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Interpolating Gamma Factors in Families

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Interpolating Gamma Factors in Families

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In this thesis, we extend the results of Jacquet, Piatetski-Shapiro, and Shalika [JPSS83] to construct interpolated local zeta integrals and gamma factors attached to families of admissible generic representations of $GL_n(F)$ where F is a p-adic field. Our families are parametrized by the spectrum of an ℓ-adic coefficient ring where $\ell \neq p$.

To show the importance of gamma factors, we prove a converse theorem in families, which says that suitable collections of interpolated gamma factors of pairs uniquely determine a family of representations, up to supercuspidal support.

To prove the converse theorem we re-prove a classical vanishing Lemma, originally due to Jacquet and Shalika, in the setting of families. This is done by extending the geometric methods of Bushnell and Henniart to families, via Helm's theory of the integral Bernstein center.

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Chapter 1

Introduction

1.1 Summary of Results

The analytic theory of local constants for $GL(n)$ was developed by Godement, Jacquet, Piatetski-Shapiro, and Shalika in [GJ72, JPSS79, JPSS83]. In this work we reconsider this theory from an entirely algebraic point of view, extending it to ℓ-adic algebraic families of representations of $GL_n(F)$ where F is a finite extension of $\mathbb{Q}_p$, with $\ell \neq p$. In the first portion of the thesis we construct local zeta integrals in families and prove they satisfy a functional equation. There is a term in the functional equation, called the gamma factor, which remains constant as the zeta integrals vary within a representation. In the second portion of the thesis we prove a local converse theorem for ℓ-adic families, which roughly states that families of representations are uniquely determined by their gamma factors. This converse theorem extends the methods of Henniart and Bushnell-Henniart in [Hen93, BH03] to the integral setting.

A family of $GL_n(F)$-representations means an $A[GL_n(F)]$-module V where A is a Noetherian commutative ring in which p is invertible. For many aspects of the theory, p-power roots of unity are required, so in this thesis A is always a $W(k)$-algebra, where k is an algebraically closed field of characteristic

ℓ and $W(k)$ denotes the Witt vectors of k (recall that $W(\overline{\mathbb{F}_\ell}) \cong \widehat{\mathbb{Z}_\ell^{nr}}$). This is also the setting of ℓ-adic Galois deformations. Given $\mathfrak{p}$ in $\mathrm{Spec}(A)$ with residue field $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, the fiber $V|_\mathfrak{p} := V \otimes \kappa(\mathfrak{p})$ gives a classical representation over $\kappa(\mathfrak{p})$.

In this paper we consider admissible generic $A[GL_n(F)]$-modules which are co-Whittaker (Definition 2.5.2). Each fiber of a co-Whittaker family admits a unique surjection onto an irreducible space of Whittaker functions. Co-Whittaker families are those families attached to continuous Galois deformations $\mathrm{Gal}(\overline{F}/F) \to GL_n(A)$ by the local Langlands correspondence in families, which is conjectured by Emerton and Helm in [EH12]. Emerton and Helm conjecture the existence of a map from the set of continuous Galois deformations over $W(k)$-algebras A to the set of co-Whittaker $A[G]$-modules (in the setting where A is complete, local, reduced, and ℓ-torsion free). This map is uniquely characterized by requiring that it interpolate (a dualized generic version of) classical local Langlands in characteristic zero ([EH12, Thm 6.2.1]). Their definition is motivated by global constructions: the smooth dual of the $\ell \neq p$ tensor factor of Emerton's ℓ-adically completed cohomology ([Eme11]) is an example of a co-Whittaker module.

Henniart's local converse theorem ([Hen93]) says that irreducible generic representations of $GL_n(F)$ over $\mathbb{C}$ are uniquely determined up to isomorphism by their gamma factors. This implies that the bijections of the classical local Langlands correspondence are uniquely determined by identities of gamma factors or, alternatively, of both the L- and epsilon-factors (see, for example, [Jia13]). However, L- and epsilon-factors are absent from the the local Langlands correspondence in families. Therefore it is natural to ask whether it is possible to attach L- and epsilon-factors to co-Whittaker families such as those appearing in local Langlands in families, in a way that interpolates the L- and epsilon factors at each point. Moreover, could such interpolated local constants satisfy a local converse theorem in families?

Over $\mathbb{C}$, L-factors $L(\pi, s)$ arise as the greatest common denominator of the zeta integrals $\Psi(W, s; j)$ of a representation π as W varies over the space $\mathcal{W}(\pi, \psi)$ of Whittaker functions (see Sections 2.2, 3.1 for definitions). Epsilon-factors $\epsilon(\pi, s, \psi)$ are the constant of proportionality in a functional equation relating the modified zeta integral $\frac{\Psi(W,s)}{L(\pi,s)}$ to its pre-composition with a Fourier transform. They are "constant" in the sense that they do not depend on the input W. In this thesis, we will replace the complex variable $q^{-s+\frac{n-1}{2}}$ appearing in [GJ72, JPSS79, JPSS83, Hen93] with the formal variable X. We consider these objects as formal series and use purely algebraic methods.

It appears difficult to construct L-factors in a way compatible with arbitrary change of coefficients. To see this, consider the following simple example: if q is the order of the residue field of F, let $q \equiv 1 \mod \ell$. In this setting, there are $\chi_1,\ \chi_2 : F^\times \to W(k)^\times$ smooth characters such that χ_1 is unramified but χ_2 is ramified, and such that $\chi_1 \equiv \chi_2 \mod \ell$. Naively following the classical procedure (see for example [BH06, 23.2]) for finding a generator of the fractional ideal of zeta integrals, we get $L(\chi_i, X) \in W(k)(X)$ and find that $L(\chi_1, X) = \frac{1}{1-\chi_1(\varpi_F)X}$, and $L(\chi_2, X) = 1$. Now let A be the Noetherian local ring $\{(a,b) \in W(k) \times W(k) : a \equiv b \mod \ell\}$, which has two characteristic zero points $\mathfrak{p}_1$, $\mathfrak{p}_2$ and a maximal ideal ℓA. Let π be the $A[F^\times]$-module A, with the action of $F^\times$ given by $x \cdot (a,b) = (\chi_1(x)a, \chi_2(x)b)$. Interpolating $L(\chi_1, X)$ and $L(\chi_2, X)$ would mean finding an element $L(\pi, X)$ in $A[[X]][X^{-1}]$ such that $L(\pi, X) \equiv L(\chi_i, X) \mod \ell$ for $i = 1, 2$, but such a task is impossible because $L(\chi_1, X)$ and $L(\chi_2, X)$ are different mod ℓ.

On the other hand, zeta integrals themselves seem to be much more well-behaved with respect to specialization, so they form the central focus of our inquiry. Classically, zeta integrals form elements of the quotient field $\mathbb{C}(X)$ of $\mathbb{C}[X, X^{-1}]$. Since A is not in general a domain, we must identify, for more arbitrary coefficient rings A, the correct fraction ring in which our naive generalization of zeta factors live. This is the subject of Chapter 3, where we prove the following theorem:

Theorem 1.1.1. *Suppose A is a Noetherian $W(k)$-algebra. Let S be the multiplicative subset of $A[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. Then if V is admissible, Whittaker-type, and finitely generated as an $A[G]$-module, $Z(W, X; j)$ lies in the fraction ring $S^{-1}(A[X, X^{-1}])$ for all $W \in \mathcal{W}(V, \psi)$ and for $0 \leq j \leq n-2$.*

The proof of this rationality property in the setting of representations over a field relies on a useful decomposition of a Whittaker function into "finite functions ([JPSS79, Prop 2.2]). In the setting of rings, such a structure theorem is lacking, but certain elements of its proof can be translated into purely representation-theoretic questions on the finiteness of the $(n-1)$st Bernstein-Zelevinsky derivative (in fact, more broadly, on the admissibility of Jacquet functors, as discussed in Chapter 5). This finiteness property is proved in Chapter 2. In Chapter 3 it is combined with a simple translation property of the zeta integrals to deduce the result (see §3.2).

In the classical setting, zeta integrals still satisfy a functional equation which does not involve dividing by the L-factor. The constant of proportionality in this functional equation is called the gamma-factor and, in situations where the L-factor makes sense, equals $\epsilon(\pi, X, \psi)\frac{L(\pi^\iota, \frac{q^{n-2}}{X})}{L(\pi, X)}$. In Chapter 3, we prove that gamma-factors interpolate in ℓ-adic families (see §3.3 for details on the notation):

Theorem 1.1.2. *Suppose A is a Noetherian $W(k)$-algebra and suppose V is a co-Whittaker $A[G]$-module. Then there exists a unique element $\gamma(V, X, \psi)$ of $S^{-1}(A[X, X^{-1}])$ such that*

$$Z(W, X; j)\gamma(V, X, \psi) = Z(\widetilde{w'W}, \frac{q^{n-2}}{X}; n-2-j)$$

for any $W \in \mathcal{W}(V, \psi)$ and for any $0 \leq j \leq n-2$.

Analogous to the results of Bernstein and Deligne in [BD84] for smooth representations over $\mathbb{C}$, Helm shows in [Hel12a, Thm 10.8] that the category $\mathrm{Rep}_{W(k)}(G)$ has a decomposition into full subcategories known as blocks. In this paper we construct for each block a gamma factor which is universal in the sense that it gives rise via specialization to the gamma factor for any co-Whittaker module in that block. We will now state this result more precisely.

Each block of the category $\mathrm{Rep}_{W(k)}(G)$ corresponds to a primitive idempotent in the Bernstein center $\mathcal{Z}$, which is defined as the ring of endomorphisms of the identity functor. It is a commutative ring whose elements consist of collections of compatible endomorphisms of every object, each such endomorphism commuting with all morphisms. Choosing a primitive idempotent e of $\mathcal{Z}$, the ring $e\mathcal{Z}$ is the center of the subcategory $e \cdot \mathrm{Rep}_{W(k)}(G)$ of representations satisfying $eV = V$. The ring $e\mathcal{Z}$ has an interpretation as the ring of regular functions on an affine algebraic variety over $W(k)$, whose k-points are in bijection with the set of unramified twists of a fixed conjugacy class of cuspidal supports in $\mathrm{Rep}_k(G)$. See [Hel12a] for details. In [Hel12b], Helm determines a "universal co-Whittaker module" with coefficients in $e\mathcal{Z}$, denoted here by $e\mathfrak{W}$, which gives rise to any co-Whittaker module via specialization (see Proposition 2.6.1 below).

By applying our theory of zeta integrals to $e\mathfrak{W}$ we construct in Chapter 3 a gamma factor $\Gamma(e\mathfrak{W}, X, \psi)$ which is universal in the following sense:

Theorem 1.1.3. *Suppose A is any Noetherian $W(k)$-algebra, and suppose V is a primitive co-Whittaker $A[G]$-module. Then there is a primitive idempotent e, a homomorphism $f_V : e\mathcal{Z} \to A$, and an element $\Gamma(e\mathfrak{W}, X, \psi) \in S^{-1}(e\mathcal{Z}[X, X^{-1}])$ such that $\gamma(V, X, \psi) = f_V(\Gamma(e\mathfrak{W}, X, \psi))$.*

Chapter 4 develops a version of the Godement-Jacquet theory of local zeta integrals for ℓ-adic families. Godement-Jacquet zeta integrals arise as the natural generalization of the local Euler factors of Hecke's L-function, which were the subject of Tate's thesis. Whereas Hecke's local L-factors are Mellin transforms of characters, the Godement-Jacquet local zeta integrals are Mellin transforms of matrix coefficients of representations of $GL_n(F)$. Whereas Chapter 3 focuses on the so-called "standard" zeta integrals by interpolating spaces of Whittaker functions, Chapter 4 focuses on the Godement-Jacquet integrals formed with matrix coefficients of the representation.

The primary motivation for interpolating Godement-Jacquet local zeta integrals in families is a potential application to the construction of p-adic L-functions for unitary Shimura varieties by Harris-Li-Skinner in [HLS06]. In their introduction, Harris, Li, and Skinner predict that an integral analogue of the Godement-Jacquet local zeta integrals would be necessary to have a construction which took into account ramification at primes not dividing p. Because the Eisenstein measure they use to construct the p-adic L-function does not take into account this ramification, the Euler factors at these primes are omitted. Note that in this thesis the role of ℓ and p is opposite from that in [HLS06] (i.e. ℓ and p are swapped).

We now state our results (see Section 1.3 below for notation):

Theorem 1.1.4. *Let A be a Noetherian $W(\overline{\mathbb{F}_\ell})$-algebra, (π, V) a co-Whittaker $A[G]$-module, and $\mathcal{C}(\pi)$ its space of matrix coefficients. Let S be the multiplicative subset of $A[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. Then:*

1. *$Z(f, \Phi, X)$ lives in the fraction ring $S^{-1}(A[X, X^{-1}])$ for all $f \in \mathcal{C}(\pi)$,*

$\Phi \in C_c^\infty(M_n(F), A)$.

2. *There exists a unique element* $\gamma(V, X, \psi)$ *of* $S^{-1}A[X, X^{-1}]$ *such that*

$$Z(f, \Phi, X)\gamma(V, X, \psi) = Z(f^\vee, \widehat{\Phi}, \frac{q^{n-2}}{X})$$

for all $f \in \mathcal{C}(\pi)$, $\Phi \in C_c^\infty(M_n(F), A)$.

The strategy for proving Theorem 1.1.4 is similar to the method of Jacquet, Piatetski-Shapiro, and Shalika in [JPSS79]. In [JPSS79, 4.3] the authors to connect the Godement-Jacquet zeta integrals $Z(f, \Phi, s)$, where f is a matrix coefficient and Φ a Schwartz function, to their "new" integrals $\Psi(W, s)$ where W is a Whittaker function. Over $\mathbb{C}$, they prove that the ideal of Godement-Jacquet zeta integrals $Z(f, \Phi, X)$ is equal to the ideal of their integrals $\Psi(W, X)$.

Using formal commutative algebra, we can exploit [JPSS79, 4.3] to prove our theorem in the case where A is reduced and ℓ-torsion free and V is absolutely irreducible at minimal primes. More precisely, by combining the characteristic zero result with the results of Chapter 3 on interpolating $\Psi(W, X)$ in families, we get a rationality result for $Z(f, \Phi, X)$ up to multiplication by a scalar in A which is a non-zerodivisor. Since power series rings form torsion-free A-modules, this scalar can be removed. The functional equation is deduced from the characteristic zero setting.

To go beyond the setting where A is reduced and ℓ-torsion free, we again employ the methods of Chapter 3. Using the theory of the integral Bernstein center, whose components are shown in [Hel12a] to be reduced and ℓ-torsion free, and whose "universal" co-Whittaker module $e\mathfrak{W}$ is absolutely irreducible at minimal primes (Prop 4.2.1 below), we can realize the zeta integrals as base-changes of those attached to the universal co-Whittaker representation.

In Chapter 5 we extend the results of Chapter 3 to Rankin-Selberg convolutions. The philosophy of Langlands dictates that there should be an operation $\times$ on automorphic representations that corresponds to tensor product $\otimes$ of Galois representations. Even when it is not known how to construct $\pi_1 \times \pi_2$, it is possible to construct its local constants, using the Rankin-Selberg convolution integral. On the local level, for general linear groups, this boils down to integrating products of Whittaker functions, after restricting the argument to the group of lower rank.

The local Rankin-Selberg integrals have two inputs, a Whittaker function W for a representation V, and a Whittaker function W' for a representation V'. In the setting of families, there is no need to restrict ourselves to the situation where V and V' have the same coefficient ring. Therefore, in Chapter 5, the base ring is taken to be $R := A \otimes_{W(k)} B$ where A and B are Noetherian $W(k)$-algebras, and V is an $A[GL_n(F)]$-module and V' is a $B[GL_m(F)]$-module.

The local Rankin-Selberg formal series $\Psi(W, W', X)$ and gamma factors $\gamma(V \times V', X, \psi)$ are constructed in Sections 5.1 and 5.2 for co-Whittaker modules by proving a rationality result and functional equation. The complex variable $q^{-s+\frac{n-m}{2}}$ appearing in [JPSS83] is replaced with the formal variable X.

As in Chapters 3 and 4, the formal series $\Psi(W, W', X)$ will define an element of the fraction ring $S^{-1}(R[X, X^{-1}])$ where S is the multiplicative subset of $R[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. Again, this ring enables us to relate the objects on either side of the functional equation, and also implies that $\Psi(W, W', X)$ will specialize to a rational function at each fiber.

The proofs of rationality and the functional equation follow a similar pattern to the results for the $GL(n) \times GL(1)$ case, which is the subject of Chapter 3, although the theory of derivatives does not play a role in the $GL(n) \times GL(m)$ case. We discover in Chapter 5 that the rationality result can be proved entirely using Jacquet functors. In fact, Chapter 5 implies all the results contained in Chapter 3, and never uses the Bernstein-Zelevinsky derivative. Even though the Bernstein-Zelevinsky derivatives are not used to prove rationality when $m > 1$, the finiteness results of §2.4 should be useful in extending the functional equation beyond the co-Whittaker setting, i.e. to all admissible, Whittaker type, and G-finite families.

In Chapter 6 we prove a $GL(n) \times GL(n-1)$ local converse theorem. By this we mean a result along the following lines: given V_1 and V_2 representations of $GL_n(F)$, if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all representations V' of $GL_{n-1}(F)$, then V_1 and V_2 are the same. Typically V_1, V_2, and V' are irreducible admissible generic complex representations, and "the same" means isomorphic. In this setting it is a conjecture of Jacquet that it should suffice to let V' vary over representations of $GL_{\lfloor \frac{n}{2} \rfloor}(F)$, or in other words a $GL(n) \times$

$GL(\lfloor \frac{n}{2} \rfloor)$ converse theorem should hold.

In the setting of families, we deal with admissible generic representations whose coefficient rings are more general, and these families are not typically irreducible, so “the same” will mean that V_1 and V_2 have the same supercuspidal support, or equivalently, the same Whittaker space. Over families, there arises a new dimension to the local converse problem: determining the smallest coefficient ring over which the twisting representations V' can be taken while still having the theorem hold. Before stating our results, we develop the notion of supercuspidal support.

In the setting of co-Whittaker families, the classical notion of supercuspidal support for representations over a field does not exist. However, the following result of Helm suggests a generalization of the definition of supercuspidal support:

Theorem 1.1.5 ([Hel12b], Thm 2.2)**.** *Let κ be a $W(k)$-algebra that is a field and let Π_1, Π_2 be two absolutely irreducible representations of G over κ which live in the same block of the Bernstein center. By Schur's lemma there are maps $f_1, f_2 : \mathcal{Z} \to \kappa$ giving the action of the Bernstein center on Π_1 and Π_2. Then Π_1 and Π_2 have the same supercuspidal support if and only if $f_1 = f_2$*

Since any co-Whittaker $A[G]$-module V satisfies Schur's lemma, there is a map $f_V : \mathcal{Z} \to \mathrm{End}_G(V) \xrightarrow{\sim} A$, and we call this map the supercuspidal support of V. We remark that two co-Whittaker $A[G]$-modules have the same supercuspidal support if and only if they have the same Whittaker space.

We are now in a position to state our converse theorem:

Theorem 1.1.6. *Let A be a finite-type $W(k)$-algebra which is reduced and ℓ-torsion free, and let $\mathcal{K} = \mathrm{Frac}(W(k))$. Suppose V_1 and V_2 are two co-Whittaker $A[GL_n(F)]$-modules. There is a finite extension $\mathcal{K}'$ of $\mathcal{K}$ such that, if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all absolutely irreducible generic integral representations V' of $GL_{n-1}(F)$ over $\mathcal{K}'$, then V_1 and V_2 have the same supercuspidal support (equivalently, $\mathcal{W}(V_1, \psi) = \mathcal{W}(V_2, \psi)$).*

Thus, in the finite-type, reduced, and ℓ-torsion free setting, our converse theorem shows it suffices to take the coefficient ring of the twisting representations V' to be no larger than the ring of integers in a finite extension of $\mathcal{K}$. The equality of gamma factors can only occur if V_1 and V_2 live in the same block of the category $\mathrm{Rep}_{W(k)}(GL_n(F))$, and the finite extension $\mathcal{K}'$ appearing in our converse theorem depends only on this block. Finding the smallest possible extension $\mathcal{K}'$ for each block will be the subject of future investigation.

If E is a finite extension of $\mathcal{K} = \mathrm{Frac}(W(k))$ with ring of integers $\mathcal{O}_E$, a representation over E is called integral if it has a $GL_n(F)$-stable $\mathcal{O}_E$-lattice L. If V is an absolutely irreducible generic integral representation of $GL_n(F)$ over E, then in particular its sublattice L is co-Whittaker ([EH12, 3.3.2 Prop], [Vig96, I.9.7]), and the supercuspidal support of L determines V up to isomorphism.

Thus our converse theorem gives as a special case the following integral converse theorem:

Corollary 1.1.7. *Let V_1, V_2 be two absolutely irreducible generic integral representations of $GL_n(F)$ over E. There is a finite extension $\mathcal{K}'/\mathcal{K}$ such that, if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all absolutely irreducible generic integral representations V' of $GL_{n-1}(F)$ over $\mathcal{K}'$, then $V_1 \cong V_2$.*

In Chapter 6 we prove this converse theorem following the method of Henniart in [Hen93] and Jacquet, Piatetski-Shapiro, and Shalika in [JPSS79, Thm 7.5.3]. By employing the functional equation, we establish an equality on the level of Whittaker functions, and this suffices to determine the supercuspidal support for a co-Whittaker family.

There is a key lemma in the setting of complex representations which is more subtle in families. If $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}$ and ψ is a nondegenerate character of N, this key lemma says that given any smooth compactly supported function H on $GL_n(F)$ with $H(ng) = \psi(n)H(g)$, the vanishing of H is detected by the convolutions of H with the Whittaker functions of a sufficiently large collection of representations. This result was originally proven by Jacquet, Piatetski-Shapiro, and Shalika over $\mathbb{C}$ ([JPSS81, Lemme 3.5]) by using harmonic analysis to decompose a representation as the direct integral of irreducible representations. A purely algebraic analogue of this decomposition was obtained by Bushnell and Henniart in 2003 ([BH03]) by viewing the representation as a sheaf on the spectrum of the Bernstein center. As an application of these algebraic techniques, Bushnell and Henniart give a new proof of this key vanishing lemma ([BH03]).

It has been observed by Vigneras in the ℓ-modular setting [Vig98, Vig04] and more recently by Helm in the integral setting [Hel12a, Hel12b] that this algebraic approach to Fourier theory and Whittaker models applies to representations over coefficient rings other than $\mathbb{C}$. In Section 6.4 we apply basic techniques from algebraic geometry to the spectrum of the integral Bernstein center, to prove the vanishing theorem (and thus the converse theorem) in the case when A is a finite-type $W(k)$-algebra which is reduced and ℓ-torsion free.

1.2 Relationship to Other Work

The question of interpolating local constants in ℓ-adic families has already been investigated in a simple case by Vigneras in [Vig00]. For supercuspidal representations of $GL_2(F)$ over $\overline{\mathbb{Q}_\ell}$, Vigneras notes in [Vig00] that it is known that epsilon factors define elements of $\overline{\mathbb{Z}_\ell}$, and she proves that for two supercuspidal integral representations to be congruent modulo ℓ it is necessary and sufficient they have epsilon factors which are congruent modulo ℓ (we call a representation with coefficients in a local field E integral if it stabilizes an $\mathcal{O}_E$-lattice). The classical epsilon and gamma factors are equal in the supercuspidal case, so when the specialization of an ℓ-adic family at a characteristic zero point is supercuspidal, the gamma factor we construct in this paper specializes to the epsilon factor of [JPSS79, Vig00].

Since two representations V_1, V_2 over $\mathcal{O}_E$ which are congruent mod $\mathfrak{m}_E$ define a family $V_1 \times_{\overline{V}} V_2$ over the connected $W(k)$-algebra $\mathcal{O}_E \times_{k_E} \mathcal{O}_E$, Theorems

1.1.1 and 1.1.2 give the following corollary (implying the "necessary" part of the result in [Vig00]):

Corollary 1.2.1. *Let K denote the fraction field of $W(k)$. If π and π' are absolutely irreducible integral representations of $GL_n(F)$ over a coefficient field E which is a finite extension of K, then:*

1. *$\gamma(\pi, X, \psi)$ and $\gamma(\pi', X, \psi)$ have coefficients in the fraction ring $S^{-1}(\mathcal{O}_E[X, X^{-1}])$.*

2. *If $\mathfrak{m}_E$ is the maximal ideal of $\mathcal{O}_E$, and $\pi \equiv \pi' \mod \mathfrak{m}_E$, then*

$$\gamma(\pi, X, \psi) \equiv \gamma(\pi', X, \psi) \mod \mathfrak{m}_E.$$

An ℓ-modular version of the Godement-Jacquet theory was accomplished by Mínguez in [M1́2] for irreducible smooth representations with coefficients in a field of characteristic ℓ in the case where ℓ is banal. In this paper we achieve similar results, without the banal hypothesis, and for the more general setting of co-Whittaker families over any Noetherian $W(\overline{\mathbb{F}_\ell})$-algebra A.

The Rankin-Selberg convolutions in this paper generalize recent results on Rankin-Selberg convolutions in the ℓ-modular setting by Kurinczuk and Matringe in [KM14]. As opposed to the ℓ-modular L-factor in [KM14], the analogue of the L-factor for ℓ-adic families does not seem to behave well. Because of this, we focus here only on the local integral factors $\Psi(W, W', X)$ and the gamma factor.

In [BK00], Braverman and Kazhdan propose a general framework for constructing a gamma factor for any irreducible *complex* representation of a reductive group over a nonarchimedean local field. For GL_n, they interpret the Fourier transform as a certain type of distribution, which can also be viewed as an element γ of the total quotient ring of the Bernstein center $\mathcal{Z}$. They show [BK00, Theorem 8.11] that for an irreducible representation π, their gamma factor evaluated at π equals (up to a sign) the gamma factor from [JPSS79, JPSS83] of any generic representation containing π. This suggests that our universal gamma factor (Theorem 1.1.3) might have an expression in terms of a distribution.

Using geometric arguments Cogdell and Piatetski-Shapiro obtain a rationality result and functional equation for deformations of Rankin-Selberg local constants of representations [CPS10, Proposition 3.2]. Their coefficient ring is a polynomial ring over the complex numbers.

Converse theorems in the complex setting have a long history dating back to Hecke, and for $GL(n)$ in the local setting have been studied over the complex numbers by Chen, Cogdell, Henniart, Jacquet, Langlands, Piatetski-Shapiro, Shalika, among others ([JL70, JPSS79, JPSS83, Hen93, CPS99, Che06, JNS13]), and in characteristic ℓ by Vigneras [Vig00].

The "sufficient" part of the result in [Vig00] is an ℓ-modular converse theorem in the cuspidal case for $GL(2)$. To generalize it beyond this case, one would need a full local converse theorem in the ℓ-modular setting, where $A = k$ is ℓ-torsion. It appears that applying the approach of Bushnell-Henniart

[BH03] in the ℓ-torsion setting (as we do in this thesis in the torsion-free case) would require further knowledge of the relationship between Whittaker models and the Bernstein center modulo ℓ.

Since gamma factors of pairs determine supercuspidal supports, they determine the action of the Bernstein center on the category. Thus, the methods of this paper may shed light on the ring structure of the integral Bernstein center. Investigations along these lines will be carried out in future research.

1.3 Notation and Conventions

We will let F be a finite extension of $\mathbb{Q}_p$, let q be the order of its residue field, and let k be an algebraically closed field of characteristic ℓ, where $\ell \neq p$ is an odd prime. The letter G or G_n will always denote the group $GL_n(F)$. We will denote by $W(k)$ the ring of Witt vectors over k. The assumption that ℓ is odd is made so that $W(k)$ contains a square root of q. When $\ell = 2$ all the arguments presented will remain valid, after possibly adjoining a square root of q to $W(k)$. Throughout the paper A will always be a Noetherian commutative ring which is a $W(k)$-algebra, with additional ring theoretic conditions in various sections of the paper, and for a prime $\mathfrak{p}$ we denote by $\kappa(\mathfrak{p})$ the residue field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

For a group H, we denote by $\mathrm{Rep}_A(H)$ the category of smooth representations of H over the ring A, i.e. $A[H]$-modules for which every element is stabilized by an open subgroup of H. We will sometimes drop the subscript and write $\mathrm{Rep}(H)$ to mean $\mathrm{Rep}_A(H)$, and even when this category is not mentioned, all representations are presumed to be smooth. An $A[H]$-module is admissible if for every compact open subgroup U, the set of U-fixed vectors is finitely generated as an A-module.

If K is a normal subgroup of a group H, and θ is a character of the group K, we denote by θ^h the character given by $\theta^h(k) = \theta(hkh^{-1})$, $k \in K$, $h \in H$. If V is a representation of H, we denote by $V_{H,\theta}$ the quotient $V/V(H,\theta)$ where $V(H,\theta)$ is the sub-A-module generated by elements of the form $hv - \theta(h)v$ for $h \in H$ and $v \in V$.

Given a standard parabolic subgroup P of $GL_n(F)$ (i.e. a subgroup consisting of block upper triangular matrices), it has a unipotent radical N (of strictly block upper triangular matrices) such that $P = MN$ for a subgroup M of block diagonal matrices, called a standard Levi subgroup. The functor $V \mapsto V_{N,\mathbf{1}}$ (after restricting to P) is called the Jacquet functor associated to M, and we denote this functor by J_M. An $A[G]$-module V is called *cuspidal* if $J_M V = 0$ for all Levi subgroups $M \neq G$. J_M has a right adjoint, given by parabolic induction, which takes a representation $V \in \mathrm{Rep}_A(M)$, inflates it to a representation of $P = MN$ by letting N act trivially, and then taking the induced representation $\text{c-Ind}_P^G V$. This functor is denoted by i_P^G. It also applies when G is an arbitrary reductive group.

This adjunction implies ([Vig96, II.2.3]), given $V \in \mathrm{Rep}_A(G)$, that V is cuspidal if and only if all G-homomorphisms from V to a parabolic induction $\text{c-Ind}_{MN}^G W$ are zero for all $W \in \mathrm{Rep}(M)$, $M \neq G$. If A is a field, then a simple $A[G]$-module is called *supercuspidal* if it is not isomorphic to a subquotient of $\text{c-Ind}_{MN}^{G} W$ for any $W \in \text{Rep}(M)$, $M \neq G$. If A is a field of characteristic zero, cuspidal representations are supercuspidal.

We denote by N_n the subgroup of G_n consisting of all unipotent upper-triangular matrices. Let $\psi : F \to W(k)^\times$ be an additive character of F with $\ker \psi = \mathfrak{p}$. Then ψ defines a character on any subgroup of $N_n(F)$ by

$$(u)_{i,j} \mapsto \psi(u_{1,2} + \cdots + u_{n-1,n});$$

we will abusively denote this character by ψ as well. Note that in Proposition 2.1.2 we construct a character $\tilde{\psi}$ slightly differently. We say that a smooth $A[G]$-module V is *generic* if $V_{N,\psi} \neq 0$.

For each $m \leq n$, we let G_m denote $GL_m(F)$ and embed it in G via $\left(\begin{smallmatrix} G_m & 0 \\ 0 & I_{n-m} \end{smallmatrix}\right)$. We let $\{1\} = P_1 \subset \cdots \subset P_n$ denote the mirabolic subgroups of $G_1 \subset \cdots \subset G_n$, which are given by

$$P_m := \left\{ \left(\begin{smallmatrix} g_{m-1} & x \\ 0 & 1 \end{smallmatrix}\right) : g_{m-1} \in G_{m-1},\ x \in F^{m-1} \right\}.$$

We also have the unipotent upper triangular subgroup

$$U_m := \left\{ \left(\begin{smallmatrix} I_{m-1} & x \\ 0 & 1 \end{smallmatrix}\right) : x \in F^{m-1} \right\}$$

of P_m such that $P_m = U_m G_{m-1}$. In particular, $U_m \xrightarrow{\sim} F^{m-1}$. Note that this is different from the groups $N(m)$ defined in Proposition 2.1.2.

Consider the identity functor in the category $\text{Rep}_{W(k)}(G)$. An endomorphism of this functor is a natural transformation from the functor to itself, meaning a collection of endomorphisms z_V, one for each object V in

$\mathrm{Rep}_{W(k)}(G)$, such that $\alpha \circ z_V = z_{V'} \circ \alpha$ for any morphism $\alpha : V \to V'$ between any two objects V, V'. The integral Bernstein center, which will always be denoted by $\mathcal{Z}$, is defined as the ring of endomorphisms of the identity functor (see the discussion preceding Theorem 1.1.3). If V is $\mathrm{Rep}_A(G)$, then it is also in $\mathrm{Rep}_{W(k)}(G)$, and we frequently use the Bernstein decomposition of $\mathrm{Rep}_{W(k)}(G)$ to interpret properties of V.

If A has a nontrivial ideal I, then $I \cdot V$ is an $A[H]$-submodule of G, which shows that most content would be missing if we only considered irreducible families, meaning simple $A[H]$-modules. Thus conditions appear throughout the paper which in the traditional setting are implied by irreducibility:

Definition 1.3.1. V in $\mathrm{Rep}_A(H)$ will be called

1. *Schur* if the natural map $A \to \mathrm{End}_{A[G]}(V)$ is an isomorphism;
2. *G-finite* if V is finitely generated as an $A[G]$-module.
3. *primitive* if there exists a primitive idempotent e in the Bernstein center $\mathcal{Z}$ such that $eV = V$.
4. *Whittaker type* if $V_{N_n,\psi}$ is free of rank one as an A-module (if A is a field, this is satisfied if V is irreducible and generic).

We say a ring is connected if it has connected spectrum or, equivalently, no nontrivial idempotents. For example, any local ring or integral domain is connected. Note that if A is connected Corollary 2.6.2 implies all co-Whittaker

$A[G]$-modules are primitive. All co-Whittaker $A[G]$-modules are necessarily a finite direct sum of modules, each factor forming a co-Whittaker module over a different direct factor of A.

Given a smooth $A[G]$-module V, we denote by $V^\vee$ the submodule of $\mathrm{Hom}_A(G, A)$ consisting of functions which are smooth with respect to the right translation action of G (i.e. fixed by some compact open subgroup), it is called the contragredient representation

Given a smooth $A[G]$-module V we can consider its space of matrix coefficients $\mathcal{C}(V)$, which is defined as the A-module generated by functions $\gamma_{v\otimes v^\vee}$ for $v \in V$, $v^\vee \in V^\vee$ where

$$\begin{aligned} \gamma_{v\otimes v^\vee} : G &\to A \\ g &\mapsto \langle v^\vee, gv\rangle \end{aligned}$$

It is possible to define a Haar integral on the space $C_c^\infty(G, A)$ of smooth compactly supported functions $G \to A$. The group G contains a compact open subgroup H_1 whose pro-order is invertible in $W(k)$; for example take $I + \varpi_F M_n(\mathcal{O}_F)$ where I is the identity matrix, ϖ_F is a uniformizer of F, and $M_n(\mathcal{O}_F)$ denotes the space of $n \times n$ matrices with entries in the ring of integers $\mathcal{O}_F$. Choosing any filtration $\{H_i\}_{i\geq 1}$ of compact open subgroups such that $[H_1 : H_i]$ is invertible in A, we can set $\mu^\times(H_i) = [H_1 : H_i]^{-1}$. This automatically defines integration on the characteristic functions of the H_i, and one can check ([Vig96, I.2.3]) that extending by linearity gives a well-defined Haar integral on $C_c^\infty(G, A)$. In fact for each choice of H_1 there is a unique Haar measure normalized so $\mu^\times(H_1) = 1$. For a function ϕ on G we frequently abbreviate $\int_G \phi(x) d\mu^\times(x)$ by $\int \phi(x) d^\times x$, or even $\int \phi(x) dx$ when the Haar measure is understood.

Similarly, we can define a Haar measure μ on group $M_n(F)$ of $n \times n$ matrices over F, with values in A; we will then abbreviate an integral $\int \Phi(x) d\mu(x)$ by $\int \Phi(x) dx$. For a function $\Phi \in C_c^\infty(M_n(F), A)$, we denote by $\widehat{\Phi}(x)$ its Fourier transform:

$$\widehat{\Phi}(x) = \int_{M_n(F)} \Phi(y) \psi(\mathrm{tr}(xy)) dy.$$

As in the complex case, for all functions $\Phi \in C_c^\infty(M_n(F), A)$, we have $\widehat{\Phi} \in C_c^\infty(M_n(F), A)$, and $\Phi \mapsto \widehat{\Phi}$ defines an automorphism of $C_c^\infty(M_n(F), A)$. After fixing a square root of q in $W(k)$, we can suppose that μ is self-dual, meaning it satisfies $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$.

Chapter 2

Representation Theoretic Background

2.1 Co-invariants and Derivatives

As in [EH12, BZ77], we define the following functors with respect to the character ψ.

$$
\begin{aligned}
\Phi^+ &: \mathrm{Rep}_A(P_{n-1}) \to \mathrm{Rep}_A(P_n) \\
&\quad V \mapsto \text{c-Ind}_{P_{n-1}U_n}^{P_n} V \ (\text{with } U_n \text{ acting via } \psi) \\
\hat{\Phi}^+ &: \mathrm{Rep}(P_{n-1}) \to \mathrm{Rep}(P_n) \\
&\quad V \mapsto \mathrm{Ind}_{P_{n-1}U_n}^{P_n} V \ (\text{with } U_n \text{ acting via } \psi) \\
\Phi^- &: \mathrm{Rep}(P_n) \to \mathrm{Rep}(P_{n-1}) \\
&\quad V \mapsto V/V(U_n, \psi) \\
\Psi^+ &: \mathrm{Rep}(G_{n-1}) \to \mathrm{Rep}(P_n) \\
&\quad V \mapsto V \ (\text{with } U_n \text{ acting trivially}) \\
\Psi^- &: \mathrm{Rep}(P_n) \to \mathrm{Rep}(G_{n-1}) \\
&\quad V \mapsto V/V(U_n, \mathbf{1})
\end{aligned}
$$

Note that we give these functors the same names as the ones originally defined in [BZ76], but we use the non-normalized induction functors, as in [BZ77, EH12], because they are simpler for our purposes.

As observed in [EH12], these functors retain the basic adjointness properties proved in [BZ77, §3.2]. This is because the methods of proof in [BZ76, BZ77] use properties of l-sheaves which carry over to the setting of smooth $A[G]$-modules where A is a Noetherian $W(k)$-algebra.

Proposition 2.1.1 ([EH12],3.1.3)**.** *(1) The functors* Ψ^-, Ψ^+, Φ^-, Φ^+, $\hat{\Phi}^+$ *are exact.*

(2) Φ^+ *is left adjoint to* Φ^-, Ψ^- *is left adjoint to* Ψ^+, *and* Φ^- *is left adjoint to* $\hat{\Phi}^+$.

(3) $\Psi^-\Phi^+ = \Phi^-\Psi^+ = 0$

(4) $\Psi^-\Psi^+$, $\Phi^-\hat{\Phi}^+$, *and* $\Phi^-\Phi^+$ *are naturally isomorphic to the identity functor.*

(5) For each V *in* $\mathrm{Rep}(P_n)$ *we have an exact sequence*

$$0 \to \Phi^+\Phi^-(V) \to V \to \Psi^+\Psi^-(V) \to 0.$$

(6) (Commutativity with Tensor Product) If M *is an* A*-module and* F *is* Ψ^-, Ψ^+, Φ^-, Φ^+, *or* $\hat{\Phi}^+$, *we have*

$$F(V \otimes_A M) \cong F(V) \otimes_A M.$$

For $1 \leq m \leq n$ we define the mth derivative functor

$$(-)^{(m)} := \Psi^-(\Phi^-)^{m-1} : \mathrm{Rep}(P_n) \to \mathrm{Rep}(G_{n-m}).$$

We extend this to a functor $\mathrm{Rep}(G_n) \to \mathrm{Rep}(G_{n-m})$ by first restricting representations to P_n and then applying $(-)^{(m)}$; this functor is also denoted $(-)^{(m)}$. We use the convention that the zero'th derivative functor $(-)^{(0)}$ is the identity.

We can describe the derivative functor $(-)^{(m)}$ more explicitly by using the following lemma on the transitivity of coinvariants:

Lemma 1 ([BZ76] §2.32)**.** *Let H be a locally profinite group, θ a character of H, and V a representation of H. Suppose H_1, H_2 are subgroups of H such that $H_1 H_2 = H$ and H_1 normalizes H_2. Then*

$$\left(V_{H_2,\theta|_{H_2}}\right)_{H_1,\theta|_{H_1}} = V_{H,\theta}.$$

Let

$$N(l) := \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & * & * & \cdots & * \\ & \ddots & & \vdots & \vdots & & & \vdots \\ & & & 1 & * & * & \cdots & * \\ & & & & 1 & * & \cdots & * \\ & & & & & & \ddots & \vdots \\ & & & & & & & 1 \end{pmatrix} \right\}$$

denote the group of matrices whose first l columns are those of the identity matrix, and whose last $n-l$ columns are those of elements of N_n (recall that N_n is the group consisting of all unipotent upper triangular matrices). For $2 \le l \le n$ we have $U_l N(l) = N(l-1)$ and U_l normalizes $N(l)$. By applying Lemma 1 repeatedly with $H_1 = U_l$, and $H_2 = N(l)$ for $n-m+1 \le l \le n$ we get an explicit description of $(\Phi^-)^m$ and $(-)^{(m)}$ for all m:

Proposition 2.1.2 ([Vig96] III.1.8)**.** *(1) As $N(n-m)$ is contained in N_n, we define ψ on $N(n-m)$ via its superdiagonal entries. Then $(\Phi^-)^m V$ is given by the coinvariants $V/V(N(n-m),\psi)$.*

(2) Define a character $\widetilde{\psi}$ *on* $N(n-m)$ *as usual via* ψ *on the last* $m-1$ *superdiagonal entries, but trivially on the* $(n-m, n-m+1)$ *entry, i.e.*

$$\widetilde{\psi}(x) := \psi(0 + x_{n-m+1,n-m+2} + \cdots + x_{n-1,n}) \text{ for } x \in N(n-m).$$

Then $V^{(m)} = V/V(N(n-m), \widetilde{\psi})$.

In particular, if $m = n$, this gives $V^{(n)} = V/V(N_n, \psi)$. Note that $V^{(n)}$ is simply an A-module.

2.2 Whittaker and Kirillov Functions

Since $\psi : N_n \to A^\times$ is a character, it defines a representation of N_n in the A-module A. Then we have by Proposition 2.1.2 that $\mathrm{Hom}_A(V^{(n)}, A) = \mathrm{Hom}_{N_n}(V, \psi)$.

Definition 2.2.1. For V in $\mathrm{Rep}_A(G_n)$, we say that V is of Whittaker type if $V^{(n)}$ is free of rank one as an A-module. As in [EH12, Def 3.1.8], if A is a field we refer to irreducible representations of Whittaker type as generic.

If V is of Whittaker type, $\mathrm{Hom}_A(V/V(N_n, \psi), A) = \mathrm{Hom}_{N_n}(V, \psi)$ is free of rank one, so we may choose a generator λ in $\mathrm{Hom}_{N_n}(V, \psi)$. For any v in V, define $W_v \in \mathrm{Ind}_{N_n}^{G_n} \psi$ as $W_v : g \mapsto \lambda(gv)$. This is called a Whittaker function and has the property that $W(nx) = \psi(n)W(x)$ for $n \in N_n$. $v \mapsto W_v$ defines a G_n-equivariant homomorphism $V \to \mathrm{Ind}_{N_n}^{G_n} \psi$. The $A[G]$-module formed by the image is independent of the choice of λ. The map $v \mapsto W_v$ is precisely the generator of $\mathrm{Hom}_{G_n}(V, \mathrm{Ind}_{N_n}^{G_n} \psi)$ corresponding to the generator λ of $\mathrm{Hom}_{N_n}(V, \psi)$ under Frobenius reciprocity.

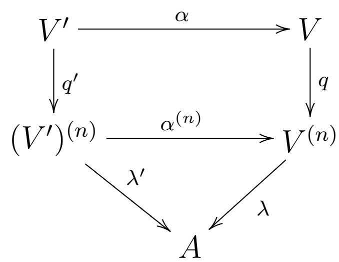
Definition 2.2.2. The image of the homomorphism $v \mapsto W_v : V \to \mathrm{Ind}_{N_n}^{G_n} \psi$ is called the space of Whittaker functions of V and is denoted $\mathcal{W}(V, \psi)$ or just $\mathcal{W}$. It carries a representation of G_n via $gW_v = W_{gv}$.

Choosing a generator of $V^{(n)}$ and allowing N_n to act via ψ, we get an isomorphism $V^{(n)} \xrightarrow{\sim} \psi$. Composing this with the natural quotient map $V \to V^{(n)}$ gives an N_n-equivariant map $V \to \psi$, which is a convenient choice of λ.

Note that the map $V \to \mathcal{W}(V, \psi)$ here is surjective but not necessarily an isomorphism, unlike the setting of *irreducible* generic representations with coefficients in a field. In other words different $A[G]$-modules of Whittaker type can have the same space of Whittaker functions:

Lemma 2. *Suppose V', V in $\mathrm{Rep}_A(G)$ are of Whittaker type, and suppose that we have a G-equivariant homomorphism $\alpha : V' \to V$ such that $\alpha^{(n)} : (V')^{(n)} \to V^{(n)}$ is an isomorphism. Then $\mathcal{W}(V', \psi)$ is the subset of $\mathcal{W}(V, \psi)$ given by $\mathcal{W}(\alpha(V'), \psi)$.*

Proof. Let $q' : V' \to V'/V'(N_n, \psi)$ and $q : V \to V/V(N_n, \psi)$ be the quotient maps. Choosing a generator for $V^{(n)}$ gives isomorphisms λ, λ' such that the following diagram commutes.



Given $v' \in V'$ we get

$$W_{\alpha(v')}(g) = \lambda(q(g\alpha v')) = \lambda((q \circ \alpha)(gv')) = \lambda'(q'(gv')) = W_{v'}(g), \quad g \in G.$$

This shows $\mathcal{W}(V', \psi) = \mathcal{W}(\alpha(V'), \psi) \subset \mathcal{W}(V, \psi)$.

$\square$

We can repeat this construction for the restriction to P_n of representations V in $\mathrm{Rep}(G_n)$ of Whittaker type. In particular, by restricting the argument of the Whittaker functions W_v to elements of P_n, we get a P_n-equivariant homomorphism $V \to \mathrm{Ind}_{N_n}^{P_n} \psi$.

Definition 2.2.3. The image of the homomorphism $V \to \mathrm{Ind}_{N_n}^{P_n} \psi : v \mapsto W_v$ is called the Kirillov functions of V and is denoted $\mathcal{K}(V, \psi)$ or just $\mathcal{K}$. It carries a representation of P_n via $pW_v = W_{pv}$.

There is a particularly important P_n-representation that naturally embeds in the restriction to P_n of any Whittaker type representation.

Definition 2.2.4. If V is in $\mathrm{Rep}(P_n)$, the P_n representation $(\Phi^+)^{n-1}V^{(n)}$ is called the Schwarz functions of V and is denoted $\mathcal{S}(V)$. For V in $\mathrm{Rep}(G_n)$ we also denote by $\mathcal{S}(V)$ the Schwarz functions of V restricted to P_n.

We gather together some of the properties of the Kirillov functions and the Schwartz functions that are well known for $\mathrm{Rep}_{\mathbb{C}}(G)$, which we will need in this paper for $\mathrm{Rep}_A(G)$.

Proposition 2.2.1. *Let V be of Whittaker type in $\mathrm{Rep}_A(P_n)$, and choose a generator of $V^{(n)}$ in order to identify $V^{(n)}$ with A. Then the following hold:*

(1) $\mathcal{S}(V) = \text{c-}\mathrm{Ind}_{N_n}^{P_n}\psi$ *and* $(\hat{\Phi}^+)^{n-1}V^{(n)} = \mathrm{Ind}_{N_n}^{P_n}\psi$.

(2) The composition

$$\mathcal{S}(V) \to V \to \mathrm{Ind}_{N_n}^{P_n}\psi$$

differs from the inclusion $\text{c-}\mathrm{Ind}_{N_n}^{P_n}\psi \hookrightarrow \mathrm{Ind}_{N_n}^{P_n}\psi$ *by multiplication with an element of $A^\times$.*

(3) The Kirillov functions $\mathcal{K}(V,\psi)$ contains $\text{c-}\mathrm{Ind}_{N_n}^{P_n}\psi$ *as a sub-$A[P_n]$-module.*

Proof. The proof in [BZ76] Proposition 5.12 (g) works to prove (1) in this context.

There is an embedding $S(V) \to V$ by Proposition 2.1.1 (5); denote by t the composition $S(V) \to V \to \mathrm{Ind}\,\psi$. Then $t^{(n)} : S(V)^{(n)} \to \mathrm{Ind}\,\psi^{(n)}$ is a nonzero homomorphism between free rank one A-modules, which we will show is an isomorphism. First, note that $t^{(n)}$ is given by multiplication with an element a of A. By Proposition 2.1.1 (6), For any maximal ideal $\mathfrak{m}$ of A, $t^{(n)} \otimes \kappa(\mathfrak{m})$ must be an isomorphism because it is a nonzero element of

$$\mathrm{Hom}_{\kappa(\mathfrak{m})}((S(V)\otimes\kappa(\mathfrak{m}))^{(n)}, (\mathrm{Ind}\,\psi\otimes\kappa(\mathfrak{m}))^{(n)}) = \kappa(\mathfrak{m}).$$

Thus a is nonzero in $\kappa(\mathfrak{m})$ for all $\mathfrak{m}$, hence a unit.

On the other hand there is the natural embedding $\text{c-Ind}\,\psi \to \text{Ind}\,\psi$, which we will denote s. Since $s^{(n)}$ is an isomorphism by [BZ77, Prop 3.2 (f)], we have $s^{(n)} = ut^{(n)}$ for some $u \in A^{\times}$. Thus, if $K := \ker(s - ut)$ then $K^{(n)} = S(V)^{(n)} = V^{(n)}$, whence $\text{Hom}_P(S(V)/K, \text{Ind}\,\psi) \cong \text{Hom}_A((S(V)/K)^{(n)}, A) = \text{Hom}_A(\{0\}, A) = 0$, which implies $s - ut \equiv 0$.

To prove (4), note that since $\mathcal{K}(V, \psi)$ is defined to be the image of the map $V \to \text{Ind}_{N_n}^{P_n} \psi$, this follows from (3). □

Proposition 2.1.2 describes the effect of the functor Φ^- explicitly on a representation V of G_n (or more precisely, its restriction to P_n). We can ask how this is reflected in the Kirillov functions of the representation. First we observe that Φ^- commutes with the functor $\mathcal{K}$:

Lemma 3. *For $0 \leq m \leq n$, we have the following equality of $A[P_m]$-modules:*

$$\mathcal{K}((\Phi^-)^{n-m}V, \psi) = (\Phi^-)^{n-m}\mathcal{K}(V, \psi).$$

Proof. The image of the P_{n-m}-submodule $V(N(m), \psi)$ in the map $V \to \mathcal{K}$ equals the submodule $\mathcal{K}(N(m), \psi)$. □

Following [CPS10], we can explicitly describe the effect of the functor Φ^- on the Kirillov functions $\mathcal{K}$. Recall that $\mathcal{K}(U_n, \psi)$ denotes the A-submodule generated by $\{uW - \psi(u)W : u \in U_n,\ W \in \mathcal{K}\}$ and $\Phi^-\mathcal{K} := \mathcal{K}/\mathcal{K}(U_n, \psi)$.

Proposition 2.2.2 ([CPS10] Prop 1.1)**.**

$$\mathcal{K}(U_n, \psi) = \{W \in \mathcal{K} : W \equiv 0 \textit{ on the subgroup } P_{n-1} \subset P_n\}.$$

Proof. The proof uses the arguments of [CPS10] Proposition 1.1, which carry over in this more general setting. It utilizes the Jacquet-Langlands criterion for an element v of a representation V to be in the subspace $V(U_{n_i}, \psi)$. This criterion remains valid even with our representations over more general coefficient rings A because all integrals are finite sums. □

As an immediate corollary, we find that Φ^- has the same effect as restriction of functions to the subgroup P_{n-1} inside P_n.

Corollary 2.2.3.

$$\Phi^-\mathcal{K} \cong \left\{W\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) : W \in \mathcal{W}(V, \psi),\ p \in P_{n-1}\right\}.$$

By applying for each $k = 1, \ldots, n-2$ the argument of Proposition [CPS10] Prop 1.1 to the P_{n-k+1} representation

$$\left\{W\left(\begin{smallmatrix} p & 0 \\ 0 & I_{k-1} \end{smallmatrix}\right) : W \in \mathcal{W}(V, \psi),\ p \in P_{n-k+1}\right\}$$

instead of to $\mathcal{K}$, we can describe $(\Phi^-)^k\mathcal{K}$:

Corollary 2.2.4. *For $k = 1, \ldots, n-1$,*

$$(\Phi^-)^k\mathcal{K} \cong \left\{W\left(\begin{smallmatrix} p & 0 \\ 0 & I_k \end{smallmatrix}\right) : W \in \mathcal{W}(V, \psi),\ p \in P_{n-k}\right\}.$$

2.3 Partial Derivatives

Given a product of groups $H_1 \times H_2$, we can define "partial" versions of the functors $\Phi^{\pm}$, $\Psi^{\pm}$ as follows: given V in $\mathrm{Rep}_A(H_1 \times H_2)$, restrict it to a representation of $H_1 = \{1\} \times H_2$, then apply the functor $\Phi^{\pm}$ or $\Psi^{\pm}$, and observe that $H_1 \times \{1\}$ acts naturally on the result, since it commutes with $\{1\} \times H_2$. More precisely:

$$\Phi^{+,2} : \mathrm{Rep}_A(G_{n-m} \times P_{m-1}) \to \mathrm{Rep}_A(G_{n-m} \times P_m)$$

$$V \mapsto \text{c-Ind}_{G_{n-m} \times P_{m-1}U_m}^{G_{n-m} \times P_m}(V), \text{ (where } \{1\} \times U_m \text{ acts via } \psi)$$

$$\hat{\Phi}^{+,2} : \mathrm{Rep}(G_{n-m} \times P_{m-1}) \to \mathrm{Rep}(G_{n-m} \times P_m)$$

$$V \mapsto \text{c-Ind}_{G_{n-m} \times P_{m-1}U_m}^{G_{n-m} \times P_m}(V)$$

$$\Phi^{-,2} : \mathrm{Rep}(G_{n-m} \times P_m) \to \mathrm{Rep}(G_{n-m} \times P_{m-1})$$

$$V \mapsto V/V(\{1\} \times U_m, \psi)$$

$$\Psi^{+,2} : \mathrm{Rep}(G_{n-m} \times G_{m-1}) \to \mathrm{Rep}(G_{n-m} \times P_m)$$

$$V \mapsto V \ \ (\{1\} \times U_m \text{ acts trivially})$$

$$\Psi^{-,2} : \mathrm{Rep}(G_{n-m} \times P_m) \to \mathrm{Rep}(G_{n-m} \times G_{m-1})$$

$$V \mapsto V/V(\{1\} \times U_m, \mathbf{1})$$

Because $H_1 \times \{1\}$ commutes with $\{1\} \times H_2$, we immediately get

Lemma 4. *The analogue of Proposition 2.1.1 holds for $\Phi^{+,2}$, $\hat{\Phi}^{+,2}$, $\Phi^{-,2}$, $\Psi^{+,2}$, and $\Psi^{-,2}$.*

Definition 2.3.1. We define the functor $(-)^{(0,m)} : \mathrm{Rep}_A(G_{n-m} \times G_m) \to \mathrm{Rep}_A(G_{n-m})$ to be the composition $\Psi^{-,2}(\Phi^{-,2})^{m-1}$.

The proof of the following Proposition holds for $W(k)$-algebras A:

Proposition 2.3.1 ([Zel80] Prop 6.7, [Vig96] III.1.8)**.** *Let $M = G_{n-m} \times G_m$. For $0 \leq m \leq n$ the m'th derivative functor $(-)^{(m)}$ is the composition of the Jacquet functor $J_M : \mathrm{Rep}(G_n) \to \mathrm{Rep}(G_{n-m} \times G_m)$ with the functor $(-)^{(0,m)} : \mathrm{Rep}_A(G_{n-m} \times G_m) \to \mathrm{Rep}_A(G_{n-m})$.*

Lemma 5. *Let V be in $\mathrm{Rep}_A(G_{n-m} \times G_m)$. Then V contains an A-submodule isomorphic to $V^{(0,m)}$.*

Proof. The image of the natural embedding $(\Phi^{+,2})^{m-1}\Psi^{+,2}(V^{(0,m)}) \to V$, which is given by Proposition 4 (5), will be denoted $\mathcal{S}^{0,2}(V)$. By Proposition 4 (4), the natural surjection $V \to V^{(0,m)}$ restricts to a surjection $\mathcal{S}^{0,2}(V) \to V^{(0,m)}$. By Proposition 4 (6), the map of A-modules $\mathcal{S}^{0,2}(V) \to V^{(0,m)}$ arises from the map $(\Phi^{+,2})^{m-1}\Psi^{+,2}(A) \to A$ by tensoring over A with $V^{(0,m)}$. Take any element that maps to the identity in $(\Phi^{+,2})^{m-1}\Psi^{+,2}(A) \to A$, and consider the A-submodule it generates. Tensoring over A with $V^{(0,m)}$ gives the desired submodule. □

2.4 Finiteness Results

In this subsection we gather certain finiteness results involving derivatives, most of which are well-known when A is a field of characteristic zero.

Let H be any topological group containing a decreasing sequence $H_0 \supset H_1 \supset \cdots$ of open subgroups whose pro-order is invertible in A, and which forms a neighborhood base of the identity in H. If V is a smooth $A[H]$-module we may define a projection $\pi_i : V \to V^{H_i} : v \mapsto \int_{H_i} hv$ for a Haar measure on H_i where H_i has total measure 1. The A-submodules $V_i := \ker(\pi_i) \cap V^{H_{i+1}}$ then satisfy $\bigoplus_i V_i = V$.

Lemma 6 ([EH12] Lemma 2.1.5, 2.1.6)**.** *A smooth $A[H]$-module V is admissible if and only if each A-module V_i is finitely generated. In particular, quotients of admissible $A[H]$-modules by $A[H]$-submodules are admissible.*

Thus the following version of the Nakayama lemma applies to admissible $A[H]$-modules:

Lemma 7 ([EH12] Lemma 3.1.9)**.** *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and suppose that M is a submodule of a direct sum of finitely generated A-modules. If $M/\mathfrak{m}M$ is finite dimensional then M is finitely generated over A.*

If V is admissible, then it is G-finite if and only if $V/\mathfrak{m}V$ is G-finite. To see this, take $S \subset V/\mathfrak{m}V$ an $(A/\mathfrak{m})[H]$-generating set, let W be the $A[H]$-span of a lift to V. Since V/W is admissible, we can apply Nakayama to each factor $(V/W)_i$ to conclude $V/W = 0$.

Proposition 2.4.1 ([EH12] 3.1.7)**.** *Let κ be a $W(k)$-algebra which is a field, and V an absolutely irreducible admissible representation of G_n. Then $V^{(n)}$ is zero or one-dimensional over κ, and is one-dimensional if and only if V is cuspidal.*

Proposition 2.4.2. *Let κ be a $W(k)$-algebra which is a field. If V is a smooth $\kappa[G]$-module, then V is admissible and G-finite if and only if V is finite length over $\kappa[G]$.*

Proof. Suppose V is admissible and G-finite. If κ were algebraically closed of characteristic zero (resp. characteristic ℓ), this is [BZ77, 4.1] (resp. [Vig96, II.5.10(b)]). Otherwise, let $\overline{\kappa}$ be an algebraic closure, then $V \otimes \overline{\kappa}$ is finite length, so V is finite length.

If V is finite length, so is $V \otimes_{\kappa} \overline{\kappa}$. Over an algebraically closed field of characteristic different from p, irreducible representations are admissible ([BZ77, 3.25],[Vig96, II.2.8]). Since admissibility is preserved under taking extensions $V \otimes \overline{\kappa}$ being finite length implies it is admissible, hence V is admissible. Thus we can reduce proving G-finiteness to proving that, given any exact sequence of admissible objects, $0 \to W_0 \to V \to W_1 \to 0$ where W_0 and W_1 are G-finite, then V is G-finite. But there is a compact open subgroup U such that W_0 and W_1 are generated by W_0^U and W_1^U, respectively. It follows that that V is generated by V^U. □

Lemma 8. *Let κ be a $W(k)$-algebra which is a field. If V is an absolutely irreducible $\kappa[G_n]$-module, then for $m \geq 0$, $V^{(m)}$ is finite length as a $\kappa[G_{n-m}]$-module.*

Proof. We follow [Vig96, III.1.10]. Choose a set of irreducible cuspidals such that $V \subset \pi_1 \times \cdots \times \pi_r$. The Liebniz formula for derivatives says that $(\pi_1 \times \pi_2)^{(t)}$ has a filtration whose successive quotients are $\pi_1^{(t-i)} \times \pi_2^{(i)}$. Its proof, given in [BZ77, §7], carries over in this generality. Then $V^{(m)} \subset (\pi_1 \times \cdots \times \pi_r)^{(m)}$, which is finite length by induction, using Proposition 2.4.1 combined with the Liebniz formula. □

Proposition 2.4.3 ([Hel12b] Prop 9.15)**.** *Let M be a standard Levi subgroup of G. If V in $\mathrm{Rep}_A(G)$ is admissible and primitive, then $J_M V$ in $\mathrm{Rep}_A(M)$ is admissible.*

Corollary 2.4.4. *If A is a local Noetherian $W(k)$-algebra and V is admissible and G-finite, then $V^{(m)}$ is admissible and G-finite for $0 \leq m \leq n$.*

Proof. Let $M = G_{n-m} \times G_m$. By Proposition 2.3.1, $V^{(m)} = (J_M V)^{(0,m)}$, so by Lemma 5, there is an embedding $V^{(m)} \to J_M V$ of A-modules. Admissibility and G-finiteness mean V is generated over $A[G]$ by vectors in V^K for some compact open subgroup K. Since V^K is finite dimensional, eV^K is nonzero for only a finite set of primitive idempotents e of the Bernstein center, but each submodule eV is also G-stable, showing that $eV \neq 0$ for at most finitely many primitive idempotents e of the integral Bernstein center. Therefore, Proposition 2.4.3 applies, and we have embedded $V^{(m)}$ in an admissible module. Thus by Lemma 7, we are reduced to proving the result for $\overline{V} := V/\mathfrak{m}V$. Since $\overline{V}$ is admissible and G-finite, and $A/\mathfrak{m}$ is characteristic ℓ, Lemma 2.4.2 shows $\overline{V}$ is finite length, therefore it follows from Lemma 8 that $\overline{V}^{(m)}$ is finite length. Applying Lemma 2.4.2 once more, we have the result. □

Loosely speaking, the $(n-1)$st derivative describes the restriction of a G_n-representation to a G_1-representation (see Corollary 2.2.4). The next result shows that this restriction intertwines a finite set of characters:

Theorem 2.4.5. *If A is a local $W(k)$-algebra and V in $\mathrm{Rep}_A(G)$ is admissible and G-finite, then $V^{(n-1)}$ is finitely generated as an A-module.*

Proof. By Lemma 7 and Corollary 2.4.4 it is sufficient to show that $\overline{V}^{(n-1)}$ is finite over the residue field κ. We know $\overline{V}^{(n-1)}$ is G-finite and admissible by Corollary 2.4.4, hence finite length as a $\kappa[G_1]$-module by Proposition 2.4.2. Since G_1 is abelian, all composition factors are 1-dimensional, so $\overline{V}^{(n-1)}$ being finite length implies it is finite dimensional over κ. □

Since the hypotheses of being admissible and G-finite are preserved under localization by Proposition 2.1.1 (6), we can go beyond the local situation:

Corollary 2.4.6. *Let A be a Noetherian $W(k)$-algebra and suppose that V is admissible and G-finite. Then for every $\mathfrak{p}$ in $\mathrm{Spec}\,A$, $V_{\mathfrak{p}}^{(n-1)}$ is finitely generated as an $A_{\mathfrak{p}}$-module.*

2.5 Co-Whittaker $A[G]$-Modules

In this subsection we define co-Whittaker representations and show that every admissible $A[G]$-module V of Whittaker type contains a canonical co-Whittaker subrepresentation.

Definition 2.5.1 ([Hel12b] 3.3)**.** Let κ be a field of characteristic different from p. An admissible smooth object U in $\mathrm{Rep}_{\kappa}(G)$ is said to have essentially

AIG dual if

1. it is finite length as a $\kappa[G]$-module,
2. its cosocle $\cos(U)$ is absolutely irreducible generic,
3. $\cos(U)^{(n)} = U^{(n)}$

(the cosocle of a module is its largest semisimple quotient).

U having essentially AIG dual is also equivalent to $U^{(n)}$ being one-dimensional over κ and having the property that $W^{(n)} \neq 0$ for any nonzero quotient $\kappa[G]$-module W (see [EH12, Lemma 6.3.5]).

Definition 2.5.2 ([Hel12b] 6.1)**.** An object V in $\mathrm{Rep}_A(G)$ is said to be co-Whittaker if it is admissible, of Whittaker type, and $V \otimes_A \kappa(\mathfrak{p})$ has essentially AIG dual for each $\mathfrak{p}$.

For co-Whittaker modules,

$$\begin{aligned}\mathcal{W}(V \otimes \kappa(\mathfrak{p}), \psi \otimes \kappa(\mathfrak{p})) &= \mathcal{W}(\cos(V \otimes \kappa(\mathfrak{p})), \psi \otimes \kappa(\mathfrak{p})) \\ &\cong \cos(V \otimes \kappa(\mathfrak{p})), \quad \text{for every } \mathfrak{p} \in \mathrm{Spec}(A)\end{aligned}$$

so last part of the definition of co-Whittaker modules implies $\mathcal{W}(V, \psi) \otimes \kappa(\mathfrak{p})$ is absolutely irreducible for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

Proposition 2.5.1 ([Hel12b] Prop 6.2)**.** *Let V be a co-Whittaker $A[G]$-module. Then the natural map $A \to \mathrm{End}_{A[G]}(V)$ is an isomorphism.*

Lemma 9. *Suppose V is admissible and, for all primes $\mathfrak{p}$, any non-generic quotient of $V \otimes \kappa(\mathfrak{p})$ equals zero (for example, a co-Whittaker module). Then V is generated over $A[G]$ by a single element.*

Proof. Let x be a generator of $V^{(n)}$, and let $\tilde{x} \in V$ be a lift of x. If V' is the $A[G]$-submodule of V generated by $\tilde{x}$, then $(V/V')^{(n)} = 0$. Since any non-generic quotient of $V \otimes \kappa(\mathfrak{p})$ equals zero, $(V/V') \otimes \kappa(\mathfrak{p}) = 0$ for all $\mathfrak{p}$. Since V/V' is admissible, we can apply Lemma 7 to the local rings $A_{\mathfrak{p}}$ to conclude V/V' is finitely generated, then apply ordinary Nakayama to conclude it is zero. □

Thus every co-Whittaker module is admissible, Whittaker type, G-finite (in fact G-cyclic), and Schur. In particular it satisfies the hypotheses of Theorems 3.1.2, 5.1.1, below. Moreover, every admissible Whittaker type representation contains a canonical co-Whittaker submodule:

Proposition 2.5.2. *Let V in $\mathrm{Rep}_A(G)$ be admissible of Whittaker type. Then the sub-$A[G]$-module*

$$T := \ker(V \to \prod_{\{U \subset V:\ (V/U)^{(n)}=0\}} V/U)$$

is co-Whittaker.

Proof. $(V/T)^{(n)} = 0$ so T is Whittaker type. Since V is admissible so is T. Let $\mathfrak{p}$ be a prime ideal and let $\overline{T} := T \otimes \kappa(\mathfrak{p})$. We show that $\cos(\overline{T})$ is absolutely irreducible and generic. By its definition, $\cos(\overline{T}) = \bigoplus_j W_j$ with W_j an irreducible $\kappa(\mathfrak{p})[G]$-module. Since the map $\overline{T} \to \bigoplus_j W_j$ is a surjection and $(-)^{(n)}$ is exact and additive, the map $(\overline{T})^{(n)} \to \bigoplus_j W_j^{(n)}$ is also a surjection. Hence $\dim_{\kappa(\mathfrak{p})}(\bigoplus_j W_j^{(n)}) \leq \dim_{\kappa(\mathfrak{p})}(\overline{T}^{(n)})$. Since T is Whittaker type and $\overline{T}^{(n)} = \overline{T^{(n)}}$ is nonzero, there can only be one j such that $W_j^{(n)}$ is potentially nonzero. On the other hand, suppose some $W_j^{(n)}$ were zero, then W_j is a quotient appearing in the target of the map

$$\overline{V} \to \prod_{\{U \subset \overline{V}:\ (\overline{V}/U)^{(n)}=0\}} \overline{V}/U,$$

hence as a quotient of $\overline{T}$ it would have to be zero, a contradiction. Hence precisely one W_j is nonzero. Now applying [EH12, 6.3.4] with A being $\kappa(\mathfrak{p})$ and V being $\cos(\overline{T})$, we have that $\mathrm{End}_G(\cos(\overline{T})) \cong \kappa(\mathfrak{p})$ hence absolutely irreducible. It also shows that $\cos(\overline{T})^{(n)} = W_j^{(n)} \neq 0$. Hence $\overline{T}^{(n)} = \cos(\overline{T})^{(n)}$. By Lemma 9, $\overline{T}$ is $\kappa(\mathfrak{p})[G]$-cyclic; since it is admissible, it is finite length by Lemma 2.4.2. $\square$

2.6 The Integral Bernstein Center

If A is a Noetherian $W(k)$-algebra and V is an $A[G]$-module, then in particular V is a $W(k)[G]$-module and we can use the Bernstein decomposition of $\mathrm{Rep}_{W(k)}(G)$ to study V. We can now recall the construction of the universal co-Whittaker module $\mathfrak{W}$:

Definition 2.6.1 ([Hel12b]). Let $\psi : N_n \to W(k)^\times$ be a nontrivial character, let $\mathfrak{W}$ be the $W(k)[G]$-module $\text{c-Ind}_{N_n}^{G_n} \psi$. If e is a primitive idempotent of $\mathcal{Z}$, the representation $e\mathfrak{W}$ lies in the block $e\,\mathrm{Rep}_{W(k)}(G)$, and we may view it as an object in the category $\mathrm{Rep}_{e\mathcal{Z}}(G)$.

With respect to extending scalars from $e\mathcal{Z}$ to A, the module $e\mathfrak{W}$ is "universal" in the following sense:

Proposition 2.6.1 ([Hel12b] Thm 6.3)**.** *Let A be a Noetherian $e\mathcal{Z}$-algebra. Then $e\mathfrak{W}\otimes_{e\mathcal{Z}}A$ is a co-Whittaker $A[G]$-module. Conversely, if V is a primitive co-Whittaker $A[G]$ module in the block $e\,\mathrm{Rep}_{W(k)}(G)$, and A is an $e\mathcal{Z}$-algebra via $f_V : e\mathcal{Z} \to A$, then there is a surjection $\alpha : \mathfrak{W} \otimes_{A,f_V} A \to V$ such that $\alpha^{(n)} : (\mathfrak{W} \otimes_{A,f_V} A)^{(n)} \to V^{(n)}$ is an isomorphism.*

If we assume A has connected spectrum (i.e. no nontrivial idempotents), then the map $f_V : \mathcal{Z} \to A$ would factor through a map $e\mathcal{Z} \to A$ for some primitive idempotent e, hence:

Corollary 2.6.2. *If A is a connected Noetherian $W(k)$-algebra and V is co-Whittaker, then V must be primitive for some primitive idempotent e.*

Regardless of whether A is connected, it is always assumed to be Noetherian. Since $\mathcal{Z}$ decomposes as a direct product, one factor for each block, any map $\mathcal{Z} \to A$ must factor through a finite product of direct factors (spectra of Noetherian rings have finitely many irreducible components). In particular, $e(\mathfrak{W} \otimes_{\mathcal{Z}} A) \neq 0$ for only finitely many primitive idempotents e. Thus we have the following corollary:

Corollary 2.6.3. *Let A be a Noetherian $\mathcal{Z}$-algebra. Then $\mathfrak{W} \otimes_{\mathcal{Z}} A$ is a co-Whittaker $A[G]$-module. Conversely, if V is a co-Whittaker $A[G]$ module, and A is a $\mathcal{Z}$-algebra via $f_V : \mathcal{Z} \to A$, then there is a surjection $\alpha : \mathfrak{W} \otimes_{\mathcal{A}, f_V} A \to V$ such that $\alpha^{(n)} : (\mathfrak{W} \otimes_{\mathcal{A}, f_V} A)^{(n)} \to V^{(n)}$ is an isomorphism.*

Chapter 3

Local Zeta Integrals in Families

In Sections 3.1 and 3.2 of this chapter, we use the representation theory of Chapter 2 to define zeta integrals and prove they satisfy an appropriate rationality property.

In Sections 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, we use the zeta integrals of previous sections to construct a gamma factor, then realize the gamma factor as the unique constant of proportionality in a functional equation involving zeta integrals.

3.1 Definition of the Zeta Integrals

We will first write down the definition of the zeta integrals which is the analog of that in [JPSS79], and then check that the definition makes sense.

Definition 3.1.1. For $W \in \mathcal{W}(V, \psi)$ and $0 \leq j \leq n-2$, let X be an indeterminate and define

$$\Psi(W, X; j) = \sum_{m \in \mathbb{Z}} (q^{n-1}X)^m \int_{x \in F^j} \int_{a \in U_F} W\left[\begin{pmatrix} \varpi^m a & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-j-1} \end{pmatrix}\right] d^\times a dx,$$

and $\Psi(W, X) = \Psi(W, X; 0)$

We first have a lemma about the support of general Whittaker functions restricted to $G_1 \subset G_n$, which shows us that $\Psi(W, X; 0)$ defines an element of $A[[X]][X^{-1}]$.

Lemma 10. *Let W be any element of $\operatorname{Ind}_{N_n}^{G} \psi$. Then there exists an integer $N < 0$ such that $W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)$ is zero for $v_F(a) < N$. Moreover if W is compactly supported modulo N_n, then there exists an integer $L > 0$ such that $W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)$ is zero for $v_F(a) > L$*

Proof. There is some integer j such that $\begin{pmatrix} 1 & \mathfrak{p}^j & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$ stabilizes W. Letting x be in $\mathfrak{p}^j$, we then have

$$\begin{aligned} W\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} &= W\left(\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}\right) \\ &= \psi\begin{pmatrix} 1 & ax & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} W\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \end{aligned}$$

Whenever $v_F(a)$ is negative enough that ax lands outside of $\ker \psi = \mathfrak{p}$, we get that $\psi\begin{pmatrix} 1 & ax & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$ is a nontrivial p-power root of unity ζ in $W(k)$, hence $1 - \zeta$ is the lift of something nonzero in the residue field k, and defines an element of $W(k)^{\times}$. This shows that $W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right) = 0$. $\square$

Just as in [JPSS79], the next two lemmas show that $\Psi(W, X; j)$ defines an element of $A[[X]][X^{-1}]$ when $0 < j < n - 2$ by reducing it to the case of $\Psi(W, X; 0)$.

Lemma 11 ([JPSS79] Lemma 4.1.5)**.** *Let H be a function on G, locally fixed under right translation by G, and satisfying $H(ng) = \psi(n)H(g)$ for $g \in G$,*

$n \in N_n$. Then the support of the function on F^j given by

$$x \mapsto H\left[\begin{pmatrix} a & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-j-1} \end{pmatrix}\right]$$

is contained in a compact set independent of $a \in F^\times$.

Corollary 3.1.1. *If ρ denotes right translation $(\rho(g)\phi)(x) = \phi(xg)$, and U is the unipotent radical of the standard parabolic subgroup of type $(1, n-1)$, then there is a finite set of elements $u_1, \dots, u_r$ of U such that*

$$\Psi(W, X; j) = \sum_{k=1}^{r} \Psi(\rho({}^t u_i)W, X; 0)$$

for any $W \in \operatorname{Ind}_{N_n}^G \psi$.

In [JPSS79], the zeta integrals form elements of the field $\mathbb{C}((X)) = \mathbb{C}[[X]][X^{-1}]$ of formal Laurent series and it is proved that in fact they are elements of the subfield $\mathbb{C}(X)$ of rational functions. Whereas $\mathbb{C}((X))$ is the fraction field of the domain $\mathbb{C}[[X]]$ and $\mathbb{C}(X)$ is the fraction field of the domain $\mathbb{C}[X, X^{-1}] \subset \mathbb{C}((X))$, our rings $A[[X]][X^{-1}]$ and $A[X, X^{-1}]$ are not in general domains. The first main result of this paper is determining the sense in which the zeta integrals $\Psi(W, X; j)$ form "rational" functions in a ring similar to $\mathbb{C}(X)$. It turns out the correct analogue is a fraction ring $S^{-1}(A[X, X^{-1}])$ for a particularly simple multiplicative subset S of $A[X, X^{-1}]$:

Theorem 3.1.2. *Suppose A is a Noetherian $W(k)$-algebra. Let S be the multiplicative subset of $A[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. Then if V is admissible, Whittaker type, and G-finite, $\Psi(W, X; j)$ lies in $S^{-1}A[X, X^{-1}]$ for all W in $\mathcal{W}(V, \psi)$ for $0 \leq j \leq n-2$.*

In particular, the result holds if V is primitive and co-Whittaker, as in Theorem 1.1.1. We record a clarifying observation about the fraction ring $S^{-1}(A[X, X^{-1}])$ appearing in Theorem 3.1.2:

Observation 1. *Let S_0 be the multiplicative subset of $A[X, X^{-1}]$ consisting of Laurent polynomials whose first and last coefficient is 1, and let S_1 be the multiplicative subset of $A[X, X^{-1}]$ consisting of monic polynomials in $A[X]$ whose constant term is a unit. Then the rings $S^{-1}(A[X, X^{-1}])$, $S_0^{-1}(A[X, X^{-1}])$, and $S_1^{-1}(A[X, X^{-1}])$ are uniquely isomorphic.*

We now give a brief overview of the proof of Theorem 3.1.2, which will occupy the remainder of this section. As in [JPSS79], Lemma 3.1.1 shows it suffices to prove the result when $j = 0$. The key idea is then as follows: the zeta integrals $\Psi(W, X)$ are completely determined by the values $W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)$ for $a \in F^\times$. As W ranges over $\mathcal{W}(V, \psi)$, the set of these values is equivalent data to the P_2-representation $(\Phi^-)^{n-2}\mathcal{K}$. The rationality of $\Psi(W, X)$ will reduce to the finiteness of the quotient $\mathcal{K}^{(n-1)}$, or more generally for $V^{(n-1)}$.

3.2 Proof of Rationality

Denote by τ the right translation representation of G_1 on $\mathcal{K}^{(n-1)}$. Let B be the commutative A-subalgebra of $\mathrm{End}_A(\mathcal{K}^{(n-1)})$ generated by $\tau(\varpi)$ and $\tau(\varpi^{-1})$, where ϖ is a uniformizer of F. It follows from Corollary 2.4.6 that $\mathcal{K}_{\mathfrak{p}}^{(n-1)}$ is finitely generated over $A_{\mathfrak{p}}$. For every $\mathfrak{p}$ of $\mathrm{Spec}\, A$, the inclusion $B_{\mathfrak{p}} \subset \mathrm{End}(\mathcal{K}^{(n-1)})_{\mathfrak{p}} \hookrightarrow \mathrm{End}(\mathcal{K}_{\mathfrak{p}}^{(n-1)})$, shows $B_{\mathfrak{p}}$ is finitely generated as an $A_{\mathfrak{p}}$-

module.

Lemma 12. *B is finitely generated as an A-module.*

Proof. B is the image of the map $A[X, X^{-1}] \to \mathrm{End}_A(\mathcal{K}^{(n-1)})$ which sends X to $\tau(\varpi)$. $B_\mathfrak{p}$ is the image of the localized map $A_\mathfrak{p}[X, X^{-1}] \to \left(\mathrm{End}_A(\mathcal{K}^{(n-1)})\right)_\mathfrak{p}$, which is finitely generated. This implies that for every $\mathfrak{p}$ in $\mathrm{Spec}\, A$, $\tau(\varpi)$ and $\tau(\varpi^{-1})$ satisfy monic polynomials $s_\mathfrak{p}(X)$, $t_\mathfrak{p}(X)$ with coefficients in $A_\mathfrak{p}$. Since $s_\mathfrak{p}$ and $t_\mathfrak{p}$ have finitely many coefficients there exists a global section $f_\mathfrak{p} \notin \mathfrak{p}$ such that $s_\mathfrak{p}(X)$, $t_\mathfrak{p}(X)$ lie in $A_{f_\mathfrak{p}}[X]$. The open subsets $D(f_\mathfrak{p})$ cover $\mathrm{Spec}\, A$ and we can take a finite subset $\{f_1, \ldots, f_n\} \subset \{f_\mathfrak{p}\}$ such that $(f_i) = 1$. Since $\tau(\varpi)$ and $\tau(\varpi^{-1})$ satisfy monic polynomials over A_{f_i}, we have that B_{f_i} is finitely generated over A_{f_i} for each i. It follows that B is finitely generated over A. □

Since B is finitely generated over A, $\tau(\varpi)$ and $\tau(\varpi^{-1})$ satisfy monic polynomials $c_0 + c_1X + \ldots c_{r-1}X^{r-1} + X^r$ and $b_0 + b_1X + \cdots + b_{s-1}X^{s-1} + X^s$ respectively. The degrees r and s of these polynomials must be nonzero because $\tau(\varpi)$ and $\tau(\varpi^{-1})$ are units in the ring B. Adding these together we have

$$\begin{aligned} 0 &= \tau(\varpi)^{-s} + b_{s-1}\tau(\varpi)^{-s+1} + \cdots + b_1\tau(\varpi)^{-1} + b_0 \\ &\quad + c_0 + c_1\tau(\varpi) + \ldots c_{r-1}\tau(\varpi)^{r-1} + \tau(\varpi)^r. \end{aligned}$$

Hence $\tau(\varpi)$ satisfies a Laurent polynomial whose first and last coefficients are units.

The final ingredient in proving rationality is the following transformation property.

Lemma 13. *$\Psi(\varpi^n W, X) = X^{-n}\Psi(W, X)$ for any $W \in \mathcal{W}(V, \psi)$, and any integer n.*

Proof of Lemma. Let b_m be the coefficient $\int_{U_F} W(\varpi^m u) d^\times u$. Then $\Psi(\varpi^n W, X)$ is $\sum_{m\in\mathbb{Z}} X^m b_{m+n}$, which can be rewritten $X^{-n}\sum_{m\in\mathbb{Z}} X^{m+n} b_{m+n}$. If m ranges over all integers, then so does $m + n$. □

Deducing Theorem 3.1.2. The representation $\mathcal{K}^{(n-1)}$ is formed by restricting the right translation representation on $(\Phi^-)^{n-2}\mathcal{K}$ from P_2 to G_1, then taking the quotient by the G_1-stable submodule $(\Phi^-)^{n-2}\mathcal{K}(U_2, \mathbf{1})$. By Corollary 2.2.4, the right translation representation on $(\Phi^-)^{n-2}\mathcal{K}$ is given by translations of the *restricted* Kirillov functions $W|_{\left(\begin{smallmatrix} x & 0 \\ 0 & I \end{smallmatrix}\right)}$, denoted $W(x)$ for short. As an endomorphism of the quotient module $\mathcal{K}^{(n-1)}$, $\tau(\varpi)$ satisfies a polynomial $X^n - a_{n-1}X^{n-1} - \cdots - a_1 X - a_0$ (in fact we can take a_0 to be -1). Hence for any restricted Kirillov function $W(x)$ we have

$$\varpi^n W(x) = \sum_{i=0}^{n-1} a_i \varpi^i W(x) + W_1(x),$$

for some element W_1 of $((\Phi^-)^{n-2}\mathcal{K})(U_2, \mathbf{1})$. Therefore we get a relation

$$\Psi(\varpi^n W, X) = \sum_{i=0}^{n-1} a_i \Psi(\varpi^i W, X) + \Psi_1(X)$$

with $\Psi_1(X)$ being a Laurent polynomial by Lemma 10. Using Lemma 13, then multiplying through by X^n and rearranging we get

$$\Psi(W, X) - \left(\sum_{i=0}^{n-1} a_i X^{n-i}\Psi(W, X)\right) = \Psi_1(X),$$

which gives $\Psi(W,X)(1-\sum_{i=0}^{n-1} a_i X^{n-i}) = X^n \Psi_1(X)$, proving that $\Psi(W,X)$ lies in $S^{-1}A[X,X^{-1}]$, since a_0 is a unit. □

3.3 Contragredient Whittaker Functions

We define an analogue of the Fourier transform of a Whittaker function W; the functional equation will relate the zeta integral of W to that of its transform. We will need the following two matrices:

$$w = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ & \vdots & & \\ (-1)^{n-1} & \cdots & 0 & 0 \end{pmatrix}, \qquad w' = \begin{pmatrix} (-1)^n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & (-1)^{n-2} \\ 0 & 0 & \cdots & (-1)^{n-3} & 0 \\ \vdots & & \vdots & & \\ 0 & (-1)^0 & \cdots & 0 & 0 \end{pmatrix}$$

For any element W of $\operatorname{Ind}_{N_n}^G \psi$, define the transform $\widetilde{W}$ of W as $\widetilde{W}(g) := W(wg^\iota)$, where $g^\iota :=^t g^{-1}$.

Observation 2. *If V is of Whittaker type, then for $v \in V$, $\widetilde{W_v}$ is an element of $\operatorname{Ind}_N^G \psi$ because*

$$\widetilde{W}(ng) = W(w(ng)^\iota) = W(wn^\iota w^{-1} wg^\iota) = \psi(wn^\iota w^{-1})W(wg^\iota) = \psi(n)\widetilde{W}(g).$$

Thus $\Psi(\widetilde{w'W}, X; j)$ lands in $A[[X]][X^{-1}]$ by Lemma 10. In this section we state the second main result and recover the rationality properties of Section 2.2 for $\Psi(\widetilde{w'W}, X; j)$. Our second main result is as follows:

Theorem 3.3.1. *Suppose A is a Noetherian $W(k)$-algebra. Let S denote the multiplicative subset of $A[X,X^{-1}]$ consisting of Laurent polynomials whose first and last coefficients are units. Suppose V in $\operatorname{Rep}_A(G)$ is co-Whittaker and* *primitive. Then there exists a unique element* $\gamma(V, X, \psi)$ *of* $S^{-1}A[X, X^{-1}]$ *such that for any* $W \in \mathcal{W}(V, \psi)$ *we have*

$$\Psi(W, X; j)\gamma(V, X, \psi) = \Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; n - 2 - j)$$

for $0 \leq j \leq n - 2$.

The proof of Theorem 3.3.1 is in Section 3.8. We now verify that the quantity $\Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; n - 2 - j)$ lives in the ring $S^{-1}A[X, X^{-1}]$.

Proposition 3.3.2. *Suppose* V *in* $\mathrm{Rep}_A(G)$ *is admissible, Whittaker type, G-finite, Schur, and primitive. Let* V^ι *denote the smooth* $A[G]$*-module whose underlying* A*-module is* V *and whose* G*-action is given by* $g \cdot v = g^\iota v$*. Then* V^ι *is also admissible, Whittaker type, G-finite, and Schur.*

Proof. Let l denote left translation, so $(l(w)\phi)(x) = \phi(wx)$ for a morphism ϕ. Consider the map $\mathrm{Hom}_{N_n}(V, \psi) \to \mathrm{Hom}_{N_n}(V^\iota, \psi)$ given by $\lambda \mapsto \widetilde{\lambda}$, where $\widetilde{\lambda} : x \mapsto \lambda(wx)$. We have

$$\widetilde{\lambda}(n \cdot v) = \lambda(wn^\iota v) = \lambda(wn^\iota w^{-1}wv) = \psi(n)\widetilde{\lambda}(v),$$

which shows $\widetilde{\lambda}$ indeed defines an element of $\mathrm{Hom}_{N_n}(V^\iota, \psi)$. Since $w^2 = (-1)^{n-1}I_n$, it is an isomorphism of A-modules.

Admissibility, G-finiteness, and Schur-ness all hold for V^ι since $g \mapsto g^\iota$ is a topological automorphism of the group G. Since V is Schur, A must be connected, hence V must be primitive since it is Schur. □

In particular, $(V^\iota)^{(n)} = V^\iota / V^\iota(N_n, \psi)$ is free of rank one and we may define $(\widetilde{W})_v(g) = \widetilde{\lambda}(g^\iota v)$ and take $\mathcal{W}(V^\iota, \psi)$ to be the A-module $\{(\widetilde{W})_v : v \in V^\iota\}$ as before. Note that this is precisely the same as $\{\widetilde{(W_v)} : v \in V\}$. We record this simple observation as a Lemma:

Lemma 14. *If λ is a generator of $\mathrm{Hom}_{N_n}(V, \psi)$ then $\widetilde{\lambda} : x \mapsto \lambda(wx)$ is a generator of $\mathrm{Hom}_{N_n}(V^\iota, \psi)$ and defines $\mathcal{W}(V^\iota, \psi)$. There is an isomorphism of G-modules $\mathcal{W}(V, \psi) \to \mathcal{W}(V^\iota, \psi)$ sending W to $\widetilde{W}$.*

Thus all the hypotheses for the results of the previous sections, in particular Theorem 3.1.2, apply to V^ι whenever they apply to V, so we get $\Psi(\widetilde{w'W}, X; j)$ is in $S^{-1}A[X, X^{-1}]$. Now we can make the substition $\frac{q^{n-2}}{X}$ for X in the ratio of polynomials $\Psi(\widetilde{w'W}, X; j)$ to get $\Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; j)$. It will again be in $S^{-1}A[X, X^{-1}]$ because this process swaps the first and last coefficients in the denominator (and q is a unit in A since q is relatively prime to ℓ).

3.4 Zeta Integrals and Tensor Products

The goal of this subsection is to check that the formation of zeta integrals commutes with changing the base ring A. For any A-algebra $f : A \to B$, denote by $\psi_A \otimes B$ the free rank one B-module with action given by the character $f \circ \psi$. The group action on $V \otimes_A B$ is given by acting in the first factor. We let i denote the map $V \to V \otimes_A B$. Proposition 2.1.1 (6), gives the following lemma.

Lemma 15. *(1) If V is of Whittaker type, so is $V \otimes_A B$.*

(2) Let λ *generate* $\mathrm{Hom}_{A[N]}(V,\psi)$ *as an A-module. Then* $\lambda \otimes id$ *is a generator of* $\mathrm{Hom}_{B[N]}(V \otimes B, \psi \otimes B)$.

(3) Let $W_{v\otimes b}(g) := (f \circ \lambda)(gv) \otimes b$ *define elements of* $\mathcal{W}(V \otimes B, \psi \otimes B)$*. Then* $f \circ W_v = W_{i(v)}$ *for any* $v \in V$.

From the definition of integration given in §1.3, it follows that if Φ_k is the characteristic function of some H_k, then $\int (f \circ \Phi_k) d(f \circ \mu^\times) = (f \circ \mu^\times)(H_k) = f\left(\int \Phi_k d(\mu^\times)\right)$. It follows from the definitions that $(f \circ \widetilde{W})(x) = \widetilde{f \circ W}(x)$.

Corollary 3.4.1. *Let* F *denote the map of formal Laurent series rings*

$$A[[X]][X^{-1}] \to B[[X]][X^{-1}]$$

induced by f*, then we have*

$$F\left(\Psi(W_v, X; j)\right) = \Psi(F \circ W, X; j) = \Psi(W_{i(v)}, X; j) \tag{3.1}$$

$$F\left(\Psi(\widetilde{w'W}, X; j)\right) = \Psi(F \circ \widetilde{w'W}, X; j) = \Psi(w'\widetilde{(F \circ W)}, X; j) \tag{3.2}$$

for any W *in* $\mathcal{W}(V,\psi)$*, and for* $0 \leq j \leq n-2$.

Since the elements $v \otimes 1$ generate $V \otimes B$ as a B-module, the elements $W_{i(v)}$ generate $\mathcal{W}(V \otimes B, \psi \otimes B)$ over B. The linearity of the zeta integrals and the Fourier transform give the next proposition.

Proposition 3.4.2. *Suppose there exists an element* $\gamma(V, X, \psi) \in A[[X]][X^{-1}]$ *satisfying a functional equation as in Theorem 3.3.1 for all* $W_v \in \mathcal{W}(V,\psi)$*. Then the element* $f(\gamma(V, X, \psi)) \in B[[X]][X^{-1}]$ *satisfies the functional equation for all* $W \in \mathcal{W}(V \otimes B, \psi \otimes B)$.

In other words if $f : A \to B$ is an A-algebra and we know that $\gamma(V, X, \psi)$ exists, then $\gamma(V \otimes_A B, X, \psi \otimes_A B)$ is $f(\gamma(V, X, \psi))$.

3.5 Construction of the Gamma Factor

We define the gamma factor to be what it must in order to satisfy the functional equation of Theorem 3.3.1 for a single, particularly simple Whittaker function W_0. We seek a W_0 such that $\Psi(W, X; 0)$ is a unit in $S^{-1}A[X, X^{-1}]$.

By Proposition 2.2.1, we can realize $\text{c-Ind}_{U_2}^{P_2} \psi$ as a submodule of

$$\mathcal{K}((\Phi^-)^{n-2}V, \psi).$$

By Lemma 3, we have that $\text{c-Ind}_{U_2}^{P_2} \psi \subset (\Phi^-)^{n-2}\mathcal{K}$. Since $\text{c-Ind}_{U_2}^{P_2} \psi$ is isomorphic to $C_c^\infty(F^\times, A)$ via restriction to G_1 (recall that $C_c^\infty(F^\times, A)$ denotes the locally constant compactly supported functions $F^\times \to A$), we find the following:

Proposition 3.5.1. *Suppose V in $\text{Rep}_A(G)$ is of Whittaker type. Then the characteristic function of U_F^1 is realized as a restricted Whittaker function $W_0\left(\begin{smallmatrix} g & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)$ for some W_0 in $\mathcal{W}(V, \psi)$.*

From now on, we use the symbol W_0 to denote a choice of element in $\mathcal{W}(V, \psi)$ whose restriction to $\left(\begin{smallmatrix} g & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)$ is the characteristic function of U_F^1. Notice that $\Psi(W_0, X)$ is simply $\int_{U_F} W_1\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) d^\times a = \mu^\times(U_F^1) = 1$. Since we want our gamma factor to satisfy the functional equation for W_0, we are left with no choice:

Definition 3.5.1 (The Gamma Factor)**.** Let A be any Noetherian $W(k)$-algebra and suppose V in $\mathrm{Rep}_A(G)$ is of Whittaker type. We define the gamma factor of V with respect to ψ to be the element of $S^{-1}A[X, X^{-1}]$ given by $\gamma(V, X, \psi) := \Psi(\widetilde{w'W_0}, \frac{q^{n-2}}{X}; n-2)$.

We will show in the following sections that this gamma factor satisfies the functional equation *for all* $W \in \mathcal{W}(V, \psi)$. Note that uniqueness follows from the functional equation: if γ and γ' both satisfy the functional equation for all Whittaker functions, then choose a Whittaker function W_0 such that $\Psi(W_0, X) = 1$ and get $\gamma = \Psi(\widetilde{w'W_0}, \frac{q^{n-2}}{X}, n-2) = \gamma'$. In particular, this will imply that our construction of the gamma factor does not depend on the choice of W_0.

3.6 Functional Equation for Characteristic Zero Points

Suppose the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ has characteristic zero, then the reduction modulo $\mathfrak{p}$ of $\Psi(W, X; j)$ forms an element of $\overline{\kappa(\mathfrak{p})}(X)$. Since $\kappa(\mathfrak{p})$ is an extension of $\mathrm{Frac}(W(k))$, $\overline{\kappa(\mathfrak{p})}$ is an uncountable algebraically closed field of characteristic zero. Thus we may fix an embedding of fields $\mathbb{C} \hookrightarrow \overline{\kappa(\mathfrak{p})}$.

The proof of [JPSS83, Thm 2.7(iii)(2)] (which occurs in [JPSS83, §2.11]) carries over verbatim to the setting where π and π' are representations of G over any field containing $\mathbb{C}$, hence for representations over $\overline{\kappa(\mathfrak{p})}$. Then, the reduction modulo $\mathfrak{p}$ of $\Psi(W, X; j)$ is precisely the integral $\Psi(s, W; j)$ of [JPSS79, §4.1], after replacing the complex variable $q^{-s+\frac{n-1}{2}}$ with the indeterminate X.

Moreover, according to [JPSS83, Thm 2.7(iii)(2)], there exists a unique element, which we will call $\gamma_{\mathfrak{p}}(s, V\otimes\overline{\kappa(\mathfrak{p})}, \psi)$, in $\overline{\kappa(\mathfrak{p})}(q^{-s})$ such that *for all* $W \in \mathcal{W}(V_{\mathfrak{p}}\otimes\overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}})$ and for all $j \geq 0$,

$$\Psi(1-s, \widetilde{w'W}; n-2-j) = \gamma_{\mathfrak{p}}(s, V\otimes\overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}})\Psi(s, W, j).$$

The change of variable $s \mapsto 1-s$ can be re-written as $-s+\frac{n-1}{2} \mapsto s-\frac{n-1}{2}+n-2$ which in terms of X is $X \mapsto \frac{q^{n-2}}{X}$ so in terms of X their functional equation translates to

$$\Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; n-2-j) = \gamma_{\mathfrak{p}}(V_{\mathfrak{p}}\otimes\overline{\kappa(\mathfrak{p})}, X, \psi_{\mathfrak{p}})Z_{\mathfrak{p}}(W, X; j),$$

$W \in \mathcal{W}(V_{\mathfrak{p}}\otimes\overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}})$. Thus we have shown:

Lemma 16. *Suppose V is admissible of Whittaker type, and G-finite. For each prime $\mathfrak{p}$ of A with residue characteristic zero, there exists a unique element $\gamma_{\mathfrak{p}}(V\otimes\kappa(\mathfrak{p}), X, \psi_{\mathfrak{p}})$ in $\kappa(\mathfrak{p})(X)$ such that for all W in $\mathcal{W}(V\otimes\kappa(\mathfrak{p}), \psi_{\mathfrak{p}})$ and for $0 \leq j \leq n-2$ we have*

$$\Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; n-2-j) = \gamma_{\mathfrak{p}}(X, V\otimes\kappa(\mathfrak{p}), \psi_{\mathfrak{p}})\Psi(W, X; j).$$

Moreover, $\gamma_{\mathfrak{p}}(V\otimes\kappa(\mathfrak{p}), X, \psi_{\mathfrak{p}}) = \gamma(V, X, \psi) \mod \mathfrak{p}$ *by uniqueness in [JPSS79].*

3.7 Proof of Functional Equation When A is Reduced and ℓ-torsion Free

In the case that A is reduced and ℓ-torsion free, we get a slightly stronger result than that of Theorem 3.3.1.

Theorem 3.7.1. *If A is a Noetherian $W(k)$-algebra and A is reduced and ℓ-torsion free, then the conclusion of Theorem 3.3.1 holds for any V in* $\mathrm{Rep}_A(G)$ *which is G-finite, admissible, and of Whittaker type.*

In particular, when A is reduced and ℓ-torsion free, the conclusion of Theorem 3.3.1 holds when V is co-Whittaker.

Proof. Let $\mathfrak{p}$ be any characteristic zero prime, and let $f_{\mathfrak{p}} : A \to \kappa(\mathfrak{p})$ be reduction modulo $\mathfrak{p}$. Corollary 3.4.1 and Lemma 16 tell us that

$$f_{\mathfrak{p}}\left(\gamma(V,X,\psi)\Psi(W,X) - \Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; n-2)\right) = 0$$

for *any* W in $\mathcal{W}(V,\psi)$, not just W_0. This shows that the difference

$$\gamma(V,X,\psi)\Psi(W,X) - \Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; n-2)$$

is in the intersection of all characteristic zero primes of A.

When A is reduced its zero divisors are the union of its minimal primes. Thus in this situation it is ℓ-torsion free as a $W(k)$-algebra if and only if all minimal primes have residue characteristic zero. When A is reduced and ℓ-torsion free, the intersection of all characteristic zero primes of A equals the nilradical which is zero. Thus, the functional equation holds for any W in $\mathcal{W}(V,\psi)$.

We quickly get uniqueness by the following argument. If there were another element γ' satisfying the same property, then it would satisfy the functional equation in $\kappa(\mathfrak{p})$ for all $W_{i(v)}$ by reduction. By Proposition 3.4.2, this implies it satisfies the functional equation for all W in $\mathcal{W}(V \otimes \overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}})$. By Lemma 16, elements of $\overline{\kappa(\mathfrak{p})(X)}$ satisfying this property are unique, hence $f_{\mathfrak{p}}(\gamma') = \gamma_{\mathfrak{p}}(V, X, \psi)$, i.e. $f_{\mathfrak{p}}(\gamma(V, X, \psi) - \gamma') = 0$ for all primes $\mathfrak{p}$ of A. Again, since the nilradical of A is zero, this means $\gamma' = \gamma(V, X, \psi)$.

We get rationality by observing that whenever V is admissible of Whittaker type, it has a canonical co-Whittaker submodule T by Proposition 2.5.2, which is primitive if V is primitive. Since $\gamma(T, X, \psi)$ satisfies the functional equation for all W in $\mathcal{W}(T, \psi)$, we must have $\gamma(T, X, \psi) = \gamma(V, X, \psi)$ by the construction of the gamma factor. But $\gamma(T, X, \psi)$ is in $S^{-1}A[X, X^{-1}]$ by Theorem 3.1.2, which holds for primitive co-Whittaker modules. $\square$

3.8 Universal Gamma Factors

When V is co-Whittaker, we can remove the hypothesis that A is reduced and ℓ-torsion free by using the theory of the universal co-Whittaker module developed in [Hel12b]. Proposition 3.7.1 associates to the universal co-Whittaker module a gamma factor, which gives rise via specialization to gamma factors for arbitrary co-Whittaker modules. We start by referring to a theorem of Helm:

Theorem 3.8.1 ([Hel12a] Thm 12.1). *Any block $e\mathcal{Z}$ of the Bernstein center of $\mathrm{Rep}_{W(k)}(G)$ is a finitely generated (hence Noetherian), reduced, ℓ-torsion free $W(k)$-algebra.*

By Proposition 2.6.1, $e\mathfrak{W}$ is co-Whittaker, and since it is clearly primitive, and thus all the hypotheses of Proposition 3.7.1 are satisfied. Hence there exists a unique gamma factor in $S^{-1}(e\mathcal{Z}[X, X^{-1}])$, which we will denote $\Gamma(e\mathfrak{W}, X, \psi)$, satisfying the functional equation for all W in $\mathcal{W}(e\mathfrak{W}, \psi)$.

Proof of Theorem 3.3.1. It suffices to prove theorem in the case when V is co-Whittaker (see, for example, Lemma 30 below). Since we are assuming V is primitive and co-Whittaker, there is a (unique) primitive idempotent e of $\mathcal{Z}$ such that we have a ring homomorphism $f_V : e\mathcal{Z} \to \mathrm{End}_G(V) \xrightarrow{\sim} A$ and a surjection of $A[G]$-modules $e\mathfrak{W} \otimes_{f_V} A \to V$ which preserves the top derivative. Proposition 3.4.2 then tells us that $f_V(\Gamma(e\mathfrak{W}, X, \psi)) = \gamma(e\mathfrak{W} \otimes_{\mathcal{A}, f_V} A, X, \psi)$, with $\gamma(e\mathfrak{W} \otimes_{\mathcal{A}, f_V} A, X, \psi)$ as in Definition 3.5.1. Since $\Gamma(e\mathfrak{W}, X, \psi)$ satisfies the functional equation for all W in $\mathcal{W}(e\mathfrak{W}, \psi)$, we can apply Proposition 3.4.2 again to conclude that $\gamma(e\mathfrak{W} \otimes A, X, \psi)$ satisfies the functional equation for all W in $\mathcal{W}(e\mathfrak{W} \otimes A,\ \psi)$. Since $e\mathfrak{W} \otimes A$ has a surjection onto V preserving the top derivative, Lemma 2 tells us that $\mathcal{W}(V, \psi) = \mathcal{W}(e\mathfrak{W} \otimes A, \psi)$. We can therefore choose the same Whittaker function W_0 of Definition 3.5.1 for both V and $e\mathfrak{W} \otimes A$, so $\gamma(V, X, \psi) = \gamma(e\mathfrak{W} \otimes A, X, \psi)$, and thus satisfies the functional equation for all W in $\mathcal{W}(V, \psi)$. Note that since $\Gamma(e\mathfrak{W}, X, \psi)$ is in $S^{-1}(e\mathcal{Z}[X, X^{-1}])$, its image in f_V is in the corresponding fraction ring of $A[X, X^{-1}]$. This proves Theorem 3.3.1. □

By the functional equation, this must be the gamma factor of Definition 3.5.1, and by §3.4 it commutes with change of base ring. We can extend the uniqueness and rationality result to a larger class of representations, though with a slightly more restrictive functional equation:

Corollary 3.8.2. *Let V be admissible of Whittaker type and let T be its canonical co-Whittaker submodule. Then there exists a unique gamma factor $\gamma(V, X, \psi)$ in $S^{-1}(A[X, X^{-1}])$ which equals $\gamma(T, X, \psi)$, and satisfies the functional equation for all W in $\mathcal{W}(T, \psi)$.*

Proof. When V is admissible of Whittaker type, Proposition 2.5.2 tells us that V has a canonical co-Whittaker sub-$A[G]$-module T. We have just shown that its gamma factor $\gamma(T, X, \psi)$ must satisfy the functional equation for all W in $\mathcal{W}(T, \psi)$. We can then apply Proposition 2, taking $\alpha : T \to V$ to be the inclusion map, to conclude that $\mathcal{W}(T, \psi) \subset \mathcal{W}(V, \psi)$. Since this subset $\mathcal{W}(T, \psi)$ is the space of Whittaker functions of the Whittaker type representation T, it is a G-invariant subspace. The coefficients of the series $\Psi(\widetilde{w'W_0}, \frac{q^{n-2}}{X}; n-2)$ in Definition 3.5.1 will always be determined by G-translates of the Whittaker function W_0, which occurs already in $\mathcal{W}(T, \psi)$ so by definition we get that $\gamma(T, X, \psi) = \gamma(V, X, \psi)$. In particular, this tells us that $\gamma(V, X, \psi)$ lies in $S^{-1}A[X, X^{-1}]$ and satisfies the functional equation for all W in $\mathcal{W}(T, \psi)$. □

By fixing a particular block $e\,\mathrm{Rep}_{W(k)}(G)$, and considering only the co-Whittaker $A[G]$-modules in that block (i.e. the ones primitive with $eV = V$), we can make precise the sense in which we have created a universal gamma factor:

Corollary 3.8.3. *Suppose A is any Noetherian $W(k)$-algebra, and suppose V is a co-Whittaker $A[G]$-module in the subcategory $e\,\mathrm{Rep}_{W(k)}(G)$ of $\mathrm{Rep}_{W(k)}(G)$. Then there is a homomorphism $f_V : e\mathcal{Z} \to A$ and*

$$\gamma(V, X, \psi) = f_V(\Gamma(e\mathfrak{W}, X, \psi)).$$

Moreover, $\gamma(V, X, \psi)$ satisfies a functional equation for all W in $\mathcal{W}(V, \psi)$.

Again, we can broaden the class of representations at the cost of a more restrictive functional equation:

Theorem 3.8.4. *Suppose A is any Noetherian $W(k)$-algebra, and suppose V is an admissible $A[G]$-module of Whittaker type in the subcategory $e\,\mathrm{Rep}_{W(k)}(G)$. Then there is a homomorphism $f_V : e\mathcal{Z} \to A$ and the gamma factor of Corollary 3.8.2 equals $f_V(\Gamma(e\mathfrak{W}, X, \psi))$.*

Proof. We define f_V to be the homomorphism $e\mathcal{Z} \to \mathrm{End}_G(T) \xrightarrow{\sim} A$ where T is the canonical co-Whittaker submodule of Proposition 2.5.2. Since T lies in a single block $e\mathcal{Z}$, $e\mathfrak{W} \otimes_{f_V, e\mathcal{Z}} A$ surjects onto T, and we have $f_V(\Gamma(e\mathfrak{W}, X, \psi) = \gamma(T, X, \psi)$ (Prop 2), and since T injects into V (with top derivative preserved), again by Prop 2, we have

$$f_V(\Gamma(e\mathfrak{W}, X, \psi)) = \gamma(e\mathfrak{W} \otimes_{e\mathcal{Z}, f_V} A, X, \psi) = \gamma(T, X, \psi) = \gamma(V, X, \psi).$$

□

Chapter 4

Godement-Jacquet Zeta Integrals in Families

This chapter is devoted to proving Theorem 1.1.4, described in the introduction.

4.1 Exploiting the irreducible case in characteristic zero

In this subsection we assume that A is reduced and ℓ-torsion free, and that $V \otimes_A \kappa(\mathfrak{p})$ is absolutely irreducible. Therefore the minimal prime ideals of A have residue fields with characteristic zero, and the diagonal map

$$A \to \prod_{\mathfrak{p} \text{ minimal}} \kappa(\mathfrak{p})$$

is an embedding of A into its total quotient ring. In this section we will prove the following weaker version of Theorem 1.1.4:

Proposition 4.1.1. *Let A be a Noetherian $W(k)$-algebra which is reduced and ℓ-torsion free, V a primitive co-Whittaker $A[G]$-module such that $V \otimes \kappa(\mathfrak{p})$ is absolutely irreducible for every minimal prime $\mathfrak{p}$. Let $\mathcal{C}(V)$ be its space of matrix coefficients and let S be the multiplicative subset of $A[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. Then:*

1. *$Z(f, \Phi, X)$ lives in the fraction ring $S^{-1}(A[X, X^{-1}])$ for all $f \in \mathcal{C}(\pi)$, $\Phi \in C_c^\infty(M_n(F), A)$.*

2. *There exists a unique element $\gamma(V, X, \psi)$ of $S^{-1}A[X, X^{-1}]$ such that*

$$Z(f, \Phi, X)\gamma(V, X, \psi) = Z(f^\vee, \widehat{\Phi}, \frac{q^{n-2}}{X})$$

for all $f \in \mathcal{C}(\pi)$, $\Phi \in C_c^\infty(M_n(F), A)$.

After choosing an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$, we can apply the following results of [JPSS79] to $\mathcal{W} \otimes_A \overline{\kappa(\mathfrak{p})}$, where $\mathfrak{p}$ is a minimal prime.

Lemma 17 (Thm 4.3, [JPSS79])**.** *Suppose $\kappa(\mathfrak{p})$ has characteristic zero and $V \otimes_A \kappa(\mathfrak{p})$ is absolutely irreducible generic, then we have the following equality of fractional ideals of $\overline{\kappa(\mathfrak{p})}[X, X^{-1}]$:*

$$\begin{aligned} &\{\Psi(W_\mathfrak{p}, X) : W_\mathfrak{p} \in \mathcal{W}(V \otimes \overline{\kappa(\mathfrak{p})}, \psi)\} \\ = &\{Z(f, \Phi, X) : f \in \mathcal{C}(\pi \otimes \overline{\kappa(\mathfrak{p})}), \Phi \in C_c^\infty(M_n(F), \overline{\kappa(\mathfrak{p})}\},\end{aligned}$$

Here $\Psi(W_\mathfrak{p}, X)$ denotes the "new" zeta integral of [JPSS79, 4.1.1], or equivalently those defined in Definition 3.1.

Since formation of zeta integrals commutes with change of base ring by §3.4, we have, given $f \in \mathcal{C}(\pi)$, that $Z(f \mod \mathfrak{p}, \Phi \mod \mathfrak{p}, X) = Z(f, \Phi, X) \mod \mathfrak{p}$. In particular for each minimal prime $\mathfrak{p}$ there exists $W_\mathfrak{p} \in \mathcal{W}(V \otimes \kappa(\mathfrak{p}), \psi)$ such that $Z(f, \Phi, X) = \Psi(W_\mathfrak{p}, X)$ in $\kappa(\mathfrak{p})(X)$.

We now use the following isomorphism:

Lemma 18. *Let $K = \prod_{\mathfrak{p}\ minimal} \kappa(\mathfrak{p})$ be the total quotient ring of A. The map $V \otimes_A K \to \prod_{i=1}^{r}(V \otimes_A \kappa(\mathfrak{p}_i))$ is an isomorphism of K-modules.*

Proof. The map is given by $v \otimes (a_1, \ldots, a_r) \mapsto (v \otimes a_1, \ldots, v \otimes a_r)$ on simple tensors and extended by K linearity.

If $v \otimes a_i$ is zero for each i, then $v \otimes (0, \ldots, a_i, \ldots, 0)$ equals zero as an element of $V \otimes (0 \times \cdots \times \kappa(\mathfrak{p}_i) \times \cdots \times 0) \subset V \otimes K$. Hence

$$v \otimes (a_1, \ldots, a_r) = \sum_i v \otimes (0, \ldots, a_i, \ldots, 0) = \sum_i 0,$$

so we would in this situation have $v \otimes (a_1, \ldots, a_r) = 0$, thus the map is injective.

Surjectivity is even easier. Given $(v_1 \otimes a_1, \ldots, v_r \otimes a_r)$, something that maps to it is $\sum_i v_i \otimes (0, \ldots, a_i, \ldots, 0) \mapsto \sum_i (0, \ldots, v_i \otimes a_i, \ldots, 0)$. $\square$

Now if T is the set of non-zerodivisors in A, we have an isomorphism

$$T^{-1}\mathcal{W} \xrightarrow{\sim} \prod_i (\mathcal{W} \otimes \kappa(\mathfrak{p}_i)).$$

Thus there exists $W' \in \mathcal{W}$ and $a_{f,\Phi} \in T$ such that

$$W := \frac{W'}{a_{f,\Phi}} \mapsto (W_{\mathfrak{p}_1}, \ldots, W_{\mathfrak{p}_r}).$$

Hence, $a_{f,\Phi}W$ is in the image of $\mathcal{W} \to T^{-1}\mathcal{W}$. Thus we can choose an element $W' \in \mathcal{W}$ which maps to $a_{f,\Phi}W \in T^{-1}\mathcal{W}$.

Lemma 19. *Let A be reduced and ℓ-torsion free, and suppose V is admissible, G-finite, and Whittaker type. Suppose further that $V \otimes_A \kappa(\mathfrak{p})$ is absolutely* *irreducible for every minimal prime $\mathfrak{p}$. Then there is an $a_{f,\Phi} \in A$, which is not a zero divisor, such that $a_{f,\Phi}Z(f,\Phi,X)$ equals $\Psi(W',X)$, and thus lives in $S^{-1}A[X,X^{-1}]$ by Theorem 3.1.2.*

Proof. Since the operator Z is A-linear, we have

$$Z(a_{f,\Phi}f,\Phi,X) = a_{f,\Phi}Z(f,\Phi,X).$$

We show $\Psi(W',X) \equiv Z(a_{f,\Phi}f,\Phi,X) \mod \mathfrak{p}$ for all minimal primes $\mathfrak{p}$, and then the lemma follows because A is reduced and ℓ-torsion free. By construction we have that W' maps to $a_{f,\Phi}W$ in $\mathcal{W} \to T^{-1}\mathcal{W}$. Moreover, since W maps to $(W_{\mathfrak{p}_1},\ldots,W_{\mathfrak{p}_r})$ in $T^{-1}\mathcal{W} \xrightarrow{\sim} \prod_i(\mathcal{W}\otimes\kappa(\mathfrak{p}_i))$ our Whittaker function W' maps to $(a_{f,\Phi,\mathfrak{p}_1}W_{\mathfrak{p}_1},\ldots,a_{f,\Phi,\mathfrak{p}_r}W_{\mathfrak{p}_r})$ where $a_{f,\Phi,\mathfrak{p}}$ denotes the reduction $a_{f,\Phi} \mod \mathfrak{p}$. Since the zeta integrals mod $\mathfrak{p}_i$ are $\kappa(\mathfrak{p}_i)$-linear, we thus have $\Psi(W',X) \equiv Z(a_{f,\Phi}f,\Phi,X) \mod \mathfrak{p}_i$ □

Lemma 20. *Suppose $g(X)$ is an element of $A[[X]][X^{-1}]$ and there exists an element r of A which is not a zero divisor such that $r \cdot g(X)$ lands in $S^{-1}(A[X,X^{-1}])$. Then $g(X)$ lives in $S^{-1}(A[X,X^{-1}])$.*

Proof. There exists $s(X) \in S$ such that $r \cdot g(X)s(X) \in A[X,X^{-1}]$. Since $s(X) \in A[X,X^{-1}]$ by definition, $g(X)s(X) \in A[[X]][X^{-1}]$. Thus $g(X)s(X)$ has a tail $h(X) \in A[[X]]$ which is a power series such that $r \cdot h(X) = 0$. Since r is not a zero divisor, this implies $h(X) = 0$, so $g(X)s(X) \in A[X,X^{-1}]$. □

We will now make use of the gamma factor $\gamma(V, X, \psi)$ for the integrals $\Psi(W, X) \in S^{-1}(A[X, X^{-1}])$ (constructed in Chapter 3) to deduce the rfunctional equation of $Z(f, \Phi, X)$.

First, we note that $S^{-1}(A[X, X^{-1}])$ is stable under the change of variable $X \mapsto \frac{q^{n-2}}{X}$ and therefore it is valid to consider the functional equation within this ring. In other words, since $Z(f^\vee, \widehat{\Phi}, X)$ lives in $S^{-1}(A[X, X^{-1}])$ (where $\widehat{\Phi}$ denotes the Fourier transform), the rational function $Z(f^\vee, \widehat{\Phi}, \frac{q^{n-2}}{X})$ also defines an element of $S^{-1}(A[X, X^{-1}])$.

By Theorem 3.7, there exists a unique element

$$\gamma(V, X, \psi) \in S^{-1}(A[X, X^{-1}])$$

which, for all $W \in \mathcal{W}(V, \psi)$, satisfies

$$\Psi(W, X)\gamma(V, X, \psi) = \Psi(\widetilde{w'W}, \frac{q^{n-2}}{X}; n-2).$$

Moreover, this gamma factor commutes with specialization and thus reduces mod characteristic zero primes $\mathfrak{p}$ to the gamma factor of [JPSS79, Thm 4.5] (§3.4). But according to [JPSS79, Thm 4.5], the gamma factor for the functional equation of Ψ equals the gamma factor for the functional equation of Z, in the characteristic zero setting. Thus we have

$$Z(f, \Phi, X)\gamma(V, X, \psi) - Z(f^\vee, \widehat{\Phi}, \frac{q^{n-2}}{X}) = 0 \mod \mathfrak{p}$$

for all minimal primes $\mathfrak{p}$, and hence by again invoking that the intersection of all characteristic zero primes in A equals zero, we have proven part 2 of Proposition 4.1.1.

4.2 Generic Irreducibility

Following the method of Chapter 3, we can remove the hypothesis that A is reduced and ℓ-torsion free by using the machinery of the integral Bernstein center $\mathcal{Z}$ and the "universal" co-Whittaker module $e\mathfrak{W}$ of §2.5.

However, to apply the results of §4.1 we need $e\mathfrak{W}$ to be irreducible at minimal primes. In other words, we need to know that co-Whittaker modules are "generically" irreducible, in the sense of algebraic geometry, i.e. that irreducibility holds in Zariski open subsets. This is a property that is widely known in representation theory, but we must verify it in our setting. It will also be used later in the proof of Theorem 6.3.1.

Proposition 4.2.1. *Suppose $\mathfrak{p}$ is a minimal prime of $e\mathcal{Z}$. Then $e\mathfrak{W} \otimes_{e\mathcal{Z}} \kappa(\mathfrak{p}')$ is absolutely irreducible for all $\mathfrak{p}'$ in an open neighborhood of $\mathfrak{p}$.*

Proof. Let $\Pi := e(\text{c-Ind}\,\psi)$. We begin by showing that the locus of points $\mathfrak{p}$ such that $\Pi \otimes \kappa(\mathfrak{p})$ is reducible is contained in a closed subset. For a ring R and K a compact open subgroup let $\mathcal{H}(G, K, R)$ be the algebra of smooth compactly supported functions $G \to R$ which are K-fixed under right translation. $\mathcal{H}(G, K, e\mathcal{Z})$ and Π form sheaves of $\text{Spec}(e\mathcal{Z})$-modules, and following [Ber93, IV.1.2], the map $P_K : \mathcal{H}(G, K, e\mathcal{Z}) \to \text{End}_{e\mathcal{Z}}(\Pi^K)$ which sends h to $\Pi(h)$ is a morphism of sheaves. $\Pi|_{\mathfrak{p}} := \Pi \otimes \kappa(\mathfrak{p})$ is irreducible if and only if, for any K, $(\Pi|_{\mathfrak{p}})^K$ is either zero or irreducible over $\mathcal{H}(G, K, \kappa(\mathfrak{p}))$. Supposing $\Pi|_{\mathfrak{p}}$ is reducible, there exists a K such that $(\Pi|_{\mathfrak{p}})^K$ is nonzero and reducible. Since $(\Pi|_{\mathfrak{p}})^K$ is a finite dimensional $\kappa(\mathfrak{p})$ vector space, a proper $\mathcal{H}(G, K, \kappa(\mathfrak{p}))$-stable subspace gives a proper submodule of the endomorphism ring containing the image of $P_K \otimes \kappa(\mathfrak{p})$. The set of points $\mathfrak{p}$ where $(P_K)_{\mathfrak{p}}$ fails to be surjective is contained in the support of the finitely generated $e\mathcal{Z}$-module $\frac{\Pi^K}{\mathrm{Im}(P_K)}$, which is closed. For any such point $\mathfrak{p}$, $(\Pi|_{\mathfrak{p}})^K = (\Pi^K)|_{\mathfrak{p}}$ must then be reducible by Schur's lemma. It remains to show that there is at least one point where we have irreducibility.

Suppose $e = e_{[L,\pi]}$ is the idempotent corresponding to the mod ℓ inertial equivalence class $[L,\pi]$ in the Bernstein decomposition of $\mathrm{Rep}_{W(k)}(G)$ (see [Hel12b]). By [Hel12a, Prop 11.1], $e\mathcal{Z} \otimes_{W(k)} \overline{\mathcal{K}} \cong \prod_{M,\pi'} \mathcal{Z}_{\overline{\mathcal{K}},M,\pi'}$ where M,π' runs over inertial equivalence classes of $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)$ whose mod ℓ inertial supercuspidal support equals (L,π), and $\mathcal{Z}_{\overline{\mathcal{K}},M,\pi'}$ denotes the center of $\mathrm{Rep}_{\overline{\mathcal{K}}}(G)_{M,\pi'}$. The ring $\mathcal{Z}_{\overline{\mathcal{K}},M,\pi'}$ is a Noetherian normal domain. Since $e\mathcal{Z}$ is reduced and ℓ-torsion free, none of its minimal primes contain ℓ. Inverting ℓ, this decomposition gives isomorphisms

$$\prod_{\mathfrak{p}\text{ minimal}} [(e\mathcal{Z})/\mathfrak{p}] \otimes_{W(k)} \overline{\mathcal{K}} \cong e\mathcal{Z} \otimes_{W(k)} \overline{\mathcal{K}} \cong \prod_{M,\pi'} \mathcal{Z}_{\overline{\mathcal{K}},M,\pi'}.$$

In particular, for each minimal prime there exists M,π' such that $[(e\mathcal{Z})/\mathfrak{p}]\otimes_{W(k)} \overline{\mathcal{K}} \cong \mathcal{Z}_{\overline{\mathcal{K}},M,\pi'}$. Hence $\overline{\kappa(\mathfrak{p})} = \mathrm{Frac}(\mathcal{Z}_{\overline{\mathcal{K}},M,\pi'})$.

Given such an M,π', we have by [BD84] that

$$\mathcal{Z}_{\overline{\mathcal{K}},M,\pi'} := \mathcal{Z}(\mathrm{Rep}_{\overline{\mathcal{K}}}(G)_{M,\pi'}) \cong (\overline{\mathcal{K}}[M/M^\circ]^H)^{W(\pi')},$$

where M° is the subgroup generated by all the compact subgroups, which equals the set of $m \in M$ with $\det m \in U_F$. Let $\Psi(M)$ be the linear algebraic group over $\overline{\mathcal{K}}$ of unramified characters of M, in other words the ring $\overline{\mathcal{K}}[M/M^\circ]$. Then by [Ber93], if π' is our given supercuspidal representation of $\mathcal{K}$, then $i_P^G(\pi'\otimes\chi)$ is irreducible for χ a generic $\overline{\mathcal{K}}$ point of $\overline{\mathcal{K}}[M/M^\circ]$. Let $\mathfrak{q}$ be a point of $e\mathcal{Z}$ lying under the point χ. Since $i_P^G(\pi'\otimes\chi)$ is cuspidal, $(i_P^G(\pi'\otimes\chi))^{(n)}$ is one dimensional and therefore we have a map $e(\text{c-Ind}\,\psi)\otimes_{\mathcal{Z}_{M,\pi'}}\kappa(\mathfrak{q})\to i_P^G(\pi'\otimes\chi)$ coming from reciprocity. Since $i_P^G(\pi'\otimes\chi)$ is absolutely irreducible this map is surjective. The kernel K of this map must be zero by the following reasoning. By [EH12, Cor 3.2.14] all the Jordan-Holder constituents of an essentially AIG representation over $\overline{\mathcal{K}}$ have the same supercuspidal support, so the same is true for representations with essentially AIG dual. Therefore, if K were nonzero it would have all Jordan-Holder constituents having the same supercuspidal support as $i_P^G(\pi'\otimes\chi)$, in particular those constituents would be irreducible and equivalent to $i_P^G(\pi'\otimes\chi)$. But then $K^{(n)}$ is nonzero, which contradicts the fact that $e(\text{c-Ind}\,\psi)\otimes_{\mathcal{Z}_{M,\pi'}}\kappa(\mathfrak{q})\to i_P^G(\pi'\otimes\chi)$ is a G-surjection of generic representations. Hence $e(\text{c-Ind}\,\psi)\otimes\kappa(\mathfrak{q})$ is absolutely irreducible. This proves the claim. □

4.3 Rationality and Functional Equation

To deduce Theorem 1.1.4 we will need the following lemma:

Lemma 21. *Given a G-surjection $\alpha: V\to U$, there is an inclusion $\mathcal{C}(U)\subset\mathcal{C}(V)$.*

Proof. Choose v mapping to u. Let $\gamma_{u^\vee\otimes u}$ be in $\mathcal{C}(U)$. Then $u^\vee$ defines an element $\tilde{u}^\vee$ of $V^\vee$ via $\tilde{u}^\vee : x \mapsto u^\vee(\alpha(x))$. Now $\gamma_{\tilde{u}^\vee \otimes v}$ doesn't depend on the choice of v mapping to u. Suppose v' were a different lift of u:

$$\begin{aligned}\gamma_{\tilde{u}^\vee\otimes v}(g) - \gamma_{\tilde{u}^\vee\otimes v'}(g) &= \langle gv - gv', \tilde{u}^\vee\rangle \\ &= \langle \alpha(gv - gv'), u^\vee\rangle \\ &= 0.\end{aligned}$$

To see it is an inclusion, we note that every value $\gamma_{\tilde{u}^\vee\otimes v}(g)$ is achieved already by $\gamma_{u^\vee\otimes u}$:

$$\langle gv, \tilde{u}^\vee\rangle = \langle gu, u^\vee\rangle = \gamma_{u^\vee\otimes u}(g),$$

hence $\gamma_{\tilde{u}^\vee\otimes v}$ is completely determined. □

Now we will use Proposition 4.2.1 in conjunction with Theorem 3.8.1 to deduce Theorem 1.1.4.

Deducing Theorem 1.1.4. By Proposition 4.1.1, $Z(f, \Phi, X)$ lives in the fraction ring $S^{-1}e\mathcal{Z}[X, X^{-1}]$ for every $f \in \mathcal{C}(e\mathfrak{W})$, and every $\Phi \in C_c^\infty(M_n(F), e\mathcal{Z})$, and satisfies the functional equation

$$Z(f, \Phi, X)\gamma(e\mathfrak{W}, X, \psi) = Z(f^\vee, \widehat{\Phi}, \frac{q^{n-2}}{X})$$

for all f, Φ.

Now suppose V is an arbitrary primitive co-Whittaker module defined over an arbitrary Noetherian $W(k)$-algebra A, as in the statement of Theorem

1.1.4. Since the formation of zeta integrals and gamma factor commutes with change of base ring (§3.4.1), we have

$$Z(f \circ f_V, \Phi \circ f_V, X)\gamma(e\mathfrak{W} \otimes_{e\mathcal{Z}, f_V} A, X, \psi) = Z(f^\vee \circ f_V, \widehat{\Phi} \circ f_V, \frac{q^{n-2}}{X})$$

for all $f \in \mathcal{C}(e\mathfrak{W})$, $\Phi \in C_c^\infty(M_n(F), e\mathcal{Z})$.

Lemma 22. *As f ranges over $\mathcal{C}(e\mathfrak{W})$ and Φ ranges over $C_c^\infty(M_n(F), e\mathcal{Z})$, $f \circ f_V$ ranges over $\mathcal{C}(e\mathfrak{W} \otimes_{f_V} A)$ and $\Phi \circ f_V$ ranges over $C_c^\infty(M_n(F), A)$.*

Hence for all $f \in \mathcal{C}(e\mathfrak{W} \otimes A)$, $\Phi \in C_c^\infty(M_n(F), A)$, we get that

$$Z(f, \Phi, X) = f_V(Z(f', \Phi', X)$$

lives in $S^{-1}A[X, X^{-1}]$, and by Corollary 3.8.3,

$$Z(f, \Phi, X)\gamma(V, X, \psi) = Z(f^\vee, \widehat{\Phi}, \frac{q^{n-2}}{X}).$$

By Lemma 21, $\mathcal{C}(V) \subset \mathcal{C}(e\mathfrak{W} \otimes A)$. Thus we have a fortiori that $Z(f, \Phi, X)$ lives in $S^{-1}A[X, X^{-1}]$ for all $f \in \mathcal{C}(V)$ and

$$Z(f, \Phi, X)\gamma(V, X, \psi) = Z(f^\vee, \widehat{\Phi}, \frac{q^{n-2}}{X})$$

for all $f \in \mathcal{C}(V)$, $\Phi \in C_c^\infty(M_n(F), A)$. □

Chapter 5

Two Ranks and Two Coefficient Rings: Rankin-Selberg Convolutions

5.1 Rationality of Rankin-Selberg Formal Series

Let A and B be Noetherian $W(k)$-algebras and let $R = A \otimes_{W(k)} B$. Let V and V' be $A[G_n]$- and $B[G_m]$-modules, respectively, where $m < n$. Suppose both V and V' are of Whittaker type. For $W \in \mathcal{W}(V, \psi)$ and $W' \in \mathcal{W}(V', \psi)$, we define the formal series with coefficients in R:

$$\Psi(W, W', X) := \sum_{r \in \mathbb{Z}} \int_{N_m \backslash \{g \in G_m : v(\det g) = r\}} \left(W\left(\begin{smallmatrix} g & 0 \\ 0 & I_{n-m} \end{smallmatrix} \right) \otimes W'(g) \right) X^r dg$$

and for $0 \leq j \leq n - m - 1$, define

$$\Psi(W, W', X; j) :=$$
$$\sum_{r \in \mathbb{Z}} \int_{M_{j,m}(F)} \int_{N_m \backslash \{g \in G_m : v(\det g) = r\}} \left(W \begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j} \end{pmatrix} \otimes W'(g) \right) X^r dg dx$$

With $\Psi(W, W', X; 0) = \Psi(W, W', X)$.

Lemma 23. *The formal series $\Psi(W, W', X; j)$ has finitely many nonzero powers of X^{-1}, thus forms an element of $R[[X]][X^{-1}]$.*

Proof. Since [JPSS83, Lemma 6.2] is valid in this context, the proof proceeds exactly as in §3.1 after applying the Iwasawa decomposition. The Iwasawa decomposition works in this setting after choosing an appropriate Haar measure, as shown in [KM14, Cor 2.9]. □

Theorem 5.1.1. *Suppose A and B are Noetherian $W(k)$-algebras, V is an $A[G_n]$-module, and V' is a $B[G_m]$-module, both admissible, of Whittaker type and finitely generated over $A[G_n]$ and $B[G_m]$ respectively. Define S to be the multiplicative subset of $R[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. For any $W \in \mathcal{W}(V, \psi)$, $W' \in \mathcal{W}(V', \psi)$, the formal series $\Psi(W, W', X; j)$ lives in $S^{-1}(R[X, X^{-1}])$.*

The remainder of this section is devoted to proving Theorem 5.1.1.

As in [JPSS83] it suffices to consider only the $j = 0$ integral. Using the Iwasawa decomposition as in [JPSS79, JPSS83, KM14], it suffices to prove the theorem when the integration is restricted to the torus T_m:

$$\sum_{r\in\mathbb{Z}} \int_{\{a\in T_m : v(\det a)=r\}} \left(W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-m}\end{smallmatrix}\right) \otimes W(a)\right) X^{v(\det a)} da$$

We parametrize the torus T_m by

$$\prod_{i=1}^{m} F^{\times} \to T_m : (a_1, \ldots, a_n) \mapsto \begin{pmatrix} a_1\cdots a_m & & & \\ & a_2\cdots a_m & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} =: a.$$

In the setting of representations over a field, there is a useful decomposition of any Whittaker function into “finite” functions, which quickly leads to a rationality result ([JPSS79, JPSS83, KM14]). In the setting of rings, such a structure theorem is lacking, but certain elements of its proof can be used to prove rationality.

Consider the exterior product representation

$$\mathcal{W} := \mathcal{W}(V, \psi) \otimes \mathcal{W}(V', \psi) \text{ in } \operatorname{Rep}_R(G_n \times G_m).$$

There is a natural surjection of R-modules

$$\mathcal{W} \to C^\infty(T_m, R)$$

mapping $W \otimes W'$ to the restriction $W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-m} \end{smallmatrix}\right) \otimes W'(a)$. This map is the restriction of functions from $G_n \times G_m$ to the subgroup $T_m \stackrel{\Delta}{\hookrightarrow} T_m \times T_m \hookrightarrow G_n \times G_m$, where Δ denotes the diagonal embedding and $T_m \hookrightarrow G_n$ is the embedding of T_m within the upper-left block of G_n. Denote by $\mathcal{V}$ the image of this restriction map, in other words the A-module generated by

$$\{W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-m} \end{smallmatrix}\right) \otimes W'(a) : a \in T_m, W \in \mathcal{W}(V, \psi), W' \in \mathcal{W}(V', \psi)\}.$$

Let $v : F \to \mathbb{Z}$ denote the p-adic valuation. Given a function ϕ on T_m, we say that $\phi(a) \to 0$ uniformly as $v(a_i) \to \infty$ if there exists $N > 0$ such that $v(a_i) \geq N$ implies $\phi(a) = 0$. Define

$$\mathcal{V}_i := \{\phi \in \mathcal{V} : \phi(a) \to 0 \text{ uniformly as } v(a_i) \to \infty\}.$$

For $i \leq m$ let $M_n(i)$ (resp. $M_m(i)$) denote the standard Levi subgroup $G_i \times G_{n-i}$ (resp. $G_i \times G_{m-i}$), and let $N_n(i)$ (resp. $N_m(i)$ denote its unipotent radical.

Lemma 24. *Let θ_i denote the composition $\mathcal{W} \to \mathcal{V} \to \mathcal{V}/\mathcal{V}_i$. Then the submodule $\mathcal{W}(N_n(i) \times N_m(i), \mathbf{1})$ is contained in $\ker(\theta_i)$.*

Proof. By definition $\mathcal{W}(N_n(i) \times N_m(i), \mathbf{1})$ is the submodule of $\mathcal{W}$ generated by elements $(n, n')\phi - \phi$ where $(n, n') \in N_n(i) \times N_m(i)$ and $\phi \in \mathcal{W}$. If $x \in G_n$ and and $x' \in G_m$ are any unipotent upper triangular matrices, we can apply ψ to (x, x') in $G_n \times G_m$ by embedding in G_{n+m} as usual, so $\psi(x, x') = \psi(x)\psi(x')$. Moreover, by the definition of $\mathcal{W}$, $\phi(xg, x'g') = \psi(x, x')\phi(g, g')$ for $g \in G_n$, $g' \in G_m$.

For $a = \begin{pmatrix} a_1 \cdots a_m & & & \\ & a_2 \cdots a_m & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} \in T_m$, $n = \begin{pmatrix} I_i & y \\ 0 & I_{n-i} \end{pmatrix} \in N_n(i)$, and $n' = \begin{pmatrix} I_i & y' \\ 0 & I_{m-i} \end{pmatrix} \in N_m(i)$, we get

$$\begin{aligned} (n, n')\phi(a, a) &= \phi(ana^{-1}a, an'a^{-1}a) \\ &= \psi(ana^{-1})\psi(an'a^{-1})\phi(a, a) \\ &= \psi \begin{pmatrix} I_i & x \\ 0 & I_{n-i} \end{pmatrix} \psi \begin{pmatrix} I_i & x' \\ 0 & I_{m-i} \end{pmatrix} \phi(a, a) \end{aligned}$$

where x is an $i \times (n-i)$ matrix whose bottom left entry is $a_i y_{i,1}$ and x' is an $i \times (m-i)$ matrix whose bottom left entry is $a_i y'_{i,1}$. Therefore, this expression equals

$$\psi(a_i x_{i,1})\psi(a_i x'_{i,1})\phi(a, a').$$

This shows that for $v(a_i)$ sufficiently large, $(n, n')\phi(a, a) - \phi(a, a)$ equals zero. $\square$

Lemma 25.

$$\frac{\mathcal{W}}{\mathcal{W}(N_n(i) \times N_m(i), \mathbf{1})} \cong J_{M_n(i)}\mathcal{W}(V, \psi) \otimes J_{M_m(i)}\mathcal{W}(V', \psi)$$

Proof. Given an $(A \otimes B)[G_n \times G_m]$-module X such that $N_n(i) \times N_m(i)$ acts trivially on X. Since $N_n(i) \times \{1\}$ and $\{1\} \times N_m(i)$ also act trivially, any $G_n \times G_m$-equivariant map

$$\phi : \mathcal{W} \to X$$

satisfies

$$\begin{aligned} \phi((nW - W) \otimes W') = 0, &\qquad n \in N_n(i), W \in \mathcal{W}(V, \psi), W' \in \mathcal{W}(V', \psi) \\ \phi(W \otimes (n'W' - W')) = 0, &\qquad n' \in N_m(i), W \in \mathcal{W}(V, \psi), W' \in \mathcal{W}(V', \psi). \end{aligned}$$

This shows that ϕ factors through the quotient maps

$$\mathcal{W} \longrightarrow J_{M_n(i)}\mathcal{W}(V) \otimes \mathcal{W}(V') \longrightarrow J_{M_n(i)}\mathcal{W}(V) \otimes J_{M_m(i)}\mathcal{W}(V').$$

It follows that $J_{M_n(i) \times M_m(i)}\mathcal{W}$ and $J_{M_n(i)}\mathcal{W}(V) \otimes J_{M_m(i)}\mathcal{W}(V')$ satisfy the same universal property. □

Hence, we've shown that the map θ_i factors through the Jacquet restriction

$$J_{M_n(i)}\mathcal{W}(V, \psi) \otimes J_{M_m(i)}\mathcal{W}(V', \psi).$$

Let $\rho_i(\varpi)$ denote right translation of a function by the diagonal matrix with ϖ in the first i diagonal entries:

$$\begin{pmatrix} \varpi & & & & & \\ & \ddots & & & & \\ & & \varpi & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Note that if we're considering functions on the torus T_m parametrized as $\prod_{i=1}^{m} F^\times$ as above, this translates to

$$\big(\rho_i(\varpi)\phi\big)(a_1,\dots,a_m) = \phi(a_1,\dots,a_{i-1},a_i\varpi,a_{i+1}\dots,a_m).$$

Lemma 26. *Let V and V' be admissible and G-finite. Let B_i be the R-subalgebra of $\mathrm{End}_R(\mathcal{V}/\mathcal{V}_i)$ generated by $\rho_i(\varpi)$. Then B_i is finitely generated as a module over R.*

Proof. For any i, the operator $\rho_i(\varpi)$ defines a linear endomorphism of the spaces $J_{M_n(i)}\mathcal{W}(V,\psi)$ and $J_{M_m(i)}\mathcal{W}(V',\psi)$, and so acts diagonally on their tensor product. For each i it preserves the kernel of the surjective map

$$J_{M_n(i)}\mathcal{W}(V,\psi)\otimes J_{M_m(i)}\mathcal{W}(V',\psi)\to \mathcal{V}/\mathcal{V}_i$$

so in particular the sub-algebra of $\mathrm{End}_R(\mathcal{V}/\mathcal{V}_i)$ generated by $\rho_i(\varpi)$ equals the subalgebra of $\mathrm{End}_R(J_{M_n(i)}\mathcal{W}(V,\psi)\otimes J_{M_m(i)}\mathcal{W}(V',\psi))$ generated by $\rho_i(\varpi)$.

But we have an injection

$$\mathrm{End}_{A[M_n(i)]}(J_{M_n(i)}\mathcal{W}(V,\psi))\otimes \mathrm{End}_{B[M_m(i)]}(J_{M_m(i)}\mathcal{W}(V',\psi))\hookrightarrow$$
$$\mathrm{End}_{R[M_n(i)\times M_m(i)]}(J_{M_n(i)}\mathcal{W}(V,\psi)\otimes J_{M_m(i)}\mathcal{W}(V',\psi))$$

as R-modules, and the subalgebra B_i we're considering lands inside the smaller space.

Because V is admissible and G-finite, there are at most finitely many primitive orthogonal idempotents e such that $eV\neq 0$. Thus, Proposition 2.4.3

says that $J_{M_n(i)}\mathcal{W}(V,\psi)$ is an admissible $A[M_n(i)]$-module. It is a finite-type $A[M_n(i)]$-module by [BZ77, Prop 3.13(e)], whose proof relies only on the fact that, if $P_n(i)$ is the parabolic subgroup $M_n(i)N_n(i)$, then $P\backslash G$ is compact. Hence we can take a finite set $\{w_i\}$ of $A[M_n(i)]$ generators and a sufficiently small compact open subgroup U of $M_n(i)$ which fixes them all. Any $A[M_n(i)]$-equivariant endomorphism is uniquely determined by its values on $\{w_i\}$. On the other hand, $M_n(i)$-equivariance means such an endomorphism preserves U-invariance, and the U-fixed vectors are finitely generated, therefore it is uniquely determined via A-linearity from a finite set of values. This shows that the algebra $\mathrm{End}_{A[M_n(i)]}(J_{M_n(i)}\mathcal{W}(V,\psi))$ is finitely generated as an A-module, hence its sub-algebra defined by B_i is also finitely generated. The same is true for $J_{M_m(i)}\mathcal{W}(V',\psi)$, hence their tensor product is finitely generated as a module over $A\otimes B$. □

Given $1 \leq j \leq m$, define $\mathcal{V}^j$ (resp. $\mathcal{V}_i^j$) to be the submodule of $C^\infty(T_j,R)$ given by $\{\phi|_{T_j} : \phi \in \mathcal{V}$ (resp. $\phi \in \mathcal{V}_i$)$\}$.

Lemma 27. *There exist monic polynomials $f_1,\dots,f_m$ in $R[X]$, which have unit constant term, such that, for any $j = 1,\dots,m$, $f_1(\rho(\varpi_1))\cdots f_j(\rho(\varpi_j))$ maps $\mathcal{V}^j$ into $\cap_{i\leq j}\mathcal{V}_i^j$.*

Proof. Proving the lemma means showing that, given $W \in \mathcal{V}$, there exist $N_1,\dots,N_j$ sufficiently large that

$$\left(f_1(\rho_1(\varpi))\cdots f_j(\rho_j(\varpi))W\right)(a_1,\dots,a_j) = 0$$

whenever *any* a_i satisfies $v(a_i) > N_i$, for $i \leq j$. The set $\{N_1, \ldots, N_j\}$ must depend only on W.

We proceed by induction on m. If $m = 1$ then $\cap_i \mathcal{V}_i = \mathcal{V}_1$, so this follows directly from the R-module finiteness of $\langle \rho_1(\varpi) \rangle \subset \mathrm{End}_R(\mathcal{V}/\mathcal{V}_i)$. The constant term is a unit because $\rho(\varpi)$ is invertible.

Assume the lemma is true for $m-1$. Fix $W(a_1, \ldots, a_m)$ an element of $\mathcal{V}$. Since $\rho_m(\varpi)$ is an integral element of the ring $\mathrm{End}(\mathcal{V}/\mathcal{V}_m)$ and $\rho_m(\varpi)$ is invertible, there exists a monic polynomial $f_m(X)$ with unit constant term such that $f_m(\rho_m(\varpi)) = 0$ in $\mathrm{End}(\mathcal{V}/\mathcal{V}_m)$, in other words there exists N_m such that

$$\big(f_m(\rho_m(\varpi))W\big)(a_1, \ldots, a_m) = 0$$

whenever $v(a_m) > N_m$.

Now fix $b \in F^\times$ and define $\phi_b : \prod_{i=1}^{m-1} F^\times \to R$ to be the function

$$\phi_b : (a_1, \ldots, a_{m-1}) \mapsto \big(f_m(\rho_m(\varpi))W\big)(a_1, \ldots, a_{m-1}, b).$$

Note that $\phi_b \equiv 0$ when $v(b) > N_m$.

We can apply the induction hypothesis to $\mathcal{V}^{m-1}$ to conclude there exist polynomials $f_1, \ldots, f_{m-1}$ in $R[X]$ satisfying the required conditions, such that for any $\phi \in \mathcal{V}^{m-1}$, there are large integers $N_1(\phi), \ldots, N_{m-1}(\phi)$, depending on ϕ, such that

$$\Big(f_1(\rho_1(\varpi)) \cdots f_{m-1}(\rho_{m-1}(\varpi))\phi\Big)(a_1, \ldots, a_{m-1}) = 0$$

whenever any one of $a_1, \ldots, a_{m-1}$ satisfies $v(a_i) > N_i(\phi)$.

Since ϕ_b is the restriction of a product of Whittaker functions to T_{m-1} by construction, we can apply this specifically to ϕ_b: there exist large integers $N_1(b)$, ..., $N_{m-1}(b)$, depending on b, such that

$$\Big(f_1(\rho_1(\varpi)) \cdots f_{m-1}(\rho_{m-1}(\varpi)) f_m(\rho_m(\varpi))W\Big)(a_1, \ldots, a_{m-1}, b) = 0$$

whenever any one of $a_1, \ldots, a_{m-1}$ satisfies $v(a_i) > N_i(b)$.

We wish to show that we can choose the N_i's independently of b. But, since $\phi_b \equiv 0$ for $v(b) > N_m$, and ϕ_b also vanishes when $v(b) << 0$ by Lemma 23, we have that ϕ_b is only nonzero when b is confined to a compact subset of $F^\times$. In particular, since $f_m(\rho_m(\varpi))W$ is locally constant in each variable (in particular its last variable), there are only finitely many distinct functions ϕ_b as b ranges over this compact set. Thus, the sets $\{N_i(b) : b \in F^\times\}$ are finite for each i and we can choose N_i to be $\max\{N_i(b) : b \in F^\times\}$.

Therefore, we have

$$\Big(f_1(\rho_1(\varpi)) \cdots f_m(\rho_m(\varpi))W\Big)(a_1, \ldots, a_m) = 0$$

whenever $v(a_i) > N_i$ for $i = 1, \ldots, m$, as desired. □

We can now deduce rationality of $\Psi(W, W', X)$ as follows. Slightly abusively, we use the symbols W and W' to denote elements of $\mathcal{V}$, so everything is already restricted to T_m. First we apply $\Psi(-, -, X)$ to both sides of the following equation:

$$f(\rho_1(\varpi)) \cdots f_m(\rho_m(\varpi))(W \otimes W') = W_0,$$

for $W \otimes W' \in \mathcal{V}$ and $W_0 \in \cap_i \mathcal{V}_i$. In particular, $\Psi(W_0, X) \in R[X, X^{-1}]$, so we have a polynomial on the right hand side.

Since the integrands on the left side are functions of T_m, we have the transformation property

$$\Psi(\rho_1(\varpi)^{t_1} \cdots \rho_m(\varpi)^{t_m}(W \otimes W'), X) = X^{t_1+2t_2+\cdots mt_m}\Psi(W, W', X).$$

Now, given the polynomials f_i in Lemma 27, we can define the multivariate polynomial

$$f(X_1, \dots, X_m) := f_1(X_1)f_2(X_2)\cdots f_m(X_m)$$

in $R[X_1, \cdots, X_m]$. Then, we have shown that $\widetilde{f}(X)\Psi(W, W', X) \in R[X, X^{-1}]$ where $\widetilde{f}$ is the image of f in the map

$$R[X_1, \dots, X_m] \to R[X]$$

$$X_i \mapsto X^i.$$

Since $\widetilde{f}$ lies in S, this proves the theorem.

Remark 1. *When $A = B$ we can take the image of the zeta integrals in the map $S_R^{-1}(R[X, X^{-1}]) \to S_A^{-1}(A[X, X^{-1}])$ induced by the map $R \to A$: $a_1 \otimes a_2 \mapsto a_1a_2$ and recover the rationality result that would be desired when both representations live over the same coefficient ring.*

5.2 Functional Equation and Other Properties

As in Chapter 3 we will construct the gamma factor to be what it must in order to satisfy the functional equation for one particular Whittaker function, and then show that the functional equation is satisfied for all Whittaker functions. We will make repeated use of the following Lemma:

Lemma 28. *If A and B are reduced ℓ-torsion free $W(k)$-algebras, then $A \otimes_{W(k)} B$ is also a reduced and ℓ-torsion free $W(k)$-algebra.*

Proof. Being ℓ-torsion-free is equivalent to being flat as a module over $W(k)$. Since the tensor product of two flat modules is again flat, we have that $A \otimes_{W(k)} B$ is ℓ-torsion free.

To show reducedness first observe that a flat $W(k)$-algebra C is reduced if and only if $C \otimes_{W(k)} \mathcal{K}$ is reduced, where $\mathcal{K}$ denotes $\mathrm{Frac}(W(k))$. To see this note that R embeds in the localization $S^{-1}R$ where $S = W(k) \setminus \{0\}$, and thus an element $\frac{r}{s}$ in the localization is nilpotent if and only if r is nilpotent.

Applying this to the flat $W(k)$-algebra $S = A \otimes_{W(k)} B$, it suffices to prove that $(A \otimes_{W(k)} B) \otimes_{W(k)} \mathcal{K}$ is reduced. But this equals

$$(A \otimes_{W(k)} \mathcal{K}) \otimes_{\mathcal{K}} (B \otimes_{W(k)} \mathcal{K}).$$

We can now apply [Bou07, Ch 5, §15, Thm 3] which says that the tensor product of reduced algebras over a characteristic zero field is again reduced. □

Lemma 29. *For V in $\mathrm{Rep}_A(G_n)$ and V' in $\mathrm{Rep}_B(G_m)$ both of Whittaker type, there exist W in $\mathcal{W}(V, \psi)$ and W' in $\mathcal{W}(V', \psi)$ such that $\Psi(W, W', X) = 1$.*

Proof. The proof follows that in [JPSS83, (2.7) p.394]. If M denotes the standard parabolic of size $(m-1,1)$, let $K = GL_m(\mathcal{O}_F)$ the maximal compact subgroup of G_m, let Z_m denote the scalar matrices. Let $P_m^{(r)}$ and $M^{(r)}$ denote the subsets of matrices with determinant having valuation r. Recall that $G_m = MK$ ([BZ77, 3.6 Lemma]), and $M = P_m Z_m$. Therefore, we have

$$\Psi(W, W', X) = \sum_r c_r(W, W') X^r,$$

where

$$c_r(W, W') = \int_K \int_{N_m \backslash M^{(r)}} W\left(\begin{smallmatrix} mk & 0 \\ 0 & I_{n-m} \end{smallmatrix}\right) \otimes W'(mk)\, dm\, dk$$

Given $\phi \in \text{c-Ind}_{N_n}^{P_n} \psi$, and ϕ' in $\text{c-Ind}_{N_m}^{P_m} \psi$, Proposition 2.2.1 tells us we can choose W, W' so that $W|_{P_n} = \phi$ and $W'|_{P_m} = \phi'$. Suppose K' is a compact open subgroup of G_m, with p-power index in K, such that W' is invariant on the right under K'. Take ϕ to be the characteristic function of the subset $P_m^{(0)} K'$ of P_n (modulo N_m). Then if $r = 0$,

$$c_r(W, W') = [K : K'] \int_{N_m \backslash P_m^{(0)}} (1 \otimes \phi'(p))\, dp,$$

and if $r > 0$, $c_r(W, W') = 0$. Since $[K : K']$ is a unit in R, we may choose ϕ' so that $\int_{N_m \backslash P_m^{(0)}} \phi'(p)\, dp$ equals $[K : K']^{-1}$. □

Let $w_{n,m} = \text{diag}(I_{n-m}, w_m)$ where w_m is the antidiagonal $m \times m$ matrix with 1's on the diagonal.

Theorem 5.2.1. *Suppose A and B are Noetherian $W(k)$-algebras, and suppose V, V' are co-Whittaker $A[G_n]$- and $B[G_m]$-modules respectively. Then* *there exists a unique element* $\gamma(V \times V', X, \psi)$ *of* $S^{-1}(R[X, X^{-1}])$ *such that*

$$\Psi(W, W', X; j)\gamma(V \times V', X, \psi)\omega_{V'}(-1)^{n-1}$$
$$= \Psi(w_{n,m}\widetilde{W}, \widetilde{W}', \frac{q^{n-m-1}}{X}; n-m-1-j)$$

for any $W \in \mathcal{W}(V, \psi)$*,* $W' \in \mathcal{W}(V', \psi)$ *and for any* $0 \leq j \leq n-m-1$.

Our notation in this theorem is slightly different from [JPSS83], and follows [CPS10, 2.1 Thm].

Proof. It suffices to prove the theorem for *primitive* co-Whittaker modules by Lemma 30 below.

First, we construct the gamma factor as in Section 3.5, using Lemma 29 in place of Proposition 3.5.1.

Second, we prove the functional equation in the case where A and B are reduced ℓ-torsion free $W(k)$-algebras. Since the zeta integrals $\Psi(W, W', X; j)$ all live in $S^{-1}R[X, X^{-1}]$ we can make sense of both sides of the functional equation. Using Lemma 28, we get that the coefficient ring $A \otimes_{W(k)} B$ is reduced and ℓ-torsion free, hence its minimal primes are are characteristic zero primes and they have trivial intersection. Each characteristic zero point $R \to \kappa$ gives characteristic zero points of A and B. For each such point we can take an algebraic closure $\overline{\kappa}$ and choose an embedding $\mathbb{C} \hookrightarrow \overline{\kappa} \cong$, then apply the arguments of [JPSS83] to $V \otimes \overline{\kappa}$ and $V' \otimes \overline{\kappa}$. In this way, the argument used to prove Theorem 3.7.1 carries over completely to the setting of gamma factors of pairs $\gamma(V \times V', X, \psi)$, where X replaces the variable $q^{-s+\frac{n-m}{2}}$.

Third, we focus on removing the hypothesis that A is reduced and ℓ-torsion free. To do this we, we mimic the arguments of Section 3.8, and consider the action of the Bernstein center on V and V' and how they are dominated by the base-changes of a universal co-Whittaker module.

Let $\mathcal{Z}$ be the center of $\mathrm{Rep}_{W(k)}(G_n)$, let $\mathcal{Z}'$ be the center of $\mathrm{Rep}_{W(k)}(G_m)$. It is proved in [Hel12a] that for primitive idempotents e and e' in $\mathcal{Z}$ and $\mathcal{Z}'$ respectively, $e\mathcal{Z}$ and $e'\mathcal{Z}'$ are reduced and ℓ-torsion free $W(k)$-algebras. Lemma 28 implies that $e\mathcal{Z} \otimes_{W(k)} e'\mathcal{Z}'$ is reduced and ℓ-torsion free, so in particular the hypotheses of the theorem hold for the pair of representations $e\mathfrak{W}_n$ and $e'\mathfrak{W}_m$. We thus define the universal gamma factor $\Gamma(e\mathfrak{W}_n \times e'\mathfrak{W}_m, X, \psi) \in S^{-1}(e\mathcal{Z} \otimes e'\mathcal{Z}')[X, X^{-1}]$.

Now, given primitive co-Whittaker modules V in $e\,\mathrm{Rep}_{W(k)}(G_n)$ and V' in $e'\,\mathrm{Rep}_{W(k)}(G_m)$ over any coefficient rings A and B which are Noetherian $W(k)$-algebras, we have supercuspidal supports $f_V : e\mathcal{Z} \to A$ and $f_{V'} : e'\mathcal{Z}' \to B$ such that $e\mathfrak{W}_n \otimes_{e\mathcal{Z}, f_V} A$ dominates V and $e'\mathfrak{W}_m \otimes_{e'\mathcal{Z}', f_{V'}} B$ dominates V'.

Because the formation of zeta integrals and gamma factors commute with change of base ring, the image of $\Gamma(e\mathfrak{W}_n \times e'\mathfrak{W}_m, X, \psi)$ in the map $S^{-1}(e\mathcal{Z} \otimes e'\mathcal{Z}')[X, X^{-1}] \to S^{-1}R[X, X^{-1}]$ induced by $f_V \otimes f_{V'}$ equals $\gamma(V \times V', X, \psi)$.

Since $e\mathfrak{W}_n \otimes_{e\mathcal{Z}, f_V} A$ dominates V, they have the same Whittaker spaces, and thus share all the same zeta integrals, and the same goes for V'. Therefore, $\gamma(V \times V', X, \psi)$ satisfies the functional equation for all $W \in \mathcal{W}(V, \psi)$ and all

W' in $\mathcal{W}(V',\psi)$. $\square$

Lemma 30. *Suppose e and f (resp. e' and f') are distinct primitive idempotents of $\mathcal{Z}(G_n)$ (resp. $\mathcal{Z}(G_m)$). Then for co-Whittaker modules V, V' as in Theorem 5.2.1 we have*

$$\gamma((eV+fV)\times V',X,\psi)=\gamma(eV\times V',X,\psi)+\gamma(fV\times V',X,\psi)$$
$$\gamma(V\times(e'V'+f'V'),X,\psi)=\gamma(V\times e'V',X,\psi)+\gamma(V\times f'V',X,\psi).$$

Proof. The statement of the lemma is that $\Psi(eW+fW,W',X)$ satisfies a functional equation as in Theorem 5.2.1, with functional constant $\gamma(eV\times V',X,\psi)+\gamma(fV\times V',X,\psi)$, and similarly for $\Psi(W,e'W+f'W',X)$, and that this functional constant determines a unique element of $S^{-1}(R[X,X^{-1}])$. Uniqueness follows from uniqueness at each component.

$$\begin{aligned}
&\Psi(eW+fW,W'X;j)\big(\gamma(eV\times V',X,\psi)+\gamma(fV\times V',X,\psi)\big)\omega_{V'}(-1)^{n-1}\\
&=\Psi(eW,W',X;j)\big(\gamma(eV\times V',X,\psi)+\gamma(fV\times V',X,\psi)\big)\omega_{V'}(-1)^{n-1}\\
&\quad+\Psi(fW,W',X;j)\big(\gamma(eV\times V',X,\psi)+\gamma(fV\times V',X,\psi)\big)\omega_{V'}(-1)^{n-1}\\
&=\Psi(eW,W,X;j)\gamma(eV\times V',X,\psi)\omega_{V'}(-1)^{n-1}\\
&\quad+\Psi(fW,W'X;j)\gamma(fV\times V',X,\psi)\omega_{V'}(-1)^{n-1}\\
&=\Psi(w_{n,m}\widetilde{eW},\widetilde{W'},q^{n-m-1}/X;n-m-1-j)\\
&\quad+\Psi(w_{n,m}\widetilde{fW},\widetilde{W'},q^{n-m-1}/X;n-m-1-j)\\
&=\Psi(w_{n,m}f\widetilde{W+e}W,\widetilde{W'},q^{n-m-1}/X;n-m-1-j),
\end{aligned}$$

where the second equality is because $ef=0$ in A (identifying e, f with their images $f_V(e)$, $f_V(f)$ in A), so the power series $\Psi(eW,W',X)\gamma(fV\times V',X,\psi)$

and $\Psi(fW, W', X)\gamma(eV \times V', X, \psi)$ must be zero. A similar argument shows additivity in the second factor.

Letting W and W' be as in Lemma 29, we have $\Psi(eW + fW, W', X) = e + f$, hence

$$\begin{aligned} &e\gamma(eV \times V', X, \psi) + f\gamma(fV \times V', X, \psi) \\ &\quad = \gamma(eV \times V', X, \psi) + \gamma(fV \times V', X, \psi) \\ &\quad = \Psi(w_{n,m}f\widetilde{W + e}W, \widetilde{W'}, q^{n-m-1}/X; n-m-1-j), \end{aligned}$$

which lives in $S^{-1}(R[X, X^{-1}])$. □

Corollary 5.2.2. *For $V \in \mathrm{Rep}_A(G_n)$ and $V' \in \mathrm{Rep}_B(G_m)$ co-Whittaker modules and A, B any Noetherian $W(k)$-algebras, $\gamma(V, V', X)$ is a unit in $S^{-1}(R[X, X^{-1}])$ and*

$$\gamma(V \times V', X, \psi)^{-1} = \gamma(V^\iota \times (V')^\iota, \frac{q^{n-m-1}}{X}, \psi^{-1}).$$

Proof. Let W and W' be the Whittaker functions guaranteed by Lemma 29. The original functional equation reads

$$\Psi(W, W', X)\gamma(V_i \times V', X, \psi)\omega_\tau(-1)^t = \Psi(\widetilde{W}, \widetilde{W'}, \frac{q^{n-m-1}}{X}).$$

Replacing X with $\frac{q^{n-m-1}}{X}$ we have

$$\Psi(W, W', \frac{q^{n-m-1}}{X})\gamma(V_i \times V', \frac{q^{n-m-1}}{X}, \psi)\omega_\tau(-1)^t = \Psi(\widetilde{W}, \widetilde{W'}, X).$$

Now multiplying through by $\gamma(V_i^\iota \times (V')^\iota, X, \psi^{-1})\omega_{(V')^\iota}(-1)^t$ and noticing that $\omega_{(V')^\iota} = \omega_{V'}^{-1}$ we get:

$$\Psi(W, W', \frac{q^{n-m-1}}{X})\gamma(V_i \times V', \frac{q^{n-m-1}}{X}, \psi)\gamma(V_i^\iota \times (V')^\iota, X, \psi^{-1})$$
$$= \Psi(W, W', \frac{q^{n-m-1}}{X}),$$

By Lemma 29 we have $\gamma(V_i \times V', \frac{q^{n-m-1}}{X}, \psi)\gamma(V_i^\iota \times (V')^\iota, X, \psi^{-1}) = 1.$ □

Chapter 6

A Converse Theorem for $GL(n) \times GL(n-1)$

6.1 Statement of the Theorem

Recall that for a co-Whittaker module V, the supercuspidal support of V is by definition the map $f_V : \mathcal{Z} \to \mathrm{End}_G(V) \to A$. The main result of this chapter is that the collection of gamma factors of pairs uniquely determines the supercuspidal support of a co-Whittaker family.

Theorem 6.1.1. *Let A be a finite-type $W(k)$-algebra which is reduced and ℓ-torsion free, and let $\mathcal{K} = \mathrm{Frac}(W(k))$. Suppose V_1 and V_2 are two co-Whittaker $A[G_n]$-modules. There is a finite extension $\mathcal{K}'$ of $\mathcal{K}$ with ring of integers $\mathcal{O}$ such that, if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all co-Whittaker $\mathcal{O}[G_{n-1}]$-modules V', then $f_{V_1} = f_{V_2}$.*

Remark 2.

1. *Because of the control achieved in Theorem 6.3.1, it suffices to take in the statement of Theorem 6.1.1 only those co-Whittaker modules V' such that $V' \otimes_{\mathcal{O}} \mathcal{K}'$ is absolutely irreducible.*
2. *The equality of gamma factors implies that V_1 and V_2 must live in the same block of the category $\mathrm{Rep}_{W(k)}(GL_n(F))$. The finite extension $\mathcal{K}'$ depends only on this block.*

6.2 Supercuspidal Support and Whittaker Models

In this section we investigate the connection between Whittaker spaces and supercuspidal support.

Lemma 31. *Suppose V_1 and V_2 are co-Whittaker modules. Then $\mathcal{W}(V_1, \psi) = \mathcal{W}(V_2, \psi)$ if and only if $f_{V_1} \equiv f_{V_2}$.*

Proof. It follows from Lemma 32 below that $f_{V_1} = f_{\mathcal{W}(V_1,\psi)} = f_{\mathcal{W}(V_2,\psi)} = f_{V_2}$. □

Lemma 32. *Suppose we have two co-Whittaker modules V_1 and V_2 such that V_1 dominates V_2. Then $f_{V_1} \equiv f_{V_2}$.*

Proof. Suppose $\phi : V_1 \to V_2$ is the dominance map. Choosing a cyclic $A[G]$-generator $v_1 \in V_1$, then $\phi(v_1)$ is an $A[G]$-generator of V_2 since its image in $V_2^{(n)}$ is a generator. Denote by v_1' the image of v_1 in $V_1 \to V_1^{(n)}$. We have v_1' generates $V_1^{(n)}$ and $\phi^{(n)}(v_1')$ generates $V_2^{(n)}$.

If z is an element of $\mathcal{Z}$, then $z_{V_1} \in \mathrm{End}_G(V_1)$ sends v_1 to $f_{V_1}(z)v_1$, where $f_{V_1}(z) \in A$. By definition, the action of the Bernstein center is functorial, hence commutes with the morphism ϕ, thus

$$z_{V_2}(\phi(v_1)) = \phi(f_{V_1}(z)v_1) = f_{V_1}(z)\phi(v_1).$$

Since $\phi(v_1)$ is an $A[G]$-generator of V_2, z_{V_2} is completely determined by where it sends $\phi(v_1)$. This shows that the map

$$f_{V_2} : \mathcal{Z} \to \mathrm{End}_G(V_2) \to A$$

given by $z \mapsto z_{V_2} \mapsto f_{V_2}(z)$ exactly equals the map f_{V_1}.

Second proof: use Lemma 33 below. $\square$

Lemma 33. *If V is a co-Whittaker module with supercuspidal support $f_V : \mathcal{Z} \to \mathrm{End}_G(V) \to A$, then the map f_V equals its "derivative" $f_V^{(n)} : \mathcal{Z} \to \mathrm{End}_A(V^{(n)}) \to A$ given by $z \mapsto z_V^{(n)} \mapsto A$.*

Proof. By definition, given $z \in \mathcal{Z}$ the endomorphism z_V is translation by the scalar $f_V(z)$. The derivative morphism $z_V^{(n)} : V^{(n)} \to V^{(n)}$ is translation by that same scalar. The map $\mathrm{End}_A(V^{(n)}) \to A$ is given by choosing a generator (it is free of rank one) and looking at the translation that an endomorphism defines. $\square$

Remark 3. *Any nonzero G-equivariant homomorphism between co-Whittaker modules which preserves the top derivative is a surjection.*

6.3 Proof of Converse Theorem

For two $W(k)$-algebras A,B, $\phi_1 \in \text{c-Ind}_N^G \psi_A$ and $\phi_2 \in \mathrm{Ind}_N^G \psi_B^{-1}$ we denote by $\langle \phi_1, \phi_2 \rangle$ the element

$$\int_{N\backslash G} \phi_1(x) \otimes \phi_2(x) dx \ \in A \otimes_{W(k)} B$$

and let $\mathcal{K} = \mathrm{Frac}\, W(k)$. At the heart of the proof of the converse theorem will lie the following result, which is proved in §6.4

Theorem 6.3.1. *Suppose A is a finite-type, reduced, ℓ-torsion free $W(k)$-algebra. Suppose $H \neq 0$ is an element of* c-Ind ψ_A*. Then there exists a finite* *extension $\mathcal{K}'$ of $\mathcal{K}$ with ring of integers $\mathcal{O}$ and an absolutely irreducible generic integral $\mathcal{K}'$ representation U' with integral structure U, such that there is a Whittaker function $W \in \mathcal{W}(U^\vee, \psi_{\mathcal{O}}^{-1})$ satisfying $\langle H, W\rangle \neq 0$ in $A \otimes_{W(k)} \mathcal{O}$.*

The rest of this section is devoted to proving Theorem 6.1.1, assuming Theorem 6.3.1. Let V_1 and V_2 be co-Whittaker with G-homomorphisms $\omega_i : V_i \to \operatorname{Ind}_N^G \psi$. Let $\mathcal{S}(V_i)$ denote the sub-$A[P_n]$-module of V_i consisting of Schwartz functions of V_i.

Lemma 34. *Consider the sub-$A[P_n]$-modules $\omega_i(\mathcal{S}(V_i))$ of $\operatorname{Ind}_N^G \psi$. If $r_P : \operatorname{Ind}_N^G \psi \to \operatorname{Ind}_N^P \psi$ denotes the map given by restriction of functions, then $r_P(\omega_1(\mathcal{S}(V_1))) = r_P(\omega_2(\mathcal{S}(V_2)))$.*

Proof. Let $\omega_{i,P}$ be the maps $V_i|_P \to \operatorname{Ind}_N^P \psi$ guaranteed by genericity. Then we have $r_P \circ \omega_i = \omega_{i,P}$ from the definitions.

By Proposition 2.2.1 (2), we have $\omega_{1,P}(\mathcal{S}(V_1)) = \omega_{2,P}(\mathcal{S}(V_2)) = \text{c-}\operatorname{Ind}_N^P \psi$ as subsets of $\operatorname{Ind}_N^P \psi$. This proves the claim. $\square$

Proposition 6.3.2. *Suppose the gamma factors are equal as in Theorem 6.1.1. Take $W_1 \in \omega_1(\mathcal{S}(V_1))$ and $W_2 \in \omega_2(\mathcal{S}(V_2))$ such that $r_P(W_1) = r_P(W_2)$, then $W_1 = W_2$ as elements of $\operatorname{Ind}_N^G \psi$.*

Proof. The proof follows [Hen93].

Let $\mathfrak{S}$ be the subspace of $\mathcal{W}(V_1, \psi) \times \mathcal{W}(V_2, \psi)$ consisting of pairs (W_1, W_2) such that $r_{G_m}(W_1) = r_{G_m}(W_2)$, where r_{G_m} denotes restriction to the subgroup G_m of G_n (with $m = n-1$). By the preceeding discussion this is nonempty. Let $(W_1, W_2) \in \mathfrak{S}$. Then

$$\Psi(W_1, W', X) = \Psi(W_2, W', X)$$

for all $W' \in \mathcal{W}(V', \psi_{\mathcal{O}}^{-1})$ as V' varies over all co-Whittaker $\mathcal{O}[G_{r-1}]$-modules.

By assumption, $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all such V', whence the equality of the products:

$$\Psi(W_1, W', X)\gamma(V_1 \times V', X, \psi) = \Psi(W_2, W', X)\gamma(V_2 \times V', X, \psi).$$

Applying the functional equation with $j = 0$ and $m = n-1$ we thus conclude that

$$\Psi(\widetilde{W_1}, \widetilde{W'}, \frac{q^{n-m-1}}{X}) = \Psi(\widetilde{W_2}, \widetilde{W'}, \frac{q^{n-m-1}}{X}),$$

and furthermore

$$\Psi(\widetilde{W_1}, \widetilde{W'}, X) = \Psi(\widetilde{W_2}, \widetilde{W'}, X).$$

For each integer m, denote by H_m the function on G_m given by

$$H_m(g) = 0 \quad \text{if} \quad v_F(\det g) \neq m$$
$$H_m(g) = \widetilde{W_1}\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right) - \widetilde{W_2}\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{if} \quad v_F(\det g) = m$$

Then the equality of formal Laurent series

$$\Psi(\widetilde{W_1}, \widetilde{W'}, X) = \Psi(\widetilde{W_2}, \widetilde{W'}, X)$$

implies that, for each m, we have

$$\int_{N_m \backslash G_m} H_m(g) \otimes \widetilde{W'}(g) dg = 0$$

for all W' in the Whittaker spaces $\mathcal{W}(V', \psi_{\mathcal{O}})$ of all co-Whittaker $\mathcal{O}[G]$-modules V'.

Now suppose V' has the property that $V' \to V' \otimes_{\mathcal{O}} \mathcal{K}'$ is an embedding and $V' \otimes \mathcal{K}'$ is absolutely irreducible. Then $(V')^{\vee} \otimes \mathcal{K}' \cong (V' \otimes \mathcal{K}')^{\vee}$, and by [BZ76, Thm 7.3], $(V' \otimes \mathcal{K}')^{\vee} \cong (V' \otimes \mathcal{K}')^{\iota}$, where $(-)^{\iota}$ means pre-composing the G action with $g \mapsto^t g^{-1}$. Thus $\mathcal{W}((V' \otimes \mathcal{K}')^{\vee}, \psi_{\mathcal{K}'}^{-1}) = \mathcal{W}((V' \otimes \mathcal{K}')^{\iota}, \psi_{\mathcal{K}'}^{-1})$, so given $W^{\vee} \in \mathcal{W}((V')^{\vee}, \psi_{\mathcal{O}}^{-1})$, there is an integer s such that $\varpi^s W^{\vee}$ is given by an element $\widetilde{W}$ in $\mathcal{W}((V')^{\iota}, \psi_{\mathcal{O}}^{-1})$. Therefore

$$\varpi^s \langle H_m, W^{\vee} \rangle = \langle H_m, \varpi^s W^{\vee} \rangle = \langle H_m, \widetilde{W} \rangle = 0,$$

which implies $\langle H_m, W^{\vee} \rangle = 0$ since $A \otimes_{W(k)} \mathcal{O}$ is flat over $\mathcal{O}$ (i.e. ϖ-torsion free).

Therefore we can apply the contrapositive of Theorem 6.3.1 to conclude that each H_m is identically zero, for all m. Hence

$$\widetilde{W_1} \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix} \right) \equiv \widetilde{W_2} \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix} \right).$$

Let $\widetilde{\mathfrak{S}}$ be the subspace of $\mathcal{W}(V_1^{\iota}, \psi^{-1}) \times \mathcal{W}(V_2^{\iota}, \psi^{-1})$ consisting of pairs (U_1, U_2) whose restrictions to $G_m \subset G_n$ are equal. Then we have shown that $(\widetilde{W_1}, \widetilde{W_2}) \in \widetilde{\mathfrak{S}}$. In fact, the following result is true:

Lemma 35. *Let W_1 be in $\mathcal{W}(V_1, \psi)$ and W_2 be in $\mathcal{W}(V_2, \psi)$. Then (W_1, W_2) is in $\mathfrak{S}$ if and only if $(\widetilde{W_1}, \widetilde{W_2})$ is in $\widetilde{\mathfrak{S}}$.*

Proof of Lemma 35. We have just proved one direction. By Lemma 5.2.2, our hypothesis on the equality of gamma factors is equivalent to the equality of the gamma factors

$$\gamma(V_1^\iota \times (V')^\iota, X, \psi^{-1}) = \gamma(V_2^\iota \times (V')^\iota, X, \psi^{-1})$$

for all $(V')^\iota$. Since $(-)^\iota$ is an exact covariant functor which is additive in direct sums, commutes with base-change, and induces an isomorphism between Whittaker spaces, $V \mapsto V^\iota$ preserves the property of being co-Whittaker and V^ι, $(V')^\iota$ are again co-Whittaker. Thus the entire argument works replacing V_i with V_i^ι and V' with $(V')^\iota$ to get the converse implication. □

We now continue with the proof of Proposition 6.3.2. If we let G_n act diagonally on $\mathcal{W}(V_1, \psi) \times \mathcal{W}(V_2, \psi)$ and on $\mathcal{W}(V_2^\iota, \psi^{-1}) \times \mathcal{W}(V_2^\iota, \psi^{-1})$, both by right translation, then the subgroup P_n stabilizes the subspaces $\mathfrak{S}$ and $\widetilde{\mathfrak{S}}$. To see this note that for $g \in G_m$ and $u \in U_n$ we have $W_i(gu) = W_i(gug^{-1}g) = \psi^g(u)W_i(g)$, so uW_i's restriction to G_m is completely determined.

If ρ denotes right translation, a short calculation shows

$$\rho(g^\iota)\widetilde{W}(x) = \widetilde{gW}(x).$$

Combining this with the lemma above, it follows that $\mathfrak{S}$ is stable under ${}^tP$ as well.

Hence $\mathfrak{S}$ is stable under the group generated by P and ${}^tP$. But this group contains all elementary matrices, hence contains all of $SL_n(F)$. On the other hand, this group also contains matrices of any determinant. Hence for any $a \in F^\times$ it contains all matrices in $GL_n(F)$ with determinant a; in other words this group equals G.

Therefore $\mathfrak{S}$ is stable under the action of all of G_n. Thus given W_1 and W_2 such that $r_P(W_1) = r_P(W_2)$ we have that $r_P(gW_1) = r_P(gW_2)$ for any $g \in G_n$ so we have $gW_1(1) = gW_2(1)$, i.e. $W_1(g) = W_2(g)$ for all $g \in G_n$. This concludes the proof of Proposition 6.3.2. □

Corollary 6.3.3. *If the gamma factors are equal as in Theorem 6.1.1,*

$$\omega_1(\mathcal{S}(V_1)) = \omega_2(\mathcal{S}(V_2)).$$

Proof. Given W_1 in the left side, there exists W_2 such that $r_P(W_1) = r_P(W_2)$. The previous lemma then implies $W_1 = W_2 \in \omega_2(\mathcal{S}(V_2))$ which shows one containment. The argument to show the opposite containment is identical. □

Corollary 6.3.4. *Suppose the gamma factors are equal as in Theorem 6.1.1, then $\mathcal{W}(V_1, \psi) = \mathcal{W}(V_2, \psi)$.*

Proof. Since V_i is co-Whittaker and surjects onto $\mathcal{W}(V_i, \psi)$, we have that $\mathcal{W}(V_i, \psi)$ is also co-Whittaker. In particular, $\mathcal{W}(V_i, \psi)$ is generated over $A[G]$ by the $A[P]$ submodule consisting of its Schwartz functions, which is the same as $\omega_i(\mathcal{S}(V_i))$. But if the gamma factors are equal we have shown that $\omega_1(\mathcal{S}(V_1)) = \omega_2(\mathcal{S}(V_2))$ and hence this lives inside $\mathcal{W}(V_1, \psi) \cap \mathcal{W}(V_2, \psi)$, the intersection taken within $\mathrm{Ind}_N^G \psi$. Hence $\mathcal{W}(V_1, \psi)$ and $\mathcal{W}(V_2, \psi)$ contain a common $A[G]$-module generating set, hence are equal. □

Combining Corollary 6.3.4 and Lemma 31 gives Theorem 6.1.1.

6.4 Proof of The Vanishing Theorem

This section is devoted to the proof of Theorem 6.3.1. Let $\psi : N \to W(k)^\times$ be an additive character, and let $\mathcal{Z}$ denote the center of $\mathrm{Rep}_{W(k)}(G_n)$. Denote ψ_A by $\psi \otimes_{W(k)} A$, then $\text{c-Ind}_N^G \psi_A \cong (\text{c-Ind}_N^G \psi) \otimes_{W(k)} A$.

There exists a primitive idempotent e in $\mathcal{Z}$ such that $eH \neq 0$. Moreover, there is some compact open subgroup K such that $e_K eH = eH$, where e_K is the projector $V \to V^K$. Letting $e' = e * e_K * e$, we have $e'H = eH \neq 0$.

Let $R := e\mathcal{Z} \otimes_{W(k)} A$. The $W(k)$-module

$$e'(\text{c-Ind}\, \psi_A) \cong e'(\text{c-Ind}\, \psi) \otimes_{W(k)} A$$

carries the structure of an R-module, by considering it as an external tensor product. For convenience denote the R-module $e'(\text{c-Ind}\, \psi) \otimes_{W(k)} A$ by $\mathfrak{M}$.

Lemma 36. *$\mathfrak{M}$ is finitely generated and torsion-free as an R-module. In particular, $\mathfrak{M}$ embeds in a free R-module.*

Proof. Since $e(\text{c-Ind}\, \psi)$ is admissible as an $e\mathcal{Z}$-module ([Hel12b]), $\mathfrak{M}$ is finitely generated as an R-module.

Next, note that $e'(\text{c-Ind}\,\psi)$ is torsion-free as an $e\mathcal{Z}$-module. This follows from its torsion-free-ness at characteristic zero primes. Since A and $e\mathcal{Z}$ are both reduced and flat over $W(k)$, the ring R is reduced and flat over $W(k)$. Now, a module over a reduced ring is torsion-free if and only if it can be embedded in a free module [Wie92, 1.5,1.7]. Thus we focus on showing that $\mathfrak{M}$ can be embedded in a free R-module.

Since $e(\text{c-Ind}\,\psi)$ is torsion-free over $e\mathcal{Z}$ there is an embedding of $e\mathcal{Z}$-modules

$$e'(\text{c-Ind}\,\psi) \to (e\mathcal{Z})^r$$

for some r. Since $W(k) \to A$ is flat, $e\mathcal{Z} \to R$ is flat, since flatness is preserved under base-change. Now tensor this embedding with R to get a map of R-modules

$$\mathfrak{M} \cong e'(\text{c-Ind}\,\psi) \otimes_{e\mathcal{Z}} R \to (e\mathcal{Z})^r \otimes_{e\mathcal{Z}} R \cong R^r,$$

where the first isomorphism is the canonical one

$$e'(\text{c-Ind}\,\psi) \otimes_{W(k)} A \cong \left(e'(\text{c-Ind}\,\psi) \otimes_{e\mathcal{Z}} e\mathcal{Z}\right) \otimes_{W(k)} A \cong e'(\text{c-Ind}\,\psi) \otimes_{e\mathcal{Z}} R.$$

But since flatness is preserved by base change, A being flat over $W(k)$ implies R flat over $e\mathcal{Z}$. Hence, the map $\mathfrak{M} \to R^r$ is an embedding, so $\mathfrak{M}$ is torsion-free over R. □

Lemma 37. *The set*

$$\{\mathfrak{q} \in \mathrm{Spec}(R) : eH \in \mathfrak{q}\mathfrak{M}\}$$

is contained in a closed subset V of $\operatorname{Spec}(R)$ *such that* $V \neq \operatorname{Spec}(R)$. *Moreover, this closed subset does not contain the generic fiber* $\{\mathfrak{q} \in \operatorname{Spec}(R) : \ell \notin \mathfrak{q}\}$.

Proof. From Lemma 36, there is an embedding $\mathfrak{M} \subset R^r$, so $\mathfrak{q}\mathfrak{M} \subset \mathfrak{q}^r$. Thus if $eH = (h_1, \ldots, h_n)$ is in $\mathfrak{q}\mathfrak{M}$, each h_i is in $\mathfrak{q}$. Hence $\mathfrak{q}$ is in the closed set $V := V(h_1) \cap \cdots \cap V(h_n)$. But $V \neq \operatorname{Spec}(R)$ because some h_i is nonzero (so there is some minimal prime not containing h_i, by reducedness). □

Thus there is some nonempty open subset $D \subset \operatorname{Spec}(R)$ in the generic fiber consisting of points $\mathfrak{q}$ such that $eH \notin \mathfrak{q}\mathfrak{M}$.

Lemma 38. *Let K be an infinite field and let B be any infinite subset of K. Then the set of points $(X_1 - b_1, \ldots, X_n - b_n)$ such that $b_i \in B$ is dense in* $\operatorname{Spec}(K[X_1, \ldots, X_n])$.

Proof. We proceed by induction on n. If $n = 1$, we can show that every principal open subset intersects the set of points $\{(X - b)\}$. If $f \in K[X]$ were nonzero, then f could not be divisible by $(X - b)$ for infinitely many b, whence there are points $(X - b)$ in $D(f)$.

Suppose the result holds for $n-1$. We denote by S the subset of points $(X_1 - b_1, \ldots, X_n - b_n)$, and choose an arbitrary f nonzero in $K[X_1, ..., X_n]$ and consider $V = V(f)$ the set of prime ideals containing f. It suffices to show that S cannot be contained in V. Consider the map $K[X_1, ..., X_n] \to K[X_1, ..., X_{n-1}]$ given by $X_n \mapsto b$ for some $b \in B$. This gives the closed immersion $H \to \mathbb{A}^n_K$ of the hyperplane $H := \{X_n = b\}$. By the induction hypothesis the subset T of points $(X_1 - b_1, ..., X_{n-1} - b_{n-1}, X_n - b)$ is dense in H. Suppose V contains S, then $V \cap H \supset S \cap H \supset T$, meaning $V \cap H = H$. Since b was arbitrary we've shown that V contains every one of the distinct hyperplanes $\{X_n = b\}$ for $b \in B$. In particular this means each $X_n - b$ divides f, which is impossible. □

Proposition 6.4.1. *Let $\mathcal{K}$ be $\operatorname{Frac} W(k)$ and e be a primitive idempotent of $\mathcal{Z}$. There is a finite extension $\mathcal{K} \subset \mathcal{K}'$, depending only on e, with rings of integers $\mathcal{O}'$ such that the set of points $\mathfrak{p} = \ker(e\mathcal{Z} \xrightarrow{f} \mathcal{O}')$ for some map $f : e\mathcal{Z} \to \mathcal{O}'$ is dense in $\operatorname{Spec}(e\mathcal{Z})[\frac{1}{\ell}]$.*

Proof. First, note that the proof in Lemma 38 carries over for polynomial rings with any number of the variables X_i inverted.

By [Hel12a, Prop 11.1], $e\mathcal{Z} \otimes_{W(k)} \overline{\mathcal{K}} \cong \prod_{M,\pi'} \mathcal{Z}_{\overline{\mathcal{K}},M,\pi'}$ where $\mathcal{Z}_{\overline{\mathcal{K}},M,\pi'}$ denotes the center of $\operatorname{Rep}_{\overline{\mathcal{K}}}(G)_{M,\pi'}$. From [BD84] we know

$$\mathcal{Z}_{\overline{\mathcal{K}},M,\pi'} \cong (\overline{\mathcal{K}}[M/M^\circ]^H)^{W(\pi')}.$$

Thus there exists a complete system primitive orthogonal idempotents $\{f_{M,\pi'}\}$ summing to 1 in $e\mathcal{Z} \otimes_{W(k)} \overline{\mathcal{K}}$, such that

$$f_{M,\pi'} e\mathcal{Z} \otimes_{W(k)} \overline{\mathcal{K}} \cong (\overline{\mathcal{K}}[M/M^\circ]^H)^{W(\pi')}.$$

But $f_{M,\pi'}$ lives in $e\mathcal{Z} \otimes \mathcal{K}_i$ for some finite extension $\mathcal{K}_i$ of $\mathcal{K}$, so there is a natural map $f_{M,\pi'} e\mathcal{Z} \otimes_{W(k)} \mathcal{K}_i \to (\mathcal{K}_i[M/M^\circ]^H)^{W(\pi')}$ of $\mathcal{K}_i$-algebras (this natural map is described, for example, in [Ber93, p. 74 Rmk]). When tensored over $\mathcal{K}_i$

with $\overline{\mathcal{K}}$, this map becomes an isomorphism. This implies $f_{M,\pi'}e\mathcal{Z}\otimes_{W(k)}\mathcal{K}_i \to (\mathcal{K}_i[M/M^\circ]^H)^{W(\pi')}$ is an isomorphism. Since $e\mathcal{Z}[\frac{1}{\ell}] \to e\mathcal{Z}\otimes_{W(k)}\mathcal{K}_i$ is faithfully flat, we have a finite list $\mathcal{K}_1,\dots,\mathcal{K}_s$ of finite extensions of $\mathcal{K}$ such that there is a continuous surjection $\bigsqcup_i \mathrm{Spec}(\mathcal{K}_i[X_1^{\pm 1},\dots,X_n^{\pm 1}]) \to \mathrm{Spec}(e\mathcal{Z}[\frac{1}{\ell}])$.

Lemma 38 tells us that, for each i, the set of primes $(X_1-b_1,\dots,X_n-b_n)$ for $b_i\in\mathcal{O}_i^\times$ is dense in $\mathrm{Spec}(\mathcal{K}_i[X_1^{\pm 1},\dots,X_n^{\pm 1}])$. Such a map is the base change to $\mathcal{K}_i$ of a surjective map $\mathcal{O}_i[X_1^{\pm 1},\dots,X_n^{\pm 1}]\to\mathcal{O}_i$. In other words the set of prime ideals $\mathfrak{p}_{f_i}$ occurring as the kernel of a map $f_i:\mathcal{O}_i[X_1^{\pm 1},\dots,X_n^{\pm 1}]\to\mathcal{O}_i$ is dense in the generic fiber of $\mathrm{Spec}(\mathcal{O}_i[X_1^{\pm 1},\dots,X_n^{\pm 1}])$.

Thus for each i the set of $\mathfrak{p}_{f_i}$ is dense in the generic fiber of each component of the disjoint union. Since the image of a dense set under a surjective continuous map is dense, we have a dense set of points in the generic fiber of $\mathrm{Spec}(e\mathcal{Z})$, each of which is valued in $\mathcal{O}_i$. Let $\mathcal{K}'$ be the smallest extension of $\mathcal{K}$ containing all the extensions $\mathcal{K}_i$, and let $\mathcal{O}'$ be its ring of integers. Our dense set is contained in the set of $\mathcal{O}'$-valued points, since each $\mathcal{O}_i$ embeds in $\mathcal{O}'$. $\square$

Since the algebra $W(k)\to A$ is flat and finite type, the natural map $e\mathcal{Z}\to R$ is flat and finite type. Let $\phi:\mathrm{Spec}(R)\to\mathrm{Spec}(e\mathcal{Z})$ be the map of spectra induced by $e\mathcal{Z}\to R$. Since these rings are Noetherian, ϕ is open, so $\phi(D)$ forms an open subset of $\mathrm{Spec}(e\mathcal{Z})$. Moreover, since D intersects the generic fiber of $\mathrm{Spec}(R)$, $\phi(D)$ intersects the generic fiber of $\mathrm{Spec}(e\mathcal{Z})$, and therefore contains a generic point of $\mathrm{Spec}(e\mathcal{Z})$. By definition, all points $\mathfrak{p}$ in $\phi(D)$ satisfy $eH\notin\mathfrak{p}\mathfrak{M}$.

By Proposition 4.2.1, we thus have an open neighborhood of this generic point consisting of points $\mathfrak{p} \in \mathrm{Spec}(e\mathcal{Z})$ such that $eH \notin \mathfrak{p}\mathfrak{M}$ and $e(\text{c-Ind}\,\psi)\otimes_{e\mathcal{Z}} \kappa(\mathfrak{p})$ is absolutely irreducible.

Let $\mathcal{O}'$ be the complete DVR, which is finite over $W(k)$, appearing in the conclusion of Proposition 6.4.1. Proposition 6.4.1 now allows us to conclude there exists an $\mathcal{O}'$-valued point $f : e\mathcal{Z} \to \mathcal{O}'$ with $\mathfrak{p} := \ker(f) \in \mathrm{Spec}(e\mathcal{Z})$ satisfying:

1. $eH \notin \mathfrak{p}\mathfrak{M}$
2. The fiber $e(\text{c-Ind}\,\psi) \otimes_{e\mathcal{Z}} \kappa(\mathfrak{p})$ is absolutely irreducible.

We will now use this point $\mathfrak{p}$ to construct a Whittaker function as in Theorem 6.3.1.

Define $\mathcal{O} := e\mathcal{Z}/\mathfrak{p} \subset \mathcal{O}'$. The ring $\mathcal{O}$ is an ℓ-torsion free $W(k)$-algebra which is an integral domain, occuring as an intermediate extension $W(k) \subset \mathcal{O} \subset \mathcal{O}'$. Since $W(k) \subset \mathcal{O}'$ is a finite extension of complete DVR's, $\mathcal{O}$ is a complete DVR, finite over $W(k)$. Let $g : e\mathcal{Z} \to \mathcal{O}$ be the surjective map given by reduction modulo $\mathfrak{p}$. Consider the map p given by

$$p : e(\text{c-Ind}\,\psi) \longrightarrow \frac{e\,\text{c-Ind}\,\psi}{\mathfrak{p}(e\,\text{c-Ind}\,\psi)}.$$

Denote by p_A the map

$$p_A : e(\text{c-Ind}\,\psi_A) \longrightarrow \frac{e\,\text{c-Ind}\,\psi_A}{\mathfrak{p}(e\,\text{c-Ind}\,\psi_A)},$$

then we have $p_A(eH) \neq 0$ by construction.

Let $A' = \mathcal{O} \otimes_{W(k)} A$ and let $g_A : R \to A'$ be the base change to A of $g : e\mathcal{Z} \to \mathcal{O}$. Define

$$\begin{aligned} U :=& e(\text{c-Ind}\,\psi) \otimes_{e\mathcal{Z},g} \mathcal{O} = \frac{e\,\text{c-Ind}\,\psi}{\mathfrak{p}(e\,\text{c-Ind}\,\psi)} \in \text{Rep}_{\mathcal{O}}(G) \\ U_A :=& e(\text{c-Ind}\,\psi_A) \otimes_{R,g_A} A' = \frac{e\,\text{c-Ind}\,\psi_A}{\mathfrak{p}(e\,\text{c-Ind}\,\psi_A)} \in \text{Rep}_{A'}(G). \end{aligned}$$

Note that $U_A = U \otimes_{W(k)} A$. Let $U_A^\vee$ be the smooth A'-linear dual of U_A and $U^\vee$ be the smooth $\mathcal{O}$-linear dual of U.

Since $p_A(eH) \neq 0$ we can choose $v_A^\vee \in U_A^\vee$ such that $\langle v_A^\vee, p_A(eH)\rangle \neq 0$ in A'.

In [Hel12a], Helm gives a decomposition of $\text{Rep}_{W(k)}(G)$ into full subcategories known as blocks. Thus any object V has a canonical decomposition into a direct sum of objects, one for each block. These blocks are parametrized by inertial equivalence classes of pairs $[L, \pi]$, where L is a standard Levi subgroup and π is a supercuspidal $k[L]$-module (see [Hel12a, Def 3.3] for the definition of inertial equivalence). The block $\text{Rep}_{W(k)}(G)_{[L,\pi]}$ is the full subcategory of objects all of whose simple subquotients have mod ℓ inertial supercuspidal support given by the pair (L, π). (For the definition of mod ℓ inertial supercuspidal support, see [Hel12a, Def 4.12]). Each block corresponds to a primitive idempotent $e_{[L,\pi]}$ of $\mathcal{Z}$ projecting V onto its largest direct summand living in $\text{Rep}_{W(k)}(G)_{[L,\pi]}$, and any primitive idempotent cuts out a block. Note that the contragredient $\pi^\vee$ is also supercuspidal. We define:

$$e^*_{[L,\pi]} := e_{[L,\pi^\vee]}.$$

Lemma 39. *Let e be a primitive idempotent of $\mathcal{Z}$. Then*

1. *for any $V \in \mathrm{Rep}_{W(k)}(G)$, $(eV)^\vee = e^*V^\vee$*
2. *given $\theta \in \text{c-Ind}_N^G \psi$ and $\eta \in \mathrm{Ind}_N^G \psi^{-1}$, we have*

$$\langle e\theta, \eta \rangle = \langle \theta, e^*\eta \rangle.$$

Proof. By definition, $e^*V^\vee$ (resp. eV) is the largest direct summand of $V^\vee$ (resp. of V) all of whose simple $W(k)[G]$-subquotients have mod-ℓ inertial supercuspidal support isomorphic to $(L, \pi^\vee)$ (resp. (L, π)). All the simple subquotients of $(eV)^\vee$ occur as the duals of simple subquotients of eV. Thus by the duality theorem for parabolic induction, the simple subquotients of $(eV)^\vee$ have supercuspidal support $(L, \pi^\vee)$. Since $(eV)^\vee$ a direct summand of $V^\vee$, and it is the largest with this property, we have $(eV)^\vee = e^*V^\vee$.

To prove the second part, recall that the pairing $\langle , \rangle$ on $\text{c-Ind}\,\psi \times \mathrm{Ind}\,\psi^{-1}$ induces a G-equivariant isomorphism $\mathrm{Ind}\,\psi^{-1} \xrightarrow{\sim} (\text{c-Ind}\,\psi)^\vee$, and therefore an isomorphism $e^*\,\mathrm{Ind}\,\psi^{-1} \xrightarrow{\sim} e^*(\text{c-Ind}\,\psi)^\vee = (e\,\text{c-Ind}\,\psi)^\vee$. □

We identify $e^*\,\mathrm{Ind}\,\psi_{A'}^{-1}$ with the A'-linear dual of $e(\text{c-Ind}\,\psi_A)$ and identify $e^*\,\mathrm{Ind}\,\psi_{\mathcal{O}}^{-1}$ with the $\mathcal{O}$-linear dual of $e(\text{c-Ind}\,\psi)$. We formulate:

Lemma 40. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Hom}_{e\mathcal{Z}[G]}(e\,\text{c-Ind}\,\psi, U) \otimes_{W(k)} A & \longrightarrow & \mathrm{Hom}_{R[G]}(e\,\text{c-Ind}\,\psi_A, U_A) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{O}[G]}(U^\vee, e^*\,\mathrm{Ind}\,\psi_{\mathcal{O}}^{-1}) \otimes_{W(k)} A & \longrightarrow & \mathrm{Hom}_{A'[G]}((U_A)^\vee, e^*\,\mathrm{Ind}\,\psi_{A'}^{-1})
\end{array}$$

Proof. Since $U^\vee \otimes_{\mathcal{O}} A = U_A^\vee$ and $(e\,\text{c-Ind}\,\psi_{\mathcal{O}}) \otimes_{W(k)} A = e\,\text{c-Ind}\,\psi_{A'}$, the horizontal arrows are maps of A-modules given by sending $\phi \otimes 1$ to the map $[h \otimes a \mapsto \phi(h) \otimes a]$. The top horizontal map is injective because U is finitely generated over $e\mathcal{Z}[G]$. The downward arrows are defined by $\phi \mapsto \phi_*$ where ϕ_* takes a map to its precomposition with ϕ.

We now show commutativity:

$$\begin{array}{ccc} p \otimes a & \longmapsto & [\phi : h \otimes b \mapsto p(h) \otimes ab] \\ \downarrow & & \downarrow ? \\ p_* \otimes a & \longmapsto & [u^\vee \otimes b \mapsto p_*(u^\vee) \otimes ab] \end{array}$$

So we must check that $\phi_*(u^\vee \otimes b)$ and $p_*(u^\vee) \otimes ab$ are equal as elements of $(e'\,\text{c-Ind}\,\psi_A)^\vee \cong (e'\,\text{c-Ind}\,\psi) \otimes A$. But given $h \in e'\,\text{c-Ind}\,\psi$ and c in A we have $\phi_*(u^\vee \otimes b)(h \otimes c) = (u^\vee \otimes b)(p(h) \otimes ac) = u^\vee(p(h)) \otimes abc$. On the other hand we have $(p_*(u^\vee) \otimes ab)(h \otimes c) = u^\vee(p(h)) \otimes abc$, as desired. $\square$

The map $p_A \in \text{Hom}_{R[G]}(e\,\text{c-Ind}\,\psi_A, U_A)$ is in the image of the top horizontal map since it is the base change $p \otimes 1$. Thus $(p_A)_*$ equals $p_* \otimes 1$. Since $v_A^\vee$ is in $U^\vee \otimes_{\mathcal{O}} A$ we can expand it as $v_A^\vee = \sum_i v_i^\vee \otimes a_i$ with $v_i^\vee \in U^\vee$ and

$a_i \in A$. Then we have

$$\begin{aligned} 0 \neq \langle v_A^\vee, p_A(eH)\rangle &= \langle (p_A)_*(v_A^\vee), eH\rangle \\ &= \langle (p_* \otimes 1)(v_A^\vee), eH\rangle \\ &= \langle (p_* \otimes 1)(\sum_i v_i^\vee \otimes a_i), eH\rangle \\ &= \langle \sum_i p_*(v_i^\vee) \otimes a_i, eH\rangle \\ &= \sum_i a_i \langle p_*(v_i^\vee), eH\rangle \end{aligned}$$

This implies that not all the terms $\langle p_*(v_i^\vee), eH\rangle$ are zero. Therefore

$$\langle p_*(v_i^\vee), eH\rangle \neq 0$$

for some i. Since $p_* : U^\vee \to e^* \operatorname{Ind} \psi_{\mathcal{O}}^{-1}$ is $\mathcal{O}[G]$-linear, it is a (the) map to the Whittaker space of $U^\vee$, so $p_*(v_i^\vee)$ defines an element of $\mathcal{W}(U^\vee, \psi^{-1})$. U is a co-Whittaker $\mathcal{O}[G]$-module by Proposition 2.6.1, as it equals $e(\text{c-Ind}\,\psi) \otimes_{e\mathcal{Z},g} \mathcal{O}$. By Lemma 39 we conclude that $\langle p_*(v_i^\vee), eH\rangle = \langle e^* p_*(v_i^\vee), H\rangle = \langle p_*(v_i^\vee), H\rangle$ is nonzero.

To show that U satisfies all the requirements of Theorem 6.3.1, and $p_*(v_i^\vee)$ is the required Whittaker function, the only thing left to check is that U is absolutely irreducible after inverting ℓ. If ϖ is a uniformizer of $\mathcal{O}$, the fact that $e(\text{c-Ind}\,\psi) \otimes_{e\mathcal{Z}} \kappa(\mathfrak{p})$ is absolutely irreducible precisely means $U[\frac{1}{\varpi}]$ is absolutely irreducible, which is true by construction. The map $U \to U[\frac{1}{\varpi}]$ is an embedding because both $\mathcal{O}$ and $e(\text{c-Ind}\,\psi)$ are ℓ-torsion free.

Hence the $W(k)[G]$-module U and the Whittaker function $p_*(v_i^\vee)$ satisfy the conclusion of Theorem 6.3.1.

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