# TIME-DOMAIN STEADY-STATE TORQUE CALCULATION OF VOLTAGE-SOURCE PULSE-WIDTH-MODULATED INVERTER FED INDUCTION MOTORS

## PART I: THEORETICAL ANALYSIS

Prepared by

J. S. Hsu

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Center for Electromechanics
The University of Texas at Austin
Balcones Research Center
Blldg. 133, EME 1.100
Austin, TX 78758-4497
(512) 471-4496

# TIME-DOMAIN STEADY-STATE TORQUE CALCULATION OF VOLTAGE-SOURCE PULSE-WIDTH-MODULATED INVERTER FED INDUCTION MOTORS

## PART I: THEORETICAL ANALYSIS

John S. Hsu (Htsui), Senior Member
Center for Electromechanics
The University of Texas at Austin
Austin, Texas 78712

**Abstract** - Evaluation of torque pulsation associated with the harmonics of pulse width modulated (PWM) inverter-fed drives is important for a quiet and smooth operation. This paper discusses an analytical method for the steady state torque calculation of the voltage source PWM inverter fed induction motors. Equations derived from the 1-2-0 coordinate system are used. A sample calculation is included for the illustration of practical application.

## INTRODUCTION

The PWM inverter fed induction motors offer a number of advantages over other types of drives. The construction of induction motors is simple and robust with minimal maintenance requirements. The rapid development of advanced microcontrollers and fast switching power electronic devices enables various optimizations of adjustable speed induction-motor drives. The currents or voltages of either voltage source or current source PWM inverter fed induction motors generally contain a certain amount of harmonics. These harmonics produce torque pulsations that affect the noise and smoothness of the drive. There is a trade-off between the smoother torque and the increase of commutation losses. A smoother torque corresponds to an increase in the number of pulses per cycle. In order to evaluate the torque pulsation [1,2] associated with harmonics of a PWM inverter fed drive (especially at low speed) the technique for the steady state torque calculation is important.

Apart from the initial switching on, pulse-width modulation involves a series of switching transients. The numerical method [3] can be used to investigate the steady state torque solution. Numerical method is the logical approach in a nonlinear situation. However, for the steady state solution that requires a full attenuation of the initial switching-on transient, additional computation time is required by the numerical method for a certain period to reach to the steady state. Furthermore, the numerical processes introduce certain errors of various types; small computation steps are normally needed for accuracy. These small steps inevitably further prolong the computation time for the steady-state solution. Various computer aided design programs (such as the PWLIB program developed by Bowes et. al. [5]) have been reported for the analyses of the computation of PWM drives.

The frequency-domain technique, based on Fourier analysis, gives a clear picture of the harmonic content of the PWM waveform. However, the torque calculated for each individual order of harmonics is smooth. Special treatment would be required to evaluate the torque pulsations, and the evaluation can be time consuming. The time-domain technique generally takes less time for the computation of the current and torque [1,4,5,6,7].

There is a rich body of knowledge in the classical, induction-motor equations derived by Park [8], Stanley [9], Lyon [10], and others through various coordinate systems that were mainly used to solve problems associated with sinusoidal supplies [11-15]. This paper uses these classical equations for induction motors with a nonsinusoidal, continuously switched supply.

The simple approach described in this paper shows the analytical logic to reach to the steady state of the voltage source PWM induction motors without a lengthy attenuation calculation. Using this method, the steady-state current can be obtained analytically. Consequently, the flux associated with the current and voltage can be calculated. Once the current and flux are known, the torque can be obtained through the proper products of current and flux.

## ASSUMPTIONS

The analysis is based on the following assumptions:

a. Carrier wave is synchronized with the fundamental output wave. The negative half of voltage pulses in one cycle is the mirror image of the positive half.

b. The rotor and load inertia is large enough to hold a constant speed. The experimental result of a six pulse, current-source drive [1] shows that even at low frequency (5 Hz), with a small oscillation of speed, the calculated torque under this assumption is still acceptable.

c. Saturated motor parameters are used, otherwise the motor magnetic circuit is considered to be linear.

d. Core, friction, windage, and stray-load losses are not considered in the analysis.

e. The commutation time is negligible to assume the Heaviside unit-function nature of the voltage pulses.

## ANALYSIS

This paper uses the 1-2-0 coordinate system [10, 14] and the per-unit values for the analysis. Since the value of coordinate 2 is always the conjugate complex number of the value of coordinate 1, attention can be drawn mainly to the values referring to coordinate 1. Resultant equations established in the available literature, such as the transformation from a, b, c values to 1, 2, 0 values, or vice versa [10, 14] and the definition of the operational impedances [14] are mentioned and used directly without repeating the derivations.

## Per-Unit Values

The reactances of the induction motor are normally measured under the supply frequency with sinusoidal voltage. For adjustable frequency, the inductance can be considered as a constant at different frequencies, but the reactance (a product of $2\pi f$ and the inductance) changes.

The unit (or base) value of time is $(1/2\pi f)$; therefore, the per-unit value of time, t, is given in radians. As far as the equations in this paper are concerned, there is no restriction for selecting the unit values based on either the input or the output of the electrical machine. However, since the calculation is mainly for motors, the shaft output is chosen as the unit power for convenience. The unit values under a particular frequency can be defined as follows:

$$\text{Unit power [watts]} = 746 \bullet \text{hp} = \text{motor output power}$$

$$\text{Unit voltage [volts]} = \text{phase voltage}$$

$$\text{Unit current [amps]} = \frac{\text{unit power}}{\text{number of phases} \bullet \text{unit voltage}}$$

$$\text{Unit impedance } [\Omega] = \frac{\text{unit voltage}}{\text{unit current}}$$

$$\text{Unit torque [N - m]} = \frac{\text{unit power} \bullet \frac{p}{2}}{2\pi f}$$

The symbols used are defined in the following input data:

- $p$ = number of poles
- $f$ = line frequency (Hz)
- $X_{ls}$ = stator leakage reactance (per unit value)
- $X_{lr}$ = rotor leakage reactance (per unit value)
- $R_s$ = stator resistance (per unit value)
- $R_r$ = rotor resistance (per unit value)
- $X_m$ = magnetizing reactance (per unit value)
- $E$ = dc source half voltage(per unit value)
- $N$ = shaft speed (rpm)

## Switching Functions and Conversions

Figure 1 shows a voltage source inverter connected to an induction motor. The power-electronic switching devices are represented by switches. The switching function for phase A is defined as

$S_A = 1$ when Sw1 is on and Sw2 is off

$S_A = -1$ when Sw1 is off and Sw2 is on

The $S_B$ and $S_C$ for phase B and phase C are also defined similarly.

The phase voltages of a delta-connected winding are

$$v_A = E\,(S_A - S_B)$$
$$v_B = E\,(S_B - S_C)$$
$$v_C = E\,(S_C - S_A) \qquad (1)$$

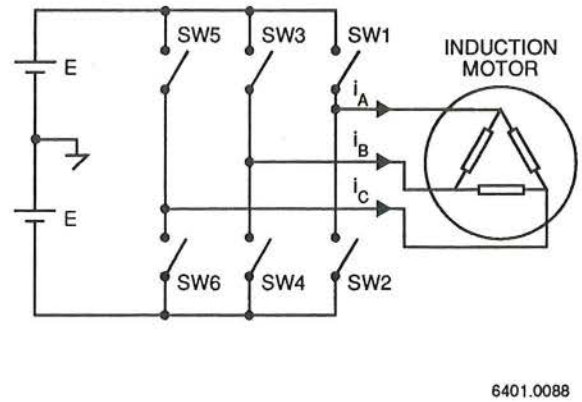


Fig. 1. Voltage source inverter connected to induction motor

The conversions between a,b,c and 1,2,0 coordinates [10,14] are given below:

$$v_1 = \frac{1}{3}\left(v_A + a\,v_B + a^2 v_C\right) \qquad v_A = v_1 + v_2 + v_0$$
$$v_2 = \frac{1}{3}\left(v_A + a^2 v_B + a\,v_C\right) \quad \text{or} \quad v_B = a^2 v_1 + a\,v_2 + v_0$$
$$v_0 = \frac{1}{3}\left(v_A + v_B + v_C\right) \qquad v_C = a\,v_1 + a^2 v_2 + v_0 \qquad (2)$$

$$\text{where } a = e^{\frac{j2\pi}{3}} = -0.5 + j\,0.866$$

Relating switching functions to the 1, 2, 0 voltage values gives

$$v_1 = \frac{E}{3}\left[(S_A - S_B) + a(S_B - S_C) + a^2(S_C - S_A)\right]$$

$$v_2 = \frac{E}{3}\left[(S_A - S_B) + a^2(S_B - S_C) + a(S_C - S_A)\right]$$

and

$$v_0 = \frac{E}{3}\left[S_A - S_B + S_B - S_C + S_C - S_A\right] = 0 \qquad (3)$$

It is worth mentioning that the current and flux of coordinate 0 do not produce torque.

For a three-phase bridge inverter there are a total of eight switch combinations, the corresponding values of voltage $v_1$ for these switch combinations are listed in Table 1.

Table 1. Eight switch combinations

| | Switch Combination | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| c | 0 | 0 | I | II | III | IV | V | VI |
| $S_a$ | –1 | 1 | 1 | 1 | –1 | –1 | –1 | 1 |
| $S_b$ | –1 | 1 | –1 | 1 | 1 | 1 | –1 | –1 |
| $S_c$ | –1 | 1 | –1 | –1 | –1 | 1 | 1 | 1 |
| $v_1$ | 0 | 0 | $A\,b$ | $A\,b^2$ | $A\,b^3$ | $A\,b^4$ | $A\,b^5$ | $A\,b^6$ |

where

$$A = \frac{E}{3}(2 - 2a)$$

and

$$b = e^{\frac{j\pi}{3}} = 0.5 + j\,0.866$$

Since $b^3 = -1$, it is observed from the above table that the value of $v_1$ of switch combination I equals the negative value of $v_1$ of switch combination IV. A similar relationship can be observed between switch combinations II and V, and switch combinations III and VI. Once the value of $v_1$ is obtained, either from equation (3) or directly from equation (2), the calculation of torque can be proceeded further as given in the following sections.

## Voltage Equations

The principle of superimposition can be applied to PWM induction-motor drives. For instance, a voltage pulse with a $\delta$ span as shown in Fig. 2a can be considered as the result of a negative voltage step superimposed on the previous positive voltage step after a time span as shown in Fig. 2b. The motor is initially switched on through a voltage-source inverter without any residual flux or current in the motor. The subsequent voltage step shown in Fig. 3a is considered to be a voltage increment (Fig. 3b) to the previous voltage step, or is the result of two new components (a new opposite step having the same amplitude as the previous step and the new voltage step) that are superimposed to the previous continuous step as shown in Fig. 3c. All the voltage increments are treated mathematically as Heaviside unit functions. As far as the new voltage increment is concerned there is no initial residual flux or current associated with it. The increments in current and flux caused by this new voltage increment are added to the attenuated values of the previous fluxes and currents for the resultant values. Using $\Delta$ to represent increment, the equations of the voltage increment and flux-linkage increment in Heaviside expressions are:

$$\Delta v_1 = P\,\Delta\psi_1 + r_s\,\Delta i_1 \tag{4}$$

and

$$\Delta\psi_1 = X(P - j\omega)\,\Delta i_1 \tag{5}$$

where

$\omega$ = rotor per-unit speed

$P$ = parameter used to solve a Laplace transform

When applying the Laplace transform to solve a differential equation, P also stands for d/dt of a function while the initial value of the function is 0. $X(P - j\omega)$ from [14] at a constant speed is the operational impedance.

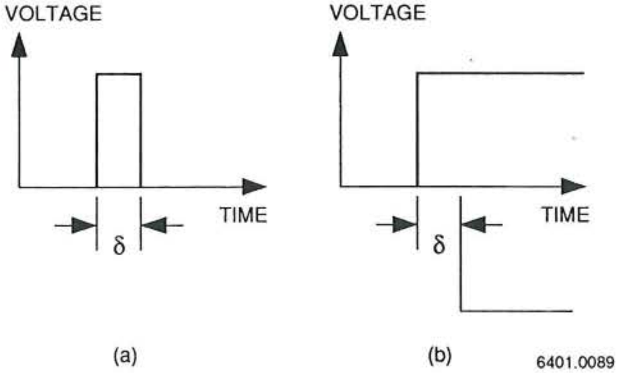


Fig. 2. Principle of superimposition

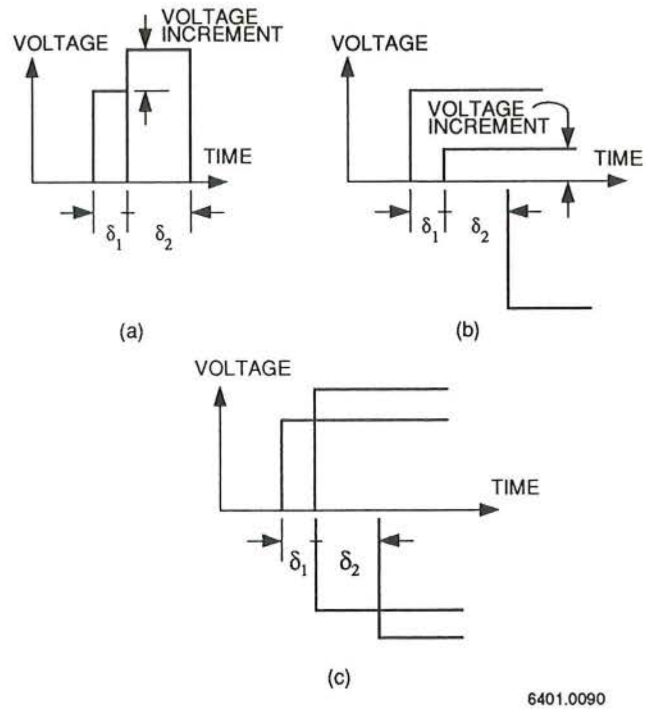


Fig. 3. Superimposition of adjacent pulse

$$X(P - j\omega) = \frac{X_{ss}\left[\tau(P - j\omega) + \frac{1}{T_2}\right]}{P - j\omega + \frac{1}{T_2}} \tag{6}$$

$$\omega = \frac{\pi N p}{60\,\omega_s} \qquad \tau = 1 - \frac{X_m^2}{X_{ss} X_{rr}}$$

$$X_{ss} = X_{ls} + X_m \qquad \omega_s = 2\pi f$$

$$T_1 = \frac{X_{ss}}{R_s} \qquad X_{rr} = X_{lr} + X_m$$

$$T_2 = \frac{X_{rr}}{R_r}$$

## Currents

When the increment of voltage $\Delta v_1$ is a flat step, the inverse Laplace-transform solution [14,16] of the current $i_1$ can be obtained from (4) and (5).

$$\Delta i_1 = \frac{-j\omega + \frac{1}{T_2}}{X'\alpha\beta}\Delta v_1 + \frac{\alpha - j\omega + \frac{1}{T_2}}{\alpha X'(\alpha - \beta)} e^{\alpha t}\Delta v_1 - \frac{\beta - j\omega + \frac{1}{T_2}}{\beta X'(\alpha - \beta)} e^{\beta t}\Delta v_1$$

$$= C_1\,\Delta v_1 + C_2\,e^{\alpha t}\Delta v_1 + C_3\,e^{\beta t}\Delta v_1 \tag{7}$$

where

$$X' = \tau X_{ss}$$

$$\alpha = -\frac{1}{T_m} + j\,\omega_1$$

$$\beta = -\frac{1}{T_s} + j\,\omega_2$$

$$\frac{1}{T_m} = \frac{1}{2\,\tau}\left(\frac{1}{T_1} + \frac{1}{T_2} - \frac{1}{T_0}\right)$$

$$\frac{1}{T_s} = \frac{1}{2\,\tau}\left(\frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_0}\right)$$

$$\frac{1}{\tau\,T_0} - j\,\omega_0 = \sqrt{\frac{4\,(1-\tau)}{\tau^2\,T_1\,T_2} + \left(\frac{1}{\tau\,T_2} - \frac{1}{\tau\,T_1} - j\,\omega\right)^2}$$

$$\omega_1 = 0.5\,(\omega - \omega_0)$$

$$\omega_2 = 0.5\,(\omega + \omega_0)$$

$$\omega_1 + \omega_2 = \omega$$

$$C_1 = \frac{-j\,\omega + \frac{1}{T_2}}{X'\,\alpha\,\beta}$$

$$C_2 = \frac{\alpha - j\,\omega + \frac{1}{T_2}}{\alpha\,X'\left(\alpha - \beta\right)}$$

$$C_3 = \frac{\beta - j\,\omega + \frac{1}{T_2}}{\beta\,X'\left(\beta - \alpha\right)}$$

At $t = 0$, the value of current $\Delta i_1$ calculated from (7) is 0. This fits the physical nature of an inductive load.

For a given voltage step, the resultant current $i_1$ is the summation of the present and all the previous attenuated current increments. Assuming the negative half of voltage steps in one cycle is the mirror image of the positive half, the resultant current during the $m^{th}$ time span of voltage $(v_1)_m$ is

$$\begin{aligned}(i_1)_m &= C_1\,(v_1)_m + C_2\,(v_1)_m\,e^{\alpha t} + C_3\,(v_1)_m\,e^{\beta t} \\ &+ C_2\left\{\frac{1}{\left(1 + e^{0.5\,\alpha T}\right)}\sum_{n=\left(m-\frac{K}{2}\right)}^{n=(m-1)}\left[\left(e^{\alpha\,\delta_n} - 1\right)(v_1)_n\;e^{\alpha\lambda_n}\right]\right\}e^{\alpha t} \\ &+ C_3\left\{\frac{1}{\left(1 + e^{0.5\,\beta T}\right)}\sum_{n=\left(m-\frac{K}{2}\right)}^{n=(m-1)}\left[\left(e^{\beta\,\delta_n} - 1\right)(v_1)_n\;e^{\beta\lambda_n}\right]\right\}e^{\beta t}\end{aligned} \tag{8}$$

where

- $T$ = period which equals $2\pi$ in per unit value
- $t$ = per-unit time measured from the beginning of the time span of $(v_1)_m$.
- $K$ = The number of voltage steps $v_1$ per period (or per cycle)
- $\lambda_n$ = time interval between the $n^{th}$ voltage step to the $m^{th}$ voltage step.

The terms $\left(e^{\beta\,\delta_n} - 1\right)$ and $\left(e^{\alpha\,\delta_n} - 1\right)$ indicate that the superimposition shown in Fig. 3c is applied. The denominators with $\left(1 + e^{0.5\,\beta\,T}\right)$ and $\left(1 + e^{0.5\,\alpha\,T}\right)$ refer to the assumption that the negative half of voltage steps in one cycle is the mirror image (or inverse in sign) of the positive half.

Figure 4 shows that $\gamma_m$ is the time in radians of the beginning of the $m^{th}$ time span, $\delta_m$. Within the range of $n = (m - K/2)$ to $n = (m - 1)$, certain numbers of n may be less than 1. Therefore, when $n < 1$, the data corresponding to n, such as $\delta$, $v_1$, and $\gamma$, can be looked up from the step $(n + K)$, which is one period ahead of n. For instance, when $n = 0$ is less than 1 and out of range of the data array, the data can be looked up from the step $(n + K) = (0 + K) = K$, which is one period ahead of 0 and is the last step of a cycle. Under this situation, the time $\gamma_n$ is adjusted back $2\pi$. This makes $\gamma_n = \gamma_{(n+K)} - 2\pi$. $\lambda_n$ can be obtained from $\lambda_n = \gamma_m - \gamma_{(n+1)}$

When the steps of $(v_1)_m$ are given, $(i_1)_m$ can be written as

$$\begin{aligned}(i_1)_m &= C_1\,(v_1)_m + \left[C_2\,(v_1)_m + (K_2)_m\right]e^{\alpha t} \\ &+ \left[C_3\,(v_1)_m + (K_3)_m\right]e^{\beta t}\end{aligned} \tag{9}$$

where

$$(K_2)_m = C_2\left\{\frac{1}{\left(1 + e^{0.5\,\alpha T}\right)}\sum_{n=\left(m-\frac{K}{2}\right)}^{n=m-1}\left[\left(e^{\alpha\,\delta_n} - 1\right)(v_1)_n\;e^{\alpha\lambda_n}\right]\right\} \tag{10}$$

$$(K_3)_m = C_3\left\{\frac{1}{\left(1 + e^{0.5\,\beta T}\right)}\sum_{n=\left(m-\frac{K}{2}\right)}^{n=m-1}\left[\left(e^{\beta\,\delta_n} - 1\right)(v_1)_n\;e^{\beta\lambda_n}\right]\right\} \tag{11}$$

$i_2$ is the complex conjugate of $i_1$.

**Flux Linkages**

The resultant flux linkage $(\psi_1)_m$ corresponding to the $m^{th}$ time span of voltage $(v_1)_m$ can be obtained from the voltage equation

$$(v_1)_m = R_s(i_1)_m + \frac{d(\psi_1)_m}{dt}$$

or

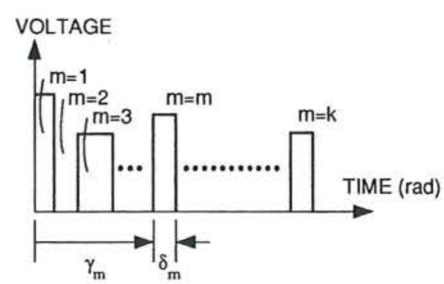


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Fig. 4. $\gamma_m$ is the time in radians of the beginning of the $m^{th}$ time span $\delta_m$

$$(\psi_1)_m = \int_0^t (v_1)_m \, dt - \int_0^t R_s (i_1)_m \, dt + (\psi_1)_{(m-1)\text{ at } \delta_{(m-1)}}$$

$$= (1 - C_1 R_s)(v_1)_m t - [C_2(v_1)_m + (K_2)_m] \, R_s \frac{(e^{\alpha t} - 1)}{\alpha}$$

$$- [C_3(v_1)_m + (K_3)_m] \, R_s \frac{(e^{\beta t} - 1)}{\beta} + (\psi_1)_{(m-1)\text{ at } \delta_{m-1}} \quad (12)$$

The value of $(\psi_1)_1$ at the end of $\delta_{(1)}$ can be obtained from

$$(\psi_1)_{1 \text{ at } \delta_1} = \frac{-1}{K} \sum_{m=2}^{m=K} \sum_{n=2}^{n=m} \Big\{ (1 - C_1 R_s)(v_1)_n \delta_n - [C_2(v_1)_n + (K_2)_n] \, R_s \frac{(e^{\alpha \delta_n} - 1)}{\alpha} - [C_3(v_1)_n + (K_3)_n] \, R_s \frac{(e^{\beta \delta_n} - 1)}{\beta} \Big\} \quad (13)$$

The phase A flux linkage is calculated in a like manner to that shown in (1):

$$\psi_a = \psi_1 + \psi_2 + \psi_0 \quad (14)$$

where $\psi_2$ is the complex conjugate of $\psi_1$, and $\psi_0$ is 0.

## Torques

The torque equation for the 1-2-0 coordinate system can be derived or obtained directly from [12]

$$\text{Torque} = j\,2\,[\, i_2 \psi_1 - i_1 \psi_2 \,] \quad (15)$$

Substituting the current calculated from (9) and the flux linkage calculated from (12) into (15) gives the torque in the time-domain format. Equation (15) can also be expressed through the a-b-c coordinate system in per-unit values as

$$\text{Torque} = \frac{2}{3\sqrt{3}} \left[ i_a(\psi_c - \psi_b) + i_b\,|\psi_a - \psi_c) + i_c(\psi_b - \psi_a) \right] \quad (16)$$

# SAMPLE CALCULATION

## PWM Scheme and Simplification of Calculation due to symmetry

The arbitrarily-chosen sample is a single chip, microprocessor based, PWM control scheme [17]. Although synchronization between carrier wave and fundamental signal wave is unnecessary in this kind of control, the sample used here is synchronized. The steps of $v_1$, their corresponding time spans $\delta$, and the beginning time, $\gamma$, of each $\delta$ are calculated and given in the following tables. The number, K, of voltage steps per cycle in this example is 288. The voltage $V_1$ in a half cycle is the mirror image of the adjacent half cycle that consists of 144 voltage steps. The values of time spans, $\delta$, repeat every 1/6 cycle (i.e. every 48 voltage steps). The line voltage across A and B corresponding to Table 2 is shown in Fig. 5.

## Motor Parameters

The unit torque and the phase parameters of a three phase, ten hp, four pole, squirrel cage induction motor are derived from the 60-Hz values and given in Table 3.

The per-unit line voltage across terminals A and B vs. $\gamma$ at 50 Hz is shown in Fig. 5. The corresponding locus of ($v_1$ * $\delta$) is plotted in Fig. 6. Two similar line voltages and locus of ($v_1$ *$\delta$) for 4 Hz with extremely low number of pulses per cycle are shown in Figs. 7 and 8 respectively. The roundness of this locus reflects the smoothness of the rotating flux.

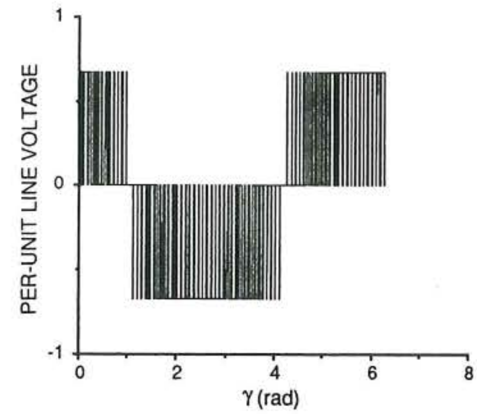


Fig. 5. Line AB per-unit voltage vs. $\gamma$ at 50 Hz

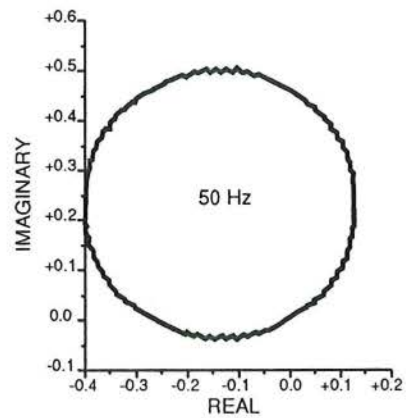


Fig. 6. Locus of ($v_1$ *$\delta$)

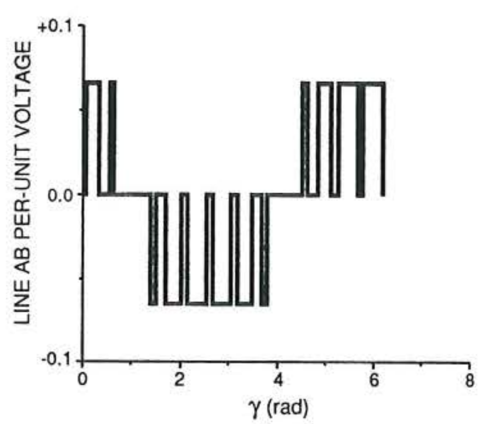


Fig. 7. Line AB per-unit voltage vs. $\gamma$ at 4 Hz

Table 2. Values of $\gamma$, $\delta$, and $v_1$ for each step of a sample calculation

| m | $\gamma$ | $\delta$ | $v_1$ |
|---|---|---|---|
| 1 | 0.0000 | 0.0134 | 0 |
| 2 | 0.0134 | 0.0589 | $A\ b$ |
| 3 | 0.0722 | 0.0030 | $A\ b^2$ |
| 4 | 0.0753 | 0.0120 | 0 |
| 5 | 0.0873 | 0.0130 | 0 |
| 6 | 0.1002 | 0.0554 | $A\ b$ |
| 7 | 0.1556 | 0.0091 | $A\ b^2$ |
| 8 | 0.1647 | 0.0098 | 0 |
| 9 | 0.1745 | 0.0123 | 0 |
| 10 | 0.1888 | 0.0515 | $A\ b$ |
| 11 | 0.2383 | 0.0151 | $A\ b^2$ |
| 12 | 0.2534 | 0.0084 | 0 |
| 13 | 0.2618 | 0.0113 | 0 |
| 14 | 0.2731 | 0.0472 | $A\ b$ |
| 15 | 0.3203 | 0.0210 | $A\ b^2$ |
| 16 | 0.3413 | 0.0078 | 0 |
| 17 | 0.3491 | 0.0103 | 0 |
| 18 | 0.3593 | 0.0425 | $A\ b$ |
| 19 | 0.4018 | 0.0267 | $A\ b^2$ |
| 20 | 0.4285 | 0.0078 | 0 |
| 21 | 0.4363 | 0.0092 | 0 |
| 22 | 0.4455 | 0.0375 | $A\ b$ |
| 23 | 0.4830 | 0.0322 | $A\ b^2$ |
| 24 | 0.5153 | 0.0083 | 0 |
| 25 | 0.5236 | 0.0083 | 0 |
| 26 | 0.5319 | 0.0322 | $A\ b$ |
| 27 | 0.5642 | 0.0375 | $A\ b^2$ |
| 28 | 0.6017 | 0.0092 | 0 |
| 29 | 0.6109 | 0.0078 | 0 |
| 30 | 0.6187 | 0.0267 | $A\ b$ |
| 31 | 0.6454 | 0.0425 | $A\ b^2$ |
| 32 | 0.6879 | 0.0102 | 0 |
| 33 | 0.6981 | 0.0078 | 0 |
| 34 | 0.7059 | 0.0210 | $A\ b$ |
| 35 | 0.7269 | 0.0472 | $A\ b^2$ |
| 36 | 0.7741 | 0.0113 | 0 |
| 37 | 0.7854 | 0.0084 | 0 |
| 38 | 0.7938 | 0.0151 | $A\ b$ |
| 39 | 0.8089 | 0.0515 | $A\ b^2$ |
| 40 | 0.8604 | 0.0123 | 0 |
| 41 | 0.8727 | 0.0098 | 0 |
| 42 | 0.8825 | 0.0091 | $A\ b$ |
| 43 | 0.8916 | 0.0554 | $A\ b^2$ |
| 44 | 0.9470 | 0.0130 | 0 |
| 45 | 0.9599 | 0.0120 | 0 |
| 46 | 0.9719 | 0.0030 | $A\ b$ |
| 47 | 0.9750 | 0.0589 | $A\ b^2$ |
| 48 | 1.0339 | 0.0133 | 0 |
| 49 | 1.0472 | 0.0134 | 0 |
| 50 | 1.0606 | 0.0589 | $A\ b^2$ |
| 51 | 1.1194 | 0.0030 | $A\ b^3$ |
| 52 | 1.1225 | 0.0120 | 0 |
| 53 | 1.1345 | 0.0130 | 0 |
| 54 | 1.1474 | 0.0554 | $A\ b^2$ |
| 55 | 1.2028 | 0.0091 | $A\ b^3$ |
| 56 | 1.2119 | 0.0098 | 0 |
| 57 | 1.2217 | 0.0123 | 0 |
| 58 | 1.2340 | 0.0515 | $A\ b^2$ |
| 59 | 1.2855 | 0.0151 | $A\ b^3$ |
| 60 | 1.3006 | 0.0084 | 0 |
| 61 | 1.3090 | 0.0113 | 0 |
| 62 | 1.3203 | 0.0472 | $A\ b^2$ |
| 63 | 1.3675 | 0.0210 | $A\ b^3$ |
| 64 | 1.3885 | 0.0078 | 0 |
| 65 | 1.3963 | 0.0103 | 0 |
| 66 | 1.4065 | 0.0425 | $A\ b^2$ |
| 67 | 1.4490 | 0.0267 | $A\ b^3$ |
| 68 | 1.4757 | 0.0078 | 0 |
| 69 | 1.4835 | 0.0092 | 0 |
| 70 | 1.4927 | 0.0375 | $A\ b^2$ |
| 71 | 1.5302 | 0.0322 | $A\ b^3$ |
| 72 | 1.5625 | 0.0083 | 0 |
| 73 | 1.5708 | 0.0083 | 0 |
| 74 | 1.5791 | 0.0322 | $A\ b^2$ |
| 75 | 1.6114 | 0.0375 | $A\ b^3$ |

| m | $\gamma$ | $\delta$ | $v_1$ |
|---|---|---|---|
| 76 | 1.6489 | 0.0092 | 0 |
| 77 | 1.6581 | 0.0078 | 0 |
| 78 | 1.6659 | 0.0267 | $A\ b^2$ |
| 79 | 1.6926 | 0.0425 | $A\ b^3$ |
| 80 | 1.7351 | 0.0102 | 0 |
| 81 | 1.7453 | 0.0078 | 0 |
| 82 | 1.7531 | 0.0210 | $A\ b^2$ |
| 83 | 1.7741 | 0.0472 | $A\ b^3$ |
| 84 | 1.8213 | 0.0113 | 0 |
| 85 | 1.8326 | 0.0084 | 0 |
| 86 | 1.8410 | 0.0151 | $A\ b^2$ |
| 87 | 1.8561 | 0.0515 | $A\ b^3$ |
| 88 | 1.9076 | 0.0123 | 0 |
| 89 | 1.9199 | 0.0098 | 0 |
| 90 | 1.9297 | 0.0091 | $A\ b^2$ |
| 91 | 1.9388 | 0.0554 | $A\ b^3$ |
| 92 | 1.9942 | 0.0130 | 0 |
| 93 | 2.0071 | 0.0120 | 0 |
| 94 | 2.0191 | 0.0030 | $A\ b^2$ |
| 95 | 2.0222 | 0.0589 | $A\ b^3$ |
| 96 | 2,0811 | 0.0133 | 0 |
| 97 | 2.0944 | 0.0134 | 0 |
| 98 | 2.1078 | 0.0589 | $A\ b^3$ |
| 99 | 2.1666 | 0.0030 | $A\ b^4$ |
| 100 | 2.1697 | 0.0120 | 0 |
| 101 | 2.1817 | 0.0130 | 0 |
| 102 | 2.1946 | 0.0554 | $A\ b^3$ |
| 103 | 2.2500 | 0.0091 | $A\ b^4$ |
| 104 | 2.2591 | 0.0098 | 0 |
| 105 | 2.2689 | 0.0123 | 0 |
| 106 | 2.2812 | 0.0515 | $A\ b^3$ |
| 107 | 2.3327 | 0.0151 | $A\ b^4$ |
| 108 | 2.3478 | 0.0084 | 0 |
| 109 | 2.3562 | 0.0113 | 0 |
| 110 | 2.3675 | 0.0472 | $A\ b^3$ |
| 111 | 2.4147 | 0.0210 | $A\ b^4$ |
| 112 | 2.4357 | 0.0078 | 0 |
| 113 | 2.4435 | 0.0103 | 0 |
| 114 | 2.4537 | 0.0425 | $A\ b^3$ |
| 115 | 2.4962 | 0.0267 | $A\ b^4$ |
| 116 | 2.5229 | 0.0078 | 0 |
| 117 | 2.5307 | 0.0092 | 0 |
| 118 | 2.5399 | 0.0375 | $A\ b^3$ |
| 119 | 2.5774 | 0.0322 | $A\ b^4$ |
| 120 | 2.6097 | 0.0083 | 0 |
| 121 | 2.6180 | 0.0083 | 0 |
| 122 | 2.6263 | 0.0322 | $A\ b^3$ |
| 123 | 2.6586 | 0.0375 | $A\ b^4$ |
| 124 | 2.6961 | 0.0092 | 0 |
| 125 | 2.7053 | 0.0078 | 0 |
| 126 | 2.7131 | 0.0267 | $A\ b^3$ |
| 127 | 2.7398 | 0.0425 | $A\ b^4$ |
| 128 | 2.7823 | 0.0102 | 0 |
| 129 | 2.7925 | 0.0078 | 0 |
| 130 | 2.8003 | 0.0210 | $A\ b^3$ |
| 131 | 2.8213 | 0.0472 | $A\ b^4$ |
| 132 | 2.8685 | 0.0113 | 0 |
| 133 | 2.8798 | 0.0084 | 0 |
| 134 | 2.8882 | 0.0151 | $A\ b^3$ |
| 135 | 2.9033 | 0.0515 | $A\ b^4$ |
| 136 | 2.9548 | 0.0123 | 0 |
| 137 | 2.9671 | 0.0098 | 0 |
| 138 | 2.9769 | 0.0091 | $A\ b^3$ |
| 139 | 2.9860 | 0.0554 | $A\ b^4$ |
| 140 | 3.0414 | 0.0130 | 0 |
| 141 | 3.0543 | 0.0120 | 0 |
| 143 | 3.0663 | 0.0030 | $A\ b^3$ |
| 143 | 3.0694 | 0.0589 | $A\ b^4$ |
| 144 | 3.1283 | 0.0133 | 0 |
| 145 | 3.1416 | 0.0134 | 0 |
| 146 | 3.1549 | 0.0589 | $-A\ b$ |
| 147 | 3.2138 | 0.0030 | $-A\ b^2$ |
| 148 | 3.2169 | 0.0120 | 0 |
| 149 | 3.2289 | 0.0130 | 0 |

| m | $\gamma$ | $\delta$ | $v_1$ |
|---|---|---|---|
| 150 | 3.24183 | 0.0554 | $-A\ b$ |
| 151 | 3.2972 | 0.0091 | $-A\ b^2$ |
| 152 | 3.3063 | 0.0098 | 0 |
| 153 | 3.3161 | 0.0123 | 0 |
| 154 | 3.3284 | 0.0515 | $-A\ b$ |
| 155 | 3.3799 | 0.0151 | $-A\ b^2$ |
| 156 | 3.3950 | 0.0084 | 0 |
| 157 | 3.4034 | 0.0113 | 0 |
| 158 | 3.4147 | 0.0472 | $-A\ b$ |
| 159 | 3.4619 | 0.0210 | $-A\ b^2$ |
| 160 | 3.4829 | 0.0078 | 0 |
| 161 | 3.4907 | 0.0103 | 0 |
| 162 | 3.5009 | 0.0425 | $-A\ b$ |
| 163 | 3.5434 | 0.0267 | $-A\ b^2$ |
| 164 | 3.5701 | 0.0078 | 0 |
| 165 | 3.5779 | 0.0092 | 0 |
| 166 | 3.5871 | 0.0375 | $-A\ b$ |
| 167 | 3.6246 | 0.0322 | $-A\ b^2$ |
| 168 | 3.6569 | 0.0083 | 0 |
| 169 | 3.6652 | 0.0083 | 0 |
| 170 | 3.6735 | 0.0322 | $-A\ b$ |
| 171 | 3.7058 | 0.0375 | $-A\ b^2$ |
| 172 | 3.7433 | 0.0092 | 0 |
| 173 | 3.7525 | 0.0078 | 0 |
| 174 | 3.7603 | 0.0267 | $-A\ b$ |
| 175 | 3.7870 | 0.0425 | $-A\ b^2$ |
| 176 | 3.8295 | 0.0102 | 0 |
| 177 | 3.8397 | 0.0078 | 0 |
| 178 | 3.8475 | 0.0210 | $-A\ b$ |
| 179 | 3.8685 | 0.0472 | $-A\ b^2$ |
| 180 | 3.9157 | 0.0113 | 0 |
| 181 | 3.9270 | 0.0084 | 0 |
| 182 | 3.9354 | 0.0151 | $-A\ b$ |
| 183 | 3.9505 | 0.0515 | $-A\ b^2$ |
| 184 | 4.0020 | 0.0123 | 0 |
| 185 | 4.0143 | 0.0098 | 0 |
| 186 | 4.0241 | 0.0091 | $-A\ b$ |
| 187 | 4.0332 | 0.0554 | $-A\ b^2$ |
| 188 | 4.0886 | 0.0130 | 0 |
| 189 | 4.1015 | 0.0120 | 0 |
| 190 | 4.1135 | 0.0030 | $-A\ b$ |
| 191 | 4.1168 | 0.0589 | $-A\ b^2$ |
| 192 | 4.1755 | 0.0133 | 0 |
| 193 | 4.1888 | 0.0134 | 0 |
| 194 | 4.2021 | 0.0589 | $-A\ b^2$ |
| 195 | 4.2610 | 0.0030 | $-A\ b^3$ |
| 196 | 4.2641 | 0.0120 | 0 |
| 197 | 4.2761 | 0.0130 | 0 |
| 198 | 4.2890 | 0.0554 | $-A\ b^2$ |
| 199 | 4.3444 | 0.0091 | $-A\ b^3$ |
| 200 | 4.3535 | 0.0098 | 0 |
| 201 | 4.3633 | 0.0123 | 0 |
| 202 | 4.3756 | 0.0515 | $-A\ b^2$ |
| 203 | 4.4271 | 0.0151 | $-A\ b^3$ |
| 204 | 4.4422 | 0.0084 | 0 |
| 205 | 4.4506 | 0.0113 | 0 |
| 206 | 4.4619 | 0.0472 | $-A\ b^2$ |
| 207 | 4.5091 | 0.0210 | $-A\ b^3$ |
| 208 | 4.5301 | 0.0078 | 0 |
| 209 | 4.5379 | 0.0103 | 0 |
| 210 | 4.5481 | 0.0425 | $-A\ b^2$ |
| 211 | 4.5906 | 0.0267 | $-A\ b^3$ |
| 212 | 4.6173 | 0.0078 | 0 |
| 213 | 4.6251 | 0.0092 | 0 |
| 214 | 4.6343 | 0.0375 | $-A\ b^2$ |
| 215 | 4.6718 | 0.0322 | $-A\ b^3$ |
| 216 | 4.7041 | 0.0083 | 0 |
| 217 | 4.7124 | 0.0083 | 0 |
| 218 | 4.7207 | 0.0322 | $-A\ b^2$ |
| 219 | 4.7530 | 0.0375 | $-A\ b^3$ |
| 220 | 4.7905 | 0.0092 | 0 |

| m | $\gamma$ | $\delta$ | $v_1$ |
|---|---|---|---|
| 221 | 4.7997 | 0.0078 | 0 |
| 222 | 4.8075 | 0.0267 | $-A\ b^2$ |
| 223 | 4.8342 | 0.0425 | $-A\ b^3$ |
| 224 | 4.8767 | 0.0102 | 0 |
| 225 | 4.8869 | 0.0078 | 0 |
| 226 | 4.8947 | 0.0210 | $-A\ b^2$ |
| 227 | 4.9157 | 0.0472 | $-A\ b^3$ |
| 228 | 4.9629 | 0.0113 | 0 |
| 229 | 4.9742 | 0.0084 | 0 |
| 230 | 4.9826 | 0.0151 | $-A\ b^2$ |
| 231 | 4.9977 | 0.0515 | $-A\ b^3$ |
| 232 | 5.0492 | 0.0123 | 0 |
| 233 | 5.0615 | 0.0098 | 0 |
| 234 | 5.0713 | 0.0091 | $-A\ b^2$ |
| 235 | 5.0804 | 0.0554 | $-A\ b^3$ |
| 236 | 5.1358 | 0.0130 | 0 |
| 237 | 5.1487 | 0.0120 | 0 |
| 238 | 5.1607 | 0.0030 | $-A\ b^2$ |
| 239 | 5.1638 | 0.0589 | $-A\ b^3$ |
| 240 | 5.2226 | 0.0133 | 0 |
| 241 | 5.2360 | 0.0134 | 0 |
| 242 | 5.2493 | 0.0589 | $-A\ b^3$ |
| 243 | 5.3082 | 0.0030 | $-A\ b^4$ |
| 244 | 5.3113 | 0.0120 | 0 |
| 245 | 5.3233 | 0.0130 | 0 |
| 246 | 5.3362 | 0.0554 | $-A\ b^3$ |
| 247 | 5.3916 | 0.0091 | $-A\ b^4$ |
| 248 | 5.4007 | 0.0098 | 0 |
| 249 | 5.4105 | 0.0123 | 0 |
| 250 | 5.4228 | 0.0515 | $-A\ b^3$ |
| 251 | 5.4743 | 0.0151 | $-A\ b^4$ |
| 252 | 5.4894 | 0.0084 | 0 |
| 253 | 5.4978 | 0.0113 | 0 |
| 254 | 5.5091 | 0.0472 | $-A\ b^3$ |
| 255 | 5.5563 | 0.0210 | $-A\ b^4$ |
| 256 | 5.5773 | 0.0078 | 0 |
| 257 | 5.5851 | 0.0103 | 0 |
| 258 | 5.5953 | 0.0425 | $-A\ b^3$ |
| 259 | 5.6378 | 0.0267 | $-A\ b^4$ |
| 260 | 5.6645 | 0.0078 | 0 |
| 261 | 5.6723 | 0.0092 | 0 |
| 262 | 5.6815 | 0.0375 | $-A\ b^3$ |
| 263 | 5.7190 | 0.0322 | $-A\ b^4$ |
| 264 | 5.7513 | 0.0083 | 0 |
| 265 | 5.7596 | 0.0083 | 0 |
| 266 | 5.7679 | 0.0322 | $-A\ b^3$ |
| 267 | 5.8002 | 0.0375 | $-A\ b^4$ |
| 268 | 5.8377 | 0.0092 | 0 |
| 269 | 5.8469 | 0.0078 | 0 |
| 270 | 5.8547 | 0.0267 | $-A\ b^3$ |
| 271 | 5.8814 | 0.0425 | $-A\ b^4$ |
| 272 | 5.9239 | 0.0102 | 0 |
| 273 | 5.9341 | 0.0078 | 0 |
| 274 | 5.9419 | 0.0210 | $-A\ b^3$ |
| 275 | 5.9629 | 0.0472 | $-A\ b^4$ |
| 276 | 6.0101 | 0.0113 | 0 |
| 277 | 6.0214 | 0.0084 | 0 |
| 278 | 6.0298 | 0.0151 | $-A\ b^3$ |
| 279 | 6.0449 | 0.0515 | $-A\ b^4$ |
| 280 | 6.0964 | 0.0123 | 0 |
| 281 | 6.1087 | 0.0098 | 0 |
| 282 | 6.1185 | 0.0091 | $-A\ b^3$ |
| 283 | 6.1276 | 0.0554 | $-A\ b^4$ |
| 284 | 6.1830 | 0.0130 | 0 |
| 285 | 6.1959 | 0.0120 | 0 |
| 286 | 6.2079 | 0.0030 | $-A\ b^3$ |
| 287 | 6.2110 | 0.0589 | $-A\ b^4$ |
| 288 | 6.2698 | 0.0133 | 0 |

Table 3. Parameters of a three phase, ten hp, four pole, squirrel cage induction motor

| f (Hz) | 60 | 50 | 4 |
|---|---|---|---|
| Unit Torque (Nm) | 39.58 | 47.49 | 593.65 |
| Unit Current (A) | 13.24 | 13.24 | 13.24 |
| Unit Voltage (V) | 187.79 | 187.79 | 187.79 |
| Unit Impedance (Ω) | 14.18 | 14.18 | 14.18 |
| $X_{ls}$ | 0.0503 | 0.0419 | 0.0034 |
| $X_{lr}$ | 0.0503 | 0.0419 | 0.0034 |
| $X_m$ | 1.4102 | 1.1752 | 0.0940 |
| $R_s$ | 0.0395 | 0.0395 | 0.0395 |
| $R_r$ | 0.0240 | 0.0240 | 0.0240 |
| E | 0.4000 | 0.3334 | 0.0333 |
| N (r/min) | | 1450.0 | 115.0 |

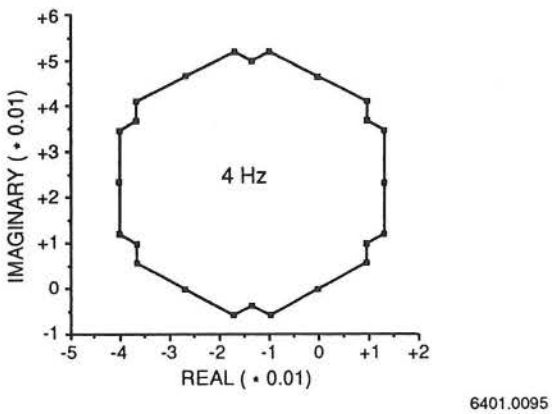


Fig. 8. Locus of ($v_1 * \delta$)

## Currents

The current $(i_1)_m$ is calculated from (9). The complex conjugate values of $(i_1)_m$ is $(i_2)_m$ .

## Flux Linkages

The flux linkage $(\psi_1)_m$ is calculated per equation (12).

## Torque

The torque is calculated per equation (15). The torque, flux, and line currents of the 50 Hz case are plotted in Fig. 9. The relatively smooth torque is a contrast to that of the 4-Hz case shown in Fig. 10. The steady-state current and flux for this case shown in Fig. 11 with extremely low number of pulses per cycle are also pulsating.

## CONCLUSIONS

Evaluation of torque pulsation associated with the harmonics of PWM inverter-fed drives is important for a quiet and smooth operation. This paper discusses an analytical method for the calculation of the voltage-source PWM inverter fed induction motors. Equations derived from the 1-2-0 coordinate system are used. A sample calculation is included for the illustration of practical application.

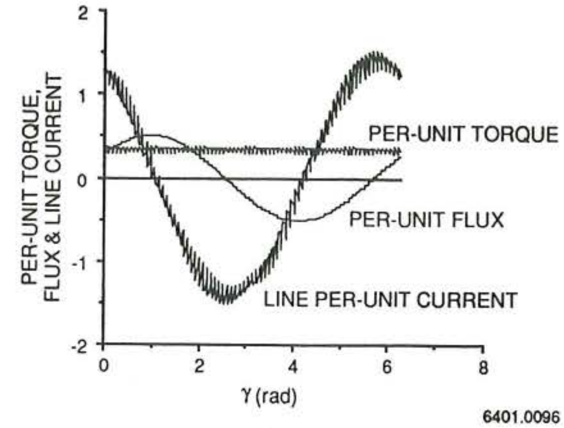


Fig. 9. Steady per-unit torque, flux, and line current vs. γ at 1,450 rpm (50 Hz)

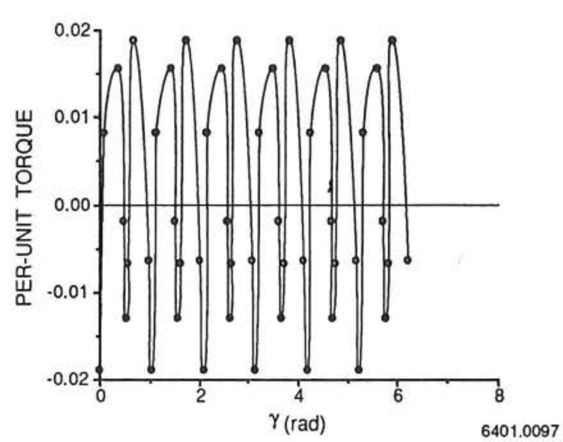


Fig. 10. Steady per-unit torque vs. γ at 115 rpm (4 Hz)

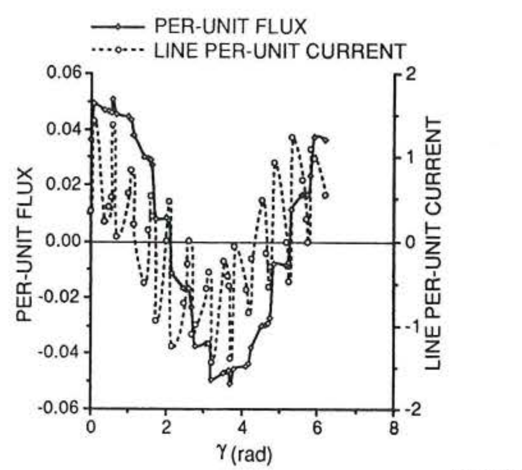


Fig. 11. Steady per-unit flux and line current vs. γ at 115 rpm (4 Hz)

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**John S. Hsu** (or Htsui) (M '64, SM '90) was born in China. He received a BS degree from Tsing-Hua University, Beijing, China, and a PhD degree from Bristol University, England, in 1959 and 1969, respectively. He joined the Electrical and Electronics Engineering Department of Bradford University, England, serving there for nearly two years.

After his arrival in the United States in 1971, he worked in research and development areas for Emerson Electric Company and later for Westinghouse Electric Corporation. He served as head of the Rotating Machines and Power Electronics Program, Center for Energy Studies, The University of Texas at Austin for over four years. Presently, he is the manager of the Industrial Drive Program at the Center for Electromechanics at The University of Texas at Austin.

Dr. Hsu is a chartered engineer in the United Kingdom and a registered professional engineer in Texas, Missouri, and New York.