CONCERNING R.L. MOORE'S AXIOM 5_1

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Dean of the Graduate School.

April 30, 1935.

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THESIS

Presented to the Faculty of the Graduate School of
The University of Texas in Partial Fulfil-
ment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

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Austin, Texas

June, 1934

PREFACE

In any space satisfying Axioms 0, 1, 2, 3, 4, and 5 of R.L. Moore's Foundations of Point Set Theory a large body of Plane Analysis Situs theorems hold true. Nevertheless, not every such space is a subset of the plane even if completely separable. However, Moore has shown that if such a space satisfies certain additional axioms, it is a subset of a plane. This problem has also been studied by Leo Zippin and, more recently, by J.H. Roberts. In this paper the problem is again attacked, and the treatment emphasizes the rather peculiar role played by connectedness. As a matter of fact, connectedness enters into all of the axioms used by the author except Axioms 0, 1, and 6.

In Part I a certain space is studied which, although not necessarily a subset of a plane, nevertheless, possesses many of the properties of a plane. However, if the space of Part I is completely separable or metric, it is shown to be a subset of a plane or a sphere.

Definitions of terms peculiar to the treatment are given in the text. For the definition of terms not defined, the reader is refered to R.L. Moore's Foundations of Point Set Theory. Many of the notational conventions of this work are also used here without explanation.

I wish to thank Professor R.L. Moore for attracting my attention to mathematics, for suggesting the present problem, and for many helpful suggestions in the course of its development.

CONTENTS

PART I

CONSEQUENCES OF AXIOMS 0-4 AND 5^*_1

R. L. Moore's Axiom 5_1--"If P is a point of a region R, there exists a connected domain D containing P and bounded by a compact continuum T such that D+T is a subset of R.[1]-- suggests the following two axioms:

Axiom 5^1_1: If P is a point of a region R, there exists in R a domain D containing P such that the boundary of D is compact,[2] and

Axiom 5^2_1: If P is a point of a region R, there exists in R a domain D containing P such that the boundary of D is connected.[3]

There is, however, a certain amount of similarity between these last two asioms. For suppose that a space satisfies Moore's Axioms 0, 1, and 2 and Axiom 5^1_1. Then, if P is a point of a region R, there exists a region R' containing P and lying together with its boundary in R. And by Axiom 5^1_1 there exists in R' a domain D containing P and whose boundary β is compact. But β may be covered by a

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[1]Moore, R.L.: Foundations of Point Set Theory, American Mathematical Society Colloquium Publications, Vol. XIII, New York, 1932.

[2]A space satisfying this axiom is said to be locally peripherally compact.

[3]A space satisfying this aciom is said to be locally peripherally connected.

a finite collection of connected domains each lying in R but not containing P or having P for a limit point. Hence, the following theorem holds true and will be used hereafter as

Axiom 5‡: If P is a point of a region R, there exists in R a domain D containing P whose boundary is a subset of the sum of a finite number of continua lying in R-D.

In a sense Axiom 5‡ is common to both Axioms 5† and 5‡ in that it follows as a theorem from either of them in the presence of Axioms 0, 1, and 2.

Now let S be a space satisfying Moore's Axioms 0, 1, 2, and 4 and Axiom 5‡.[4]

Theorem. If no point separates the point A from the

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[4]Axiom 0. Every region is a point set.

Axiom 1. There exists a sequence $G_1, G_2, G_3, \ldots$ such that (1) for each n, G_n is a collection of regions covering S, (2) for each n, G_{n+1} is a subcollection of G_n, (3) if R is any region whatsoever, X is a point of R and Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X then $\overline{g}$ is a subset of (R-Y)+X, (4) if $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets such that, for each n, M_n contains M_{n+1} and, for each n, there exists a region g_n of the collection G_n such that M_n is a subset of $\overline{g}_n$, then there is at least one point common to all the point sets of the sequence $M_1, M_2, M_3, \ldots$.

Axiom 2. If P is a point of a region R there exists a non-degenerate connected domain containing P and lying wholly in R.

Axiom 4. If J is a simple closed curve, S-J is the sum of two mutually separated connected point sets such that J is the boundary of each of them.

point B, there exists a simple closed curve J containing A and B.[5]

In the proof that is to follow no use will be made of Axioms 3 or 4, and no use will be made of Axiom 5† except at the points A and B.

Proof. Let AB denote an arc from A to B. Then, if R is a region containing A, suppose that there exists no point O of AB·R such that no point X of the arc AO of AB separates A from O in R. Then there exists a sequence α of points X of AB such that the sequence α converges to A and if X and Y are points of α and X follows Y in α, then X separates A from Y in R. Hence, for each point X of α except the first, R-X is the sum of two domains U_X and V_X such that V_X contains A and U_X contains every point of α preceding X in α. But the point X does not separate A from B; so there exists an arc from A to B lying in S-X, and in this arc there exists an arc segment $\widehat{PE}$ lying in R such that AB·PE=P and E belongs to the boundary of R.[6] Therefore, if for no X of α is P identical with A, there exists a sequence X_1, X_2, X_3, ... of points of α and a sequence P_1E_1, P_2E_2, P_3E_3, ... of arcs such that

[5]See R.L. Moore's Foundations of Point Set Theory, p. 124 (this work will be refered to hereafter as Foundations), and G.T. Whyburn, "On the Cyclic Connectivity Theorem", Bull. Am. Math. Soc., Vol. 37, 1931, pp.429-433.

[6]This notation will be used throughout to denote the arc minus its end points.

for each n (1) $AB \cdot P_nE_n = P_n$, (2) E_n belongs to the boundary of R, (3) $\widehat{P_nE_n}$ is a subset of R, and (4) X_{n+1} separates $\sum_{i=1}^{n} \widehat{P_iE_i}$ from $\sum_{i=n+1}^{\infty} P_iE_i+A$ in R. But (1), (2) and (3) lead to a contradiction of (4), for since there exists a domain D lying in R and containing A whose boundary is a subset of the sum of a finite collection Δ of continua lying in R-D, there exists a continuum M of Δ lying in R-A such that M does not contain more than a finite number of the points X_1, X_2, X_3, ... but for infinitely many values of n, M contains a point of P_nE_n. Therefore, either there exists an arc PE which is a subset of an arc from P to B such that $PE \cdot AB = P = A$, or if R is any region whatsoever containing A, there exists a point O of $AB \cdot R$ such that no point of the interval AO of AB separates A from O in R.

Suppose the latter. Let R_1, R_2, R_3, ... be a monotonic sequence of regions closing down on A and O_1, O_2, O_3, ... be a sequence of points of AB such that for each n (1) the arc AO_n is a subset of R_n, (2) no point X of AO_n separates A from O_n in R_n. For each n and each point X of AO_n there exists an arc T_{xn} lying in R_n and having only its end points in common with AB such that the segment S_{xn} of AB between these end points contains X and is a subset of R_n. For each m there exists a finite collection H_n of the segments S_{xn} covering the arc $O_{n+1}O_{n+2}$ of AB. But each

$S_{xn}+T_{xn}$ is a simple closed curve lying in R_n. Hence, there exists a sequence J_1, J_2, J_3, ... of simple closed curves such that (1) for each m J_m and J_{m+1} have at least two points in common and (2) if R is a region containing A there exists a number δ such that if m>δ then J_m is a subset of R. Hence, by Theorem 35 of Chapter II of Foundations there exists in ΣJ_m a simple closed curve J_A containing A and a segment of AB.

If, on the other hand, there exists an arc PE which is a subset of an arc PB from P to B such that PE·AB=P=A, then there exists in PB an arc PX_1 such that P and X_1 are the only points PX_1 has in common with AB. But PX_1+AX_1(of AB) forms a simple closed curve containing A and some segment of AB.

So in either case there exists a simple closed curve J_A containing A and some segment OF of AB. Likewise, there exists a simple closed curve J_B containing B and a segment YE of AB. Now for each point X of the arc FY of AB, there exists an arc T_x such that T_x has only its end points in common with AB and the segment S_x of AB lying between these end points contains X. Some finite collection of these segments S_x covers the arc FY. Since every S_x+T_x is a simple closed curve, there exists a sequence of simple closed curves $J_1=J_A$, J_2, J_3, .. , $J_k=J_B$ such that for each

$n<k$ J_n contains at least two points of J_{n+1}. By Theorem 35 of Chapter II of Foundations there exists in the sum of these simple closed curves a simple closed curve J containing A and B.

The example as indicated in the figure is a subset of a plane with the shaded portions removed. Except for those points of the arc XAY different from A, the boundaries of these shadded portions are not removed. It will be easily seen that the result is a connected, connected im kleinen inner limiting subset of the plane and hence, satisfies Axioms 0, 1, and 2.[7] Also Axiom 5_1^* is satisfied at every point except A. As a matter of fact, the space is locally compact at every point except A. It is also true that if X is a point and R is a region containing A, there exists in R an arc (or a simple closed curve) separating A from X. Still there exists no simple closed curve containing A and B. So the above theorem does not hold true in this space.

Theorem 2. The space S is either acyclic or contains no cut point.[8]

[7] See Theorem 9 of Chapter II of Foundations.

[8] Leo Zippin has done something of this nature in his paper, "On Continuous Curves and the Jordon Curve Theorem", Amer. Jour. Math., Vol. 52, 1930, pp. 331-350.

Proof. Suppose that S contains a simple closed curve J. Then it is evident from Axiom 4 that S is connected and that no point of J separates S. Suppose that some point X separates S. Let AX denote an arc with end points A and X and such that $AX \cdot J = A$. Let M denote the set of cut points of S that belong to AX and let B denote the first point of $\overline{M}$ in the order from A to X. B is a cut point of S. For if it is not, then B is a sequential limit point of a sequence of points B_1, B_2, B_3, ... of M and at the same time lies on a simple closed curve C containing an arc segment of AX which contains B; hence, C contains points of the sequence B_1, B_2, B_3, ... which contradicts Axiom 4. But since $\widehat{AB}$ of AX contains no point of M, no point separates A from B in S and therefore, there exists a simple closed curve containing A and B which again contradicts Axiom 4.

It is not the purpose of this paper to treat the acyclic case;[9] so we shall at this point assume R.L. Moore's

Axiom 3: If O is a point, S-O is connected.

Theorem 3. If A and B are distinct points, there exists a simple closed curve separating A from B.

Proof. There exists a simple closed curve J such that J is the sum of two arcs AXB and AYB. S-J is the sum of two connected domains I and E each having J for its boundary. Hence, there exists two arcs X'O'Y' and X*O*Y* such that (1) the points X' and X* lie on the segment $\widehat{AXB}$

[9] Such a space is a Regular (Menger) Curve.

and the points Y' and Y* lie on the segment $\widehat{AYB}$, and (2) the segment $\widehat{X'O'Y'}$ is a subset of I and the segment $\widehat{X^*O^*Y^*}$ is a subset of E. The simple closed curve formed by the sum of the four arcs X'O'Y', X*O*Y*, X'X*(of AXB), and Y'Y* (of AYB) separates A from B in S.

Theorem 4. If A, B, and ω are three distinct points, there exists a simple closed curve separating A from B+ω.

Proof. Let BXω denote an arc from B to ω not containing A. For each point P of BXω there exists a simple closed curve J_P separating P from A. So there exists a finite number of simple closed curves J_1, J_2, J_3, ... , J_n such that if P is a point of BXω, some one of them separates P from A. By Theorem 13 of Chapter III of <u>Foundations</u> J_1+ J_2+ .. +J_n contains a simple closed curve J separating A from the arc BXω and hence, from B+ω.

Theorem 5. If P and ω are distince points not belonging to the closed and compact point set H, there exists a simple closed curve separating P from H+ω.

Proof. For each point X of H there exists a simple closed curve J separating P from X+ω. Since H is closed and compact, there exists a finite number of simple closed curves J_1, J_2, J_3, ... J_n such that if X is any point of H, some one of these separates P from X+ω. For each integer $i \leq n$ let T_i denote an arc from J_1 to J_i and let M denote the compact continuum $J_1+J_2+..+J_n+T_1+T_2+..+T_n$. The continuum M separates P from H+ω. Now for each point X of M let C denote a simple closed curve separating X from P and let C_1, C_2, C_3, ... , C_j denote a finite collection of these whose interiors with respect to P as the point at infinity cover M. By Theorem 13 of Chapter III of Foundations $C_1+C_2+..+C_j$ contains a simple closed curve separating P from M and hence, separating P from H+ω.

With the help of Theorem 5, the arguments of R.L. Moore with slight modifications prove that a number of the theorems of Chapter IV of Foundations hold true in this space. Of these the following will be used:

Theorem 6. If the points A and B belong to different components of the closed and compact point set M, there exists a simple closed curve separating A from B and containing no point of M. (Theorem 10 of Foundations, Chapter IV).

Theorem 7. If α and β are two connected point sets and neither of the two mutually exclusive closed and compact point sets H and K separates α from β, then H+K does not separate α from β. (Theorem 16).

Theorem 8. If the common part of the two closed and compact point sets H and K is a continuum and neither H nor K separates the point A from the point B, then H+K does not separate A from B. (Theorem 18).

Theorem 9. No arc separates S. (Theorem 19).

Theorem 10. If the points A and B are separated from each other by the closed and compact point set M then they are separated from each other by a continuum which is a subset of M and which contains no proper subset that separates A from B. (Theorem 24).

Theorem 11. If P is a point of a domain D, there exists a region R containing P such that if a simple closed curve J is a subset of R, one of the complementary domains of J is a subset of D.

Proof. Let Q denote a domain lying in D and containing P such that the boundary of Q is a subset of the sum of a finite collection Δ of continua lying in D-Q. By Axiom 3 S-P is a connected domain. By Theorem 1 of Chapter II of Foundations there exists an arc joining any two points of S-P and in particular when the points belong to diffenent continua of the collection Δ. Hence, the boundary of Q is a subset of a continuum M lying in S-P. Since M is closed and does not contain P, there exists a region R which contains P and no point of M but which is a subset of Q. If a simple closed curve J is a subset of R, both of its complementary domains contain points of Q. If both of these domains contain points of S-D, then both of them must contain points of M. And in this case M must contain some point of J. But M contains no point of J.

Hence, one of the complementary domains of J is a subset of D.

Theorem 12. If D is a complementary domain of a compact continuous curve M and P is a point of the boundary of D, then K=D+P is a connected, connected im kleinen inner limiting set.

Proof. It is evident that K is a connected inner limiting set. It is furthermore evident that K is connected im kleinen at every one of its points with the possible exception of P. Suppose that K is not connected im kleinen at P. Then there exists a domain Q containing P such that the component of Q·K, C, that contains P is not open with respect to K at the point P. Hence, there exists a sequence α of points of K converging to P such that each point of α belongs to some component of Q·K but not to C, and no two points of α belong to the same component of Q·K. There exists a point O of K not belonging to Q, for otherwise Q·K=K would be connected and open with respect to K at P. For each point X of the sequence α let XO denote an arc from X to O lying in D, and let Y denote the first point in the order from X to O that XO has in common with the boundary of Q.

Now let R be a region containing P and lying in Q such that if J is a simple closed curve lying in R, one of its complementary domains is a subset of Q, and let Q_1

and Q_2 denote domains lying together with their boundaries in R and containing P such that Q_1 contains $\overline{Q}_2$ and β_2, the boundary of Q_2, is a subset of the sum of a finite collection Δ of continua lying in Q_1-Q_2. For infinitely many points X of the sequence α the interval XY of the arc XO contains a point of some one of the continua of the finite collection Δ. Hence, there exists a continuum H of Δ such that for infinitely many points X of α the arc XY contains a point of H. Now let D_H and D_P denote connected domains lying together with their boundaries in R and containing H and P respectively such that $\overline{D}_H \cdot \overline{D}_P = 0$.

For each point X of α such that X lies in D_P and the arc XY contains some point of H, let TX denote an arc lying in D_P such that T belongs to M but the segment $\widehat{TX}$ is a subset of D and let Z denote a point of $XY \cdot H$ such that the interval XZ of the arc XY is a subset of R. Now D_H lies in Q and contains points of more than one component of $Q \cdot K$ and hence, contains some point of M. For each Z let ZW denote an arc lying in D_H such that the segment $\widehat{ZW}$ is a subset of R and W is a point of M. Thus, for some infinite subsequence $X_1, X_2, X_3, \ldots$ of α there correspond sequences of points $T_1, T_2, T_3, \ldots$ and $W_1, W_2, W_3, \ldots$ such that (1) the points $T_1, T_2, T_3, \ldots$ belong to $D_P \cdot M$, (2) the points $W_1, W_2, W_3, \ldots$ belong to $D_H \cdot M$, (3) for each n $T_nX_n+X_nZ_n+Z_nW_n$ contains an arc T_nW_n lying in R, and (4) if $m \neq n$, $\overset{\frown}{T_m W_m}$ and $\overset{\frown}{T_n W_n}$ are mutually exclusive and lie in different components of $Q \cdot K$. Since M is compact and a continuous curve, there exists two mutually exclusive connected subsets d_P and d_H of $M \cdot R$ which are open with respect to M such that for infinitely many values of n and in particular for the three different values a, b, and c of n, T_n and W_n lie in d_P and d_H respectively. Let $T_a T_b$ and $T_a T_c$ denote arcs lying in d_P and $W_a W_b$ and $W_a W_c$ denote arcs lying in d_H. There exist three arcs AO_1B, AO_2B, and AO_3B lying in $T_aT_b + T_aW_a + W_aW_b$, $T_aT_b + T_bW_b + W_aW_b$, and $T_aT_c + T_cW_c + W_aW_c$ respectively and having their end points A and B in common. Therefore, by Theorem 5 of Chapter III of <u>Foundations</u>, the sum of two of these arcs, say AO_1B and AO_3B, forms a simple closed curve J lying in R and whose interior with respect to some point ω of the boundary of Q contains the segment of the third arc, namely, $\overset{\frown}{AO_2B}$. But the interior of J is a subset of Q and contains no point of the component of $Q \cdot K$ which contains points of the segment $\overset{\frown}{AO_2B}$ and has limit points in the boundary of Q. This is a contradiction, and K is connected im kleinen at P.

The following theorems follow immediately from Theorem 12 together with Theorem 10 of Chapter II of <u>Foundations</u>.

Theorem 13. Every point of a simple closed curve is accessable from either of its complementary domains.

Theorem 14. If T is an arc, every point of T is accessable from S-T.

Furthermore, a number of the intuitive propositions concerning abutting and crossing arcs hold true. Although some of these will be used in arguments to follow, they will not be stated and the reader is referen to Chapter IV, Theorems 28-32, of Foundations for their precise statement and proof. Some of these proofs must be modified, however, to be valid for the set of axioms used here.

Theorem 15. An arc is accessable from both sides at any interior point.

Theorem 16. If the arc AB is a subset of the connected domain D and D-AB is connected, then D contains a simple closed curve J separating A from B.

Proof. Since the arc AB is accessable from both sides at an interior point O, there exist two arcs EO and FO abutting on AB from different sides such that neither EO nor FO has any point except O in common with the arc AB. The arcs EO and FO contain points E_1 and F_1 respectively, such that the arcs E_1O and F_1O lie in D. In D-AB there exists an arc E_1F_1 and in $E_1O+E_1F_1+F_1O$ there exists a simple closed curve J lying in D and separating A from B.

Theorem 17. If P is a point of a connected domain D, then there exists a region R containing P such that if X is a point of R-P, X lies in a simple domain which together with its boundary is a subset of D.

Proof. If D is S, the theorem is evident from Theorem 3. Suppose that the boundary of D is not vacuous and that the theorem is false. Then there exists a sequence α of points X converging to P such that X does not lie in a simple domain which together with its boundary is a subset of D. Let Δ denote a monotonic sequence of connected domains Q closing down on P. Let Q_1 denote an element of Δ lying in D such that any simple closed curve lying in Q_1 has one of its complementary domains in D. It is clear from Axioms 3 and 5_1^* that Q_1-P contains only a finite number of components. One of these components, say C_1, contains two points X_1 and X_2 of α. Let T_1 denote an arc from X_1 to X_2 lying in C_1. Let Q_2 denote the first element of Δ which follows Q_1 and contains no point of T_1. One of the components of Q_2-P contains two points X_3 and X_4 of α and an arc T_2 from X_3 to X_4. Let Q_3 denote the first element of Δ not containing a point of T_1+T_2. Then Q_3-P contains an arc T_3 whose end points are points of α. If this process is continued, one may construct a sequence of mutually exclusive arcs T_1, T_2, T_3, ... converging to P such that, for each n, T_n is a subset of Q_n-P and the end points of T_n are points of α. By the preceding theorem, for each n, T_n separates Q_1. For each n one of the components of Q_1-T_n does not contain P and in this component there exists an arc segment W_n containing no point of $T_1+T_2+..+T_{n-1}$. Now since $P+T_1+T_2+..+T_n$ does not separate space (Theorems 9 and 7), there exists an arc segment W_n' lying in $Q_1-(P+T_1+T_2+..+T_n)$ having one end point in W_n and the other end point in the boundary of Q_1. For each n let L_n denote the component of $Q_1-(P+T_1+T_2+..+T_n)$ which contains W_n+W_n'. Then for each n T_n separates L_n from P in Q_1, and $\overline{L}_n$ contains points of both T_n and the boundary of Q_1. Now by Axiom 5_1^* there exists in Q_1 a domain U which contains P and whose boundary is a subset of the sum of a finite number of continua lying in Q_1-U. There is a continuum H of this set which, for infinitely many values j of n, contains a point of L_j while U contains T_j. But if k is the smallest value of j, $P+H+\Sigma L_j$ is a connected point set in Q_1-T_k containing P and L_k. This is a contradiction.

Definition. If P is a point and there exists a sequence of simple domains closing down on P, then P is said to be a simple point. A non-simple point is said to be an edge point.

Theorem 18. The set of simple points is everywhere dense.

Proof. Suppose that R is a region. With the help of repeated applications of Theorem 17 it may be shown that there exists a sequence D_1, D_2, D_3, ... of simple domains lying in R such that for each n (1) D_n contains $\overline{D}_{n+1}$ and (2) $\overline{D}_n$ lies in some region of the collection G_n of Axiom 1. By (4) of Axiom 1 there exists a point P common to $\overline{D}_1$, $\overline{D}_2$, D_3, ... and hence, to D_1, D_2, D_3, ... and by (3) of Axiom 1 P is the only common point. The point P is simple and lies in R.

Theorem 19. No completely separable point set contains uncountably many edge points.

Proof. Let H denote the collection of all edge points belonging to the completely separable point set M. If X is a point of H, there exists no sequence of simple domains dlosing down on X. Hence, for each point X of H there exists an integer n_x such that no region of G_{n_x} of Axiom 1 contains a simple domain containing X. If H is uncountable, there exist an integer k and an uncountable subset K of H, such that if X is a point of K, n_x=k. Since M is completely separable, K has a point P of condensation. Let D denote a connected domain containing P and lying in a region of G_k of Axiom 1. By Theorem 17 there exists in D a region R containing P such that every point X of R-P is contained in a simple domain lying in D. This is clearly a contradiction.

Theorem 20. If M is a countable set of simple points, S-M is arc-wise connected.

This theorem may be proved without using Axioms 4 and 5_1^* by a modification of R.L. Moore's proof of Theorem 1 of Chapter II of Foundations.[10] The kernal of this proof is the construction of a sequence C_1, C_2, C_3, ... of simple dhains of connected domains from a point A to a point B such that the common part of the point sets $C_1^* > C_2^* C_3^* >$ ...[11] is an arc from A to B. Now for each n C_n^* is a connected domain, and it is easy to see that no finite number of simple points disconnects a connected domain. Hence, if (1) A and B are any two points of S-M, (2) $M=P_1+P_2+P_3+$ ... and (3) for each n the chain C_n is constructed so that C_n^* contains no point of the set $P_1+P_2+..+P_n$, then the arc from A to B of Moore's construction will not contain any point of M.

Theorem 21. If M is a countable set of simple points, S-M is cyclicly connected.

Proof. If A and B are any two points of S-M, then by Theorem 20 there exists an arc AXB in S-M. With the help of Theorems 7, 9, and 12 one may show that S-$\widehat{AXB}$ is a space

[10] See also J.R. Kline, "Concerning the Complement of a Countable Infinity of Point Sets of a Certain Type," Bull. Amer. Math. Soc., Vol. 23, 1917, pp. 290-292.

[11] C* denotes the sum of the elements of C.

satisfying Axioms 0, 1, 2, and 3. Hence, by Theorem 20 there exists an arc AYB lying in $S-(M+\widehat{AXB})$. AXB+AYB forms a simple closed curve lying in S-M and containing A+B.

Theorem 22. No compact set of edge points separates two simple points from each other.

Proof. Suppose that A and B are two simple points and M is a compact set of edge points. Suppose that n is any fixed integer. Then there exists from A to B a simple chain of simple domains such that each element of the chain is a subset of some region of G_n of Axiom 1. For suppose the sontrary. Let H_A denote the set of all points X such that there exists from A to X a chain of simple domains whose elements are each a subset of some region of G_n. Since A is a simple point, H_A is a connected domain. This domain does not contain B, and its boundary β is a subset of M. Hence, β is a closed and compact point set which separates A from B. By Theorem 10 and Axiom 3, β contains a non-degenerate continuum separating A from B. Hence, M is uncountable which contradicts Theorem 19.

Now with the help of Theorem 9 of Chapter II of <u>Foundations</u> and Theorem 7 one may show that a simple domain is a space satisfying Axioms 0, 1, 2, 3, 4, and 5_1^*, and that the complementary domain of a point is also such a space.

Let $M=P_1+P_2+P_3+\ldots$. Let C_1 denote a simple chain of simple domains from A to B such that (1) each element of C_1 is a subset of some region of G_1 and (2) C_1^* does not contain P_1. Now the boundary of the domain C_1^* is a closed and compact point set not separating A from B; so there exists a simple closed curve separating it from A+B (Theorem 5 in modification). Hence, there exists in C_1^* a simple domain D_1 containing A+B. Let C_2 denote a simple chain of simple domains from A to B such that if c is an element of C_2, (1) $\bar{c}$ is a subset of some element of C_1, (2) c is a subset of some region of G_2 and (3) c does not contain P_2. This process may be continued and by Theorem 80 of Chapter I of *Foundations* $C_1^* \cdot C_2^* \cdot C_3^* \cdot \ldots$ is a continuum. Hence, M does not separate A from B.

As a matter of fact, the above process may be carried out along the lines of the proof of Theorem 1 of Chapter II of *Foundations*, so as to show that S-M is arc-wise and even cyclicly connected.

That Theorem 22 does not remain true if the condition of the compactness of M is omitted is shown by the following example. Let space consist of all points of the number plane whose ordinates are not zero together with those ordinates are zero but whose abscissas are rational. If (X_0,Y_0) is a point whose ardinate Y_0 is not zero, then for each number $r<|Y_0|$ let $V_r(X_0,Y_0)$ denote the point set consisting of (X_0,Y_0) together with all points (X,Y) which lie at a distance less than r from (X_0,Y_0). No V_r contains a point of the X-axis. Let P_1, P_2, P_3, ... denote the points of the X-axis whose abscissas are rational. For each integer m let R_{m1} denote the point set consisting of P_m together with the interiors of the teo isocles triangles with vertices at P_m and bases parallel to the X-axis, such that (1) the angle at the vertex of each is one radian, (2) the altitude h_m of each is less than or or equal to 1/m and (3) for each $i\neq m$ $\bar{R}_m \cdot \bar{R}_i=0$. In general for each m and n' let $R_{mn'}$ satisfy the conditions imposed on R_{m1} except that the vertex angle in 1/n' radians and the altitude is less than h_m/n'. The point sets $V_r(X,Y)$ and $R_{mn'}$ shall be called regions, and for each integer n G_n of Axiom 1 shall denote the collection of all of the point sets V_r with $r<1/n$ together with all of the point sets $R_{mn'}$ with $n'>n$. It can be shown that Axioms 0, 1, 2, 3, 4, and 5_1^* hold true in this space. Furthermore, the space is completely separable and locally compact at all but a countable no-where-dense set of points, namely, the set of edge points. This set of points actually separates the lower half of the space from the upper half of the space. Hence, the above theorem does not hold true when A is in the lower half of the space and B is in the upper half of the space, the point set H_A of the above argument consisting of all the points below the X-axis.

Definition. If a domain is connected and contains one of the complementary domains of each simple closed curve lying in it, then it is said to be simply connected.

Definitions. Suppose that P is a point of a domain R and Δ is a collection of continua whose sum separates P from the boundary of R such that each continuum of Δ lies in a component of R-P whose boundary contains P but no component of R-P contains more than one element of Δ. Then the collection Δ is said to be minimal with respect to R and P. And if D is a domain containing P whose boundary is a subset of Δ*, then Δ is said to surround D minimally with respect to R and P.

Theorem 23. If Δ is a collection of continua which surrounds a connected domain D minimally with respect to a domain R and a point P, then (a) each component of R-P whose boundary contains P contains one and only one element of Δ, and (b) no component of D-P has boundary points in more than one element of Δ.

Proof. (a) By definition if C is a component of R-P whose boundary contains P, then C contains not more than one element of Δ. But since S-P is connected, the boundary of C also contains a point of the boundary of R; hence, C must contain at least one element of Δ. Therefore, C contains one and only one element of Δ.

(b) Suppose that there exists a component C' of D-P which has boundary points in more than one element of Δ. Let C denote the component of R-P which contains C'. Then C contains more than one element of Δ. But since D is connected, the boundary of C contains P. This is a contradiction of (a).

Theorem 24. If P is a point of a domain R, not S, there exist in R a connected domain D containing P and a finite collection Δ of continua such that Δ surrounds D minimally with respect to R and P.

Proof. Let D' denote a domain lying in R and containing P whose boundary is a subset of the sum of the elements of a finite collection Δ' of continua lying in R-D'. Let N denote the sum of all the continua of Δ' which lie in components of R-P whose boundaries contain P. If H and K are components of N lying in the same component C of R-P, let T denote an arc in C from a point of J to a point of K. Let Δ denote the collection of all components of the point set obtained by adding N to the sum of the arcs T. The collection Δ minimally separates P from the boundary of R. Let D denote the component of R-Δ^* that contains P. Then, the boundary of D is a subset of Δ^*, and hence, Δ surrounds D minimally with respect to R and P.

Theorem 25. If D is a connected domain and there exists a collection Δ of continua such that Δ surrounds D minimally with respect to a domain R and a point P, then D is simply connected.

Proof. Suppose, on the contrary, that there exists in D a simple closed curve J such that neither of its complementary domains I and E are subsets of D. Since J lies in D, both I and E contain points of D and, therefore, points of the boundary β of D. Let T_I and T_E denote arcs irreducible from J to β·I and β·E respectively which lie in S-P and let T denote an interval of J·(S-P) from T_I to T_E. The continuum T_I+T+T_E lies in R-P and contains points of two different continua of Δ. This is a contradiction.

Theorem 26. If P is a point of a region R, there exists in R a simply connected domain D containing P.

Theorem 26 is a consequence of Theorems 24 and 25.

Definitions. If D_1 and D_2 are domains such that D_1 contains $\bar{D}_2$ and $\widehat{T}$ is a segment lying in $D_1-\bar{D}_2$ having one end point of the boundary of D_1 and the other on the boundary of D_2, then $\widehat{T}$ is said to cross $D_1-\bar{D}_2$. Furthermore, if C is the component of $D_1-\bar{D}_2$ which contains $\widehat{T}$, then $\widehat{T}$ is said to cross C.

Definition. If D_1 and D_2 are domains such that D_1 contains $\bar{D}_2$, C is a somponent of $D_1-\bar{D}_2$, and $\widehat{T}_1$, $\widehat{T}_2$, and $\widehat{T}_3$

are segments crossing C, then $\widehat{T}_2$ is said to be between $\widehat{T}_1$ and $\widehat{T}_3$ in C if $\widehat{T}_2$ separates $\widehat{T}_1$ from $\widehat{T}_3$ in C.

Theorem 27. Suppose that D_1 and D_2 are simply connected domains such that D_1 contains $\overline{D}_2$ but no simple domain lying in D_1 contains D_2, C is a component of $D_1-\overline{D}_2$, and $\widehat{T}_1$, $\widehat{T}_2$, $\widehat{T}_3$ are three mutually exclusive segments crossing C. Then one and only one of these segments is between the other two in C.

Proof. That not more than one of these segments lies between the other two in C is an immediate consequence of a well known theorem.

Suppose that no one of them is between the other two in C. Then either (1) $C-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3)$ contains a component W shich has limit points in each of the three segments, or (2) $C-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3)$ contains three mutually exclusive components W_{12}, W_{13}, and W_{23} having limit points in the segments indicated by their subscripts.

Case I. Suppose (1). Then there exist in W three mutually exclusive segments $\widehat{AA_1}$, $\widehat{AA_2}$, and $\widehat{AA_3}$ having a common end point A in W such that A_1, A_2, and A_3 belong to $\widehat{T}_1$, $\widehat{T}_2$, and $\widehat{T}_3$ respectively. For each i (i=1,2,3) let O_i and P_i denote the end points of T_i which belong to the boundaries of D_1 and D_2 respectively. Since D_2 is connected, there exist in $D_1-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3+AA_1+AA_2+AA_3)$ three mutually exclusive segments $\widehat{BB_1}$, $\widehat{BB_2}$, and $\widehat{BB_3}$ having a common end point B in D_2 and having their other end points B_1, B_2, and B_3 in the intervals A_1P_1 of T_1, A_2P_2 of T_2, and A_3P_3 of T_3 respectively. If AB_nB denotes AA_n+A_nB(of A_nP_n) $+BB_n$ (n=1,2,3), then $\widehat{AB_1B}$, $\widehat{AB_2B}$, and $\widehat{AB_3B}$ are mutually exclusive segments. By Theorem 5 of Chapter III of Foundations two of these arcs, say AB_1B and AB_3B, form a simple closed curve J whose interior I with respect to O_1 contains the segment from A to B of the remaining one. Since D_1 is simply connected, I is a subset of D_1. But T_2 contains B_2 and no point of J and is, therefore, a subset of I. Hence, I contains O_2. This is a contradiction.

Case II. Suppose (2). There exists segments $\widehat{A_1B_3}$, $\widehat{B_2A_3}$, and B_1A_2 lying in W_{13}, W_{23}, and W_{12} respectively and having end points on the arcs T_1, T_2, and T_3 as indicated by the subscripts in the notation. The point set $A_1B_3+A_1B_1$(of T_1)$+B_1A_2+A_2B_2$(of T_2)$+B_2A_3+A_3B_3$(of T_3) is a simple closed curve J lying in D_1-D_2. Since D_1 is simply connected, one of the complementary domains I of J is a subset of D_1. Since J contains no point of D_2, I either contains D_2 or is a subset of $D_1-\overline{D}_2$. By hypothesis D_1 contains no simple domain containing D_2. Hence, I is a subset of $C-(\widehat{T}_1+\widehat{T}_2+\widehat{T}_3)$ and has limit points in each of the three segments $\widehat{T}_1$, $\widehat{T}_2$, and $\widehat{T}_3$, which is Case I again.

Lemma. If the compact continuum K does not separate the point A from the point B and G is a finite collection of compact continua such that (1) the common part of any two elements of G is a subset of K and (2) if H is any element of G, H does not separate A from B and H·K is connected, then K+G* does not separate A from B.

This lemma may be established by a finite number of applications of Theorem 8.

Theorem 28. If AXB is an arc and J is a simple closed curve separating A from B, then J+AXB contains a simple closed curve C separating A from B such that C·AXB is connected.

Proof. Let A' and B' denote the first and last points respectively that the arc AXB has in common with J. Let AYB denote an arc from A to B such that AXB·AYB=A+B. Let G denote the collection of all simple closed curves C in J+AXB such that C·AXB is connected and C·AYB is not vacuous. The collection G is finite. Let K denote A'B'+(J-J·G*). The point set K is a continuum containing no point of AYB; hence, K does not separate A from B. Since the common part of any two elements of G is a subset of K, and the common part of K with an element of G is connected, therefore, if no simple closed curve of the collection G separates A from B, then by the preceding lemma K+G* does not separate A from B. Therefore, some element C of G separates A from B.

Theorem 29. Suppose that D_1, D_2, and D_3 are simply connected domains containing the point P such that (1) D_3 contains $\overline{D}_2$, (2) D_3 is surrounded minimally with respect to D_1 and P by a finite collection, and (3) D_1 contains no simple domain containing P. Then if C is a component of $D_1-\overline{D}_2$, $\widehat{T}$ is an arc segment crossing C, and J is a simple closed curve separating T from P, J contains two segments $\widehat{T}_a$ and $\widehat{T}_b$ which cross C such that $\widehat{T}$ is between them in C.

Proof. Let R denote a connected domain lying in $D_1-D_1\cdot J$ and containing the end point A of T which is on the boundary of D_2. Now in $R+D_2$ there exists an arc AP from A to P, such that $J\cdot AP$ is a subset of an interval $A'P'$ of AB lying in D_2. By Theorem 25 $J+AP$ contains a simple closed curve J' separating A from P such that $J'+AP$ is connected. $J'\cdot AP$ is a subset of $A'P'$ and is, therefore a subset of D_2. Since D_1 is simply connected but contains no simple domain containing P, J' contains a point O not belonging to D_1. $J'-(J'\cdot AP+O)$ is the sum of two mutually exclusive segments $\widehat{OX}$ and $\widehat{OY}$ where X and Y belong to AP (and Y belongs to the interval XP of AP) and the arcs OX and OY abutt on AP from different sides. Both OX and OY contain segments which cross $D_1-\overline{D}_2$. Let $\widehat{T}_a$ denote the first segment of OX in the order from X to O that crosses $D_1-\overline{D}_2$, and let $\widehat{T}_b$ denote the first segment of OY in the order from Y to O which crosses $D_1-\overline{D}_2$. Now C contains an element of Δ and if U denotes the component of D_1-P which contains C, U has no point in common with any other component of $D_1-\overline{D}_2$ because Δ is a minimal collection with respect to D_1 and P. Hence, $\widehat{T}_a$ and $\widehat{T}_b$ cross C. Let X' and Y' denote the end points of T_a and T_b respectively which are on the boundary of D_2. The intervals XX' of OX and YY' of OY lie in D_1.

Suppose that $\widehat{T}$ is not between $\widehat{T}_a$ and $\widehat{T}_b$ in C. Then there exists an arc T_1 in C irreducible from $\widehat{T}_a$ to $\widehat{T}_b$ not intersecting $\widehat{T}$. Let W denote the last point that OX has in common with T_1 and let Z denote the last point that OY has in common with T_1. The intervals WX of OX and ZY of OY are subsets of T_a+XX' and T_b+YY' respectively, and hence, are subsets of D_1. The arcs WX and ZY abutt on AP from different sides. Now $J''=J'\cdot AP+WX+WZ(\text{of } T_1)+ZY$ is a simple closed curve lying in D_1 not containing P, or a point of T, or a point of $\widehat{PY}$ of AP. Since D_1 is simply connected, one of the complementary domains, I, of J'' is a subset of D_1. But since D_1 does not contain a simple domain containing P, both P and T must be subsets of the other complementary domain E of J''. Now AX contains an arc which abutts on J" from the side opposite PY of AP, and since both A and $\widehat{PY}$ are in D, AX must intersect J". Let A_1 and A_2 denote the first and last points respectively that $\widehat{AX}$ has in common with J". $\widehat{AA_1}$ and $\widehat{A_2X}$ of AX lie in E and I respectively. The segment $\widehat{AX}$ does not intersect J'; so $\widehat{AX}\cdot J''$ is a subset of T_1. Hence, $\widehat{AA_1}$ and $\widehat{A_2X}$ approach T_1 from different sides. Let R' denote a connected domain lying in D_1 and containing A but no point of $T_1+T_a+T_b$. Then, in $R'+D_2$ there exists an arc FG irreducible from AA_1 to A_2X. The simple closed curve $J_3=FG+GA_2(\text{of } A_2X)+A_1A_2(\text{of } T_1)+A_1F(\text{of } AA_1)$ lies in D_1, contains no point of T_a+T_b, but crosses T_1. Hence, T_a and T_b lie in different complementary domains of J_3. This is impossible, for since D_1 is simply connected, one complementary domain of J_3 is a subset of D_1.

Theorem 30. In order that space be metric, it is necessary and sufficient that space be completely separable.

Proof. That a space satisfying Axioms 0 and 1 is metric if it is completely separable has been shown by R.L. Moore.[12] Hence, this space is metric if it is completely separable.

It will now be shown that if space is metric, it is completely separable. Let P denote an edge point and let

[12] See Foundations p. 459 and p. 464.

R denote a region containing P but containing no simple domain containing P. Since R is not S, by Theorems 24 and 25 there exists in R three simply connected domains D_1, D_2, and D_3 such that (1) D_3 is surrounded minimally with respect to D_1 and P by a collection Δ of continua, and (2) D_3 contains $\bar{D}_2$. D_1 contains no simple comain containing P. Let C denote a component of $D_1 - \bar{D}_2$.

Suppose that there are two mutually exclusive arcs T_1 and T_{-1} crossing C. Let α' denote a well ordered sequence (whose first element is T_1) of all arcs T which cross C such that either (1) T is T_1, or (2) T_1 lies between T_{-1} and T in C. Let α denote a subsequence of α' such that (1) the first element of α is T_1, (2) if an element T of α is not T_1, then T is the first element of α' which neither intersects a preceding element of α nor lies between two preceding elements of α, and (3) every element of α' either intersects an element of α or lies between T_1 and some element of α in C. Then, between any element of α and the next following element in α there is no element of α.

The sequence α is countable. For suppose that α is uncountable. For each arc T of α let d_T denote the distance from T to the next element of α. There exists a number ϵ and an uncountable subsequence α_ϵ of α such that if T belongs to α_ϵ, $d_T > \epsilon$+ Let T_2 denote the first element of α such that T is preceded in α by infinitely many elements of α_ϵ. Let L denote an arc from T_1 to T_2 lying in C. Infinitely many elements of α_ϵ are between T_1 and T_2 in C and therefore, intersects L. Consequently, there exists in C a connected domain D of diameter less than $\epsilon/2$ which contains points of two different elements T and T' of α_ϵ, such that T precedes T' in α. Then, the first element of α which follows T in α is either T' or lies between T and T' in C and intersects D. This is a contradiction.

The same process may be followed for the collection of all arcs T such that T_{-1} lies between T_1 and T in C. Hence, there exists a countable collection G of arcs crossing C, such that any arc which crosses C wither intersects an arc of G or lies between two arcs of G. This was on the assumption that there exists two mutually exclusive arcs crossing C. It is evident that G exists if this is not the case, for there exists at least one arc which crosses C.

For each pair of arcs G let L denote an arc in C which contains points of both of them, and let M_c denote the sum of all these arcs together with the sum of the elements of G. Then if T is any arc which crosses C, T intersects M_c. Since each component C of $D_1-\overline{D}_2$ contains an element of Δ, there are only a finite number of them. Thus, if the above construction is carried out in each one of them, $M=\overline{\Sigma M_c}$ is a closed and completely separable subset of $\overline{D}_1-P$ which separates P from the boundary of D_1. Let Q denote the component of $\overline{D}_1-M$ which contains P. Then Q contains P and lies in R, and the boundary of Q is completely separable. Thus, if P is an edge point of a region R, there exists in R a domain containing P whose boundary is separable. This is also true if P is a simple point of a region R. Therefore, space is locally peripherally separable.[13] By a theorem of the author's such a connected, connected im kleinen metric space is completely separable.[14]

Theorem 31. If space is completely separable and P is a point, there exists a sequence of simple domains Q_1, Q_2, Q_3, ... bounded by simple closed curves J_1, J_2, J_3, ... respectively, such that (1) for each n Q_n contains $\overline{Q}_{n+1}$, and (2) if M is a closed and compact point set not containing P, there exists an integer n such that $\overline{Q}_n$ does not contain a point of M.

Proof. For each point X of S-P there exists a simple domain D_X containing X such that $\overline{D}_X$ does not contain P. Theorem 3. Let G denote the collection of all such domains. G covers S-P, and since S is completely separable, G contains a countable subcollection $G'=D_1$, D_2, D_3, ... covering S-P. For each n let C_n denote the boundary of D_n. Let J_1 denote the boundary of a simple domain containing P. For each n let J_n denote a simple closed curve separating

P from the closed and compact point set $C_1+C_2+..+C_{n-1}+J_1+J_2+..+J_{n-1}$ (Theorem 5), and let Q_n denote the complementary domain of J_n which contains P. Then, Q_1, Q_2, Q_3, ... is the required sequence of simple domains. For if M is any closed and compact point set, there exists an integer k such that $D_1+D_2+..+D_k$ covers M. Hence, if $n>k$, $\overline{Q}_n$ contains no point of M.

Theorem 32. If space is completely separable and R is a region containing an edge point P but containing no simple domain containing P, then R contains a connected domain D whose boundary is a subset of the sum of the elements of a finite collection Δ of open curves which surrounds D minimally with respect to R and P, such that if J is a simple closed curve lying in $D+\Delta^*-P$ whose common part with Δ^* is connected then one of the complementary domains of J is a subset of D.

Proof. Let D_1, D_2, and D_3 denote simply connected domains lying in R and containing P such that (1) D_3 contains $\overline{D}_2$, (2) D_3 is surrounded minimally with respect to D_1 and P by a finite collection Δ' of continua lying in D_1-D_3. By Theorem 31 there exists a sequence of simple domains Q_1, Q_2, Q_3, ... bounded by simple closed curves J_1, J_2, J_3, ... respectively, such that (1) for each n Q_n contains $\overline{Q}_{n+1}$ and (2) if M is a closed and compact subset of S-P, there exists an integer n such that $\overline{Q}_n$ contains no point of M. Let C denote a component of $D_1-\overline{D}_2$ and let G denote the collection of all arcs T such that T crosses C and for some n is a subset of J_n. If T and T' are arcs of G, $\widehat{T}\cdot\widehat{T'}=0$. Since D_1 is simply connected but contains no simple domain containing P, for each n J_n contains a point of $S-D_1$. Furthermore, let AP denote an arc lying in D_2 and in the component of D_1-P which contains C; then there exists an integer $\overline{n}$ such that $\overline{D}_{\overline{n}}$ does not contain A; so if $n>\overline{n}$, contains a point of the segment $\widehat{AP}$ and a point of $S-D_1$, and therefore contains at least two arcs T of G. However, for each n J_n contains at most a finite number of arcs of G. Hence, G is countable. Furthermore, suppose that L is an arc lying in C from one arc of G to another arc of G. There exists an integer $\overline{n}$ such that if $n>\overline{n}$, Q_n contains no point of L. Hence, between any two arcs of G there are only a finite number of arcs of G. Furthermore, each arc of G is between some two arcs of G. For, if T is an arc of G, there exists an integer n such that J_n separates T from P. By Theorem 29 J_n contains two arcs which cross C such that T is between them in C.

Let α denote a well ordered sequence whose elements are the elements of G. Let N_1 and N_0 denote the first two elements of G. Let N_2 denote the first element of G such that N_1 is between N_0 and N_2 in C. Let N_{-1} denote the element of G such that N_0 is between N_{-1} and N_1 in C. This process may be continued. The sequences N_1, N_2, ... , N_ω, ... and N_0, N_{-1}, N_{-2}, ... , $N_{-\omega}$, ... may or may not be simple sequences, but since each is countable, the first contains a simple countable sub-sequence T_1, T_2, T_3, ... running through it and the second contains a simple subsequence T_0, T_{-1}, T_{-2}, ... running through it, such that if T is any arc crossing C, there exists an integer n such that T is between T_n and T_{-n} in C and such that the linear order of these arcs in C is the same as the order in the sequences.

Since space is metric (Theorem 30) and C contains one and only one of the continua of Δ', there exists in C a connected domain C' such that (1) C' contains this element of Δ' and (2) $\overline{C'}$ is a subset of C. For each pair of consecutive integers (positive or negative), a and b with $a<b$, let U_a denote an arc irreducible from T_a to T_b lying in C' and let M_a denote the interval of T_a between the end points of U_a and U_{a-1} in T_a. The point set ΣM_a is closed, since each M_a is for some n a closed subset of $C \cdot J_n$, for each n J_n contains only a finite number of the arcs of G, and the limiting set of J_1, J_2, J_3, ... is the point P. Also ΣU_a is closed; for suppose that O is a

limit point of the set ΣU_a not belonging to it. O belongs to $\bar{C}'$ and therefore, to C. There exists a connected domain V which lies in C, contains O, but does not intersect more than one of the simple closed curves $J_1, J_2, J_3, \ldots$; and an infinite collection H of segments $\widehat{U_a}$ such that if $\widehat{U_a}$ is a segment of H, $\widehat{U_a}$ intersects V. Hence, V+H* is a connected subset of C not intersecting more than one of the arcs $T_1, T_2, T_3, \ldots, T_o, T_{-1}, T_{-2}, \ldots$ but having limit points in infinitely many of them. This is a contradiction. So $L=\Sigma(M_a+U_a)$ is closed and is, because of its method of construction, an open curve.

Now suppose that T is any arc crossing C. There exists an integer n such that T lies between T_n and T_{-n} in C. Hence, T intersects that interval of L which is erreducible from T_n to T_{-n}. Since each component C of $D_1-\bar{D}_2$ contains an element of Δ', there are only a finite number of them. Hence, if these components are $C_1, C_2, \ldots, C_j$ and for each $i \leq j$, L_i denotes an open curve obtained from the above method of construction and lying in C_i, then $L_1+L_2+\ldots+L_j$ is closed and separates D_2 from the boundary of D_1. Since no component of D_1-P contains more that one element of Δ', no such component contains more than one of the open curves $L_1, L_2, \ldots, L_j$. Hence, the collection $\Delta=L_1, L_2, \ldots L_j$ is minimal with respect to D_1 and P.

Let D denote the component of $S-\Delta^*$ which contains P. D contains D_2, and a segment of each element of G having one and only one of its ends from some i in L_1. Therefore Δ surrounds D minimally with respect to D_1 and P.

Let AP denote an arc from P to a point A of L_1 lying in D except for A, and let W denote any interval of L_1 which contains P. By Theorems 5 and 28 there exists a simple closed curve J separating W from P such that $J \cdot AP$ is connected. If J does not intersect L_1, by Theorem 25 one of the complementary domains I of J is a subset of D. Since J separates A from P, I contains P. But this is impossible, since R contains no simple comain containing P. Hence, J contains a point O of S-D such that $J-(O+J \cdot AP)$ is the sum of two segments $\widehat{OX}$ and $\widehat{OY}$ which abutt on AP from different sides. Since the component of D_1-P which contains AP-P contains only one element of Δ', this component contains only one open curve of the collection Δ, namely, L_1. Hence, both OX and OY intersects L_1 and contains arcs XX' and YY' respectively which lie in D except for the points X' and Y' which lie on L_1. Let J' denote the simple closed curve XX'+X'Y'(of L_1)+YY'+XY(of AP). J' crosses the segment $\widehat{AP}$. Let I' denote the complementary domain of J' which lies in D_1. The domain I' contains no point of Δ^* except possibly points of L_1. But if I' contains a point of L_1, I' must contain a ray of points of L_1. This is impossible because for each n Q_n intersects every ray of L_1, but since J' does not contain P, there exists an integer n such that $\overline{Q}_n$ contains no point of J'+I'. Therefore, I' is a subset of D. Furthermore, if J' does not contain W, J' does not contain A. And since J' crosses AP, J' separates A from P. Thus, I would be a simple domain lying in R and containing P which is contrary to hypothesis. Hence, J contains W and every point of W is on the boundary of E. Therefore, the boundary of D is the point set Δ^*.

This argument also shows that if J is a simple closed curve lying in $D+\Delta^*-P$ and $J\cdot\Delta^*$ is connected, one of the complementary domains of J is a subset of D.

Definition. A domain D of the type shown to exist in Theorem 32 is said to be a pseudo-simple domain with respect to R and P or simply, a pseudo-simple domain.

Theorem 33. A pseudo-simple domain is simply connected.

Theorem 33 is a consequence of Theorem 25.

Theorem 34. If D is a pseudo-simple domain with respect to a region R and a point P, and J is a simple closed curve lying in $\overline{D}$ such that the common part of J and β, the boundary of D, is connected, then one of the complementary domains of J is a subset of D.

Proof. If J does not contain P, by the definition of a pseudo-simple domain one of its complementary domains is a subset of D. Now suppose that J contains P and that neither of its complementary domains is a subset of D. Since $J\cdot\beta$ is connected, $J-J\cdot\beta$ is a segment $\widehat{APB}$. Some open curve L of β contains $J\cdot\beta$ and since $\beta-L$ does not separate P from the boundary of R, L contains a limit point O of $S-\bar{D}$. Let T denote an interval of L containing O and $J\cdot\beta$. Let C denote a simple closed curve separating T from P and whose common part with J is the sim of two connected subsets of the intervals AP of APB and BP of APB respectively (apply Theorem 28 twice). Let E_1F_1 and E_2F_2 denote arcs of C, such that $C=C\cdot\widehat{AP}+C\cdot\widehat{BP}+E_1F_1+E_2F_2$

Suppose that both $\widehat{E_1F_1}$ and $\widehat{E_2F_2}$ intersect β. Then they must each intersect L, for if this were not the case some open curve of $\beta-L$ would be joined to L by a segment lying in D-P. Both of them have end points in $\widehat{AP}$ and $\widehat{BP}$ respectively. Let E_1' and F_1' denote the first and last points that $\widehat{E_1F_1}$ has in common with L and let E_2' and F_2' denote the first and last points that $\widehat{E_2F_2}$ has in common with L. The intervals $E_1'F_1'$ and $E_2'F_2'$ of L lie in different complementary domains of J. Let $C'=C\cdot\widehat{AP}+\widehat{E_1E_1'}(\text{of } E_1F_1)+E_1'F_1'(\text{of } L)+\widehat{F_1'F_1}(\text{of } E_1F_1)+C\cdot\widehat{BP}+E_2E_2'(\text{of } E_2F_2)+E_2'F_2'(\text{of } L)+\widehat{F_2'F_2}(\text{of } E_2F_2)$. C' separates T from P, because $\widehat{AP}$ crosses

C and therefore, crosses C'. Let I denote the interior of C' with respect to P. Now I is composed of an arc segment of L which contains O, together with the interiors I_1 and I_2 of the simple closed curves $E_1'E_2'$(of L)+$E_1'E_1E_2E_2'$(of C') and $F_1'F_2'$(of L)+$F_1'F_1F_2F_2'$(of C') respectively. But I_1 and I_2 are subsets of D since D is simply connected, and hence, O is not a limit point of S-$\overline{D}$. This is a contratiction.

So one of the segments, $\widehat{E_1F_1}$ and $\widehat{E_2F_2}$, say $\widehat{E_1F_1}$, does not intersect β. E_1F_1 is a subset of D. Furthermore, if now I denotes the domplementary domain of J which contains $\widehat{E_1F_1}$, I-$\widehat{E_1F_1}$ is a subset of D since it is the sum of two simple domains known to lie in D. Thus, I is a subset of D.

Theorem 35. If D is a pseudo-simple domain with respect to a region R and a point P and OX and O'X' are arcs lying in D except for their end points X and X' which lie in an open curve of β, the boundary of D, then the arcs OX and O'X' abutt on L from the same side.

Proof. Suppose the contrary. Then L contains an arc AXX'B containing the points X and X' as interior points such that the arcs OX and O'X' abutt on AXX'B from different sides. Let FF' denote an arc in D irreducible from OX to O'X'. Then the arcs AX and BX' of AXX'B approach the simple closed curve J=FX(of OX)+XX'(of AXX'B)+F'X'(of O'X')+FF' from different sides. Hence, A lies in one complementary domain of J, and B lies in the other. But this is impossible, for since J$\cdot\beta$ is connected and J is a subset of $\overline{D}$, one of the complementary domains of J is a subset of D.

Theorem 36. If D is a pseudo-simple domain with respect to a region R and a point P and OX and O'X' are arcs lying in S-$\overline{D}$ except for their end points X and X' which lie in an open curve L of β, the boundary of D, then the arcs OX and O'X' abutt on L from the same side.

Proof. Let FF' denote an arc lying in D except for the end points F and F' which lie on L such that the interval FF' of L contains X and X' as interior points. Then J=FF'+FF'(of L) is a simple closed curve one of whose complementary domains is a subset of D. The other complementary domain of J contains the arc segments $\widehat{OX}$ and $\widehat{O'X'}$. Hence, the arcs approach FF' of L, and therefore L, from the same side.

Theorem 37. If D is a pseudo-simple domain with respect to a region R and a point P and OX and O'X' are arcs which have their end points X and X8 in an open curve L of β, the boundary of D, but have no other point in common with β, and approach L from different sides, then one of these arcs lies in D and the other lies in S-$\overline{D}$, and every point of β is accessable from D and some component of S-$\overline{D}$.

Theorem 37 is a consequence of the two preceding theorems and Theorem 15.

Theorem 38. If D is a connected domain and J is a simple closed curve lying in S-D, then one of the complementary domains of J is a subset of S-$\overline{D}$.

Proof. Let I denote the interior of J with respect to a point ω of D. Suppose that X is a point of $\overline{D}$ and belongs to I. Then X+D is a connected point set containing ω and a point of I but no point of J. This is a contradiction.

Theorem 39. If X is a point on the open curve L of the boundary β of a pseudo-simple domain D with respect to a region R and a point P, and M is a closed and compact point set not containing X, then there exists a simple closed curve lying in S-(D+M) whose common part with β is an interval of L containing X as an interior point, and whose interior with respect to P contains no point of M.

Proof. Let XP denote an arc from X to P lying except for X in D and let J denote a simple closed curve separating X from P such that J·XP is a connected subset of D. J contains a point of S-D. Let O denote the first point that XP has in common with J and let AOB denote the interval of J containing O and lying in D except for its end points A and B which lie on L. The interval AB of L contains X because one of the complementary domains of the simple closed curve AOB+AB is a subset of D and since it can not contain P, it must contain $\widetilde{OX}$ of XP. Let AZB denote the arc of J such that J=AXB+AZB. Let B_o denote the first point that AZB has in common with the ray XB of L; let A_1 denote the next preceding point that AZB has in common with β if it is not in L; let L_1 denote the open curve of β containing A_1; and let B_1 denote the first point AZB has in common with L_1. Let A_2 denote the next preceding point that AZB has in common with β if it is not in L; let L_2 denote the open curve of β containing A_2; and let B_2 denote the first point AZB has in common with L_2. This process can be comtinued (for a finite number of steps) until for some integer n the next point that AZB has in common with β preceding B_n is a point A_o of L. Since A_o precedes B_o in AZB, A_o belongs to the ray XA of L. Let J_1 denote the simple closed curve A_oXB_o(of L)+ B_oA_1(of AZB)+A_1B_1(of L_1)+B_1A_2(of AZB)+..+$B_{n-1}A_n$(of AZB)+ A_nB_n(of L_n)+B_nA_o(of AZB). It is clear that if A_1 exists, J_1 lies in S-D, because any arc segment lying in D having its end points in different open curves of β contains P and J_1 does not contain P. Now in case J_1=A_oXB_o(of L)

$+B_0A_0$(of AZB) suppose that $\widehat{B_0A_0}$ lies in D. Then J_1 lies in $\overline{D}$ and $\beta \cdot J_1$ is connected; so one of the complementary domains, say I, of J_1 is a subset of D. Let P'X denote an arc lying in I except for X which is on the boundary of I. Now I does not contain P; hence, I does not contain PX and therefore PX and P'X abutt on AXB from different sides. But since both PX and P'X lie in D, this contradicts Theorem 37. Thus, in any case J_1 is a subset of S-D. By Theorem 38 one of the complementary domains I_1 of J_1 is a subset of $S-\overline{D}$. Hence, I_1 is the interior of J_1 with respect to P. Let X' denote a point of $J_1-J_1 \cdot L$ and let J_2 denote a simple closed curve separating X from P+M+X' (Theorem 5), and let I_2 denote the interior of J_2 with respect to P. By Theorem 11 of Chapter III of <u>Foundations</u> J_1+J_2 contains a simple closed curve J_3 such that (1) I_3, the interior of J_3 with respect to P, is a subset both of I_1 and of I_2, and hence of S-(D+M), and (2) $J_3 \cdot L$ is an arc containing X as an interior point. Let A'X'B' denote an arc lying in I_3 except for points A' and B' which belong to $J_3 \cdot \widehat{A_0X}$ and $J_3 \cdot \widehat{B_0X}$ respectively. Then $J_4=$ A'XB(of L)+A'X'B' is a simple closed curve lying in S-(D+M) whose interior with respect to P is a subset of I_3 and hence, contains no point of M. Furthermore, the common part of J_4 with β is the arc A'XB' which is an interval of L of which X is an interior point.

Theorem 40. If D is a pseudo-simple domain with respect to a region R and a point P, then $S-\overline{D}$ is connected.

Proof. The boundary of D is the sum of the elements of a finite collection Δ of open curves $L_1, L_2, \ldots, L_j$. For each point X of Δ^* let J_x denote a simple closed curve such that J_x is a subset of S-D and $J_x \cdot \Delta^*$ is for some $n \leq j$ an interval T_x of L_n containing X as an interior point. Let I_x denote the component of J_x which does not contain P. I_x is a subset of $S-\overline{D}$. Now for each $n \leq j$ let $D_n = \Sigma I_x$ for X on L_n. All the arc segments $J_x - T_x$ for X on L_n lie in $S-\overline{D}$ and abutt on L_n from the same side. Hence, the domains I_x overlap, so that D_n is a connected subset of $S-\overline{D}$. Furthermore, if T is any arc in S-D which abutts on L_n at a point X, it abutts on L_n from the same side as $J_x - T_x$, and hence, contains points of I_x and D_n. So for each $n \leq j$ only one component of $S-\overline{D}$ has boundary points on L_n.

Now suppose that $S-\overline{D}$ is not connected. Then there exists two mutually exclusive domains H and K such that $S-\overline{D}=H+K$. Since space is connected, both H and K have boundary points in Δ^*, but for no $n \leq j$ does L_n contain boundary points of both H and K. Hence, $\overline{H} \cdot \Delta^*$ and $\overline{K} \cdot \Delta^*$ are mutually exclusive closed point sets. Let AB denote an arc in S-P irreducible from $\overline{H} \cdot \Delta^*$ to $\overline{K} \cdot \Delta^*$. Now since each

D_n, $n \leq j$, belongs either to H or to K, $\Delta^* = \overline{H} \cdot \Delta^* + \overline{K} \cdot \Delta^*$; so AB contains only the two points A and B of Δ^*, and these points belong to different open curves of the collection Δ. Therefore, since AB does not contain P, the arc segment $\widehat{AB}$ lies in $S - \overline{D}$. Hence, AB contains a point of I_A and a point of I_B. But, since I_A and I_B belong to H and K respectively, this is a contradiction.

Theorem 41. If D is a pseudo-simple domain and J is a simple closed curve lying in $\overline{D}$, then one of the complementary domains of J is a subset of D.

This theorem is a consequence of Theorems 38 and 40.

Corollary. If D is a pseudo-simple domain with respect to a region R and a point P whose boundary if the sum of the elements of a finite collection Δ of open curves, then the boundary of each component of D-P consists of P and one of the open curves of Δ.

Definition. A domain D is said to be internally simple provided that for each point X of D there exists a simple domain I_x containing X and lying together with its boundary in D.

Theorem 42. If D is a pseudo-simple domain with respect to a region R and a point P, then each component of D-P is internally simple.

Proof. Let U denote a somponent of D-P and let X denote a point of U. Let XP denote an arc from A to P lying except for P in U. By Theorems 3 and 28 there exists a simple closed curve J crossing XP. If J intersects the boundary of D, then J contains an arc AYB crossing XP and lying except for the points A and B of L in D. Let J_1 denote the simple closed curve AYB+AB(of L). One of the complementary domains of J_1 is a subset of D and contains X. Let J_x denote a simple closed curve separating J_2+P from X. If J does not intersect the boundary of D, let J_x denote J. Then, one of the complementary domains I_x of J_x contains X and I_x+J_x is a subset of U. Hence, U is internally simple.

Theorem 43. If D is a pseudo-simple domain with respect to a region R and a point P, S-$\overline{D}$ is internally simple.

Proof. Let Δ denote the collection of mutually exclusive open curves whose sum is the boundary of D, and let L denote an element of Δ. Let X denote a point of S-$\overline{D}$, and let AX denote an arc irreducible from L to X and containing no point of Δ^*-L (Theorem 40). Suppose that S-($\overline{D}$+AX) is the sum of two mutually exclusive domains H and K. Because no arc separates space, both H and K have boundary points in Δ^*. Furthermore, because of Theorem 39 no element of Δ-L contains points of both $\overline{H}\cdot\Delta^*$ and $\overline{K}\cdot\Delta^*$

not so either of the rays of L from A contain points of both $\overline{H}\cdot\Delta^*$ and $\overline{K}\cdot\Delta^*$. Let AP denote an arc in D+A from A to P. Since AX+AP does not separate space, there exists an arc T irreducible from $\overline{H}\cdot\Delta^*$ to $\overline{K}\cdot\Delta^*$ in S-(AX+AP). Now the arc segment $\widehat{T}$ is a subset of S-$\overline{D}$, for if $\widehat{T}$ were a subset of D it would have one end on one ray of L from A and the other end on the other ray from A without intersecting AP, which is impossible (Theorem 34). But it is also impossible for $\widehat{T}$ to lie in S-$\overline{D}$, for in this case H and K would have boundary points in the same open curve of Δ. Hence, S-($\overline{D}$+AX) is connected.

Let EYF denote an arc lying except for Y in S-($\overline{D}$+AX) and crossing AX at the point Y. Since S-($\overline{D}$+AX) is connected, there exists an arc EZF lying in it from E to F. EYF+EZF contains a simple closed curve J crossing AX at the point Y and the interior I of J with respect to P is a simple domain I containing X such that $\overline{I}$ is a subset of S-$\overline{D}$.

Theorem 44. If (1) D_1 is a pseudo-simple domain with respect to a region R_1 and a point P, (2) D_2 is a pseudo-simple domain with respect to a region $R_2<D_1$ and the point P, and (3) C is a component of $D_1-\overline{D}_2$, then C is an internally simple, simply connected domain.

Proof. C is simply connected; for it a simple closed curve J is a subset of C, the interior I of J with respect to P is a subset of D_1 (Theorem 33) and also of $S-\overline{D}_2$ (Theorem 38), and therefore, I is a subset of C. If X is a point of C, there exist a simple domain I_1 containing X such that $\overline{I}_1$ is a subset of D_1-P (Theorem 42) and a simple domain I_2 containing X such that $\overline{I}_2$ is a subset of $S-\overline{D}_2$ (Theorem 43). By Theorem 12 of Chapter III of <u>Foundations</u> I_1+I_2 contains a simple domain I_x which contains X and whose boundary is a subset of the sum of the boundaries of I_1 and I_2. Therefore, $\overline{I}_x$ is a subset of C, and C is internally simple.

Theorem 45. If D is an internally simple, simply connected domain and AB is an arc lying in D except for its end points which belong to the boundary of D, then $D-D\cdot AB$ is the sum of two mutually exclusive, internally simple, simply connected domains, each having AB on its boundary.

Proof. For each point X of $\widehat{AB}$ let I_x denote a simple domain containing X such that I_x+J_x(the boundary of I_x) is a subset of D. Let C denote a simple closed curve separating X from $A+B+J_x$ (Theorem 5). $C+\widehat{AB}$ contains a simple closed curve C' separating X from $A+B+J_x$ whose common part with AB is the sum of two continua. Let A'XB' denote the interval of AB containing X such that $A'XB'\cdot C'$

$=A'+B'$. One of the complementary domains I' of C' is a subset of D and contains the arc segment $\widehat{A'XB'}$. $I'-\widehat{A'XB'}$ is the sum of two simple domains I_{x1} and I_{x2}, one on one side of AB and the other on the other side of AB. How if the subscripts 1 and 2 are so used so that all the domains I_{x1} for X on $\widehat{AB}$ are on the same side of AB and all the domains I_{x2} are on the other side of $\widehat{AB}$, $D_1=\Sigma I_{x1}$ and $D_2=\Sigma I_{x2}$ are two connected domains which are subsets of D. Now each point O of $D-\widehat{AB}$ may be joined to a point of $\widehat{AB}$ by an arc OE lying in $D-\widehat{AB}$ except for its one end point E, and $\widehat{OE}$ must intersect either D_1 or D_2. If for each i (i=1,2) H_i denotes the set of all points O such that the arc segment $\widehat{OE}$ intersects D_i, then H_1 and H_2 are connected domains and $H_1+H_2=D-\widehat{AB}$. But if D is simply connected, no arc segment lying in D can abutt on $\widehat{AB}$ from different sides since neither A nor B belongs to D. Hence, H_1 and H_2 are mutually exclusive.

Now suppose that J is a simple closed curve lying in H_1. One of the complementary domains I of J is a subset of D. Therefore, I does not contain any point of AB. But I has limit points in H_1, namely, the points of J, and hence contains points of H_1. Consequently, I is a subset of H_1, and therefore, H_1 is simply connected. Likewise, H_2 is simply connected.

Let X denote a point of H_1. Since D is internally simple, there exists a simple domain I_x^1 containing X and lying together with its boundary J_x^1 in D. By Theorem 5 there exists a simple closed curve J_x separating X from J_x^1+AB. J_x is a subset of H_1 and contains X. Consequently, H_1 is internally simple and likewise, H_2 is internally simple. It is clear that AB is on the boundary of each of them.

Theorem 46. If D is an internally simple, simply connected domain, J is a simple closed curve lying in D, and AB is an arc irreducible from J to the boundary of D, then D-(J+AB) is the sum of two mutually exclusive, internally simple, simply connected domains one having J for its boundary and the other having for its boundary, the boundary of D plus J+AB.

This theorem may be established by methods similar to those used in proving Theorem 44.

Theorem 47. If D is a pseudopsimple domain with respect to a region R and a point P and Ab is an arc lying in S-$\overline{D}$ except for the points A and B which lie on the boundary of D, then S-($\overline{D}$+AB) is the sum of two connected domains whose boundaries have as their common part the arc AB.

Proof. From Theorems 38, 40, 43, and 45 that S-($\overline{D}$+AB) is the sum of two connected domains whose boundaries contain AB in their common part is easily seen. And it is clear from Theorem 39 that this common part can not contain a point of the boundary of D different from A and B.

Theorem 48. If under the hypothesis of Theorem 44, AXB is an arc lying in C except for the points A and B which lie on the boundary of C, then $C-\widehat{AXB}$ is the sum of two connected domains whose boundaries have as their common part the arc AXB.

This theorem may be proved by an argument similar to that of the preceding theorem using Theorems 44 and 45 and a theorem similar to Theorem 39 which may be established in a manner similar to that used to prove Theorem 39.

Definition. Suppose that X_1, X_2, X_3, and X_4 are points of β, the boundary of a pseudo-simple domain with respect to a region R and a point P. Then X_1+X_3 is said to ordinally separate X_2 from X_4 on β if there exists a simple closed curve J whose common part with β is X_1+X_3 and which separates X_2 from X_4.

Theorem 49. If D is a pseudo-simple domain with respect to a region R and a point P, β is the boundary of D, and X_1, X_2, X_3, and X_4 are four distinct points of β, then one pair of them ordinally separates the other two on β, and if X_1+X_3 ordinally separates X_2 from X_4 on β, X_2+X_4

ordinally separates X_1 from X_3 on β, every simple closed curve J containing a point of D and a point of $S-\overline{D}$, such that $J \cdot \beta = X_1 + X_3$ separates X_2 from X_4, but X_1+X_2 does not ordinally separate X_3 from X_4 on β.

Proof. Let X_1AX_2 and X_1BX_2 denote arcs lying except for X_1 and X_2 in D and $S-\overline{D}$ respectively. Suppose that $J' = X_1AX_2 + X_1BX_2$ does not separate X_3 from X_4. Now let AX_3 and BX_3 denote arcs lying except for X_3 in $S-\overline{D}$ respectively, such that $AX_3 \cdot J' = A$ and $BX_3 \cdot J' = B$. With respect to X_4 as the point at infinity the interior of one of the simple closed curves formed by one pair of the three arcs AX_1B, AX_2B, and AX_3B contains the other arc segment. Hence, one pair of the points X_1, X_2, X_3, and X_4 ordinally separates the other two on β.

Now suppose that X_1+X_3 ordinally separates X_2 from X_4 on β. Then there exists by a construction similar to that of the above paragraph two arcs X_2AX_4 and X_2BX_4 lying except for X_2 and X_4 in D and $S-\overline{D}$ respectively whose sum is a simple closed curve separating X_1 from X_3. Now suppose that J is a simple closed curve containing a point of D and a point of $S-\overline{D}$ whose common part with β is X_1+X_3. Then J is the sum of two arcs $X_1A'X_3$ and $X_1B'X_3$ lying except for X_1 and X_3 in D and $S-\overline{D}$ respectively. Let C denote the simple closed curve $X_2AX_4+X_2BX_4$. By a double application of Theorem 28 $C+\widehat{X_1A'X_3}+\widehat{X_1B'X_3}$ contains a simple closed curve C' whose common part with $\widehat{X_1A'X_3}$ is connected and whose common part with $\widehat{X_1B'X_3}$ is also connected and which separates X_2 from X_4. It is clear now that J separates X_2 from X_4.

Now suppose that X_1+X_2 ordinally separates X_3 from X_4 on β. Then there would exist a simple closec curve J having the properties of J in the preceding paragraph which does not separate X_2 from X_4. This is impossible if X_1+X_3 ordinally separates X_2 from X_4.

Arguments of a similar nature show that if order of β is interputed by the notion of ordinal separation, then

Theorem 50. If β is the boundary of a pseudo-simple domain, the points on β have a cuclic order which preserves the ordinary order on any open curve component of β.

Theorem 51. Suppose that D_1 and D_2 are pseudo-simple domains with respect to regions R_1 and R_2 and a point P respectively, and R_2 is a subset of D_1; and X_1, X_2, X_3, and X_4 and Y_1, Y_2, Y_3, and Y_4 are the end points of the mutually exclusive arcs T_1, T_2, T_3, and T_4 which cross $D_1-\overline{D}_2$. Then if X_1+X_3 ordinally separates X_2 from X_3 of β_1, the boundary of D_1, then Y_1+Y_3 ordinally separates Y_2 from Y_4 on β_2, the boundary of D_2, and conversely.

Proof. Let X_1AX_3 denote an arc lying except for the points X_1 and X_3 in $S-\overline{D}_1$ and let Y_1BY_3 denote an arc lying except for the points Y_1 and Y_3 in D_2. Let H denote the simple closed curve $T_1+X_1AX_3+T_3+Y_1BY_3$. Since X_1+X_3 ordinally separates X_2 from X_4 on β_1, J separates X_2 from X_4. But J does not intersect T_2 or T_4 and hence, separates Y_2 from Y_4. Hence, Y_1+Y_3 ordinally separates Y_2 from Y_4 on β_2. Likewise, the converse is true.

PART II

CONSEQUENCES OF AXIOMS 0-4, 5* AND 6

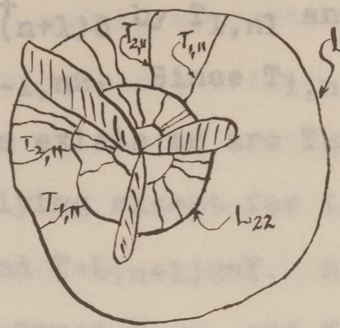
Axiom 6: S is completely separable.

About each edge point of space it is possible to construct what will be called a radial web skeleton. Suppose that P is an edge point. Then there exist a sequence of regions R_1, R_2, R_3 ... closing down on P, a sequence of domains D_1, D_2, D_3, ... such that for each n D_n is a pseudo-simple domain with respect to the region R_n and the point P which contains R_{n+1}, and a sequence of simple domains Q_1, Q_2, Q_3, ... bounded by simple closed curves J_1, J_2, J_3, ... respectively, such that (1) for each n Q_n contains $\overline{Q}_{n+1}$ and (2) if M is a closed and compact point set not containing P, there exists an integer n such that $\overline{Q}_n$ does not contain a point of M. For each n the boundary β_n of D_n is the sum of a finite collection Δ_n of open curves, L_{n1}, L_{n2}, L_{n3}, .. , L_{nq_n}. For each pair of integers n and $q \leq q_n$, let C_{nq} denote the component of $D_n - \overline{D}_{n+1}$ which has L_{nq} on its boundary. By the method of Theorem 32 there exists a double sequence of arcs $T_{1,nq}$, $T_{2,nq}$, $T_{3,nq}$, .. , $T_{-1,nq}$, $T_{-2,nq}$, $T_{-3,nq}$, ... such that for each integer m, positive or negative, $T_{m,nq}$ belongs to J_i for some integer i, $i > |m|$.

Thus $i \to \infty$ as $|m| \to \infty$ and the double sequence has no sequential limit set. For each pair of integers n and q, $q \leq q_n$, let $X_{m,nq}$ and $Y_{m,nq}$ denote $T_{m,nq} \cdot \beta_n$ and $T_{m,nq} \cdot \beta_{n+1}$ $(m=1,2,3,\ldots$ and $-1,-2,-3,\ldots)$[13] respectively, and let Z_{nq} denote a point of L_{nq} between $X_{1,nq}$ and $X_{-1,nq}$ on L_{nq}. Z_{nq} determines two rays of L_{nq}. Let L_{nq}^{+} denote the one containing $X_{1,nq}, X_{2,nq}, X_{3,nq} \ldots$ and L_{nq}^{-} denote the one containing $X_{-1,nq}, X_{-2,nq}, X_{-3,nq}, \ldots$.

Now if (1) certain arcs out of each sequence of the types $T_{1,nq}, T_{2,nq}, \ldots$ and $T_{-1,nq}, T_{-2,nq}, \ldots$ are discarded, (2) an arbitrary cyclic order on β_1 is chosen and trans-fered from β_1 to β_2, from β_2 to β_3, etc., by the arcs of (1) using Theorem 51, and (3) the subscript notation is suitably changed in accordance with changes (1) and (2), then (a) for each integer n $L_{n1}^{+}, L_{n2}^{-}, L_{n2}^{+}, \ldots, L_{nq_n}^{+}, L_{n1}^{-}$; $Z_{n1}, X_{1,n1}, X_{2,n1}, \ldots, \ldots X_{-2,n2}, X_{-1,n2}, Z_{n2}, X_{1,n2}, X_{2,n2}, \ldots, , Z_{nq_n}, X_{1,nq_n}, X_{2,nq_n}, \ldots, \ldots X_{12,n1}, X_{-1,n1}$ have on β_n the cyclic order indicated; (b) if m, n, and q are integers, $n>0$ and $0<q\leq q_n$, $X_{m,nq}$ belongs to L_{nq}^{+} for each positive integer m but to L_{nq}^{-} for each negative integer m; (c) if m, n, and q are integers, $n>0$ and $0<q\leq q_n$, for some integer $r\geq q$ $Y_{m,nq}$ belongs to the interval of $L_{(n+1)r}^{+}$ from $X_{m,(n+1)r}$ to $X_{m+1,(n+1)r}$ if $m>0$ and for some integer $r'\geq q$

[13] The use of the comma after m indicates that its range is both positive and negative, but otherwise only positive.



$Y_{m,nq}$ belongs to the interval of $L^{-}_{(n+1)r'}$ from $X_{m,(n+1)r'}$ to $X_{m-1,(n+1)r'}$ if $m<0$ such that if $q=1$, $r'=1$.

Let $Ws=P+\Sigma L_{nq}+\Sigma T_{m,nq}$ $(n>0$ and $0<q\leq q_n)$. Ws is a closed subset of R_1 and is called a radial web skeleton about P. Ws will depend, of course, on the above construction and no attempt will be made to give it a definition independent of its construction or to state any of its properties other than those of the following theorems, which in most cases are evident.

Theorem 52. If R is a region containing an edge point P, R contains a radial web skeleton about P.

Proof. Let R_1 of the above construction be R.

Definition. Ws is said to connect L^{+}_{nq} to $L^{+}_{(n+1)r}$ if $T_{1,nq}$ contains points of both of them, and Ws is said to connect $L^{-}_{nq'}$ to $L^{-}_{(n+1)r'}$ if $T_{-1,nq'}$ contains points of both of them.

Theorem 53. Suppose that Ws is a radial web skeleton about an edge point P and Ws connects L^{+}_{nq} to $L^{+}_{(n+1)r}$ and $L^{-}_{nq'}$ to $L^{-}_{(n+1)r'}$. Then if $q'=q+1$, $r'=r+1$ and conversely.

Proof. For definiteness suppose that L^{+}_{n1} is connected to $L^+_{(n+1)3}$ by $T_{1,n1}$ and that L^-_{n2} is connected to $L^+_{(n+1)5}$ by $T_{-1,n2}$. Since $T_{1,n1}+T_{-1,n2}+P$ does not separate space, there exists an arc T from $L_{(n+1)4}$ to β_n (the boundary of D_n) lying except for its end points in $D_n-\bar{D}_{n+1}$. Let $T\cdot\beta_n$ $=X$ and $T\cdot L_{(n+1)2}=Y$. Since in the cyclic order of (a) Y is between $Y_{1,n1}$ and $Y_{-1,n2}$ of β_{n+1}, X is between $X_{1,n1}$ and $X_{-1,n2}$ on β_n (Theorem 51). Hence, X is on either the subray of L^+_{n1} from $X_{1,n1}$ or the subray of L^-_{n2}. Suppose that X belongs to L^+_{n1}. A similar argument applies to the other case. Since the sequence of arcs $T_{1,n1}$, $T_{2,n1}$, ... has no sequential limit set, there is some one of them, say $T_{m,n1}$, having no point in common with T such that $X_{m,n1}$ is between X and $X_{-1,n2}$ on β_n. But $Y_{m,n1}$ must belong to $L^+_{(n+1)3}$ and is not between Y and $Y_{-1,n2}$ on β_{n+1}, which contradicts Theorem 51.

Since no use was made of the fact that $T\cdot(T_{1,n1}+T_{-1,n2})=0$ except for notational simplicity, a similar argument will show that the converse holds true, that is, if $r'=r+1$, then $q'=q+1$.

Definition. L^+_{nq} and $L^-_{n(q+1)}$ ($q\leq q_n$ with q+1 interputed as 1 when $q=q_n$) are said to be adjacent rays in β_n.

Theorem 54. Suppose that Ws is a radial web skeleton and that L^+_n and L^-_n are adjacent rays in β_n of Ws; then (1) the two rays L^+_{n+1} and L^-_{n+1} which are connected by Ws to L_n^+ and L_n^- respectively are adjacent rays in β_{n+1}, (2) if Ws connects either L_n^+ or L_n^- to a ray in β_{n-1}, Ws connects L_n^+ and L_n^- to two adjacent rays in β_{n-1}.

This theorem is a consequence of the construction and the preceding theorem.

Theorem 55. If Ws is a radial web skeleton and β_{nq} is the boundary of C_{nq}, $q \leq q_n$, then $\beta_{nq} \cdot \beta_{n+1}$ is the sum of a finite collection H_{nq} of consecutive rays in β_{n+1} between two extreme rays L^- and L^+ and all other rays in β_{n+1} between L^- and L^+ in the order of (a), and (2) L^- and L^+ are the only rays of H_{nq} which are connected by Ws to a ray in β_n.

Theorem 56. If Ws is a radial web skeleton and L^+ and L^- are adjacent rays in β_{n+1} of Ws neither of which are connected by Ws to a ray in β_n, then there exists integers n and q such that β_{nq} of C_{nq} contains L^+ and L^-, and any point of L^+ may be joined to any point of L^- by an arc lying except for its end points in C_{nq}.

Theorem 57. If Ws is a radial web skeleton and neither $L^+_{(nq)}$ nor $L^-_{(n[q+1])}$ is connected to a ray of β_{n-1} by Ws, then there exists a sequence of mutually exclusive arcs $U_1(nq)$, $U_2(nq)$, $U_3(nq)$, ... [14] such that for each

[14]This ( ) notation is used to restrict n and q to values for which the hypothesis of Theorem 57 is satisfied.

$m>0$ $U_m(nq)$ is an arc from $X_{m,nq}$ to $X_{-m,nq}$ lying except for these two points in $D_{n-1}-\bar{D}_n$ ($D_0=S$) such that the sequence has no sequential limiting set.

Proof. Let C denote the component of $D_{n-1}-\bar{D}_n$ whose boundary contains $L^+_{(nq)}+L^-_{(n[q+1])}$, and let L denote $C\cdot\beta_{n-1}$.[15] For each point X of C let I_x denote a simple domain containing X and lying together with its boundary J_x in C. Since space is completely separable, there exists a countable sequence I_1, I_2, I_3, ... of these covering C. For each integer $m>0$ let M_m denote an arc irreducible from J_m, the boundary of I_m, to L lying except for one end point in C. By Theorem 46 $C-C(J_m+M_m)$ is the sum of two commected domains, I_m and E_m, where the boundary of E_m consists of J_m+M_m together with the boundary of C. For each m let $U'_m(nq)$ denote an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these end points in E_m. By Theorem 48 $U'_m(nq)$ separates C into two internally simple, simply connected domains. One of these domains V'_m has for its boundary $U'_m(nq)$ plus the ray of $L^+_{(nq)}$ from $X_{m,nq}$ and the ray of $L^-_{(n[q+1])}$ from $X_{-m,(q+1)}$, and contains no point of I_m If the arcs $\{U'_m(nq)\}$ are mutually exclusive, then they form the sequence $U_1(nq)$, $U_2(nq)$, $U_3(nq)$, ...

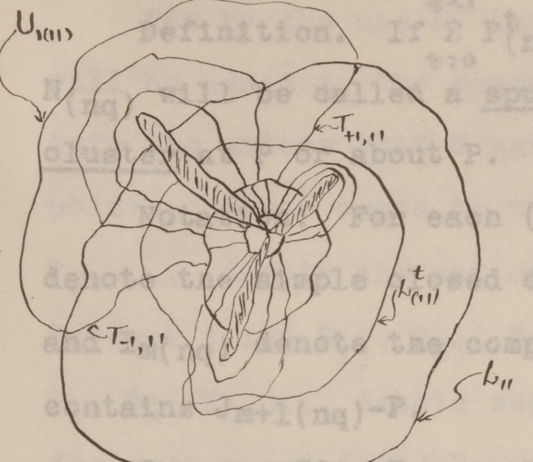
15When n=1 $C\cdot\beta_{n-1}$ is to be interpreted as a point of $L^-_{(nq)}$.

If this is not the case, let $U_{1(nq)}=U'_{1(nq)}$ and suppose that $U_{1(nq)}\cdot U'_{2(nq)}$ is not vacuous. Let A_1B_1 and A_2B_2 denote intervals of $U_{1(nq)}$ and $U'_{2(nq)}$ respectively which contain $U_{1(nq)}\cdot U'_{2(nq)}$. Let J denote a simple closed curve lying in C and separating $A_1B_1+A_2B_2$ from $X_{2,nq}+X_{-2,n(q+1)}$ such that $J\cdot A_2B_{2,nq}$(of $U'_{2(nq)}$) is connected and $J\cdot B_2X_{-2,(n[q+1])}$(of $U'_{2(nq)}$) is connected (Theorem 28 and 44 and Theorem 13 of Chapter III of Foundations). Now since $\widehat{A_2X_2}_{,nq}$ and $B_2X_{-2,n(q+1)}$ are subsets of V_1 and $\widehat{A_1X_1}_{,nq}$ and $B_1X_{-1,n(q+1)}$ contain no points of V'_2, J contains an arc AB lying in $V_1\cdot V'_2$ except for A and B irreducible from $\widehat{A_2X_2}_{,nq}$ to $\widehat{B_2X}_{-2,n(q+1)}$. Let $U_{2(nq)}=AB+AX_{2,nq}$(of $A_2X_{2,nq}$)$+BX_{-2,n(q+1)}$ (of $B_2X_{-2,n(q+1)}$) and let V_2 denote the component of $V_1-\widehat{U}_{2(nq)}$ which has the ray of $L^+_{(nq)}$ from $X_{2,nq}$ and the ray of $L^-_{(n[q+1])}$ from $X_{-2,n(q+1)}$ of its boundary. V_2 is a subset of $V_1\cdot V'_2$. This process may be continued and the required sequence of arcs constructed, for it is clear that since $V_1>V_2>V_3>\dots$ and $V_1\cdot V_2\cdot V_3\cdot\dots=0$, the sequence of arcs has no sequential limit set. It is to be noted that if K is any compact point set whatsoever, only a finite number of the arcs of the sequence have points in common with K.

Now on a radial web skeleton it is possible to construct what will be called a radial web. Let Ws denote a radial web skeleton about an edge point P. If $X_{m,nq}$ is a point of a ray in β_n of Ws which is not connected to a ray in β_{n-1} by Ws, let $N_{m,(nq)}$ denote $\sum_{r=n}^{\infty} T_{m,rq_r} + \sum_{r=n+1}^{\infty} M_{m,rq_r} + P$, where $q_1=q$ and for each $r>n$ q_r is such, so that L_{rq_r} contains $Y_{m,(r-1)q_{r-1}}$, and M_{m,rq_r} denotes the interval of L_{rq_r} between $Y_{m,(r-1)q_{r-1}}$ and X_{m,rq_r}. By (c) of the construction of Ws no two of these sets (of the type M_{m,rq_r}) have a point in common. $N_{m,(nq)}$ is an arc from $X_{m,nq}$ to P. For each integer triplet m (positive or negative), $n>0$ and $0<q\leq q_n$, there exists a continuous reversible transformation of the point set $T_{m,nq}-Y_{m,nq}$ into the number set $1/n \geq t > 1/(n+1)$. Let $X^t_{m,nq}$ denote the point of $T_{m,nq}$ whose transform is the number t, $0<t<1$. Then, of course, $X_{m,nq}=X^{1/n}_{m,nq}$. Let t_1, t_2, t_3, ... denote a countable set of numbers everywhere dense on $0<t<1$ and which contains all the reciprocals in this range. For each pair of positive integers n and q, $q\leq q_n$, for each t, such that $1/n \geq t > 1/(n+1)$ and $t=t_i$ for some i, let L^t_{nq} denote an open curve lying in C_{nq} such that (1) $L^t_{nq}=L_{nq}$ if $t=1/n$, (2) L^t_{nq} intersects $T_{m,nq}$ in the point $X^t_{m,nq}$, and (3) if t and t' are different numbers of the set t_1, t_2, t_3, ... , $L^t_{nq}\cdot L^{t'}_{nq}=0$. This sequence can be constructed by the use of Theorem 13 and repeated applications of Theorem 45 constructing the curves one at a time in the order of assending subscripts of t_i.

For example, suppose that $1/n \geq t_1 > 1/(n+1)$ and that 5 is the smallest value of i larger than 1 such that $1/n \geq t_i > 1/(n+1)$. Applying Theorem 45 twice, $C_{nq}-(T_{1,nq}+T_{-1,nq})$ contains a domain $C_{o,nq}$ having both $T_{1,nq}$ and $T_{-1,nq}$ on its boundary; and since an arc is accessible from both sides, $C_{o,nq}$ contains an arc segment $\widehat{T}_{o1}$ from $X^{t_1}_{1,nq}$ to $X^{t_1}_{-1,nq}$. $C_{o,nq}-T_{o1}$ contains a domain containing an arc segment $\widehat{T}_{o5}$ from $X^{t_5}_{1,nq}$ to $X^{t_5}_{-1,nq}$, etc., for the points, t_7, t_{18}, ... , which satisfy the condition, $1/n \geq t_i > 1/(n+1)$. Then this is repeated in each of the components of $C_{nq}-\sum_{n=-\infty}^{\infty} T_{m,nq}$ and the arcs added together in the obvious fashion to give the open curves of the type L^{t}_{nq}.

Now for each pair of integers n and q, $n>0$ and $q \leq q_n$, such that L^{+}_{nq} is not connected by Ws to a ray in β_{n-1}, let $U_1(nq)$, $U_2(nq)$, $U_3(nq)$, ... denote a sequence of mutually exclusive arcs such that for each $m>0$ $U_{m(nq)}$ is an arc from $X_{m,nq}$ to $X_{-m,n(q+1)}$ lying except for these two points in $D_{n-1}-\overline{D}_n$, $(D_o=S)$, and $U_{1(nq)}$, $U_{2(nq)}$, $U_{3(nq)}$, ... has no sequential limit set (Theorem 57). Let $\beta^{t_i}=\sum_{q=1}^{q_n} L^{t_i}_{nq}$ and let $W=Ws+\Sigma\beta^{t_i}+(\text{complete})\Sigma U_m(nq)$. W is called a <u>radial web</u> about P, and Ws is said to be the skeleton of W. Here again, W dependends not only on the above construction, but also on the construction of Ws. Consequently, no attempt will be made to give it a definition independent of its con-



struction. Many of the obvious properties of W will not be stated, and in the statement of the few theorems to follow concerning radial webs, the above construction will be implicitly assumed as a part of the hypothesis.

For two valuse t' and t" of t_1, t_2, t_3, ... (t"<t') let $\beta^{t't''}_{m(nq)}$ denote the point set consisting of the interval of $N_{m,(nq)}$ between $X^{t'}_{m,nq}$ and $X^{t''}_{m,nq}$ together with the two positive or negative rays of $L^{t'}_{nq}$ and $L^{t''}_{nq}$ from these points depending upon whether m is positive or negative. It is clear that $\beta^{t't''}_{m,(nq)}$ separates space into two mutually exclusive domains, such that $I^{t't''}_{m,(nq)}$, the one of these two not containing P, is the sum of an infinite number of simple domains and has $\beta^{t't''}_{m,(nq)}$ for its complete boundary. For m>0 let $R^{t't''}_{m(nq)} = I^{t't''}_{m,(nq)} + I^{t't''}_{-m,(n[q+1])}$. Let $P^{t}_{(nq)}$ (t here is any number between 0 and 1) denote the collection of all domains $R^{t't''}_{m(nq)}$ where t'>t>t". $P^{t}_{(nq)}$ will be hereafter refered to as an <u>ideal point</u>.

Definition. If $\sum_{t>0}^{t<1} P^t_{(nq)}$ is denoted by $N_{(nq)}$, then $N_{(nq)}$ will be called a <u>spur</u> and $\Sigma N_{(nq)}$ will be called a <u>cluster</u> at P or about P.

Notation: For each (nq) and each m>0 let $J_{m(nq)}$ denote the simple closed curve $N_{m,(nq)}+U_{m(nq)}+N_{-m,(n[q+1])}$ and $I_{m(nq)}$ denote the complementary domain of $J_{m(nq)}$ which contains $J_{m+1(nq)}-P$.

Theorem 58. For each (nq) the sequence $J_{1(nq)}$, $J_{2(nq)}$, $J_{3(nq)}$, ... has P for its sequential limit set and $I_{1(nq)} \cdot I_{2(nq)} \cdot I_{3(nq)} \cdot \ldots = 0$.

Theorem 59. For each m $W - W \cdot \Sigma I_{m(nq)}$ is a closed and compact point set.

Theorem 60. If $P^t_{(nq)}$ is an ideal point and M is any closed and compact point set, there exists a number δ and an integer $\overline{m}$, such that $0<t'-t''<\delta$, $m>\overline{m}$, and $R^{t't''}_{m(nq)}$ belongs to $P^t_{(nq)}$, then $\overline{R}^{t't''}_{m(nq)}$ contains no point of M.

Theorem 61. If M is a compact point set and $\beta^{t't''}_{m(nq)}$ contains no point of M, then there exists an integer $\overline{m}$, such that if $m'>\overline{m}$, then the arc of $J_{m'(nq)}$ which is irreducible from the positive half of $\beta^{t't''}_{m'(nq)}$ to the negative half of $\beta^{t't''}_{m'(nq)}$ contains no point of M and is a subset of D_{n-1}.

Now by the use of these radial webs enough ideal points will be added to the space S to make the new space S' satisfy the axioms that S satisfies but which contains no edge points. Since space is completely separable, let P_1, P_2, P_3, ... denote the set of all edge points of S. For each integer j let R_{1j}, R_{2j}, R_{3j}, ..., D_{1j}, D_{2j}, D_{3j}, ... and W_1, W_2, W_3, ... denote sequences of point sets such that for each n and j (1) R_{nj} is a region of $G_{(n+j)}$ of Axiom I and the sequence $\{R_{nj}\}$ closes down on P, (2) D_{nj} is a pseudo-simple domain with respect to R_{nj} and P_j containing $R_{(n+1)j}$, (3) W_j is a radial web about P_j whose skeleton lies in R_{1j}, and (4) if $j'<j$ and $R_{1j'}$ contains a point of D_{1j}, then for some m and n (a) $\overline{R}_{1j'}$ is a subset of $R-\overline{D}_{nj}$ and (b) contains no point of $W_j-\Sigma I_{m(nq)}$ of W_j.

Since space is completely separable, there exists a countable collection G of simple domains such that if P is a simple point and R is a region containing P, some element of G contains P and lies together with its boundary in R. For each integer i let H_i denote the collection of all the domains D such that either (1) D is a simple domain of G which is a subset of a region of G_i of Axiom I, or (2) for some pair of numbers n and j, $(n+j\geq i)$, D is the pseudo-simple domain D_{nj}, or (3) for some triplet of integers m, n, and q and some pair of numbers t' and t", $m\geq i$ and $0<t'-t''\leq 1/i$, $D=R^{t'\,t''}_{m(nq)}$ of W_j for some j.

Definition: If for some i, D is a domain of H_i and for some j, $P^t_{(nq)}$ is an ideal point defined from W_j, then D is said to <u>inclose</u> $P^t_{(nq)}$ if some domain of $P^t_{(nq)}$ together with its boundary is a subset of D.

Definition: Let S' be a space in which point is to be interpreted as meaning either a point of S or one of the ideal points defined by W_j for some j; and in which region is to be interpreted as meaning a point set D' consisting of the points of some domain D of H_i, for some i, together with the ideal points which D incloses. A point of S' which is also a point of S will be called an "ordinary" point. Regions in S' will be refered to as G'-regions.

Theorem 62. In S point and limit point are unchanged by the above definition of point and region for S'.

Theorem 63. Every point of S' is a limit point of S.

Theorem 64. If R' is a G'-region, then $\overline{G'} = \overline{G' \cdot S} = \overline{\overline{G'} \cdot S}$.

Definition: For each i let G'_i denote the collection of all regions g' of S' such that g'·S=D is an element of H_i. g' is said to be defined by D or obtained from D.

Theorem 65. If J is a simple closed curve of "ordinary" points, then S'-J is the sum of two mutually exclusive connected domains each having J for its boundary.

Proof: S-P is the sum of two mutually exclusive point sets H and K each having J on its boundary. Let $H' = \overline{H} - \overline{H} \cdot J$ and $K' = K - \overline{K} \cdot J$. Suppose that H' has a point A in common with K'. A is an "ideal" point $P^t_{(nq)}$. Since J is compact, there exists a region containing P which was obtained from a domain $R^{t't''}_{m(nq)}$ such that $\overline{R}^{t't''}_{m(nq)}\cdot S$ contains no point of J. But $R^{t't''}_{m(nq)}$ contains points of both H and K. Hence, $I^{t't''}_{m,(nq)} + \beta^{t't''}_{m,(nq)}$ is a subset of H (or K) and $I^{t't''}_{-m,(n[q+1])} + \beta^{t't''}_{-m,(n[q+1])}$ is a subset of K (or H). By THeorem 57 there exists a sequence of mutually exclusive arcs U_1, U_2, U_3, ... lying in S irreducible from $\beta^{t't''}_{m,(nq)}$ to $\beta^{t't''}_{-m,(n[q+1])}$ and having no sequential limiting set in S. This is a contradiction, since each of these arcs must intersect J. Hence H' and K' are mutually exclusive connected domains each having J for its complete boundary.

Theorem 66. For each i G'_i is a countable collection of regions covering S' and containing H''_{n+i}.

Theorem 67. If $P_{(nq)}$ is an ideal point defined by the web W_j, there exists an integer k such that no region of G'_k contains $P^t_{(nq)}$ except those regions of G'_k defined from domains belonging to $P^t_{(nq)}$.

Proof. There exists an integer k_o such that if R is a region of G_{k_o} which contains P_j, R is a subset of D_{nj}. Now if g' is a region of G'_{k_o} such that g'·S is a simple domain in S and g' contains $P^t_{(nq)}$, then $\overline{g}'\cdot S$ contains P_j and hence, is a subset of a region R of G_{k_o} which contains P_j. Therefore, $\overline{g}'\cdot S$ is a subset of D_{nj} and contains some domain of $P^t_{(nq)}$. This is impossible for since P_j is an edge point, g'·S does not contain P_j but does contain points of different components of $D_{nj}-P_j$, namely, points of the positive and negative halves of the boundary of some domain of $P^t_{(nq)}$. So if g' is a region of G'_{k_o} which was obtained from a simple domain of $\{H_i\}$ or from a pseudo-simple domain of W_j, g' does not contain $P^t_{(nq)}$.

Now by (4) if j'>j no region defined from $W_{j'}$, either of the pseudo-simple domain type or the $R^{t't''}_{m(nq)}$ type contains $P^t_{(nq)}$. Now $K = P_1 + P_2 + P_3 + \ldots + P_{j-1}$ is a closed and compact point set and by Theorem 58 there exists an integer m such that $I_{m(nq)} + J_{m(nq)}$ of W_j contains no point of K. Further, for each integer j'<j, there exists an integer $k_{j'}$ such that no region of $G_{kj'}$ which contains $P_{j'}$ contains any point of $I_{m(nq)} + J_{m(nq)}$ of W_j, and there exists another integer $l_{j'}$ such that $\sum_{n=1}^{k_i} \sum_{q=1}^{q_n} (I_{l_j(nq)} + J_{l_j(nq)})$ of $W_{j'}$ contains no point of $I_{m(nq)} + J_{m(nq)}$ of W_j. Now let k denote the largest of the integers $l_{j'}$, k_o, k_1, k_2, ... , k_{j-1}, l_1, l_2, l_3, ... , l_{j-1}. Then if g' is a region of G'_k but not a region of the type $R^{t't''}_{m(nq)}$ defined by W_j, then g' does not contain $P^t_{(nq)}$, and it is evident from the construction that if a region of the $R^{t't''}_{m(nq)}$ type defined by W_j contains $P^t_{(nq)}$, it belongs to $P^t_{(nq)}$ since no two regions of this type containing points of different spurs of the same cluster have a point in common. Hence, every region G'_k which contains $P^t_{(nq)}$ was defined from some domain of $P^t_{(nq)}$.

Theorem 68. If A and B are two distinct points of S', there exists an integer k such that if g' is a region of G_k' containing A, g' does not contain B.

Proof: Case I. Suppose both A and B are "ordinary" points. Then there exists an integer k' such that if R is a region of $G_{k'}'$ which contains A, $\overline{R}$ does not contain B. Now if g' is a region of G_k' which contains A and g'·S is a subset of a region R of $G_{k'}$, then g'·S does not contain B and hence, g' does not contain B. Now the only regions of $G_{k'}'$ which are not defined from domains which are possibly not subsets of regions of $G_{k'}'$ are those defined from domains of the type $R_{m(nq)}^{t't''}$ of W_j where $t'' \geq 1/k'$ and $j<k'$. By an argument similar to that of the last paragraph of the preceding theorem there exists an integer k" such that none of these regions contain A. Hence, if k is the larger of the two integers k' and k", no region of G_k' which contains A contains B.

Case II. Suppose that B is an "ideal" point and that A is an "ordinary" point. Then A is a compact point set and by Theorem 60 there exists a G'-region R' containing B such that $\overline{R}'$·S does not contain A. But S-$\overline{R}'$·S is a domain of S containing A and there exists a number k' such that if R is a region of $G_{k'}$ containing A, then $\overline{R}$ is a subset of S-$\overline{R}'$·S. Hence, by the argument of Case I there exists an integer k such that if g' is a region of G_k', g' does not contain B.

Case III. Suppose that A is an "ideal" point and that B is an "ordinary" point. Interchange the A and B of Case II.

Case IV. Suppose that both A and B are "ideal" points. By Theorem 67 there exists a number k such that every region of G'_k which contains B was defined from an element of B. Hence, if A and B belong to different spurs, no region of G'_k which contains A also contains B. Further, if $A = P^{t'}_{(nq)}$ and $B = P^{t''}_{(nq)}$ and $1/k < |t'-t''|$, then although A and B both belong to the same spur no region of G'_k which contains A contains B.

Theorem 68. S' satisfies Axiom I.

Proof: It is clear that parts (1) and (2) are satisfied (Theorem 66), and slight changes in the above argument will show that part (3) is also satisfied. Furthermore, part (4) of Axiom I is satisfied. For suppose that M_1, M_2, M_3, ... is a sequence of closed point sets such that for each i M_i contains M_{i+1} and is a subset of a region g'_i of G'_i.

Case I. Suppose that for infinitely many values i_1, i_2, i_3, ... of i, $g'_{i_i} \cdot S$ is either a pseudo-simple domain of S used in the definition of W_{j_i} or is a domain of the type $R^{t't''}_{m(nq)}$ defined by W_{j_i} where $j_i > j_{i'}$, and $W_{j_i} \neq W_{j_{i'}}$ if $i \neq i'$. In either case $\overline{g}'_i \cdot S$ is a subset of the pseudo-simple domain D_{1j_i} of S. Hence, $g'_{i_{i+1}} \cdot S$ and therefore, $R_{1j_{i+1}}$ contain points of D_{1j_i} and by (4) R_{1j_i} contains $\overline{R}_{1j_{i+1}} \cdot S$. But since S satisfies part (4) of Axiom I, S contains a point P which is common to R_{1j_1}, R_{1j_2}, R_{1j_3}, ... Therefore P belongs to the sequential limit set of $g^j_{i_1}$, $g^j_{i_2}$, $g^j_{i_3}$, ... and M_1, M_2, M_3, ... Hence, since for each i M_i is closed and contains M_{i+1}, P is common to M_1, M_2, M_3, ...

Case II. Suppose that for infinitely many values i_1, i_2, i_3, ... of i, there exists a web W_j such that for each i $g^j_{i_i} \cdot S$ is either a pseudo-simple domain of S used in the definition of W_j or is a domain of the type $R^{t't''}_{m(nq)}$ defined by W_j. If infinitely many of the regions g_{i_1}, g_{i_2}, g_{i_3}, ... were defined from pseudo-simple domains of S, then it is that P_j must belong to M_i for each i. If this is not the case, we shall assume for simplicity that for each i $g^j_{i_i} \cdot S$ is $R^{t't''}_{m(nq)}$ for some value of the superscripts and subscripts. But since no two regions defined from domains of the type $R^{t't''}_{m(nq)}$ which contain points of different spurs of the same cluster have a point in common, (nq) does not vary with i. As $i \to \infty$, $m \to \infty$ and $|t'-t''| \to 0$. Now since for each i g_{i_i} contains points of g_{i_1}, there exists a number t, $0<t<1$, such that as $i \to \infty$, $t' \to t$ and $t'' \to t$. Hence $P^t_{(nq)}$ is common to M_1, M_2, M_3, ...

Case III. When neither Case I or II occurs, we may assume for simplicity that for each i $g^j_i \cdot S$ is a simple domain of S. Then for each i $\overline{g}^j_i \cdot S$ is a subset of a region

R_i of G_i. For each i, $(\overline{g}_1^i \cdot S) \cdot (\overline{g}_2^i \cdot S) \cdot \ldots \cdot (\overline{g}_i^i \cdot S) = K_i$ is a point set closed with respect to S such that K_i contains K_{i+1} and K_i is a subset of R_i. Since S satisfies part (4) of Axiom I, there is a point P of S which for each i belongs to K_i. Hence, P belongs to the sequential limit set of $M_1, M_2, M_3, \ldots$ and therefore, for each i P belongs to M_i.

Theorem 70. If J is a simple closed curve of "ordinary" points, I is one of its complementary domains, the points A and B separate the points C and D on J and AXB is an arc of "ordinary" points such that $\widehat{AXB}$ is a subset of I, then I-$\widehat{AXB}$ is the sum of two simple domains I_1 and I_2, such that the boundary of I_1 is the simple closed curve AXB+ACB(of J) and the boundary of I_2 is the simple closed curve AXB+ADB (of J).

Since the theorem holds for S it can be shown to hold for S' with the help of Theorem 65.

Theorem 71. Theorem 70 remains true if the arc AXB contains only one "ideal" point.

Proof: Suppose that X is the "ideal" of AXB. Let W_j denote the web used in the definition of the domains of X, and let $N_{(nq)}$ denote the spur of "ideal" points which contains X.

Case I. Suppose that there exists a monotonic sequence $R_1', R_2', R_3', \ldots$ of G'-regions defined from domains $R_1, R_2, R_3, \ldots$ belonging to X such that for each m both

$\widehat{AX}$ and $\widehat{BX}$ the same half of the boundary β_m of R_m in S. Now for each m let AX_nB denote an arc of "ordinary" points composed of the arcs AA_m of AX and BB_m of BX plus $A_mX_mB_m$ of β_m such that $A_mX_mB_m$ has only the two points A_m and B_m in common with AXB. Now the sequential limiting set of $A_1X_1B_1$, $A_2X_2B_2$, $A_3X_3B_3$, ... is X. So there exists integers n_1, n_2, n_3, ... such that for each m (1) $AA_{m_{m+1}}$ and $BB_{m_{m+1}}$ contain AA_{m_m} and BB_{m_m} respectively, (2) $I-\widehat{AX_{m_m}B} = I_{xm}+E_{xm}$ with respect to X as ω (Theorem 70), and (3) I_{xm+1} contains I_{xm}. Let $I_x = \sum_{n=1}^{\infty} I_{xm}$ and $E_x = \prod_{m=1}^{\infty} E_{xm}-X$. Then I_x and E_x are the domains I_1 and I_2 of the conclusion of the theorem.

Case II. Suppose that there exists a monotonic sequence R_1', R_2', R_3', ... of G'-regions defined from domains R_1, R_2, R_3, ... belonging to X such that for each m $\widehat{AX}$ and $\widehat{BX}$ intersect different halves of the boundary β_m of R_m in S. The argument here is as that of Case I with $A_mX_mB_m$ a subset of $\beta_m + J_{m'(nq)}$(of W_j)$- P_j$, for some m'>m, instead of from β_m alone. Let the component of $N_{(nq)}-X$ which does not have P_j as a limit point be denoted by M. Then the sequential limit set of $A_1X_1B_1$, $A_2X_2B_2$, $A_3X_3B_3$, ... is M + X. Let $I_x = \sum_{m=1}^{\infty} I_{xm}$ and $E_x = \prod_{m=1}^{\infty} E_{xm}-(M+X)$. Then I_x and E_x are the domains of the conclusion of the theorem.

Theorem 72. Theorem 70 remains true if the arc AXB contains only a finite number of "ideal" points.

Proof: The method of proof will be that of mathematical induction. Theorem 71 shows that this theorem holds when the number of "ideal" points in AXB is only one. We shall suppose that the theorem holds for j-1 or less points and show that it holds for j points. Suppose that $AXB \cdot (S'-S) = Y_1 + Y_2 + Y_3 + \ldots + Y_j$. Let $N_{(nq)}$ denote a spur containing one of these "ideal" points and let Y_k denote that one of them on $N_{(nq)}$ which is fartherest on $N_{(nq)}$ from its end point. Now apply the argument of the preceding theorem where X is now Y_k and the arcs AX_mB contains j-1 or less points for which Theorem 72 holds by supposition.

Theorem 73. Theorem 70 remains true regardless of the composition of the arc ACB.

Proof: For each integer i let K_i denote a finite collection of G'-regions of G_i' covering AXB such that (1) if g' is an element of K_i, $\overline{g}'$ does not contain C+D, (2) only one element of K_i contains A and only one element of K_i contains B and all other elements of K_i are together with their boundaries subsets of I, (3) each element of K_{i+1} is together with its boundary a subset of some element of K_i. There exists a sequence of mutually exclusive arcs $A_1X_1B_1$, $A_2X_2B_2$, $A_3X_3B_3$, ... such that for each integer i (1) $A_iX_iB_i$, is a subset of the sum of the boundaries of the elements of K_i, (2) $A_iX_iB_i \cdot J = A_i + B_i$, (3) Lim A_i =A and Lim B_i=B, (4) I-$\widehat{AXB}$ contains $\widehat{A_iX_iB_i}$, (5) I-$\widehat{A_iX_iB_i}$ contains $\widehat{A_{i'}X_{i'}B_{i'}}$+ $\widehat{AXB}$ if i'>i, and (6) Lim $A_iX_iB_i$= AXB. Now for each i, let

$I-A_iX_iB_i=I_{xi}+E_{xi}$ with respect to X as ω. Then $I_x = \sum_{i=1}^{\infty} I_{xi}$ and $E_x = \prod_{i=1}^{\infty} E_{xi}-AXB$ are the two domains I_1 and I_2 of the theorem.

Theorem 74. If J is a simple closed curve containing an arc T of "ordinary" points, then S'-J is the sum of two mutually exclusive connected domains each having J for its boundary.

Proof: T contains an "ordinary" simple point X. There exists a simple domain containing X whose boundary C is composed of "ordinary" points and which contains no point of J-T. C contains an arc AYB having only its ends in common with J such that AXB of J is a subset of T. The simple closed curve C' =AXB+AYB is composed of "ordinary" points and if I' denotes the complementary domain of C' that contains $J-AXB=\widehat{AZB}$, $I'-\widehat{AZB}=I+I_1$, where the boundary of I is J and the boundary of I_1 is AYB+AZB. But $I_1+I'+AYB=E$ is a simple domain and contains no point of I. Thus S'-J=I+E.

Theorem 75. If J is a simple closed curve, S'-J is the sum of two mutually exclusive connected domains each having J for its boundary.

Proof: J contains uncountably many "ordinary" points and hence J contains a simple "ordinary" point X and another point Y. Let R_1', R_2', R_3', ... denote a sequence of G'-regions closing down on X such that for each i the boundary of R_i' is a simple closed curve J_i of "ordinary" points.

There exists an infinite sequence of integers i_1, i_2, i_3, $\ldots i_i$, ... such that for each k (1) $\overline{R}^{i}_{i_i}$ does not contain Y, (2) J_{i_i} contains an arc $A_iX_iB_i$ having the following properties: (a) $A_iX_iB_i \cdot J = A_i + B_i$, (b) Lim $A_iX_iB_i = X$, and (c) $C_i = A_iX_iB_i + A_iX_iB_i$ (of J) is a simple closed curve one of whose complementary domains E_{xi} contains X and the other I_{xi} contains $\widehat{A_{i-1}X_{i-1}B_{i-1}} + I_{x(i-1)}$. Then $I_x = \sum_{i=1}^{\infty} I_{xi}$ and $E_x = \prod_{i=1}^{\infty} E_{xi} - X$ are two mutually exclusive connected domains each having J for its boundary.

Theorem 76. S' satisfies Axiom IV.

This is a restatement of Theorem 75.

Theorem 77. S is homeomorphic with a subset of a completely separable space S' which satisfies Axioms 0, 1, 2, 3, 4, and Axiom 5'[16]

This theorem may be established with the help of Theorems 69 and 75. Axioms 2, 3, and 5' follow from the fact that every region is a simple domain.

Theorem 78. S is homeomorphic with a subset of a plane or a sphere.

Proof: J.H. Roberts[17] has shown that S' of Theorem 77 is homeomorphic with a subset of a plane or a sphere.

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[16]Axiom 5': If P is a point of a region R, there exists a simple domain containing P which lies together with its boundary in R.

[17]Roberts, J.H., "Concerning compact continua in certain spaces of R.L. Moore," Bull. Amer. Math. Soc., Vol. 39, 1933, pp. 615-621.

Some of the preceding results will now be sumarized for metric spaces.

Theorem 79. A locally connected, complete metric space in which the Jordon Curve Theorem[18] is satisfied non-vacuously and in which Axiom 5_1^* holds true, is a cyclicly connected subset of a plane (if not compact) or a sphere (if compact).

Proof: A locally connected complete metric space satisfies Axioms 0, 1, and 2; and such a space in which Axiom 5_1^* holds true satisfies Axioms 4 and 3 if the Jordon Curve Theorem holds true non-vacuously (Theorem 2). Thus the above theorem follows from Theorem 78.

Definition: A space is said to be locally peripherally connected if it satisfies the following axiom: If P is a point of a region R, there exists in R a domain D containing P such that the boundary of D ic connected.

Theorem 80. Any regular space which is locally peripherally connected satisfies Aciom 5_1^*.

Theorem 81. A locally connected, locally peripherally connected, complete metric space in which the Jordon Curve Theorem is non-vacuously satisfied is a cyclicly connected subset of a sphere.

[18]Axiom 4.

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