategy.

- **Perfect hedge**: a self-financing trading strategy $\theta(\cdot)$ and an initial deposit p into the strategy, that guarantees to generate cash flows at T that are precisely equal to the payoff of some derivative $D(x_1, ..., x_T)$, regardless of the sample path $x_1, ..., x_T$.

If $D(\cdot)$ is perfectly replicable, then the (p, θ) are unique. If the derivative sells for some price other than p at $t = 0$, an arbitrage opportunity arises.

- **Price-weighted index**: an index (like the Dow-Jones) that corresponds to the (time-varying) portfolio weights $c_j = \frac{S_{tj}}{\sum_{k=1}^{m} S_{tk}}$, where S_{tj} is the price of stock j at the close of session t. In spite of the fact that these weights are always changing, the performance of a price-weighted index is replicated by purchasing a single share of each stock, and "letting it ride" forever. The numerical value of the index after t sessions is $\lambda(S_{t1} + \cdots + S_{tm})$, where λ is just a constant chosen to keep the index level within a "nice" interval of numbers.

- **Rebalancing rule**: a trading strategy that maintains constant proportions $\theta(x^t) \equiv c = (c_1, ..., c_m)$ of wealth in each stock j. This requires trading every period: at the start of each session t, the trader adjusts his portfolio so as to maintain the constant proportions c_j. Then the stocks move, and by the end of the session the trader no longer has exactly c_j of his wealth in each stock j. The portfolio must be rebalanced at the start of the next trading session, and so on. By contrast, a buy-and-hold strategy never trades, and so has a fluctuating proportion of wealth in each stock.

- **Self-financing trading strategy**: a function that prescribes the the proportion of wealth bet on each stock j in session $t{+}1$ after having observed the return history $x_1, ..., x_t$. The portfolio vector is denoted $\theta(x^t) = (\theta_1(x^t), \theta_2(x^t), ..., \theta_m(x^t))$, and θ is a mapping $\theta : \cup_{t=1}^{\infty}(\mathcal{R}_+^m)^t \to \Delta$. θ must also select a portfolio to use in $t = 1$ (when there is no data available), and this vector is denoted $\theta(h^0)$, where h^0 is the empty history.

- **Shannon's Demon**: In a famous lecture at MIT (attended by Samuelson), Shannon considered a stock whose price S_t, each period, either doubles or gets cut in half with 50% probability. Although $E[S_t] = S_0 \cdot 1.25^t \to \infty$, we have $E[log\, S_t] = log\, S_0$. Thus, the asymptotic per-period growth rate of the buy-and-hold investor is 0. A Kelly gambler, by contrast, uses the $50 - 50$ rebalancing rule and grows his wealth at an asymptotic rate of 6% per period. This is in spite of taking half the risk of the buy-and-hold investor. Note the amusing "buy low and sell high" tactics being employed by the Kelly bettor. Assuming the Kelly gambler starts with \$1, his expected wealth after t periods is 1.125^t, but his expected log-wealth is $0.06t$.

- **Shtarkov's Bound**: an upper bound on the superhedging cost $p(T, m)$ that corresponds to the final wealth of the best rebalancing rule in hindsight, assuming T periods and m stocks. The bound is very accurate, and simple in that it is an $m - 1^{st}$ degree polynomial in the variable $\sqrt{T}$. Shtarkov studied the number $p(T, m)$ in connection with his *universal source code* in information theory. The code is able to compress a stream of binary data to its Shannon (entropy) limit, in spite of the fact that it starts with no prior distributional information about the $0 - 1$ process. The Shannon limit is thus "universally achievable," and so too is the Kelly growth rate.

- **Super-hedge**: a self-financing trading strategy $\theta(\cdot)$ and an initial deposit p into the strategy, that guarantees to generate cash flows at T that are $\geq$ to the payoff of some derivative $D(x_1, ..., x_T)$, regardless of the sample path $x_1, ..., x_T$. If the price of the derivative were to exceed p, one could short the derivative and use the proceeds to buy a super-hedge, guaranteeing a riskless profit at T.

- **Superhedging price for a derivative** D: the smallest amount of money p for which a self-financing trading strategy θ exists such that (p, θ) is a super hedge for D. This is the sharpest upper bound on the price of D that can be given without specifying the distribution from which the path $(x_1, ..., x_T)$ is drawn.

- **Symmetric derivative**: a derivative $D(x_1, ..., x_T)$ whose final payoff is unchanged if the input vectors x_t are re-ordered. That is, $D(x_{\sigma(1)}, ..., x_{\sigma(T)}) = D(x_1, ..., x_T)$ for any permutation σ of the indices $1, ..., T$.

- **Symmetric trading strategy**: a trading strategy θ whose induced final wealth function $W_\theta(x_1, ..., x_T)$ depends only on the numerical values of the x_t, and not their order. For example, a symmetric trading strategy would not have been "tricked" by the ordering of the returns before, during, and after the crash of 1929.

- **Type class of** $j^T = (j_1, ..., j_T)$: let $n_k = n_k(j^T)$ be the number of k's that appear among the indices $j_1, ..., j_T \in \{1, ..., m\}$. If j^T is regarded as a horse race history, then, n_k is the number of races won by horse k. The numbers $(n_1, ..., n_m)$ constitute the *type* of j^T. Regarding two horse-race histories as *equivalent* if they have the same type, the set $\{1, ..., m\}^T$ gets decomposed into *type classes*. The type classes correspond to solutions of the equation $n_1 + \cdots + n_m = T$ in nonnegative integers. There are $\binom{T+m-1}{m-1} = \mathcal{O}(T^{m-1})$ possible types. The type class $(n_1, ..., n_m)$ contains $\binom{T}{n_1\, n_2 \cdots n_m}$ sample paths j^T. If $f(j^T)$ is a function that is to be summed over the sample paths, then we have the decomposition

$$\sum_{j^T} f(j^T) = \sum_{n_1+\cdots+n_m=T} \quad \sum_{j^T:type(j^T)=(n_1,...,n_m)} f(j^T). \tag{A.2}$$

- **Universal gambling scheme**: a strategy for sequential gambling on horse races that guarantees to compound its money at the same asymptotic rate as the best fixed-fraction betting system in hindsight. Of necessity, any universal portfolio will particularize to a universal gambling scheme.

- **Universal portfolio**: a non-anticipating trading strategy $\theta(\cdot)$ that compounds its money at the same asymptotic per-period rate as the best rebalancing rule in hindsight. Against *iid* returns drawn from a CDF $F(\cdot)$, a universal strategy compounds its money at the same asymptotic rate as the Kelly rebalancing rule $c^*(F)$.

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