Vertex Magic Group Edge Labelings

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Dedication

To my loving and supporting family.

Abstract

A vertex-magic group edge labeling of a graph $G(V, E)$ with $|E| = n$ is an injection from E to an abelian group Γ of order n such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. We completely characterize all Cartesian products $C_n \square C_m$ that admit a vertex-magic group edge labeling by $\mathbb{Z}_{2nm}$, as well as provide labelings by a few other finite abelian groups.

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Chapter 1

Introduction

A graph labeling is an assignment of values, traditionally integers, to the edges, vertices, or both, of a graph. Formally stated, given a graph $G = (V, E)$, a labeling of G is a function from V, E, or both, to a set of labels. In the above definition, a graph is understood to be a finite undirected simple graph. However, the notion of a graph labeling has been extended to many other generalizations of graphs. There are also many different types of graph labelings. For instance, a vertex labeling is function from V to a set of labels, and similarly, an edge labeling is a function from E to a set of labels.

Most graph labelings trace their origins to labelings presented by Alex Rosa in his 1967 paper. Rosa identified three types of labelings, which he called α-, β-, and ρ-labelings [22]. β-labelings were later renamed graceful by S.W. Golomb. This naming has remained popular to this day.

Over the years many papers have been written on a vast array of graph labeling methods. Due to the nature of the subject, there are few general results on graph labelings. Instead, many of the papers focus on particular classes of graphs and labeling methods. It is common in these papers to be given a specific construction that will ensure the given type of labeling is satisfied. It is also frequent that the same classes of graphs and labeling methods have been written about by several authors leading to some terminology being used to represent more than one concept, or different terminology meant to represent the same ideas. A thorough compilation of various graph labelings is being maintained by Gallian in his Dynamic Survey of Graph Labelings [15].

In this project we discuss a major graph labeling method known as magic labelings and its variations. We then provide our own magic labelings of a particular class of graphs using finite abelian groups instead of the traditional integer labels. Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedlacek in 1963 [23]. In general, a magic-type labeling is a labeling in which we require the sum of labels related to a vertex (for a vertex magic labeling) or to an edge (for an edge magic labeling) to be constant throughout the entire graph. This sum of labels is referred to as the weight of a vertex or an edge depending on the type of labeling.

We are motivated to look at various graph labelings by group elements because of the similar structure many graphs and groups share. The most obvious example of such being cycles and cyclic groups. In this project, we use the structure of cyclic subgroups and cosets of the group $\mathbb{Z}_{2nm}$ to construct vertex-magic edge labelings for Cartesian products of cycles $C_n \square C_m$. A vertex-magic group edge labeling of a graph $G(V, E)$ with $|E| = n$ is an injection from E to an abelian group Γ of order n such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. We completely characterize all Cartesian products $C_n \square C_m$ that admit a vertex-magic group edge labeling by $\mathbb{Z}_{2nm}$.

We give definitions and describe magic labelings in more detail in Chapter 2, before delving into our own results on vertex-magic group edge labelings.

Chapter 2

Known Results

We focus on magic-type labelings that relate to our problem in some way, shape, or form, whether that be they use group elements as labels, or are related to a product of cycles. There are many different types of products of graphs that will be discussed. We give definitions of such products here. The first and most important for our research is that of the Cartesian product.

Definition 2.0.1. The *Cartesian graph product* $G = G_1 \Box G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2 respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if either $u_1 = v_1$ and u_2 is adjacent with v_2 in G_2 or, $u_2 = v_2$ and u_1 is adjacent with v_1 in G_1.

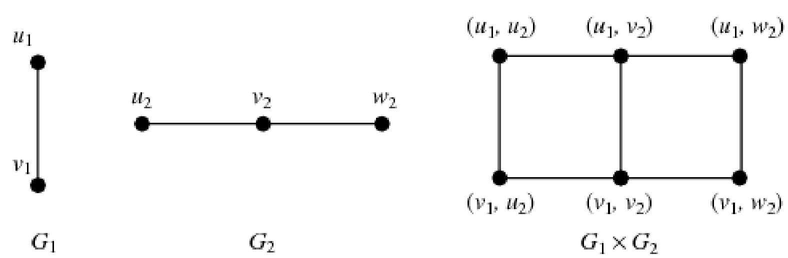


Figure 2.1: Cartesian Graph Product

A Cartesian product of cycles is then simply a Cartesian graph product where each graph is a cycle of varying length. Some other types of graph products that show up in our discussion of magic labelings are the following.

Definition 2.0.2. The *direct product (also known as the tensor product)* $G = G_1 \times G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2 respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent with v_2 in G_2.

Definition 2.0.3. The *lexicographic product* $G = G_1 \circ G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2 respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if either u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 is adjacent with v_2 in G_2.

2.1 Vertex Magic Total Labelings

Definition 2.1.1. Let $G(V, E)$ be a graph with vertex set V and edge set E. A one-to-one mapping $\lambda : V \cup E \to \{1, \ldots, |V| + |E|\}$ is called a *vertex magic total labeling* if there is a constant k such that for every vertex $v \in V$, $\lambda(v) + \sum \lambda(uv) = k$ where the sum is over all vertices u adjacent to vertex v. The constant k is called the magic constant for λ.

In [14], Froncek, Kovar, and Kovarova proved the following about vertex magic total labelings of a product of cycles:

Theorem 2.1.2. *For each $m, n \geq 3$ and n odd, there exists a vertex magic total labeling of $C_m \square C_n$ with magic constant $k = \frac{1}{2}m(15n + 1) + 2$.*

2.2 Distance Magic Labelings

Definition 2.2.1. A *distance magic labeling* of a graph $G(V, E)$ with $|V| = n$ is an injection from V to the set $\{1, 2, \ldots, n\}$ such that the sum of the labels of all neighbors of every vertex $x \in V$, called the *weight* of x is equal to the same *magic constant* μ.

Distance magic labelings of graphs have been studied by many authors. This labeling has also been called a *1-vertex magic vertex labeling* or a $\sum$*-labeling*. Rao, Singh, and Parameswaran proved in [21] the following result on Cartesian products of cycles.

Theorem 2.2.2. *The graph $C_k \Box C_m$ is distance magic if and only if $k = m$ and $k, m \equiv 2 \pmod 4$.*

Based on this notion of distance magic graphs, Froncek introduced the concept of group distance magic labeling defined in [13]. This leads us to our discussion of magic labelings by group elements.

Definition 2.2.3. A Γ-*distance magic labeling* of a graph $G(V, E)$ with $|V| = n$ is an injection from V to an abelian group Γ of order n such that the sum of labels of all neighbors of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. If a graph G is Γ-distance magic for every abelian group Γ, then G is said to be group-distance magic.

Many results on group distance magic labelings have been obtained by a variety of authors, many of which have collaborated on numerous papers on the subject. A few of their results are now provided.

Together, Cichacz and Froncek in [8] and [13] showed the following.

Theorem 2.2.4. *For an r-regular distance magic graph G on n vertices, where r is odd there does not exist an abelian group Γ of order n having exactly one involution (an element that is its own inverse) that is Γ-distance magic.*

Theorem 2.2.5. *$C_m \Box C_n$ is a $\mathbb{Z}_{mn}$-distance magic graph if and only if mn is even.*

Theorem 2.2.6. *$C_{2^n} \Box C_{2^n}$ has a $\mathbb{Z}_2^{2n}$-distance magic labeling.*

Cichacz also showed some Γ-distance magic labelings for $C_m \Box C_n$ where $\Gamma \not\approx \mathbb{Z}_{mn}$ and $\Gamma \not\approx \mathbb{Z}_2^{2n}$.

In [5], Anholcer, Cichacz, Peterin, and Tepeh proved the following.

Theorem 2.2.7. *If an r_1-regular graph G_1 is Γ_1-distance magic and an r_2-regular graph G_2 is Γ_2-distance magic, then the direct product of G_1 and G_2 is $\Gamma_1 \times \Gamma_2$-distance magic.*

Theorem 2.2.8. *If G is an r-regular graph of order n and $m = 4$ or $m = 8$ and r is even, then $C_m \Box G$ is group distance magic.*

Theorem 2.2.9. *$C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic if and only if $m \in 4, 8$, $n \in 4, 8$, or $m, n \equiv 0 \pmod 4$.*

Theorem 2.2.10. *If $m, n \not\equiv 0 \pmod 4$ then $C_m \times C_n$ is not Γ-distance magic for an abelian group Γ of order mn.*

Cichacz [9] also gave necessary and sufficient conditions for complete k-partite graphs of odd order p to be $\mathbb{Z}_p$-distance magic, as well as showed the following.

Theorem 2.2.11. *If $p \equiv 2 \pmod 4$ and k is even, then there does not exist a group Γ of order p that admits a Γ-distance magic labeling for a k-partite complete graph of order p.*

Theorem 2.2.12. *$\mathbb{K}_{m,n}$ is a group distance magic graph if and only if $n + m \not\equiv 2 \pmod 4$.*

Theorem 2.2.13. *If G is an Eulerian graph, then the lexicographic product of G and C_4 is group distance magic.*

Theorem 2.2.14. *If $m + n$ is odd, then the lexicographic product of $\mathbb{K}_{m,n}$ and C_4 is group distance magic.*

In yet another paper [10], Cichacz gave necessary and sufficient conditions for direct product of $\mathbb{K}_{m,n}$ and C_4 for $m + n$ odd and for $\mathbb{K}_{m,n} \times C_8$ to be group distance magic. Lastly Cichacz proved the following [10].

Theorem 2.2.15. *For n even and $r \geq 2$, the Cartesian product of the complete r-partite graph $\mathbb{K}_{n,n,\ldots,n}$ and C_4 is group distance magic.*

In [4], Anholcer, Chichacz, Peterin, and Tepeh introduced the notion of *balanced distance magic graphs*, and proved the following theorems.

Definition 2.2.16. A distance magic graph G with an even number of vertices is *balanced* if there exists a bijection f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that for every vertex w the following holds: If $u \in N(w)$ with $f(u) = i$, then there exists $v \in N(w), u \neq v$ with $f(v) = |V(G)| - i + 1$.

Theorem 2.2.17. *A graph G is balanced magic if and only if G is regular and $V(G)$ can be partitioned in pairs $(u_i, v_i), i \in \{1, 2, \ldots, |V(G)|/2\}$, such that $N(u_i) = N(v_i)$ for all i.*

Using this characterization, they were able to prove the following theorems.

Theorem 2.2.18. *If G is a regular graph and H is a graph not isomorphic to $\overline{K_n}$ where n is odd, then $G \circ H$ is a balanced distance magic graph if and only if H is a balanced* *distance magic graph.*

Theorem 2.2.19. *$G \times H$ is balanced distance magic if and only if one of G and H is balanced distance magic and the other one is regular.*

Theorem 2.2.20. *$C_m \times C_n$ is distance magic if and only if $n = 4$ or $m = 4$ or $m, n \equiv 0 \pmod 4$*

Theorem 2.2.21. *$C_m \times C_n$ is balanced distance magic if and only if $n = 4$ or $m = 4$.*

In [6] they proved the following:

Theorem 2.2.22. *Every balanced distance magic graph is also group-distance magic.*

Theorem 2.2.23. *The direct product of C_4 or C_8 and a regular graph is group-distance magic.*

Theorem 2.2.24. *$C_8 \times G$ is group-distance magic for any even-regular graph G.*

Theorem 2.2.25. *$C_{4s} \times C_{4t}$ is $A \times B$-distance magic for any Abelian groups A and B of order $4s$ and $4t$ respectively.*

This led them to conjecture that $C_{4m} \times C_{4n}$ is a group distance magic graph.

Theorem 2.2.26. *$C_m \times C_n$ is $\mathbb{Z}_{mn}$-distance magic if and only if $m \in \{4, 8\}$ or $n \in \{4, 8\}$ or both n and m are divisible by 4.*

Theorem 2.2.27. *$C_m \times C_n$ with orders not divisible by 4 is not Γ-distance magic for any Abelain group Γ of order mn.*

We now turn our attention to another type of magic labeling by group elements.

2.3 $\mathbb{A}$-Magic Labelings

Definition 2.3.1. For any nontrivial abelian group $\mathbb{A}$ under addition, a graph G is said to be $\mathbb{A}$-*magic* if there exists a labeling f of the edges of G with nonzero elements of $\mathbb{A}$ such that the vertex weight $w(v) = \sum f(vu)$ over all edges vu is constant.

Now at first glance, $\mathbb{A}$-magic seems to be almost identical to our notion of vertex magic group edge labeling. There is however a major difference in the two. In $\mathbb{A}$-magic labelings, there is no injection between the elements of $\mathbb{A}$ and the edges in the graph being labeled. This makes the task of obtaining magic much easier, as can be seen by the many results obtained about this form of magic. In $\mathbb{A}$-magic, there is also no 0 element.

In [27] and [28], Stanley noted that $\mathbb{Z}$-magic graphs can be viewed more generally by linear homogeneous diophantine equations. Shiu, Lam, and Sun have shown the following in [24].

Theorem 2.3.2. *The union of two edge disjoint $\mathbb{A}$-magic graphs with the same vertex set is $\mathbb{A}$-magic.*

Theorem 2.3.3. *The Cartesian product of two $\mathbb{A}$-magic graphs is $\mathbb{A}$-magic.*

Theorem 2.3.4. *The lexicographic product of two $\mathbb{A}$-magic connected graphs is $\mathbb{A}$-magic.*

Theorem 2.3.5. *For an abelian group $\mathbb{A}$ of even order, a graph is $\mathbb{A}$-magic if and only if the degrees of all of its vertices have the same parity.*

Theorem 2.3.6. *If G and H are connected and $\mathbb{A}$-magic, $G \circ H$ is $\mathbb{A}$-magic.*

Theorem 2.3.7. *If $K_{m,n}$ is $\mathbb{A}$-magic then $m, n \geq 2$ and $\mathbb{A}$ has order at least 4.*

Theorem 2.3.8. *K_n with an edge deleted is $\mathbb{A}$-magic when $n \geq 4$ and $\mathbb{A}$ has order at least 4.*

Definition 2.3.9. A *θ-graph* is a block with two non-adjacent vertices of degree 3 and all other vertices of degree 2.

Theorem 2.3.10. *All generalized θ-graphs are $\mathbb{A}$-magic when $\mathbb{A}$ has oder at least 4.*

Theorem 2.3.11. *$C_n + \overline{K_m}$ is $\mathbb{A}$-magic when $n \geq 3$ and $m \geq 2$ and $\mathbb{A}$ has order at least 2.*

Definition 2.3.12. A *wheel graph* is a graph formed by connecting a single vertex to all vertices of a cycle

Theorem 2.3.13. *Wheels are $\mathbb{A}$-magic when $\mathbb{A}$ has order at least 4.*

When the magic constant of an $\mathbb{A}$-magic graph is zero, the graph is called a *zero-sum $\mathbb{A}$-magic*. Akbari, Ghareghani, Khosrovshahi, and Zare in [1] along with Akbari, Kano, and Zare in [2] proved:

Theorem 2.3.14. *The null set $N(G)$ of a graph G (the set of all positive integers h such that G is zero-sum $\mathbb{Z}_h$-magic) of an r-regular graph G, $r \geq 3$, does not contain the numbers $2, 3$ or 4.*

Akbari, Rahnati, and Zare proved the following in [3].

Theorem 2.3.15. *If G is an even regular graph then G is zero-sum $\mathbb{Z}_h$-magic for all h.*

Theorem 2.3.16. *If G is an odd r-regular graph, $r \geq 3$ and $r \neq 5$ then $N(G)$ contains all positive integers except 2 and 4.*

Theorem 2.3.17. *If an odd regular graph is also 2-edge connected then $N(G)$ contains all positive integers except 2.*

Theorem 2.3.18. *A 2-edge connected bipartite graph is zero-sum $\mathbb{Z}_h$-magic for $h \geq 6$.*

They also determined the null set of 2-edge connected bipartite graphs, described the structure of some odd regular graphs, $r \geq 3$, that are not zero-sum 4-magic, and described the structure of some 2-edge connected bipartite graphs that are not zero-sum $\mathbb{Z}_h$-magic for $h = 2, 3, 4$. They also conjectured that every 5-regular graph admits a zero-sum 3-magic labeling.

In [20], Lee, Saba, Salehi, and Sun investigated graphs that are $\mathbb{A}$-magic where $\mathbb{A} = \mathbb{V}_4$ the Klein four-group. Many of the theorems in that paper are special cases of the results mentioned above. They also proved that the following were $\mathbb{V}_4$-magic: a tree if and only if every vertex has odd degree; the star $K_{1,n}$ if and only if n is odd; $K_{m,n}$ for all $m, n \geq 2$; the edge deleted $K_n - e$ when $n \geq 3$; even cycles with k pendant edges if and only if k is even; odd cycles with k pendant edges if and only if k is a common edge; and $C_n + \overline{\mathbb{K}_2}$; generalized theta graphs; graphs that are copies of C_n that share a common edge; and $G + \overline{K_2}$ whenever G is $\mathbb{V}_4$-magic.

In [12] Choi, Georges, and Mauro investigated $\mathbb{Z}_2^k$-magic graphs in terms of even edge-coverings, graph parity, factorability, and nowhere-zero 4-flows. They proved the following.

Theorem 2.3.19. *The minimum k such that bridgeless G is zero-sum $\mathbb{Z}_2^k$-magic is equal to the minimum number of even subgraphs that cover the edges of G, known to be at most 3.*

Theorem 2.3.20. *A bridgeless graph G is zero-sum $\mathbb{Z}_2^k$-magic for all $k \geq 2$ if and only if G has a nowhere-zero 4-flow.*

Theorem 2.3.21. *G is zero-sum $\mathbb{Z}_2^k$-magic for all $k \geq 2$ if G is Hamiltonian, bridgeless planar, or isomorphic to a bridgeless complete multipartite graph.*

They also established equivalent conditions for graphs of even order with bridges to be $\mathbb{Z}_2^k$-magic for all $k \geq 4$. In [16] Georges, Mauro, and Wang were able to use well-known results about edge-colorings to construct infinite families that are $\mathbb{V}_4$-magic but not $\mathbb{Z}_4$-magic.

2.4 Anti-Magic Labelings

A somewhat inverse notion to *magic labelings* was introduced in 1990 by Hartsfield and Ringel [17]. This being the notion of *anti-magic*. Vaguely speaking, in an anti-magic labeling, we want that the weight of every vertex is different rather than the weights all being the same as in a magic labeling. We discuss anti-magic labelings here in hopes that we can utilize the structure of these labelings to connect anti-magic components in a way to make our entire labeling magic. An anti-magic labeling of a finite simple undirected graph with p vertices and q edges is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, q\}$ such that the vertex sums are pairwise distinct, where the vertex sum at one vertex is the sum of labels of all incident edges to that vertex. A graph is called *anti-magic* if it admits and anti-magic labeling. A more formal definition is as follows:

Definition 2.4.1. For a graph $G(V, E)$ with p vertices and q edges and without any isolated vertices, an *anti-magic* edge labeling is a bijection $f : E \to \{1, 2, \ldots, q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{N}$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective.

Hartsfield and Ringel proved that $P_n (n \geq 3)$, cycles, wheels, and $K_n (n \geq 3)$ are all anti-magic. More closely related to our work are the results of Tao-Ming Wang [31]. He provided the following results.

Theorem 2.4.2. *The generalized toroidal grid graphs, i.e the Cartesian products of cycles, $C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}$, are anti-magic.*

Theorem 2.4.3. *Graphs of the form $G \square C_n$ are anti-magic it G is an r-regular anti-magic graph with $r > 1$.*

In [11], Cheng proved the following.

Theorem 2.4.4. *All Cartesian products of two or more regular graphs of positive degree are anti-magic.*

2.5 Supermagic Labelings

In the mid 1960's, Stewart introduced the following types of magic labelings very closely related to our notion of vertex-magic group edge labelings.

Definition 2.5.1. A connected graph is called *semi-magic* if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v.

Definition 2.5.2. A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic labeling.*

Definition 2.5.3. A graph is called *supermagic* if it admits a labeling of the edges by pairwise different consecutive positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex.

First note that this notion of supermagic labeling is exactly the same as our notion of vertex-magic edge labeling. Also note that any graph that admits a magic labeling of any kind using consecutive positive integers can also be magically labeled using the cyclic group of order n. This is because in the labeling by integers the weight of each element in the graph is equal to the same magic constant, and thus all weights will also be congruent modulo n when labeled the same way with the cyclic group of order n.

In [29] and [30], Stewart proved the following theorems.

Theorem 2.5.4. *K_n is magic for $n = 2$ and all $n \geq 5$.*

Theorem 2.5.5. *$K_{n,n}$ is magic for all $n \geq 3$.*

Theorem 2.5.6. *Fans F_n are magic if and only if n is odd and $n \geq 3$.*

Theorem 2.5.7. *Wheels W_n are magic for $n \geq 4$, and W_n with one spoke deleted is magic for $n = 4$ and for $n \geq 6$.*

Theorem 2.5.8. *$K_{m,n}$ is semi-magic if and only if $m = n$.*

Theorem 2.5.9. *K_n is supermagic for $n \geq 5$ if and only if $n > 5$ and $n \not\equiv 0 \pmod 4$.*

In [23], Sedlacek showed the following.

Theorem 2.5.10. *Mobius ladders M_n are supermagic when $n \geq 3$ and n is odd.*

Theorem 2.5.11. *$C_n \times P_2$ is magic, but not supermagic, when $n \geq 4$ and n is even.*

Shiu, Lam, and Lee proved the following in [25].

Theorem 2.5.12. *The composition of C_m and $\overline{K_n}$ is supermagic when $m \geq 3, m \not\equiv 0 \pmod 4$.*

Theorem 2.5.13. *If G is an r-regular supermagic graph, then so is the composition of G and $\overline{K_n}$ for $n \geq 3$.*

In [18], Ho and Lee proved the next result.

Theorem 2.5.14. *The composition of K_m and $\overline{K_n}$ is supergmagic for $m = 3$ or 5 and $n = 2$ or n odd.*

In [26], Shiu Lam, and Cheng proved the following.

Theorem 2.5.15. *For $n \geq 2$, $mK_{n,n}$ is supermagic if and only if n is even or both m and n are odd.*

In [19], Ivanco gives some constructions of supermagic labelings of regular graphs and completely characterizes supermagic regular complete multipartite graphs and supermagic cubes. We will discuss a few of his discoveries that are directly related to our topic.

In section 4 of his paper [19], Ivanco discusses supermagic labelings of the Cartesian products of cycles. He provides proof for the following theorems.

Theorem 2.5.16. *$C_n \Box C_n$ is a supermagic graph for any $n \geq 3$.*

Theorem 2.5.17. *Let $n \geq 2, k \geq 2$ be integers. Then $C_{2n} \Box C_{2k}$ is a supermagic graph.*

As mentioned above, this gave us that the Cartesian product of two cycles of even length is not only a supermagic graph, but it is also a vertex-magic group edge label-able graph. The previous two theorems led Ivanco to suggest the following conjecture:

Conjecture 2.5.1. *$C_n \Box C_k$ is a supermagic graph for any $n, k \geq 3$.*

This conjecture led us to look at the perhaps easier problem of labeling the product of two cycles using group elements, which we will discuss in the next chapter. Ivanco concludes his paper by characterizing supermagic cubes, but in doing so he first gives the following:

Theorem 2.5.18. *$C_4 \square C_4 \square C_4$ is a supermagic graph.*

If $C_4 \square C_4 \square C_4$ is a supermagic graph, then it must also be vertex-magic group edge label-able. This result in turn made us consider the possibility that any product of any number of cycles could have a group edge labeling that produces vertex-magic.

Chapter 3

Vertex Magic Group Edge Labelings

As mentioned above, we are motivated to look at various graph labelings by group elements because of the similar structure many graphs and groups share. The most obvious example of such being cycles and cyclic groups. In this section we show how to utilize these similarities, and use the structure of cyclic subgroups and cosets of the group $\mathbb{Z}_{2nm}$ to construct vertex-magic edge labelings for Cartesian products of cycles $C_n \square C_m$.

3.1 Labeling Cartesian Products of Two Cycles by the Cyclic Group $\mathbb{Z}_{2nm}$

Definition 3.1.1. A *vertex-magic group edge labeling* of a graph $G(V, E)$ with $|E| = n$ is an injection from E to an abelian group Γ of order n such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$.

We now set out to prove that any product of two cycles $C_n \square C_m$ has a vertex-magic group edge labeling by the cyclic group $\mathbb{Z}_{2nm}$.

Theorem 3.1.2. *For n, m both odd, $C_n \square C_m$ can be labeled with group elements from $\mathbb{Z}_{2nm}$ to form a vertex-magic edge labeling.*

Proof. Without loss of generality let $m \geq n$. Let x_{ij} refer to the vertex with incident edges from the i-th n-cycle, and the j-th m-cycle in our product. Cycle C_m^j then contains vertices x_{kj} for $1 \leq k \leq m$, and cycle C_n^i contains vertices x_{ik} for $1 \leq k \leq n$. We may then label our n m-cycles, $C_m^1, C_m^2, \ldots, C_m^n$, as follows. Start by labeling any edge of our first m-cycle, C_m^1 with 0 and proceed labeling every other edge with consecutive even numbers. This results in every edge being labeled since our m-cycles have odd length. We then proceed to label our cycle C_m^i in the same manner but starting with $0 + 2m(i-1)$, see Figure 3.1. Note that together these m-cycles contain every even number in $\mathbb{Z}_{2nm}$ as labels.

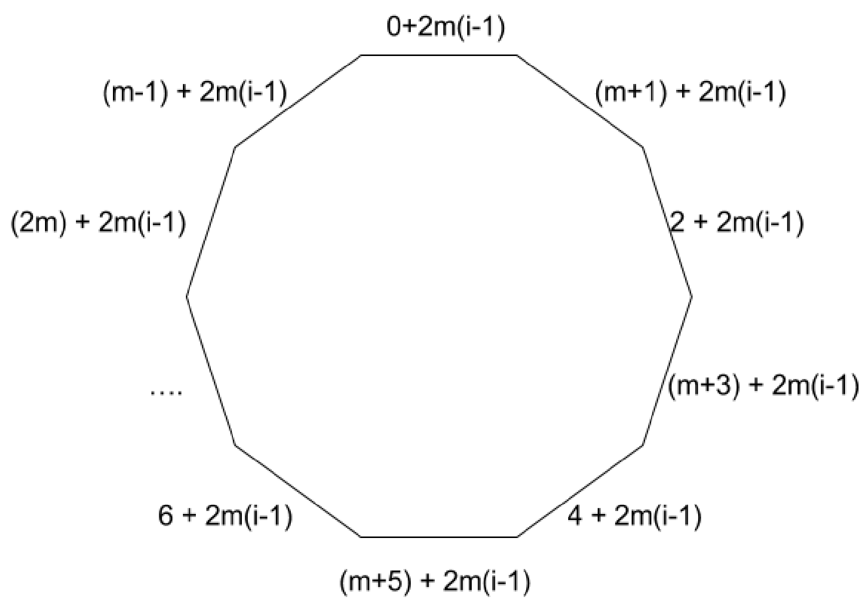


Figure 3.1: The labels on the i^{th} m-cycle

The j-th vertex in C_m^i, vertex x_{ij} has a temporary weight of $(m-1)+2(j-1)+4m(i-1)$, see Figure 3.2. Now each m-cycle has temporary vertex weights that contains one element from every even coset of the subgroup generated by $2m$, and together these m-cycles contain every even coset of this subgroup. This follows from the above temporary

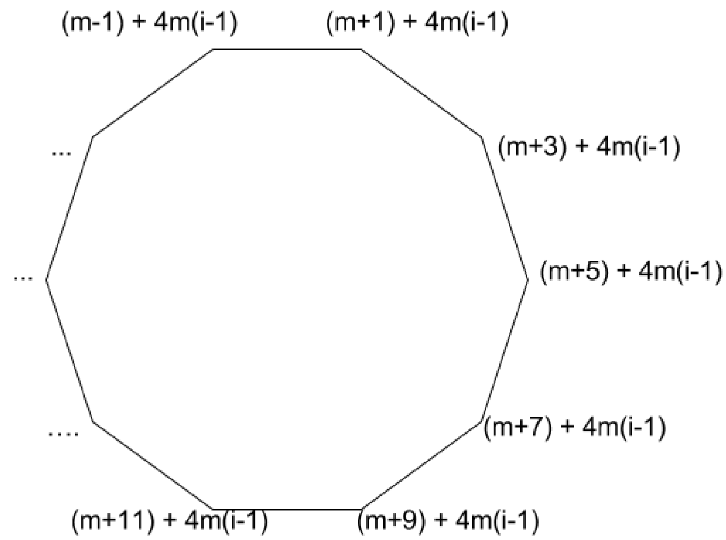


Figure 3.2: The temporary weights on the i^{th} m-cycle

weights and the property that the subgroup generated by $2m$ is isomorphic to the subgroup generated by $4m$ in $\mathbb{Z}_{2nm}$ since both n and m are odd.

Now consider labeling the m n-cycles as follows. Start by labeling any edge of our first n-cycle, C_n^1 with 1 and proceed labeling consecutive edges with $1 + 2m(i-1)$ for $1 \leq i \leq n$. We may then proceed to label our k-th n-cycle, C_n^k, with $(2k-1)+2m(i-1)$ for $1 \leq i \leq n$, see Figure 3.3.

The j-th vertex in C_n^k, x_{jk}, has temporary weight $2(2k-1) + 2m(2j-1)$, see Figure 3.4. Each n-cycle has partial weights forming an even coset of the subgroup generated by $2m$.

We now consider the following method to label $C_n \square C_m$. To simplify notation, let $l = \dfrac{n+3}{2}$. Start by labeling the horizontal m-cycles with C_m^1 labeled above, then the l-th labeled m-cycle, C_m^l then C_m^2 followed by C_m^{l+1} and so on. Each time alternating between C_m^i and C_m^{l+i}, $1 \leq i \leq \dfrac{n-1}{2}$. This method of labeling ensures that the partial weight (due to the m-cycle labels) of each vertex in a given n-cycle is a coset ascending in order by $2m$, and that the partial weights of vertices in an m-cycle are increasing by

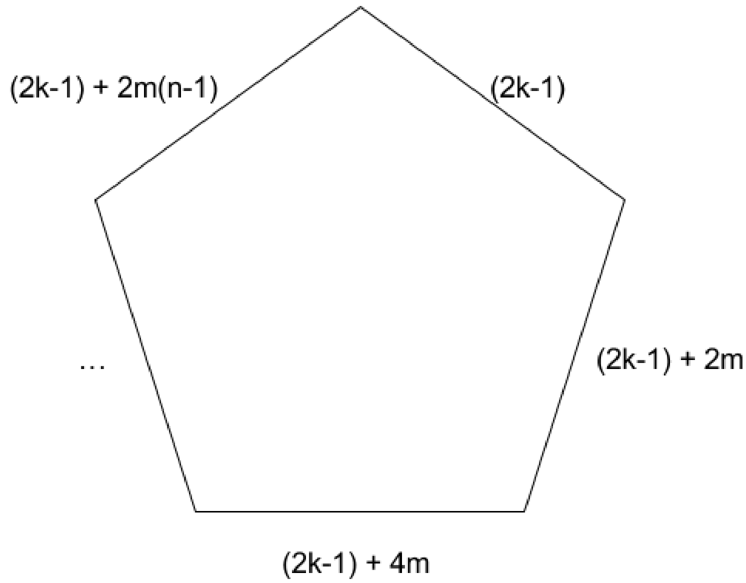


Figure 3.3: The labels on the k^{th} n-cycle

2 as we move from left to right.

In order to make the weight of each vertex congruent to 0 $\pmod{2nm}$, choose the n-cycle with partial weight $2nm-(m-1)$ to have edges incident on x_{00}, where the partial weight of the vertex due to the m-cycle is $(m-1)$. For the i-th n-cycle, C_n^i, in the cartesian product, choose the n-cycle labeled with partial weight $2nm-((m-1)+2i)$, and have the edges corresponding to this partial weight be the edges incident on the vertex x_{0i}. (This can indeed be done since every possible even weight is found on one of our n-cycles) Also note that this ensures a constant weight on this entire n-cycle since our n-cycle has labels *decreasing* (in one direction) by $2m$ and as stated above, the partial weights on our n-cycles due to our m-cycles are *increasing* by $2m$. Given this labeling, vertex x_{ij} has partial weight due to the horizontal m-cycle, $w_h(x_{ij}) = (m-1)+2i+2mj$, and partial weight due to the vertical n-cycle, $w_v(x_{ij}) = 2mn-2mj-(m-1+2i)$. This gives a total weight of vertex x_{ij}, $w(x_{ij}) = (m-1)+2i+2mj-2mj-(m-1+2i) = 2nm \equiv 0 \pmod{2nm}$. Thus the weight of every vertex is congruent to 0, and we have shown that by this construction, for n, m both odd, $C_n \square C_m$ can be labeled with group elements

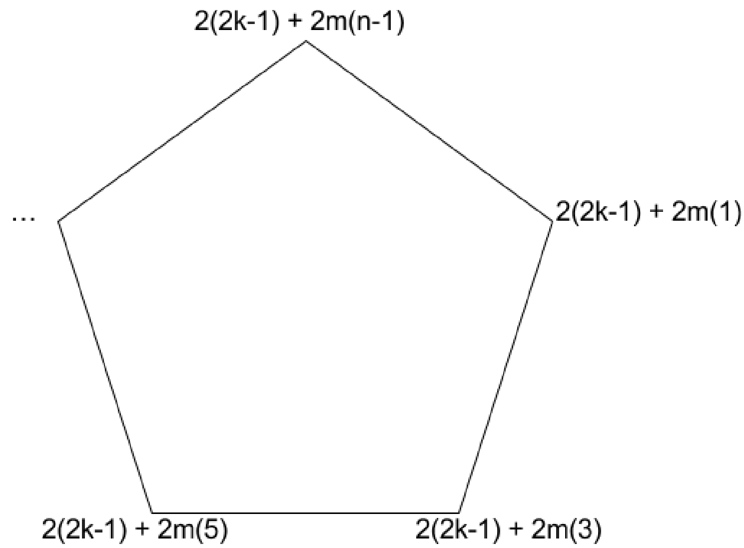


Figure 3.4: The temporary weights on the k^{th} n-cycle

from $\mathbb{Z}_{2nm}$ to form a vertex-magic edge labeling.

$\square$

Theorem 3.1.3. *For n odd and m even, $C_n \square C_m$ can be labeled with group elements from $\mathbb{Z}_{2mn}$ to form a vertex-magic edge labeling.*

Proof. The method of labeling $C_n \square C_m$ for n odd and m even is similar to that when both n and m are odd. Let x_{ij} refer to the vertex with incident edges from the i-th n-cycle, and the j-th m-cycle in our product.

Start by labeling the first edge of our first m-cycle, C_m^1, with 0 and proceed labeling every edge with consecutive even numbers. We then proceed to label C_m^i in the same manner but starting with $0+2m(i-1)$, see Figure 3.5. Note that together these m-cycles contain every even number in $\mathbb{Z}_{2nm}$ as labels.

Now each m-cycle has temporary vertex weights that contain one element from an even coset of the subgroup generated by $2m$. Note that these m-cycles may not contain every even coset of this subgroup, and some cosets may be repeated. In fact, half of the even cosets are used, and they are used exactly twice since $4m$ generates only half of the subgroup generated by $2m$ in $\mathbb{Z}_{2nm}$ when m is even. The j-th vertex in C_m^i, xij has temporary weight $2+4(j-1)+4m(i-1)$, see Figure 3.6.

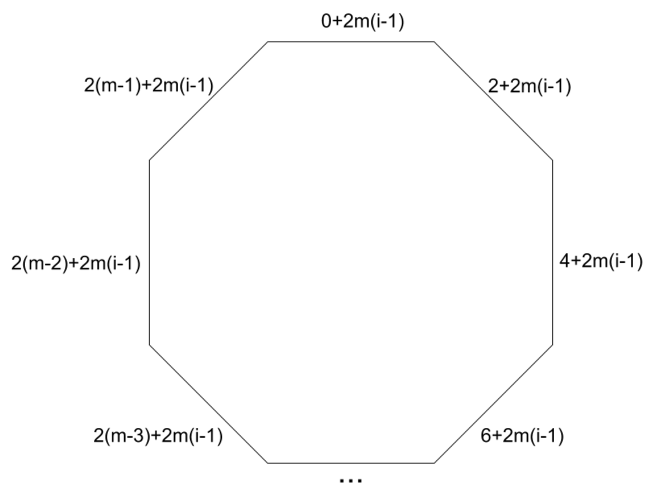


Figure 3.5: The labels on the i^{th} m-cycle

Now consider labeling the m n-cycles. Start by labeling the first edge of our first n-cycle, C_n^1, with 1 and proceed labeling consecutive edges with $1+2m(i-1)$ for $1 \leq i \leq n$. We then proceed to label our C_n^k with $(2k-1)+2m(i-1)$ for $1 \leq i \leq n$. The j-th vertex in C_n^k, x_{jk} has temporary weight $2(2k-1)+2m(2j-1)$, and each n-cycle has partial weights forming an even coset of the subgroup generated by $4m$.

We now consider the following method to label $C_n \square C_m$. To simplify notation, let $s = \dfrac{n+3}{2}$. Begin by labeling the horizontal m-cycles with C_m^1 labeled above, then C_m^s, then C_m^2, followed by the C_m^{s+1} and so on alternating between the C_m^i and $C_m^{s+(i-1)}$, $1 \leq i \leq \dfrac{n-1}{2}$. This method of labeling ensures that the partial weight (due to the m-cycle labels) of each vertex in a given n-cycle is a coset ascending in order by $2m$, and that the partial weights of vertices in an m-cycle are increasing by 4 as we move from left to right.

Now, in order to again make the weight of each vertex congruent to 0 (mod $2mn$), choose the n-cycle with partial weight $2mn-2$ to match up in the first spot, where the

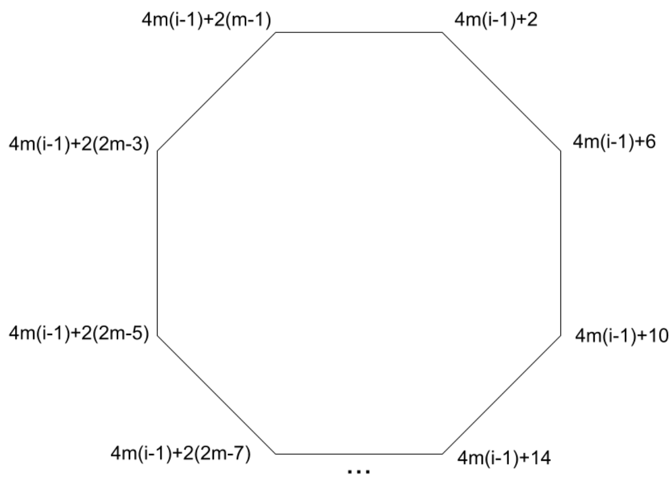


Figure 3.6: The partial weights due to the i^{th} m-cycle

partial weight of the vertex due to the m-cycle is 2. For the i-th n-cycle in the Cartesian product, choose the n-cycle labeled with partial weight $2nm - (2 + 4(i-1))$. (This can indeed be done since the every possible even weight of our m-cycle is found on one of our n-cycles, since both are generated by $4m$) Also note that this ensures a constant weight on this entire n-cycle since our n-cycle has labels *decreasing* (in one direction) by $2m$ and as stated above, the partial weights on our n-cycles due to our m-cycles are *increasing* by $2m$.

Given this labeling, vertex x_{ij} has partial weight due to the horizontal m-cycle, $w_h(x_{ij}) = 2 + 4(j-1) + 2mi$, and partial weight due to the vertical n-cycle, $w_v(x_{ij}) = 2nm - 2mi - (2 + 4(j-1))$. This gives a total weight of vertex x_{ij}, $w(x_{ij}) = 2 + 4(j-1) + 2mi + 2nm - 2mi - (2 + 4(j-1)) \equiv 0 \pmod{2nm}$. Thus the weight of every vertex is congruent to 0, and we have shown that by this construction, for n odd and m even, $C_n \Box C_m$ can be labeled with group elements from $\mathbb{Z}_{2nm}$ to form a vertex-magic edge labeling. $\Box$

By Ivanco's results [19], we know that for integers $k \geq 2$ and $t \geq 2$, $C_{2k} \Box C_{2t}$ can be labeled with consecutive positive integers to form a vertex-magic labeling. It is easy to

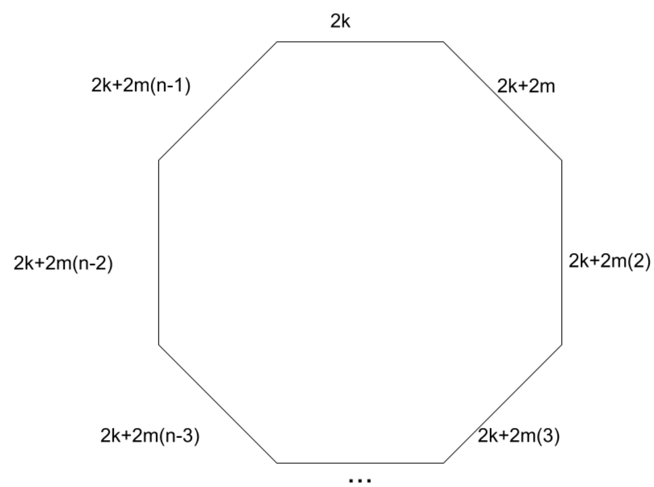


Figure 3.7: The labels on the k^{th} n-cycle

see that this same labeling will then produce a vertex-magic labeling in the cyclic group $\mathbb{Z}_{2nm}$ since all vertex weights are equal and therefore congruent mod $2nm$. Thus we obtain the following theorem. We do, however, provide our own construction similar to the ones above.

Theorem 3.1.4. *For n, m both even, $C_n \square C_m$ can be labeled with group elements from $\mathbb{Z}_{2nm}$ to form a vertex-magic edge labeling.*

Proof. Without loss of generality, suppose $m \geq n$. Let x_{ij} refer to the vertex with incident edges from the i-th n-cycle, and the j-th m-cycle in our product. We begin by labeling the n m-cycles as follows: label C_m^1 with consecutive odd integers $1, 3, 5, \ldots, 2m-1$, and continue to label C_m^i with the consecutive odd integers $1 + 2m(i-1), 3 + 2m(i-1), 5 + 2m(-1), \ldots, (2m-1) + 2m(i-1)$, $1 \leq i \leq n$. This causes the partial weights in a n-cycle, due to the m-cycles, to be increasing downward by $4m$.

Now to label our n-cycles, choose a subgroup of order $n/2$, call it H. Note this can always be done since n is even, and $n/2$ divides the order of our group $2mn$. This subgroup is generated by $4m$, since $2mn/(n/2) = 4m$. Now consider pairing even cosets

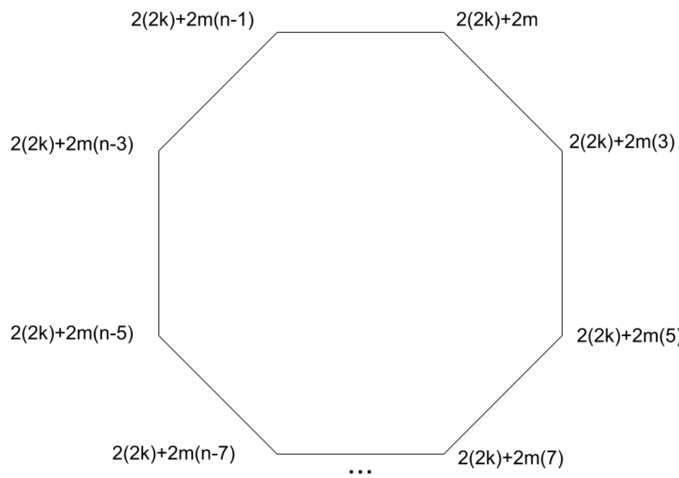


Figure 3.8: The partial weights due to the k^{th} n-cycle

(i.e the cosets $H+2k$, $0 \leq k \leq 2m-1$), as such: $H+2k, H+2m+2k$, $1 \leq k \leq m$, where $2k$ and $2m+2k$ are considered modulo $4m$. Together, each pair of cosets will be used to label one n-cycle.

We label each edge of C_n^k by alternating between increasing elements of the coset $H+2k$ and $H+2m+2k$, $1 \leq k \leq m$. That is, label every other edge with increasing elements of $H+2k$. Then starting on the edge directly following the first edge labeled, we label the remaining edges with increasing elements of $H+2m+2k$, see Figure 3.7. Then C_n^1 is labeled in order with edge labels $2, 2m, 2+4m, 2m+4m, 2+2\cdot 4m, 2m+2\cdot 4m, \ldots, 2m+(m/2)\cdot 4m$. This ensures that the partial weight of the vertices in an n-cycles is increasing (or decreasing) by $4m$ as we move through the cycle, see Figure 3.8.

Now to label $C_n \Box C_m$, label the i-th m-cycle with the labels we used for C_m^i $1 \leq i \leq n$, beginning with the smallest label used in C_m^i on the left most edge of the torus grid representing $C_n \Box C_m$ (Here the m-cycles are horizontal cycles). In this torus grid representation, we consider the vertical n-cycles to be numbered

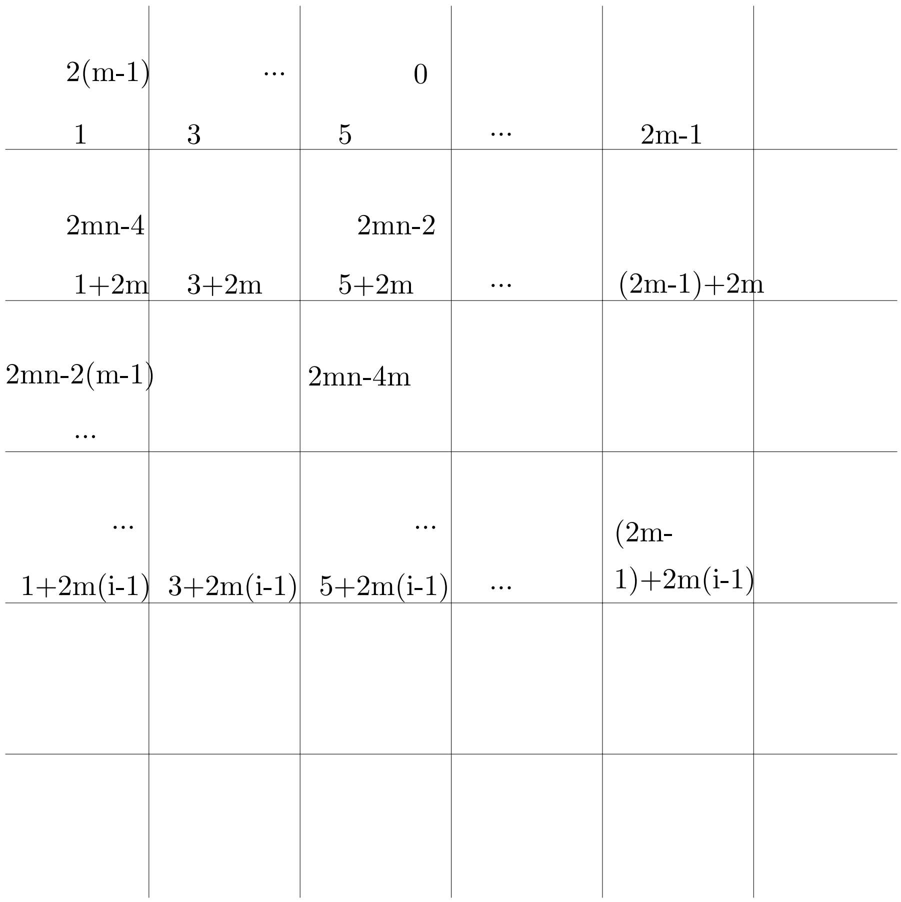


Figure 3.9: General Labeled Product of Two Even Cycles

$\overline{1}, \overline{2}, \overline{3}, \ldots, \overline{m-1}, \overline{m}$ from left to right. Using this numbering of n-cycles, we then label $\overline{(m/2)}$ with the labels used to label C_n^m, and $\overline{1}, \overline{2}, \overline{3}, \ldots, \overline{(m/2)-2}, \overline{(m/2)-1}$ with labels from $C_n^{m-1}, C_n^{m-2}, C_n^{m-3}, \ldots, C_n^{(m/2+2}, C_n^{(m/2)+1}$ respectively. Then label $\overline{m}$ with the labels used to label $C_n^{(m/2)}$, and $\overline{(m/2)+1}, \overline{(m/2)+2}, \ldots, \overline{m-1}$ with labels from $C_n^{(m/2)-1}, C_n^{(m/2)-2}, C_n^{(m/2)-3}, \ldots, C_n^1$ respectively. When labeling these n-cycles on our torus grid however, we label them in the reverse cyclic order, top to bottom, then we did when we first considered labeling the n-cycles. That is, label them in counter clockwise fashion starting with $2k$ with regards to Figure 3.7.

This method of labeling $C_n \Box C_m$ ensures that the partial weights due to the m-cycles, going from top to bottom in a column of vertices, x_{ij} for some fixed j and i ranging from 1 to n, is increasing by $4m$ while the partial weights due to the n-cycles in these same vertices will be decreasing by $4m$. Note also that the partial weights due to the m-cycles going left to right in a row of vertices, x_{ij} for some fixed i and j ranging from 1 to m, is increasing by 4, while the partial weight due to an n-cycle on these vertices is decreasing by 4, see Figure 3.1. This ensures us that the vertex weights are congruent along each row and column of vertices. Thus all vertex weights must be congruent mod $2nm$. We have thus have shown that by this construction, for n and m even, $C_n \square C_m$ can be labeled with group elements from $\mathbb{Z}_{2nm}$ to form a vertex-magic edge labeling. $\square$

By combining Theorems 3.1.2, 3.1.3, and 3.1.4, we obtain the following result.

Theorem 3.1.5. *For integers n and m, $C_n \square C_m$ can be labeled with group elements of $\mathbb{Z}_{2nm}$ to form a vertex-magic edge labeling.*

3.2 Alternative Labeling Method by the Cyclic Group $\mathbb{Z}_{2nm}$

Together, the previous three theorems prove that $C_n \square C_m$ has a vertex-magic group edge labeling by the cyclic group $\mathbb{Z}_{2nm}$. The following theorem gives one construction that allows for a magic labeling of $C_n \square C_m$ regardless of the parity of n and m. We will see later that, even though this construction works for all $C_n \square C_m$, it is intrinsically different from the three given above, and in some cases may not be as useful.

Theorem 3.2.1. *For integers n and m, $C_n \square C_m$ can be labeled with group elements of $\mathbb{Z}_{2nm}$ to form a vertex-magic edge labeling.*

Proof. Let us assume that $m \geq n$ unless stated otherwise and denote the vertices of $C_n \square C_m$ by $v_{i,j}$ with $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. By a *diagonal* of $C_n \square C_m$ we are referring to a sequence of vertices $(v_{0,j}, v_{1,j+1}, \ldots, v_{n-1,j+n-1}, v_{0,j+n}, v_{1,j+n+1}, \ldots, v_{n-1,j-1})$ of length l and their corresponding labeled edges. It is easy to observe that $l = \mathrm{lcm}(n, m)$, the least common multiple of n and m. Let $C_n^j(i)$, $0 \leq i \leq n-1$, $1 \leq j \leq m$, be the i-th edge from the top in the j-th n-cycle in the lattice grid representation of $C_n \square C_m$, and let $C_m^t(s)$, $0 \leq s \leq m-1$, $1 \leq t \leq n$, be the s-th edge from the left in the t-th m-cycle in the lattice grid representation of $C_n \square C_m$. (This lattice grid representation can be seen in Figure 3.1).

Consider $\mathbb{Z}_{2nm}$. Since the converse of Lagrange's Theorem is true for finite abelian groups, there exists a cyclic subgroup H of order $l = \mathrm{lcm}(n,m)$. Since H is cyclic, it can be generated by a single element, call it a. Then $H =< a >$. We know that there are an even number of cosets of this subgroup since the index $[G : H] = (2nm/\mathrm{lcm}(n,m)) = 2 \cdot \gcd(n,m)$.

Without loss of generality, consider the lattice grid representation of $C_n \square C_m$ where the vertical cycles are of length n and the horizontal cycles are of length m. Label the first vertical edge of the left-most n-cycle, $C_n^1(0)$ with the element 0 and proceed down the diagonal labeling the top vertical edges of the n-cycles in this diagonal, $C_n^2(1), C_n^3(2), C_n^4(3), \ldots$ with the consecutive elements of H.

Now label the horizontal edge, $C_m^1(1)$, the edge down and to the right of the vertical edge labeled 0 of the top m-cycle, with the element -1 and proceed down the diagonal labeling the right horizontal edges of the m-cycles in this diagonal, $C_m^2(2), C_m^3(3), C_m^4(4), \ldots$ with consecutive elements of the coset $H - 1$ in decreasing order. That is, in the opposite order of the previous labeled diagonal in the n-cycles.

If all $2 \cdot \gcd(n,m) = 2$ cosets have been used as labels, we are done. If not, we continue by labeling the first edge of the next n-cycle from the left, $C_n^2(0)$ with the element 1 and again proceed down the diagonal, $C_n^3(1), C_n^4(2), C_n^5(3), \ldots$ labeling the edges of the n-cycles in this diagonal with consecutive elements of the coset $H + 1$. We then have at least one more coset, $H - 2$. We label the next edge from the left in the top m-cycle, $C_m^1(2)$ with the element -2 and proceed down the diagonal labeling the edges of the m-cycles in this diagonal, $C_m^2(3), C_m^3(4), C_m^4(5), \ldots$ with consecutive elements of the coset $H - 2$ in decreasing order.

Repeat this process until all edges have been labeled, that is, we have used all $2 \cdot \gcd(n,m)$ cosets.

By this construction, given any vertex, the partial weight due to the vertical and horizontal edges forming a right angle facing up and to the left is $(i + k \cdot a) + (-(i + 1) - k \cdot a) = -1$, and those forming a right angle facing down and to the right is

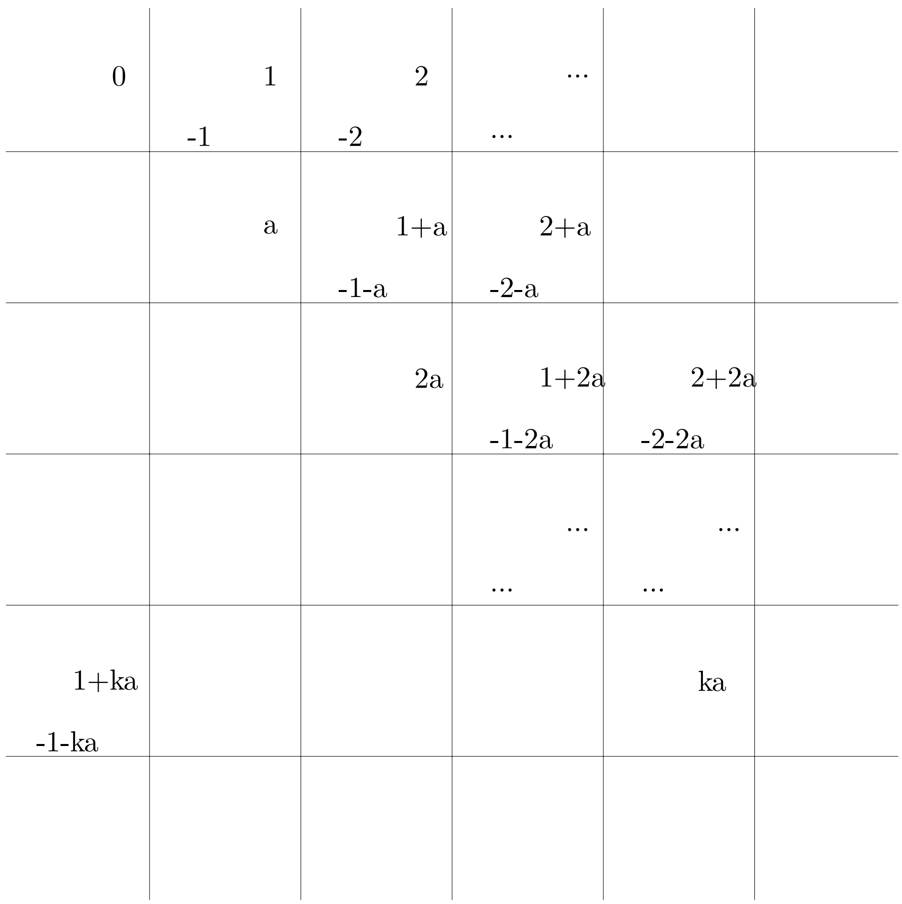


Figure 3.10: Diagonal Labeling Method

$(i+k\cdot a)+(-(i+1)-(k-1)\cdot a)=-1+a$, where $0\leq i\leq \gcd(n,m)$, is obtained from the coset $H+i$ from which the vertical edges of a given right angle were produced.

The weight of every vertex is thus $-2+a$, and we obtain a magic labeling of $C_n\Box C_m$ with group elements from $\mathbb{Z}_{2nm}$. □

Theorem 3.2.2. *The construction given in 3.2.1 is different than those in 3.1.2, 3.1.3, and 3.1.4, when* $\gcd(n,m)\geq 2$.

Proof. Note that due to the nature of construction 3.2.1, if the $\gcd(n,m)\geq 2$, that is, there is more than one diagonal in our product, then we have some vertical edges that have even labels and yet others that are odd. See Figure 3.2.

Now in construction 3.1.2, 3.1.3, and 3.1.4, this could never happen since all horizontal (or vertical) cycles were labeled with even (or odd) labels. Thus no vertical and horizontal edge could have labels of the same parity showing that the previous constructions where intrinsically different.

However, if $\gcd(n,m)=1$, then there is only one diagonal. Vertical edges are then labeled with the even elements of the subgroup $H=<a>$, and the horizontal edges are labeled with the odd elements of the coset $H-1$. It is clear to see that the labels in a vertical n-cycle are elements of a particular coset of $2m$. In this case, this labeling is the same (up to a permutation of the labels) as those given in 3.1.2, 3.1.3, and 3.1.4. □

Chapter 4

Future Work

In this chapter we present some conjectures and partial results obtained on the road to proving them.

4.1 Labeling Cartesian Product of More Than Two Cycles by the Cyclic Group $\mathbb{Z}_{2nm}$

We had hoped that we would be able to label the Cartesian product of any number of cycles with the cyclic group of order $3nmk$ to form a vertex-magic group edge labeling. To such an end we tried many different approaches. We had hoped we would be able to label a product of three cycles of any parity. If we could do so, we would be able to inductively combine our results for products of two and three cycles to obtain the product of any number of cycles. Note that the product of any number of cycles can be decomposed into a combination of products of three or two cycles. This led us to propose the following conjectures.

Conjecture 4.1.1. *$C_n \square C_m \square C_k$ has a vertex-magic group edge labeling for any n, m and k with labels from the cyclic group of order $3nmk$.*

Conjecture 4.1.2. *$C_{n_1} \square C_{n_2} \square \ldots \square C_{n_t}$ has a vertex-magic group edge labeling for any $n_1, n_2, \ldots, n_t$ with labels from the cyclic group of the correct order.*

The following are partial results obtained during our attempts to obtain a magic labeling of a product of three cycles.

4.2 Labeling Cartesian Product of Two Cycles by the Cyclic Group $\mathbb{Z}_{2nm}$ to Form a Vertex-Anti-Magic Group Edge Labeling

We now give a construction to label a product of two cycles with the group $\mathbb{Z}_{2nm}$ in such a way as to obtain anti-magic. Recall that in an anti-magic labeling, we want the weight of every vertex to be different rather than the weights all being the same as in a magic labeling.

An anti-magic labeling of a finite simple undirected graph with p vertices and q edges is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, q\}$ such that the vertex sums are pairwise distinct, where the vertex sum at one vertex is the sum of labels of all incident edges to that vertex. A graph is called *anti-magic* if it admits an anti-magic labeling.

Theorem 4.2.1. *For odd integers n and m, $C_n \Box C_m$ can be labeled with group elements of $\mathbb{Z}_{2nm}$ to form a vertex-anti-magic edge labeling.*

Proof. Let us assume that $n \leq m$ unless stated otherwise and denote the vertices of $C_n \Box C_m$ by $v_{i,j}$ with $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. By a *diagonal* of $C_n \Box C_m$ we are again referring to a sequence of vertices $(v_{0,j}, v_{1,j+1}, \ldots, v_{n-1,j+n-1}, v_{0,j+n}, v_{1,j+n+1}, \ldots, v_{n-1,j-1})$ of length l and their corresponding labeled edges. It is easy to observe that $l = \operatorname{lcm}(n, m)$, the least common multiple of n and m. Let $C_n^j(i)$, $0 \leq i \leq n-1$, $1 \leq j \leq m$, be the i-th edge from the top in the j-th n-cycle in the lattice grid representation of $C_n \Box C_m$, and let $C_m^t(s)$, $0 \leq s \leq m-1$, $1 \leq t \leq n$, be the s-th edge from the left in the t-th m-cycle in the lattice grid representation of $C_n \Box C_m$. (This lattice grid representation can be seen in Figure 3.1).

Consider $\mathbb{Z}_{2nm}$. Since the converse of Lagrange's theorem is true for finite abelian groups, there exists a cyclic subgroup H of order $l = \operatorname{lcm}(n, m)$. Since H is cyclic, it can be generated by a single element, call it a. Then $H = < a >$. We know that there are an even number of cosets of this subgroup since the index $[G : H] = (2nm/\text{lcm}(n, m)) = 2 \cdot \gcd(n, m)$.

Without loss of generality, consider the lattice grid representation of $C_n \Box C_m$ where the vertical cycles are of length n and the horizontal cycles are of length m. Label the first vertical edge of the left-most n-cycle, $C_n^1(0)$ with the element 0 and proceed down the diagonal labeling the top vertical edges of the n-cycles in this diagonal, $C_n^2(1), C_n^3(2), C_n^4(3), \ldots$ with the consecutive elements of H.

Now it may be that all vertical edges have been labeled. Note that this depends on the index of H in G, $[G : H] = (2nm/\text{lcm}(n, m)) = 2 \cdot \gcd(n, m)$, which gives the total number of cosets. If not, continue to label the first vertical edge of the next n-cycle not labeled, $C_n^2(0)$, with the element 1 and proceed down the diagonal labeling the top vertical edges of the n-cycles in this diagonal, $C_n^3(1), C_n^4(2), C_n^5(3), \ldots$ with the consecutive elements of $H + 1$. Continue in this fashion until all vertical edges have been labeled.

Now label the left-most horizontal edge of the top m-cycle, $C_m^1(0)$, with the next available element k. Note, this element may be as large as m, and proceed down the diagonal labeling the left horizontal edges of the m-cycles in this diagonal, $C_m^2(1), C_m^3(2), C_m^4(3), \ldots$ with consecutive elements of the coset $H + k$ in increasing order (or the same consecutive order that the n-cycles were labeled).

If all horizontal edges have been labeled, we are done. If not, continue by labeling the first edge not having a label in the first m-cycle with the next available label t, and proceed down the diagonal labeling the horizontal edges in the diagonal with consecutive elements of the coset $H + t$.

When all $2 \cdot \gcd(n, m)$ cosets have been used as labels, we are done.

By this construction, the weights of the vertices traversing down a diagonal are increasing by $4a$.

Recall that given an additive group of order o, and an element a, $| < ia > | = | < ja > |$ if and only if $\gcd(o, i) = \gcd(o, j)$. Now since m and n are both odd, $l = \text{lcm}(n, m)$ must

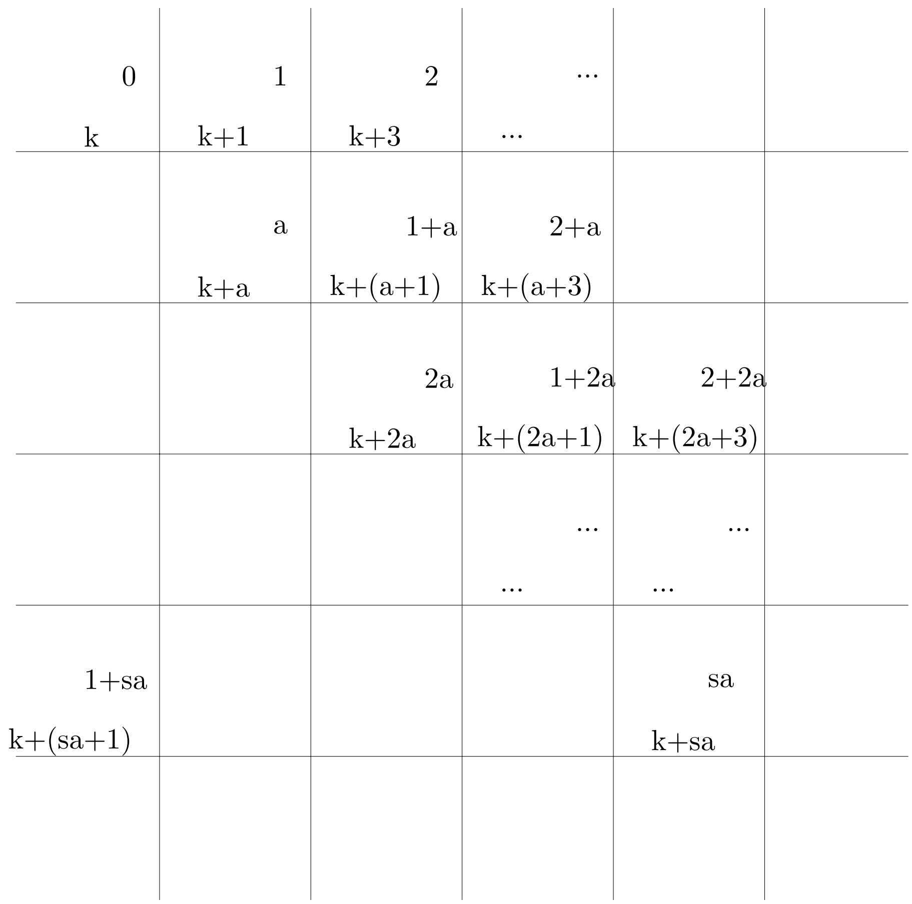


Figure 4.1: Anti-magic Labeling

be odd since l divides nm which is odd. This also implies however that $1 = \gcd(1, l) = \gcd(4, l)$. Thus the subgroup generated by a and the subgroup generated by $4a$ have the same order l.

The weight of the vertices in a diagonal then form a coset of the subgroup generated by $4a$ (or a as we explained above), and are thus all distinct. It can easily be seen that by this construction, each of the diagonals is a different coset, and thus each vertex has a distinct weight. We then have an anti-magic labeling of $C_n \Box C_m$ with group elements from $\mathbb{Z}_{2nm}$. $\Box$

4.3 Example of Vertex-Magic Group Edge Labeled Product of Three Cycles by $\mathbb{Z}_{3nmk}$

After many months of trial and error in labeling a product of three cycles, we were able to successfully label $C_5 \Box C_3 \Box C_3$ with group elements from $\mathbb{Z}_{135}$. We hope that the following method of labeling this product can be generalized to any product of three cycles.

Note that the partial weights on the nine 5-cycles directly correspond to those of a given vertex in the five different copies of $C_3 \Box C_3$. Thus connecting these five copies of $C_3 \Box C_3$ by the correct ordering of the nine 5-cycles gives us a vertex magic group edge labeling of $C_5 \Box C_3 \Box C_3$.

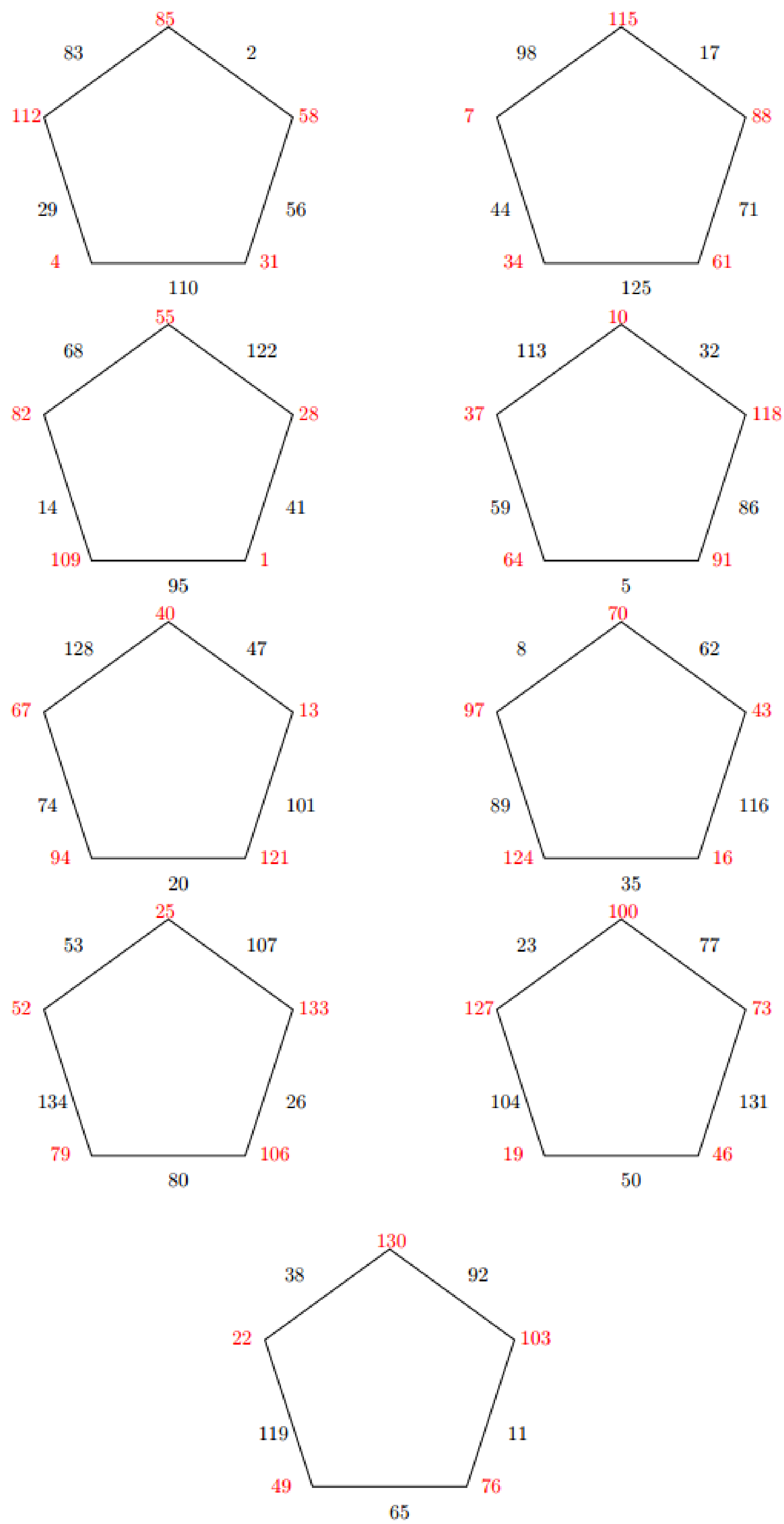
	0		15		30
31	107	1	77	16	2
	75		45		60
61	47	76	17	46	122
	105		120		90
91	32	106	92	121	62

	3		18		33
109	134	79	104	94	29
	78		48		63
4	74	19	44	124	14
	108		123		93
34	59	49	119	64	89

	6		21		36
52	26	22	131	37	56
	81		51		66
82	101	97	71	67	41
	111		126		96
112	86	127	11	7	116

	9		24		39
130	53	100	23	115	83
	84		54		69
25	128	40	98	10	68
	114		129		99
55	113	70	38	85	8

	12		27		42
73	80	43	50	58	110
	87		57		72
103	20	118	125	88	95
	117		132		102
133	5	13	65	28	35


85
83
2
112
58
29
56
4
31
110
115
98
17
7
88
44
71
34
61
125
55
68
122
82
28
14
41
109
1
95
10
113
32
37
118
59
86
64
91
5
40
128
47
67
13
74
101
94
121
20
70
8
62
97
43
89
116
124
16
35
25
53
107
52
133
134
26
79
106
80
100
23
77
127
73
104
131
19
46
50
130
38
92
22
103
119
11
49
76
65

4.4 Labeling Cartesian Products of Two Cycles by Non-Cyclic Groups

We now turn our attention to labeling the product of cycles $C_n \square C_m$ with noncyclic groups of order $2nm$. We first give a construction of how to obtain a magic labeling with the group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$, and make note that when $2, m$,and n are relatively prime, $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$ is isomorphic to the group $\mathbb{Z}_{2nm}$. Also note that if one of n, m is odd, say n, then $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_{2n}$.

Theorem 4.4.1. *For integers $m \geq 2$ and $n \geq 2$, $C_m \square C_n$ can be labeled with group elements from the group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$ to form a vertex-magic edge labeling.*

Proof. Without loss of generality consider $m \geq n$, and let G be the group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$. Let $H \leq G$ be the subgroup of order n generated by $(0, 1, 0)$. Let cosets of the subgroup H in which the 3rd coordinate is a 0 be known as 'even cosets', and let those cosets in which the 3rd coordinate is a 1 be known as 'odd cosets'. We can thus define the even coset $H + i$ to be the elements of H plus $(i, 0, 0)$ for $0 \leq i < m$. Let $\overline{H}$ be the the set containing the elements of $H + (0, 0, 1)$. We can then define the odd coset $\overline{H} + j$ to be the elements of $\overline{H}$ plus $(j, 0, 0)$ for $0 \leq j < m$.

Begin by labeling the first n-cycle with the elements of H, and continue by labeling the i-th n-cycle with the elements of the even coset $H + i$. Now when labeling the m-cycles we want to place the first element of $\overline{H}$ on the first m-cycle, and continue placing the j-th element of $\overline{H}$ on the j-th m-cycle. We then continue placing the k-th element of $\overline{H} + k$ onto the k-th m-cycle. The labelings of i-th n-cycle and j-th m-cycle can be seen in Figures 4.2 and 4.3 respectively. The temporary weights associated with each cycle are as seen in Figures 4.4 and 4.5. The previous discussion gives us an ordered representation of our cycles that we will refer to in our construction.

We now show how to label the product of cycles: Begin by fixing a vertex x_{00} in our product of $C_m \square C_n$. Let the vertex x_{00} be the intersection of the 0-th n-cycle and the 0-th m-cycle such that the partial weight due to the n-cycle is $(0, 1, 0)$ and the partial weight due to the m-cycle is $(1, 0, 0)$. Have the vertex x_{0j} be the intersection of the 0-th n-cycle and the $(n - j)$-th m-cycle in such a way that the partial weight due to the m-cycle is $(1, n-2j, 0)$. This weight indeed exists since the $(n-j)$-th m-cycle has a vertex with partial weight $(1, 2(n-j), 0)$ which is congruent to $(1, n-2j, 0)$. Note that here the labelings and thus partial weights of the n-cycle have already been fixed when we fixed the vertex x_{00}. Similarly have the vertex x_{i0} be the intersection of the $(m-i)$th n-cycle and the 0th m-cycle in such a way that the partial weight due to the n-cycle is $(2i, 1, 0)$. Note that we have thus fixed the partial weights and labelings at every vertex in our product. The vertex x_{ij} is then the intersection of the $(m-i)$-th n-cycle and the $(n-j)$-th m-cycle in such a way that the partial weight due to the n-cycle is $(2i, 2j+1, 0)$ and the partial weight due to the m-cycle is $(1-2i, n-2j, 0)$. This gives us a total weight at each vertex of $(2i, 2j+1, 0)+(1-2i, n-2j, 0) = (1, 1, 0)$. Thus we have a magic labeling of $C_m \square C_n$ with magic constant $\mu = (1, 1, 0)$. $\square$

Python code to label $C_m \square C_n$ with group elements from the group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$ to form a vertex-magic edge labeling can be found in the appendix.

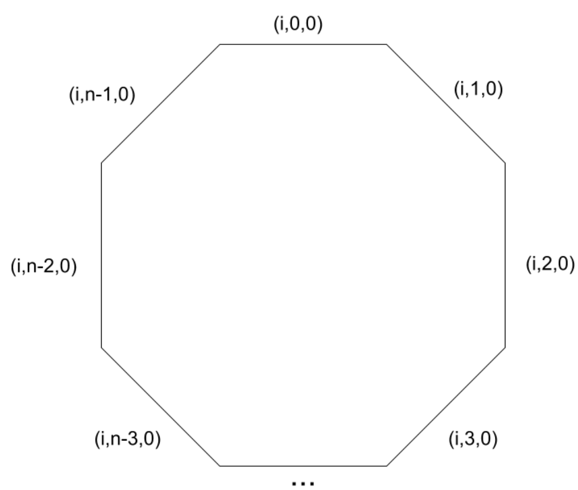


Figure 4.2: The labels on the i^{th} n-cycle

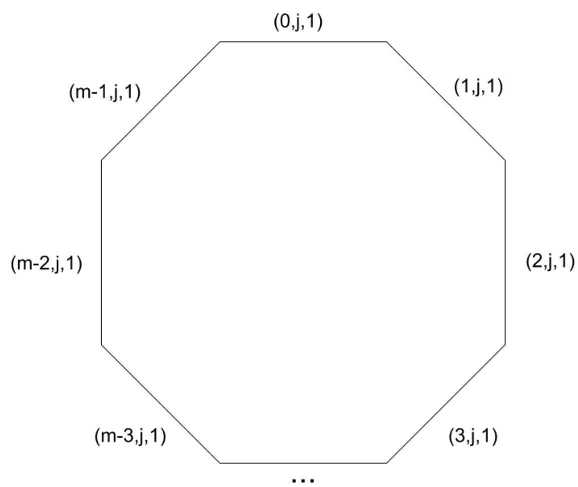


Figure 4.3: The labels on the j^{th} m-cycle

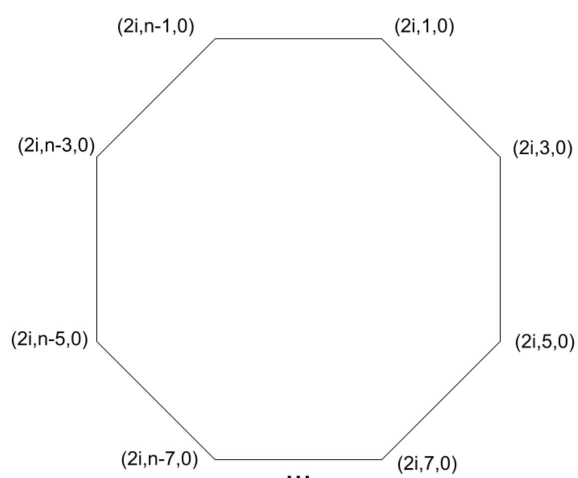


Figure 4.4: Partial weights on the i^{th} n-cycle

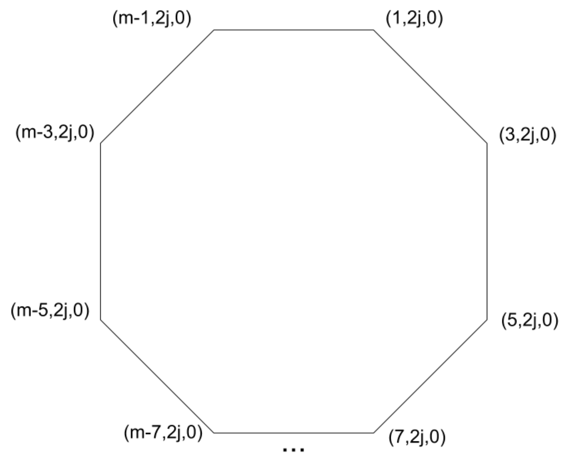


Figure 4.5: Partial weights on the j^{th} m-cycle

Throughout our research we have been able to label various products of cycles with other noncyclic group ableian groups of order $2nm$. Most methods of labeling these products have been similar to the constructions given in 3.1.2, 3.1.3, and 3.1.4, making use of a cyclic subgroup and its cosets. These findings lead us to pose the following conjecture.

Open Problem 4.4.2. *Does $C_n \square C_m$ has a vertex-magic group edge labeling for any n and m with labels from any finite abelian group of order $2nm$?*

Combining the topic of this section with our discussion of the magic labeling of any number of cycles, some natural questions arise:

Open Problem 4.4.3. *Does $C_n \square C_m \square C_k$ has a vertex-magic group edge labeling for any n, m and k with labels from any finite abelian group of order $3nmk$?*

Open Problem 4.4.4. *Does $C_{n_1} \square C_{n_2} \square \ldots \square C_{n_t}$ has a vertex-magic group edge labeling for any $n_1, n_2, \ldots, n_t$ with labels from any finite abelian group of the correct order?*

We have not found enough significant evidence for or against these claims to bolster a conjecture of them. They do however provide interesting questions for future research in this area, and are thus left to peak the interest of any researcher(s) interested in vertex magic group edge labelings.

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Appendix A

Computer Code

A.1 Code for Labeling $C_n \square C_m$ by $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$

The following is python code to produce a vertex magic group edge labeling of $C_n \square C_m$ with group elements of $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$.

```
from pprint import pprint
import sys
# trying to label Cm cross Cn with Zm x Zn x Z2



# def vectMod(v,mods):
#     l = []
#     for i,x in enumerate(v):
#         l.append(x % mods[i])
#     return tuple(l)



def vectSum(a,b,mods=None):
    l = []
    assert len(a) == len(b)
```

```
    for x in range(len(a)):
        if mods:
            l.append((a[x]+b[x]) % mods[x])
        else:
            l.append(a[x]+b[x])
    return tuple(l)

def rotateCycle(cycle,n):
    return cycle[n:]+cycle[:n]

try:
    m = int(sys.argv[1])
    n = int(sys.argv[2])
except IndexError:
    m = int(raw_input("m: "))
    n = int(raw_input("n: "))

mods = [m,n,2]
MAGIC = (1,1,0)



h = [[(i,j,0) for j in range(n)] for i in range(m)]
hb = [[(i,j,1) for j in range(n)] for i in range(m)]

nCycles = h
mCycles = [[hb[i][j] for i in range(len(hb))] for j in range(n)]
mCycles = [mCycles[-i] for i in range(len(mCycles))]
nCycles = [nCycles[-i] for i in range(len(nCycles))]

nPartialWeights = [[vectSum(nCycles[j][i],nCycles[j][i-1],mods=mods) for i in range(n)]
mPartialWeights = [[vectSum(mCycles[j][i],mCycles[j][i-1],mods=mods) for i in range(m)]
```

```
# t = nCycles[1]
# nCycles[1] = nCycles[2]
# nCycles[2] = t
#
# t = nPartialWeights[1]
# nPartialWeights[1] = nPartialWeights[2]
# nPartialWeights[2] = t



# set up (0,0)
breakBool = False
for i,x in enumerate(nPartialWeights[0]):
    for j,y in enumerate(mPartialWeights[0]):
        if vectSum(x,y,mods=mods) == MAGIC:
            nCycles[0] = rotateCycle(nCycles[0],i)
            nPartialWeights[0] = rotateCycle(nPartialWeights[0],i)
            mCycles[0] = rotateCycle(mCycles[0],j)
            mPartialWeights[0] = rotateCycle(mPartialWeights[0],j)
            breakBool = True
            break
    if breakBool:
        break

# set up (0,x)
for j,cycle in enumerate(mPartialWeights):
    breakBool = False
    for i,x in enumerate(cycle):
        if vectSum(x,nPartialWeights[0][j],mods=mods) == MAGIC:
            print "rotating m-cycle {} by {}".format(j,i)
            mCycles[j] = rotateCycle(mCycles[j],i)
```

```
            mPartialWeights[j] = rotateCycle(mPartialWeights[j],i)
            breakBool = True
            break
    if not breakBool:
        print "unable to set up (0,{})".format(j)

# set up (x,0)
for i,cycle in enumerate(nPartialWeights):
    breakBool = False
    for j,y in enumerate(cycle):
        if vectSum(y,mPartialWeights[0][i],mods=mods) == MAGIC:
            print "rotating n-cycle {} by {}".format(i,j)
            nCycles[i] = rotateCycle(nCycles[i],j)
            nPartialWeights[i] = rotateCycle(nPartialWeights[i],j)
            breakBool = True
            break
    if not breakBool:
        print "unable to set up ({},0)".format(i)

nCycles = [rotateCycle(nCycles[i],-1) for i in range(len(nCycles))]
mCycles = [rotateCycle(mCycles[i],-1) for i in range(len(mCycles))]


magicMatrix = [vectSum(nPartialWeights[i][j],mPartialWeights[j][i],mods=mods) for i in r


print "---------------------------------------------------------------------------"
print "n cycles:"
pprint(nCycles)
print "---------------------------------------------------------------------------"
print "m cycles:"
pprint(mCycles)
```

```
print "------------------------------------------------------------------------"
print "n partial weights:"
pprint(nPartialWeights)
print "------------------------------------------------------------------------"
print "m partial weights:"
pprint(mPartialWeights)
print "------------------------------------------------------------------------"
print "magic matrix:"
pprint(magicMatrix)
```