Fiberness of almost-Montesinos knots

Presented by Maggie Miller

In partial fulfillment of the requirements for graduation with the Dean's Scholars Honors Degree in Mathematics.

Cameron Gordon
Supervising Professor

Date

David Rusin
Honors Advisor in Mathematics

Date

FIBEREDNESS OF ALMOST-MONTESINOS KNOTS

MAGGIE MILLER

May 12, 2015

Abstract. In this paper we begin to classify fiberedness of "Almost-Montesinos" knots, a generalization of Montesinos knots. We employ the method used in the classification of fiberedness of Montesinos knots due to Hirasawa and Murasugi [H]. To achieve this classification, we find minimal-genus surfaces of "skew pretzel links" (a generalization of pretzel links) via sutured manifold decompositions as in [G2]. We end by stating three remaining cases.

Acknowledgements. This work could only be undertaken with the help of my advisor, Dr. Cameron Gordon of UT Austin, whose advice and suggestions have been invaluable. This undergraduate thesis is meant to satisfy the graduation requirement of the Dean's Scholars Natural Science honors program at the University of Texas at Austin.

1. Introduction and Basic Definitions

A *knot* is the image of a smooth embedding of S^1 into S^3. A *link of k-components* is the image of a smooth embedding of $\sqcup_k S^1$ into S^3. Note that a knot is a 1-component link. Two links are said to be equivalent (isotopic) if an isotopy of S^3 carries one to another. We picture a link L with a *diagram*, a generic projection of L onto an equatorial S^2 in S^3 with some of the image erased around double points to illustrate which strand lies above the other in S^3 (as in Fig. 1).

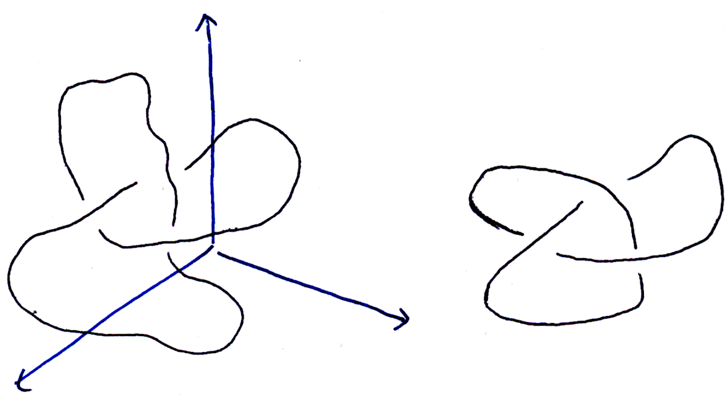


Figure 1. Left: a knot in S^3. Right: A diagram of the knot in S^2. The diagram consists of a closed curve with generic (transverse, 2-stranded) intersections. At intersections, we erase part of the "lower" strand to illustrate which strand is above the other in S^3.

Definition 1.1. An orientable surface with no closed components bounded by a link L is said to be a *Seifert surface for L* (see Fig. 2). Every link has infinitely many Seifert surfaces.

Remark 1.2. The Euler characteristic $\chi(S)$ of a Seifert surface S for L is calculated as follows: view S as the disjoint union of d disks with b bands attached to the disks. Then $\chi(S) = d - b$.

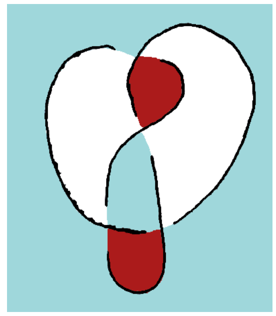


FIGURE 2. A Seifert surface S for a knot. This surface consists of a disk (light blue) with two bands (dark red) attached, each with two half-twists of opposite sign. We conclude $\chi(S) = -1$.

Definition 1.3. The *genus* $g(S)$ of a Seifert surface S is defined to be genus of the surface obtained by attaching disks to each boundary component of the surface. There is some ambiguity in the term "minimal-genus" for disconnected Seifert surfaces. For the purposes of this paper, a *minimal-genus* Seifert surface S of a link L is one that satisfies $\chi(S) \geq \chi(T)$ for any seifert surface T of L. For a connected surface S, $g(S) = (2 - \chi(S) - \#\text{ boundary components})/2$.

Definition 1.4. Given two surfaces S_1, S_2 let D_1, D_2 be $2n$-gons in S_1, S_2 respectively so that $D_1 = D_2$ lies in an equatorial sphere S^2 in S^3 while $S_1 - D_1, S_2 - D_2$ lie in different halfs of $S^3 - S^2$, and so that alternating edges of D_i are contained in the boundary of S_i while the interior of the other edges are in the interior of S_i, as pictured in Figure 3 (lower left). The Murasugi sum (or plumbing) $S = S_{1D_1} \# S_{2D_2}$ of S_1, S_2 along D_1, D_2 is the surface pictured in Figure 3 (lower right).

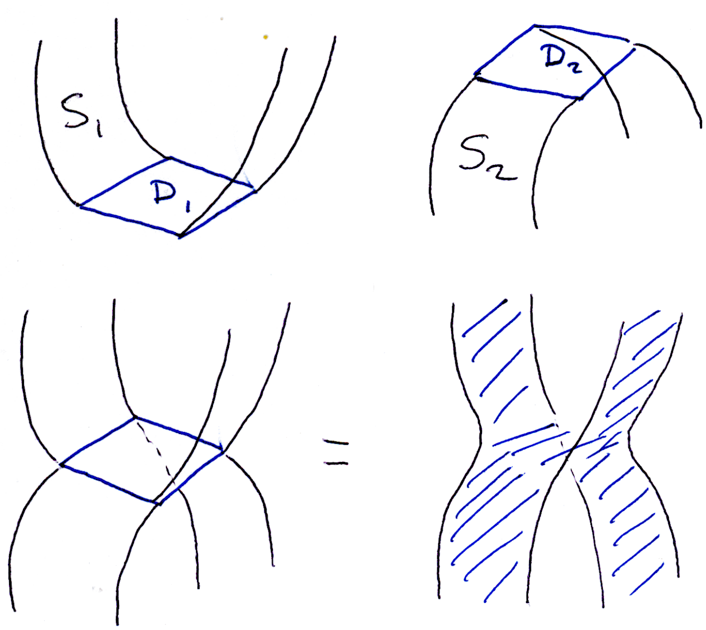


FIGURE 3. Upper left: $2n$-gon D_1 in surface S_1. Upper right: $2n$-gon D_2 in surface S_2. Lower left and right: $S_{1D_1} \# S_{2D_2}$.

Remark 1.5. The Murasugi sum has several nice properties, as shown by Gabai in [G3]:

- S is of minimal-genus if and only if both S_1, S_2 are of minimal-genus.

- S is a fiber (see 1.6) if and only both S_1, S_2 are fibers.

We will use both of these facts extensively.

Definition 1.6. A link L is said to be *fibered* if it is the binding for an open book decomposition of S^3, i.e. L bounds a Seifert surface whose interior fibers $S^3 - L$ so that near L the fibering looks like the illustration in Figure 4 (left) (an "open book").

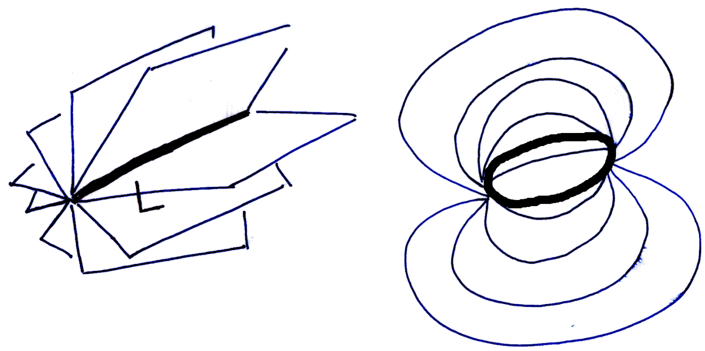


FIGURE 4. Left: a local picture of an open book decomposition with binding L. Right: an open book foliation of S^3 whose fibers are disks and with binding the unknot. We conclude that the unknot is is fibered (with fiber the disk).

Remark 1.7. We refer to Kobayashi [K] for some fundamental facts about fibered links:

Lemma 2.1 of [K]**.** Every fiber surface in a connected 3-manifold is connected.

Lemma 2.2 of [K]**.** For a surface S in a rational homology 3-sphere, with $L = \partial S$ a fibered link, the following three conditions are equivalent.

- S is a fiber surface.
- S is a minimal genus Seifert surface for L.
- S is incompressible.

These results give a strategy for determining whether or not a link L is fibered:

- Construct a minimal-genus surface for L.
- Deplumb the surface into smaller surfaces which are known to be fibers or known not to be fibers.

By Gabai [G3], to show that a surface is minimal-genus it is sufficient to deplumb the surface into surfaces which are known to be minimal-genus. Unfortunately, the natural Seifert surface from a standard diagram of a link might not be minimal-genus, and in general it is difficult to construct surfaces of minimal-genus. The classification of fiberedness is known for some families of links, in particular pretzel links [G1] and Montesinos knots [H].

Remark 1.8. Sometimes we question the fiberedness of an oriented link, i.e. ask if a link has a fiber agreeing with a specified orientation. In this paper we investigate only the fiberedness of knots, on which any orientable Seifert surface may agree with either orientation, so the point of orientation is moot.

Definition 1.9. In this paper, a *sutured manifold* is a pair (M, γ) where $M \subset S^3$ is a compact 3-manifold with boundary and $s(\gamma)$ is a set of simple closed curves (called sutures) on ∂M separating ∂M into two distinct regions ($R_+(\gamma)$ and $R_-(\gamma)$) as in Figure 5. We may visualize sutured manifolds as bounded 3-manifolds with corners in place of sutures.

Definition 1.10. Let (M, γ) be a sutured manifold, and S be a properly embedded surface in M so that $S \cap \partial M = \partial S$ and ∂S is transverse to $s(\gamma)$. Through *decomposition with respect to S* we may obtain a new sutured manifold (N, δ) where

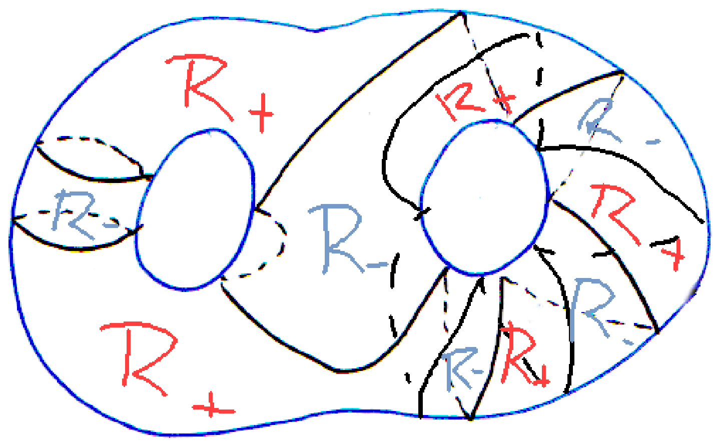


FIGURE 5. A sutured manifold in S^3.

- $N = M - \mathring{N}(S)$
- $R_+(\delta) = (R_+(\gamma) \cap M) \cup S \times \{0\}$
- $R_-(\delta) = (R_-(\gamma) \cap M) \cup S \times \{1\}$.

See Figure 6. If S is a disk, we may refer to this operation as a *disk decomposition*. If there exists a sequence of decompositions $(M_1, \gamma_1) \mapsto (M_2, \gamma_2) \mapsto \cdots \mapsto (M_n, \gamma_n)$, where ∂M_n is a disjoint collection of 2-spheres and each sphere intersects $s(\gamma_n)$ in a simple closed curve, then (M_1, γ_1) is said to be *decomposable*.

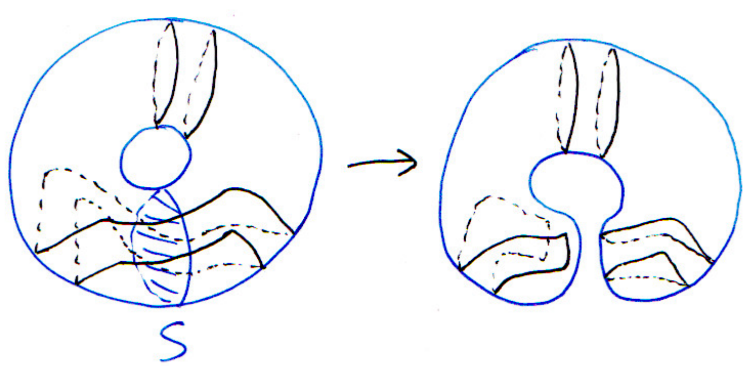


FIGURE 6. Left: A sutured manifold (M, γ) and a properly embedded disk S in M. Right: A sutured manifold (N, δ) obtained through decomposition of M with respect to S.

Definition 1.11. Let (M, γ) be a sutured manifold, and S be a properly embedded surface in $S^3 - \mathring{M}$ so that $S \cap \partial M = \partial S$ and ∂S is transverse to $s(\gamma)$. Through *C-decomposition with respect to S* (C for complementary) we may obtain a new sutured manifold (N, δ) where

- $N = M \cup \mathring{N}(s)$
- $R_+(\delta) = (R_+(\gamma) \cap M) \cup S \times \{0\}$
- $R_-(\delta) = (R_-(\gamma) \cap M) \cup S \times \{1\}$.

See Figure 7. When S is disk, we may refer to this operation as a C-disk decomposition. If there exists a sequence of C-decompositions $(M_1, \gamma_1) \mapsto (M_2, \gamma_2) \mapsto \cdots \mapsto (M_n, \gamma_n)$, where ∂M_n is a disjoint collection of 2-spheres and each sphere intersects $s(\gamma_n)$ in a simple closed curve, then (M_1, γ_1) is said to be *C-decomposable*.

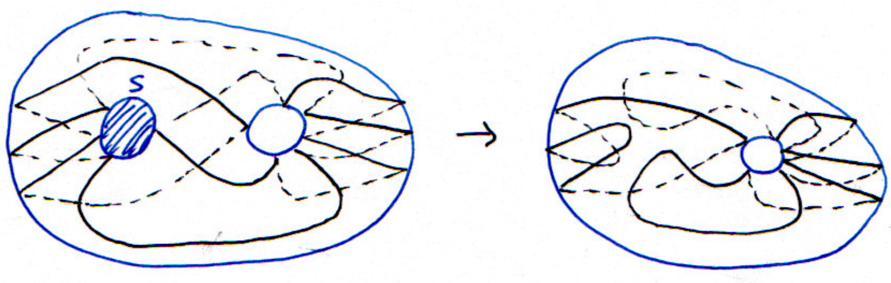


FIGURE 7. Left: A sutured manifold (M, γ) and a properly embedded disk S in $S^3 - \mathring{M}$. Right: A sutured manifold (N, δ) obtained through C-decomposition of M with respect to S.

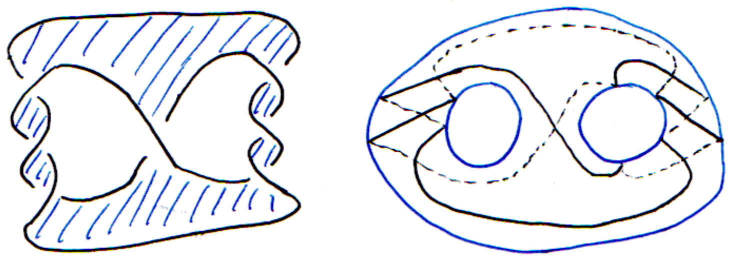


FIGURE 8. Left: a Seifert surface S. Right: The complementary sutured manifold obtained from S is easy to visualize.

Definition 1.12. Let $(M, \gamma), (N, \delta)$ be sutured manifolds so that C-disk decomposition of (M, γ) with respect to a disk D properly embedded in $S^3 - \mathring{M}$ yields (N, δ). This decomposition is called a *C-product decomposition* if $D \cap s(\gamma) = 2$ points.

Definition 1.13. Let (M_0, γ_0) be a sutured manifold. (M_0, γ_0) is said to have a *complete C-product decomposition* if there exist sutured manifolds $(M_1, \gamma_1), \ldots, (M_n, \gamma_n)$ so that (M_i, γ_i) is obtained from (M_{i-1}, γ_{i-1}) by C-product decomposition and ∂M_n is a disjoint union of 2-spheres and each sphere intersects $s(\gamma_n)$ in a simple closed curve.

Definition 1.14. Let S be a Seifert surface for a link L. The *sutured manifold obtained from S* is the sutured manifold $(S^3 - S \mathring{\times} I, L \times \{1/2\})$. It is generally easier to visualize the *complementary sutured manifold obtained from S*, i.e. $(S \times I, L \times \{1/2\})$ (See Fig. 8.) Decomposition in a sutured manifold corresponds to C-decomposition in its complementary sutured manifold.

Definition 1.15. Let S be a Seifert surface. we say that S is *decomposable* if the complementary sutured manifold obtained from S is C-decomposable. We say that S is *completely decomposable* if the complementary sutured manifold obtained from S had a complete C-product decomposition.

Remark 1.16. Sutured manifolds are a useful tool for studying surfaces of minimal genus through this following results of Gabai:

Corollary 1.29 of [G2]**.** If a Seifert surface S is decomposable, then S is a surface of minimal genus for the oriented link $L = \partial S$.

Theorem 1.9 of [G1]**.** Let S be an oriented Seifert surface for an oriented link L. Then L is fibered with fiber S if and only if the S is completely decomposable.

Throughout this paper, we will often refer to decomposition of the sutured manifold obtained from S, but will illustrate the corresponding C-decomposition of the complementary sutured manifold of the sutured manifold obtained from S: naming the first is more efficient, but visualizing the second is much easier.

2. Skew pretzel links

Remark 2.1. The main goal of this paper is to classify fiberedness of a large family of knots (referred to as “almost-Montesinos knots”). In the classification of fiberedness of Montesinos knots [H], Hirasawa and Murasugi show that certain Seifert surfaces for Montesinos knots are minimal genus by deplumbing twisted annuli until they are left with pretzel surfaces. They then refer to Gabai's classification of minimal genus pretzel surfaces in [G1]. To show that certain Seifert surfaces for almost-Montesinos knots are minimal genus, we will deplumb twisted annuli until we are left with a surface we refer to as a ”skew-pretzel surface.” Here we decide when those surfaces are minimal-genus.

A *pretzel link* $P(t_1, \ldots, t_k)$ is a link that bounds a (possibly non-orientable) surface consisting of two disks connected by k vertical twisted bands so that the i-th band has t_i half-twists, as in Figure 9. Gabai completely determined the minimal-genus surface for pretzel links in [G1]. Here we extend that classification to *skew pretzel links*, largely following Gabai's method for pretzels.

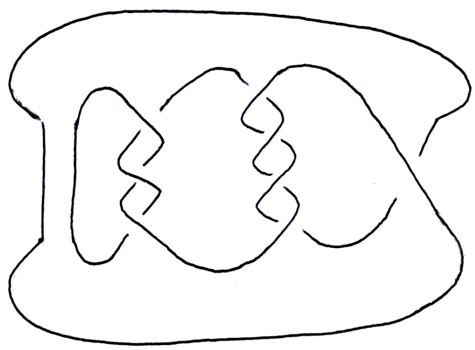


Figure 9. The pretzel link $P(0, -3, 4, -1)$.

A skew pretzel link $P(T_1, \ldots, T_k)$, where $T_i = (\delta_i; r^i_1, \ldots, r^i_{k_i})$ ($\delta_i = \pm 1$) bounds a surface consisting of two disks connected by $\sum_{i,j} r^i_j$ bands arranged as in FIgure 10 (left). The values of $r^i_1, \ldots, r^i_{k_i}$ given the number of half-twists in k_i twisted bands connecting the two disks. Each of the k_i bands crosses each of the others exactly once (as pictured) and are arranged in ascending or descending order. The sign of δ_i refers to which of the bands lies on top; we take $\delta_i = +1$ to mean that the top-most band has positive slope and $\delta_i = -1$ to mean that the top-most band has negative slope. Bands corresponding to r^i_j, r^l_m when $i \neq l$ do not cross. When $k_i = 1$ we may write $T_i = r^i_1$ instead of $T_i = (\delta_i; r^i_1)$ to avoid unnecessary semicolons. Note the skew pretzel surface $P(T_1, \ldots, T_k)$ can be arranged so that each band is vertical with $r^j_i - \delta_i$ half-twists, but with one of the disks twisted as in Figure 10 (right). To obtain the diagram of Figure 10 (right), imagine for each i grabbing the whole group of bands corresponding to the entries of T_i and twisting by 180° about a vertical axis.

Proposition 2.2. When r^i_j is odd for all i, j, and $T_1, \ldots, T_k$ satisfy the following conditions, then the skew pretzel surface for $P(T_1, \ldots, T_k)$ is of minimal genus.

- If $r^i_j - \delta_i = 0$ for some i, j, then $r^i_s - \delta_i \neq 0$ for all $s \neq j$.
- If $r^i_j - \delta_i = 0$ for some i, j and $\delta_l = -\delta_i$, then $r^l_m - \delta_l \neq 0$ for any m.
- If each k_i is even, then $(r^1_1 - \delta_1, r^1_2 - \delta_1, \ldots, r^1_{k_1} - \delta_1, r^2_1 - \delta_2, \ldots, r^k_{k_k} - \delta_k) \neq \pm(2, -2, \ldots, 2, -2)$.
- If $T_i = (+1; -1, 3, \ldots, -1, 3, -1)$ then $T_l \neq (-1; 1, -3, \ldots, 1, -3, 1)$ for any l.
- If $k = 1$, then $T_1 \neq \pm(+1; -1, 3, \ldots, 1, -3)$ or $\mp(+1; 3, -1, \ldots, 3, -1)$ and $T_1 \neq (\delta_i; n, -n + 2\delta_i)$ for any integer n.
- If $k = 2$, then $(T_1, T_2) \neq (n, -n)$ for any integer n.

Proof.
1. If $r^i_j - \delta_i = 0$ for some j deplumb annuli with $r^i_l - \delta_i$ half twists for each $l \neq j$ to obtain the skew pretzel surface with T_i replaced by δ_i. When $\delta_l = -\delta_i$, deplumb annuli with $r^l_m - \delta_l$ half-twists (note $\delta_l = -\delta_i$ and $r^i_j - \delta_i = 0$ implies $r^l_m - \delta_l \neq 0$) for each m to obtain a skew pretzel surface with T_l removed.

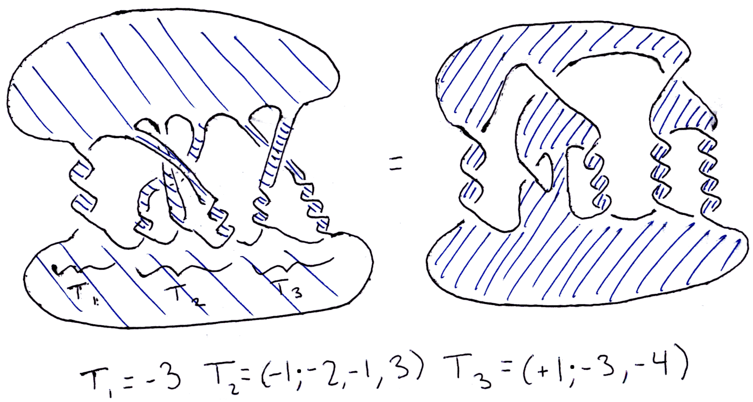
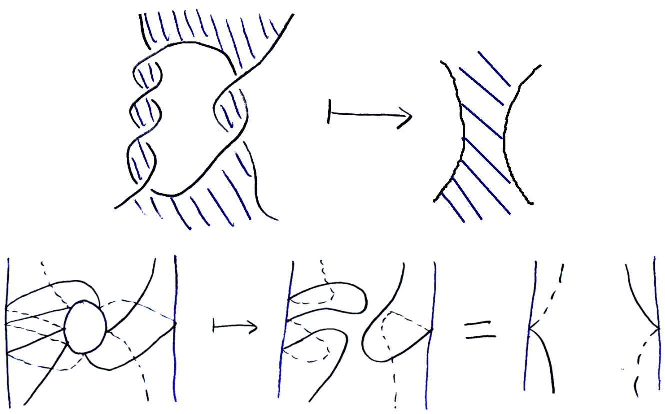


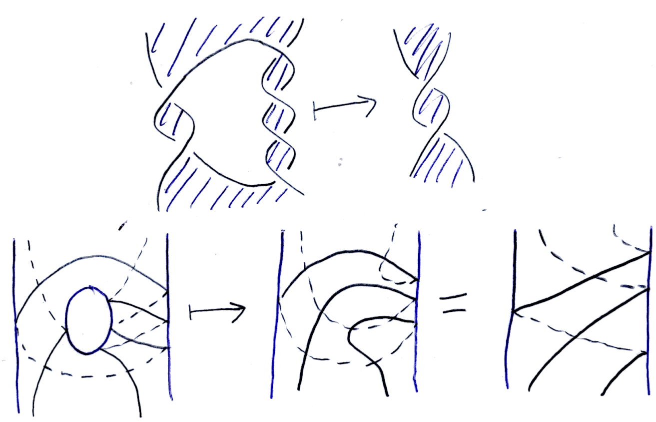
FIGURE 10. The skew pretzel surface bounded by $P(-3,(-1;-2,-1,3),(+1;-3,-4)$.

2. If $|r^i_j - \delta_i|, |r^i_{j+1} - \delta_i| \geq 2$ and $r^i_j \cdot r^i_{j+1} > 0$ then apply local disk decomposition:

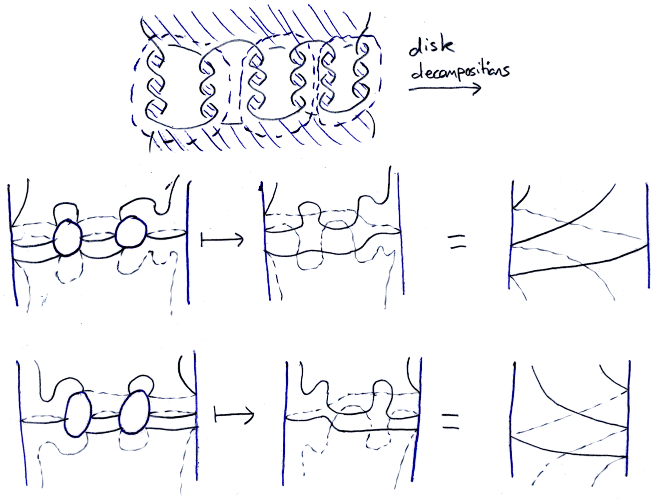


to obtain the skew pretzel surface with r^i_j, r^i_{j+1} together replaced by δ_i. Apply **1**.

3. If $|r^i_j - \delta_i| = 2$ and $|r^i_{j+1} - \delta_i| > 2$ and $r^i_j \cdot r^i_{j+1} < 0$, apply the decomposition:

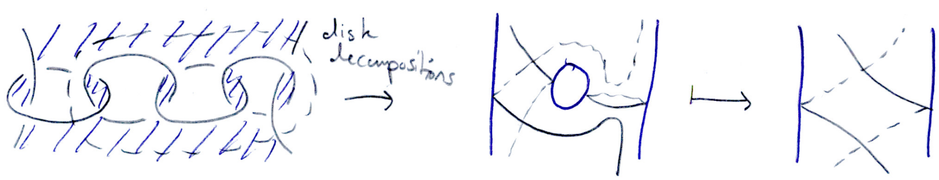


4. If $|r^i_l - \delta_i| > 2$ for all l and $r^i_l \cdot r^i_{l+1} < 0$ for all l and $k_i > 2$ is even, then decompose as in the following example:



to obtain the skew pretzel surface with $T_i = \pm 2 + \delta_i$. Choose sign so that $|\pm 2 + \delta_i| = 3$. If $k_i > 1$ is odd then apply the above and then **2**.

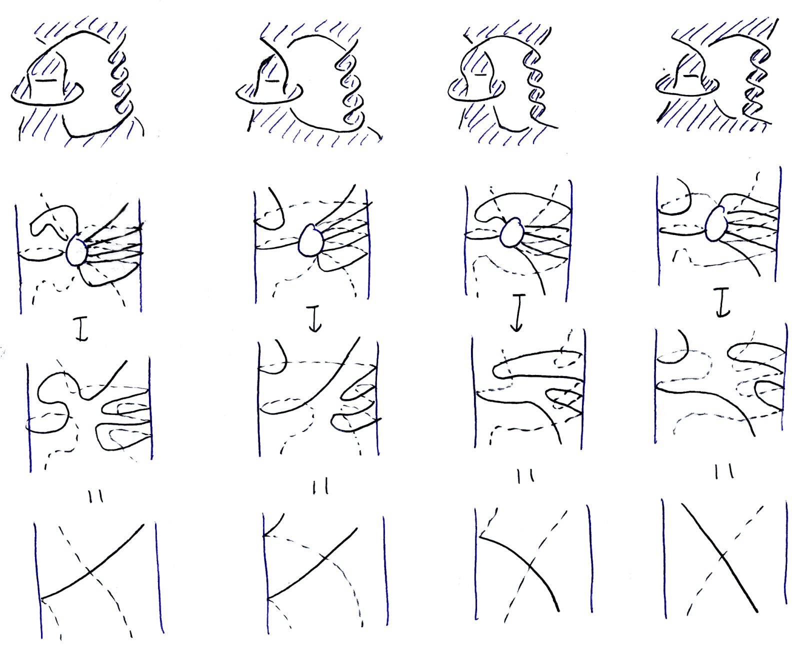
5. If $T_i = \pm(+1; -1, 3, \ldots, -1, 3, -1)$ or $\pm(+1; 3, -1, \ldots, 3, -1)$ (i.e. $(r_1^i - \delta_i, \ldots, r_{k_i}^i - \delta_i) = \pm(2, -2, \ldots, 2, -2, 2)$ then apply the decomposition:



to replace T_i with r_1^i.

6. If $T_i = \pm(+1; -1, 3, \ldots, -1, 3)$ or $T_i = \pm(+1; 3, -1, \ldots, 3, -1)$ (i.e. $(r_1^i - \delta_i, \ldots, r_{k_i}^i - \delta_i) = \pm(2, -2, \ldots, 2, -2))$ then:

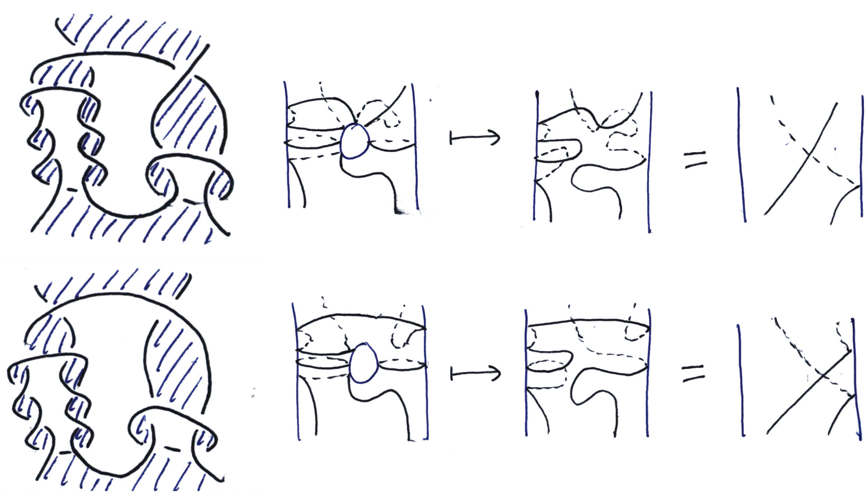
- If $T_{i+1} = n$, perform the disk decompositon:



to obtain the skew pretzel surface with T_i, T_{i+1} together replaced by ± 1. Assume **1** does not apply before the decomposition; the resulting surface satisfies the given hypotheses.
If $(r_1^{i+1} - \delta_{i+1}, \ldots, r_{k_{i+1}}^{i+1} - \delta_{i+1}) = \mp(2, -2, \ldots, 2, -2)$ then perform the disk decomposition:

to obtain the skew pretzel surface with T_i, T_{i+1} together replaced by δ_i. Assume **1** does not apply before the decomposition; the resulting surface satisfies the given hypotheses.

7. If $k_i = 2$, $|r_1^i - \delta_i|, |r_2^i - \delta_i| > 2$ and $r_1^i \cdot r_2^i < 0$, then:
- If $k = 1$ then the surface is an annulus with $r_1^i + r_2^i - 2\delta_i$. By hypothesis, $r_1^i + r_2^i - 2\delta_i \neq 0$
- If $T_{i+1} = \pm(+1; -1, 3, \ldots, -1, 3)$ or $\pm(+1; 3, -1, \ldots, 3, -1)$, then if $r_1^i \cdot r_1^{i+1} < 0$ decompose analogously to **6** case 2. If $r_1^i \cdot r_1^{i+1} > 0$ decompose as:



to obtain the skew pretzel surface with T_i, T_{i+1} together replaced by δ_i. Assume **1** does not apply before the decomposition; the resulting surface satisfies the given hypotheses.
If $T_{i+1} = n$, decompose as in **6** case 1.

Thus, the skew pretzel surface for $P(T_1 \ldots, T_k)$ yields through deplumbing and disk decomposition a disk, an annulus, or a pretzel surface $P(a_1, \ldots, a_n)$ with all a_n odd and $\{1, -1\} \not\subset \{a_1, \ldots, a_n\}$ by hypothesis (see step **5**). Then by [G2], the skew pretzel surface for $P(T_1, \ldots, T_k)$ is minimal genus.

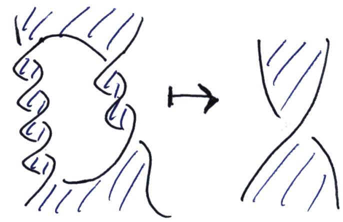
□

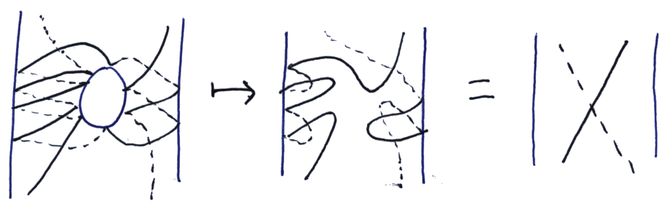
Proposition 2.3. When r_j^i is even for all i, j and $T_1, \ldots, T_k$ satisfy the following conditions, then the skew pretzel surface $P(T_1, \ldots, T_k)$ is of minimal genus.
- If $r_j^i = 0$, then $r_m^l \neq 0$ for any $l \neq m$.
- If $|r_j^i - \delta_i| = 1$, then $r_l^i + r_j^i \neq 2\delta_i$ for any $l \neq j$.
- If $k = 1$, then $T_1 \neq (\delta_1; n, -n + 2\delta_i)$ for any integer n.
- If k is even and $\delta_{2j} = \pm 1$ and $\delta_{2j+1} = \mp 1$ for each j, then there do not exist $l_1, \ldots, l_k$ so that $r_{l_{2j}}^{2j} = \pm 1$ and $r_{l_{2j+1}}^{2j+1} = \mp 2$. (Note in particular, this implies $(T_1, \ldots, T_k) \neq \pm(2, -2, \ldots, 2, -2)$.)

Proof.
1. If $r^i_j = 0$, deplumb annuli with r^l_m half-twists for each $l \neq i$ and annuli with $r^i_m - 2\delta_i$ half-twists for each $m \neq j$ to obtain a disk. By hypothesis, each deplumbed annuli has a nonzero number of half-twists.

2. If $|r^i_j - \delta_i| = 1$ and $r^i_j \neq 0$, deplumb annuli with $r^i_l + r^i_j - 2\delta_i$ half-twists to obtain the skew pretzel surface with T_i replaced by r^i_j.

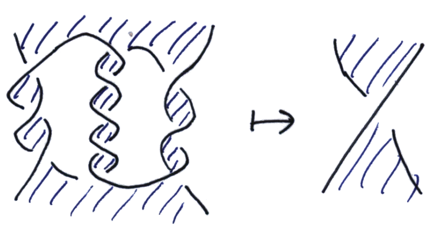
3. Assume **1** and **2** do not apply. If $(r^i_j - \delta_i) \cdot (r^i_{j+1} - \delta_i) > 0$ then apply the decomposition:





to obtain the skew pretzel surface with r^i_j, r^i_{j+1} together replaced by $r^i_j/|r^i_j| + \delta_i$. Apply **1** or **2**.

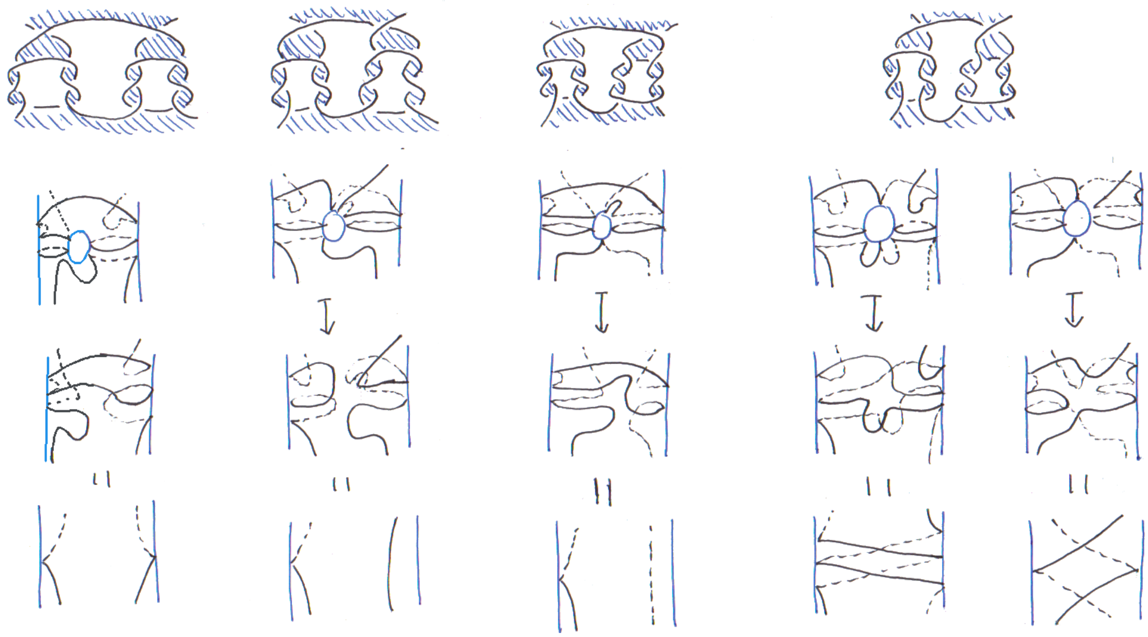
4. Assume **1** and **2** do not apply. If $|r^i_l - \delta_i| > 1$, $r^i_l \cdot r^i_{l+1} < 0$ for each l and $k_i > 1$ is odd, apply the decomposition:



to obtain the skew pretzel surface with T_i replaced with $r^i_1/|r^i_1| + \delta_i$. Apply **1** or **2**. If $k_i > 2$ is even, apply the above decompostion to $r^i_1, \ldots, r^i_{k_i-1}$ and proceed to **5**.

5. Assume **1** and **2** do not apply. If $k_i = 2$ and $r^i_1 \cdot r^2_1 < 0$, then:

- If $k = 1$, then the surface is an annulus with $r^i_1 + r^i_2 - 2\delta_i$ half-twists. By hypothesis, $r^i_1 + r^i_2 - 2\delta_i \neq 0$.
- If $k_{i+1} = 2$ and $r^{i+1}_1 \cdot r^{i+1}_2 < 0$ decompose as:



to obtain the skew pretzel surface with T_i, T_{i+1} with 0, or ± 2 if $\delta_i = \delta_{i+1}$ and $r_i^1 \cdot r_{i+1}^1 < 0$. Note we can choose the sign of ± 2. Apply **1** or **2**.

Thus, from the skew pretzel surface for $P(T_1, \ldots, T_k)$, through disk decomposition we may obtain a disk, a twisted annulus, or a pretzel surface for $P(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are even. By (the first) hypothesis $\{0, 0\} \not\subset \{a_1, \ldots, a_n\}$, and $(a_1, \ldots, a_n) \neq \pm(2, -2, \ldots, 2, -2)$ (see **5**), so by [G2] the original surface is minimal-genus. □

3. Almost-Montesinos knots

3.1. **Definition.** We say that a link L is *almost-Montesinos* if it has a diagram of the form shown in Figure 11, where β_{ij}/α_{ij} refers to the rational tangle of slope β_{ij}/α_{ij}, e_i is an integer number of half-twists, and tangles have been normalized so that $|\beta_{ij}/\alpha_{ij}| < 1$. We write a continued fraction expansion of β_{ij}/α_{ij} as

$$\beta_{ij}/\alpha_{ij} = \cfrac{1}{c_{ij}^1 - \cfrac{1}{c_{ij}^2 - \cfrac{1}{\ddots - \cfrac{1}{c_{ij}^{k_{ij}}}}}} := [c_{ij}^1, c_{ij}^2, \ldots, c_{ij}^{k_{ij}}],$$

where $c_{ij}^1, c_{ij}^2, \ldots, c_{ij}^{k_{ij}} \neq 0$. By replacing β_{ij}/α_{ij} with $\beta_{ij}/\alpha_{ij} \pm 1$ and e_i with $e_i \mp 1$, we may assume $|\beta_{ij}/\alpha_{ij}| < 1$ and β_{ij}, α_{ij} are not both odd.
If a knot K is almost-Montesinos, then it has a diagram of the form in Figure 11 where for each $i \leq m$, there exists at most one $j \leq n_i$ so that α_{ij} is even. As in [H], we say that the i-th row is of odd type if each α_{ij} is odd, and of even type if one α_{ij} is even. Note that if K is a knot, not every row may be of even type.

Remark 3.1. If K is an almost-Montesinos knot and $m = 1$, then K is a connect sum of two-bridge links. If $m = 2$, then K is Montesinos. From now on we assume $m \geq 3$.

Strict and even continued fractions. Hirasawa and Murasugi define strict continued fractions as follows: if $S = [x^1, \ldots, x^k]$ is a continued fraction, then it is *strict* if x^j is even for all odd j, $x^j x^{j+1} < 0$ when j is odd and $|x^j| = 2$, and k is even. They show [H, Prop. 2.5] that for α odd and $\alpha > 2|\beta|$, β/α has a strict continued fraction. We say that S is *even* if x^i is even for all i. It is well known that if α is odd and β is even, then β/α has a (unique) even continued fraction of even length (i.e. $\beta/\alpha = [2c^1, \ldots, 2c^{2k}]$). If α is even and β is odd, then β/α has an even continued fraction of odd length (i.e. $\beta/\alpha = [2c^1, \ldots, 2c^{2k-1}]$).

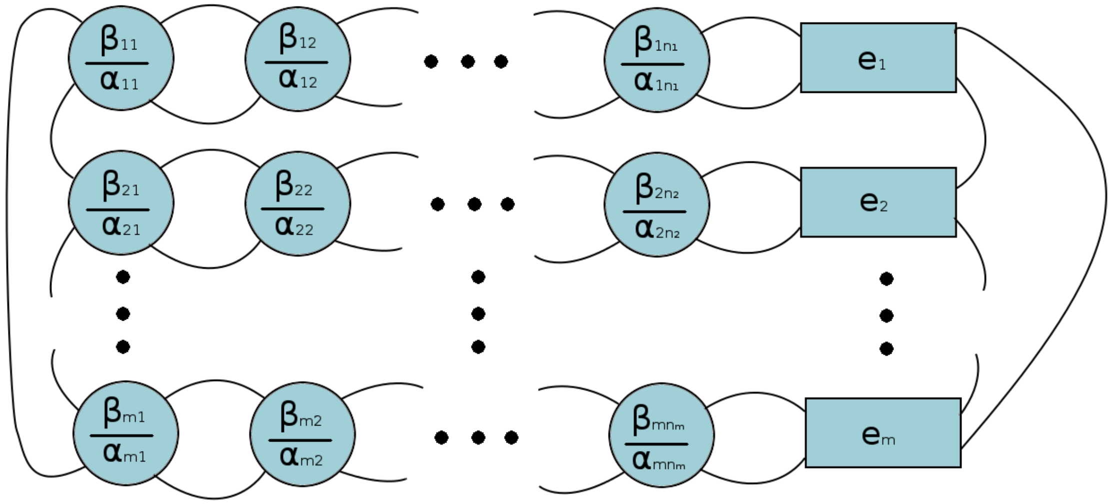


FIGURE 11. An almost-Montesinos link has $m \geq 1$ rows of rational tangles connected in series, with adjacent rows connected at the ends as shown. Note the rows might not be of equal length. Each $\beta_{ij}, \alpha_{ij}, e_i$ are integers, β_{ij}, α_{ij} are coprime, and $\alpha_{ij} > 1$.

3.2. **Classification of fiberedness of almost-Montesinos knots.**

Remark 3.2. We remind the reader of our strategy for deciding whether an almost-Montesinos knot K is fibered.

- Construct a minimal-genus Seifert surface S for K.
  - Construct a Seifert surface S for K.
  - Deplumb annuli wherever possible to obtain surface S'.
    - If S' is known to be minimal-genus, then S is minimal-genus. Here we may refer to Section 2 if S' is a skew pretzel surface.
    - Else, show S' is decomposable, so S' is of minimal genus. Conclude that S is minimal-genus.
- Decide whether S is a fiber.
  - If S' is known to be a fiber or known not to be a fiber, conclude similarly for S.
  - Else, show that S' completely decomposable or argue that S' is not completely decomposable. Conclude that S is a fiber or not a fiber, correspondingly.
- Since S is minimal-genus, K is fibered if and only if S is a fiber.

3.3. **Classification.**

Proposition 3.3. Suppose K has an almost-Montesinos diagram so that:

- e_i is odd for all i,
- $\{1, -1\} \not\subset \{e_1, \dots, e_m\}$.

Let $[c_{ij}^1, \dots c_{ij}^{k_{ij}}]$ be the even continued fraction expansion for β_{ij}/α_{ij}. Note k_{ij} is even if α_{Ij} is odd, and k_{ij} is odd if α_{ij} is even. Then K is fibered if and only if:

- $|c_{ij}^l| = 2$ for all i, j and $l \leq k_{ij}$,
- each e_i is ± 1 or ∓ 1 and for at least one i $|e_i| = 1$.

Proof. Consider the Seifert surface for K featured in Figure 12 (left). For each i, j, l, deplumb an annulus with $c_{ij}^l \neq 0$ half-twists to obtain the pretzel surface for $P(e_1, \dots, e_m)$. By Gabai [G2] this pretzel surface is minimal-genus, so the original Seifert surface for K is minimal-genus. Moreover, by Gabai [G1] the pretzel is fibered if and only if each e_i is ± 1 or ∓ 1 and for at least one i $|e_i| = 1$. Therefore, K is fibered if and only if this condition is met and each deplumbed annuli is a Hopf band (i.e. $|c_{ij}^l| = 2$ for all i, j, l.)

□

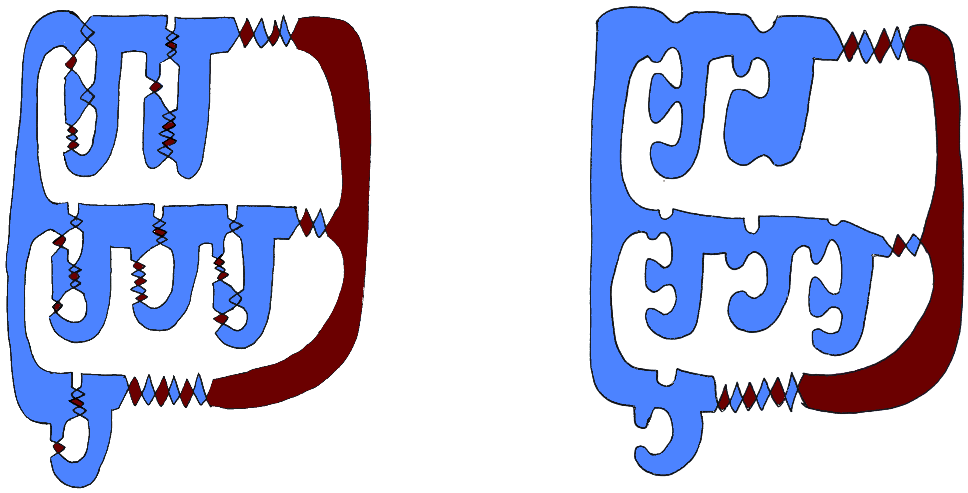


FIGURE 12. Left: a minimal genus Seifert surface for an almost-Montesinos knot with all e_i odd and $\{1,-1\} \not\subset \{e_1,\ldots,e_m\}$. Right: deplumbing annuli yields a pretzel surface.

For the following proposition, when α_{ij} is odd and $|\beta_{ij}| > \alpha_{ij}/2$, replace $\beta_{ij}/\alpha_{ij} \pm 1$ and e_i with $e_i \mp 1$ so that $|\beta_{ij} \pm \alpha_{ij}| < \alpha_{ij}/2$. When α_{ij} is even and e_i is odd, replace β_{ij}/α_{ij} with $\beta_{ij}/\alpha_{ij} \pm 1$ and e_i with $e_i \mp 1$.

Proposition 3.4. . Suppose K has an almost-Montesinos diagram with an even number of odd-type rows, e_i is even when the i-th row is of even type, $|\beta_{ij}| < \alpha_{ij}/2$ for all j when the i-th row is of odd type, $\sum_{\text{odd-type}} \sigma(e_i) \neq 0$ (where $\sigma(x) = x/|x|$) and $e_i \neq 0$ for any i.

If the i-th row is of even type, let $[c^1_{ij},\ldots,c^{k_{ij}}_{ij}]$ be the even continued fraction expansion for β_{ij}/α_{ij}. If the i-th row is of odd type, let $[c^1_{ij},\ldots,c^{k_{ij}}_{ij}]$ be the strict continued fraction expansion for β_{ij}/α_{ij}. Then K is fibered if and only if

- When the i-th row is of even type:
  - $|c^l_{ij}| = 2$ for all j, l.
- When the i-th row is of odd type:
  - $|c^1_{ij} - \sigma(c^2_{ij}) + \sigma(e_i)| = 2$ for all j,
  - $|c^l_{ij} - \sigma(c^{l+1}_{ij}) - \sigma(c^{l-1}_{ij})| = 2$ for all j, odd $l > 1$.
- $\sum^m_{\text{odd-type}} \sigma(e_i) = \pm 2$.

Proof. Consider the Seifert surface for K featured in Figure 13 (left). To obtain the surface of Figure 13 (right):

- When the i-th row is of even type:
  - deplumb annuli with c^l_{ij} half-twists for all j, l,
  - deplumb an annulus with e_i half-twists.
- When the i-th row is of odd type:
  - deplumb annuli with $c^1_{ij} - \sigma(c^2_{ij}) + \sigma(e_i)$ half-twists for all j,
  - deplumb annuli with $c^l_{ij} - \sigma(c^{l+1}_{ij}) - \sigma(c^{l-1}_{ij})$ half-twists for all j and odd $l > 1$.

Since $[c^1_{ij},\ldots,c^{k_{ij}}_{ij}]$ is a strict continued fraction when the i-th row is of odd-type, none of the deplumbed annuli have zero twists. From the resulting surface we may deplumb Hopf bands to obtain an annulus with $\sum_{\text{odd-type}} \sigma(e_i) \neq 0$ half-twists. We conclude that the original Seifert surface for K is minimal genus. Then K is fibered if and only if each deplumbed annulus is a Hopf band and $\sum_{\text{odd-type}} \sigma(e_i) = \pm 2$, which is exactly the proposition.

□

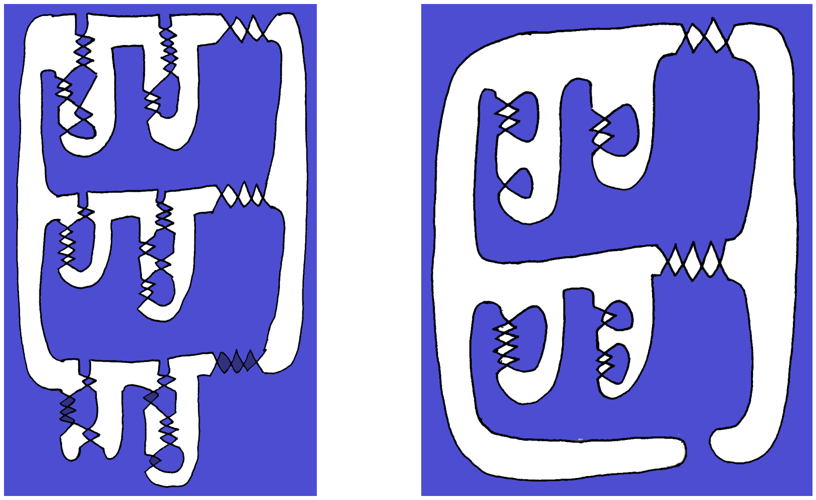


FIGURE 13. Left: a minimal genus Seifert surface for an almost-Montesinos knot with $\sum_{\text{odd row}} \sigma(e_i) \neq 0$ and $e_i \neq 0$ for all i. Right: Deplumbing annuli yields a simpler surface.

Proposition 3.5. . Suppose K has an almost-Montesinos diagram with an even number of odd-type rows. Assume $|\beta_{ij}| < \alpha_{ij}/2$ for all j when the i-th row is of odd type, $e_i \neq 0$ for any i, and $\sum_{\text{odd-type}} \sigma(e_i) = 0$. When the i-th row is of odd-type, define s_i by

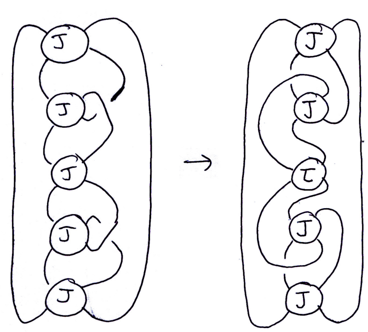
$$s_i = \begin{cases} 2e_i/|e_i| & |e_i| > 1 \\ c_{i1}^1 - \sigma(c_{i1}^2) + \sigma(e_i) & |e_i| = 1 \text{ and } n_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Assume $(s_1, \ldots, s_m)$ (where s_i is omitted when the i-th row is of even-type) is not $\pm(2, -2, \ldots, 2, -2)$. Then K is fibered if and only if

- When the i-th row is of even type:
  - $c_{ij}^l = \pm 2$ for each j, l,
  - $e_i = \pm 2$.
- When the i-th row is of odd type, $c_{ij}^l - \sigma(c_{ij}^{l-1}) - \sigma(c_{ij}^{l+1}) = \pm 2$ for all j and odd $l > 1$.
- When the i-th row is of odd type and $|e_i| > 1$, $c_{ij}^1 - \sigma(c_{ij}^2) + \sigma(e_i) = \pm 2$ for each j.
- $(s_1, \ldots, s_m) = \pm(2, -2, \ldots, 2, -2, 2, -4)$ up to cyclic permutation.

Proof. If the i-th row is of even type, let $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ be the even continued fraction expansion for β_{ij}/α_{ij} and assume e_i is even. If the i-th row is of odd type, let $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ be the strict continued fraction expansion for β_{ij}/α_{ij}.

Match the odd-type rows into nested pairs so that within each pair, one $e_i < 0$ and one $e_i > 0$. Apply the diagram move:



to each pair as in Figure 14.
Consider the Seifert surface for K featured in Figure 15 (upper left). To obtain the surface of Figure 15 (upper right):

- When the i-th row is of even type:
  - deplumb annuli with c_{ij}^l half-twists for all j, l,
  - deplumb an annulus with e_i half-twists.
- When the i-th row is of odd type:
  - deplumb annuli with $c_{ij}^l - \sigma(c_{ij}^{l-1}) - \sigma(c_{ij}^{l+1})$ half-twists for all odd $l > 1$,
  - If $|e_i| > 1$, deplumb annuli with $c_{ij}^1 - \sigma(c_{ij}^2) + \sigma(e_i)$ half-twists for all j.

Since $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ is a strict continued fraction when the i-th row is of odd-type, none of the deplumbed annuli have zero twists. From the resulting surface we may deplumb Hopf bands to obtain the skew pretzel $P(T_1, \ldots, T_m)$, where

- If the i-th row is of even type, $T_i = e_i$.
- If the i-th row is of odd type and $|e_i| > 1$, $T_i = 2e_i/|e_i|$.
- If the i-th row is of odd type and $|e_i| = 1$, $T_i = (e_i; c_{i1}^1 + e_i, c_{i2}^1 + e_i, \ldots, c_{ik_i}^1 + e_i)$.

By assumption, the s_i do not alternate in sign. Moreover, since $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ is a strict continued fraction expansion when the i-th row is of odd type, no $r_j^i - \delta_i = 1$ if $k_i > 1$. Then by Proposition 2.3, this surface is of minimal genus.

If $e_i = \pm 1$ and $k_i > 1$, then as in [G1] there exists no complete C-product decomposition of the resulting surface, so the surface is not a fiber. If $k_i = 1$ whenever $e_i = \pm 1$, then the surface is an ordinary pretzel, so the result follows from the classification of fibered pretzel links in [G1].

□

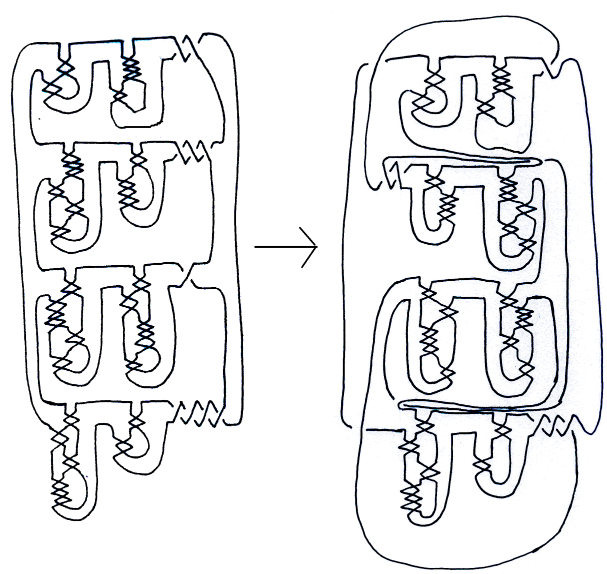


FIGURE 14. When $\sum_{\text{odd-type}} \sigma(e_i) = 0$, we obtain a new diagram.

Proposition 3.6. Suppose K has an almost-Montesinos diagram with an even number of odd-type rows and for some n, $|\beta_{nj}| < \alpha_{nj}/2$ for each j, $e_n = 0$, and the n-th row is of odd type. Then K is never fibered.

Proof. If the i-th row is of even type, assume e_i is even, and let $[c_{1j}^1, \ldots, c_{1j}^k]$ be the even continued fraction expansion for β_{ij}/α_{ij}.. When the i-th row is of odd type, take $|\beta_i| < \alpha_i/2$ and let $[c_{ij}^1, \ldots, c_{ij}^k]$ be the strict continued fraction expansion for β_{ij}/α_{ij}.

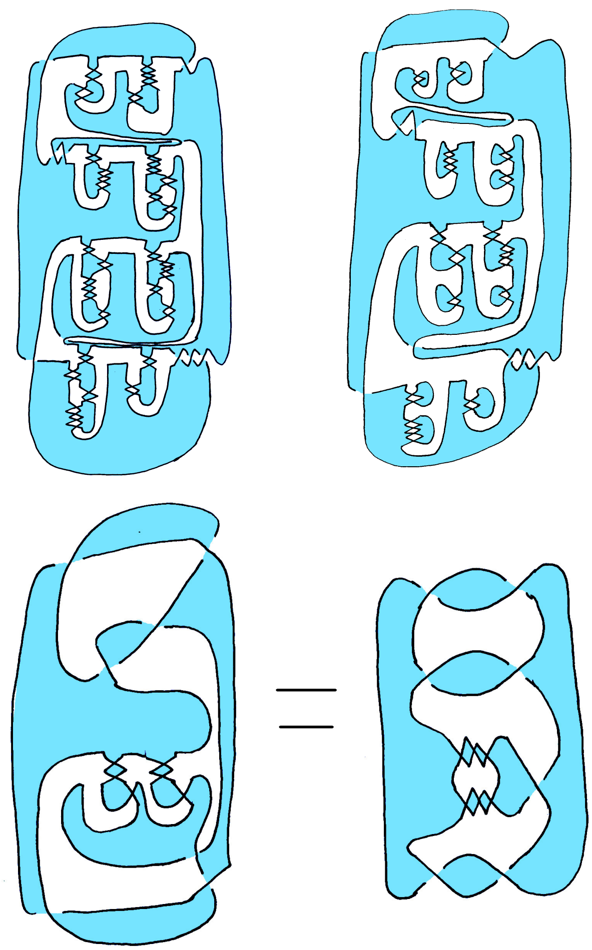


FIGURE 15. Upper left: a minimal-genus Seifert surface. Upper-right: deplumbing twisted annuli yields a new surface. Bottom: Deplumbing Hopf bands yields a skew pretzel surface.

We proceed as in Proposition 3.4. Consider the Seifert surface for K featured in Figure 16 (upper left). To obtain the surface of Figure 16 (upper right):

- When the i-th row is of even type and $e_i \neq 0$:
  - deplumb annuli with c_{ij}^l half-twists for all j, l,
  - deplumb an annulus with e_i half-twists.
- When the i-th row is of even type and $e_i = 0$:
  - deplumb annuli with c_{ij}^l half-twists for all j and $l > 1$.
- When the i-th row is of odd type and $e_i \neq 0$:
  - deplumb annuli with $c_{ij}^1 - \sigma(c_{ij}^2) + \sigma(e_i)$ half-twists for all j,
  - deplumb annuli with $c_{ij}^l - \sigma(c_{ij}^{l+1}) - \sigma(c_{ij}^{l-1})$ half-twists for all j and odd $l > 1$.
- When the i-th row is of odd type and $e_i = 0$:
  - deplumb annuli with $c_{ij}^l - \sigma(c_{ij}^{l+1}) - \sigma(c_{ij}^{l-1})$ half-twists for all j and odd $l > 1$.

Since $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ is a strict continued fraction when the i-th row is of odd-type, none of the deplumbed annuli have zero twists. From the resulting surface we may deplumb Hopf bands to obtain the surface in Figure 16 (lower left). We first show that this surface is of minimal genus:

As a scholium of Theorem 3.2 in [G2], for each odd-type row there exists a disk decomposition replacing that row with a single half-twist. Perform this decomposition on each odd-type row except the n-th (recall

$e_n = 0$) to obtain the pretzel surface $P(c_{n1}^1 - \sigma(c_{n1}^2), \ldots, c_{nk_{nj}}^1 - \sigma(c_{nk_{nj}}^2), l)$ for some odd integer l (see Fig. 16 lower right). If there is an even-type row with $e_i = 0$, it corresponds to a row of vertical bands with $a_1, \ldots a_s$ half-twists in the surface of Figure 16 (lower left). If $(a_1, \ldots a_s) \neq \pm(2, -2, \ldots, 2, -2)$, then as in [G2] there exists a disk decomposition replacing that row with a 0-,2-, or -2-half-twisted band. Perform this disk decomposition; the resulting surface is again a pretzel surface. Since $[c_{nj}^1, \ldots, c_{nj}^{k_{nj}}]$ is a strict continued fraction, $|c_{nj}^1 - \sigma(c_{nj}^2)| \geq 3$ for each j. Therefore, since the pretzel surface is of minimal genus [G2], the original Seifert surface is of minimal genus. If $(a_1, \ldots, a_s) = \pm(2, -2, \ldots, 2, -2)$ then perform a disk decomposition to reduce the n-th row to a ± 1 half-twisted band. The resulting surface is the pretzel $P(\pm 2, \mp 2, \ldots, \pm 2, \mp 2, l)$ for some even integer l. By [G2], this is a minimal genus surface, so the original Seifert surface is of minimal genus.

We now show that the surface in Figure 16 (lower left), and hence the original Siefert surface, is not a fiber:

As in Theorem 6.7 of [G1], we note when the i-th row consists of bands with $a_1, \ldots, a_s$ half-twists, if $|a_i| \geq 3$ for each i then there exists no complete C-product decomposition of the sutured manifold over the surface (to see this, note that if this is the case in every row then there is no possible C-product decomposition at all). But the n-th row consists of band with $c_{n1}^1 - \sigma(c_{n1}^2), \ldots, c_{nk_{nj}}^1 - \sigma(c_{nk_{nj}}^2)$ half-twists and $[c_{nj}^1, \ldots, c_{nj}^{k_{nj}}]$ is a strict continued fraction, so $|c_{nj}^1 - \sigma(c_{nj}^2)| \geq 3$ for each j. Therefore, the surface is not completely decomposable, so is not a fiber. Therefore, K is not fibered.

□

Proposition 3.7. Suppose K has an almost-Montesinos diagram with an even number of odd-type rows, the 1st row is of even type and $e_1 = 0$. Take e_i even when $i > 1$ and the i-th row is of even type, and assume $e_i \neq 0$. Take $|\beta_{ij}| < \alpha_{ij}/2$ for each j when the i-th row is of odd type. If $e_i = 0$ for some $i > 1$ refer to Prop 3.6. Assume $(c_{11}^1, c_{12}^1, \ldots, c_{1n_1}^1, \sum_{\text{odd-type}} \sigma(e_i)) \neq \pm(2, -2, \ldots, 2, -2)$.

Let $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ be the even continued fraction for β_{ij}/α_{ij} when the i-th row is of even type, and $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ be the strict continued fraction expansion for β_{ij}/α_{ij} when the i-th row is of odd-type. Then K is fibered if and only if:

- When the i-th row is of odd-type, for all j:
  - $c_{ij}^l - \sigma(c_{ij}^{l-1}) - \sigma(c_{ij}^{l+1}) = \pm 2$ for $l > 1$ odd,
  - $c_{ij}^1 - \sigma(c_{ij}^2) + \sigma(e_i) = \pm 2$
- When the i-th row is of even type:
  - $c_{ij}^l = \pm 2$ for all j and $l > 1$,
  - $e_i = \pm 2$ if $i > 1$,
  - $c_{ij}^1 = \pm 2$ for all j if $i > 1$.
- If $\sum_{\text{odd-type}} \sigma(e_i) = 0$, then $c_{1j}^1 = \pm 2$ for each j.
- If $\sum_{\text{odd-type}} \sigma(e_i) \neq 0$ and k_i is even then up to cyclic permutation $(c_{11}^1, c_{12}^1, \ldots, c_{1n_1}^1, \sum_{\text{odd-type}} \sigma(e_i)) = \pm(2, -2, \ldots, 2, -2, n)$ for some $n \in \mathbb{Z}$.
- If $\sum_{\text{odd-type}} \sigma(e_i) \neq 0$ and k_i is odd then up to cyclic permutation $(c_{11}^1, c_{12}^1, \ldots, c_{1n_1}^1, \sum_{\text{odd-type}} \sigma(e_i)) = \pm(2, -2, \ldots, 2, -2, 2, -4)$.

Proof. We proceed as in Proposition 3.4. Consider the Seifert surface for K featured in Figure 17 (left). To obtain the surface of Figure 17 (center):

- Deplumb annuli with c_{ij}^l half-twists for all j and $l > 1$ when the i-th row is of even type.
- Deplumb annuli with c_{ij}^1 half-twists for all j when the $i > 1$ and the i-th row is of even type.
- Deplumb annuli with e_i half-twists when $i > 1$ and the i-th row is of even type.
- Deplumb annuli with $c_{ij}^1 - \sigma(c_{ij}^2) + \sigma(e_i)$ half-twists for all j when the i-th row is of odd type.
- Deplumb annuli with $c_{ij}^l - \sigma(c_{ij}^{l+1}) - \sigma(c_{ij}^{l-1})$ half-twists for all j, odd $l > 1$ when the i-th row is of odd type.

Since $[c_{ij}^1, \ldots, c_{ij}^{k_{ij}}]$ is a strict continued fraction when the i-th row is of odd type, none of the deplumbed annuli have zero twists. From the resulting surface we may deplumb Hopf bands to obtain the Pretzel $P(c_{11}^1, \ldots, c_{1n_1}^1, \sum_{\text{odd-type}} \sigma(e_i))$ (as in Fig. 17 [right]). By assumption,

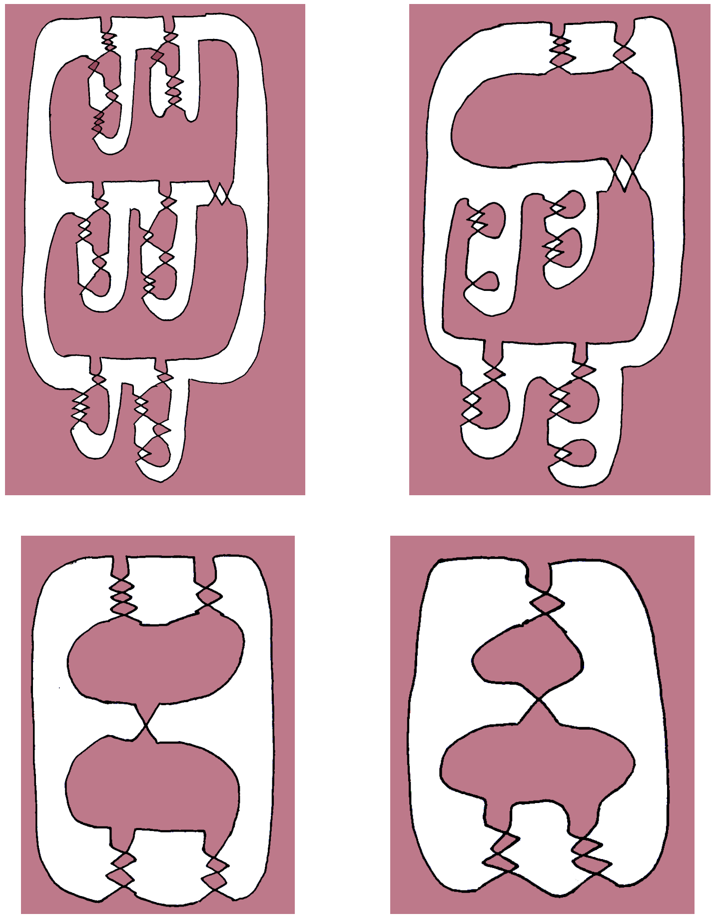


FIGURE 16. Upper left: a minimal genus Seifert surface for an almost-Montesinos knot with $e_n = 0$ for some n when n is a row of odd-type and $|\beta_{nj}| < \alpha_{nj}/2$ for each j. (Here, $n = 3$.) Upper right: Deplumbing annuli yields a simpler surface. Lower left: Deplumbing Hopf bands yields a simpler surface. Lower right: Disk decomposition yields a pretzel surface.

$(c^1_{1j}, \ldots, c^1_{1n_1}, \sum_{\text{odd-type}} \sigma(e_i)) \neq \pm(2, -2, \ldots, 2, -2)$, and we know $c^1_{1j} \neq 0$ for each j, so by Gabai [G2], the surface is minimal-genus. If $\sum_{\text{odd-type}} \sigma(e_i) = 0$ then the pretzel is a murasugi sum of k_i annuli with $c^1_{11}, \ldots, c^1_{1n_1}$ twists. Thus, the pretzel is a fiber if and only if $|c^1_{1j}| = 2$ for each j. If $\sum_{\text{odd-type}} \sigma(e_i) \neq 0$ then by Gabai [G1] the pretzel is a fiber if and only if up to cyclic permutation

$$\left(c^1_{11}, \ldots, c^1_{1n_1}, \sum_{\text{odd-type}} \sigma(e_i)\right) = \pm(2, -2, \ldots, 2, -2, n) \text{ for some } n \in \mathbb{Z} \text{ or } \pm(2, -2, \ldots, 2, -2, 2, -4).$$

□

3.4. **Remaining Cases.** This work is a preliminary draft, and we have not yet completely classified fibredness of almost-Montesinos knots. Here we state the remaining cases.

Remaining Case 3.8. *K has an almost-Montesinos diagram so that e_i is odd for all i and $\{+1, -1\} \subset \{e_1, \ldots, e_n\}$.*

Remaining Case 3.9. *K has an almost-Montesinos diagram with an even number of odd-type rows so that either:*

- *$e_i = 0$ for more than one even-type row*

or

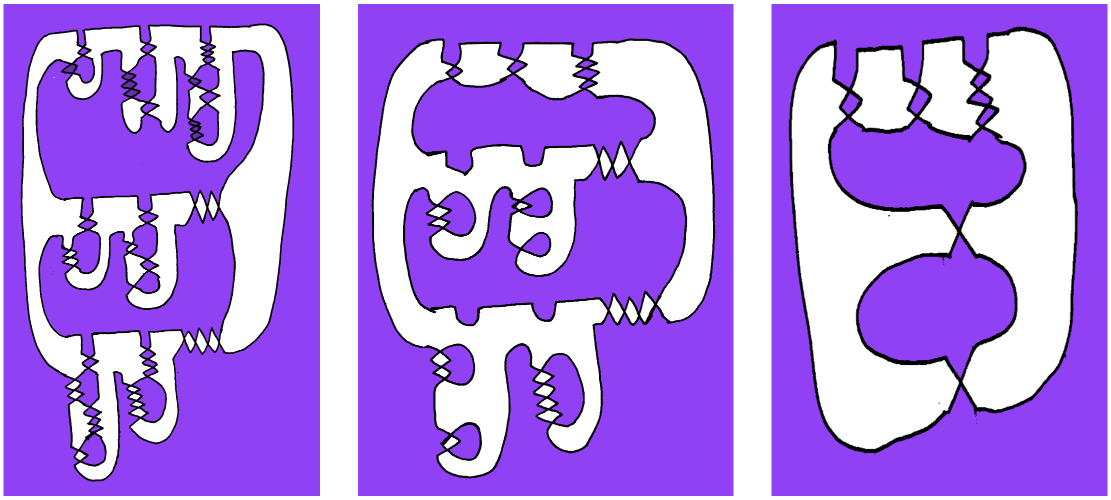


FIGURE 17. Left: a minimal genus surface for an almost-Montesinos knot K with the 1st row of even-type, $e_1 = 0$, and $\sum_{\text{odd-type}} \sigma(e_i) = 0$. Center: Deplumbing annuli yields a simpler surface. Right: Deplumbing Hopf bands yields a pretzel surface.

- $e_i = 0$ for exactly one even-type row and $(c^1_{i1}, \ldots, c^1_{in_i}, \sum_{\text{odd-type}} \sigma(e_i)) = \pm(2, -2, \ldots, 2, -2)$.

Remaining Case 3.10. *K has an almost-Montesinos diagram with an even number of odd-type rows, no $e_i = 0$, $\sum_{\text{odd-type}} \sigma(e_i) = 0$ and when s_i are defined as in Proposition 3.5 we have $(s_1, \ldots, s_m) = \pm(2, -2, \ldots, 2, -2)$.*

Remark 3.11. Note that in all propositions and remaining cases except Proposition 3.3 and Remaining Case 3.8 we assume that the almost-Montesinos knot has an even number of odd-type rows. This is not an oversight; we can reduce fiberedness of an almost-Montesinos knot with an odd number of odd-type rows to the fiberedness of an almost-Montesinos knot with an even number of odd-type rows. Given a knot K with an almost-Montesinos diagram with an odd number of odd-type rows:

- If e_i is odd for each odd-type row (when β_{ij} is even for each j), then look to Proposition 3.3 or Remaining Case 3.8.
- If the n-th row is of odd type and e_n is even, pretend the n-th row is of even type and refer to the appropriate proposition from section 3.3 or remaining case from section 3.4.

Thus, finishing Remaining Cases 3.8, 3.9, and 3.10 will complete the classification of fiberedness of almost-Montesinos knots.

REFERENCES

[G1] David Gabai, Detecting fibred links in S^3, Comment. Math. Helv. (1986), 519-555.

[G2] David Gabai, Genera of the Arborescent Links, Memoirs of the AMS **339** (1986).

[G3] David Gabai, The Murasugi sum is a natural geometric operation, Contemp. Math. **20** (1983), 131-145.

[H] Mikami Hirasawa and Kunio Murasugi, Genera and fiberedness of Montesinos knots, Pacific J. Math. (2006), 52-83.

[K] Tsuyoshi Kobayashi, Fibered links and unknotting operations, Osaka J. Math. **26** (1989), 699-742.

[R] Dale Rolfsen, Knots and Links, Mathematical Lecture Series 7, Publish or Perish Inc., Berkeley CA., 1976.

[S] John Stallings, Construction of fibered knots and links, Proc. Symp. Pure Math. AMS **32** (1978), 55-60.