The Non-Split Bessel Model on GSp(4) as an Iwahori-Hecke Algebra Module

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Dedication

To the memory of Dr. Margaret Murphy Prullage and Dr. D. L. Prullage.

Abstract

We realize the non-split Bessel model of Novodvorsky and Piatetski-Shapiro in [1] as a generalized Gelfand-Graev representation of GSp(4), as defined by Kawanaka in [2]. Our primary goal is to calculate the values of Iwahori-fixed vectors of unramified principal series representations in the Bessel model. On the path to achieving this goal, we will first use Mackey theory to realize the Bessel functional as an integral - as a result, we will reestablish the uniqueness and existence of a Bessel model for principal series representations, originally proved in [1] and by Bump, Friedberg, and Furusawa, in [3], respectively.

Inspired by the work of Brubaker, Bump, and Friedberg in [4], our method of calculation takes advantage of the connection between the Iwahori-fixed vectors in the Bessel model and a certain linear character of the Hecke algebra of GSp(4). We will also provide a detailed description of the conjectural program, originally appearing in [5], connecting characters of the Hecke algebra for a more general reductive group G with multiplicity-free models of principal series representations. In particular, we will focus on the role played by the Springer correspondence in this program.

Additionally, using the formulas we develop for the Iwahori-fixed vectors, we provide an explicit alternator expression for the spherical vector in the Bessel model which matches previous results in [3].

Contents

Chapter 1

Introduction

In the context of a group G acting on a vector space V, the term "matrix coefficient" refers to a type of function on G. In general, a matrix coefficient is any linear combination of functions of the form $\langle f, g \cdot v\rangle$ where $g \in G$, $v \in V$, and $f \in V^*$, the dual space of V. In particular, if we are in the finite-dimensional setting, with a basis $\{e_1, \ldots, e_n\}$ of V, and a basis $\{e_1^*, \ldots, e_n^*\}$ of V^*, such that $\langle e_i^*, e_j\rangle = \delta_{ij}$, then the literal matrix coefficient $\langle e_i^*, g \cdot e_j\rangle$ is an example of a matrix coefficient. For example, the character $\chi_V : g \mapsto \mathrm{tr}((g_{ij}))$ of the representation V, where (g_{ij}) denotes the matrix of g with respect to these bases, is a powerful invariant of V.

Within the theory of automorphic forms, one important place where matrix coefficients appear is in the integral expressions used to construct automorphic L-functions. Two well-studied examples of matrix coefficients that have this property are the Whittaker function and the Macdonald spherical function. Additionally, both the Whittaker function and the Macdonald spherical function have been shown to have interesting connections to the fields of algebraic combinatorics (see [6], [7]) and mathematical physics (see [8], [9], [10]).[1] In this thesis, we will eventually explain a program for generating a family of matrix coefficients with applications to the theory of automorphic forms similar to those of Whittaker functions and the Macdonald spherical function, but first we will provide an extensive study of a particular type of matrix coefficient, the Bessel functions. In particular, we will be interested in the images of Bessel functions on the

[1] We note that the connections to physics arise primarily in the setting where the group G is a reductive group over the real or complex field.

universal principal series representation of an appropriate reductive group G - the universal principal series is a generalization of principal series representations of G. In the language of matrix coefficients, Bessel functions are formed by choosing a special vector in the representation space - such as a spherical vector, or, more generally, an Iwahori-fixed vector - and by choosing the Bessel functional as the element of the dual space.

In their 1973 paper [1], Piatetski-Shapiro and Novodvorsky defined the Bessel model for irreducible admissible representations of the group $G = \mathrm{GSp}(4)$ over a p-adic field, and showed that the dimension of such an embedding is at most 1. Since then, many deep connections have been established between Bessel models and automorphic forms. For the moment, let $\mathbb{A}$ denote the ring of adeles of a number field. In 1979, Novodvorsky connected the Bessel model of an irreducible automorphic representation of $\mathrm{GSp}(4, \mathbb{A})$ to the global integral representation of a degree eight L-function for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ in [11]. Subsequent contributions in this area were made by Piatetski-Shapiro and Soudry ([12]) and Bump, Friedberg, and Furusawa ([13],[3]). More recently, Roberts and Schmidt proved, in [14], that every irreducible, admissible representation of $\mathrm{GSp}(4)$ over a p-adic field of dimension greater than one admits a Bessel functional.

In this dissertation, we will recreate the results from [1] and [3] regarding the uniqueness and existence, respectively, of the non-split Bessel model specifically for an unramified principal series representation using Mackey theory, which will allow us to realize the corresponding Bessel functional explicitly as an integral. This will, in turn, allow us to proceed to our ultimate goal of providing an explicit expression for the Iwahori-fixed vectors in the model. In particular, the formula that we develop for the spherical function agrees with the formula for the spherical function in the Bessel model on $\mathrm{SO}(5)$ established by Bump, Friedberg, and Furusawa in [3].

Along the way, we will describe how our construction of the Bessel functional fits into a conjectural program for connecting characters of the finite Hecke algebra of G with multiplicity-free models of principal series representations. This program, formulated by Brubaker, Bump, and Friedberg in [4], was motivated by the study of the Whittaker and spherical functionals, which it contains as special cases.

We will show momentarily that the most natural way to view this connection between these models and characters of the finite Hecke algebra is from the perspective of the "universal principal series." In this section, we will only provide the definitions needed to give an overview of our results; a more in-depth treatment of the universal principal series, following that provided by Haines, Kottwitz, and Prasad, in [15], will be given in Section 2.1.

Although our results in this paper will apply specifically to GSp(4) over a p-adic field, we expect them to generalize considerably, much as the examples of the Whittaker and spherical models do, and towards that end we will work in greater generality when possible. In particular, for the rest of this section, let G be a split, connected reductive group over a p-adic field F with ring of integers $\mathfrak{o}$ and uniformizer π. Let k denote the residue field $\mathfrak{o}/(\pi)$, and let q denote its cardinality. Let W denote the Weyl group of G. Let B be a Borel subgroup of G with maximal torus T and unipotent subgroup U such that $B = TU$. Let $\overline{U}$ denote the opposite unipotent of U in B. We assume that these subgroups, as well as G, are defined over $\mathfrak{o}$. Note that this means that $K = G(\mathfrak{o})$ is a maximal compact subgroup of G. Let J denote the Iwahori subgroup, which is the preimage of $B(k)$ under the canonical homomorphism $G(\mathfrak{o}) \to G(k)$. Let $\mathcal{H}$ denote the Iwahori-Hecke algebra of G, which is the $\mathbb{C}$-algebra of functions $C_c(J\backslash G/J)$, with multiplication given by convolution.

Our first definition of the universal principal series will be as the vector space $M := C_c(T(\mathfrak{o})U\backslash G/J)$, which we can regard as a right $\mathcal{H}$-module under convolution. It will also be useful to have the following alternate definition of the universal principal series:[2]

We begin by defining R to be the complex algebra $C_c(T/T(\mathfrak{o}))$. We will write π^μ to denote the element $\mu(\pi) \in T/T(\mathfrak{o})$ where μ is an element of the cocharacter group $X_*(T)$. We regard R as a left $(T/T(\mathfrak{o}))$-module via the inverse of the "universal" character $\chi_{\text{univ}} : \pi^\mu \mapsto \pi^\mu$. The universal principal series M is isomorphic to $\text{ind}_B^G(\chi_{\text{univ}}^{-1})^J$ as right $\mathcal{H}$-modules, where the representation $\text{ind}_B^G(\chi_{\text{univ}}^{-1})^J$ is composed of the J-fixed vectors of the representation $\text{ind}_B^G(\chi_{\text{univ}}^{-1})$ formed using normalized induction.

Recall that the spherical function ϕ° in $\text{ind}_B^G(\chi_{\text{univ}}^{-1})$ is the image of the characteristic function $1_{T(\mathfrak{o})UK} \in M$ under the isomorphism, which we will call η, mentioned above.

[2] We note that this definition gives more of an indication of the connection between the universal principal series and principal series representations - we will state this connection explicitly in Section 2.1.

Using the Iwahori-Bruhat decomposition, we see that we can decompose ϕ° as

$$\phi^\circ = \sum_{w \in W} \phi_w,$$

where $\phi_w := \eta(1_{T(\mathfrak{o})UwJ})$. In order to provide an explicit expression for the images of ϕ° and ϕ_w in the Bessel model, we are going to need to use the $\mathcal{H}$-structure of $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$ and its image in the model as *left* $\mathcal{H}$-modules. In order to define the appropriate $\mathcal{H}$-module structure on $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$, we first note that $M \cong \mathcal{H}$ as a free, rank one right $\mathcal{H}$-module. Using this isomorphism, along with η^{-1}, we can see that R embeds into $\mathcal{H}$. In fact, as vector spaces, $\mathcal{H} \cong R \otimes_{\mathbb{C}} \mathcal{H}_0$ where $\mathcal{H}_0 := C(J\backslash K/J)$ is the finite Hecke algebra. Eventually, we will show that the image of the universal principal series - and, hence, the images of these functions - in the Bessel model are contained in a submodule isomorphic to $V_\varepsilon := \mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon$, where ε is a certain linear character of $\mathcal{H}_0$. We will show that the Bessel model contains V_ε with multiplicity one; this fact will be crucial to our calculations.

We can see that $V_\varepsilon \cong R$ as vector spaces, so we can transfer the $\mathcal{H}$-action on V_ε to R via $v_\varepsilon \mapsto r$, where r is any element of R and v_ε is the eigenvector of $\mathcal{H}_0$ corresponding to ε. In fact, the model that V_ε appears in is dependent on the choice of normalization, and this also affects the value r in the map above. Roughly stated, a goal of Brubaker, Bump, and Friedberg is to find many examples where, if $\mathcal{L}$ is an R-valued map arising from a unique model, then there is a character ε of $\mathcal{H}_0$ and a subgroup $S \subset G$ such that the transformation properties of $\mathcal{L}$ under S imply that $\mathcal{L}$ is an $\mathcal{H}$-map from M to V_ε; a key idea here is that the models are connected to the representations of $\mathcal{H}_0$ via the Springer correspondence - we will discuss this connection further in Section 2.3.

We will see momentarily that, due to the structure of $\mathcal{H}_0$, there are not many possible choices of ε. First, we note that the set $\{T_s \mid s \text{ a simple reflection in } W\}$, where $T_s := 1_{JsJ}$, generates $\mathcal{H}_0$. This set of generators satisfies the same braid relations that the simple reflections in W satisfy, in addition to satisfying the quadratic relation

$$(T_s - q)(T_s + 1) = 0. \tag{1.0.1}$$

From (1.0.1) we see that the only possible eigenvalues for the generators of $\mathcal{H}_0$ are -1 and q. The braid relations for $\mathcal{H}_0$ then imply that we either have two or four linear characters of $\mathcal{H}_0$, depending on whether or not the Dynkin diagram for G is simply laced.

The simplest examples in this program are the Whittaker and spherical models. If we take $\mathcal{L}$ to be the R-valued spherical functional, uniquely determined up to scalar by the condition that $\mathcal{L}(\phi(gk)) = \mathcal{L}(\phi(g))$ for all $\phi \in \mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ and $k \in K$, and ε to be the trivial character on $\mathcal{H}_0$, then it was shown by Brubaker, Bump, and Friedberg (based on the work of Casselman in [16]) that $\mathcal{L}$ is an $\mathcal{H}$-intertwiner from M to V_ε; the analogous result was shown by Brubaker, Bump and Licata, where $\mathcal{L}$ is taken to be the R-valued Whittaker functional, uniquely determined up to scalar by the condition that $\mathcal{L}(\phi(g\overline{u})) = \psi(\overline{u})\mathcal{L}(\phi(g))$ for all $\phi \in \mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ and $\overline{u} \in \overline{U}$, where ψ is a non-degenerate character of $\overline{U}$, and ε is the sign character of $\mathcal{H}_0$. Most recently, in [4], Brubaker, Bump, and Friedberg showed that the Bessel functional on the doubly-laced group $\mathrm{SO}(2n+1)$ is an $\mathcal{H}$-intertwiner from M to V_ε in the manner described above; in this case, ε is the character of $\mathcal{H}_0$ that acts by -1 on long simple roots and by q on short simple roots.

In general, we start with a subgroup S of G and a linear $\mathbb{C}$-valued character ψ of S, and we look for an R-module homomorphism $\mathcal{L} : \mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1}) \to R$ such that

$$\mathcal{L}(s \cdot \phi) = \psi(s)\mathcal{L}(\phi) \text{ for all } s \in S \text{ and } \phi \in \mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1}), \tag{1.0.2}$$

where the action of G on $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ is given by right translation. In order to find $\mathcal{L}$, we will use Mackey theory. In the case where F is a finite field, Mackey theory tells us that the space of R-module homomorphisms satisfying (1.0.2) is in bijection with the vector space of functions $\Delta : G \to R$ that satisfy the equivariance properties

$$\Delta(sgb) = \psi(s)\Delta(g)\chi_{\mathrm{univ}}^{-1}(b) \tag{1.0.3}$$

for all $s \in S$, $b \in B$; here we are thinking of ψ as taking values in R, since R is a commutative $\mathbb{C}$-algebra with $\mathbb{C}$ included in it.

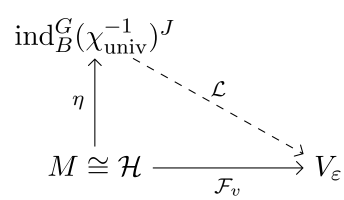
When F is a p-adic field, Mackey theory tells us that the space of R-module homomorphisms satisfying (1.0.2) is in bijection with the vector space of *distributions* satisfying (1.0.3).[3] If such a Δ exists, we get the corresponding R-module homomorphism $\mathcal{L}$ from the convolution

$$\mathcal{L}(\phi)(g) = \int_{B\backslash G} \Delta(h^{-1})\phi(hg)\, dh.$$

If such an $\mathcal{L}$ exists then the space $\mathrm{Ind}_S^G \psi$ is called a model for $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ - we say that the model is unique for $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ if the space $\mathrm{Hom}_G(\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1}), \mathrm{Ind}_S^G \psi)$ is one-dimensional, i.e. if the space of functionals satisfying (1.0.2) is one-dimensional.

Based on the formalism of [4], it can be shown that if $\mathcal{L}$ is restricted to the space of Iwahori-fixed vectors, $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$, then $\mathcal{L}$ induces a left $\mathcal{H}$-module structure on its image. In particular, the algebra R embedded in $\mathcal{H}$, as described earlier, acts on the image of $\mathcal{L}$ by translation. What's more, it turns out that the only $\mathcal{H}$-module actions on R, in which the embedded copy of R in $\mathcal{H}$ acts by translation, are those arising from the isomorphism $R \cong \mathrm{Ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$, with ε a linear character of $\mathcal{H}_0$. Putting this all together, we end up with Brubaker, Bump and Friedberg's conjecture with regards to this program:

Conjecture 1.0.1 (Rough Form)**.** *[4] Let $\mathcal{L}$ be an R-valued linear map on $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ obtained from a unique model. Then $\mathcal{L}$ is an $\mathcal{H}$-map from M to $V_\varepsilon = \mathrm{Ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$ for some choice of linear character ε of $\mathcal{H}_0$ and the following diagram commutes:*



(1.0.4)

with $v_\varepsilon := \mathcal{L}(\phi_1)$ and $\mathcal{F}_{v_\varepsilon} : h \mapsto h \cdot v_\varepsilon$ where h acts on v_ε according to the module structure on V_ε. In particular, there exists an $r \in R$ such that $r \cdot \mathcal{L}$ is an $\mathcal{H}$-map to the module V_ε with eigenvector 1.

Of course, such an $\mathcal{H}$-map $\mathcal{L}$ is guaranteed to exist since $\mathcal{F}_{v_\varepsilon}$ and η are isomorphisms; rather, the dotted line is meant to reiterate the point made earlier that we are looking for an explicit realization of this isomorphism using a subgroup S such that the transformation property (1.0.2) implies that $\mathcal{L}$ is an $\mathcal{H}$-map to V_ε.

[3] In practice, for the models that we are considering, any nonzero Δ satisfying (1.0.3) is defined on an open set, so that, in these cases, such Δ are, in fact, functions.

We will show in Section 4 that the Bessel functional on GSp(4) as defined by Piatetski-Shapiro and Novodvorsky in [1] provides another example of such an $\mathcal{L}$:

Theorem 1.0.2. *Let $G = \mathrm{GSp}(4)$ and let ε be the character of $\mathcal{H}_0$ that acts by multiplication by -1 on long simple roots and acts by q on short simple roots. Let $V_\varepsilon = \mathrm{Ind}_{\mathcal{H}_0}^{\mathcal{H}} \varepsilon$. Then the diagram (1.0.4) commutes by taking $v_\varepsilon = \pi^{\rho_\varepsilon^\vee}$, where ρ_ε is half of the sum of the long positive roots; and by taking $\mathcal{L} = \mathcal{B}$, the non-split Bessel functional as defined by Piatetski-Shapiro and Novodvorsky.*

It should be noted that the split Bessel model also gives rise to a functional fitting into Conjecture 1.0.1 - however, in this case one can show that this model is related to the sign character of $\mathcal{H}_0$.

Before we can prove this theorem, we will discuss the definition of the Bessel model on GSp(4) and then use Mackey theory to prove the existence of a Bessel model for $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ in Sections 3.1 and 3.2, respectively. We conclude Section 3.2 with an explicit realization of the Bessel functional as an integral. And then, once we have proved Theorem 1.0.2, in Section 5.1, we will use that result to calculate the images of the Iwahori-fixed vectors $\{\phi_w\}_{w\in W}$ on torus elements in the model V_ε, which has not previously appeared in the literature. In particular, we prove the following theorem:

Theorem 1.0.3. *For dominant λ and fixed w,*

$$\mathcal{B}(\pi^{-\lambda} \cdot \phi_w) = \frac{1}{m(J\pi^\lambda J)} T_w \pi^\lambda \cdot v_\varepsilon,$$

where the action of T on $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ is by right translation and where the action of $T_w\pi^\lambda$ on v_ε is the left action on v_ε appearing in the definition of $\mathcal{B}$.

Using Theorem 1.0.3, we will also be able to calculate the image of the spherical function in the model, giving a new proof of the same result from [3] (in what follows, α_1 is the short simple root of GSp(4) and α_2 is the long simple root):

Theorem 1.0.4. *[3] Let ρ be the half-sum of the positive roots of Φ, and let ρ_ε be as defined in Theorem 1.0.2. Then, for any dominant coweight λ,*

$$\mathcal{B}(\pi^{-\lambda} \cdot \phi^\circ) = \frac{\pi^{-\rho_\varepsilon^\vee}(1 - q\pi^{\alpha_2^\vee})(1 - q\pi^{(2\alpha_1+\alpha_2)^\vee})}{\pi^{\rho^\vee} \prod_{\alpha\in\Phi^+}(1 - \pi^{-\alpha^\vee})} \mathcal{A}((1 - q\pi^{\alpha_1^\vee})(1 - q\pi^{(\alpha_1+\alpha_2)^\vee})\pi^{2\rho_\varepsilon^\vee - \rho^\vee + \lambda}),$$

where the action of T on $\operatorname{ind}_B^G(\chi_{\text{univ}}^{-1})$ is by right translation and where $\mathcal{A}$ denotes the standard alternator expression $\mathcal{A}(\pi^\mu) = \sum_{w \in W}(-1)^{\ell(w)} w\pi^\mu$ with W acting on $X_*(T)$ in the usual way.

We note here that our proof of Theorem 1.0.2 does not rely on prior knowledge of the image of the spherical function in the model - in this way our method of proof differs from the proofs of similar results in [4]. Instead, we will calculate the relevant intertwining constants directly. Additionally, the proof we give does not use uniqueness of the model in an essential way, should we eventually find ourselves in a situation where the space $\operatorname{Hom}_G(\operatorname{ind}_B^G(\chi_{\text{univ}}^{-1}), \operatorname{Ind}_S^G \psi)$ is finite-dimensional instead of one-dimensional. We will discuss how this situation might arise naturally within this program when we discuss the Springer correspondence in Section 2.3, and again in Chapter 6.

In Section 5.2, we discuss the fourth character, σ, of the finite Hecke algebra of GSp(4), which acts by multiplication by q on long simple roots and -1 on short simple roots. At this time we do not have a realization of the intertwiner $\mathcal{L}$ satisfying the diagram (1.0.4) that is also defined according to a subgroup transformation, but we have matched the image of the spherical function under $\mathcal{F}_{v_\sigma}$ for σ to the image of the spherical function in the Whittaker-Orthogonal models defined by Bump, Friedberg and Ginzburg in [17]. In particular, we prove the following proposition:

Proposition 1.0.5. *Let WO be the Whittaker-Orthogonal functional on an unramified principal series representation τ of $\mathrm{SO}(6)$, such that τ is a local lifting of an unramified principal series representation of $\mathrm{Sp}(4)$. Then $\mathcal{F}_{v_\sigma}(\pi^{-\lambda} \cdot 1_{T(\mathfrak{o})UK})$ and $\mathrm{WO}(z^{-\lambda} \cdot \phi^\circ)$ agree, for any dominant coweight λ.*

Finally, we note that Piatetski-Shapiro and Novodvorsky do not provide an explicit integral formula for their functional, so part of our task in proving Theorem 1.0.2 is coming up with the correct integral formula for $\mathcal{L}$. Our method for doing this follows what we believe to be the general method for connecting models of the form $\operatorname{Ind}_S^G \psi$ to characters of $\mathcal{H}_0$, which has its genesis in [2]. We will discuss Kawanaka's conjectured method for building unique models, along with the modifications we made to it, in Section 2.2. Then, in Chapters 6 and 7, we end this thesis with a discussion of how exactly we expect Kawanaka's generalized Gelfand-Graev representations to be connected to irreducible representations of $\mathcal{H}_0$ and explain how this connection seems to have led us to a (unique) Bessel model for $G = \mathrm{GSp}(2n)$.

Chapter 2

Background

2.1 Universal Principal Series

In this section, we retain the notation from Section 1. In particular, G will be a split, connected reductive group over a p-adic field F with ring of integers $\mathfrak{o}$ and uniformizer π. B will denote a Borel subgroup of G with maximal torus T and unipotent subgroup U such that $B = TU$, and $\overline{U}$ will denote the opposite unipotent of U in B. We assume that these groups are defined over $\mathfrak{o}$; in particular, this means that $K = G(\mathfrak{o})$ is a maximal compact subgroup of G. Let J denote the Iwahori subgroup, and let $\mathcal{H}$ denote the Iwahori-Hecke algebra of G.

As mentioned in the introduction, our discussion of the universal principal series is inspired by the treatment given in [15]. We define the universal principal series M to be the vector space $C_c(T(\mathfrak{o})U\backslash G/J)$. Evidently, we can make M into a right $\mathcal{H}$-module where $\mathcal{H}$ acts by convolution. Now, observe that $T/T(\mathfrak{o})$ is isomorphic to the cocharacter group $X_*(T)$ of G under the map that sends $\mu \in X_*(T)$ to $\mu(\pi) \in T/T(\mathfrak{o})$. We will write $\mu(\pi)$ as π^μ throughout this thesis. Define $R := C_c(T/T(\mathfrak{o})) = \mathbb{C}[X_*(T)]$, and regard R as a left $(T/T(\mathfrak{o}))$-module via the inverse of the "universal" character $\chi_{\text{univ}} : \pi^\mu \mapsto \pi^\mu$. If we use normalized induction to form $\text{ind}_B^G \chi_{\text{univ}}^{-1}$, and then take its J-fixed vectors $\text{ind}_B^G(\chi_{\text{univ}}^{-1})^J$, then we can see that $M \cong \text{ind}_B^G(\chi_{\text{univ}}^{-1})^J$ as right $\mathcal{H}$-modules; explicitly we have $\eta : M \to \text{ind}_B^G(\chi_{\text{univ}}^{-1})^J$ where

$$\eta(\phi)(g) = \sum_{\mu \in X_*(T)} \delta_B(\pi^\mu)^{-1/2} \pi^\mu \phi(\pi^\mu g).$$

Here we can see the motivation for our terminology: if we're given an unramified principal series obtained from parabolic induction by a character $\chi : T/T(\mathfrak{o}) \to \mathbb{C}^\times$, then χ determines a $\mathbb{C}$-algebra homomorphism $R \to \mathbb{C}$, and

$$\mathbb{C} \otimes_R M \cong \mathrm{ind}_B^G(\chi^{-1})^J,$$

the Iwahori-fixed vectors of our original unramified principal series.

In order to gain a better understanding of the Hecke algebra $\mathcal{H}$, we are going to make use of an alternate point of view of M. First, we note that M is isomorphic to $\mathcal{H}$ as a free, rank one right $\mathcal{H}$-module; it has a $\mathbb{C}$-basis made up of the characteristic functions $1_{T(\mathfrak{o})UwJ}$ where w is an element of the affine Weyl group $\widetilde{W}$. The isomorphism from $\mathcal{H}$ to M is given by the map $h \mapsto 1_{T(\mathfrak{o})UJ} * h$. We can define a left action of $\mathcal{H}$ on M via this isomorphism: in particular, we identify $h \in \mathcal{H}$ with the endomorphism

$$h : 1_{T(\mathfrak{o})UJ} * h' \mapsto 1_{T(\mathfrak{o})UJ} * (hh').$$

Using η, we can transfer this left $\mathcal{H}$-action to $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$, so that $h \in \mathcal{H}$ sends $\phi_1 * h'$ to $\phi_1 * (hh')$, where $\phi_1 = \eta(1_{T(\mathfrak{o})UJ})$. Note that this left action identifies $\mathcal{H}$ with $\mathrm{End}_{\mathcal{H}}(M)$.

As mentioned in the introduction, the spherical function, ϕ°, will be a major figure in this thesis; we pause here to note that, as an element of $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$, the spherical function is defined as

$$\phi^\circ(g) := \delta^{-1/2}(\pi^\mu)\pi^{-\mu},$$

where $g = tuk$ is the Iwasawa decomposition of g with $u \in U$, $k \in G(\mathfrak{o})$, and $t \in T(F)$ where $t \equiv \pi^\mu \in T(F)/T(\mathfrak{o})$. As mentioned in the introduction, ϕ° can be decomposed into the sum

$$\phi^\circ = \sum_{w \in W} \phi_w,$$

where $\phi_w := \eta(1_{T(\mathfrak{o})UwJ}) \in \mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$.

Now, if we take the obvious left action of R on $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$ and transfer it via η^{-1} to M, we see that R embeds into $\mathrm{End}_{\mathcal{H}}(M)$, and hence embeds into $\mathcal{H}$. Additionally, the finite Hecke algebra $\mathcal{H}_0 = C(J\backslash K/J)$ is a subalgebra of $\mathcal{H}$, and there is a vector space isomorphism $\mathcal{H} \cong R \otimes_{\mathbb{C}} \mathcal{H}_0$. While we will often conflate $\pi^\mu \in R$ with its embedded image in $\mathcal{H}$, we would like to point out that the image of π^μ is convolution with the characteristic function $1_{J\pi^\mu J}$ only when μ is dominant. We use T_s to denote the generator 1_{JsJ} of $\mathcal{H}_0$, where s is a simple reflection in the Weyl group, W, of G. The generators of $\mathcal{H}_0$ satisfy the braid relations

$$(T_iT_j)^{m(i,j)} = (T_jT_i)^{m(i,j)},$$

where $m(i,j)$ is the order of the braid relation satisfied by the corresponding simple reflections s_i, s_j, in addition to satisfying the quadratic relation

$$(T_s - q)(T_s + 1) = 0.$$

Finally, to understand $\mathcal{H}$ in terms of these generators, we need the Bernstein relation, first proved in [18], which says that, for $\pi^\mu \in R$ and $T_s \in \mathcal{H}_0$,

$$T_s\pi^\mu = \pi^{s(\mu)}T_s + (1-q)\frac{\pi^{s(\mu)} - \pi^\mu}{1-\pi^{-\alpha^\vee}}, \tag{2.1.1}$$

where $s = s_\alpha$ for a simple root α in the root system Φ of G.

2.2 Generalized Gelfand-Graev Representations

We believe that the unique models that give rise to R-homomorphisms as described in Conjecture 1.0.1 are related to Kawanaka's construction of the "generalized Gelfand-Graev representations" (gGGr) in [2]. Although Kawanaka's results are given in the context of algebraic groups over finite fields, we believe that they can be suitably adapted for the p-adic setting. For this subsection, all notations for algebraic groups indicate points over the finite field $F = F_q$.

We will now broadly describe the method for constructing a gGGr and leave more in-depth discussions of certain parts of the construction for the subsections to follow. When constructing a gGGr, we begin with a nilpotent $\mathrm{Ad}(G)$-orbit with representative A. We then use Kirillov's orbit method to form the attached irreducible representation η_A on $\overline{U}_A$, where U_A is the unipotent radical of the associated parabolic subgroup P_A and $\overline{U}_A$ is the opposite unipotent. We denote the stabilizer of the representation η_A contained in the Levi L_A of P_A by $Z_L(A)$, and we note that $Z_L(A)$ is a reductive group. We build a representation $\eta_{A,\alpha}$ on $\overline{U}_A \rtimes Z_L(A)$ from η_A and an irreducible representation α of $Z_L(A)$. Then the gGGr associated to A is the irreducible representation $\Gamma_{A,\alpha} := \mathrm{Ind}^G_{\overline{U}_A Z_L(A)}\eta_{A,\alpha}$.

In Conjecture 2.4.5, Kawanaka offers a method of producing gGGr's that contain each unipotent representation with multiplicity one. Since the principal series representations are precisely those representations containing a B-fixed vector, Kawanaka's conjecture implies that, for each irreducible $\mathcal{H}_0$-module, there should be a unique gGGr containing it with multiplicity one. After the conjecture, Kawanaka notes that the nilpotent orbit containing A and the irreducible representation of $\mathcal{H}_0$ appearing inside it appear to be linked via the Springer correspondence.

Shifting back to the p-adic setting, we note that in [19], Mœglin and Waldspurger give a treatment of those representations - also referred to as gGGr's in [2] - that are constructed by inducing η_A from $\overline{U}_A$ up to G directly. However, when we try to construct gGGr's (as defined in the previous paragraph) in the p-adic setting, the obstacle that we encounter is that the representation $\eta_{A,\alpha}$ is not necessarily guaranteed to be a genuine representation; we are only guaranteed that it is projective. However, if η_A is a character and α is a character of $Z_L(A)$, as is the case for the Bessel model on GSp(4), then $\eta_{A,\alpha}$ will be a genuine representation.

Unlike the Whittaker model, which served as the inspiration for the definition of a Gelfand-Graev representation (see [20]), the spherical model and Bessel model are not realized directly as gGGr's. Instead, we realize these models by extending η_A from $\overline{U}_A$ to $\overline{U}_A \rtimes (Z_L(A) \cap G(\mathfrak{o}))$, and then inducing to G. In the case of the Whittaker model, $Z_L(A)$ is trivial, so it appears that this method of extending η_A to the semidirect product of $\overline{U}_A$ and $Z_L(A) \cap G(\mathfrak{o})$ is a step towards understanding the general construction of gGGr's over local fields.

2.2.1 Kirillov's Orbit Method

A complete treatment of the orbit method can be found in [21]. As originally formulated, the orbit method gives us a way to construct all unitary irreducible representations of a connected and simply connected real nilpotent Lie group G.

To construct an irrreducible unitary representation of G, we begin with an element F of $\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$ with respect to the coadjoint representation. Our next task is to find a maximal Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ *subordinate* to F, i.e. such that $[\mathfrak{h}, \mathfrak{h}] \subset \ker F$. We then define the character $\chi(\exp(X)) = e^{2\pi i F(X)}$ of $H = \exp(\mathfrak{h})$. That χ is a character of H is a consequence of the Baker-Campbell-Hausdorff formula. The representation $T_\chi = \mathrm{Ind}_H^G \chi$ can be shown to be an irreducible unitary representation of G. The Kirillov orbit method tells us that every irreducible unitary representation of G can be constructed in this manner. In particular, Kirillov proved the following theorem, listed as Theorem 7.2 in [21]:

Theorem 2.2.1. 1. *Every irreducible unitary representation T of G has the form*

$$T = \mathrm{Ind}_H^G \chi_{F,H},$$

where $H \subset G$ is a connected subgroup and $F \in \mathfrak{g}^*$.

2. *The representation $T_{F,H} = \mathrm{Ind}_H^G \chi_{F,H}$ is irreducible if and only if the Lie algebra $\mathfrak{h}$ of the group H is a subalgebra of $\mathfrak{g}$ subordinate to the functional F with maximal possible dimension.*

3. *Irreducible representations T_{F_1,H_1} and T_{F_2,H_2} are equivalent if and only if F_1 and F_2 belong to the same orbit of $\mathfrak{g}^*$.*

Switching back to the situation where G is a connected reductive linear algebraic group defined over F_q, we will say a few words about how the orbit method appears in the construction of the gGGr associated to a nilpotent $\mathrm{Ad}(G)$-orbit, according to Kawanaka in [2]: Suppose that we choose such an orbit, $\mathcal{O}$, and choose a representative A from that orbit. By Dynkin-Kostant theory, there is an $\mathfrak{sl}_2$ subalgebra with nilpositive element A whose semisimple element H is invariant under the choice of representative of $\mathcal{O}$. Exponentiating H gives us a one-parameter subgroup, $\phi(s)$, which we then use to define the associated unipotent subgroup U_A: U_A is the subgroup of G whose Lie algebra is composed of elements X such that

$$\lim_{s \to 0} \phi(s) \cdot X = 0.$$

As stated in Springer's book [22], U_A is the unipotent radical of a parabolic subgroup of G, which we call P_A. Finally, we can use the orbit method to construct an irreducible representation η_A of the opposite unipotent subgroup $\overline{U}_A$ from the coadjoint $\overline{U}_A$-orbit of the dual element A^* of A in $\overline{\mathfrak{u}}_A^*$, the dual space to $\mathrm{Lie}(\overline{U}_A)$. The construction of the associated gGGr then proceeds as described above.

2.2.2 From η_A to the Generalized Gelfand-Graev Representation $\Gamma_{A,\alpha}$

In our initial discussion of the construction of gGGr's, we introduced the notation $Z_L(A)$ for the stabilizer of the representation η_A in the Levi subgroup L_A of P_A. We will often shorten this notation to $Z_L = Z_L(A)$ when there is no risk of confusion. The motivation for this notation comes from the fact that the stabilizer of η_A in L_A is equal to the centralizer of the element A in L_A, the proof of which can be found in [23]. In order to construct a representation $\eta_{A,\alpha}$ of $\overline{U}_A Z_L$ as described above, we will use Wigner's method of little subgroups. In particular, we must start by extending η_A from $\overline{U}_A$ to $\overline{U}_A \rtimes Z_L$ and extending α from Z_L to $\overline{U}_A \rtimes Z_L$. In the case where η_A is a linear character of $\overline{U}_A$ (as it is for the Bessel model) this process is straightforward and yields a genuine representation of $\overline{U}_A Z_L$. If η_A is a higher-dimensional representation, then the method of little subgroups is only guaranteed to yield a projective representation of $\overline{U}_A Z_L$ (see the discussion preceding Proposition 2.2 in [23]). However, as proved in Proposition 2.3 in [23], with G defined over a finite field, this extension of η_A, which we denote by $\tilde{\eta}_A$, is a genuine representation. We define the representation $\eta_{A,\alpha}$ on $\overline{U}_A Z_L$ as $\eta_{A,\alpha} := \tilde{\eta}_A \otimes \alpha$ and, finally, we define the gGGr

$$\Gamma_{A,\alpha} = \mathrm{Ind}_{\overline{U}_A Z_L(A)}^{G} \eta_{A,\alpha};$$

here we abuse notation and let α refer to the extension of α to $\overline{U}_A Z_L$.

When we adapt this theory to the p-adic setting, we find that the extension of η_A to $\overline{U}_A Z_L$ is no longer guaranteed to be a genuine representation. Kawanaka offers a version of Conjecture 2.4.5 modified for this case at the end of [2], and, as mentioned previously, Mœglin and Waldspurger provide some answers to these conjectures for the gGGr's of the form $\mathrm{Ind}_{\overline{U}_A}^{G} \eta_A$. However, our results seem to be related to the more refined gGGr's induced from $\overline{U}_A Z_L$ that we have been discussing. In [4], the authors suggest that, in order to suitably adapt the more refined gGGr's to the p-adic setting, we should be using a representation of $Z_L(A) \cap G(\mathfrak{o})$ to construct $\eta_{A,\alpha}$ rather than a representation on $Z_L(A)$. Taking this tack, (as described in [4] one can realize both the spherical model and the non-split Bessel model on $\mathrm{SO}(2n+1)$, as well as the non-split Bessel model on $\mathrm{Sp}(4)$ (as we will show in this thesis).

In particular, we can realize the spherical model as a gGGr by taking $A = 0$, so that $\overline{U}_A$ is trivial and $Z_L(A) = G$; putting the trivial representation on $G(\mathfrak{o})$, gives us the spherical model. As for the non-split Bessel model on $G = \mathrm{SO}(2n+1)$, we let A be an element of the orbit corresponding to the partition $[2n-1, 1^2]$ (see 2.3.1 for further explanation), so that $\overline{U}_A$ is the unipotent subgroup generated by the root subgroups corresponding to the roots $\alpha_1, \alpha_2, \alpha_2 + \alpha_3$. In this case, $Z_L(A)$ is a non-split one-dimensional torus embedded in the central SO(3) block. If we put the trivial representation on $Z_L \cap G(\mathfrak{o})$ and form the associated gGGr, we see that we end up with the non-split Bessel model on $\mathrm{SO}(2n+1)$ as defined in [5].

2.3 The Springer Correspondence

As mentioned in Section 1, we expect the model containing V_ε as defined in Conjecture 1.0.1 to be a gGGr $\Gamma_{A,\alpha}$ where A and α are closely related to the character ε of $\mathcal{H}_0$. In particular, we believe that the Springer correspondence plays a major role in this connection between ε and $\Gamma_{A,\alpha}$. We pause here to summarize the known results again: if ε is the trivial character, then V_ε lives inside the J-fixed vectors of the spherical model; if ε is the sign character, then V_ε lives inside the J-fixed vectors of the Whittaker model; and if $G = \mathrm{SO}(2n+1)$, then if ε is the character that acts by -1 on long simple roots and by q on short simple roots, then V_ε lives inside the J-fixed vectors of the Bessel model (in this thesis we will show that this result holds for $G = \mathrm{Sp}(4)$ as well).

The Springer correspondence is a bijection between irreducible representations of W and pairs $(\mathcal{O}, \mu)$, where $\mathcal{O}$ is a nilpotent orbit of the Lie algebra and μ is an irreducible representation of $A(\mathcal{O})$, a subgroup of the G-equivariant fundamental group. The irreducible representations of W are in bijection with those of $\mathcal{H}_0$ by Tits' Deformation Theorem. Geometrically, the Springer correspondence arises from the realization of the irreducible representations of W in the top degree cohomology group of partial flag varieties. For classical groups, this bijection can be represented by a combinatorial recipe between the two sets, which is generally more useful in practice than the general definition. Since, in the rest of this thesis, we are primarily concerned with $G = \mathrm{Sp}(4)$, we will describe this recipe for type C_n. We note that the recipes for type B_n and D_n are quite similar, while type A_n is much simpler. A description of the Springer correspondence for each of these types can be found in Chapter 10 of [24] and in Chapter 13 of [25].

In this section, we will give an overview of the Springer correspondence, and then, in

Chapter 6, we will give a description of how, for a given ε, we think one might construct $\Gamma_{A,\alpha}$ so that V_ε is contained in the J-fixed vectors of this gGGr. In particular, we will start by describing the correspondence between nilpotent orbits and representations of the Weyl group W. Once we have done this, we will describe how to determine the character of $A(\mathcal{O})$ corresponding to a given representation of W.

2.3.1 The Combinatorics of Nilpotent Orbits

We will begin by offering parametrizations of both the set of nilpotent orbits of type C_n and the set of irreducible representations of W. Once we have done this, we will describe an injective map from the set of nilpotent orbits into the set of irreducible representations of W formulated by Lusztig in [26].

Recall that, for type C_n, $W \cong S_n \rtimes (\mathbb{Z}/2)^n$. As noted in Chapter 10 of [24], since $(\mathbb{Z}/2)^n$ is abelian, we can use Wigner's method of little subgroups to obtain the following parametrization of W ($|\mathbf{d}|$ will denote the sum of parts of the partition $\mathbf{d}$):

Theorem 2.3.1. *The irreducible representations of the Weyl group W of type C_n are parametrized by ordered pairs $(\mathbf{p}, \mathbf{q})$ of partitions such that $|\mathbf{p}| + |\mathbf{q}| = n$. The resulting representation has dimension*

$$\dim \pi_{(\mathbf{p},\mathbf{q})} = \binom{n}{|\mathbf{p}|}(\dim \pi_{\mathbf{p}})(\dim \pi_{\mathbf{q}}).$$

We also have

$$\pi_{(\overline{\mathbf{p}},\overline{\mathbf{q}})} \cong \pi_{(\mathbf{q},\mathbf{p})} \otimes sgn,$$

where $\overline{\mathbf{p}}$ denotes the conjugate partition of $\mathbf{p}$, and sgn denotes the sign character. The representation $\pi_{(\mathbf{p},\mathbf{q})}$ is characterized by the following property. Let V be the subspace of $\pi_{(\mathbf{p},\mathbf{q})}$ consisting of all vectors on which the first $|\mathbf{p}|$ copies of $\mathbb{Z}/2$ act trivially while the remaining $|\mathbf{q}|$ copies act by -1. Then $S_{|\mathbf{p}|} \times S_{|\mathbf{q}|}$ acts on V according to the representation $\pi_{\mathbf{p}} \times \pi_{\mathbf{q}}$.

Going forward, we will abuse notation and refer to the representation $\pi_{(\mathbf{p},\mathbf{q})}$ as $(\mathbf{p}, \mathbf{q})$. We will make the same abuse of notation with respect to our parametrization for the nilpotent orbits of $\mathfrak{sp}(2n)$, which we recall from Section 5.1 in [24]:

Theorem 2.3.2. *Nilpotent orbits in $\mathfrak{sp}(2n)$ are in one-one correspondence with the set of partitions of $2n$ in which odd parts occur with even multiplicity.*

Now, let $\mathbf{d}$ be a partition of $2n$ corresponding to an orbit of $\mathfrak{sp}(2n)$. We ensure that $\mathbf{d} = [d_1, \ldots, d_{2k}]$ has an even number of parts by calling the first part 0 if necessary, and we arrange the parts in increasing order (so that $d_i > 0$ for $i > 1$). We then define a new strictly increasing sequence of integers $(e_1, \ldots, e_{2k})$ by setting $e_i = d_i + i - 1$. Next, we use the e_i's to define two new strictly increasing sequences of integers $(f_1, \ldots, f_a)$ and $(g_1, \ldots, g_b)$, where $2f_1 + 1 < 2f_2 + 1 < \cdots < 2f_a + 1$ are the odd e_i's and $2g_1 < 2g_2 < \cdots < 2g_b$ are the even e_i's. It turns out that $a = b = k$.

Next, let $p_i = f_i - (i-1)$ and $q_i = g_i - (i-1)$ for all i. Once we have discarded any 0 parts, we find that we have an ordered pair of partitions $(\mathbf{p}, \mathbf{q})$, with $\mathbf{p} = [p_1, \ldots, p_j]$ and $\mathbf{q} = [q_1, \ldots, q_\ell]$. In particular, it is always the case that the ordered pair of partitions $(\mathbf{p}, \mathbf{q})$ that we get from this process satisfy the property that $|\mathbf{p}| + |\mathbf{q}| = n$. Thus, using Theorem 2.3.1 and Theorem 2.3.2, we can see that this method gives us a way to associate a nilpotent orbit of C_n to an irreducible representation of the Weyl group of C_n.

As an example, let's suppose that $n = 3$ and calculate the calculate the irreducible Weyl group representation corresponding to the nilpotent orbit $\mathbf{d} = [2^2, 1^2]$. In this case, we have that $(e_1, e_2, e_3, e_4) = (1, 2, 4, 5)$, and so $(f_1, f_2) = (0, 2)$ and $(g_1, g_2) = (1, 2)$. Then $(p_1, p_2) = (0, 1)$ and $(q_1, q_2) = (1, 1)$. Thus, the irreducible Weyl group representation corresponding to this nilpotent orbit is $([1], [1^2])$.

We will end this subsection by investigating the image of the map $\mathbf{d} \mapsto (\mathbf{p}, \mathbf{q})$. It will turn out that the method we will use to identify which irreducible representations $(\mathbf{p}, \mathbf{q})$ of W are hit by this map can be adapted to give the Springer correspondence, as we will show in the next subsection.

Given an ordered pair of partitions $(\mathbf{p}, \mathbf{q})$ with $|\mathbf{p}| + |\mathbf{q}| = n$, write $\mathbf{p} = [p_1, \ldots, p_k]$ and $\mathbf{q} = [q_1, \ldots, q_\ell]$ with the parts of $\mathbf{p}$ and $\mathbf{q}$ given in increasing order. For the following correspondence to be defined, we must make sure that $\mathbf{p}$ has exactly one more partition than $\mathbf{q}$. To achieve this, we simply pad the given partitions with 0's in front as necessary (we now reindex the parts of $\mathbf{p}$ and $\mathbf{q}$ to reflect these added zeros, so that $k = \ell + 1$). From these partitions we consider the symbol (originally defined by Lusztig in [27], and referred to as the "Lusztig symbol" of the representation $(\mathbf{p}, \mathbf{q})$ in the sequel)

$$\begin{pmatrix} p_1 & & p_2 + 2 & & \cdots & & p_k + 2(k-1) \\ & q_1 + 1 & & q_2 + 3 & \cdots & q_\ell + 2\ell - 1 & \end{pmatrix}.$$

If $p_1 \leq q_1 + 1 \leq p_2 + 2 \leq q_2 + 3 \leq \cdots \leq q_\ell + 2\ell - 1 \leq p_k + 2(k-1)$, then $(\mathbf{p}, \mathbf{q})$ is in the image of the injection from nilpotent orbits to irreducible representations of the Weyl group defined above. To figure out the nilpotent orbit associated to a given symbol, just decompose the symbol back into p_i's and q_j's (as in the original definition of the symbol) and then work backwards towards the partition $\mathbf{d}$ associated to a nilpotent orbit using the algorithm described in the paragraphs above. Note that in the construction of the symbol, we end up with $\mathbf{p}$ having one more part than $\mathbf{q}$ (including the zeros), even though, in our map from nilpotent orbits to irreducible W_0-representations, these two partitions were constructed to have the same number of parts. Thus, to make it possible to work backwards through the injection $\mathbf{d} \mapsto (\mathbf{p}, \mathbf{q})$, either remove p_1 from $\mathbf{p}$ if $p_1 = 0$, or add a 0 to $\mathbf{q}$ if $p_1 \neq 0$. Then $\mathbf{p}$ and $\mathbf{q}$ should have the same number of parts and it should be possible to recover the original partition $\mathbf{d}$.

As an example, consider the irreducible representation $([2], [1])$ of the Weyl group of C_3. After padding $[2]$ to $[0, 2]$, we get the symbol

$$\begin{pmatrix} 0 & & 4 \\ & 2 & \end{pmatrix}.$$

Since $0 < 2 < 4$, $([2], [1])$ is the image of some nilpotent orbit under the map defined above. Working backwards, we can see that the associated nilpotent orbit is $[4, 2]$.

2.3.2 The Springer Correspondence in Type C_n

Using the symbol of a representation of W as defined in the previous subsection, we can give a description of the Springer correspondence, following Lusztig. We begin with the observation that each irreducible representation of W can be associated to a Lusztig symbol as described in the previous section. In particular, we can see that $(\mathbf{p}_1, \mathbf{q}_1), (\mathbf{p_2}, \mathbf{q}_2)$ will have the same Lusztig symbol if and only if $\mathbf{p}_1 = \mathbf{p}_2$ and $\mathbf{q}_1 = \mathbf{q}_2$. In order to determine the nilpotent orbit $\mathbf{d}$ and representation μ of $A(\mathbf{d})$ associated to $(\mathbf{p}, \mathbf{q})$ under the Springer correspondence, we must first reorganize the Lusztig symbol of $(\mathbf{p}, \mathbf{q})$ so that the symbol satisfies the condition $p_1 \leq q_1 + 1 \leq p_2 + 2 \leq q_2 + 3 \leq \cdots \leq q_\ell + 2\ell - 1 \leq p_k + 2(k-1)$ given in the previous section. The nilpotent orbit associated to $(\mathbf{p}, \mathbf{q})$ is the orbit $\mathbf{d}$ associated to this rearranged Lusztig symbol.

In order to determine μ, we first need to be able to say what $A(\mathbf{d})$ is. In Chapter 6 of [24], the authors prove the following theorem:

Theorem 2.3.3. *In type C_n,*

$$A(\mathbf{d}) = \begin{cases} (\mathbb{Z}/2)^b & \text{if all even parts have even multiplicity} \\ (\mathbb{Z}/2)^{b-1} & \text{otherwise,} \end{cases}$$

where b is the number of distinct nonzero parts of $\mathbf{d}$.

Next, let S be the set of integers appearing with multiplicity one in the Lusztig symbol, and break S into intervals, where an interval of S is a subset

$$(i+1,\ldots,j) \subset S \text{ with } 0 \leq i < j \text{ and } i, j+1 \notin S.^{1}$

The group $A(\mathbf{d})$ is generated by elements of order 2, where each generator, a_I, corresponds to an interval I of S, with the additional relation that the sum of the generators is defined to be 0. In type C_n, μ is always a character, and the value of $\mu(a_I)$ is determined by the Lusztig symbol. In particular, $\mu(a_I) = 1$ if the associated interval I lies in the same row of the rearranged Lusztig symbol as it did in the original Lusztig symbol, and $\mu(a_I) = -1$ otherwise.

For example, consider the irreducible representation $(\emptyset, [3])$ of the Weyl group of C_3. After padding $\emptyset$ to $[0^2]$, we get the symbol

$$\begin{pmatrix} 0 & & 2 \\ & 4 & \end{pmatrix}.$$

Since this symbol does not satisfy the inequalities $p_1 \leq q_1 + 1 \leq p_2 + 2$, it is not in the image of the map from nilpotent orbits to irreducible representations of the Weyl group. Hence, we associate $\pi_{(\emptyset,[3])}$ to the nilpotent orbit that corresponds to the symbol

$$\begin{pmatrix} 0 & & 4 \\ & 2 & \end{pmatrix},$$

which we saw earlier was the nilpotent orbit $[4, 2]$. Next, we observe that $A([4, 2]) = \mathbb{Z}/2$; hence, μ is either trivial or the sign character. We see that $S = \{0, 2, 4\}$, and hence

[1] It turns out that the number of intervals is equal to the number of distinct even parts of $\mathbf{d}$.

that we have two intervals, $I_1 = (2)$, $I_2 = (4)$, giving us two generators a_1 and a_2 of $A([4,2])$. Since 2 is in the first row of the original Lusztig symbol but is in the second row of the rearranged Lusztig symbol, we have $\mu(a_1) = -1$. Thus, μ must be the sign character of $A([4,2])$, and so we see that the Springer correspondence associates $(\emptyset, [3])$ to $([4,2], sgn)$.

Chapter 3

The Bessel Model and the Bessel Functional

We return now to the setting where $G = \mathrm{GSp}(4)$, and show how the Bessel model as formulated in [1] fits into the narrative formulated at the end of the previous section before we move on to establishing our main results. We carry all of our notation through from the previous section. We will have need to realize specific elements of G, and so we will explicitly define G as

$$G := \{g \in M_4(F) \mid g^\top \Omega g = k\Omega, k \in F^\times\},$$

where

$$\Omega = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

As in the first section, we let Φ denote the root system of G, with simple roots α_1, α_2, and let s_1, s_2 denote the corresponding simple reflections in W. Let ρ denote the half-sum of the positive roots of Φ, and let Φ^+ and Φ^- denote the sets of postive and negative roots of Φ, respectively.

3.1 The Bessel Model as a Generalized Gelfand-Graev Representation

The transformation property satisfied by the Bessel model depends on the parabolic subgroup P_A of G containing the subgroup corresponding to the short simple root $-\alpha_1$. We can factor $P_A = L_A U_A$ where L_A is the Levi component of P_A, and U_A is the unipotent component of P_A, as outlined previously. In this case, the nilpotent element A can be chosen so that A is the sum of non-zero elements in the long simple root α_2-subalgebra and in the $(2\alpha_1 + \alpha_2)$-subalgebra. Let $\overline{U}_A$ denote the opposite unipotent of U_A. Let ψ_0 be a non-degenerate additive character on F^+, and let $\psi_A(u) = \psi_0(\mathrm{tr}(ru'))$ for $u \in \overline{U}_A$, where u' is the lower left 2×2 block of u and $r \in M_2(F)$. We assume that r is non-degenerate. Then the linear character ψ_A is the representation of $\overline{U}_A$ that we denoted as η_A in the previous section.

We wish to extend ψ_A to a character $\tilde{\psi}_A$ of $\overline{U}_A \rtimes Z_L$, where $Z_L = Z_L(r)$ is the centralizer of r in L_A (this is the same group as $Z_L(A)$ as described in the previous section, by duality). If we look at the orbits of the elements of $\overline{U}_A$ under the conjugation action of L_A, we see that each orbit contains an element with

$$r = \begin{pmatrix} 0 & -\omega \\ 1 & 0 \end{pmatrix}.$$

Piatetski-Shapiro and Novodvorsky show that Z_L is a torus in L_A – if it is non-split then the upper left 2×2 block of an element of Z_L looks like

$$\begin{pmatrix} \alpha & \omega\beta \\ \beta & \alpha \end{pmatrix} \in \mathrm{GL}(2).$$

We can then define $\tilde{\psi}_A(ut) = \psi_A(u)$ for $t \in Z_L$ and $u \in \overline{U}_A$. Note that this representation is the one denoted by $\tilde{\eta}_A$ in Section 2.2.2; going forward we will continue to use $\tilde{\psi}$ to denote the representation $\tilde{\psi} \otimes 1$, where 1 is the trivial character of $Z_L(A)$. Then, following the previous section, we can define the Bessel model to be $\mathrm{Ind}_{\overline{U}_A Z_L}^{G}(\tilde{\psi}_A)$.

The Bessel functional for an irreducible admissible representation θ on G is defined to be a linear functional $\mathcal{B}$ on the representation space V_θ of θ such that

$$\mathcal{B}(\theta(ut)v) = \tilde{\psi}_A(ut)\mathcal{B}(v),$$

for $v \in V_\theta$, $t \in Z_L$ and $u \in \overline{U}_A$. In particular, note that this means that $\tilde{\psi}_A$ must agree with the central character of θ. Following [3], let $Z_L(\mathfrak{o}) = Z_L \cap \mathrm{SL}(4, \mathfrak{o})$, so that Z_L is the semidirect product of the compact group $Z_L(\mathfrak{o})$ and the center of G. In particular, this means that the model $\mathrm{Ind}_{\overline{U}_A Z_L(\mathfrak{o})}^G(\tilde{\psi}_A)$ is a scalar multiple of the model defined in the preceding paragraph. Going forward, we will take this second definition as the definition of the Bessel functional, so as to be consistent with the discussion in Section 2.2.2. We want the character $\tilde{\psi}_A$ to have $\mathfrak{o}$ as its conductor, so we choose $\omega \in \mathfrak{o}$. In order to ensure that $Z_L(\mathfrak{o})$ is non-split, we choose $\omega \notin \mathfrak{o}^2$. In the next section we will discuss the existence and uniqueness of a Bessel functional for $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$. As mentioned previously, the latter condition was covered in [1] in greater generality, but the Mackey theory argument outlined in Section 3.2 has the additional benefit of suggesting an integral realization of the functional, as described in Section 1. We end this section with the statement of Novodvorsky and Piatetski-Shapiro's theorem regarding the uniqueness of the Bessel functional:

Theorem 3.1.1. *[1] Let θ be an irreducible admissible representation of the group G in a complex space V. Then the dimension of the space of all linear functionals $\mathcal{B}$ on V for which*

$$\mathcal{B}(\theta(ut)v) = \tilde{\psi}_A(ut)\mathcal{B}(v), \text{ for all } t \in Z_L(\mathfrak{o}),\ u \in \overline{U}_A,\ v \in V$$

does not exceed one.

3.2 Existence and Uniqueness of Bessel Functionals for Principal Series Representations

In this section, we will use Bruhat's extension of Mackey theory as described in [28] in order to first show that $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ admits a Bessel model, and then to give an integral realization of the corresponding Bessel functional. In particular, much of the argument used to prove this result for $\mathrm{SO}(2n+1)$ given in [29] can be applied without significant alteration, so, in the discussion to follow, we will refer the reader to the relevant results in [29] where appropriate. Before we begin, we note, per [15], that while the treatment in [29] ultimately yields a $\mathbb{C}$-valued functional on principal series representations, the method of proof applies equally well to a functional taking values in any commutative

$\mathbb{C}$-algebra, and so the fact that χ_{univ}^{-1} takes values in R does not introduce any new complications when translating results from [29].

As mentioned above, the argument that we will use to show that

$$\dim \operatorname{Hom}_G(\operatorname{ind}_B^G(\chi_{\text{univ}}^{-1}), \operatorname{Ind}_{\overline{U}_A Z_L(\mathfrak{o})}^G \tilde{\psi}_A) = 1$$

originated with Rodier in [28], and it makes use of the following theorem of Bruhat:

Theorem 3.2.1. *[30] Let G be a locally compact, totally disconnected unimodular group. Let H_1 and H_2 be two closed subgroups of G, and δ_i the module of H_i. Let τ_i be a smooth representation of H_i in the vector space E_i, π_i be the induced representation $\operatorname{Ind}_{H_i}^G \tau_i$ in the Schwarz space of τ_i.*

Then the space of all intertwining forms I of π_1 and π_2 is isomorphic to the space of $(E_1 \otimes E_2)$-distributions Δ on G such that

$$\lambda(h_1) * \Delta * \lambda(h_2^{-1}) = (\delta_1(h_1)\delta_2(h_2))^{1/2} \Delta \circ (\tau_1(h_1) \otimes \tau_2(h_2)) \tag{3.2.1}$$

where $h_i \in H_i$ and $\lambda(x)$ is the Dirac distribution in x. The correspondence between I and Δ is given by

$$I(p_1(f_1), p_2(f_2)) = \int_G dg_2 \int_G f_1(g_1 g_2) \otimes f_2(g_2)\, d\Delta(g_1), \tag{3.2.2}$$

where f_i are locally constant functions on E_i with compact support, and p_i is the projection from this space of functions to the Schwarz space of τ_i.

Let $\mathcal{D}(X, R)$ denote the space of R-distributions on a locally compact, totally disconnected space X. Following [29], we begin by noting that

$$\operatorname{Hom}_G(\operatorname{ind}_B^G \chi_{\text{univ}}^{-1}, \operatorname{Ind}_{\overline{U}_A Z_L(\mathfrak{o})}^G \tilde{\psi}_A) \cong \operatorname{Hom}_G(\operatorname{ind}_{\overline{U}_A Z_L(\mathfrak{o})}^G \tilde{\psi}_A^*, \operatorname{ind}_B^G(\chi_{\text{univ}}^{-1})^*),$$

where $\tilde{\psi}_A^*$ and $(\chi_{\text{univ}}^{-1})^*$ are the smooth contragredients of $\tilde{\psi}_A$ and χ_{univ}^{-1}, respectively. Then, by Theorem 3.2.1, this latter space is isomorphic to the subspace $\mathcal{D}_{\tilde{\psi}_A, \chi_{\text{univ}}^{-1}}(G, R)$ of $\mathcal{D}(G, R)$ of R-distributions Δ on G satisfying

$$\lambda(b) * \Delta * \lambda(h^{-1}) = \delta_B^{1/2}(b) \chi_{\text{univ}}^{-1}(b) \tilde{\psi}_A^*(h) \Delta. \tag{3.2.3}$$

for all $h \in \overline{U}_A Z_L(\mathfrak{o})$ and $b \in B$. With this condition in mind, we will use a double-coset decomposition of G to analyze $\mathcal{D}_{\tilde{\psi}_A, \chi_{\text{univ}}^{-1}}(G, R)$. The Bruhat decomposition $G =$

$\sqcup_{w\in W} Bw\overline{U}$ tells us that every element of G lies in some double coset $Bwx_{-\alpha_1}(t)\overline{U}_A Z_L(\mathfrak{o})$, where $x_{-\alpha_1}(F)$ is the $-\alpha_1$ root subgroup of G, and where $w = s_1, s_2 s_1, s_1 s_2 s_1$, or w_0, since $s_1 \in \overline{U}_A Z_L(\mathfrak{o})$. In fact, for each of these w, $wx_{-\alpha_1}(t) = x_\alpha(t)w \in Bw$, where either $\alpha = \alpha_1$ or $\alpha = \alpha_1 + \alpha_2$, so we can refine our double coset decomposition and say that every element of G lies in one of the double cosets $Bw\overline{U}_A Z_L(\mathfrak{o})$.

In a series of results, starting with Proposition 2.4, Friedberg and Goldberg show that, for a given non-zero $\Delta \in \mathcal{D}_{\tilde{\psi}_A, \chi_{\text{univ}}^{-1}}(G, R)$, Δ can only be supported on one specific double coset, and that, in addition, Δ is completely determined by its restriction to that double coset. The same thing is true in our case, and we will show that the only double coset that can serve as the support of Δ is $B\overline{U}_A Z_L(\mathfrak{o})$. Once we have done this, our proof of uniqueness of the Bessel model concludes in the same way described in [29].

The following lemma allows us to determine which of these double cosets do not satisfy the compatibility condition

$$\chi_{\text{univ}}^{-1}(b) = \tilde{\psi}_A(w^{-1}bw) \tag{3.2.4}$$

for some $b \in B$ with $h = w^{-1}bw \in \overline{U}_A Z_L(\mathfrak{o})$. By Theorem 1.9.5 in [31], any double coset that fails to satisfy (3.2.4) for all such b is not part of the support of any distribution in $\mathcal{D}_{\tilde{\psi}_A, \chi_{\text{univ}}^{-1}}(G, R)$.

Lemma 3.2.2. *If $w(\alpha) \in \Phi^+$ for any α such that $x_\alpha(t)$ is in the support of $\tilde{\psi}_A$, then every element of $\mathcal{D}_{\tilde{\psi}_A, \chi_{\text{univ}}^{-1}}(G, R)$ must vanish on $Bw\overline{U}_A Z_L(\mathfrak{o})$.*

Proof. In this case, we have $\chi_{\text{univ}}^{-1}(x_{w(\alpha)}(t)) = 1$ for all $t \in F$, but $\tilde{\psi}_A(x_\alpha(t))$ is not constant, so (3.2.4) does not hold on $Bw\overline{U}_A Z_L(\mathfrak{o})$. □

Since $w(-\alpha_2) \in \Phi^+$ for $w = s_2, s_1 s_2, s_2 s_1 s_2$, Lemma 3.2.2 tells us that every $\Delta \in \mathcal{D}^{-1}_{\tilde{\psi}_A, \chi_{\text{univ}}}(G, R)$ must vanish off of $B\overline{U}_A Z_L(\mathfrak{o})$.

In the rest of this section, we will discuss the integral realization of the Bessel functional for $\text{ind}_B^G \chi_{\text{univ}}^{-1}$, as well as the issue of convergence of the functional. We will leave the proof of the existence of a non-zero Bessel functional for Section 4. If $\text{ind}_B^G \chi_{\text{univ}}^{-1}$ has a Bessel model, then Theorem 3.2.1 tells us that, if Δ is a non-zero element of $\mathcal{D}_{\eta_A, \chi_{\text{univ}}^{-1}}(G, R)$, then the corresponding intertwining form, I, of $\text{ind}_B^G \chi_{\text{univ}}^{-1}$ and $\text{Ind}_{U_A Z_L(\mathfrak{o})}^G \tilde{\psi}_A$, is given by (3.2.2). Hence, the corresponding Bessel functional is realized as the inner integral of I, which in this case is

$$\begin{aligned}\mathcal{B}(\phi)(g) &= \int_G \phi(hg)\, d\Delta(h)\\ &= \int_{Z_L(\mathfrak{o})} \int_{\overline{U}_A} \psi(\overline{u})\phi(\overline{u}hg)\, d\overline{u}\, dh,\end{aligned}$$

with g set equal to 1.

Upon showing that this integral is non-zero in the next section, we will have proved the following theorem:

Theorem 3.2.3. *The space of Bessel functionals on* $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ *is one dimensional and there exists a unique such Bessel functional whose restriction to functions supported on the big cell* $P_A\overline{U}_A$ *is given by*

$$\mathcal{B}(\phi) = \pi^{\rho_\varepsilon^\vee} \int_{Z_L(\mathfrak{o})} \int_{\overline{U}_A} \psi(\overline{u})\phi(\overline{u}h)\, d\overline{u}\, dh.$$

Here we have normalized the Bessel functional so that the diagram (1.0.4) will commute with $v_\varepsilon = \pi^{\rho_\varepsilon^\vee}$ as in Theorem 1.0.2. Of course, we could just as easily define the outer integral to be over all of Z_L, since we can see Z_L is compact in the non-split case, so that the functional as defined is just a constant multiple of the Bessel functional we would get by integrating over all of Z_L. Note that, in the split case, Z_L is no longer compact, so that we will have to worry more about convergence of the functional in this case. Because of this, we expect the integral realization of the functional to have a slightly different form, so that, as a Hecke algebra map, it behaves like the Whittaker functional and intertwines the sign character.

We conclude this section with a brief discussion regarding the convergence of the functional $\mathcal{B}$. We will start by discussing the convergence of $\mathcal{B}(\phi)$, $\phi \in \mathrm{ind}_B^G\chi_{\mathrm{univ}}^{-1}$ in a particular completion of R, as described in Section 6.2 of [15] for the Whittaker functional. Let $\mathcal{J} = \{-\alpha_2^\vee, (-\alpha_1 - \alpha_2)^\vee, (-2\alpha_1 - \alpha_2)^\vee\}$, and let $\mathbb{C}[\mathcal{J}]$ denote the subalgebra of R generated by $\mathcal{J}$. Let $R_{\mathcal{J}}$ denote the completion of $\mathbb{C}[\mathcal{J}]$ with respect to the maximal ideal generated by $\mathcal{J}$. Our initial claim is that $\mathcal{B}(\phi) \in R_{\mathcal{J}}$. Since $\phi \in \mathrm{ind}_B^G\chi_{\mathrm{univ}}^{-1}$ is compactly supported mod B, we can see that we do not need to include any positive coroots in $\mathcal{J}$. Additionally, since $\mathcal{B}(\phi)$ is an integral over $\overline{U}_A Z_L(\mathfrak{o})$, we can see that there is no need to include $-\alpha_1^\vee$ in $\mathcal{J}$ either. In order to see that $\mathcal{B}(\phi) \in R_{\mathcal{J}}$, we apply the following lemma from [15]:

Lemma 3.2.4. *[15] Let $\mu \in X_*(T)$. Then the set $\overline{U} \cap \pi^\mu U K$ is compact.*

Finally, we observe that, due to the oscillation of the character ψ_A, all but finitely many of the coefficients of the Laurent series $\mathcal{B}(\phi)$ will vanish, which means that $\mathcal{B}(\phi)$ is indeed an element of R, not just $R_{\mathcal{J}}$. Note that, if we were to specialize χ_{univ}^{-1} to a $\mathbb{C}$-valued character on B, we could show that the resulting functional converges in $\mathbb{C}$ on elements of the corresponding principal series representation using an argument analogous to that presented in [29].

Chapter 4

The Bessel Functional as a Hecke Algebra Intertwiner

4.1 Principal Series Intertwining Operators

We will now introduce the principal series intertwining operators. These operators turn out to be closely connected to the left action of the elements of the finite Hecke algebra on $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$, and we will exploit this connection in order to show that our functional acts as a Hecke algebra intertwiner in the way predicted in Theorem 1.0.2.

Our initial goal is to define a family of intertwining operators, one for each $w \in W$, that take $\mathcal{M}$ to itself. Our first guess at such an operator

$$\mathcal{I}_w : \phi \mapsto \int_{U \cap w\overline{U}w^{-1}} \phi(w^{-1}ug)\, du,$$

does not quite work, because it does not preserve $\mathcal{M}$. As shown in [15], there is a way to extend $\mathcal{M}$ by scalars to a completion of R according to the roots

$$\Phi_w^+ := \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\},$$

such that this extension of $\mathcal{M}$ is preserved by $\mathcal{I}_w$, but instead of doing this, we choose to use normalized versions of these intertwiners, A_w, where

$$A_w := \left(\prod_{\alpha \in \Phi^+} (1 - \pi^{\alpha^\vee}) \right) \mathcal{I}_w,$$

since, using basic properties of $\mathcal{I}_w$ recorded in Lemma 1.13.1 in [15], we can see that A_w preserves $\mathcal{M}$. Now, since $A_w \in \mathrm{End}_{\mathcal{H}}(\mathcal{M})$, we can regard A_w as an element of $\mathcal{H}$ acting on the left of $\mathcal{M}$. In particular, we see that, for a simple reflection s_α, the desired relation between A_{s_α} and T_{s_α} is

$$A_{s_\alpha} = (1 - q^{-1})\pi^{\alpha^\vee} + q^{-1}(1 - \pi^{\alpha^\vee})T_{s_\alpha}. \tag{4.1.1}$$

We pause here to note that it was Rogawski in [32] who first used (4.1.1) to recover earlier results of Rodier and others on the structure of the unramified principal series representations. However, Rogawski was using (4.1.1) to recover information about the intertwining operators from his knowledge of the Hecke algebra action, which is the opposite of what we will do.

4.2 Calculating Intertwining Factors

In order to prove Theorem 1.0.2, we will use (4.1.1) to reduce the problem to understanding the interaction between the principal series intertwiners and the functional. In particular, since the Bessel functional is unique and since $\mathcal{B} \circ \mathcal{A}_{s_\alpha}$ is a Bessel functional on $\mathrm{ind}_B^G(s_\alpha \cdot \chi_{\mathrm{univ}}^{-1})$, we know that it must be a constant multiple of $s_\alpha \circ \mathcal{B}$. Hence, for each simple root α, we want to calculate $c_\alpha \in R$ such that

$$\mathcal{B} \circ A_{s_\alpha} = c_\alpha(s_\alpha \circ \mathcal{B}).$$

This turns out to be a tractable calculation, yielding the following results:

Proposition 4.2.1. *With notation as above, we have that*

$$\mathcal{B} \circ A_{s_1} = (1 - q^{-1}\pi^{\alpha_1^\vee})(s_1 \circ \mathcal{B}), \tag{4.2.1}$$

and

$$\mathcal{B} \circ A_{s_2} = (\pi^{\alpha_2^\vee} - q^{-1})(s_2 \circ \mathcal{B}). \tag{4.2.2}$$

Remark. We expect that, for the rank n case, (4.2.1) will hold for all short simple roots and (4.2.2) will hold for the long simple root.

In order to prove (4.2.1), we will need to calculate the image of the Iwahori-fixed vectors ϕ_1 and ϕ_{s_1} in the model. It turns out that the difficult part of this is calculating the intersection of our domain of integration, $\overline{U}_A Z_L(\mathfrak{o})$, with the support of these functions. The goal is to express the intersection as a set whose measure we can calculate when it appears during evaluation of the functional. As a first step towards determining these intersections, then, we note that $Z_L(\mathfrak{o})$ is contained in the Iwahori-Bruhat cells J and Js_1J. Recalling the definition of $Z_L(\mathfrak{o})$ from Section 3.1, we see that elements of the stabilizer are in J if and only if $\alpha \in \mathfrak{o}^\times$ and $\beta \in (\pi)$. Since $Z_L(\mathfrak{o}) \cap Js_1J$ is the complement of J contained in $Z_L(\mathfrak{o})$, we see that $Z_L(\mathfrak{o}) \cap Js_1J$ consists of those elements of $Z_L(\mathfrak{o})$ such that $\beta \in \mathfrak{o}^\times$. Hence, an element of $Z_L(\mathfrak{o})$ is in Js_1J if and only if $\beta \in \mathfrak{o}^\times$ and $\alpha \in \mathfrak{o}$. We choose our Haar measure so that $m(Z_L(\mathfrak{o}) \cap J) = 1$, which means that $m(Z_L(\mathfrak{o}) \cap Js_1J) = q$.

If $\overline{u}_A \in \overline{U}_A$ has Iwahori-Bruhat decomposition bwj with $b \in B$, $w \in W$, and $j \in J$, and $z \in Z_L(\mathfrak{o})$, then $\overline{u}_A z$ can be in the support of ϕ_1, BJ, only if $z \in J$ and $w = 1$, or if $z \in Js_1J$ and $w = s_1$. Likewise, $\overline{u}_A z$ can be in the support of ϕ_{s_1}, Bs_1J, only if $z \in J$ and $w = s_1$ or if $z \in Js_1J$ and $w = 1$. With this in mind, once we develop our understanding of $\overline{U}_A \cap BJ$ and $\overline{U}_A \cap Bs_1J$, we should be able to calculate $\mathcal{B}(\phi_1)$ and $\mathcal{B}(\phi_{s_1})$ easily.

In order to determine that $\overline{U}_A \cap Bs_1J$ is empty, we are going to appeal to the Iwasawa decomposition of G with respect to P_A and to the Bruhat factorization of $G(\mathfrak{o})$ with respect to $P_A(\mathfrak{o})$. In other words we will be using the "block" Iwasawa decomposition $G = P_A G(\mathfrak{o})$, which is just a more coarse version of the usual Iwasawa decomposition. The "block" Bruhat decomposition of $G(\mathfrak{o})$ is achieved analogously - we begin by defining J_A to be the preimage of $P_A(k)$ under the canonical homomorphism $G(\mathfrak{o}) \to G(k)$, so that $G(\mathfrak{o}) = J_A \sqcup J_A s_2 J_A$, which gives us the decomposition $G = P_A J_A \sqcup P_A s_2 J_A$. Since $Bs_1J \subset P_A J_A$, we will be able to use the result of the following lemma to figure out what $\overline{U}_A \cap Bs_1J$ is.

Lemma 4.2.2. *$\overline{U}_A \cap P_A J_A = \overline{U}_A \cap J_A$.*

Proof. Let $\overline{u}_A \in \overline{U}_A \cap P_A J_A$. We see that the standard argument for the rank 1 Iwahori factorization $J = (J \cap B)(J \cap \overline{U})$ can be adapted here to give $J_A = (J_A \cap P_A)(J_A \cap \overline{U}_A)$. Using this, we see that we can factor $\overline{u}_A = pj$, with $p \in P_A$ and $j \in J_A \cap \overline{U}_A$. Rewriting this as $\overline{u}_A j^{-1} = p$, we see that $\overline{u}_A j^{-1} \in \overline{U}_A \cap P_A$, so $\overline{u}_A = j \in J_A \cap \overline{U}_A$. □

Lemma 4.2.2 tells us that $\overline{U}_A \cap Bs_1J$ is contained in $\overline{U}_A \cap J_A$. In particular, note that $\overline{U}_A \cap J_A \subset J$, which means that $\overline{U}_A \cap Bs_1J$ is empty.

Our next step is to show that $\overline{U}_A \cap BJ = \overline{U}_A \cap J$. To do this, we will use the result analogous to the one above for the usual Iwahori factorization. This result follows by an argument similar to the one given in the proof of Lemma 4.2.2.

Lemma 4.2.3. *$\overline{U} \cap BJ = \overline{U} \cap J$.*

Proof. Let $\overline{u} \in \overline{U} \cap BJ$. Then, using the standard Iwahori factorization, $J = (J \cap B)(J \cap \overline{U})$, we see that we can factor $\overline{u} = bj$, with $b \in B$ and $j \in \overline{U} \cap J$. Rewriting this as $\overline{u}j^{-1} = b$, we see that, while $\overline{u}j^{-1} \in \overline{U} \cap B$, so $\overline{u} = j \in \overline{U} \cap J$. □

For our purposes, suppose that $\overline{u}_A \in \overline{U}_A \cap BJ$ in the argument above. Then, using the factorization of $\overline{u}_A$ from the proof of Lemma 4.2.3, we see that $\overline{u}_A = j$. In particular, this means that $j \in \overline{U}_A$, as well as meaning that $\overline{u}_A \in J$, so that we see that $\overline{U}_A \cap BJ = \overline{U}_A \cap J$.

With these results behind us, we now see that

$$\overline{U}_A Z_L(\mathfrak{o}) \cap BJ = (\overline{U}_A \cap J)(Z_L(\mathfrak{o}) \cap J) \tag{4.2.3}$$

and that

$$\overline{U}_A Z_L(\mathfrak{o}) \cap Bs_1J = (\overline{U}_A \cap J)(Z_L(\mathfrak{o}) \cap Js_1J). \tag{4.2.4}$$

We make the choice now to normalize our Haar measure so that $m(\overline{U}_A \cap J) = 1$. We are ready to prove Proposition (4.2.1):

Proof of Proposition 4.2.1. In order to make our calculation of c_{α_1} easier, we will evaluate $\mathcal{B} \circ A_{s_1}$ on the Iwahori-fixed vector $\phi_1 + \phi_{s_1}$. From Lemma 1.13.1 in [15], we know that

$$\mathcal{B}(A_{s_1}(\phi_1 + \phi_{s_1})) = (1 - q^{-1}\pi^{\alpha_1^\vee})\mathcal{B}(\phi_1 + \phi_{s_1}).$$

Note that, if we can show that $\mathcal{B}(\phi_1)$ and $\mathcal{B}(\phi_{s_1})$ are both invariant under the reflection s_1, then we will have proved (4.2.1). Now, it follows from (4.2.3) that $\mathcal{B}(\phi_1) = \pi^{\rho_\varepsilon^\vee}$, and hence $\mathcal{B}(\phi_1)$ is invariant under the reflection s_1. Similarly, it follows from (4.2.4) that $\mathcal{B}(\phi_{s_1}) = q\pi^{\rho_\varepsilon^\vee}$, and so we see that $\mathcal{B}(\phi_{s_1})$ is also invariant under the reflection s_1.

Next, we calculate c_{α_2}. Finding this intertwining constant is similar to the corresponding calculation for the Whittaker functional on GL(2). Let ϕ be an element of

$\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$ on which $\mathcal{B}$ is non-zero. A priori, we do not know that such an element exists - however, in our proof of (4.2.1) we showed that ϕ_1 is such a function. We see that

$$\mathcal{B}(A_{s_{\alpha_2}}\phi)(1) = \pi^{\rho_\varepsilon^\vee} \int_{Z_L(\mathfrak{o})} \int_{\overline{U}_A} \int_F \psi_A(\overline{u})\phi(s_2 x_{\alpha_2}(\tau)\overline{u}z)\, d\tau\, d\overline{u}\, dz,$$

where $x_{\alpha_2}(\tau)$, with $\tau \in F$, denotes an element of the α_2-root subgroup of G. Note that we only need to evaluate the functional at 1 in order to determine the intertwining constant. Using the rank 1 Bruhat decomposition

$$s_2 x_{\alpha_2}(\tau) = h_{\alpha_2}(\tau^{-1}) x_{\alpha_2}(\tau) x_{-\alpha_2}(\tau^{-1}),$$

where h_{α_2} is the semisimple subgroup of the embedded SL(2) triple corresponding to α_2, and excluding the point $\tau = 0$, we can rewrite this integral as

$$\int_{Z_L(\mathfrak{o})} \int_{\overline{U}_A} \int_{F^\times} \psi_A(\overline{u})\chi_{\mathrm{univ}}^{-1}(h_{\alpha_2}(\tau^{-1}))\phi(x_{-\alpha_2}(\tau^{-1})\overline{u}z)\, d\tau\, d\overline{u}\, dz.$$

After factoring $\overline{u}$ into root subgroups and performing some linear changes of variables, we find that

$$\begin{aligned} \mathcal{B}(A_{s_{\alpha_2}}\phi)(1) &= \pi^{\rho_\varepsilon^\vee} \int_{Z_L(\mathfrak{o})} \int_{\overline{U}_A} \psi_A(\overline{u})\phi(\overline{u}z) \int_{F^\times} \psi_A(-\omega\tau^{-1})\chi_{\mathrm{univ}}^{-1}(h_{\alpha_2}(\tau^{-1}))\, d\tau\, d\overline{u}\, dz \\ &= c_{\alpha_2}(s_{\alpha_2} \circ \mathcal{B}(\phi))(1), \end{aligned}$$

where

$$c_{\alpha_2} = \int_{F^\times} \psi_A(-\omega\tau^{-1})\chi_{\mathrm{univ}}^{-1}(h_{\alpha_2}(\tau^{-1}))\, d\tau.$$

This last integral can be evaluated by shells so that, after normalizing the Haar measure so that $m(x_{\alpha_2}(\mathfrak{o})) = 1$, we get the familiar Whittaker intertwining constant

$$c_{\alpha_2} = (\pi^{\alpha_2^\vee} - q^{-1}).$$

□

4.3 Proof of Theorem 1.0.2

In order to show that $\mathcal{B}$ is an $\mathcal{H}$-intertwiner as claimed in Theorem 1.0.2, we will need to know the action of T_{s_α} on $V_\varepsilon \cong R$ explicitly for simple reflections s_α. The calculation of this action follows easily from the Bernstein relation: for a basis element $\pi^\mu v_\varepsilon$, where, as before, v_ε denotes the eigenvector of $\mathcal{H}_0$ corresponding to ε, (2.1.1) gives

$$\begin{aligned} T_{s_\alpha} \cdot \pi^\mu v_\varepsilon &= \pi^{s_\alpha(\mu)}\varepsilon(T_{s_\alpha})v_\varepsilon + (1-q)\frac{\pi^{s_\alpha(\mu)} - \pi^\mu}{1-\pi^{-\alpha^\vee}}v_\varepsilon \\ &= \left(\varepsilon(T_{s_\alpha}) + \frac{1-q}{1-\pi^{-\alpha^\vee}}\right)\pi^{s_\alpha(\mu)}v_\varepsilon + \frac{q-1}{1-\pi^{-\alpha^\vee}}\pi^\mu v_\varepsilon, \end{aligned}$$

where in the second equality we have rearranged terms so that we can see how $T_{s_\alpha} \cdot \pi^\mu v_\varepsilon$ is expressed as a linear combination of $\pi^\mu v_\varepsilon$ and $\pi^{s_\alpha(\mu)}v_\varepsilon$ over R. Thus, regarding T_{s_α} as an operator on R, we see that T_{s_α} acts on $f \in R$ by

$$T_{s_\alpha} : f \mapsto \left(\varepsilon(T_{s_\alpha}) + \frac{1-q}{1-\pi^{-\alpha^\vee}}\right)f^{s_\alpha} + \frac{q-1}{1-\pi^{-\alpha^\vee}}f. \tag{4.3.1}$$

Proof of Theorem 1.0.2. The main result we need to prove is that $\mathcal{B}$ is indeed a left $\mathcal{H}$-module intertwiner from $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$ to V_ε, where ε is the character that acts by multiplication by -1 on long simple roots and acts by q on short simple roots. Once we have done this and checked that $\mathcal{F}(1_{T(\mathfrak{o})UJ}) = \mathcal{B}(\phi_1) = \pi^{\rho_\varepsilon^\vee}$, we can see that the diagram commutes since $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J \cong M \cong \mathcal{H}$. That the diagram commutes on $1_{T(\mathfrak{o})UJ}$ is a consequence of Lemma 4.2.3; as shown in the proof of Proposition 4.2.1, $\mathcal{B}(\phi_1) = \pi^{\rho_\varepsilon^\vee}$, and we observe that $\mathcal{F}(1_{T(\mathfrak{o})UJ}) = \mathcal{F}(1_{T(\mathfrak{o})UJ} * 1_J) = \pi^{\rho_\varepsilon^\vee}$.

In order to prove that $\mathcal{B}$ is a left $\mathcal{H}$-module intertwiner, it suffices to show that, for any $\phi \in \mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$,

$$\mathcal{B}(h \cdot \phi) = h \cdot \mathcal{B}(\phi),$$

for a set of generators $\{h\}$ for $\mathcal{H}$. In particular, we will choose our set of generators to be those elements of the form $\pi^\mu T_{s_\alpha}$ where $\mu \in X_*(T)$ and s_α is a simple reflection. Since π^μ acts by translation on both V_ε and $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})^J$, we can reduce to checking the equality on T_{s_α}.

From (4.1.1), we immediately see that

$$q^{-1}(1-\pi^{\alpha^\vee})\mathcal{B}(T_{s_\alpha} \cdot \phi) = \mathcal{B}(A_{s_\alpha}\phi) - (1-q^{-1})\pi^{\alpha^\vee}\mathcal{B}(\phi).$$

Now, we need to split into cases since the left action of $\mathcal{H}$ on elements of V_ε depends on the length of the simple root α. From Proposition (4.2.1), we see that

$$q^{-1}(1-\pi^{\alpha^\vee})\mathcal{B}(T_{s_\alpha} \cdot \phi) = \begin{cases} (1-q^{-1}\pi^{\alpha^\vee})(s_\alpha \circ \mathcal{B})(\phi) + (q^{-1}-1)\pi^{\alpha^\vee}\mathcal{B}(\phi) & \text{if } \alpha = \alpha_1 \\ (\pi^{\alpha^\vee} - q^{-1})(s_\alpha \circ \mathcal{B})(\phi) + (q^{-1}-1)\pi^{\alpha^\vee}\mathcal{B}(\phi) & \text{if } \alpha = \alpha_2. \end{cases}$$

Dividing by $q^{-1}(1-\pi^{\alpha^\vee})$, we see that the operator acting on $B(\phi)$ is

$$f \mapsto \frac{q}{1-\pi^{\alpha^\vee}} \begin{cases} (1-q^{-1}\pi^{\alpha^\vee})f^{s_\alpha} + (q^{-1}-1)\pi^{\alpha^\vee} f & \text{if } \alpha = \alpha_1 \\ (\pi^{\alpha^\vee} - q^{-1})f^{s_\alpha} + (q^{-1}-1)\pi^{\alpha^\vee} f & \text{if } \alpha = \alpha_2. \end{cases}$$

If we compare this with the operator that we got in (4.3.1) that described the action of T_{s_α} on R, we see that it matches it exactly in both cases, remembering that $\varepsilon(T_{s_1}) = q$ and $\varepsilon(T_{s_2}) = -1$. Thus, $\mathcal{B}(T_{s_\alpha} \cdot \phi) = T_{s_\alpha} \cdot \mathcal{B}(\phi)$ for any $\phi \in \mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$ and simple reflection s_α. $\square$

Chapter 5

Calculating Distinguished Vectors at Torus Elements

In this section, we will focus on calculating the images of distinguished vectors in unique models of the universal principal series of GSp(4). In 5.1, we will conclude our discussion of the Bessel functional with proofs of Theorem 1.0.3 and Theorem 1.0.4. Then, in Section 5.2, we will move on to discussing the connection between the Whittaker-Orthogonal models defined in [17] and the proposed $\mathcal{H}$-intertwiner corresponding to the fourth character, σ, of the finite Hecke algebra of GSp(4).

5.1 Calculating Distinguished Vectors in the Bessel Model

As mentioned in Section 1, we will use Theorem 1.0.2 to calculate the images of certain distinguished vectors under $\mathcal{B}$ on anti-dominant, integral torus elements - in particular, we will use it to prove Theorems 1.0.3 and 1.0.4. Before we can prove Theorem 1.0.3, we must first prove the following Iwahori factorization:

Lemma 5.1.1. $J = (J \cap B)(J \cap \overline{U}_A Z_L(\mathfrak{o}))$.

Proof. Using the usual Iwahori factorization, we can see that it suffices to show that the subgroup $x_{-\alpha_1}((\pi))$ of J is contained in $(J \cap B)(J \cap \overline{U}_A Z_L(\mathfrak{o}))$. To see that this is the case, observe that, for $\tau = u\pi^j$ with $u \in \mathfrak{o}^\times$ and $j > 0$, we can factor $x_{-\alpha_1}(\tau) = bh$, where

$$b = \begin{pmatrix} g & 0 \\ 0 & \det g \cdot (g')^{-1} \end{pmatrix} \text{ with } g = \begin{pmatrix} (1-\omega t^2)^{-1} & -\omega\tau(1-\omega\tau^2)^{-1} \\ 0 & 1 \end{pmatrix},$$

and

$$h = \begin{pmatrix} \gamma & 0 \\ 0 & \det \gamma \cdot (\gamma')^{-1} \end{pmatrix} \text{ with } \gamma = \begin{pmatrix} 1 & \omega\tau \\ \tau & 1 \end{pmatrix}.$$

□

proof of Theorem 1.0.3. We begin by looking at the right-hand side, $T_w\pi^\lambda \cdot v_\varepsilon$. We will use the commutativity of the diagram (1.0.4) and the dominance of λ to show that

$$\mathcal{B}(\phi_w * 1_{J\pi^\lambda J}) = T_w\pi^\lambda \cdot v_\varepsilon,$$

so that it suffices to show that

$$\mathcal{B}(\pi^{-\lambda} \cdot \phi_w) = \frac{1}{m(J\pi^\lambda J)}\mathcal{B}(\phi_w * 1_{J\pi^\lambda J}).$$

In order to see that this second equality holds, first note that $\pi^{-\lambda} \cdot \phi_w = \eta(1_{T(\mathfrak{o})UwJ\pi^\lambda})$ by definition (here we emphasize the definition of η as a vector-space isomorphism from $C_c(T(\mathfrak{o})U\backslash G)$ to $\mathrm{ind}_B^G(\chi_{\mathrm{univ}}^{-1})$). Now, if we look at $\mathcal{B}(\phi_w * 1_{J\pi^\lambda J}) = \mathcal{B}(\eta(1_{T(\mathfrak{o})UwJ} * 1_{J\pi^\lambda J}))$, we see from the definition of the convolution that

$$\mathcal{B}(\phi_w * 1_{J\pi^\lambda J}) = \int_{\overline{U}_A Z_L} \int_{J\backslash J\pi^{-\lambda}J} \int_J \tilde{\psi}(h)\eta(1_{T(\mathfrak{o})UwJ})(hj\gamma)\, dj\, d\gamma\, dh.$$

Using Lemma 5.1.1 and making the change of variables $h \mapsto hj^{-1}$, the integral above simplifies to

$$\mathcal{B}(\phi_w * 1_{J\pi^\lambda J}) = m(J\pi^\lambda J) \int_{\overline{U}_A Z_L} \tilde{\psi}(h)\eta(1_{T(\mathfrak{o})UwJ\pi^\lambda})(h)\, dh,$$

since the conductor of ψ is $\mathfrak{o}$. This is the equality that we wanted to establish.

To see that $\mathcal{B}(\phi_w * 1_{J\pi^\lambda J}) = T_w\pi^\lambda \cdot v$, we first note that

$$\phi_w * 1_{J\pi^\lambda J} = \eta((1_{T(\mathfrak{o})UJ} * T_w) * 1_{J\pi^\lambda J}) = \eta((T_w\pi^\lambda) \cdot 1_{T(\mathfrak{o})UJ}),$$

where the second equality follows because λ is dominant. Thus, using the commutativity of the diagram (1.0.4), we see that

$$\mathcal{B}(\phi_w * 1_{J\pi^\lambda J}) = T_w\pi^\lambda \cdot v_\varepsilon.$$

□

As noted at the beginning of the section, the linearity of $\mathcal{B}$ gives us the following immediate corollary regarding ϕ°:

Corollary 5.1.2. *For dominant λ,*

$$\mathcal{B}(\pi^{-\lambda} \cdot \phi^\circ) = \frac{1}{m(J\pi^\lambda J)} \sum_{w \in W} T_w \pi^\lambda \cdot v_\varepsilon.$$

In order to prove Theorem 1.0.4, we will to make use of an identity of operators on Frac(R). Recall that when we recorded the action of T_{s_α} as an operator on R in (4.3.1), it was with R regarded as a left $\mathcal{H}$-module with eigenvector 1. Our goal is to calculate the image of the spherical function in the model V_ε, and, as noted previously, R is isomorphic to V_ε under the isomorphism $1 \mapsto \pi^{\rho_\varepsilon^\vee}$. Then, extending the action of $\mathcal{H}$ to Frac(R), we realize the operator associated to T_{s_α} via this isomorphism (now regarded as an isomorphism of Frac(R)) as

$$\mathfrak{T}_{s_\alpha} := \pi^{\rho_\varepsilon^\vee} T_{s_\alpha} \pi^{-\rho_\varepsilon^\vee},$$

so that we can rewrite Corollary (5.1.2) as

$$\mathcal{B}(\pi^{-\lambda} \cdot \phi^\circ) = \frac{\pi^{-\rho_\varepsilon^\vee}}{m(J\pi^\lambda J)} \sum_{w \in W} \mathfrak{T}_w \pi^{\lambda + 2\rho_\varepsilon^\vee}.$$

Explicitly, the action of $\mathfrak{T}_{s_\alpha}$ on Frac(R) for a simple root α is given by

$$\mathfrak{T}_{s_\alpha} : f \mapsto \frac{q}{1 - \pi^{\alpha^\vee}} \begin{cases} (\pi^{\alpha^\vee} - q^{-1})\pi^{\alpha^\vee} f^{s_\alpha} + (q^{-1} - 1)\pi^{\alpha^\vee} f & \text{if } \alpha = \alpha_1 \\ (1 - q^{-1}\pi^{\alpha^\vee}) f^{s_\alpha} + (q^{-1} - 1)\pi^{\alpha^\vee} f & \text{if } \alpha = \alpha_2. \end{cases}$$

The operator identity that we will use is a deformed version of the Weyl character formula, established in [4] in a more general setting where G is only assumed to be split, connected and reductive. Let Ω denote the operator on Frac(R) given by the Weyl character formula-like expression

$$\Omega := \pi^{-\rho^\vee} \prod_{\alpha \in \Phi^+} (1 - \pi^{-\alpha^\vee})^{-1} \mathcal{A}(\pi^{-\rho^\vee}).$$

The deformation depends on the choice of character of the Hecke algebra, as described in the following theorem:

Theorem 5.1.3. *[4] If we have a character τ of $\mathcal{H}_0$ for G, then*

$$\sum_{w\in W} \mathfrak{T}_w = \left(\prod_{\alpha\in\Phi^+_{-1}} (1 - q\pi^{\alpha^\vee})\right) \Omega \left(\prod_{\alpha\in\Phi^+_{q}} (1 - q\pi^{\alpha^\vee})\right),$$

where Φ^+_{-1}, resp. Φ^+_q, are those positive roots that are the same length as the simple roots α such that $\tau(\alpha) = -1$, resp. $\tau(\alpha) = q$.

In the case of ε, $\Phi^+_{-1} = \{\alpha_2, 2\alpha_1 + \alpha_2\}$ and $\Phi^+_q = \{\alpha_1, \alpha_1 + \alpha_2\}$, and making these substitutions gives us Theorem 1.0.4. The image of the spherical function in the Bessel model evaluated on torus elements was already calculated by Bump, Friedberg and Furusawa in Corollary 1.8 in [3], and, indeed, it can be confirmed by observation that our formula matches theirs, up to normalization.[1]

5.2 Whittaker-Orthogonal Models and the Shalika character

In this section, we consider the character σ of $\mathcal{H}_0$, which was defined in Section 1 to be the character which acts by q on long simple roots and -1 on short simple roots. For each of the other three characters of $\mathcal{H}_0$ on GSp(4), we have found a subgroup $S \subset G$ such that the model formed by inducing S to G contains that character with multiplicity one - σ is the only character for which we have not found an explicit integral realization of $\mathcal{L}$ as in the diagram (1.0.4). However, even without this information, we can still say what the image of the spherical function under $\mathcal{L}$ in V_σ, evaluated on torus elements, would have to be, by the commutativity of (1.0.4) combined with Theorem 5.1.3. Upon completing this calculation, we notice that the result matches the image of the spherical function in the Whittaker-Orthogonal model (WO-model) defined by Bump, Friedberg, and Ginzburg in [17], which we record in Proposition 1.0.5.

The WO-model is defined as a representation on SO$(2n+2)$. Let $\overline{U}$ be the opposite unipotent radical of the parabolic subgroup of SO$(2n+2)$ with a Levi component that is diagonal except for a central SO(4) block and let ψ be a character of $\overline{U}$ defined as

$$\psi(\overline{u}) := \psi_0(\overline{u}_{21} + \overline{u}_{32} + \cdots + \overline{u}_{n-2,n-1} + \overline{u}_{n-1,n+1} + \overline{u}_{n-1,n+2}),$$

[1] In [3], they work with a choice of unramified principal series $\mathrm{ind}_B^G\chi$ instead of the universal principal series. The parameters denoted α_1, α_2 in [4] can be expressed as $\alpha_1^2 = \chi(\pi^{-(\alpha_1+\alpha_2)^\vee})$ and $\alpha_2^2 = \chi(\pi^{\alpha_1^\vee})$.

where ψ_0 is a nontrivial additive character of F with conductor $\mathfrak{o}$. Let $Z(\psi) \cong \mathrm{SO}(3)$ be the stabilizer of this character contained in the Levi. Then, for an unramified representation θ of $\mathrm{SO}(2n+2)$, we say that θ has a WO-model if there exists a nonzero linear functional WO on the representation space V_θ of θ such that

$$\mathrm{WO}(\theta(\overline{u}h)x) = \psi(\overline{u})\mathrm{WO}(x),$$

for $\overline{u} \in \overline{U}_\pi$, $h \in Z(\psi)$, and $x \in V_\theta$. The uniqueness of WO-models is established in Theorem 4.1 in [17]. The authors then show, in Theorem 4.2, that, if $\hat{\chi} = \mathrm{ind}_B^G(\chi)$ is an irreducible unramified principal series representation, then $\hat{\chi}$ admits a WO-model if and only if $\hat{\chi}$ is a *local lifting* of an unramified principal series representation of $\mathrm{Sp}(2n)$. We call $\hat{\chi}$ a local lifting from $\mathrm{Sp}(2n)$ if one of the Langlands' parameters of $\hat{\chi}$ is 1. The authors note that this is in conformity with Langlands' functoriality since the L-group of $\mathrm{Sp}(2n)$ is $\mathrm{SO}(2n+1)$, and that if one of the Langlands' parameters is 1 then the given conjugacy class is in the image of the inclusion of L-groups $\mathrm{SO}(2n+1) \hookrightarrow \mathrm{SO}(2n+2)$.[2]

Now, suppose that we have an unramified principal series representation of $\mathrm{SO}(6)$, $\hat{\chi} = \mathrm{ind}_B^G \chi$, where $\chi = (\chi_1, \ldots, \chi_{n+1})$ with $\chi_1, \ldots, \chi_{n+1}$ quasicharacters of $F^\times$. Let $z_i = \chi_i(\pi)$ for each i, and let $z = \chi(\pi)$. Then, if $z_{n+1} = 1$, we have the following formula from Theorem 4.3 in [17] for the image of the spherical vector of $\hat{\chi}$ under WO evaluated at torus elements π^λ, with $\lambda = (\lambda_1, \lambda_2, 0)$:

Theorem 5.2.1. *[17] Let* WO *be the WO-functional on* $V_{\hat{\chi}}$ *such that* $\mathrm{WO}(\phi^\circ) = 1$. *For* $\lambda \in X_*(T)$, *we have*

$$\mathrm{WO}(z^{-\lambda} \cdot \phi^\circ) = z_1^{-2\lambda_1} \cdot \frac{\mathcal{A}(z^{\rho^\vee} z_1^{\lambda_1}(1 - q^{-1}z_1^{-1}))}{\mathcal{A}(z^{\rho^\vee})}.$$

Remark. Note that all of the root data here are given with respect to the root system for $\mathrm{SO}(2n+2)$, with ρ defined analogously to how it was defined for $\mathrm{GSp}(4)$.

We will conflate $\hat{\chi}$ with the representation of $\mathrm{GSp}(4)$ of which it is a local lifting (and, hence, we will also conflate the spherical functions for both representations). In the following proof, let $\lambda_1, \lambda_2 \in X_*(T)$ with $\lambda_1 = (1,0)$ and $\lambda_2 = (0,1)$.

proof of Proposition 1.0.5. We begin by evaluating the two functionals at z^{λ_1} and π^{λ_1}, respectively, where λ_1 (resp. λ_2) is embedded in the cocharacter group of the torus of

[2] Recall that the L-group of $\mathrm{SO}(2n+2)$ is $\mathrm{SO}(2n+2)$.

SO(6) as $(1,0,0)$ (resp. $(0,1,0)$). In this case, we have that

$$\mathrm{WO}(z^{-\lambda}\cdot\phi^\circ) = \frac{\mathcal{A}(z^{\rho^\vee}z_1) - q^{-1}\mathcal{A}(z^{\rho^\vee})}{\mathcal{A}(z^{\rho^\vee})}.$$

In order to give explicit expressions for these alternators, we will need to make a choice of quasicharacters μ_i such that $\mu_i^2 = \chi_i$ for each i - there are two options for each μ_i, and we make one arbitrarily. Let $\xi_i = \mu_i(\pi)$ for each i. Then, evaluating these two alternator expressions, we find that

$$\mathcal{A}(\xi_1^3\xi_2) = \frac{(\xi_1^2+1)(\xi_1\xi_2+1)(\xi_1\xi_2-1)(\xi_2^2+1)(\xi_1+\xi_2)(\xi_1-\xi_2)}{\xi_1^3\xi_2^3}, \text{ and}$$
$$\mathcal{A}(\xi_1^5\xi_2) = \frac{\xi_1^4\xi_2^2+\xi_1^2\xi_2^4-\xi_1^2\xi_2^2+\xi_1^2+\xi_2^2}{\xi_1^2\xi_2^2}\cdot\mathcal{A}(\xi_1^3\xi_2).$$

Simplifying, we see that

$$\mathrm{WO}(z^{-\lambda}\cdot\phi^\circ) = \frac{\xi_1^4\xi_2^2+\xi_1^2\xi_2^4-\xi_1^2\xi_2^2+\xi_1^2+\xi_2^2-q^{-1}\xi_1^2\xi_2^2}{\xi_1^2\xi_2^2} = z_1+z_2-1+z_2^{-1}+z_1^{-1}-q^{-1}. \tag{5.2.1}$$

On the other hand, in order to calculate $\mathcal{F}(\pi^{-\lambda}\cdot 1_{T(\mathfrak{o})UK})$, where $\mathcal{F}$ is the functional from the universal principal series M to V_σ defined in the diagram (1.0.4), we can use Theorem 5.1.3 with $\Phi_{-1}^+ = \{\alpha_1, \alpha_1+\alpha_2\}$ and $\Phi_q^+ = \{\alpha_2, 2\alpha_1+\alpha_2\}$, along with the commutativity of (1.0.4). Hence, we have that

$$\begin{aligned}\mathcal{F}(\pi^{-\lambda}\cdot 1_{T(\mathfrak{o})UK}) &= \sum_{w\in W}\mathfrak{T}_w\cdot\pi^\lambda\\ &= N\cdot\frac{\pi^{2\lambda_1+\lambda_2}+\pi^{\lambda_1+2\lambda_2}-q^{-1}\pi^{\lambda_1+\lambda_2}-\pi^{\lambda_1+\lambda_2}+\pi^{\lambda_1}+\pi^{\lambda_2}}{\pi^{\lambda_1+\lambda_2}},\\ \text{where } N &= \frac{-(q^{-1}+1)(q^{-1}\pi^{\lambda_2}-\pi^{\lambda_1})(\pi^{\lambda_1+\lambda_2}-q^{-1})}{\pi^{2\lambda_1+\lambda_2}}.\end{aligned}$$

As defined in (1.0.4), $\mathcal{F}$ is not normalized so that $\mathcal{F}(1_{T(\mathfrak{o})UK}) = 1$, as WO is in Theorem 5.2.1. Indeed, we see that

$$\begin{aligned}\mathcal{F}(1_{T(\mathfrak{o})UK}) &= \sum_{w\in W}\mathfrak{T}_w\cdot 1\\ &= N.\end{aligned}$$

Thus, after normalizing $\mathcal{F}(1_{T(\mathfrak{o})UK}) = 1$, we see that

$$\mathcal{F}(\pi^{-\lambda}\cdot 1_{T(\mathfrak{o})UK}) = \pi^{\lambda_1}+\pi^{\lambda_2}-1+\pi^{-\lambda_2}+\pi^{-\lambda_1}-q^{-1},$$

which agrees with (5.2.1), indicating that the WO-functional is a lift of the proposed intertwiner corresponding to σ.

$\square$

Chapter 6

Unique Models and the Springer Correspondence

In this chapter, we assume all of the definitions, results, and notations discussed in Section 2.3. We begin by finishing up a thought from that section before describing how we expect to construct a gGGr containing V_τ in its J-fixed vectors for a given irreducible representation τ of $\mathcal{H}_0$.

As noted in Section 2.3, the trivial character and the sign character of $\mathcal{H}_0$ are connected to the spherical model and the Whittaker model, respectively, in the sense described above, and the character ε of $\mathcal{H}_0$ that acts by -1 on long simple roots and by q on short simple roots is similarly connected to the Bessel model for $G = \mathrm{SO}(2n+1)$ or $G = \mathrm{Sp}(4)$. Using the parametrization from Section 2.3, we see that these three characters correspond to $([n], \emptyset), (\emptyset, [1^n]), (\emptyset, [n])$, respectively, with the fourth character σ corresponding to $([1^n], \emptyset)$. In Section 6.1, we will describe a version of our conjectured connection between $\mathcal{H}_0$ and gGGr's in the context of $G = \mathrm{Sp}(4)$, and then, in Section 6.2, we will discuss the impact that initial investigations into extending the results of this paper to Sp(6) have had on the current state of this conjecture.

6.1 A First Pass: $G = \mathbf{GSp}(4)$

If $G = \mathrm{Sp}(4)$, the Springer correspondence can be represented in the following chart:

$(\mathbf{p}, \mathbf{q})$	$(\mathbf{d}, \mu_{\mathcal{O}})$
$(\emptyset, [1^2])$	$([1^4], 1)$
$([2], \emptyset)$	$([4], 1)$
$([1^2], \emptyset)$	$([2, 1^2], 1)$
$(\emptyset, [2])$	$([2^2], 1)$
$([1], [1])$	$([2^2], sgn)$

In particular, we note that, in this cse, we have four nilpotent orbits corresponding to the partitions $[4], [2^2], [2, 1^2]$, and $[1^4]$, and for each of these partitions, the group $A(\mathbf{d})$ is trivial except for the partition $[2^2]$, for which $A(\mathbf{d}) \cong Z/2$.

If we use Kawanaka's gGGr construction to build the Whittaker functional, we see that the nilpotent element A that we start with lives in the orbit [4] according to Theorem 2.3.3. If we do the same thing for the spherical functional, we see that we begin with an element of the orbit $[1^4]$. However, Brubaker, Bump, and Licata showed in [33] that the Whittaker functional is an intertwiner for the sign character of $\mathcal{H}_0$, which is associated to $[1^4]$ via the Springer correspondence, and Brubaker, Bump, and Friedberg showed in [5] showed that the spherical functional is an intertwiner for the trivial character of $\mathcal{H}_0$, which is associated to [4] via the Springer correspondence. This led to the conjecture that, if one starts with an irreducible representation of $\mathcal{H}_0$, then one should be able to construct a gGGr in which this representation is realized with multiplicity one from an element of the nilpotent orbit whose associated partition is the transpose of the partition associated to the nilpotent orbit associated to the original $\mathcal{H}_0$-representation via the Springer correspondence. In the example established in this paper, we see that the Bessel functional is associated to the character ε of $\mathcal{H}_0$, which, via the Springer correspondence is associated to the nilpotent orbit parametrized $[2, 1^2]$. However, the transpose of this partition is $[3, 1]$, which does not parametrize a nilpotent orbit in $\mathfrak{sp}(4)$. The issue here is that, while, for type A_{n-1}, the transpose is an order-reversing involution of the Hasse diagram for the nilpotent orbits of $\mathfrak{sl}(n)$, the analogous involution for type C_n is a bit more complicated. In particular, if $\mathbf{d}$ parametrizes a given nilpotent orbit, but $\mathbf{d}^\top$ does not, then we follow further instructions in [24] for how to manipulate $\mathbf{d}^\top$ in order to find the image of $\mathbf{d}$ in the order-reversing involution; these manipulations are referred to as the C-collapse of $\mathbf{d}^\top$. In the case of the partition $[3, 1] = [2, 1^2]^\top$, the C-collapse of this partition is $[2^2]$, which is exactly the orbit containing our original nilpotent element A in Section 3.1.

As for the fourth character that we discussed in Section 5.2, the Springer correspondence tells us that this character should be associated to the orbit $[2^2]$. Since $[2^2]$ is its own transpose, our conjecture tells us that the gGGr that should contain this character with multiplicity one is the Bessel model. However, this cannot be correct since the values for the spherical function on torus elements in the Bessel model do not agree with the values for the spherical function in a model that intertwines σ.

The search for a unique model containing the remaining irreducible representation of $\mathcal{H}_0$ - the two-dimensional representation $([1],[1])$ - also provides us with some avenues of future research. In Kawanaka's conjecture, there are no restrictions on the type of orbit from which the nilpotent element A is chosen, so we expect this connection between unique models and Hecke algebra representations to extend to higher-dimensional representations of $\mathcal{H}_0$. Attempting to do this will also require some development of the original conjecture, however, since, as seen from the table, the gGGr we would naively construct according to the conjecture would start with the nilpotent orbit $[2^2]$. It is not surprising that that this orbit seems to be overloaded, since, up to this point, we have completely ignored the group $A(\mathbf{d})$. Specifically, we would expect that the sign representation of $A([2^2])$ would be a factor in the construction of the gGGr for the representation$([1],[1])$. In fact, this seems to be borne out in Conjecture 2.4.5 in [2], where the connection between $A(\mathbf{d})$ and the gGGr appears to be realized in the extension of the character η_A from $\overline{U}_A$ to the representation $\tilde{\eta}_A$ on $Z_L(A) \ltimes \overline{U}_A$. In particular, recall that this extension is achieved by taking the tensor product of η_A and a representation of $Z_L(A)$. In each of the examples computed so far, we have had the trivial representation of $A(\mathbf{d})$ as part of the pair $(\mathbf{d}, \mu_{\mathbf{d}})$ associated to our irreducible representation of $\mathcal{H}_0$, and in each case we have taken $\tilde{\eta}_A = \eta_A \otimes 1$, but Kawanaka implies that there is a connection between the character $\mu_{\mathbf{d}}$ and the character of $Z_L(A)$ that we use in the definition of $\tilde{\eta}_A$. Figuring out this connection seems like another logical next step in this story.

6.2 The Next Step: $G = \mathbf{GSp}(6)$

We hope to extend the main results of this paper from $G = \mathrm{Sp}(4)$ to $G = \mathrm{Sp}(2n)$. In particular, we have some preliminary results that suggest that the gGGr that we believe should be connected to ε should itself be constructed using a nilpotent element A taken from the orbit $[2^n]$ and the trivial character of the stabilizer of η_A. Focusing on the case where $n = 3$, we record the Springer correspondence in the following chart:

$(\mathbf{p}, \mathbf{q})$	$(\mathbf{d}, \mu_{\mathcal{O}})$
$([3], \emptyset)$	$([6], 1)$
$([2,1], \emptyset)$	$([4, 1^2], 1)$
$([1^3], \emptyset)$	$([2, 1^4], 1)$
$([2], [1])$	$([4, 2], 1)$
$([1^2], [1])$	$([2^3], 1)$
$([1], [2])$	$([3^2], 1)$
$([1], [1^2])$	$([2^2, 1^2], 1)$
$(\emptyset, [3])$	$([4, 2], sgn)$
$(\emptyset, [2, 1])$	$([2^2, 1^2], sgn)$
$(\emptyset, [1^3])$	$([1^6], 1)$

In particular, we note that neither $([2^3], 1)$ nor $([3^2], 1)$ correspond to ε, as we might have predicted. Instead, we now believe that the path from an irreducible representation of the Hecke algebra to its associated gGGr goes through the Langlands dual group, ${}^L G$, of G (recall that both G and ${}^L G$ have the same Weyl group, so having this correspondence go through the dual group versus through G is not something that would be detectable from $\mathcal{H}$). Explicitly, our idea is that, in order to determine from which nilpotent orbit A should be chosen, we start with an irreducible representation τ of $\mathcal{H}_0$ and apply the Springer correspondence to ${}^L G$ to get the pair $(\mathbf{d}, \mu)$. We then take $\mathbf{d}'$ to be the image of $\mathbf{d}$ under the appropriate order-reversing involution of the set of nilpotent orbits, and pick A from the special orbit of G corresponding to $\mathbf{d}'$ under the bijection between the set of special nilpotent orbits of G and the set of special nilpotent orbits of ${}^L G$. In types B_n and C_n, a special nilpotent orbit is simply a nilpotent orbit $\mathbf{d}$ such that $\mathbf{d}^\top$ is also a nilpotent orbit.

In order to make this all a bit more concrete, we first take a step back and define the partial order on the set of nilpotent orbits that we referenced above as well as in the last section: geometrically, if $\mathcal{O}$ and $\mathcal{O}'$ are two nilpotent orbits, then we say that $\mathcal{O} \leq \mathcal{O}'$ if $\overline{\mathcal{O}} \subset \overline{\mathcal{O}}'$; translated to our parametrizations, we have that $\mathbf{d} \leq \mathbf{d}'$ if

$$\sum_{1 \leq j \leq k} d_j \leq \sum_{1 \leq j \leq k} d'_j \text{ for } 1 \leq k \leq N,$$

where $\mathbf{d} = [d_1, \ldots, d_N]$, $\mathbf{d}' = [d'_1, \ldots, d'_N]$ are partitions of N. This partial order on partitions is referred to as *dominance order*. Refocusing on the case where $G = \mathrm{Sp}(6)$, whence ${}^LG = \mathrm{SO}(7)$, we have the following list of special orbits, listed according to the partial order described above:

Sp(6)	SO(7)
$[6]$	$[7]$
$[4, 2]$	$[5, 1^2]$
$[3^2]$	$[3^2, 1]$
$[2^3]$	$[3, 2^2]$
$[2^2, 1^2]$	$[3, 1^4]$
$[1^6]$	$[1^7]$

(6.2.1)

The bijection between orbits of G and ${}^LG$ mentioned above is simply the one suggested by the partial ordering, which is depicted in (6.2.1). Thus, according to our revamped conjecture, we see that ε corresponds to the pair $([3^2, 1], 1)$ for ${}^LG = \mathrm{SO}(7)$. Since $[3^2, 1]$ is a special orbit, its transpose $[3, 2^2]$ is its image under the usual order-reversing involution, and we see that $[3, 2^2]$ corresponds to $[2^3]$ under the bijection between special nilpotent orbits of G and special nilpotent orbits of ${}^LG$, as desired.

We also point out that the trivial character of $\mathcal{H}_0$ still corresponds to $[1^4]$ under this new conjecture, and the sign character still corresponds to $[4]$. One can check that this new conjecture is also compatible with our results for $\mathrm{Sp}(4)$. Additionally, one can check that this conjecture is also compatible with the results of [4], in which $G = \mathrm{SO}(2n+1)$ and ${}^LG = \mathrm{Sp}(2n)$.

However, just as with our original conjecture set forth in Section 6.1, we are still completely ignoring the group $A(\mathcal{O})$ in this updated conjecture. The connection between the character $\mu_{\mathcal{O}}$ and $Z_L(A)$ is slightly more mysterious given this longer path from $\mathcal{O}$

to A, but, as explained in the last section, we can still be sure that the final version of the conjecture will prominently feature the role played by $\mu_{\mathcal{O}}$ in the construction of the associated gGGr.

Chapter 7

The Bessel Model for GSp$(2n)$

In this section, we will give a description of what we believe to be the Bessel model for $G = \mathrm{GSp}(2n)$, along with a proof that this model is a unique model for the universal principal series. We will employ the same notation as we used in Chapter 3 (extended appropriately, in some cases). In particular, we will explictly define G as

$$G := \{G \in M_{2n}(F) \mid g^\top \Omega g = k\Omega, k \in \mathbb{F}^\times,$$

where

$$\Omega = \begin{pmatrix} & -\Omega' \\ \Omega' & \end{pmatrix}$$

and Ω' is the $n \times n$ matrix with 1's on the antidiagonal. Let $\alpha_1, \ldots, \alpha_n$ denote the simple roots of the root system Φ, and let $s_1, \ldots, s_n$ denote the corresponding simple reflections in W.

7.1 The Bessel Model for GSp$(2n)$ as a Generalized Gelfand-Graev Representation

As discussed in Section 6.2, we believe that the nilpotent orbit that we will want to use to construct our gGGr is $[2^n]$; this means that the nilpotent element A will be an element of the subalgebra $\sum \alpha$, where the sum is taken over the long roots in Φ. It also means that the parabolic subgroup P_A will contain the subgroups corresponding to the negative short simple roots; we define $L_A, U_A, \overline{U}_A$ analogously to Section 3.1.

Let ψ_0 be a non-degenerate additive character on F^+, and let $\psi_A(u) = \psi_0(\mathrm{tr}(ru'))$ for $u \in \overline{U}_A$, where u' is the lower $n \times n$ block of u and $r \in M_n(F)$. We assume that r is non-degenerate.

In this case, we find that $Z_L = Z_L(A)$ is the subgroup of L_A with $\mathrm{GSO}(n)$ blocks on the diagonal according to the symmetric bilinear form Aw_0, where w_0 is the long Weyl element. We extend ψ_A to a character $\tilde{\psi}_A$ on $\overline{U}_A \rtimes Z_L$ in the usual way, and define the Bessel model to be the gGGr $\mathrm{Ind}_{\overline{U}_A Z_L(\mathfrak{o})}^{G}(\tilde{\psi}_A)$, where $Z_L(\mathfrak{o}) := Z_L \cap \mathrm{SL}(2n, \mathfrak{o})$.

Extending the GSp(4) case, we define a linear functional $\mathcal{B}$ on the representation space V_θ of an irreducible admissible representation θ of G to be a Bessel functional if

$$\mathcal{B}(\theta(ut)v) = \tilde{\psi}_A(ut)\mathcal{B}(v),$$

for $v \in V_\theta, t \in Z_L(\mathfrak{o})$, and $u \in \overline{U}_A$.

7.2 Uniqueness of Bessel Functionals on GSp$(2n)$ for Principal Series Representations

The argument for the uniqueness of $\mathcal{B}$ on GSp$(2n)$ for the universal principal series follows the same arc as the proof given in the GSp(4) case given in Section 3.2. In fact, most of the argument can be transferred over, virtually unchanged, so in this section we will highlight the only true obstacle in generalizing this result to the rank n case, which is the reorganization of the Bruhat decomposition that we will need to use. Even this task is done in analogy with how it was handled in Section 3.2, but the nature of that analogy requires some explanation.

Proposition 7.2.1. *Let $W_L = \langle s_i \rangle_{i<n}$. The group $G = \mathrm{GSp}(2n)$ can be decomposed into the disjoint union*

$$G = \bigsqcup_{w \in W/W_L} Bw\overline{U}_A Z_L \mathfrak{o}.$$

Proof. As in Section 3.2, we begin with the Bruhat decomposition

$$G = \bigsqcup_{w \in W} Bw\overline{U}.$$

This decomposition tells us that every element of G lies in at least one double coset of the form

$$Bw\overline{U}_L\overline{U}_A Z_L(\mathfrak{o}), \text{ where } \overline{U}_L = \left(\prod_{\alpha\in\Phi_L^-} u_\alpha\right),$$

and where $\Phi_L = \{\alpha \in \Phi \mid x_\alpha(F) \subset L_A\}$, and $\Phi_L^- := \Phi_L \cap \Phi^-$. In fact, since $s_i \in Z_L(\mathfrak{o})$ for each $i < n$, we can refine this decomposition and assert that each element of G lies in at least one double coset of the form $BwU_L\overline{U}_A Z_L(\mathfrak{o})$ where U_L is the product of root subgroups whose corresponding roots are in $\Phi_L^+ := \Phi_L \cap \Phi^+$, and where w is the coset representative from the group W/W_L with minimal length. Note that, at this point, our decomposition becomes disjoint again. Finally, observe that, for any such w, $wuw^{-1} \in B$, and hence

$$G = \bigsqcup_{w\in W/W_L} Bw\overline{U}_A Z_L(\mathfrak{o}).$$

□

Recall from Section 3.2 that one step in determining the uniqueness of the model is showing that the subspace of R-distributions on G satisfying (3.2.3) is one-dimensional. In particular, we noted that, according to [31], any double coset that didn't satisfy the compatibility criterion (3.2.4) could not be part of the support of any distribution satisfying (3.2.3). This result still applies to our current case, $G = \mathrm{GSp}(2n)$, and, hence, Lemma 3.2.2 still applies, its proof unaltered. Since $w(-\alpha_n) \in \Phi^+$ for $w \notin W_L$, Lemma 3.2.2 tells us that any distribution satisfying (3.2.3) must vanish off of $B\overline{U}_A Z_L(\mathfrak{o})$, implying that this substace of distributions is at most one-dimensional. From here, we see that the Bessel model on $\mathrm{GSp}(2n)$ is a unique model for the universal principal series.

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