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Exploring Methods for Finding Solutions to Polynomial Equations

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Exploring Methods For Finding Solutions to Polynomial Equations

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Report

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Dedication

I dedicate this report to my wife of twenty eight years, Gayla Beaver, who has supported and encouraged me to be the person I am today. Without her help, I would never have attempted the Masters program at the University of Texas and for that I am grateful. Throughout our married life Gayla has insisted that I persist in my endeavors and studies to improve my knowledge and teaching. I thank her for that in this dedication.

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Abstract

Exploring Methods for Finding Solutions to Polynomial Equations

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The University of Texas at Austin, 2010

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There are many methods for solving polynomial equations. Dating back to the Greek and Babylonian mathematicians, these methods have been explored throughout the centuries. The introduction of the Cartesian Coordinate Plane by Rene Descartes greatly enhanced the understanding of what the solutions actually represent.

The invention of the graphing calculator has been a tremendous aid in the teaching of solutions of polynomial equations. Students are able to visualize what these solutions represent graphically. This report explores these methods and their uses.

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Chapter 1: Introduction

Typical secondary curricula introduce the basic concept of functions with first order linear equations. Teachers invest great time, effort, and creativity as they convey the ideas of obtaining solutions and the meanings behind those solutions. In solving linear equations, the concept of balancing an equation can be made concrete and tangible by allowing students to experience the phenomenon with a balance scale. There exists a seemingly endless supply of manipulatives which students can use to visualize most fundamental mathematics concepts. However, making the transition from solving these basic first order equations to higher order polynomials can prove quite challenging for both student and teacher. It is at this point in mathematics when one must move from the concrete to the abstract. Whereas the balance scale provides a core foundational understanding of solving linear equations, the zero product theorem is not so easily demonstrated. While the basic idea of the theorem itself seems attainable, using this as a method to solve quadratic equations is not always readily understood by students. Additionally, because so much emphasis is placed on finding the one solution to a first order equation, students often have great difficulty grasping the concept that more than one solution may exist to a given equation.

The introduction of the graphing calculator has been a tremendous aid in allowing the student to actually see the points where the function crosses the x-axis. Unfortunately, making the connection from the viewing window to the written polynomial equation requires students make the abstract connection between the zero product theorem and x-intercepts. While solving a linear equation can be done without much mathematical depth, solving quadratics demands students have a fairly in depth understanding of what it means to be a function, the relationship between the equation and its domain and range, and how the function itself produces the points associated with the graph. The students are now able to really connect the solutions obtained by the quadratic formula, completing the square, and factoring to what they see in the viewing window. Fortunately, once a student reaches this level of understanding, the transition to cubic, quartic, and higher degree equations proves far less taxing.

Once certain concepts are understood by the student, the x-intercepts are easily seen as the inputs into a polynomial function that generate the number 0 as the output, or solutions of the polynomial equation $P(x) = 0$. Specifically these concepts are:

i. what it means to be a function
ii. the meaning of *Domain* and *Range* of a function
iii. all inputs of the *Domain* generate outputs in the *Range* that are associated with points on the graph of the function

Similarly, cubic, quartic, and higher degree equations are seen in the same manner. The methods used to find the solution to quadratic equations, such as completing the square, factoring, and the quadratic formula, are taught in first year Algebra classes, but a connection (or meaning) comes to fruition when the actual graphs are seen.

Another problem arises when solutions to quadratics are complex numbers that are not real numbers. It is hard for the student to visualize what is taking place since the complex solutions are not often seen or are not easy to graph. The quadratic formula yields these complex solutions, but what is their meaning? And when discussing the solutions to a cubic equation, how do these solutions appear in the graph of the polynomial associated with that equation?

Because most of what is discussed in secondary mathematics education in relation to polynomial functions and their graphs is "local behavior," finding the zeros of a polynomial is crucial to the understanding of the function. The previously discussed methods for finding solutions and what they mean are subjects addressed in this report. Even though each of these methods is of very little value in themselves, when used in combination with each other, a function's behavior can be discovered. Further use of methods learned in calculus such as the derivative of a function or relative extrema, are also helpful but not discussed here.

Polynomials of a second degree, or quadratics, take three forms, each of which has its own purpose. The three forms and their use are:

Name	**Form**	**Use**
Standard Form	$f(x) = ax^2 + bx + c$	easy to generate points, especially the y-intercept
Vertex Form	$f(x) = c(x-a)^2 + b$	easy to find vertex (a, b) and graph
Root Form	$f(x) = a(x-r_1)(x-r_2)$	easy to find the x-intercepts (r_1, 0) and (r_2, 0)

When graphing the parabola generated by a quadratic, the vertex form is preferred. Once the student has manipulated the equation into this form, a simple chart with the vertex in the center entry and two values of *x* to the left and right (or above/below the *y* in the case of horizontal parabolas) is sufficient to see its shape.

If the graph of the parabola intersects the *x*-axis, these are referred to as the real zeros of the function. In the case of parabolas that do not cross the x-axis, the numbers that can be substituted for *x* that generate the number 0 for y are still complex solutions.

Consider a parabola whose equation in vertex form is

$$f(x) = c(x-a)^2 + b \,.$$

As mentioned earlier, the vertex is (a, b). Letting $f(x) = 0$ and solving for x,

$$0 = c(x - a)^2 + b$$

$$-\frac{b}{c} = (x - a)^2$$

$$\pm\sqrt{-\frac{b}{c}} = x - a$$

$$x = a \pm \sqrt{-\frac{b}{c}}\,.$$

Depending on the algebraic signs of b and c, the parabola will have real or complex roots, but the spacing about the axis of symmetry is the same. In Figure 1, the roots are $\alpha \pm i\beta$, where α the x-coordinate, or *abscissa* of the vertex, and β is half the length of the chord determined by the horizontal line $y = 2b$ and where b is the y-coordinate, or *ordinate,* of the vertex. The chord is referred to as the *latus rectum* and passes through the focus of the parabola. As the parabola is shifted down, these become real roots and the horizontal line is simply the x-axis, as illustrated in the transition from Figure 1 to Figure 2 below.

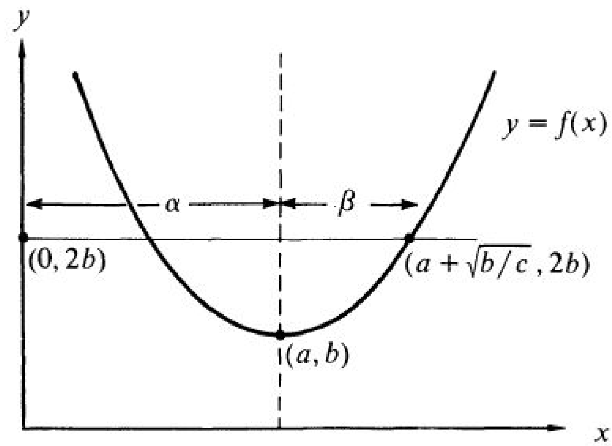


Figure 1. Graph of $f(x) = c(x-a)^2 + b$ [**6**, p. 248]

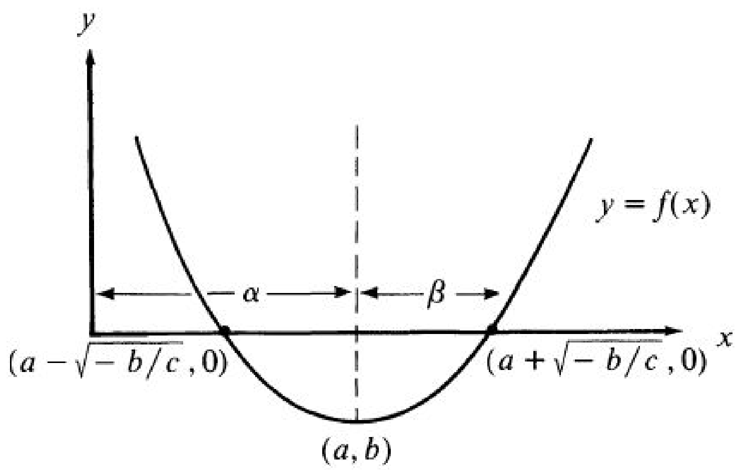


Figure 2. Graph of $f(x)$ shifted *3b* units down **[2,** p**.** 248]

A simple example would be to consider the parabola with vertex (0, -1) and $c = 1$

$$f(x) = x^2 - 1$$

and with real roots at $(\pm 1, 0)$. If the parabola was shifted up two units to have a new vertex at (1, 0), but *c* remains the same, the new complex roots are $(\pm i, 0)$.

Chapter 2: A Geometrical Approach

Long before Rene Descartes introduced the Cartesian Coordinate Plane, the Greek mathematicians found solutions to quadratic equations geometrically. A few of these methods will be discussed here.

A well known theorem in mathematics resulting from triangle similarity states that in a right triangle the altitude drawn to the hypotenuse is the geometric mean between the segments into which it is divided. In the proportion $\frac{a}{b} = \frac{b}{c}$, b is the geometric mean of *a* and *c*.

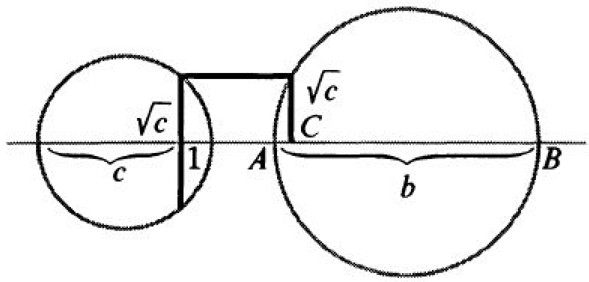
Simple geometric constructions can be made using a straight edge and compass, as shown in Figure 3. Then some use of simple algebra and geometry provides visual solutions that are generated for the following quadratic equations:

i. $x^2 - bx + c = 0$

ii. $x^2 + bx + c = 0$

iii. $x^2 - bx - c = 0$

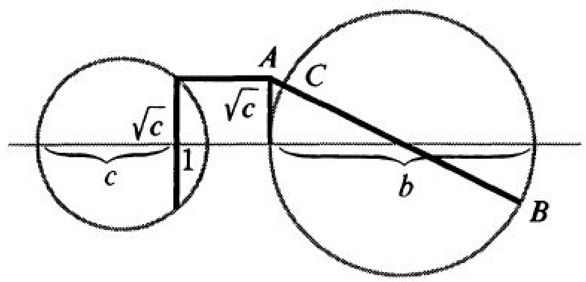
iv. $x^2 + bx - c = 0$



Roots

1. $x_1 = AC,\ x_2 = CB$
2. $x_1 = -AC,\ x_2 = -CB$

Figure 3. Euclidean constructions for i. and ii. [**4**, p. 363]



Roots

3. $x_1 = AB,\ x_2 = -AC$
4. $x_1 = -AB,\ x_2 = AC$

Figure 4. Euclidean Constructions for iii. and iv. [**4**, p. 363]

The first equation (i.) can be verified by setting $x_1 = AC$ and $x_2 = BC$. The previously mentioned theorem then states

$$\frac{AC}{\sqrt{c}} = \frac{\sqrt{c}}{BC}$$

or

$$c = AC \cdot BC.$$

But since

$$x_1 = AC \quad \text{and} \quad BC = b - x_1,$$

it follows that

$$x_1(b - x_1) = c$$

$$bx_1 - x^2 = c$$

$$x^2 - bx_1 + c = 0.$$

Similarly the second equation ii. can be solved. However, equations iii. and iv. make use of another well known theorem stating that the square of a tangent segment drawn from a point in the exterior of a circle is equal to the product of the secant segment and the outer secant segment.

Carlyle's Method.

In the early 1800's another method was suggested using the intersection of a circle with the x-axis to show solutions to the equation $x^2 + bx + c = 0$. For if the equation of a particular circle with a diameter having endpoints at (0, 1) and *(-b,c)* given by $x^2 + y^2 + bx - (1+c)y + c = 0$ is evaluated for y = 0, the equation simplifies to the desired quadratic.

To illustrate, consider the quadratic equation

$$x^2 + 2x - 3 = 0 .$$

If the circle whose diameter has endpoints at (0, 1) and (-2, -3) is graphed, the roots of the quadratic equation are also the abscissas of the points of intersection of the circle with the x-axis as shown in Figure 5. In this case $x_1 = -3$ and $x_2 = 1$.

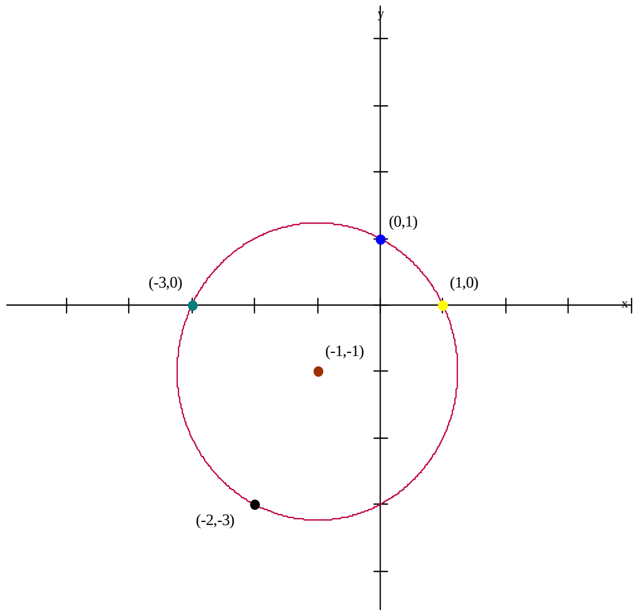


Figure 5. Graph of $(x+1)^2+(y+1)^2=5$

It is customary to solve a quadratic equation by graphing its corresponding quadratic function as a parabola and locating points where the graph crosses the x-axis. Hornsby suggests an alternate method which incorporates the graph of the "parent function" $y = x^2$ and somewhat simplifies the process. [**3**, p.364] Solving the quadratic equation

$$x^2+bx+c=0$$

is equivalent to finding the points of intersection of the line $y=-bx-c$. This phenomenon is easily seen because if the left side of the equation is solved for x^2, the equation becomes

$$x^2=-bx-c.$$

In the example used earlier, $y = x^2 + 2x - 3$, the points of interest are the intersection of $y_1 = -2x + 3$ with $y_2 = x^2$, i.e. (-3, 9) and (1, 1) as shown in Figure 6 below:

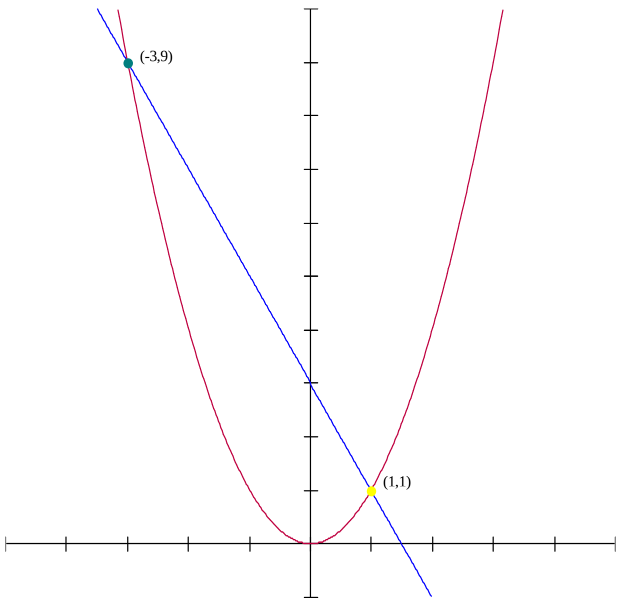


Figure 6. $y_1 = -2x + 3;\ y_2 = x^2$

The abscissas of these points are the solutions to the original quadratic equation.

Chapter 3: Iteration as a Technique

An alternate method of solving polynomial equations is offered by Butts with some interesting results. [**2**] The process will be demonstrated here with a quadratic even though use of the Quadratic Formula would be an easier method of solution. Consider the function

$$P(x) = x^2 - 2x - 2\,.$$

The goal is to find roots of the polynomial, that is the *x*-values that make *P(x)* = 0 or, the *x*-values that satisfy the equation

$$x^2 - 2x - 2 = 0\,.$$

Instead of proceeding with the Rational Root Theorem, Butts suggests putting the equation in an alternate form and applying iteration to this equation to find a *fixed point*. A fixed point, x_0, is a particular value of *x* such that $f(x_0) = x_0$.

One alternate form could be

$$x = \frac{x^2 - 2}{2}\ .$$

Iteration is the process by which a *seed* (or initial) value x_0 is substituted into the right side of this equation, then the result $f(x_0)$ is then substituted for *x* until the two values are identical, the aforementioned fixed point. For if the output (functional value) is equal to the input (*x*-value) accurate to any desired decimal approximation, that particular value satisfies the alternate form and therefore will satisfy the original form, yielding the desired result. If a seed value converges to a fixed point, the root has been found. However, sometimes the seed value will diverge to $\pm\infty$ in which case iteration fails. To ensure convergence, the following definition and ensuing theorem guarantee intervals of convergence.

Definition. The *magnification factor* MF of *f(x)* at $x = x_o$ is

$$MF(f(x_o)) = \lim_{\varepsilon \to 0} \frac{f(x_o+\varepsilon) - f(x_o-\varepsilon)}{2\varepsilon}.$$

This definition leads us to the theorem: The Fixed Point Iteration Algorithm converges if $|MF(f(x))| < 1$ for values of x near the seed value x_0" [**2,** p. 5]. In this example,

$$|MF(f(x_o))| = \lim_{\varepsilon \to 0} \frac{f(x_o+\varepsilon) - f(x_o-\varepsilon)}{2\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\left[\frac{(x_o+\varepsilon)^2 - 2}{2}\right] - \left[\frac{(x_o-\varepsilon)^2 - 2}{2}\right]}{2\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{x_o{}^2 + 2\varepsilon x_o + \varepsilon^2 - 2 - x_o{}^2 + 2\varepsilon x_o - \varepsilon^2 + 2}{2\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{4\varepsilon x_o}{4\varepsilon}$$

$$= x_o,$$

so that convergence is guaranteed if $|x_o| < 1$, *i.e.*, $-1 < x_o < 1$.

The theorem doesn't necessarily determine the smallest interval of convergence, merely one that guarantees convergence. Choosing the seed value to be 0, it can be verified after 42 iterations with a scientific calculator that the sequence converges to approximately 0.732051, a root of the polynomial accurate to six decimal places. If a seed value of 3 is chosen, the sequence diverges to ∞ and no root is found.

There exist three types of fixed points; attracting, repelling, and neutral. The iteration method only finds roots when a fixed point is an attracting fixed point.

Chapter 4: Finding Roots Of Higher Degree Polynomials

Finding the roots of higher degree polynomials is much more difficult than finding the roots of linear or quadratic functions. A few theorems and properties make the process easier.

i. If r is a root of a polynomial function, then $(x-r)$ is a factor of the polynomial,

ii. Any polynomial function with real coefficients can be written as the product of linear factors $(x-r)$ and quadratic factors (ax^2+bx+c) which are irreducible over the real numbers.

A quadratic factor that is irreducible over the reals is a quadratic function with no real zeros; equivalently those that have a negative discriminant.

Another helpful theorem is Descartes' Rule of Signs, which states that the number of variations in (algebraic) signs throughout the polynomial determines the number of positive roots that the function will have. The Rule will not tell where the polynomial's roots are, but will tell how many to expect when finding them. Consider the following polynomial in its original form:

$$f(x) = 2x^5 - x^4 + 2x^3 + 4x^2 - x + 3.$$

Without concern for the actual values of the coefficients themselves, notice that the algebraic signs change four times. Thus there will be, at most, four positive roots. Conveniently, the roots come in "pairs," so if one positive root is found, finding another positive root is in order.

To find the number of negative roots, *f(-x)* is generated:

$$f(-x) = 2(-x)^5 - (-x)^4 + 2(-x)^3 + 4(-x)^2 - (-x) + 3$$

$$f(x) = -2x^5 - x^4 - 2x^3 + 4x^2 + x + 3\,.$$

Since there is only one sign change, this polynomial has exactly one negative root. So once that root is found, looking for another negative root becomes moot. Therefore, there are 4, 2, or 0 positive roots and exactly 1 negative root. To illustrate the "nature of the roots," that is, the number and type (real or complex), a chart is generated.

Table 1. Number and nature of possible roots

Number of positive real roots	Number of negative real roots	Number of complex roots
4	1	0
2	1	2
0	1	4

The Rational Root Theorem also can be used as an aid to finding roots of a polynomial function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\ .$$

Assuming the coefficients are all integers and a root of the polynomial is rational, the numerator of the root is always a factor of a_0 and the denominator is a factor of a_n. In the example above, if any rational roots exist, they must come from the "bank" of

$$\pm 1, \pm 3, \pm \frac{1}{2}, \ or \ \pm \frac{3}{2}.$$

As stated earlier, each of these theorems is not all that helpful alone, but when used in combination with one another, finding roots becomes easier. The idea is to use these tools in combination in order to find a rational root. This is usually done by synthetic division, then compressing the equation and repeating the process until the polynomial is expressed as the product of linear factors or irreducible quadratic factors that can be solved using the quadratic formula. Students sometimes find this process laborious and tedious. If, however, a student realizes some important facts about the process, there comes a realization that even though a root is not found by synthetic division, it is not a waste of time. The remainder from synthetic division is synonymous with the functional value, so it still yields a point on the graph. Also, if $f(0)$ is positive but $f(1)$ is negative, a logical choice to try next would be $\frac{1}{2}$; since the polynomial is a function its graph ***must*** cross the x-axis between 0 and 1. That is, the graph cannot "go around" that portion because then it would not be a function. Of course, there is always the possibility that the root may be irrational.

Taking the Rational Root Theorem to a higher level, Redmond [**7**] proves a theorem that states "If $P(x)$ is a polynomial in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \qquad n \geq 2$$

where a_0, a_n, and $f(1)$ are all three ***odd*** numbers, then $f(x)$ has no rational roots." [**7**] While this does not apply to the previous function $(a_n = 2)$, let's consider the example Redmond gives:

$$f(x) = x^5 + 7x^4 - 28x^3 + 125x^2 + x - 3275$$

in which a_5 = 1, a_0 = -3275, and $f(1)$ = -3169, all ***odd*** numbers. Redmond assures that this particular polynomial has no ***rational*** roots. An easier example, one which will be discussed later, is

$$f(x) = x^2 + x + 1$$

a_2 = 1, a_0 = 1, and f(1) = 3 (all ***odd*** numbers) which has as its roots $\{\frac{-1 \pm i\sqrt{3}}{2}\}$, obviously ***complex*** roots.

As seen earlier, the Rational Root Theorem simply narrows the search for rational roots of the polynomial. In many cases a great majority of roots can be eliminated before attempting the tedious process of synthetic division. Barrs, J. Braselton, and L. Braselton offer the Imaginary Rational Root Theorem, which narrows the search for imaginary roots. [**1**] The theorem states:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

is an *nth* degree polynomial function with integer coefficients. If $x = \alpha \pm \beta i = \frac{p}{r} \pm \frac{q}{r} i$ are rational imaginary zeros of *P(x)*, where α and $\beta \neq 0$ are rational, *p*, *q*, and *r* are integers, then r^2 is a divisor of a_n and $p^2 + q^2$ is a divisor of a_0.

To illustrate this theorem, consider the following polynomial function:

$$P(x) = x^4 + 2x^3 + 6x^2 + 2x + 5. \tag{1.1}$$

The possible rational roots are 1, -1, 5, and -5. Synthetically dividing with these possible roots yields remainders of 16, 8, 1040, and 520, respectively, therefore there are no rational roots. An application of the imaginary rational root theorem indicates that since $1 = 0^2 + 1^2$ and $5 = 1^2 + 2^2$, the possible complex rational roots would be given by $1 \pm 2i, -1 \pm 2i,$ and $\pm i$ with corresponding factors $x^2 - 2x + 5, x^2 + 2x + 5,$ and $x^2 + 1$. The chart below gives the results when P(x) is divided by each of these potential factors.

Table 2. Results of dividing (1.1) by quadratic factors with rational complex zeros.

Possible Zero	Possible Factor	Quotient	Remainder
$1 \pm 2i$	$x^2 - 2x + 5$	$x^2 + 4x + 9$	-40
$-1 \pm 2i$	$x^2 + 2x + 5$	$x^2 + 1$	0
$\pm i$	$x^2 + 1$	$x^2 + 2x + 5$	0

Therefore,

$$x^4 + 2x^3 + 6x^2 + 2x + 5 = (x^2 + 1)(x^2 + 2x + 5)$$

and

$$x^4+2x^3+6x^2+2x+5=0 \text{ if } x=\pm i \quad or \quad x=-1\pm 2i.$$

Another interesting (but perhaps not so useful) theorem is afforded by Luthar. [**5**] According to Luthar, Luddhar's Theorem states "If $P(x)=ax^3+bx^2+cx+d$, with *a*, *b*, *c*, and *d* integers, $a\neq 0$, $b\neq 0$, the function has a rational root if there exist non-zero integers *l*, *m*, *p*, and *q* such that $c = l + m$, $b = p + q$, $\frac{lm}{d} = p$, and $\frac{pq}{a}=l$. That rational root is given by $-\frac{l}{p}$ [**5**, p. 107]. The following example illustrates Luddhar's Theorem.

Let

$$P(x)=6x^3+2x^2+15x+5.$$

Since $\frac{6}{2}=\frac{15}{5}$, the function can be written as

$$P(x)=2x^2(3x+1)+5(3x+1)$$

$$=(3x+1)(2x^2+5)$$

which yields a rational root of $-\frac{1}{3}$. The theorem and its process essentially amount to "factoring by grouping" for this example, but the next problem illustrates its worth. Consider a polynomial that is not factorable by grouping:

$$P(x)=x^3+6x^2+11x+6.$$

Select two integers whose sum is 11, the coefficient of the term involving x. For example, let l = 9 and m = 2. Now p and q must be determined. So

$$p = \frac{lm}{d} = \frac{9 \cdot 2}{6} = 3$$

so that

$$q = b - p = 6 - 3 = 3.$$

Checking the other condition on q, namely,

$$q = \frac{la}{p} = \frac{9 \cdot 1}{3} = 3;$$

thus Luddhars's conditions are satisfied. Consequently, the original function can now be written as

$$P(x) = x^3 + 3x^2 + 3x^2 + 9x + 2x + 6$$

$$= x^2(x+3) + 3x(x+3) + 2(x+3)$$

$$= (x+3)(x^2 + 3x + 2)$$

$$= (x+3)(x+1)(x+2).$$

Consider a quadratic function with complex roots given by $a \pm bi$. Using a well known fact that the quadratic function with roots r_1 and r_2 is given by

$$f(x) = x^2 - (r_1 + r_2)x + (r_1 \cdot r_2)$$

$$= x^2 - [(a+bi)+(a-bi)]x + [(a+bi)(a-bi)]$$

$$= x^2 - 2ax + a^2 + b^2.$$

This equation can be easily transformed by completing the square into

$$y - b^2 = (x-a)^2$$

from which it is evident that, if OA and AP are measured as shown in Figure 7. The complex roots will be given by $a \pm ib = OA \pm i\sqrt{AP}$.

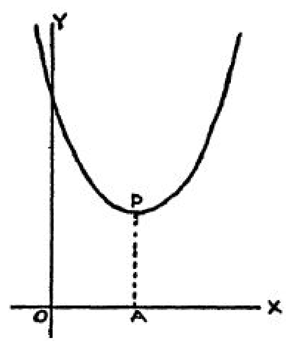


Figure 7. $f(x) = x^2 - 2ax + a^2 + b^2$

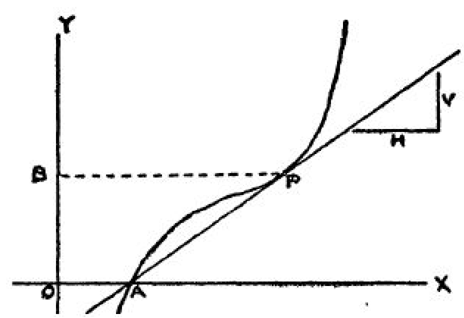


Figure 8. $f(x) = (x-a)(x^2 - 2bx + b^2 + c^2)$

To illustrate, consider the following quadratic function:

$$f(x) = x^2 - 2x + 4$$

or

$$f(x) = x^2 - 2 \cdot 1 \cdot x + 1^2 + (\sqrt{3})^2 ,$$

where $a = 1$, $b = \sqrt{3}.$ It is easily shown by completing the square that the function can be written in vertex form as

$$f(x) = (x-1)^2 + 3 .$$

The vertex of this parabola is (1, 3) and OA = 1 while AP = 3. According to Yanosik [**8**], the complex roots would be given by

$$OA \pm i\sqrt{AP} = 1 \pm i\sqrt{3} .$$

Using the quadratic formula to generate the roots, this fact is seen to be true. Similarly, the cubic function given by

$$f(x) = (x-a)(x^2 - 2bx + b^2 + c^2)$$

will yield the real root a and the complex roots $b \pm ic$ by the following method. As shown in Figure 8, a line is drawn through $A(a, 0)$ tangent to the graph at P. BP is measured and the slope m of the tangent line is calculated. Yanosik suggests (and proves in another article) the complex roots are

$$b \pm ic = BP \pm i\sqrt{m} \,.$$

An example to illustrate follows. The roots of the aforementioned quadratic will help in the illustration. Take the cubic function given by:

$$f(x) = x^3 + 8$$

$$= (x+2)(x^2 - 2x + 4)$$

The graphs of this function and the corresponding tangent are shown in Figure 9.

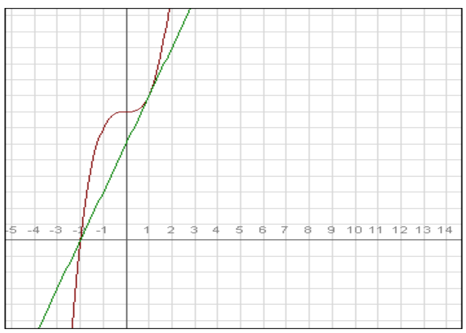


Figure 9. Graphs of $f(x) = x^3 + 8; g(x) = 3x + 6$

From the graph (and the factored form of the function), it can be seen that the real root is –2. If Yanosik is correct, the complex roots are given by

$$b \pm ic = BP \pm i\sqrt{m} \,,$$

indicating that *BP* = 1 and the slope of the tangent is 3. The line shown tangent in Figure 9 indeed has a slope of 3 and is tangent at the point (1, 9) yielding *BP* = 1.

In a previously mentioned polynomial function, the real roots of the polynomial seem to "disappear" as the graph is shifted up. For example, Figure 10 shows a graph of the following function:

$$f(x) = x^2 + x - 1$$

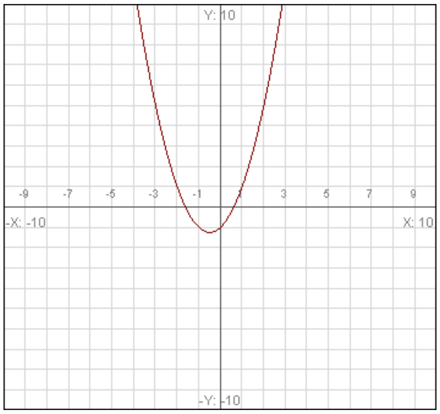


Figure 10. Graph of $f(x) = x^2 + x - 1$.

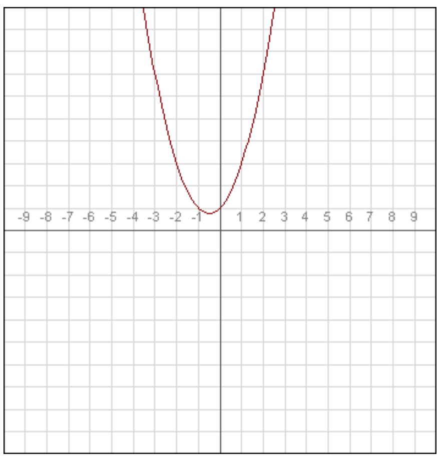


Figure 11. Graph of $f(x) = x^2 + x + 1$.

Notice that the function has real roots at $x = \dfrac{-1 \pm \sqrt{5}}{2}$. However when the function with equation

$$f(x) = x^2 + x + 1$$

is graphed (see Figure 11), the roots become imaginary.

Complex numbers are defined to be expressions of the form $a + bi$ treated as residues to the *modulus* $i^2 + 1$. *Modulus* in this sense is used in the same way as modulus in a clock. That is, a time of 20 o'clock is considered to be 8 o'clock (mod 12). It's the residue, or remainder, when 20 is divided by 12. In using the Rational Root Theorem mentioned earlier, when a possible root is "checked" by synthetic division, if the remainder is zero, the number is a root. However if there is a remainder other than zero, that remainder is the same as the functional value at that value of *x*. Therefore, it is a point on the graph of the polynomial.

Similarly, this phenomenon occurs with polynomial residues to the modulus $y^2 + 1$. If any polynomial in *y* (with real coefficients) is divided by $y^2 + 1$ the remainder is of the form $a + by$ with *a* and *b* being real numbers. The polynomial mentioned before

$$P(x) = x^2 + x + 1$$

has no real solutions, but

$$x^2 + x + 1 \equiv 0 \pmod{y^2 + 1}$$

has solutions $x = \frac{-1 \pm y\sqrt{3}}{2}$ because if $x = \frac{-1 + y\sqrt{3}}{2}$ then

$$x^2 + x + 1 = \left(\frac{-1 + y\sqrt{3}}{2}\right)^2 + \left(\frac{-1 + y\sqrt{3}}{2}\right) + 1$$

$$= \frac{1-2y\sqrt{3}+3y^2}{4} + \frac{-2+2y\sqrt{3}}{4} + \frac{4}{4}$$

$$= \frac{1-2y\sqrt{3}+3y^2-2+2y\sqrt{3}+4}{4}$$

$$= \frac{3+3y^2}{4}$$

$$= \frac{3}{4}(1+y^2) \equiv 0 \pmod{y^2+1}.$$

Long and Hern use modulus surfaces to explain this phenomenon. In Figure 10, the graph shows the real roots mentioned earlier. [**4**] Figure 12 shows taking the absolute value of the function, gives somewhat of an insight as to the appearance of the modulus surface associated with this function as seen in Figure 13. The "low points" of the surface signify the real roots. If the function is shifted up two units, the surface is that shown in Figure 14. The appearance of the two are similar, but seem to be rotated horizontally $\frac{\pi}{2}$ radians.

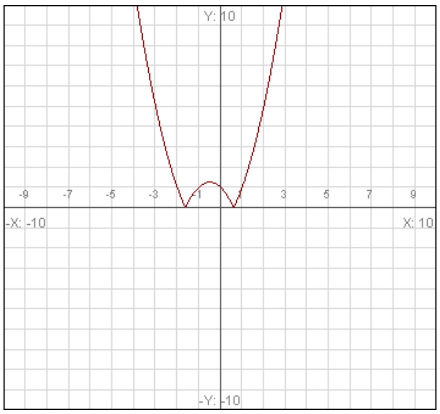


Figure 12. $g(x) = \left|x^2 + x - 1\right|$

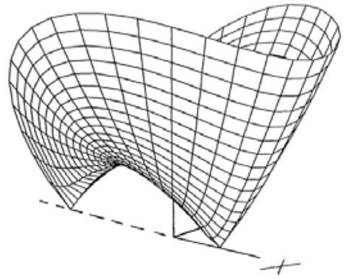


Figure 13. $f(x) = \left|x^2 + x - 1\right|$

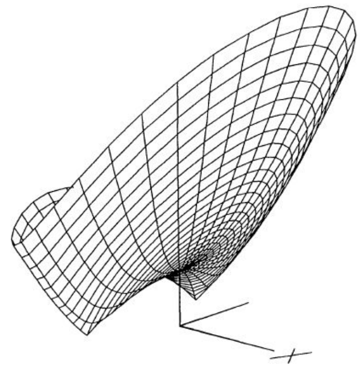


Figure 14. $f(x) = \left|x^2 + x + 1\right|$

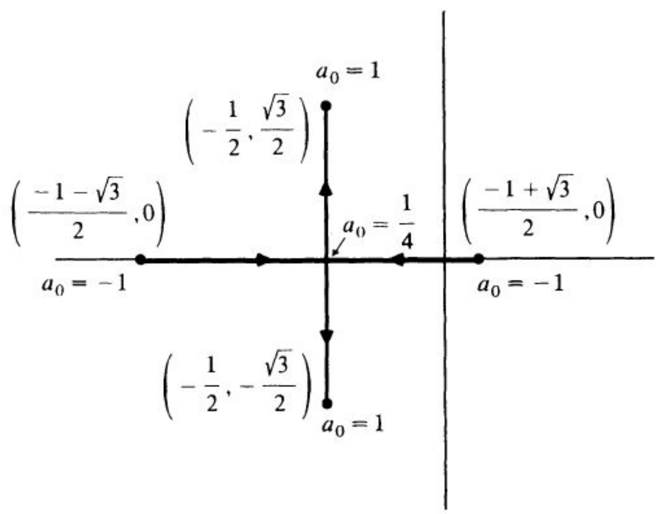


Figure 15. Paths of zeros of $x^2 + x + a_0$ as a_0 varies over $[-1,1]$.

The zeros of a polynomial are continuous functions of the coefficients of that polynomial. Hence as the coefficient a_0 of the polynomial $z^2 + z + a_0$ changes continuously from –1 to +1 on the real axis, the positions of the low points on the corresponding modulus surfaces move along a continuous path along the complex plane with the beginning and end positions as shown in Figure 15. Since the zeros are given by $\frac{-1\pm\sqrt{1-4a_0}}{2}$, they will remain real until $a_0 > \frac{1}{4}$ at which point they become complex. When $a_0 = \frac{1}{4}$, the polynomial $z^2 + z + \frac{1}{4}$ is a perfect square and has a real zero at $x = -\frac{1}{2}$. Considering cubic functions of the nature $f(x) = x^3 + a_0$, as a_0 varies over an interval $[-\delta, \delta]$, it is seen that the roots appear to be at the endpoints of the "spokes of a wheel," with a rotation of $\frac{\pi}{3}$ radians, depending on whether δ is positive or negative as shown in Figure 16. This phenomenon is consistent with the concepts illustrated by DeMoivre's Theorem.

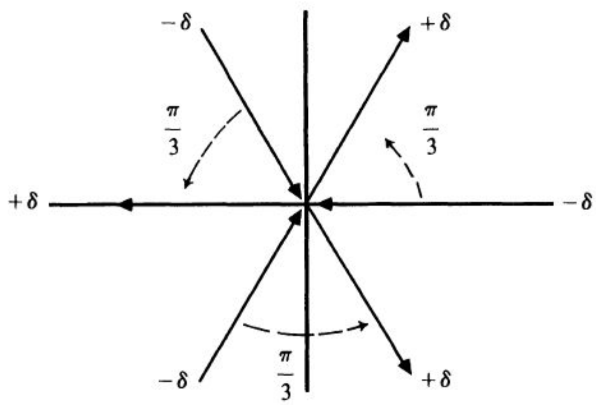


Figure 16. Paths of zeros near a triple zero of $x^3 + a_0$ for a_0 in $[-\delta, \delta]$

Chapter 5: Conclusion

Several methods for solving polynomial equations have been introduced in this report. Depending on the nature of the polynomial and what is known, some methods are more appropriate than others. The underlying fact is that the more methods that are known, the more proficient one becomes at solving these equations.

Most secondary mathematics teachers agree that the introduction of the graphing calculator has greatly aided in student learning. While some might argue that the understanding of algorithms and processes is essential to advancement in mathematics, the introduction of the graphing calculator into lessons can assist in enhancing the understanding of these concepts.

References

1. Sharon Barrs, James Braselton, Lorraine Braselton, A Rational Root Theorem for Imaginary Roots, *The College Mathematics Journal* **34**, No. 5, (2003) 280-382.

2. Thomas Butts, Fixed Point Iteration – An Interesting Way to Begin a Calculus Course, *The Two-Year College Mathematics Journal* **12** (1991) 2-7.

3. E John Hornsby, Jr. Geometrical and Graphical Solutions of Quadratic Equations, *The College Mathematics Journal* **21** (1990) 362-369.

4. Cliff Long and Thomas Hern, Graphing the Complex Zeros of Polynomials Using Modulus Surfaces, *The College Mathematics Journal* **20** (1989) 98-105.

5. R.S. Luthar, Luddhar's Method of Solving a Cubic Equation with a Rational Root, *The Two-Year College Mathematics Journal* **11** (1980) 107-110.

6. Alec Norton and Benjamin Lotto, Complex Roots Made Visible¸*The College Mathematics Journal* (1984) 248-249.

7. Don Redmond, Finding Rational Roots of Polynomials, *The College Mathematics Journal* **20** (1989) 139-141.

8. George Yanosik, Graphical Solutions for Complex Roots of Quadratics, Cubics, and Quartics, *National Mathematics Magazine* **17** (1943) 147-150.

Vita

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