The Dissertation Committee for Thomas Joseph Mainiero certifies that this is the approved version of the following dissertation:

Beyond Wild Walls there is Algebraicity and Exponential Growth (of BPS indices)

Committee:

Andrew Neitzke, Supervisor

Jacques Distler, Co-supervisor

Willy Fischler

Vadim Kaplunovsky

Sean Keel

Beyond Wild Walls there is Algebraicity and Exponential Growth (of BPS indices)

by

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To Thelma and Louise: May the rest of your days be filled with pungent smells and garden hoses.

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I feel it only appropriate to thank my officemates and my friends; I will tactfully avoid naming you individually, fearing my thesis may be played off the stage by an orchestra. However I will make an exception for Dan Carney, whose lively discussions and deep questions made me actually look forward to working in a windowless box.

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Preface

This thesis is concerned with surprising phenomena in the BPS spectrum of theories of class $S[A]$: a large class of four-dimensional $\mathcal{N} = 2$ field theories that include familiar Lagrangian field theories, such as $SU(N)$ super Yang Mills, along with other more exotic field theories that may not have a Lagrangian description. A particularly useful way to study such theories are by their spectrum of BPS states – states that are invariant under half of the supersymmetries and, as a representation-theoretic corollary, behave nicely under small deformations of moduli. Thanks to large developments by Gaiotto-Moore and Neitzke (referenced copiously throughout the main body of this thesis), the *BPS index*– which is a weighted count of BPS states– has been made a fairly accessible quantity to compute in theories of class $S[A]$.

The first part of this thesis is devoted to studying some rather unexpected phenomena in theories of class $S[A]$, in particular: pure $SU(3)$ super Yang Mills. It is shown that this theory contains regions on its moduli space of vacua where the BPS index grows exponentially with the charge of the state. By naïve thermodynamic arguments, this type of phenomenon was originally believed by Greg Moore to be impossible according to his "no wild wall crossing" conjecture. The walls referenced here refer to (real) codimension 1 loci on the moduli space of vacua–more precisely a component called the *Coulomb branch*– where the BPS index suddenly jumps. The presence of certain "wild" walls on the Coulomb branch would allow for the BPS indices to be well-behaved (without exponential growth) on one side of the wall, to "wild" (exponential growth) on the other. As already hinted, the "no wild wall crossing" conjecture is disproven by our explicit counterexample of $SU(3)$ super Yang mills; the presence of exponential degeneracies, and their associated wild walls is the primary physical content of the first part. This part of the thesis is largely contained in [31] and was co-authored with Dmitri Galakhov, Pietro Longhi, Greg Moore, and Andrew Neitzke. In particular, these authors have contributed the majority of the content in §1, and 4 - 8 along with appendices App. D - E.

The second part of this thesis is devoted to a phenomenon we call *algebraicity*: generating series for BPS indices, which are a priori formal series, are actually algebraic functions over the rational numbers. This phenomenon is hinted at in the first part where the *spectral network* technique for computing BPS indices produces algebraic equations for BPS index generating series. The sort of algebraic equations derived in the first part were previously known in the mathematical literature to be algebraic equations for generating series that encode a subset of the Donaldson-Thomas invariants (essentially a rigorous geometric definition for BPS indices) for a quiver called the m-Kronecker quiver. In the second part we present another example of an algebraic function satisfied by generating series for m-Kronecker DT-invariants – which also occur as BPS indices in $SU(3)$ super Yang Mills; this example is novel to both mathematicians and physicists. These examples are explicit manifestations of the very general algebraicity phenomenon (mentioned above) in Theories of class $S[A]$. In the mathematics community, Kontsevich and Soibelman have announced an indirect understanding of algebraicity for a large class of Donaldson Thomas invariants; however, spectral network machinery provides an independent argument of algebraicity for BPS indices in theories of class $S[A]$. The advantage of the latter approach is that the argument is a direct/constructive one, making it clear that explicit algebraic equations can always be algorithmically derived through spectral network machinery.

There are many reasons why one should be excited about algebraicity, one being the likelihood of deep implications for the geometry of moduli spaces of vacua (in particular, the generalized cluster-like structures discovered by Gaiotto-Moore-Neitzke). The least abstract implication, however, is relevant to our observations in the first part of the thesis: *algebraicity leads to exponential growth of BPS indices with the charge*. Only in the simplest of possible scenarios–in particular, when the generating series is a rational function with zeros and poles at roots of unity– do BPS indices *not* grow exponentially with the charge. The complicated nature of spectral networks in theories of class $S[A]$ with rank $K > 1$ (e.g. $SU(K)$ super Yang Mills) suggests that typically algebraic generating series are not typically polynomials; in this sense, the nice behaviour of the BPS spectrum for theories of rank 1 may be rather exceptional. In the final part of this thesis we provide a constructive proof of how algebraic generating series lead to exponential growth; in the process we give an explicit way to extract precise asymptotics of BPS indices given the algebraic equations satisfied by generating series.

Beyond Wild Walls there is Algebraicity and Exponential Growth (of BPS indices)

Thomas Joseph Mainiero, Ph.D.
The University of Texas at Austin, 2015

Supervisor: Andrew Neitzke
Co-supervisor: Jacques Distler

The BPS spectrum of pure SU(3) four-dimensional super Yang-Mills with $\mathcal{N} = 2$ supersymmetry (a theory of class $S[A]$) exhibits a surprising phenomenon: there are regions of the Coulomb branch where the growth of BPS-indices with the charge is exponential. We show this using spectral networks and, independently, using wall-crossing formulae and quiver methods. The technique using spectral networks hints at a general property dubbed *algebraicity*: generating series for BPS-indices in theories of class $S[A]$ (a class of $\mathcal{N} = 2$ four-dimensional field theories) are secretly algebraic functions over the rational numbers. Kontsevich and Soibelman have an independent understanding of algebraicity using indirect techniques, however, spectral networks give a distinct reason for algebraicity with the advantage of providing explicit algebraic equations obeyed by generating series; along these lines, we provide a novel example of such an algebraic equation, and explore some relationships to Euler characteristics of Kronecker quiver stable moduli. We conclude by proving that exponential asymptotic growth is a corollary of algebraicity, leading to the slogan "there are either finitely many BPS indices or exponentially many" (in theories of class $S[A]$).

Contents

Part I

Wild Wall Crossing[1]

[1]As mentioned in the preface, this part of the thesis is largely contained in [31] and was co-authored with Dmitri Galakhov, Pietro Longhi, Greg Moore, and Andrew Neitzke. In particular, these authors have contributed the majority of the content in §1, and 4 - 8 along with appendices App. D - E.

Chapter One: Introduction & Conclusion

One good reason to investigate the BPS spectra of four-dimensional $\mathcal{N} = 2$ field theories is that one might discover new phenomena in field theory. This part of the thesis demonstrates an example of such a new phenomenon.

In the past few years there has been much progress in understanding the BPS spectra of $\mathcal{N} = 2, d = 4$ theories. For recent reviews see [46, 45, 12]. These methods have been particularly powerful when applied to the theories of class $S[A_1]$. As one example, the spectrum generator technique of [29] gives an algorithm which can — in principle — give the BPS spectrum of any theory of class $S[A_1]$ anywhere on its Coulomb branch. Advances in quiver technology have also been very effective in investigating this class of theories [6, 13]. In contrast, theories associated to higher rank gauge groups, such as theories of class $S[A_{K-1}]$ for $K > 2$, have been less explored.

It has been noted by various authors that theories of class $S[A_{K-1}]$ for $K > 2$ could have higher spin BPS states, beyond the familiar hypermultiplets and vectormultiplets which occur in theories of class $S[A_1]$. One result of ours is that this expectation is indeed correct: higher spin BPS multiplets do occur at some points of the Coulomb branch in one explicit theory of class $S[A_2]$, namely the pure $d = 4$, $\mathcal{N} = 2$, $SU(3)$ theory.

In addition, we find a much more surprising phenomenon: theories of class S can have *wild BPS spectra*, i.e. at some points of the Coulomb branch, the number of BPS states with mass $\leq M$ grows *exponentially* with M. The main result of this part of the thesis is two independent demonstrations, in Sections 3 and 4, that wild spectra appear in the pure $d = 4, \mathcal{N} = 2, SU(3)$ theory.

As explained in Section 7 below, this exponential growth is physically a bit surprising. Indeed, the existence of a conformal fixed point defining the 4d theory, plus dimensional analysis, implies that the degeneracy of BPS states at energy E in finite volume V cannot grow faster than $\exp[\text{const} \times V^{1/4}E^{3/4}]$. On the other hand, here we are finding that the spectrum of BPS 1-particle states grows like $\exp[\text{const} \times E]$. The resolution of this puzzle must lie in the difference between BPS 1-particle states and states in the finite volume Hilbert space; we propose that the size of the objects represented by the BPS 1-particle states grows with E, so that for any fixed V, most of the BPS 1-particle states simply do not fit into the finite-volume Hilbert space. Indeed, in Section 7, using Denef's picture of BPS bound states, we demonstrate directly that their size does indeed grow with E. The invalid exchange of large E and large V limits when accounting for field theory entropy should perhaps serve as a cautionary tale.

Here is the fundamental idea which we use to find wild BPS degeneracies. Suppose we have an $\mathcal{N} = 2$ theory and a point of the Coulomb branch in which the spectrum contains two BPS hypermultiplets, of charges γ and γ', and no bound states thereof — i.e. we have the BPS degeneracies $\Omega(\gamma) = 1$, $\Omega(\gamma') = 1$, $\Omega(a\gamma + b\gamma') = 0$ for all other $a, b \geq 0$. Then suppose we move on the Coulomb branch to a point where the central charges Z_γ and $Z_{\gamma'}$ have the same phase. Such a point lies on a wall of marginal stability. On the other side of the wall, the spectrum includes bound states with charge $a\gamma + b\gamma'$ for various a, b. Their precise degeneracies can be determined by the Kontsevich-Soibelman wall-crossing formula, and indeed depend *only* on the integer $m = \langle \gamma, \gamma' \rangle$. For this reason we call the collection of BPS states thus generated an "m-cohort."

The cases $m = 1$ and $m = 2$ occur already in the theories of class $S[A_1]$. For $m = 1$ an m-cohort contains only a single bound state; for $m = 2$ an m-cohort contains an infinite set of hypermultiplets plus a single vector multiplet. In either case, at any rate, one does not get wild degeneracies. In contrast, for $m > 2$ the wall-crossing formula shows that an m-cohort does contain wild degeneracies. Indeed, even if one restricts attention to charges of the form $n(\gamma + \gamma')$, one already has exponential growth. This is explained and made precise in Proposition 3.3.3, Section 5.3, and Section 15.2 below. With this in mind, for any $m > 2$, we will say that a theory contains "m-wild degeneracies" if its BPS spectrum contains an m-cohort.

The BPS degeneracies arising in m-cohorts have been studied at some length in the mathematics literature because they arise as Donaldson-Thomas invariants attached to the m-Kronecker quiver in one region of its stability parameter space. The latter have been intensively studied in [49, 50, 48, 47, 55, 56]. One interesting feature noted there is that for $m > 2$, the phases of the central charges of BPS states in an m-cohort are *dense* in some arc of the circle.

This discussion motivates two approaches to the problem of exhibiting wild degeneracies in a physical theory. Our first approach goes via the "spectral networks" of [27, 28]: rather than studying the wall-crossing directly, we make a guess about the kind of spectral networks which *could* arise from wall-crossing involving two hypermultiplets with arbitrary $m = \langle\gamma, \gamma'\rangle$. For $m = 1$ the network we draw looks like a saddle, which motivates an equine terminology: our networks are built from constituents we call "horses" (defined in Section. 3.1, Figure 3.3, and detailed in Appendix B), glued together to form "m-herds." See Figure 3.4 for some examples. We show moreover that m-herds indeed occur in physical spectral networks at some particular points of the Coulomb branch of the $SU(3)$ theory: see Figure 3.5 for the evidence. The general rules of spectral networks, combined with Proposition 3.1 and Proposition 3.2 below, lead to the following formula for the BPS spectrum for charges of the form $n(\gamma + \gamma') := n\gamma_c$ in the wild region. We first form a generating function $P_m(z)$ related to the BPS spectrum by

$$P_m(z) = \prod_{n=1}^{\infty}(1 - (-1)^{mn} z^n)^{n\Omega(n\gamma_c)/m}. \tag{1.1}$$

Then, Proposition 3.1 states that $P_m(z)$ is a solution of the algebraic equation (3.2), which we reproduce here:

$$P_m = 1 + z\,(P_m)^{(m-1)^2}. \tag{1.2}$$

This equation had been identified previously by Kontsevich and Soibelman [40] and by Gross and Pandharipande [32], as the one governing the generating function of BPS degeneracies of an m-cohort, for charges of the form $n(\gamma + \gamma')$. It follows that if we have an m-herd ($m > 2$) somewhere in our theory, then our theory does contain at least the part of an m-cohort corresponding to charges of the form $n(\gamma + \gamma')$. In particular, if the theory contains an m-herd, then it does contain wild degeneracies. Since we have found m-herds at some points of the Coulomb branch in the pure $SU(3)$ theory, we conclude that we indeed have wild degeneracies in that theory.

The algebraic equation (1.2) is an instance of a more general phenomenon. It has been observed by Kontsevich that the generating functions of Donaldson-Thomas invariants are often solutions of algebraic equations. In fact, for the Kronecker quiver this has been proved [38]. Our analysis via spectral networks produces the algebraic equation (1.2) directly. Moreover, we expect that this will happen more generally, as we explain in Appendix E; thus spectral networks seem to be a natural framework for explaining Kontsevich's observation.

Our second method of demonstrating the existence of wild spectra uses wall-crossing more directly. Namely, in Section 4.2 we exhibit a path on the Coulomb branch which begins in a strong coupling chamber with a finite set of BPS states, and leads to a wall-crossing between two hypermultiplet charges γ, γ' with $\langle \gamma, \gamma' \rangle = 3$. As we have discussed above, the existence of such a path directly implies the existence of wild spectra. In fact this gives more than we got from the spectral network: it shows that there is a whole 3-cohort in the spectrum. In Section 5.2 we perform some nontrivial checks of this statement by factorizing the spectrum generator derived from the known finite spectrum in a strong coupling chamber. In Section 5.3 we also check numerically the exponential growth of the BPS degeneracies for sequences of charges of the form $n(a\gamma + b\gamma')$, $n \to \infty$, for various values of a, b.

In Section 6 we discuss the behavior of the "BPS quivers" of the $SU(3)$ theory along the path found in Section 4.2. It turns out that the Kronecker 3-quiver is in fact a subquiver of the BPS quiver, after one has performed suitable mutations and made a suitable choice of half-plane to define simple roots. We similarly argue that for *all* $m \geq 3$ (not only $m = 3$) there are Kronecker m-subquivers and corresponding m-wild spectra on the Coulomb branch of the $SU(3)$ theory.

In Section 8 we discuss a few open problems and questions raised by the present work. The remaining appendices address more technical points of spectral networks.

Chapter Two: Brief Review of Spectral Networks

In this section we give a brief review of the spectral network machinery and its use for computing BPS spectra in $\mathcal{N} = 2$, $d = 4$ theories of class S. For a more complete discussion we refer the reader to [27]. For a more informal (but incomplete) review see [45].

2.1 The Setting

Recall that the $\mathcal{N} = 2$, $d = 4$ theories of class S are specified by three pieces of data [29, 24]:

1. A Lie algebra $\mathfrak{g}$ of ADE type (as in [27] the following discussion assumes $\mathfrak{g} = A_{K-1}$),
2. a compact Riemann surface C with punctures at points $\mathfrak{s}_1, \cdots, \mathfrak{s}_n \in C$,
3. a collection of defect operators D located at the punctures.

To shed some light on this collection of data, we note that such theories can be constructed via a partial twist (preserving eight supercharges) of the $\mathcal{N} = (2, 0)$, $d = 6$ theory $S[\mathfrak{g}]$ defined on $\mathbb{R}^{3,1} \times C$. The defect operators D are codimension-2 defects located at $\mathbb{R}^{3,1} \times \{\mathfrak{s}_1\}, \cdots, \mathbb{R}^{3,1} \times \{\mathfrak{s}_n\}$. A four-dimensional $\mathcal{N} = 2$ field theory is produced after integrating out the degrees of freedom along C and is labeled $S[\mathfrak{g}, C, D]$.

We now present some useful definitions.

Definitions

1. The *Coulomb branch* $\mathcal{B}$ of $S[\mathfrak{g}, C, D]$ is the set of tuples $(\phi_2, \cdots, \phi_K)$ of holomorphic r-differentials ϕ_r with singularities at $\mathfrak{s}_1, \cdots, \mathfrak{s}_n \in C$ prescribed by the defect operators D.
2. Let $u = (\phi_2, \cdots, \phi_r) \in \mathcal{B}$ and denote the holomorphic cotangent bundle of C as $\mathcal{T}^*C$. Then the *spectral cover* is a K-sheeted branched cover $\pi_u : \Sigma_u \to C$,

where Σ_u is the subvariety[1]

$$\Sigma_u := \{\lambda \in \mathcal{T}^*C : \lambda^K + \sum_{r=2}^{K} \phi_r \lambda^{K-r} = 0\} \subset \mathcal{T}^*C, \tag{2.1}$$

and the projection π_u is the restriction of the standard projection $\mathcal{T}^*C \to C$.

3. As $\Sigma_u \subset \mathcal{T}^*C$ it carries a natural holomorphic 1-form which is just the restriction of the tautological (Liouville) 1-form. In the spirit of its tautological nature we abuse notation and denote this 1-form λ_u.

Often we will work over a fixed $u \in \mathcal{B}$; so eventually the index u will be dropped where there is no ambiguity.

Spectral Cover Crash Course

Let us make some observations about the spectral cover. First, the fibers are given by

$$\pi_u^{-1}(z) = \{\lambda(z) \in \mathcal{T}_z^*C : \lambda(z)^K + \sum_{r=2}^{K} \phi_r(z)\lambda^{K-r}(z) = 0\},$$

i.e. the roots of the defining polynomial of Σ_u at the point z. Generically, $\pi_u^{-1}(z)$ consists of K distinct roots, although at particular values of z (branch points) two or more roots may coincide. In fact, letting $C' = C - \{\text{branch points}\}$, $\pi_u|_{C'}$ is a K-fold (unramified) cover of C'.

If $\pi_u|_{C'}$ is a non-trivial cover, the roots do not fit together into global holomorphic 1-forms on C as they undergo monodromy around branch points. However, restricted to the complement of a choice of branch cuts on C, the cover is trivializable: a projection of K distinct sheets onto the complement. Each sheet is the graph traced out by a root of the defining polynomial; such roots are distinct holomorphic differential forms. A choice of trivialization of the restricted cover is a bijective map between the set of K sheets and the set $\{1, 2, \ldots, K\}$, or equivalently, a labeling of the roots of the defining polynomial from 1 to K.

Definitions

[1] Σ_u is also called the Seiberg-Witten curve.

1. Make a suitable choice of branch cuts for the branched cover $\pi_u : \Sigma_u \to C$. The complement of these branch cuts in C will be denoted by C^c.

2. A choice of trivialization of $\pi^{-1}(C^c) \to C^c$ will be denoted by a labeling of the roots of the defining polynomial for Σ, i.e. a labeling $\lambda_i \in H^0(C^c; K)$, $i = 1, \ldots, K$, where each λ_i (a holomorphic 1-form on C^c) is a distinct root of the defining polynomial for Σ. Note that this gives us a labeling of sheets: the ith sheet is the graph of λ_i in $\mathcal{T}^*C$. If we wish, we can extend the $\lambda_i(z)$ to branch points z to speak of "collisions" of sheets.

3. For later convenience, we define

$$\lambda_{ij} := \lambda_i - \lambda_j \in H^0(C^c; K).$$

As in [27] we will assume that all branch points are simple, i.e. at most two sheets of Σ collide at any z.

Definition A branch point of type ij $(i, j \in 1, \cdots, K)$ is a point $z \in C$ where the ith and jth sheets of Σ_u collide, i.e, $\lambda_i(z) = \lambda_j(z)$.

The data of the full spectral cover can be recovered after trivializing by specifying the monodromy around all branch points, and all closed cycles of C. We assume *simple ramification*: in a neighborhood around each branch point, the spectral cover looks like the branched cover $z \mapsto z^2$ of the disk. Thus, for a simple closed curve surrounding a branch point of type ij, there is a $\mathbb{Z}/2\mathbb{Z}$ monodromy

$$\lambda_i \leftrightarrow \lambda_j.$$

Monodromy around an arbitrary closed cycle of C may permute the sheets in a more complicated fashion.

BPS objects in $S[A_{K-1}, C, D]$

Theories of class S admit a zoo rich in BPS species, each of which has a different classical description from the point of view of the six-dimensional geometry of $\mathbb{R}^{3,1} \times C$. Our ultimate interest in this thesis is in the 4D (vanilla) BPS states, but the power behind the spectral network machine draws heavily on the symbiosis between these different species; so we take a moment to project each of them into the spotlight.

BPS Strings and "vanilla" 4D BPS states

4D BPS states in the four-dimensional $\mathcal{N} = 2$ theory arise from extended objects in the 6D description: BPS strings. In the effective IR description, at a point $u \in \mathcal{B}$, BPS strings wrap closed paths p on the branched cover $\Sigma_u \subset \mathcal{T}^*C \to C$. The resulting states are classified by their homology classes $\gamma = [p] \in H_1(\Sigma_u; \mathbb{Z})$ in the sense that there is a natural grading of the Hilbert space of BPS strings as

$$\mathcal{H}^{\text{BPS}}(u) = \bigoplus_{\gamma \in H_1(\Sigma_u;\mathbb{Z})} \mathcal{H}(\gamma; u),$$

commuting with the action of the super-Poincaré group.

Definition The *charge lattice* of 4D BPS states at a point $u \in \mathcal{B}$ is $\Gamma_u = H_1(\Sigma_u; \mathbb{Z})$. It is equipped with an antisymmetric pairing $\langle \cdot, \cdot \rangle : \Gamma_u \times \Gamma_u \to \mathbb{Z}$ given by the intersection form on $H_1(\Sigma_u; \mathbb{Z})$.

To count the number of BPS states of a particular charge γ we recall a major celebrity of this thesis: the second helicity supertrace (a.k.a. the "BPS degeneracy" or "BPS index")

$$\Omega(\gamma; u) = -\frac{1}{2}\text{Tr}_{\,\mathcal{H}(\gamma;u)}(2J_3)^2(-1)^{2J_3},$$

where J_3 is any generator of the rotation subgroup of the massive little group. This index is piecewise constant on $\mathcal{B}$, jumping across real codimension-1 walls of marginal stability on $\mathcal{B}$ where two BPS states with linearly independent charges $\gamma, \gamma' \in \Gamma_u$ have central charges of the same phase: $\arg(Z_\gamma) = \arg(Z_{\gamma'})$. To compute this index we will not rely on its definition as a supertrace, but instead utilize the geometric methods of the spectral network machine.

Remarks

1. On $\mathcal{B}$ there may be (complex) codimension-1 loci where a cycle of Σ_u degenerates. Let $\mathcal{B}^* = \mathcal{B} - \{\text{degeneration loci}\}$. Then the collection $\widehat{\Gamma} = \{\Gamma_u\}_{u \in \mathcal{B}^*}$ forms a local system of lattices $\widehat{\Gamma} \to \mathcal{B}^*$. This local system is often equipped with a non-trivial monodromy action.

2. As mentioned previously, we will often drop the subscript $u \in \mathcal{B}$ as we will often be working over a single point on the Coulomb branch, or choosing a local trivialization of the local system of lattices on some open set.

3. Strictly speaking, the lattice of charges Γ_u is not quite $H_1(\Sigma_u;\mathbb{Z})$ [29]; in the theory we consider in this thesis, though, Γ_u is just a sublattice of $H_1(\Sigma_u;\mathbb{Z})$, and for our considerations there is no harm in replacing Γ_u by $H_1(\Sigma_u;\mathbb{Z})$. (If we considered the theory with $\mathfrak{g} = \mathfrak{gl}(K)$ instead of $\mathfrak{g} = \mathfrak{sl}(K)$ then the charge lattice would be literally $H_1(\Sigma_u;\mathbb{Z})$.)

4. From the four-dimensional point of view, Γ_u is the lattice of electric/magnetic and flavor charges in the IR effective abelian gauge theory defined at u.

Fix $u \in \mathcal{B}$. The central charge and mass of a string $p : S^1 \to \Sigma$ are[2]

$$Z_p = \frac{1}{\pi}\int_p \lambda,$$
$$M_p = \frac{1}{\pi}\int_p |\lambda|.$$

With this, the BPS condition $|Z_p| = M_p$ is given by

$$\int_p \lambda = e^{i\vartheta}\int_p |\lambda| \tag{2.2}$$

for some $\vartheta = \arg(Z_p) \in \mathbb{R}/2\pi\mathbb{Z}$. The value of ϑ specifies which four-dimensional BPS subalgebra is preserved. We can rewrite this condition in more useful form; indeed, let v_p denote a vector field along the path p, then (2.2) is true iff

$$\mathrm{Im}\left[e^{-i\vartheta}\langle\lambda, v_p\rangle\right] = 0. \tag{2.3}$$

Solitons

The theory $S[\mathfrak{g}, C, D]$ is equipped with a special set of BPS surface defect operators $\{\mathbb{S}_z\}_{z\in C}$ parameterized (in the UV[3]) by points on C. In the IR, for fixed $u \in \mathcal{B}$, the operator $\mathbb{S}_z$ possesses finitely many massive vacua labeled by the set $\pi^{-1}(z)$ (with $\pi = \pi_u$). Letting $z \in C'$, then *solitons* are BPS states[4] bound to the defect $\mathbb{S}_z$, which interpolate between two different vacua. Classically, they are given by oriented

[2] The integral $\int_p \lambda$ is only a function of the class $[p] \in H_1(\Sigma;\mathbb{Z})$; hence, the central charge reduces to a function $\Gamma \to \mathbb{C}$.

[3] In the six-dimensional UV description, the operator $\mathbb{S}_z$ attached to a point $z \in C$ is a surface defect which intersects C at a single point.

[4] After insertion of $\mathbb{S}_z$ there are four remaining supercharges. A BPS soliton preserves two supercharges.

paths $s : [0,1] \to \Sigma$ with endpoints $s(0)$, $s(1) \in \pi^{-1}(z)$; furthermore, each such path satisfies a BPS condition that we will now describe.

Consider a soliton path s such that, after choosing a trivialization, s only runs along sheets i and j and such that the projection $s_C := \pi \circ s$ is a connected open path on C. Let v_{s_C} be a vector field along the path s_C. Then, the BPS condition is the differential equation

$$\operatorname{Im}\left[e^{-i\vartheta}\langle \lambda_{ij}, v_{s_C}\rangle\right] = 0 \tag{2.4}$$

for some fixed angle ϑ. For more complicated solitons that travel along more than two sheets, we can break the soliton up into a concatenation of partial solitons running along various pairs of sheets; each partial soliton involved in the concatenation must satisfy (2.4) where ij is replaced by the relevant pair of sheets, and ϑ is the same for each partial soliton. Hence, the BPS condition for solitons leads to a system of $\binom{K}{2}$ differential equations on C' (one for each disjoint pair of sheets). For such a soliton s, broken into partial solitons $\{s^r\}_{r=1}^{L}$ as decribed above, its central charge and mass are

$$\begin{aligned} Z_s &= \sum_{r=1}^{L} \frac{1}{\pi} \int_{s_C^r} \lambda_{ij} \\ M_s &= \sum_{r=1}^{L} \frac{1}{\pi} \int_{s_C^r} |\lambda_{ij}|. \end{aligned} \tag{2.5}$$

We can now identify the angle as the phase of the central charge, $\vartheta = \arg(Z_s)$, and indeed the BPS condition is equivalent to $M_s = |Z_s|$.

As with 4D BPS states, solitons also carry a charge, but now given by a relative homology class as they are open paths.

Let $z \in C'$; choose a labeling of the K points in $\pi^{-1}(z) \in C'$.

Definition

1. Let $z_l \in \pi^{-1}(z)$ denote the pre-image of $z \in C'$ on the lth sheet. Then a soliton is of type ij if it is given by a path that begins on z_i and ends on z_j. We also refer to such solitons as *ij-solitons*.

2. $\Gamma_{ij}(z,z)$ is the set of charges of ij-solitons, i.e.

$$\Gamma_{ij}(z,z) := \{[a] \in H_1(\Sigma, \{z_i\} \cup \{z_j\}; \mathbb{Z}) : a \text{ is a 1-chain with } \partial a = z_j - z_i\}.$$

3. The total set of soliton charges is

$$\Gamma(z,z) := \bigsqcup_{i,j=1}^{K} \Gamma_{ij}(z,z).$$

Remarks

1. A soliton s can be extended by "parallel transporting" its endpoints. Indeed, let s be a soliton of type ij with $s(0), s(1) \in \pi^{-1}(z)$. Now, given a path $q : [0,1] \to C'$ from z to z', let $q\{n\}$ denote the lift of q to the nth sheet of Σ defined by lifting the initial point $q(0)$ to sheet n; then one can define the transported path,

$$\mathbb{P}_q s = q\{j\} \star s \star q^{-}\{i\} \tag{2.6}$$

where $\star$ denotes concatenation of paths, and $q^{-}\{i\}$ is $q\{i\}$ with reversed orientation. The resulting path on Σ has endpoints in $\pi^{-1}(z')$. If s is an ij soliton, then the path $\mathbb{P}_q s$ is a soliton iff q satisfies (2.4) for the same pair of sheets ij.

2. $\bigcup_{z\in C'} \Gamma(z,z) \to C'$ is a local system over C': for any path $q : [0,1] \to C$ there is a parallel transport map $P_q : \Gamma(q(0), q(0)) \to \Gamma(q(1), q(1))$, induced by the map $\mathbb{P}_q$ defined above, and only depending on the homotopy class of q relative to the endpoints. (Henceforth we abbreviate this as "rel endpoints.")

3. If there is an extension of an ij-soliton through a branch cut emanating from an ij branch point, it becomes a ji-soliton. More generally, if a soliton passes through any branch cut, its type is permuted according to the permutation of sheets across the branch cut.

Just as with 4D vanilla BPS states, for each $a_z \in \Gamma(z,z)$, there is an index $\mu(a_z) \in \mathbb{Z}$ that counts BPS solitons of charge a_z. Again, this can be defined as a supertrace over an appropriate BPS subspace, however, we will compute it via geometric methods. Using the parallel transport map described in the remarks above, this BPS index is also stable along extensions of solitons at generic $z \in C'$; [5] it jumps only at points $z \in C'$ where solitons of different types exist and interact. This motivates the following (notation-simplifying) definitions.

[5] In the sense that if s is an ij soliton, and q is a sufficiently small path on Σ satisfying (2.4), then $\mu([s]) = \mu(P_q[s])$

Definitions

1. A soliton $s : [0,1] \to \Sigma$ is said to be of *phase* θ if it satisfies the BPS condition[6] (2.4) for $\vartheta = \theta$.

2. A point $z \in C$ is said to *support* a soliton of phase ϑ if there exists a soliton s with $[s] \in \Gamma(z, z)$ and $\mu([s]) \neq 0$. A path p on C supports a family of solitons of phase ϑ if each point on p supports a soliton fitting into a 1-parameter family of solitons of phase ϑ. When the phase ϑ is clear from context, occasionally we will just say that p supports a family of solitons.

3. Let $p \subset C$ be a path on C supporting a family of solitons of phase ϑ extending a soliton s_0 with charge $a_z = [s_0] \in \Gamma(z, z)$. With an abuse of notation, occasionally a will denote any one of the parallel transports of a_z along the path p.

4. Let $z \in p$ and $a_z \in \Gamma(z, z)$. If the index $\mu(a_z)$ is constant for any soliton in the family generated by parallel transports of $a_z \in \Gamma(z, z)$ along $p \subset C$, then we will denote the index by $\mu(a, p) \in \mathbb{Z}$.

Framed 2D-4D States

We consider one final BPS construction: the framed 2D-4D states. Given $\vartheta \in \mathbb{R}/(2\pi\mathbb{Z})$, z_1, $z_2 \in C$, and $\wp$ a path on C from z_1 to z_2, one can associate two surface defects $\mathbb{S}_{z_1}$ and $\mathbb{S}_{z_2}$, along with a supersymmetric interface $L_{\wp,\vartheta}$ between these two surface defects.[7] The interface is supersymmetric in the sense that it preserves two out of the four supercharges preserved by the surface defects; the parameter ϑ controls which two are preserved. Framed 2D-4D states are the vacua of the theory after insertion of this defect.

Geometrically, such a state is represented by a path $f : [0,1] \to \Sigma$ such that there exists a finite subdivision of times

$$[0,1] = [0,t_1] \cup [t_1,t_2] \cup \cdots \cup [t_{N-1},1]$$

and, with respect to this subdivision:

[6]Thought of as a system of equations on each "partial soliton" as described above.

[7]From the four-dimensional perspective, $L_{\wp,\vartheta}$ is a line defect extended along $\mathbb{R}^{0,1}$ and living on the interface between $\mathbb{S}_{z_1}$ and $\mathbb{S}_{z_2}$.

- $f|_{[0,t_1]}$ and $f|_{[t_N,1]}$ have images in $\pi^{-1}(\wp)$ (in particular, the path begins on a lift of z_1 and ends on a lift of z_2).

- If $1 < i < N-2$, then $f_{[t_i,t_{i+1}]}$ is either a soliton of phase ϑ, or has image in $\pi^{-1}(\wp)$.

When f is projected to C the resulting path looks like $\wp$ with finitely many diversions to solitons (and back) along the way. In [26], such a path f is referred to as a *millipede with body* $\wp$ *and phase* ϑ.

Similar to solitons, we can classify framed 2D-4D states by their values in a set of charges given by relative homology classes $[f]$, for f a millipede; as the geometric description above suggests, now the relative cycles can have boundaries on pre-images of two different points on C.

Definition Let $\wp : [0,1] \to C$ with $\wp(0) = z$ and $\wp(1) = w$; with a choice of labeling of sheets above $\pi^{-1}(z)$ and $\pi^{-1}(w)$, let z_i (resp. w_i) be a point on the ith sheet in $\pi^{-1}(z)$ (resp. $\pi^{-1}(w)$). Then, the set of charges of framed 2D-4D states corresponding to $\wp$ is

$$\Gamma(z,w) := \bigsqcup_{i,j=1}^{K} \{[a] \in H_1(\Sigma, \{z_i\} \cup \{z_j\}; \mathbb{Z}) : a \text{ is a 1-chain with } \partial a = w_j - z_i \} .$$

Furthermore, for each $a \in \Gamma(z,w)$ we define the counting index $\overline{\underline{\Omega}}(L_{\wp,\vartheta}, a)$ that, once again, can be defined via a supertrace over an appropriate Hilbert space, but we will only utilize its interpretation from a geometric perspective.

Remark It is believed that the theory obtained after insertion of the defect $L_{\wp,\vartheta}$ only depends on the homotopy class (rel boundary) of $\wp$. This homotopy invariance is the key ingredient that ties the story of spectral networks together.

Adding a Little Twist

Before proceeding to the definition of the $\mathcal{W}_\vartheta$ networks, we make an important technical detour. As discussed in [27], the indices $\mu(a)$ and $\overline{\underline{\Omega}}(L_{\wp,\vartheta}, a)$ are only well-defined up to a sign, due to potential integer shift ambiguities in the fermion number operators that enter their definitions. To correct these ambiguities globally over all regions of parameter space, it suffices to construct (geometrically motivated) $\mathbb{Z}/2\mathbb{Z}$

extensions of Γ and $\Gamma(z,w)$. First, a bit of notation that will be used throughout this section and part of Appendix B.

Definition Let S be a real surface, then $\xi^S : \widetilde{S} \to S$ is the unit tangent bundle projection to S.

The map $\xi^S_* : H_1(\widetilde{S};\mathbb{Z}) \to H_1(S;\mathbb{Z})$ has a kernel generated by the homology class that has a representative winding once around some fiber.

Definition Let $H \in H_1(\widetilde{\Sigma};\mathbb{Z})$ denote the homology class represented by a 1-chain that winds once around a fiber of $\widetilde{\Sigma} \to \Sigma$, then

$$\widetilde{\Gamma} := H_1(\widetilde{\Sigma};\mathbb{Z})/\left(2H\right).$$

We abuse notation and denote the image of H in $\widetilde{\Gamma}$ by H again.

It follows that $\widetilde{\Gamma}$ is a $\mathbb{Z}/2\mathbb{Z}$ extension of Γ, i.e there is an exact sequence of abelian groups,

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\Gamma} \to \Gamma \to 0.$$

Similarly, for framed states and solitons we define extended charge sets. First we pass through an intermediate construction.

Definition Let $\widetilde{\pi} : \widetilde{\Sigma} \to \widetilde{C}$ be the restriction of $d\pi : T\Sigma \to TC$ to the unit tangent bundle. For fixed $\widetilde{z}$, $\widetilde{w} \in \widetilde{C}$, choose a labeling of sheets above $\pi^{-1}(z)$ and $\pi^{-1}(w)$; let z_i (resp. w_i) be a point on the ith sheet in $\pi^{-1}(z)$ (resp. $\pi^{-1}(w)$), then define

$$\begin{aligned} G_{ij}(\widetilde{z},\widetilde{w}) &:= \left\{[a] \in H_1(\widetilde{\Sigma},\{\widetilde{z}_i\}\cup\{\widetilde{w}_j\};\mathbb{Z}) : a \text{ is a 1-chain with } \partial a = \widetilde{w}_j - \widetilde{z}_i\right\}, \\ G(\widetilde{z},\widetilde{w}) &:= \bigsqcup_{i,j=1}^{K} G_{ij}(\widetilde{z},\widetilde{w}). \end{aligned} \tag{2.7}$$

Remark $G(\widetilde{z},\widetilde{w})$ is equipped with an $H_1(\widetilde{\Sigma};\mathbb{Z})$ action given by the addition of a closed cycle (at the level of chains).

This allows us to make the following definition,

Definition

$$\widetilde{\Gamma}(\widetilde{z},\widetilde{w}) := G(\widetilde{z},\widetilde{w})/\langle 2H\rangle. \tag{2.8}$$

Sometimes it is useful to view $\widetilde{\Gamma}(\widetilde{z},\widetilde{w})$ as a disjoint union of quotients of G_{ij}:

Definition

$$\widetilde{\Gamma}_{ij}(\widetilde{z},\widetilde{w}) := G_{ij}(\widetilde{z},\widetilde{w})/\langle 2H\rangle.$$

So we may write,

$$\widetilde{\Gamma}(\widetilde{z},\widetilde{w}) := \bigsqcup_{i,j=1}^{K} \widetilde{\Gamma}_{ij}(\widetilde{z},\widetilde{w}).$$

Remark

$\widetilde{\Gamma}(\widetilde{z},\widetilde{w})$ is equipped with a $\widetilde{\Gamma}$ action, descending from addition of a closed cycle with a relative cycle. For $\gamma \in \widetilde{\Gamma}$ and $a \in \widetilde{\Gamma}(\widetilde{z},\widetilde{w})$ we will denote this action by $\gamma : a \mapsto a+\gamma = \gamma + a$. In fact, for any ordered pair ij, $\widetilde{\Gamma}_{ij}(\widetilde{z},\widetilde{w})$ is a torsor for $\widetilde{\Gamma}$.

$\widetilde{\Gamma}(\widetilde{z},\widetilde{w})$ carries an extra $\mathbb{Z}/2\mathbb{Z}$'s worth of "winding" information in the sense that $\widetilde{\Gamma}(\widetilde{z},\widetilde{w}) \overset{\text{proj}}{\to} \Gamma(z,w)$ is a principal $\mathbb{Z}/2\mathbb{Z}$ bundle, with proj given by forgetting lifts[8], and the $\mathbb{Z}/2\mathbb{Z}$ action given by adding H.

Now, a soliton is a smooth curve on Σ; furthermore, the tangent vectors at the endpoints (which lie on disjoint sheets) of a soliton are oppositely oriented in the sense that their pushforwards to C are oppositely oriented.

Definition Let $\widetilde{z} \in \widetilde{C}$, then $-\widetilde{z} \in \widetilde{C}$ is the unit tangent vector pointing in the opposite direction to $\widetilde{z}$.

Remark To every soliton (represented by a smooth path) there is a natural lifted charge in $\widetilde{\Gamma}(\widetilde{z},-\widetilde{z})$ that descends from the relative homology class of the soliton's tangent framing lift.

[8]More precisely, proj is the map descending from the induced map on relative homology $(\xi^{\Sigma})_* : H_1\left(\widetilde{\Sigma}, \pi^{-1}(\widetilde{z}) \cup \pi^{-1}(\widetilde{w}); \mathbb{Z}\right) \to H_1\left(\Sigma, \pi^{-1}(z) \cup \pi^{-1}(w); \mathbb{Z}\right)$ where $z = \left(\pi \circ \xi^{\Sigma}\right)(\widetilde{z})$ and $w = \left(\pi \circ \xi^{\Sigma}\right)(\widetilde{w})$.

We introduce one final piece of technology. First, note that for each $\widetilde{z} \in \widetilde{C'}$ there is a disjoint union of K lattices inside of the set $\widetilde{\Gamma}(\widetilde{z},\widetilde{z})$:

$$\bigsqcup_{i=1}^{K} \widetilde{\Gamma}_{ii}(\widetilde{z},\widetilde{z}) \subset \widetilde{\Gamma}(\widetilde{z},\widetilde{z}).$$

Any representative of an element in $\widetilde{\Gamma}_{ii}(\widetilde{z},\widetilde{z})$ has zero boundary, hence, is actually a cycle. Indeed,there is a canonical "basepoint forgetting" isomorphism of lattices $\widetilde{\Gamma}_{ii}(\widetilde{z},\widetilde{z}) \cong \widetilde{\Gamma}$ for each $i = 1, \cdots, K$, descending from the identity map at the level of chain representatives. This allows us to define the *closure* map.

Definition

$$\mathrm{cl} : \bigcup_{i=1}^{K} \widetilde{\Gamma}_{ii}(\widetilde{z},\widetilde{z}) \to \widetilde{\Gamma}$$

is the map which acts on each component by the "basepoint-forgetting" map described above.

Now, due to the sign ambiguity in μ and $\underline{\overline{\Omega}}$ then, naïvely, only their absolute values are well-defined: i.e. we have functions,

$$\begin{aligned} \mu_{\geq 0} &: \bigcup_{z \in C'} \Gamma(z,z) \to \mathbb{Z}_{\geq 0} \\ \underline{\overline{\Omega}}_{\geq 0}(\wp, \cdot) &: \bigcup_{(z,w) \in C' \times C'} \Gamma(z,w) \to \mathbb{Z}_{\geq 0}. \end{aligned}$$

However, with our "lifted charge" definitions, we can lift $\mu_{\geq 0}$ to a function $\mu : \bigcup_{\widetilde{z} \in \widetilde{C'}} \widetilde{\Gamma}(\widetilde{z}, -\widetilde{z}) \to \mathbb{Z}$ such that $\forall a \in \bigcup_{\widetilde{z} \in \widetilde{C'}} \widetilde{\Gamma}(\widetilde{z}, -\widetilde{z})$,

$$\begin{aligned} |\mu(a)| &= \mu_{\geq 0}(\xi_*^{\Sigma} a) \\ \mu(a+H) &= -\mu(a). \end{aligned} \tag{2.9}$$

Similarly, fixing a path $\wp$ on C, the framed BPS degeneracies lift to well-defined functions $\underline{\overline{\Omega}}(L_{\wp,\vartheta}, \cdot) : \bigcup_{(\widetilde{z},\widetilde{w}) \in \widetilde{C'} \times \widetilde{C'}} \widetilde{\Gamma}(\widetilde{z},\widetilde{w}) \to \mathbb{Z}$ such that $\forall a \in \bigcup_{(\widetilde{z},\widetilde{w}) \in \widetilde{C'} \times \widetilde{C'}} \widetilde{\Gamma}(\widetilde{z},\widetilde{w})$,

$$\begin{aligned} \left| \underline{\overline{\Omega}}\left(L_{\wp,\vartheta}, a\right) \right| &= \underline{\overline{\Omega}}_{\geq 0}\left(L_{\wp,\vartheta}, \xi_*^{\Sigma} a\right) \\ \underline{\overline{\Omega}}\left(L_{\wp,\vartheta}, a+H\right) &= -\underline{\overline{\Omega}}\left(L_{\wp,\vartheta}, a\right). \end{aligned} \tag{2.10}$$

2.2 The $\mathcal{W}_\vartheta$ Networks

Using (2.4), we can produce a concrete picture of (the projections to C of) ij-solitons on the curve C. This motivates the following definitions.

Definition Fix $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, for each (ordered) pair of sheets ij we define a (real) oriented line field $l_{ij}(\vartheta)$ on C^c given at every $z \in C^c$ by

$$l_{ij,z}(\vartheta) := \left\{ v \in T_z C : \mathrm{Im}\left[e^{-i\vartheta} \langle \lambda_{ij}, v \rangle \right] = 0 \right\},$$

with v positively oriented if $\mathrm{Re}\left[e^{-i\vartheta} \langle \lambda_{ij}, v \rangle \right] > 0$.

Given an integral curve p of $l_{ij}(\vartheta)$, the orientation of $l_{ij}(\vartheta)$ tells us how to lift the curve back to a curve p_Σ on Σ.

Definition Any integral curve p (on C') of $l_{ij}(\vartheta)$ has a lift to a curve p_Σ on Σ defined as the union of $p\{i\}$ (the lift of p to the ith sheet), and $p^-\{j\}$ (the lift of p to the jth sheet, reversing orientation).

Remarks

- Fix $z_* \in C^c$ and take a neighborhood U of z_* that does not contain any branch cuts of type ij. Then for each ordered pair ij we can define local coordinates $w_{ij} : U \to \mathbb{C}$ by
$$w_{ij}(z) = \int_{z_*}^{z} (\lambda_i - \lambda_j) \,. \tag{2.11}$$
In these coordinates, the integral curves of $l_{ij}(\vartheta)$ are precisely the straight lines of inclination ϑ.

- Note that the line field $l_{ji}(\vartheta)$ is just $l_{ij}(\vartheta)$ with reversed orientation.

- On a cycle surrounding a branch point of type ij, the monodromy action induces $\lambda_{ij} \mapsto \lambda_{ji} = -\lambda_{ij}$; hence, $l_{ij}(\vartheta) \mapsto l_{ji}(\vartheta)$ (i.e., the line field orientation reverses when passing through a branch cut extending from a branch point.)

We can finally define the (real) codimension-1 networks of interest.

Definition

$$\mathcal{W}_\vartheta = \bigcup_{\text{ordered pairs } ij} \left\{ p : \begin{array}{c} p \text{ is an integral curve of } l_{ij}(\vartheta) \\ \text{and } p \text{ supports a soliton of phase } \vartheta \end{array} \right\} \subset C'.$$

The network $\mathcal{W}_\vartheta$ is composed of individual integral curve segments, which may interact and join each other at vertices on C'.

Definitions

1. An individual integral curve segment on $\mathcal{W}_\vartheta$ is called a *street*.[9] A *street of type* ij is a street that is an integral curve of $l_{ij}(\vartheta)$.
2. A joint is a point on C' where two or more streets of different types meet.

The upshot of all these constructions is that now we have a solidified picture of solitons via a network on C'. Indeed, we can lift $\mathcal{W}_\vartheta$ to a graph on $\text{Lift}(\mathcal{W}_\vartheta) \subset \Sigma$ by taking the union of the lifts (as defined above) p_Σ of each street p. Then an ij soliton of phase ϑ traces out a path supported on $\text{Lift}(\mathcal{W}_\vartheta)$, and begins and ends on points z_i, z_j that are lifts to the ith and jth streets (respectively) of a point z on a street of type ij. In particular, an ij street of $\mathcal{W}_\vartheta$ represents the endpoints of a set of solitons of type ij.

From a constructive viewpoint, however, the reader may feel unsatisfied as we have not yet defined how to determine the condition that $p \subset \mathcal{W}_\vartheta$ actually *supports* a soliton of phase ϑ, i.e. $\mu(a,p) \neq 0$, for some a the charge of a soliton of phase ϑ. To fill this void we remark that there are exactly three integral curves of $l_{ij}(\vartheta) \cup l_{ji}(\vartheta)$ emerging from each ij-branch point. On each such integral curve p there is a family of solitons represented by "small" paths: for $z \subset p$ and $z_i, z_j \in \pi^{-1}(z)$ lifts of z to sheet i and sheet j (respectively), there is a soliton supported on p_Σ traveling from $z_i \in \Sigma$, through the ramification point on Σ, to $z_j \in \Sigma$. Such solitons become arbitrarily light as z approaches the branch point. Furthermore, as argued in [27], letting a be the (lifted) charge of any soliton in this family, we assign

$$\mu(a,p) = +1. \tag{2.12}$$

Terminology The "light" solitons described in the previous paragraph will be called *simpletons*.

[9] In [27] these were also referred to as S-walls.

We defer the problem of determining the soliton indices μ on all other streets until the appropriate definitions are developed in the next section; for now it will suffice to say that, with this condition, the soliton indices on all other streets can be determined via a set of algebraic equations.

$\mathcal{K}$-walls and Degenerate Networks

Of particular interest in this thesis will be $\mathcal{W}_\vartheta$ networks of a very special type.

Definition A street $p \subset \mathcal{W}_\vartheta$ is *two-way* if it consists of a coincident ij-street and a ji-street. Equivalently, p supports ij-solitons and ji-solitons. A street that is not two-way is called *one-way*. A network that contains a two-way street is said to be *degenerate*.

We adopt the following convention in order to keep track of the individual directions of the constituent one-way streets of a two-way street.

Convention Let p be a two-way street consisting of coincident ij and ji-streets, then we will say p is of type ij and assign it the orientation of its constituent ij-street. (Or, equivalently, we will say p is of type ji and assign it the orientation of its constituent ji-street.)

As described in [27], sec. 6.2, for generic values of ϑ, the network $\mathcal{W}_\vartheta$ will only contain one-way streets due to a bifurcation behavior of integral curves near branch points. However, at critical values $\vartheta_c \in \mathbb{R}/\mathbb{Z}$, an ij street will collide with a ji street and the network $\mathcal{W}_{\vartheta_c}$ will contain two-way streets. Now we make an important claim:

$$\mathcal{W}_\vartheta \text{ contains a two-way street} \Rightarrow \exists \text{ a homologically non-trivial closed loop on } \Sigma \\ \text{satisfying (2.3) for some phase } \vartheta \in \mathbb{R}/2\mathbb{Z}.$$

To see this, fix a point $z \in p \subset C$ on any two-way street p; without loss of generality we will say p is of type ij. Then z supports a soliton of type ij and a soliton of type ji, both of the same phase ϑ; the concatenation of these two paths yields a closed loop l satisfying (2.3) for the phase ϑ. Moreover, this loop is homologically non-trivial. Indeed, the period of l is just the sum of the periods of the two solitons forming it. However, both have periods (central charges) of the same phase; so the sum must be nonzero.

Thus, via the claim, a degenerate network automatically leads to a possible 4D BPS state of charge $[l] \in \Gamma$; in fact, there are possible BPS states of charges $n[l]$, $n \in \mathbb{Z}_{>0}$. All that remains is to determine the BPS indices $\Omega(n[l])$ which, as expressed more explicitly below, are computable from the soliton data supported on $\mathcal{W}_\vartheta$.

In practice, degenerate networks can be found by looking for discontinuous changes in the topology of $\mathcal{W}_\vartheta$ as ϑ is varied. Indeed, if a region $R \subset \mathbb{R}/\mathbb{Z}$ does not contain any degenerate networks then, as ϑ is varied continuously in R, the network $\mathcal{W}_\vartheta$ also varies continuously (in the sense described in [27]). However, if the region R contains a single critical angle ϑ_c, the bifurcation of integral curves near a branch point induces a discontinuous change in the topology of $\mathcal{W}_\vartheta$ as ϑ is varied[10] past ϑ_c. (If we consider the parameter space of ϑ and the Coulomb branch then the locus where degenerate networks appear defines $\mathcal{K}$-*walls*.)

Formal Variables

In order to construct the generating functions that keep track of various BPS degeneracy indices, it is helpful to construct spaces of formal variables with some algebraic structure.

First suppose that for fixed $\vartheta \in S^1$, the set of homology classes of any non-trivial closed loop on Σ satisfying (2.3) are contained in a finitely generated, free submonoid $M_\vartheta \subset \Gamma$. Note that, by the discussion in the previous section, $M_\vartheta \neq \{0\}$ if and only if $\mathcal{W}_\vartheta$ is degenerate. For the degenerate networks of interest in this thesis, we will always have $M_\vartheta = \mathbb{Z}_{\geq 0}\gamma$ for a single $\gamma \in \Gamma$. It is not hard to show that the relative homology classes of any two solitons of type ij supported at a point $z \in C$ differ by an element of M_ϑ or $-M_\vartheta$. For the rest of this section we will drop occurences of ϑ from our notation and write M instead of M_ϑ.

Definition

1. $\overline{M} \leq \widetilde{\Gamma}$ is the preimage of M under the quotient map $\widetilde{\Gamma} \to \Gamma$; it is also a monoid.

[10] However, there may be an accumulation point of critical angles as in the picture of the vector multiplet when $K = 2$ (see [27]). Around such an accumulation point the topology of $\mathcal{W}_\vartheta$ rapidly changes, and there is no open region containing the accumulation point where the topology smoothly varies. Even "worse," as we will see, the critical angles can densely fill an open interval.

2. $\mathbb{Z}[\overline{M}]$ is the monoid algebra of $\overline{M}$: it is the commutative ring of $\mathbb{Z}$-coefficient polynomials in the formal variables X_γ, $\gamma \in \overline{L}$, subject to the relations

$$X_0 = 1,$$
$$X_H = -1,$$
$$X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}.$$

Now, for any finitely generated, free monoid, there is a natural descending filtration on its monoid algebra given a set of generators; hence, one can define a corresponding ring of formal series by taking a limit of the inverse system formed by quotienting with respect to such a filtration. Although $\overline{M}$ is not free, because it is a relatively mild extension of the free and finitely generated M, we can easily find a descending filtration on $\overline{M}$ and, hence, make sense of the ring of formal series.

Definition $\mathbb{Z}[\![\overline{M}]\!]$ is the commutative ring of formal series as described above.

We can make $\mathbb{Z}[\![\overline{M}]\!]$ explicit in the following manner: let $(\alpha_i)_{i=1}^r \subset \Gamma$ is a basis for M so that $(\widetilde{\alpha}_i)_{i=1}^r \cup \{H\}$ forms a basis for $\overline{M}$ (where $\widetilde{\alpha}_i \in \widetilde{\Gamma}$ is the standard lift of $\alpha_i \in \Gamma$ as defined in App. D); then as an *abelian group* we can identify $\mathbb{Z}[\![\overline{M}]\!]$ with $\mathbb{Z}[\![x_1, \cdots, x_r, x_H]\!]/(x_H + 1) \cong \mathbb{Z}[\![x_1, \cdots, x_r, x_H]\!]$ by the map $X_{\widetilde{\alpha}_i} \mapsto x_i$ and $X_H \mapsto x_H = -1$. The product structure on $\mathbb{Z}[\![\overline{M}]\!]$ is not the product structure on $\mathbb{Z}[\![x_1, \cdots, x_r, x_H]\!]$, but rather a $\mathbb{Z}/2\mathbb{Z}$-twisted version of it; this is due to the fact that the standard lift does not define a homomorphism $\Gamma \to \widetilde{\Gamma}$.

To define a product structure on formal variables in 2D-4D/soliton charges, we note that there is a partially defined "addition" operation.

Definitions

1. Let $a \in \widetilde{\Gamma}(\widetilde{z}_1, \widetilde{z}_2)$, then

$$\text{end}(a) = \widetilde{z}_2$$
$$\text{start}(a) = \widetilde{z}_1.$$

2. Let $a, b \in \bigcup_{\widetilde{z},\widetilde{w}\in\widetilde{C}} \widetilde{\Gamma}(\widetilde{z}, \widetilde{w})$, then if $\text{end}(a) = \text{start}(b)$ there is a well-defined operation (concatenation of paths) $a + b \in \bigcup_{\widetilde{z},\widetilde{w}\in\widetilde{C}} \widetilde{\Gamma}(\widetilde{z}, \widetilde{w})$ descending from the usual addition of relative homology cycles.

With this we can define the space of interest.

Definition The *polynomial homology path algebra* $\mathcal{A}^{\text{poly}}$ is the non-commutative $\mathbb{Z}[\![\overline{M}]\!]$-algebra whose

1. underlying module is given by the abelian group of $\mathbb{Z}$-coefficient polynomials in the formal variables X_a, $a \in \bigcup_{\widetilde{z},\widetilde{w}\in\widetilde{C}} \widetilde{\Gamma}(\widetilde{z},\widetilde{w})$ such that for $\gamma \in \overline{L}$,
$$X_\gamma X_a = X_{a+\gamma} = X_{\gamma+a};$$
2. product structure is defined by the following: for any a, $b \in \bigcup_{\widetilde{z},\widetilde{w}\in\widetilde{\Sigma}} \widetilde{\Gamma}(\widetilde{z},\widetilde{w})$
$$X_a X_b = \begin{cases} X_{a+b}, & \text{if } \text{end}(a) = \text{start}(b) \\ 0, & \text{otherwise} \end{cases}.$$

With some care, one can define an enlargement of the polynomial homology path algebra that forms a $\mathbb{Z}[\![\overline{M}]\!]$-algebra. Its elements are $\mathbb{Z}$-coefficient formal series (as opposed to polynomials) in the variables X_a, $a \in \bigcup_{\widetilde{z},\widetilde{w}\in\widetilde{C}} \widetilde{\Gamma}(\widetilde{z},\widetilde{w})$.

Terminology The *homology path algebra* is the $\mathbb{Z}[\![\overline{M}]\!]$-algebra described above.

There are two important subalgebras.

Definition

1. $\mathcal{A}_S$ is the $\mathbb{Z}[\![\overline{L}]\!]$-subalgebra generated by formal variables in soliton charges $a \in \bigcup_{\widetilde{z}\in\widetilde{C}} \widetilde{\Gamma}(\widetilde{z},-\widetilde{z})$.
2. $\mathcal{A}_C$ is the (commutative) $\mathbb{Z}[\![\overline{L}]\!]$-subalgebra generated by formal variables in $a \in \bigcup_{\widetilde{z}\in\widetilde{C}} \bigsqcup_{i=1}^{K} \widetilde{\Gamma_{ii}}(\widetilde{z},\widetilde{z})$.

The closure map can be easily extended to $\mathcal{A}_C$.

Definition
$$\text{cl} : \mathcal{A}_C \to \mathbb{Z}[\![\overline{M}]\!]$$
is the linear extension of the map
$$\text{cl}(X_a) = X_{\text{cl}(a)}.$$

We now define generating functions for BPS indices.

Definition For each path $\wp$ from $z \in C$ to $w \in C$ that represents an interface $L_{\wp,\vartheta}$, we associate the framed generating function

$$F(\wp, \vartheta) := \sum_{a_* \in \Gamma(z,w)} \underline{\overline{\Omega}}(L_{\wp,\vartheta}, a) X_a \in \mathcal{A},$$

where $a \in \widetilde{\Gamma}(\widetilde{z}, \widetilde{w})$ is a lift of the charge $a_* \in \Gamma(z, w)$ such that $\widetilde{z}$ and $\widetilde{w}$ are the unit tangent vectors at the ends of $\wp$.

For each street p of type ij, we associate two *soliton generating functions*: $\Upsilon(p)$, that encodes the indices of solitons of type ij, and $\Delta(p)$, that encodes the indices of solitons of type ji.

Definition Let $z \in p \subset C$, then we define

$$\Upsilon_z(p) := \sum_{a_* \in \Gamma_{ij}(z,z)} \mu(a) X_a \in \mathcal{A}_S \tag{2.13}$$

$$\Delta_z(p) := \sum_{b_* \in \Gamma_{ji}(z,z)} \mu(b) X_b \in \mathcal{A}_S, \tag{2.14}$$

where $a \in \widetilde{\Gamma}_{ij}(\widetilde{z}, -\widetilde{z})$, $b \in \widetilde{\Gamma}_{ji}(-\widetilde{z}, \widetilde{z})$ denote respective lifts of $a_* \in \Gamma_{ij}(z, z)$ and $b_* \in \Gamma_{ji}(z, z)$, for $\widetilde{z} \in \widetilde{C}$ the unit tangent vector agreeing with the orientation of p at the point $z \in C$.[11]

Definition From the soliton generating functions on a street p, we can define the *street factor*,

$$\begin{aligned} Q(p) &:= \mathrm{cl}\left[1 + \Upsilon_z(p)\Delta_z(p)\right] \\ &= 1 + \sum_{a_* \in \Gamma_{ij}(z,z), b_* \in \Gamma_{ji}(z,z)} \mu(a)\mu(b) X_{\mathrm{cl}(a+b)} \in \mathbb{Z}[\![\overline{M}]\!]. \end{aligned}$$

where $z \in C$ is any point on p.

[11] As $\widetilde{\Gamma}(\widetilde{z}, -\widetilde{z})$ is a principal $\mathbb{Z}/2\mathbb{Z}$ bundle over $\Gamma(z, z)$, there are two possible lifts of a_* related by addition of H. Via $X_H = -1$ along with (2.9) and (2.10), the definition of $\Upsilon_z(p)$ is independent of the choice of lift. This argument also applies to $\Delta_z(p)$.

Remark As the notation suggests, $Q(p)$ is independent of the choice of point z. This follows as the index $\mu(a)$ is constant as any charge a is parallel transported along any path supported on $p \subset C$. By the same reasoning, for any $z, z' \in p$, $\Upsilon_z(p)$ and $\Upsilon_{z'}(p)$ are related by applying an appropriate parallel transport map [12] (similarly for $\Delta_z(p)$ and $\Delta_{z'}(p)$).

And now for the punchline.

Computing $\Omega(n\gamma_c)$

The power of the spectral network machine can be summarized with the following squiggly arrows:

Jumping of Framed 2D-4D Spectrum + Homotopy Invariance of $L_{\wp,\vartheta}$	$\overset{(A)}{\rightsquigarrow}$	Soliton Spectrum	$\overset{(B)}{\rightsquigarrow}$	(Vanilla) 4D spectrum.

To understand (A): the framed generating function $F(\wp, \vartheta)$ is piecewise constant in the sense that as the endpoints of $\wp$ are varied on $C - \mathcal{W}_\vartheta$, then $F(\wp, \vartheta)$ does not vary in $\mathcal{A}$; however, if an endpoint of $\wp$ is varied across a street of $\mathcal{W}_\vartheta$, then $F(\wp, \vartheta)$ will jump in a manner depending on the spectrum of solitons located on that street. Indeed, $F(\wp, \vartheta)$ is the sum of the charges of "millipedes," and as the "body" $\wp$ of each such millipede crosses the street p, then the millipede can gain an extra leg by detouring along a soliton supported along p; hence, the spectrum of 2D-4D states (represented by millipedes) will jump. To reproduce the soliton spectrum we utilize the homotopy invariance of the operator $L_{\wp,\vartheta}$ to equate the different jumps of $F(\wp, \vartheta)$ across different, but homotopic (rel endpoints), paths $\wp$. The resulting equations are equivalent to conditions on the soliton generating functions. These conditions, combined with the simpleton input data (2.12), allow us to completely determine the soliton generating functions, which encapsulate the soliton spectrum.

To describe (B), let $\Gamma_c \subset \Gamma$ be the lattice of charges γ with $e^{-i\vartheta_c} Z_\gamma \in \mathbb{R}_-$; then the degenerate network $\mathcal{W}_{\vartheta_c}$ captures all of the 4D BPS states carrying charges $\gamma \in \Gamma_c$. Their spectrum can be extracted from the generating functions $Q(p)$. But, first we have to deal with a technical point.

[12] For this reason, the point z in soliton generating functions is often dropped as in the calculations of Appendix B.

Definitions

1. For every curve q on a surface S, there is a canonical "lift" $\widehat{q}$ to a curve on $\widetilde{S}$, given by the tangent framing.

2. For each $\gamma \in \Gamma$, we define another lift $\widetilde{\gamma} \in \tilde{\Gamma}$ by the following rule. First, represent γ as a union of smooth closed curves β_m on Σ. Then $\widetilde{\gamma}$ is the sum of $\widehat{\beta}_m$, shifted by $\left(\sum_{m\leq n} \delta_{mn} + \#(\beta_m \cap \beta_n)\right) H$ (of course, because we work modulo $2H$, all that matters here is whether this sum is odd or even.)

One can check directly (see Appendix D) that $\tilde{\gamma}$ so defined is independent of the choice of how we represent γ as a union of β_m; this requirement is what forced us to add the tricky-looking shift.

Then, for each street p, we factorize $Q(p)$ as a product:

Definition

$$Q(p) = \prod_{\gamma\in\Gamma_c} (1 - X_{\widetilde{\gamma}})^{\alpha_\gamma(p)}. \tag{2.15}$$

This representation determines the coefficients $\alpha_\gamma(p)$.

Definition Let $\boldsymbol{p}_\Sigma \in C_1(\Sigma;\mathbb{Z})$ be the one-chain corresponding to the lift p_Σ, then we define[13]

$$L(\gamma) := \sum_{\text{streets p}} \alpha_\gamma(p)\boldsymbol{p}_\Sigma \in C_1(\Sigma;\mathbb{Z}). \tag{2.16}$$

Now, as shown in [27], the magic of this definition is that $L(\gamma)$ is actually a 1-cycle satisfying the BPS condition (2.2) for $\vartheta = \vartheta_c$. [14] Let us make the further assumption that Γ_c is a rank-1 lattice, which holds automatically off of the walls of marginal stability on $\mathcal{B}$, then it follows that both γ and $[L(\gamma)]$ are multiples of a choice of generator $\gamma_c \in \Gamma_c$. With this in mind, the journey to the end of the squiggly arrow (B) follows by analyzing the jumping of $F(\wp,\vartheta)$, but now as ϑ is varied across the

[13] Note that the sum over streets in (2.16) reduces to a sum over two-way streets; indeed, $Q(p) \neq 1$ iff p is two-way.

[14] This last comment follows from the fact that $\int_{p_\Sigma} \lambda = \int_p \lambda_{ij} \in e^{i\vartheta_c}\mathbb{R}_{<0}$ for any street p of type ij.

critical angle ϑ_c (fixing $\wp$). The resulting analysis (see [27], sec. 6) leads us to the desired result:

$$[L(\gamma)] = \Omega(\gamma)\gamma, \qquad \gamma \in \Gamma_c, \tag{2.17}$$

from which all BPS indices of 4D BPS states with central charge phase ϑ_c can be computed.

Abstract Spectral Networks

It is possible to abstract the properties of the $\mathcal{W}_\vartheta$ networks in order to draw networks on C that do not necessarily come from integral curves of (2.4). It is not necessary to give a precise list of the properties here, and we instead refer the interested reader to Section 9 of [27]. There, the abstracted networks are particularly useful for defining the "non-abelianization map" between moduli spaces of flat $GL(1)$-bundles on Σ, and flat $GL(K)$-bundles on C. In this thesis, however, our interest in abstract spectral networks will be in constructions of *potential* $\mathcal{W}_\vartheta$ networks. Indeed, the m-herds mentioned in the introduction, and introduced in Section 3.1, are examples of abstract networks on an arbitrary curve C. By searching the parameter space of the pure $SU(3)$ theory, where $C = S^1 \times \mathbb{R}$ and $K = 3$, it turns out that a large subset of m-herds actually arise as $\mathcal{W}_\vartheta$ networks at various points on the Coulomb branch.

Chapter Three: Spectral network analysis of a wild point on the Coulomb branch

3.1 Horses and Herds

We begin by describing a sequence of spectral networks that may arise in the hypothetical wall-crossing between two BPS hypermultiplets of charges γ, $\gamma' \in \Gamma$ such that $\langle \gamma, \gamma' \rangle = m$. Indeed, assume at some point on the Coulomb branch there are two BPS states (occurring at different phases) such that the degenerate network associated to each state has a single two-way street given by a simple curve passing through two branch points of the same type (frame (A) of Figs. 3.1-3.2); such spectral networks are associated to BPS hypermultiplets. Now, assume that there exists a marginal stability wall on the Coulomb branch associated to the (central charge phase) crossing of these two hypermultiplets (and no other BPS states). On the other side of the wall, a possible bound state of charge $\gamma + \gamma'$ may be formed (where γ, γ' are the charges of the original hypermultiplets). Figs. 3.1-3.2 depict three hypothetical snapshots along a path passing through the wall of marginal stability for the cases $m = 1, 3$; frame (C) depicts a guess at the appearance of the degenerate network associated to the bound state of charge $\gamma + \gamma'$. After drawing such pictures for progressively higher m, and given a sufficient dose of mildly-confused staring, one will begin to notice that the (two-way streets of) networks associated to the bound state of charge $\gamma + \gamma'$ can be decomposed into m-components that look like "extended" saddles; as they are the generalization of saddles we have no choice but to call each such component a "horse."

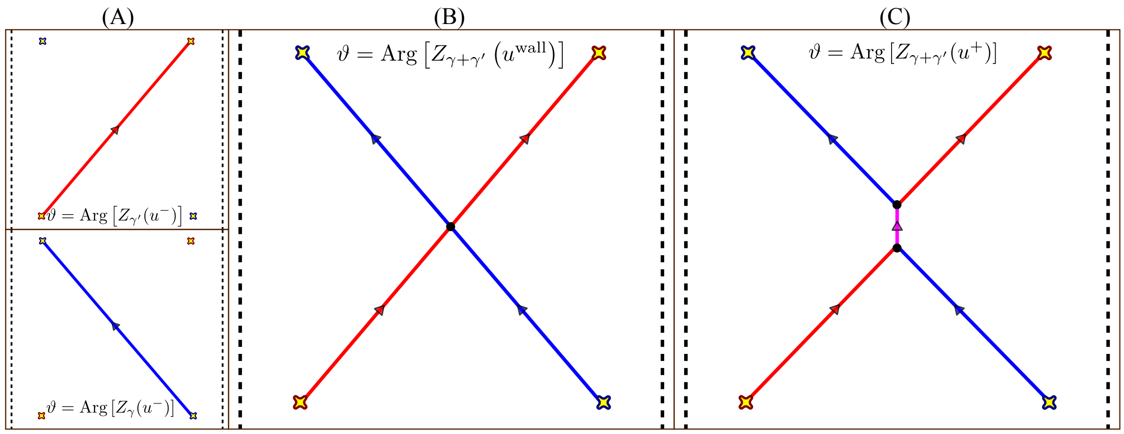


Figure 3.1: A hypothetical wall-crossing of two hypermultiplets with charges γ, γ' such that $\langle\gamma,\gamma'\rangle = 1$. Streets of type 12 are shown in red, 23 in blue, and 13 in fuchsia; only two-way streets are depicted. Arrows denote street orientations according to the convention described in Section 2.2. Yellow crosses denote branch points. Arrows denote the direction of solitons of type 12, 23, or 13 (according to the street). The black dotted lines are identified to form the cylinder. (A): The two hypermultiplet networks at a point u^- just "before" the wall of marginal stability. (B): The hypermultiplet networks at a point u^{wall} on the wall of marginal stability and at phase $\vartheta = \arg\left[Z_\gamma(u^{\mathrm{wall}})\right] = \arg\left[Z_{\gamma'}(u^{\mathrm{wall}})\right] = \arg\left[Z_{\gamma+\gamma'}(u^{\mathrm{wall}})\right]$. (C): Slightly "after" the wall at a point u^+, a BPS bound state of charge $\gamma+\gamma'$ is born and a two-way street of type 13 "grows" as one proceeds away from the wall.

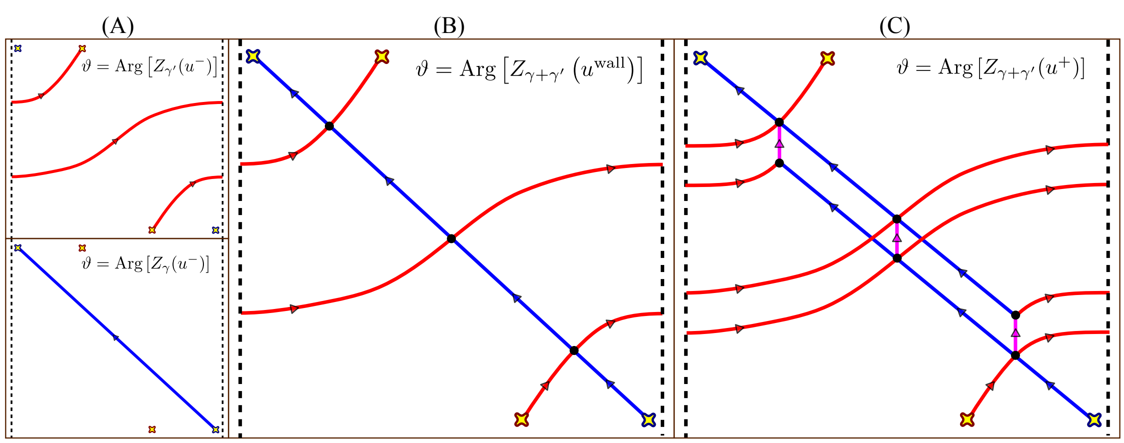


Figure 3.2: A hypothetical wall-crossing of two hypermultiplets with charges γ, γ' such that $\langle\gamma,\gamma'\rangle = 3$. The story is similar to that described in the caption of Fig. 3.1.

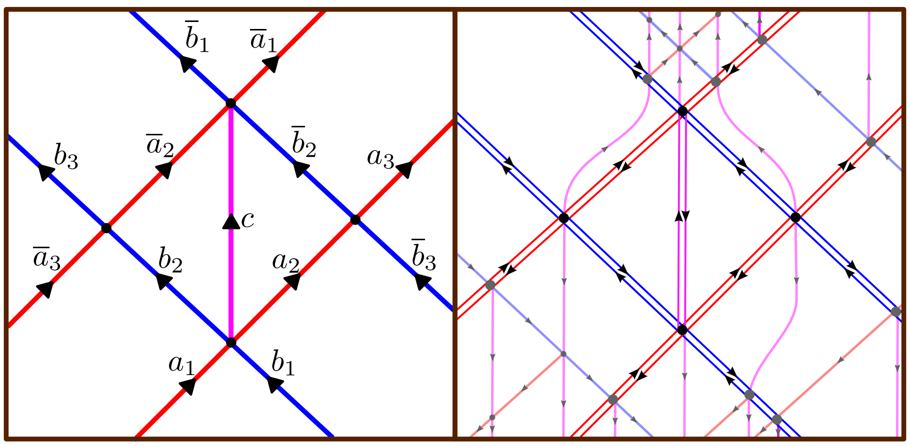


Figure 3.3: *Left Frame*: Two-way streets of a horse on some open disk U; the solid streets depicted are capable of being two-way; one-way streets are not shown. The sheets of the cover $\Sigma \to C$ are (locally) labeled from 1 to $K \geq 3$. Red streets are of type 12, blue streets are of type 23, and fuchsia streets are of type 13. We choose an orientation for this diagram such that all streets "flow up." *Right Frame*: A relatively simple example of a horse with one-way streets shown as partially transparent and two-way streets resolved (using the "British resolution", cf. Appendix A or [27]). One can imagine horses with increasingly intricate "backgrounds" of one-way streets.

Definitions

1. A *horse street* $p \in \{a_1, a_2, a_3, b_1, b_2, b_3, c, \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{b_1}, \overline{b_2}, \overline{b_3}\}$ is one of the streets of Fig. 3.3 (left frame).

2. Let N be a spectral network (subordinate to some branched cover $\Sigma \to C$) and $U \subset C'$ be an open disk region. Then $U \cap N$ is a *horse* if a subset of its streets can be identified with Fig. 3.3 in a way such that:

    a) every two-way street is a horse street,

    b) there is always a two-way street identified with the street labeled c.

We can reconstruct the two-way streets of the full spectral network by gluing m horses back together. This leads us to the following working definition (a more complete definition is provided in Appendix B), which we extend to any curve C.

Working Definition Given a collection of m horses, let $p^{(l)}$ denote a horse street on the lth horse ($l = 1, \cdots, m$). A spectral network on a curve C is an m-herd if its two-way streets are generated by gluing together m horses using the following relations:

$$\begin{aligned} a_1^{(l)} &= a_3^{(l-1)} \\ b_1^{(l)} &= b_3^{(l-1)} \\ \overline{a_1}^{(l)} &= \overline{a_3}^{(l+1)} \\ \overline{b_1}^{(l)} &= \overline{b_3}^{(l+1)}, \end{aligned} \tag{3.1}$$

and such that $a_1^{(1)}$, $b_1^{(1)}$, $\overline{a_1}^{(m)}$, and $\overline{b_1}^{(m)}$ are connected to four distinct branch points.

Remark It can be shown from our definition that a 1-herd (which consists of a single horse) is just a saddle. Indeed, a small computation will show that $Q(p)$ is nontrivial ($Q(p) \neq 1$) only for $p = a_1$, $\overline{a_1}$, b_1, $\overline{b_1}$, and c; this leads us to the picture of a saddle extending from four branch points (pictured in the top left corner of Fig. 3.4).

An advantage of the decomposition into horses is computability: a horse should be thought of as a scattering machine which takes inflowing solitons, and regurgitates outgoing solitons as well as all spectral data "bound" to the horse.[1] The combinatorial

[1]See Appendix B.3 for a the precise and explicit description of the horse as a scattering machine.

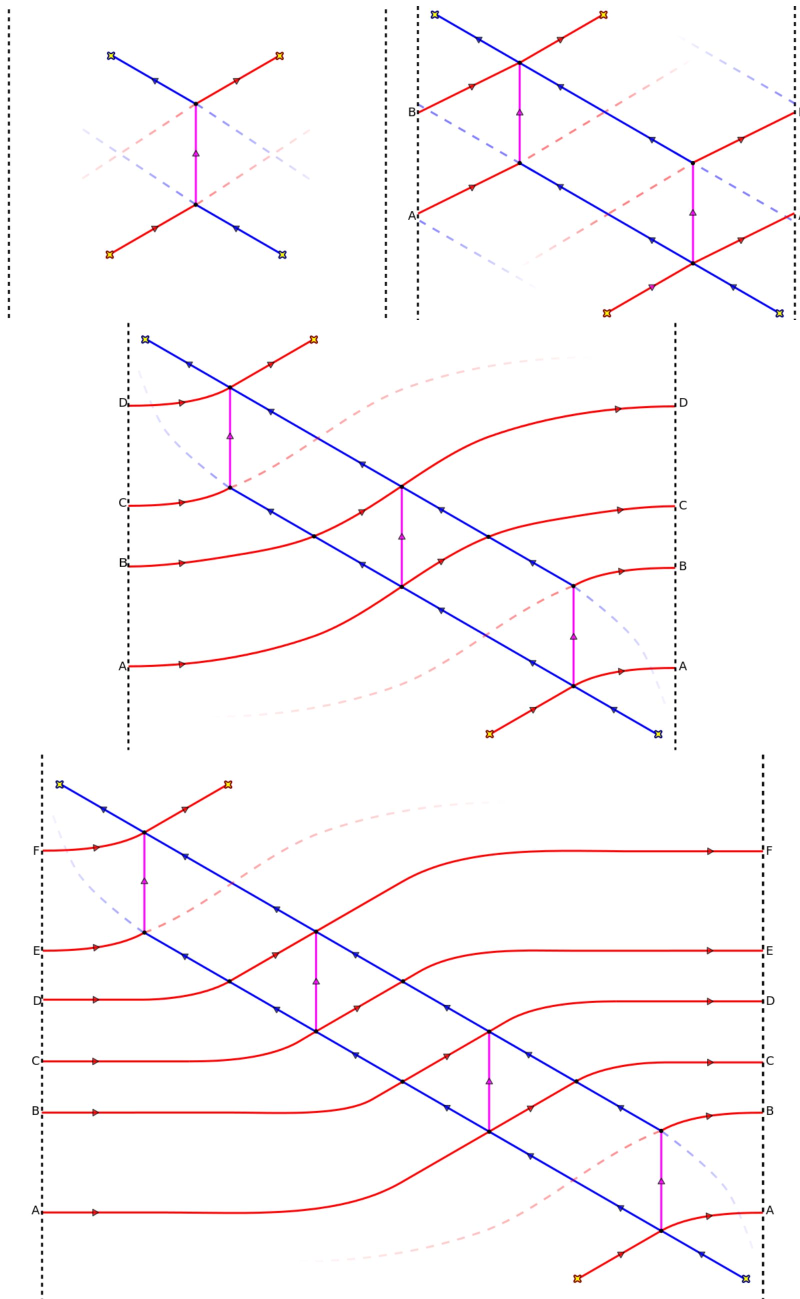


Figure 3.4: The first four herds on the cylinder. Solid streets are two-way; dotted, transparent streets are streets of Fig. 3.3 that happen to be only one-way as indicated by Prop. 3.1.1. The black dotted lines are identified to form the cylinder and capital Latin letters are placed on either side to aid in the identification of streets. Top row (from left to right): The 1-herd (saddle) and 2-herd. The middle row shows a 3-herd and the bottom row shows a 4-herd.

problem of computing the BPS degeneracies $\Omega(n\gamma_c)$, $n \geq 1$, using spectral network machinery, is then greatly simplified and explicit results can be obtained for all $m \geq 1$. In fact, we have the following.

Proposition 3.1.1. *Let N be an m-herd, then $Q(p)$ for all two-way streets p on N are given in terms of powers of a single generating function P_m satisfying the algebraic equation*

$$P_m = 1 + z\,(P_m)^{(m-1)^2}\,, \tag{3.2}$$

where $z = (-1)^m X_{\tilde{\gamma}+\tilde{\gamma}'}$ for some $\gamma, \gamma' \in H_1(\Sigma;\mathbb{Z})$ such that $\langle\gamma,\gamma'\rangle = m$. In particular, adopting the notation $Q(p,l) := Q(p^{(l)})$,

$$\begin{aligned} P_m &= Q(c,l) \\ (P_m)^{m-l} &= Q(a_2,l) = Q(b_2,l) = Q(a_3,l) = Q(b_3,l) \\ (P_m)^{l-1} &= Q(\overline{a_2},l) = Q(\overline{b_2},l) = Q(\overline{a_3},l) = Q(\overline{b_3},l) \\ (P_m)^{m-l+1} &= Q(a_1,l) = Q(b_1,l) \\ (P_m)^{l} &= Q(\overline{a_1},l) = Q(\overline{b_1},l). \end{aligned} \tag{3.3}$$

for $l = 1, \cdots, m$.

Proof. See Appendix B.6 for the full calculational proof. □

The precise cycle $\gamma_c = \gamma + \gamma'$ that appears depends on the embedding of N in C as a graph. Further, as shown at the end of Appendix B.6, there are cycles representing γ and γ' that look like the charges of simple "saddle-connection" hypermultiplets. Indeed, the cycle representing either γ or γ' projects down to a path on C that runs between two distinct branch points of the same type. These are precisely the (hypothetical) hypermultiplets whose wall-crossing motivated the construction of m-herds.[2]

Remarks

- A street p is two-way iff $Q(p) \neq 1$. Thus, (3.3) states that on the first ($l = 1$) and last ($l = m$) horses, some streets depicted in Fig. 3.3 are only one-way.

[2]The representative cycles discussed here, however, do not live entirely on $\mathrm{Lift}(N) \subset \Sigma$. Roughly speaking representatives of γ, γ' are given by the lifts of paths running along the $a_i, \overline{a_i}$ and b_i, $\overline{b_i}$ respectively, but these do not define closed paths on Σ without running through at least one street of type 13.

- When $m = 1, 2$, (3.2) has easily derivable solutions:

$$P_1 = 1 + z, \tag{3.4}$$

$$P_2 = (1 - z)^{-1}. \tag{3.5}$$

  For a saddle ($m = 1$), this result, combined with (3.3), states that there are five two-way streets; each such two-way street p is equipped with a generating function $Q(p) = 1 + z$, as originally derived in [27].

3.2 Connection with Kontsevich-Soibelman, Gross-Pandharipande

The algebraic equation (3.2) and relevant solutions appear in a conjecture by Kontsevich and Soibelman (KS) [40], later proven by Reineke [47] and generalized by Gross-Pandharipande (GP) [32]. A series solution of (3.2) can be obtained using the Lagrange formula for reversion of series and the result for $m > 1$ is [40]:

$$\begin{aligned} P_m &= \sum_{n=0}^{\infty} \frac{1}{1 + (m^2 - 2m)n} \binom{(m-1)^2 n}{n} z^n, \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{1}{(m-1)^2 n} \binom{(m-1)^2 n}{n} z^n\right]. \end{aligned} \tag{3.6}$$

To describe the connection between our result and that of KS and GP, we review the generalized conjecture of GP, briefly adopting their notation in [32].

The algebraic equation (3.2) appears in [32]. [3] There, the object of study is a group of (formal 1-parameter familes of) automorphisms of the torus $\mathbb{C}^* \times \mathbb{C}^*$ generated by $\theta_{(a,b),f}$ that are defined by

$$\theta_{(a,b),f}(x) = f^{-b} \cdot x, \quad \theta_{(a,b),f}(y) = f^{a} \cdot y$$

where x and y are coordinate functions on the two factors of $\mathbb{C}^* \times \mathbb{C}^*$, $(a, b) \in \mathbb{Z}^2$, and f is a formal series of the form

$$f = 1 + x^a y^b \left[t f_1(x^a y^b) + t^2 f_2(x^a y^b) + \cdots\right], \; f_i(z) \in \mathbb{C}[z].$$

[3] A different, but related, algebraic equation on the quantity $(P_m)^m$ was originally stated by Kontsevich and Soibelman in [40].

Alternatively we may say $f \in \mathbb{C}[x, x^{-1}, y, y^{-1}][[t]]$ (i.e. f is a formal power series in t with coefficients Laurent polynomials in x and y). Such automorphisms preserve the holomorphic symplectic form

$$\omega = (xy)^{-1} dx \wedge dy.$$

Now, letting

$$S_q = \theta_{(1,0),(1+tx)^q}, \quad T_r = \theta_{(0,1),(1+ty)^r},$$

we consider the commutator

$$T_r^{-1} \circ S_q \circ T_r \circ S_q^{-1} = \overrightarrow{\prod} \theta_{(a,b),f_{(a,b)}} \tag{3.7}$$

where the product on the right hand side is over primitive vectors $(a, b) \in \mathbb{Z}^2$ (i.e. $\gcd(a, b) = 1$) such that $a, b > 0$, and the order of the product is taken with increasing slope a/b from left to right. The conjecture of Gross-Pandharipande involves the slope 1 term of (3.7).

Conjecture (Gross-Pandharipande)
For arbitrary (q, r), the slope 1 term $\theta_{(1,1),f_{(1,1)}}$ in (3.7) is specified by

$$f_{1,1} = \left(\sum_{n=0}^{\infty} \frac{1}{(qr - q - r)n + 1} \binom{(q-1)(r-1)n}{n} t^{2n} x^n y^n \right)^{qr}.$$

The case $q = r$ was first conjectured by KS, and later proven by Reineke. Now, letting

$$\mathcal{P}_{q,r} = \sum_{n=0}^{\infty} \frac{1}{(qr - q - r)n + 1} \binom{(q-1)(r-1)n}{n} t^{2n} x^n y^n, \tag{3.8}$$

For general q, r, Gross and Pandharipande noted that $\mathcal{P}_{q,r}$ satisfies the equation

$$t^2 xy \left(\mathcal{P}_{q,r}\right)^{(q-1)(r-1)} - \mathcal{P}_{q,r} + 1 = 0; \tag{3.9}$$

so that $f_{1,1}$ is an algebraic function (over $\mathbb{Q}(t, x, y)$).

In the case $q = r = m$, the equation (3.9) and solution (3.8) bear striking similarity to (3.2) and (3.6), which motivates identifying $t^2 xy = z$ in hopes of identifying $\mathcal{P}_{m,m}$ with P_m.

To motivate the identification $t^2 xy = z$, we turn our attention back to the original motivation for our definition of m-herds: they are expected to arise after two hypermultiplets of charges γ, γ', with $\langle\gamma,\gamma'\rangle = m$, cross a wall of marginal stability. If m-herds do arise in this manner, then in the resulting wall-crossing formula we should expect the P_m to be related to the generating function for the KS transformations attached to the charges $n(\gamma+\gamma')$, $n > 0$. We now go about unpacking the identification of such a wall crossing formula with (3.7).

Assume on one side of the wall $\arg(Z_\gamma) < \arg(Z_{\gamma'})$, then the wall crossing formula reads (see Section 5.1)

$$\mathcal{K}_\gamma \mathcal{K}_{\gamma'} = \mathcal{K}_{\gamma'} \left[\prod_{(a,b)\in\mathbb{Z}^2} (\mathcal{K}_{a\gamma+b\gamma'})^{\Omega(a\gamma+b\gamma')} \right] \mathcal{K}_\gamma \tag{3.10}$$

where all products are taken in order of increasing central charge phase (when read from left to right) and the $\mathcal{K}_\alpha$ are transformations on a twisted Poisson algebra of functions on the torus $T = \Gamma \otimes_{\mathbb{Z}} \mathbb{C}^\times$, i.e. the space of functions generated by polynomials in formal variables $Y_\alpha, \alpha \in \Gamma$ equipped with twisted product given by

$$Y_\alpha Y_\beta = (-1)^{\langle\alpha,\beta\rangle} Y_{\alpha+\beta}. \tag{3.11}$$

T is equipped with a holomorphic symplectic form induced by the symplectic pairing on Γ; it is equivalently given by the holomorphic Poisson bracket

$$\{Y_\alpha, Y_\beta\} = \langle\alpha,\beta\rangle Y_\alpha Y_\beta. \tag{3.12}$$

Now, the $\mathcal{K}_\alpha$ are symplectomorphisms that act as

$$\mathcal{K}_\alpha : Y_\beta \mapsto (1 - Y_\alpha)^{\langle\alpha,\beta\rangle} Y_\beta. \tag{3.13}$$

For $\langle\gamma,\gamma'\rangle = m$, it follows that

$$\begin{aligned} &\mathcal{K}_\gamma : Y_\gamma \mapsto Y_\gamma, &&\mathcal{K}_{\gamma'} : Y_\gamma \mapsto (1 - Y_{\gamma'})^{-m} Y_\gamma, \\ &\mathcal{K}_\gamma : Y_{\gamma'} \mapsto (1 - Y_\gamma)^m Y_{\gamma'}, &&\mathcal{K}_{\gamma'} : Y_{\gamma'} \mapsto Y_{\gamma'}. \end{aligned} \tag{3.14}$$

We identify the torus $\mathbb{C}^\times \times \mathbb{C}^\times$ of Gross-Pandharipande by the subtorus of T generated by

$$x := Y_\gamma$$
$$y := Y_{\gamma'};$$

then by (3.14) we have[4]

$$\mathcal{K}_\gamma = \theta_{(1,0),(1-x)^m} = S_m$$
$$\mathcal{K}_{\gamma'} = \theta_{(0,1),(1-y)^m} = T_m.$$

Furthermore, noting that

$$x^a y^b = (-1)^{\langle a\gamma, b\gamma'\rangle} Y_{a\gamma+b\gamma'} = (-1)^{mab} Y_{a\gamma+b\gamma'}, \tag{3.15}$$

then

$$\begin{aligned}\mathcal{K}_{a\gamma+b\gamma'} &: x = Y_\gamma \mapsto (1 - Y_{a\gamma+b\gamma'})^{-mb} Y_\gamma = (1-(-1)^{mab}x^a y^b)^{mb} x \\ &: y = Y_{\gamma'} \mapsto (1 - Y_{a\gamma+b\gamma'})^{-ma} Y_{\gamma'}^{ma} = (1-(-1)^{mab}x^a y^b)^{-ma} y;\end{aligned}$$

giving the identification

$$\mathcal{K}_{a\gamma+b\gamma'} = \theta_{(a,b),(1-(-1)^{mab}x^a y^b)^m}.$$

On the right hand side of (3.10) $\arg(Z_\gamma) > \arg(Z_{\gamma'})$ and so the phase ordered product is equivalent to ordering by increasing slope a/b from left to right. This completes the identification of (3.10) with (3.7). Matching the slope 1 terms in both equations,

$$\theta_{(1,1),f_{1,1}} = \prod_{n\geq 1} \left(\mathcal{K}_{n\gamma_c}\right)^{\Omega(n\gamma_c)},$$

where $\gamma_c := \gamma + \gamma'$; in terms of generating functions, this is equivalent to the statement[5]

$$f_{1,1} = \prod_{n\geq 1} \left[(1-(-1)^{mn}z^n)^m\right]^{n\Omega(n\gamma_c)}.$$

Equivalently, as $f_{1,1} = (\mathcal{P}_{m,m})^{m^2}$,

$$(\mathcal{P}_{m,m})^m = \prod_{n\geq 1} (1-(-1)^{mn}z^n)^{n\Omega(n\gamma_c)}. \tag{3.16}$$

Now assume that the generating function P_m, derived in the context of spectral networks, *is* the generating function $\mathcal{P}_{m,m}$, derived in the context of wall crossing; then, given the exponents $\{\alpha_n\}_{n\geq 1}$ of the factorization of P_m (see (3.18)), (3.16) predicts spectral network techniques will show $\Omega(n\gamma_c) = m\alpha_n/n$. As we will see, this prediction is confirmed with Prop. 3.3.1.

[4] To make the identification with S_m and T_m we evaluate the formal (time) parameter at $t = -1$. Alternatively, we could set $-tx = Y_\gamma$ and $-ty = Y_{\gamma'}$.

[5] To see this, let $g_n = (1-(-1)^{mn}(xy)^n)^m$, then $\mathcal{K}_{n\gamma_c} = \theta_{(n,n),g_n} = \theta_{(1,1),(g_n)^n}$; furthermore, as $\theta_{(1,1),(g_n)^n}$ fixes the product xy: $\theta_{(1,1),(g_n)^n} \circ \theta_{(1,1),(g_l)^l} = \theta_{(1,1),(g_n)^n(g_l)^l}$.

3.3 Herds of horses are wild (for $m \geq 3$)

Definition For each two-way street p, define the sequence $((\alpha_n(p,l))_{n\geq 1} \subset \mathbb{Z}$ via

$$Q(p,l) = \prod_{n=1}^{\infty} \left(1 - (-1)^{mn} z^n\right)^{\alpha_n(p,l)} . \tag{3.17}$$

We also define the sequence of integers $\{\alpha_n\}_{n\geq 1}$ via

$$P_m = \prod_{n=1}^{\infty} (1 - (-1)^{mn} z^n)^{\alpha_n}. \tag{3.18}$$

By Prop. 3.1.1, we can express all $\alpha_n(p,l)$ as multiples of α_n. [6]

Remark The choice of signs $(-1)^{mn}$ follows from our convention of factorization, defined by (2.15), in terms of formal variables in the image of $Y_\gamma \mapsto X_{\tilde{\gamma}}$ (which forms an embedding of the twisted algebra of Y_γ, $\gamma \in \Gamma$, as subalgebra of $\mathbb{Z}[\Gamma]$ as detailed in Appendix D). By Prop. 3.1.1, $z^n = (-1)^{mn} X_{n\tilde{\gamma}_c}$ for some $\gamma_c \in \Gamma$, leading to the choice of signs in (3.17).

Proposition 3.3.1.

$$[L(n\gamma_c)] = m\alpha_n\gamma_c \in H_1(\Sigma; \mathbb{Z}).$$

Proof (sketch). A rough argument goes as follows. Note that, using Prop. 3.1.1 and the definition of $L(n\gamma_c)$ in (2.16), we have

$$\begin{aligned} L(n\gamma_c) &= \sum_{l=1}^{m} \sum_{p^{(l)}} \alpha_n(p,l) \boldsymbol{p}^{(l)} \\ &= \alpha_n \sum_{l=1}^{m} \Big\{ \boldsymbol{c}^{(l)} + (m-l) \left(\boldsymbol{a_2}^{(l)} + \boldsymbol{a_3}^{(l)} + \boldsymbol{b_2}^{(l)} + \boldsymbol{b_3}^{(l)} \right) \\ &+ (l-1) \left(\overline{\boldsymbol{a_2}}^{(l)} + \overline{\boldsymbol{a_3}}^{(l)} + \overline{\boldsymbol{b_2}}^{(l)} + \overline{\boldsymbol{b_3}}^{(l)} \right) + \\ &+ (m-l+1) \left(\boldsymbol{a_1}^{(l)} + \boldsymbol{b_1}^{(l)} \right) + l \left(\overline{\boldsymbol{a_1}}^{(l)} + \overline{\boldsymbol{b_1}}^{(l)} \right) \Big\} . \end{aligned} \tag{3.19}$$

[6] The radius of convergence R of the series in equation (3.2) is $\log R = -c_m$, where c_m is given in equation (3.23); in particular $R < 1$. Therefore, the product expansion is only a formal expansion and is not absolutely convergent; otherwise, it would predict that all the singularities of $d \log P$ sit on the unit circle.

Each term in this sum can be split up into a sum of words of the form

$$\boldsymbol{a}_1^{(1)} + \boldsymbol{b}_1^{(1)} + (\cdots) + \overline{\boldsymbol{a}}_1^{(m)} + \overline{\boldsymbol{b}}_1^{(m)},$$

Each such word represents a closed cycle on the lift of the m-herd to a graph on Σ, and is homologous[7] to γ_c. As $\boldsymbol{a}_1^{(1)}$, $\boldsymbol{b}_1^{(1)}$, $\overline{\boldsymbol{a}}_1^{(m)}$, $\overline{\boldsymbol{b}}_1^{(m)}$ all come with multiplicity m in (3.19), then there are m such words and the proposition follows. A full proof, using brute-force homology calculations, can be found in Appendix B.8. □

Via (2.17), the immediate result of Prop. 3.3.1 is that

$$\Omega(n\gamma_c) = \frac{m\alpha_n}{n};$$

so all that remains is to compute α_n. For the cases $m = 1, 2$: using (3.4) and (3.5) we immediately have[8]

$$\alpha_n = \begin{cases} \delta_{n,1}, & \text{if } m = 1 \\ -\delta_{n,1}, & \text{if } m = 2 \end{cases} \Rightarrow \Omega(n\gamma_c) = \begin{cases} \delta_{n,1}, & \text{if } m = 1 \\ -2\delta_{n,1}, & \text{if } m = 2 \end{cases}. \tag{3.20}$$

More generally, we can find an explicit form for α_n by taking the log of both sides of (3.18), matching powers of z, and applying Möbius inversion to derive

$$\alpha_n = \frac{1}{n} \sum_{d|n} (-1)^{md+1} \mu\left(\frac{n}{d}\right) \frac{1}{(d-1)!} \left[\frac{d^d}{dz^d} \log(P_m)\right]_{z=0},$$

where μ is the Möbius mu function. Using (3.6),

$$\alpha_n = \frac{1}{(m-1)^2 n} \sum_{d|n} (-1)^{md+1} \mu\left(\frac{n}{d}\right) \binom{(m-1)^2 d}{d}, \; m \geq 2.$$

Corollary 3.3.2. *For $m \geq 2$,*

$$\Omega(n\gamma_c) = \frac{m}{(m-1)^2 n^2} \sum_{d|n} (-1)^{md+1} \mu\left(\frac{n}{d}\right) \binom{(m-1)^2 d}{d}. \tag{3.21}$$

[7]This homological equivalence can be shown using explicit calculations of the form shown in Appendix B.8. For the reader that wishes to avoid excruciating detail: sufficient staring at some simple examples will suffice.

[8]The case $m = 1$ (i.e. the saddle) was also computed in [27].

This agrees with the result of Reineke[9] in the last section of [47]. A table of the values of $\Omega(n\gamma_c)$ is provided in Appendix B.9 for $1 \leq n, m \leq 7$. From this explicit result, we can deduce the large n asymptotics for the non-trivial[10] case $m \geq 3$.

Proposition 3.3.3. *Let $m \geq 3$, then as $n \to \infty$,*

$$\Omega(n\gamma_c) \sim (-1)^{mn+1} \left(\frac{1}{m-1} \sqrt{\frac{m}{2\pi(m-2)}} \right) n^{-5/2} e^{c_m n}, \tag{3.22}$$

where c_m is the constant

$$c_m = (m-1)^2 \log\left[(m-1)^2\right] - m(m-2)\log\left[m(m-2)\right]. \tag{3.23}$$

Proof. Restricting n to be an element of an infinite subsequence of primes, the sum over divisors simplifies and the claimed asymptotics (restricted to this subsequence) follow immediately using Stirling's asymptotics and (3.21). See Appendix C for a full proof. □

3.4 Herds in the pure $SU(3)$ theory

Now, finally, let us exhibit some points of the Coulomb branch of the pure $SU(3)$ theory where m-herds actually occur in spectral networks $\mathcal{W}_\vartheta$.

In the pure $SU(3)$ theory, the curve C is $\mathbb{CP}^1$ with two defects. It is natural to view it topologically as the cylinder $\mathbb{R} \times S^1$. Moreover, the spectral curve (4.1) has 4 branch points. Thus, the pictures of actual spectral networks in this theory look much like the "hypothetical" spectral networks we considered in Figures 3.1, 3.2.

In particular, consider the parameters

$$u_2 = -3, \quad u_3 = \frac{95}{10} \tag{3.24}$$

(in the notation of (4.1).) At this point, in accordance with the discussion of Section 3.1, we consider two charges γ, γ' supporting BPS hypermultiplets, represented simply by paths connecting pairs of branch points across the cylinder, as in the left side of

[9] Reineke showed (in our notation) $\Omega(n\gamma_c) = \frac{1}{(m-2)n^2} \sum_{d|n} (-1)^{md+1} \mu(n/d) \binom{(m-1)^2 d - 1}{d}$. To translate between results, we use the observation that $\binom{(m-1)^2 d}{d} = \frac{(m-1)^2}{m(m-2)} \binom{(m-1)^2 d - 1}{d}$.

[10] In the case $m = 2$, using the identity $\sum_{d|n} \mu(d) = \delta_{n,1}$ in (3.21) reproduces the result $\Omega(n\gamma_c) = -2\delta_{n,1}$ of (3.20).

Figure 3.2. In particular they have $\langle \gamma, \gamma' \rangle = 3$. By numerically computing the appropriate contour integrals we find that these charges have $Z_\gamma = 7.244 - 9.083i$, $Z_{\gamma'} = 20.980 - 40.148i$.

Now, our proposal in Section 3.1 was that when we have two such hypermultiplets, there will be a wall of marginal stability in the Coulomb branch when Z_γ and $Z_{\gamma'}$ become aligned, and on one side of that wall, the spectral network at the phase $\vartheta = \arg Z_{\gamma+\gamma'}$ will contain a 3-herd. So, we plot the spectral network at phase $\vartheta = \arg Z_{\gamma+\gamma'}$, and find Figure 3.5. Comparing with Figure 3.4, we see that the two-way streets in this network make up a 3-herd as desired.[11]

Moving u_3 in the positive real direction, we have similarly found a 4-herd, a 5-herd and a 6-herd. It is natural to conjecture that one can similarly obtain m-herds for any m in this way. Of course, at a fixed point in the Coulomb branch it is in general possible that there could be m-herds for many different values of m at different values of ϑ.

In any case, the existence of 3-herds in the pure $SU(3)$ theory is already enough to show that the analysis of the last few sections is not only a theoretical exercise: the wild BPS degeneracies we found there indeed occur in the $\mathcal{N} = 2$ supersymmetric pure $SU(3)$ Yang-Mills theory!

[11] In particular, our point (3.24) is on the side of the wall where the 3-herd exists. The wall of marginal stability where the 3-herd disappears can be reached by moving u_3 in the negative real direction.

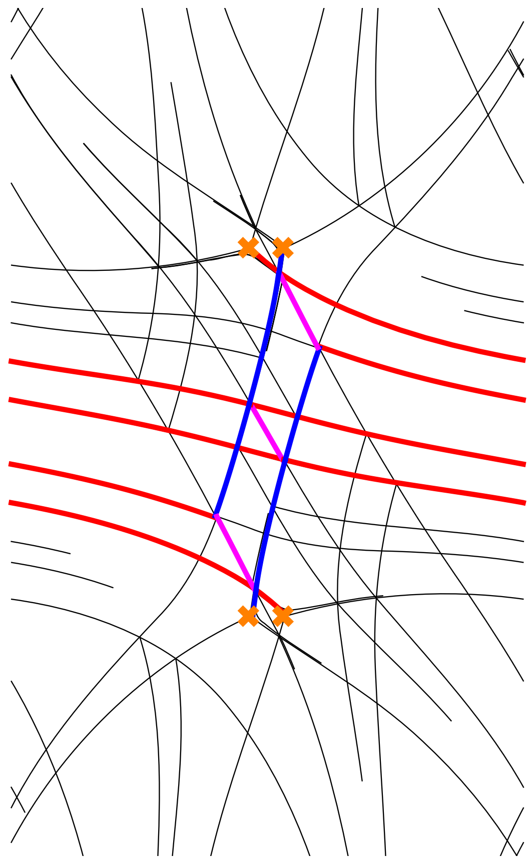


Figure 3.5: The spectral network $\mathcal{W}_\vartheta$ which occurs in the pure $SU(3)$ theory at the point (3.24) of the Coulomb branch. The phase ϑ has been chosen very close to the critical phase $\vartheta = \arg Z_{\gamma+\gamma'}$. Here we represent the cylinder C as the periodically identified plane, i.e., the left and right sides of the figure should be identified. Streets which become two-way at $\vartheta = \arg Z_{\gamma+\gamma'}$ are shown in thick red, blue and fuchsia. We do not show the whole network but only a cutoff version of it, as described in [27].

Chapter Four: Wild regions for pure $SU(3)$ theory from wall-crossing

In the previous section we exhibited an example of a class of spectral networks that lead to the m-wild degeneracies of slope $(1, 1)$. An explicit point on the Coulomb branch of the pure $SU(3)$ theory which produces such a spectral network for $m = 3$ was given in equation (3.24) above.

In the present section we start anew, and use wall crossing and quiver techniques to give an alternative demonstration that wild degeneracies exist on the Coulomb branch of the pure $SU(3)$ theory.

4.1 Strong Coupling Regime of the Pure $SU(3)$ Theory

The spectral curve Σ of pure $SU(3)$ SYM theory is

$$\lambda^3 - \frac{u_2}{z^2}\,\lambda + \left(\frac{1}{z^2} + \frac{u_3}{z^3} + \frac{1}{z^4}\right) = 0. \tag{4.1}$$

It is a branched three-sheeted covering of the cylinder C, with six ramification points. There are four branch points corresponding to two-cycles of S_3, and there are also ramifications at the irregular singularities at $0, \infty$, with associated permutations of the sheets given by three-cycles.

In the strong coupling region, i.e. at small values of the moduli u_2, u_3, the BPS spectrum is finite; so the spectral network evolves in a rather simple fashion. As a concrete example we choose $u_2 = 0.7$, $u_3 = 0.4i$; then varying ϑ from 0 to π we encounter six degenerate networks containing finite webs, which are depicted in Figure 4.1.

We assign to these cycles the charges γ_1, γ_2, $\gamma_2 + \gamma_4$, $\gamma_1 + \gamma_3$, γ_3, γ_4, Figure 4.2 shows the charge assignments with the basis cycles resolved.

The mutual intersections of cycles can be read off Figure 4.2, and are summarized

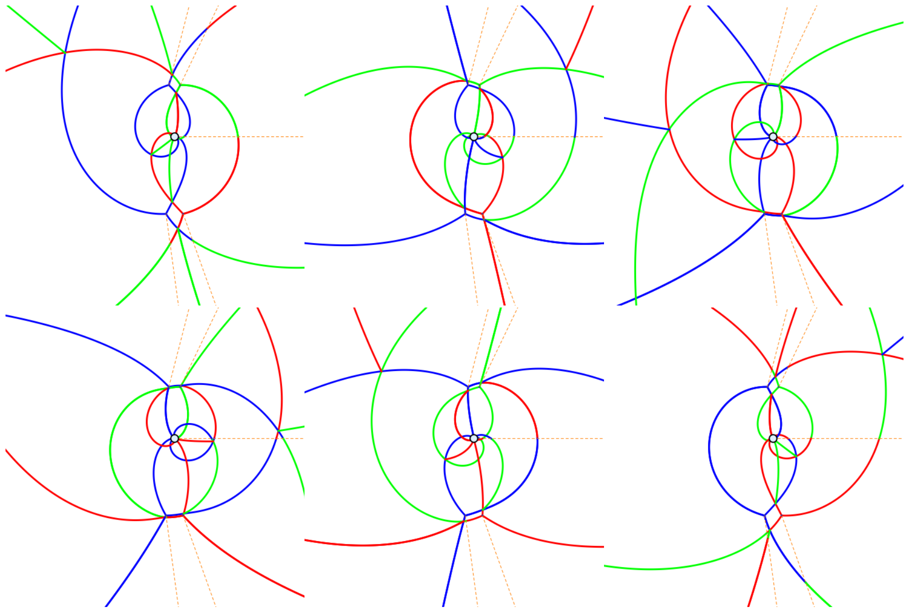


Figure 4.1: The six hypermultiplets in the strong coupling chamber: from the top left, the flips corresponding to $\gamma_1, \gamma_2, \gamma_1 + \gamma_3, \gamma_2 + \gamma_4, \gamma_3, \gamma_4$. $Arg\, Z_{\gamma_1} < Arg\, Z_{\gamma_2} < Arg\, Z_{\gamma_3} < Arg\, Z_{\gamma_4}$. Here we represent the cylinder C as the punctured plane.

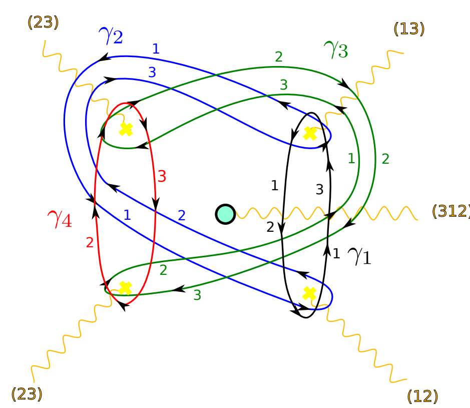


Figure 4.2: The labeling of finite networks. We only show the four basis hypermultiplets $\gamma_1, \ldots, \gamma_4$. The trivialization is indicated by the branch cuts (wavy lines, the associated permutations of sheets are also specified), the sheets on which the cycles run are indicated explicitly. Here we represent the cylinder C as the punctured plane.

by the following pairing matrix $P_{ij} = \langle \gamma_i, \gamma_j \rangle$

$$P = \begin{pmatrix} 0 & -2 & 1 & 0 \\ 2 & 0 & -2 & 1 \\ -1 & 2 & 0 & -2 \\ 0 & -1 & 2 & 0 \end{pmatrix}. \tag{4.2}$$

For a video showing the evolution of the spectral network through an angle of π, see [1].

4.2 A path on the Coulomb branch

We now consider a straight path on the Coulomb branch of the pure $SU(3)$ theory, parameterized by

$$\begin{aligned} wwc_chapter/u_2(t) &= (u_2^{(f)} - u_2^{(i)})t + u_2^{(i)}, \\ u_3(t) &= (u_3^{(f)} - u_3^{(i)})t + u_3^{(i)}, \end{aligned} \tag{4.3}$$

with $t \in [0, 1]$ and

$$\begin{aligned} u_2^{(i)} &= 0.7, \quad u_3^{(i)} = 0.4i \quad \text{(strong coupling chamber)} \\ u_2^{(f)} &= 0.56 - 0.75i, \quad u_3^{(f)} = 2 + 1.52i \quad \text{(wild chamber)} \end{aligned} \tag{4.4}$$

As discussed above, the spectrum in the strong coupling chamber is known (see for example [27]) to consist of six hypermultiplets. As we move along our path we cross several walls of marginal stability, with consequent jumps of the BPS spectrum. In order to study the evolution of the BPS spectrum, we must track explicitly the evolution of central charges. Variation of the moduli also induces changes in the geometry of the Seiberg-Witten curve Σ, therefore in computing the central charges at different points one must take care of deforming the cycles in a way compatible with the flat parallel transport of the local system $\widehat{\Gamma} \to \mathcal{B}^*$. Starting from the point studied in Section 4.1, the evolution of branch points can be tracked on C, as shown in Figure 4.3.

4.3 Cohorts in pure $SU(3)$

As the moduli cross walls of marginal stability, the BPS spectrum jumps according to a regular pattern. At a wall $\text{MS}(\gamma, \gamma')$ for two populated hypermultiplets, with

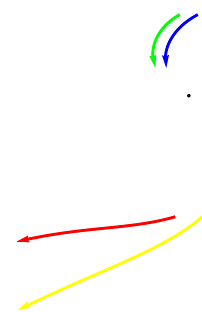


Figure 4.3: The picture shows the projection of the Seiberg-Witten curve on C. The four arrows show the progression of the four branch points as we vary $u_{2,3}$ along the path of equation (4.3). The black dot is the singularity at $z = 0$. The central charges have been computed numerically using Mathematica and, as a check, they evolve smoothly along the path (see [2]).

$\langle \gamma, \gamma' \rangle = m > 0$, the KS wall crossing formula predicts

$$\mathcal{K}_{\gamma'}\mathcal{K}_{\gamma} =: \prod_{a,b\geq 0} \mathcal{K}_{a\gamma+b\gamma'}^{\Omega(a\gamma+b\gamma')} : \tag{4.5}$$

where the normal ordering symbols : : on the right hand side indicate that factors are ordered according to the phases of central charges, phase-ordering on the right hand side is the opposite of that on the left-hand side. We refer to the spectrum on the right hand side as the *cohort* generated by γ, γ', and will occasionally denote it by $\mathcal{C}_m(\gamma, \gamma')$. An important fact to note about cohorts, following from the linearity of the central charge homomorphism, is that

$$\arg Z_{\gamma'} < \arg Z_{a\gamma+b\gamma'} < \arg Z_{\gamma}, \qquad \forall a, b \geq 0 \tag{4.6}$$

for moduli corresponding to the right hand side of (4.5).

Quite generally, the wall-crossing of two hypermultiplet states with pairing m can be analyzed in terms of the corresponding m-Kronecker quiver (see [15, 6]), from this perspective the degeneracies of an m-cohort correspond to Euler characteristics of moduli spaces of (semi)stable quiver representations.

Cohort structures $\mathcal{C}_m$ with $m = 1, 2$ are known exactly. Examples of such cohorts have been encountered a number of times in the literature [39, 25, 29, 26, 32], and are common in A_1 theories of class $\mathcal{S}$. For later convenience, we recall the structure of the $m = 2$ cohort in figure 4.4.

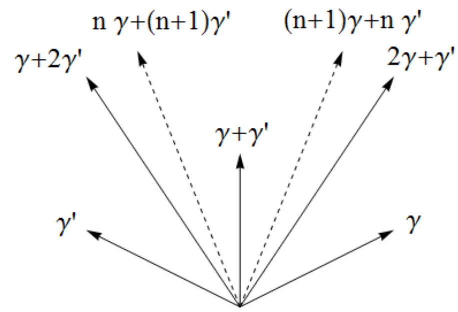


Figure 4.4: The populated BPS rays of the $m = 2$ cohort (a schematic depiction of central charges in the complex plane). The state with charge $\gamma + \gamma'$ is a BPS vectormultiplet ($\Omega = -2$), surrounded by two infinite towers of hypermultiplets ($\Omega = 1$), represented by dashed arrows.

As we start moving along our path on the Coulomb branch, from $t = 0$ to $t = 1$, several cohorts are created. The first wall of marginal stability is $\mathrm{MS}(\gamma_1 + \gamma_3, \gamma_2 + \gamma_4)$, with $\langle \gamma_1 + \gamma_3, \gamma_2 + \gamma_4 \rangle = 2$, thus a $\mathcal{C}_2$ cohort is generated. As we proceed along the path, other BPS states undergo wall-crossing, generating other $\mathcal{C}_2$ cohorts. As shown in Fig. 4.5, first γ_1 generates a cohort with γ_2, then γ_3, γ_4 generate a similar cohort, finally another $m = 2$ cohort is generated by wall crossing of γ_1 and $\gamma_2 + \gamma_4$. At this point, *i.e.* within a chamber around $t = 0.95$, the spectrum can be schematically summarized as the union of four $\mathcal{C}_2$ cohorts

$$\mathcal{C}_2(\gamma_2 + \gamma_4, \gamma_1 + \gamma_3) \cup \mathcal{C}_2(\gamma_2, \gamma_1) \cup \mathcal{C}_2(\gamma_4, \gamma_3) \cup \mathcal{C}_2(\gamma_1, \gamma_2 + \gamma_4) \tag{4.7}$$

consisting of four vectormultiplets, and infinite towers of hypermultiplets.

Proceeding further along our path, we encounter another wall of marginal stability: $\gamma_2 + \gamma_4$ undergoes wall-crossing with $2\gamma_1 + \gamma_2$ generating a new cohort with $m = 3$. This phenomenon has not been studied before, and deserves a detailed analysis. We anticipate here that this cohort contains distinctive new features, such as a wealth of higher spin states and a cone of densely populated BPS rays.

It is worth stressing that merely finding a point on the Coulomb branch where $Z_{\gamma_2+\gamma_4}$ approaches $Z_{2\gamma_1+\gamma_2}$ is hardly sufficient to claim that such wall-crossing happens. In addition one must make sure that such rays are populated. This is certainly the case in our example. Another important requirement is the absence of populated rays between $Z_{\gamma_2+\gamma_4}$ and $Z_{2\gamma_1+\gamma_2}$, as we approach their mutual wall of marginal stability. We claim that there aren't any, based on two independent facts. First, at values

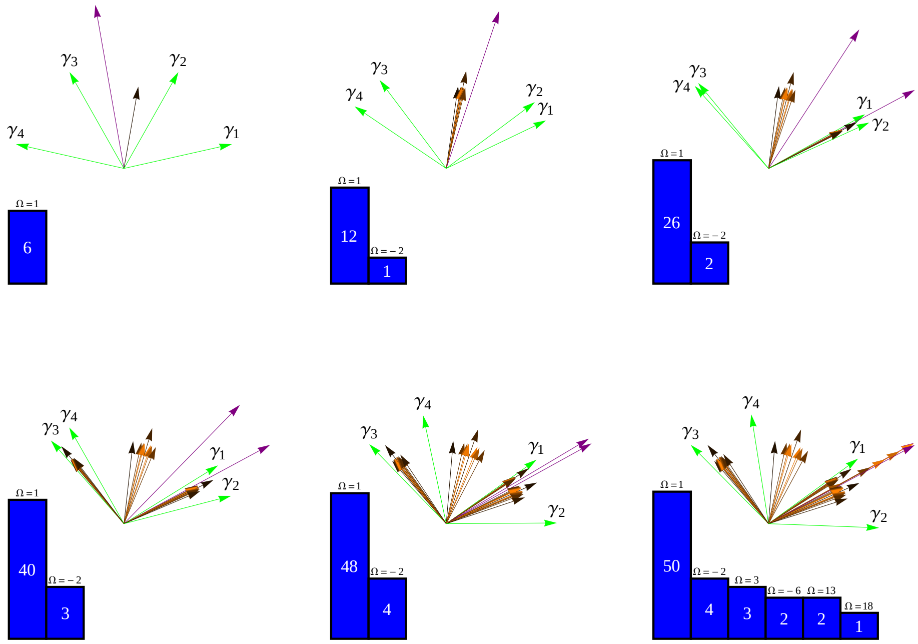


Figure 4.5: The evolution of the spectrum is illustrated. Green arrows represent the basis hypermultiplets, the purple arrows are $\gamma_2+\gamma_4$ and $2\gamma_1+\gamma_2$, the two states that generate the $m=3$ cohort. For other charges, increasing length denotes higher $|\Omega|$ and lighter shades denote larger charges. First picture: the strong coupling chamber. Second picture: the states $\gamma_1+\gamma_3$ and $\gamma_2+\gamma_4$ have crossed and created a $\mathcal{C}_2$ cohort. Third picture: γ_2 and γ_1 cross and create another cohort. Fourth picture: the cohort generated by γ_3, γ_4. Fifth picture: $\gamma_2+\gamma_4$ and γ_1 have crossed and created a cohort. In the sixth picture $\gamma_2+\gamma_4$ and $2\gamma_1+\gamma_2$ have crossed, generating wild degeneracies.. For a video showing the full evolution of the spectrum along our path, see [3].

of the moduli just before $MS(\gamma_2+\gamma_4, 2\gamma_1+\gamma_2)$, the spectral network shows simple, smooth evolution for $\arg Z_{2\gamma_1+\gamma_2} < \vartheta < \arg Z_{\gamma_2+\gamma_4}$, see [4]. Second, our explicit path on the Coulomb branch – together with property (4.6) of cohorts – guarantees that all boundstates created so far fall outside of the cone bounded by the central charges of $2\gamma_1+\gamma_2$, $\gamma_2+\gamma_4$: indeed if a populated boundstate were between $\gamma_2+\gamma_4$ and $2\gamma_1+\gamma_2$, it would have to be one of the following

- a boundstate of $2\gamma_1+\gamma_2$ with a charge counterclockwise of $\gamma_2+\gamma_4$
- a boundstate of $\gamma_2+\gamma_4$ with a charge clockwise of $2\gamma_1+\gamma_2$

- a boundstate of two charges lying respectively counterclockwise of $\gamma_2 + \gamma_4$ and clockwise of $2\gamma_1 + \gamma_2$
- a boundstate due to one of the antiparticles

All these possibilities are clearly ruled out by our explicit choice of path. Our analysis relies on the numerical evaluation of central charges at several points on the Coulomb branch, video [2] shows the smooth evolution of central charges of basis hypermultiplets along the path, ensuring that integration contours have been adapted suitably. Another important check is the following: at fixed u_2, u_3 we tune the spectral network to the phase of central charges (as predicted numerically), and we check that there are indeed degenerate networks.

4.4 Wall-crossings with intersections $m > 3$

So far we have encountered an MS wall of two hypermultiplets with intersection pairing 3, but there is nothing special about $m = 3$. The path proposed in (4.3) can be extended through walls of marginal stability with $m = 4, 5$, and higher. The strategy is simply to look for a direction on the Coulomb branch, along which the ray $\gamma_2+\gamma_4$ sweeps across the infinite tower of hypermultiplets with charges $(n+1)\gamma_1+n\gamma_2$.

For example, moving along a straight line from $(u_2^{(f)}, u_3^{(f)})$ to

$$u_2^{(4)} = 0.56 - 0.75i, \qquad u_3^{(4)} = 2.00 + 1.99i \tag{4.8}$$

induces wall-crossing of $\gamma_2+\gamma_4$ with $3\gamma_1+2\gamma_2$, with intersection $\langle\gamma_2+\gamma_4, 3\gamma_1+2\gamma_2\rangle = 4$. In this chamber the spectrum gains a new $m = 4$ cohort, described by the 4-Kronecker quiver.

Proceeding further, along a straight segment, to

$$u_2^{(5)} = 0.56 - 0.75i, \qquad u_3^{(5)} = 2.00 + 2.52i \tag{4.9}$$

we cross the marginal stability wall of $\gamma_2 + \gamma_4$ and $4\gamma_1 + 3\gamma_2$, with intersection $\langle\gamma_2 + \gamma_4, 4\gamma_1 + 3\gamma_2\rangle = 5$ generating an $m = 5$ cohort.

In the same spirit, we have checked numerically that there is a path along which $\gamma_2 + \gamma_4$ crosses all hypermultiplets with charges $(m-1)\gamma_1 + (m-2)\gamma_2$, with pairings

$$\langle\gamma_2 + \gamma_4, (m-1)\gamma_1 + (m-2)\gamma_2\rangle = m \tag{4.10}$$

hence generating an infinite tower of cohorts. The situation gets very complicated, as these cohorts will widen and start overlapping with each other, inducing further wild wall crossing.[1]. It is worth stressing that, by the same reasoning outlined for the wall-crossing of $\gamma_2+\gamma_4$ with $2\gamma_1+\gamma_2$, there are no populated states between $\gamma_2+\gamma_4$ and $(m-1)\gamma_1+(m-2)\gamma_2$ immediately before the point where they cross. This crucial fact guarantees that in this region of the Coulomb branch m-cohorts are generated, for arbitrarily high m.

Finally, we remark that a natural question arises as to whether analogous wall-crossings happen where the integer m is negative. In fact, there is a simple physical argument that such wall-crossings cannot happen on Coulomb branches of physical theories, it goes as follows. Let us consider two charges γ_1, γ_2 with $\langle\gamma_1,\gamma_2\rangle<0$; we would like to investigate whether there could be a chamber of the Coulomb branch, bounded by $MS(\gamma_1,\gamma_2)$, where

- $\arg Z_{\gamma_2} > \arg Z_{\gamma_1}$
- $\Omega(\gamma)=1$ for $\gamma\in\{\pm\gamma_1,\pm\gamma_2\}$
- $\Omega(\gamma)=0$ for all other combinations $\gamma=a\gamma_1+b\gamma_2$.

If these conditions were realized, we would be in a situation in which the spectrum generator (defined below eq. (5.1)) contains a factor $\mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1}$, and we stress that there would be no other $\mathcal{K}$ factors between $\mathcal{K}_{\gamma_2}$ and $\mathcal{K}_{\gamma_1}$.
We claim that this cannot happen: under sufficiently general conditions, near a wall $MS(\gamma_1,\gamma_2)$ we expect Denef's multicenter equations (for the case under consideration, they are reported below in (7.6)) to provide a reliable description of the boundstates. It is immediately evident from such description that, in the case of negative $\langle\gamma_1,\gamma_2\rangle=m$, on the side of $MS(\gamma_1,\gamma_2)$ where $\arg Z_{\gamma_2}>\arg Z_{\gamma_1}$, there will be stable boundstates of γ_1 with γ_2 populating rays between those of Z_{γ_1} and Z_{γ_2}. In particular, inside the spectrum generator, the factors $\mathcal{K}_{\gamma_2}$ and $\mathcal{K}_{\gamma_1}$ are *necessarily* separated by other factors $\mathcal{K}_{a\gamma_1+b\gamma_2}$, for $a,b>0$, violating the conditions formulated above.

[1]As explained in the next section, the spectrum is best studied via the *spectrum generator* technique introduced in [29]. This technique is straightforwardly applicable whenever comparing two points on the Coulomb branch, such that the lattice basis vectors have corresponding central charges all contained within a half-plane. When instead one or more of the central charges exit the half-plane, one needs to account for that by suitably modifying the spectrum generator. While moving from strong coupling into these *wilder* regions, we actually incur in such a situation.

Nevertheless, it makes sense to ask what the prediction of the KSWCF would be. To learn something interesting, it is actually sufficient to consider the motivic version of the primitive WCF (see [18]). From such formula, the protected spin character associated to $\gamma_1 + \gamma_2$ has the simple expression

$$\Omega(\gamma_1 + \gamma_2; y) := Tr_{\mathfrak{h}_m}(y)^{2J_3}(-y)^{2I_3} = \frac{y^m - y^{-m}}{y - y^{-1}} \tag{4.11}$$

corresponding (not uniquely)[2] to the following exotic representations of $so(3) \oplus su(2)_R$

$$\mathfrak{h}_m = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right) \oplus (1, 0) & m = -1 \\ \left(0, \frac{1}{2}\right) & m = -2 \\ \left(\frac{-m-2}{2}, \frac{1}{2}\right) \oplus \left(\frac{-m-3}{2}, 0\right) & m \leq -3 \end{cases} . \tag{4.12}$$

Since the no-exotics theorem is in fact fairly well established for pure $SU(K)$ gauge theories [13], this further supports the argument that such wall-crossings cannot occur on the Coulomb branch.

[2] Albeit necessarily involving exotic representations.

Chapter Five: Some Numerical Checks on the $m = 3$ Wild Spectrum

The discussion of Section 4 is sufficient to prove that there are wild degeneracies on the Coulomb branch of the pure $SU(3)$ theory. However, since this phenomenon is somewhat novel, we have checked the results using the "spectrum generator" in some relevant regions of the Coulomb branch. This section explains those checks.

5.1 The spectrum generator technique

According to the KSWCF, the phase-ordered product

$$A(\sphericalangle) = :\prod_{\gamma,\, \arg Z_\gamma \in \sphericalangle} \mathcal{K}_\gamma^{\Omega(\gamma)} : \tag{5.1}$$

is invariant across walls of marginal stability provided no occupied BPS rays cross into or out of the angular sector $\sphericalangle$. Considering an angle of π corresponds to a choice of the "half plane of particles". Once this choice is made, $A(\pi)$ is called a [1] *spectrum generator* and denoted $\mathbb{S}$ [29].

The idea of the "spectrum generator technique" is that if - through some means or other - one can compute $A(\pi)$, then, by factorization one can deduce the spectrum (after computing the phase ordering of the Z_γ at that point). For example in [29] an algorithm is given for computing $A(\pi)$ without an *a priori* knowledge of the spectrum. Here our strategy will be a little different. We will derive the spectrum generator in the strong coupling chamber, where the spectrum can be easily read off from the spectral network or from quiver techniques. We then use wall-crossing to argue that $A(\pi)$ is unchanged along a particular path in the Coulomb branch (described in Section 4) to the wild region. Then we factorize the spectrum generator at points along that path.

An effective technique for factorizing $\mathbb{S}$ is the following. Let $\{\gamma_i\}_{i=1,\ldots,k}$ be a basis for the lattice of charges Γ, and let $\gamma = \sum a_i\gamma_i$, with $a_i > 0$. Define the height

[1] Several equivalent choices are related by how one chooses the half-plane in the complex plane of central charges.

$|\gamma| := \sum_i a_i$, and $\mathbb{S}^{(r)} =: \prod_{\gamma,|\gamma|\leq r} \mathcal{K}_\gamma^{\Omega(\gamma)} :$[2]. The full spectrum generator $\mathbb{S}$ can then be factorized by studying its action on the basis formal variables[3] Y_{γ_i} for increasing values of r, by employing

$$Y_{-\gamma_i}(\mathbb{S} - \tilde{\mathbb{S}}^{(r)})Y_{\gamma_i} = -\sum_{|\gamma'|=r+1} \langle \gamma_i, \gamma' \rangle \Omega(\gamma') Y_{\gamma'} + \ldots \tag{5.2}$$

where $\tilde{\mathbb{S}}$ represents the factorization of the spectrum generator under study. The ellipses contain terms with $Y_\gamma, |\gamma| > r + 1$.

5.2 Factorizing the spectrum generator

The spectrum in the strong coupling region can be obtained via spectral network techniques, as discussed in Section 4.1. According to the results presented there, the spectrum generator is

$$\mathbb{S} = \mathcal{K}_{\gamma_4}\mathcal{K}_{\gamma_3}\mathcal{K}_{\gamma_2+\gamma_4}\mathcal{K}_{\gamma_1+\gamma_3}\mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1}, \tag{5.3}$$

in agreement with [27, 6].

We now fix a point on our path

$$u_2 = 0.56 - 0.73i, \quad u_3 = 1.94 + 1.49i, \tag{5.4}$$

corresponding to the situation exhibited in (4.7) immediately before the wall $\mathrm{MS}(\gamma_2 + \gamma_4, 2\gamma_1 + \gamma_2)$. The central charges corresponding to the simple roots are

$$\begin{aligned} Z_{\gamma_1} &= 8.42972 + 6.00549i & Z_{\gamma_2} &= 4.83278 - 0.0226871i \\ Z_{\gamma_3} &= -7.30679 + 7.50651i & Z_{\gamma_4} &= -0.504898 + 2.53401i\,, \end{aligned} \tag{5.5}$$

[2]Recall that the ordering depends crucially on the position u on the Coulomb branch, hence we should really write $\mathbb{S}^{(r)}(u)$. To lighten the notation we do not indicate the u-dependence.

[3]i.e., it is sufficient to work with formal variables corresponding to a choice of simple roots for the lattice of charges. The choice of simple roots must be consistent with the choice of half-plane that comes with the spectrum generator.

the factorization of the spectrum generator up to $|\gamma| = 21$ reads[4]

$$
\begin{aligned}
&\mathcal{K}_{\gamma_3}\mathcal{K}_{2\gamma_3+\gamma_4}\mathcal{K}_{3\gamma_3+2\gamma_4}\mathcal{K}_{4\gamma_3+3\gamma_4}\mathcal{K}_{5\gamma_3+4\gamma_4}\mathcal{K}_{6\gamma_3+5\gamma_4}\mathcal{K}_{7\gamma_3+6\gamma_4}\mathcal{K}_{8\gamma_3+7\gamma_4}\mathcal{K}_{9\gamma_3+8\gamma_4}\\
&\mathcal{K}_{10\gamma_3+9\gamma_4}\mathcal{K}_{11\gamma_3+10\gamma_4}\mathcal{K}^{-2}_{\gamma_3+\gamma_4}\mathcal{K}_{10\gamma_3+11\gamma_4}\mathcal{K}_{9\gamma_3+10\gamma_4}\mathcal{K}_{8\gamma_3+9\gamma_4}\mathcal{K}_{7\gamma_3+8\gamma_4}\mathcal{K}_{6\gamma_3+7\gamma_4}\\
&\mathcal{K}_{5\gamma_3+6\gamma_4}\mathcal{K}_{4\gamma_3+5\gamma_4}\mathcal{K}_{3\gamma_3+4\gamma_4}\mathcal{K}_{2\gamma_3+3\gamma_4}\mathcal{K}_{\gamma_3+2\gamma_4}\mathcal{K}_{\gamma_4}\mathcal{K}_{\gamma_1+\gamma_3}\mathcal{K}_{2\gamma_1+\gamma_2+2\gamma_3+\gamma_4}\\
&\mathcal{K}_{3\gamma_1+2\gamma_2+3\gamma_3+2\gamma_4}\mathcal{K}_{4\gamma_1+3\gamma_2+4\gamma_3+3\gamma_4}\mathcal{K}_{5\gamma_1+4\gamma_2+5\gamma_3+4\gamma_4}\mathcal{K}^{-2}_{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\mathcal{K}_{4\gamma_1+5\gamma_2+4\gamma_3+5\gamma_4}\\
&\mathcal{K}_{3\gamma_1+4\gamma_2+3\gamma_3+4\gamma_4}\mathcal{K}_{2\gamma_1+3\gamma_2+2\gamma_3+3\gamma_4}\mathcal{K}_{\gamma_1+2\gamma_2+\gamma_3+2\gamma_4}\mathcal{K}_{\gamma_1}\mathcal{K}_{2\gamma_1+\gamma_2+\gamma_4}\mathcal{K}_{3\gamma_1+2\gamma_2+2\gamma_4}\\
&\mathcal{K}_{4\gamma_1+3\gamma_2+3\gamma_4}\mathcal{K}_{5\gamma_1+4\gamma_2+4\gamma_4}\mathcal{K}_{6\gamma_1+5\gamma_2+5\gamma_4}\mathcal{K}_{7\gamma_1+6\gamma_2+6\gamma_4}\mathcal{K}^{-2}_{\gamma_1+\gamma_2+\gamma_4}\mathcal{K}_{6\gamma_1+7\gamma_2+7\gamma_4}\\
&\mathcal{K}_{5\gamma_1+6\gamma_2+6\gamma_4}\mathcal{K}_{4\gamma_1+5\gamma_2+5\gamma_4}\mathcal{K}_{3\gamma_1+4\gamma_2+4\gamma_4}\mathcal{K}_{2\gamma_1+3\gamma_2+3\gamma_4}\mathcal{K}_{\gamma_1+2\gamma_2+2\gamma_4}\mathcal{K}_{\gamma_2+\gamma_4}\\
&\mathcal{K}_{2\gamma_1+\gamma_2}\mathcal{K}_{3\gamma_1+2\gamma_2}\mathcal{K}_{4\gamma_1+3\gamma_2}\mathcal{K}_{5\gamma_1+4\gamma_2}\mathcal{K}_{6\gamma_1+5\gamma_2}\mathcal{K}_{7\gamma_1+6\gamma_2}\mathcal{K}_{8\gamma_1+7\gamma_2}\mathcal{K}_{9\gamma_1+8\gamma_2}\\
&\mathcal{K}_{10\gamma_1+9\gamma_2}\mathcal{K}_{11\gamma_1+10\gamma_2}\mathcal{K}^{-2}_{\gamma_1+\gamma_2}\mathcal{K}_{10\gamma_1+11\gamma_2}\mathcal{K}_{9\gamma_1+10\gamma_2}\mathcal{K}_{8\gamma_1+9\gamma_2}\mathcal{K}_{7\gamma_1+8\gamma_2}\mathcal{K}_{6\gamma_1+7\gamma_2}\\
&\mathcal{K}_{5\gamma_1+6\gamma_2}\mathcal{K}_{4\gamma_1+5\gamma_2}\mathcal{K}_{3\gamma_1+4\gamma_2}\mathcal{K}_{2\gamma_1+3\gamma_2}\mathcal{K}_{\gamma_1+2\gamma_2}\mathcal{K}_{\gamma_2}
\end{aligned}
\tag{5.6}
$$

The spectrum exhibits four $m = 2$ cohorts, as expected from the discussion of Section 4.3: they include four vectormultiplets (with $\Omega = -2$), accompanied by infinite towers of hypermultiplets.

On the other side of the $m = 3$ wall, at

$$u_2 = 0.56 - 0.75i,\ u_3 = 2.00 + 1.52i, \tag{5.7}$$

central charges read

$$
\begin{aligned}
Z_{\gamma_1} &= 8.52337 + 6.18454i & Z_{\gamma_2} &= 4.89813 - 0.18347i\\
Z_{\gamma_3} &= -7.43876 + 7.53531i & Z_{\gamma_4} &= -0.410809 + 2.59321i.
\end{aligned}
\tag{5.8}
$$

[4]Color code: The factors in blue come from the hypermultiplets of the strong coupling chamber. The factors in red come from vectormultiplets. The remaining factors in black are hypermultiplets created by the wall-crossing from the strong coupling chamber.

The spectrum generator, up to $|\gamma| = 21$, is

$$\begin{aligned}
&\mathcal{K}_{\gamma_3}\mathcal{K}_{2\gamma_3+\gamma_4}\mathcal{K}_{3\gamma_3+2\gamma_4}\mathcal{K}_{4\gamma_3+3\gamma_4}\mathcal{K}_{5\gamma_3+4\gamma_4}\mathcal{K}_{6\gamma_3+5\gamma_4}\mathcal{K}_{7\gamma_3+6\gamma_4}\mathcal{K}_{8\gamma_3+7\gamma_4}\mathcal{K}_{9\gamma_3+8\gamma_4}\\
&\mathcal{K}_{10\gamma_3+9\gamma_4}\mathcal{K}_{11\gamma_3+10\gamma_4}\mathcal{K}^{-2}_{\gamma_3+\gamma_4}\mathcal{K}_{10\gamma_3+11\gamma_4}\mathcal{K}_{9\gamma_3+10\gamma_4}\mathcal{K}_{8\gamma_3+9\gamma_4}\mathcal{K}_{7\gamma_3+8\gamma_4}\mathcal{K}_{6\gamma_3+7\gamma_4}\\
&\mathcal{K}_{5\gamma_3+6\gamma_4}\mathcal{K}_{4\gamma_3+5\gamma_4}\mathcal{K}_{3\gamma_3+4\gamma_4}\mathcal{K}_{2\gamma_3+3\gamma_4}\mathcal{K}_{\gamma_3+2\gamma_4}\mathcal{K}_{\gamma_4}\mathcal{K}_{\gamma_1+\gamma_3}\mathcal{K}_{2\gamma_1+\gamma_2+2\gamma_3+\gamma_4}\\
&\mathcal{K}_{3\gamma_1+2\gamma_2+3\gamma_3+2\gamma_4}\mathcal{K}_{4\gamma_1+3\gamma_2+4\gamma_3+3\gamma_4}\mathcal{K}_{5\gamma_1+4\gamma_2+5\gamma_3+4\gamma_4}\mathcal{K}^{-2}_{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\mathcal{K}_{4\gamma_1+5\gamma_2+4\gamma_3+5\gamma_4}\\
&\mathcal{K}_{3\gamma_1+4\gamma_2+3\gamma_3+4\gamma_4}\mathcal{K}_{2\gamma_1+3\gamma_2+2\gamma_3+3\gamma_4}\mathcal{K}_{\gamma_1+2\gamma_2+\gamma_3+2\gamma_4}\mathcal{K}_{\gamma_1}\mathcal{K}_{2\gamma_1+\gamma_2+\gamma_4}\mathcal{K}_{3\gamma_1+2\gamma_2+2\gamma_4}\\
&\mathcal{K}_{4\gamma_1+3\gamma_2+3\gamma_4}\mathcal{K}_{5\gamma_1+4\gamma_2+4\gamma_4}\mathcal{K}_{6\gamma_1+5\gamma_2+5\gamma_4}\mathcal{K}_{7\gamma_1+6\gamma_2+6\gamma_4}\mathcal{K}^{-2}_{\gamma_1+\gamma_2+\gamma_4}\mathcal{K}_{6\gamma_1+7\gamma_2+7\gamma_4}\\
&\mathcal{K}_{5\gamma_1+6\gamma_2+6\gamma_4}\mathcal{K}_{4\gamma_1+5\gamma_2+5\gamma_4}\mathcal{K}_{3\gamma_1+4\gamma_2+4\gamma_4}\mathcal{K}_{2\gamma_1+3\gamma_2+3\gamma_4}\mathcal{K}_{\gamma_1+2\gamma_2+2\gamma_4}\mathcal{K}_{2\gamma_1+\gamma_2}\\
&\mathcal{K}_{6\gamma_1+4\gamma_2+\gamma_4}\mathcal{K}^{3}_{10\gamma_1+7\gamma_2+2\gamma_4}\mathcal{K}^{3}_{4\gamma_1+3\gamma_2+\gamma_4}\mathcal{K}^{-6}_{8\gamma_1+6\gamma_2+2\gamma_4}\mathcal{K}^{68}_{10\gamma_1+8\gamma_2+3\gamma_4}\mathcal{K}^{13}_{6\gamma_1+5\gamma_2+2\gamma_4}\\
&\mathcal{K}^{68}_{8\gamma_1+7\gamma_2+3\gamma_4}\mathcal{K}^{18}_{6\gamma_1+6\gamma_2+3\gamma_4}\mathcal{K}^{3}_{2\gamma_1+2\gamma_2+\gamma_4}\mathcal{K}^{-6}_{4\gamma_1+4\gamma_2+2\gamma_4}\mathcal{K}^{-84}_{8\gamma_1+8\gamma_2+4\gamma_4}\mathcal{K}^{68}_{6\gamma_1+7\gamma_2+4\gamma_4}\\
&\mathcal{K}^{13}_{4\gamma_1+5\gamma_2+3\gamma_4}\mathcal{K}^{68}_{6\gamma_1+8\gamma_2+5\gamma_4}\mathcal{K}^{18}_{6\gamma_1+9\gamma_2+6\gamma_4}\mathcal{K}^{3}_{2\gamma_1+3\gamma_2+2\gamma_4}\mathcal{K}^{-6}_{4\gamma_1+6\gamma_2+4\gamma_4}\\
&\mathcal{K}^{3}_{4\gamma_1+7\gamma_2+5\gamma_4}\mathcal{K}_{2\gamma_1+4\gamma_2+3\gamma_4}\mathcal{K}_{\gamma_2+\gamma_4}\mathcal{K}_{3\gamma_1+2\gamma_2}\mathcal{K}_{4\gamma_1+3\gamma_2}\mathcal{K}_{5\gamma_1+4\gamma_2}\mathcal{K}_{6\gamma_1+5\gamma_2}\mathcal{K}_{7\gamma_1+6\gamma_2}\\
&\mathcal{K}_{8\gamma_1+7\gamma_2}\mathcal{K}_{9\gamma_1+8\gamma_2}\mathcal{K}_{10\gamma_1+9\gamma_2}\mathcal{K}_{11\gamma_1+10\gamma_2}\mathcal{K}^{-2}_{\gamma_1+\gamma_2}\mathcal{K}_{10\gamma_1+11\gamma_2}\mathcal{K}_{9\gamma_1+10\gamma_2}\mathcal{K}_{8\gamma_1+9\gamma_2}\\
&\mathcal{K}_{7\gamma_1+8\gamma_2}\mathcal{K}_{6\gamma_1+7\gamma_2}\mathcal{K}_{5\gamma_1+6\gamma_2}\mathcal{K}_{4\gamma_1+5\gamma_2}\mathcal{K}_{3\gamma_1+4\gamma_2}\mathcal{K}_{2\gamma_1+3\gamma_2}\mathcal{K}_{\gamma_1+2\gamma_2}\mathcal{K}_{\gamma_2},
\end{aligned} \tag{5.9}$$

where $\mathcal{K}$-factors in green are those of the newborn $m = 3$ cohort. Notice the large values of Ω.

Both formulae (5.6), (5.9) can be recast in more suggestive forms by adopting the notation[5]

$$\Pi^{(n,m)}(a,b) := \left(\prod_{k\nearrow n}^{\infty} \mathcal{K}_{(k+1)a+kb}\right) \mathcal{K}^{-2}_{a+b} \left(\prod_{\ell\searrow m}^{\infty} \mathcal{K}_{\ell a+(\ell+1)b}\right) \tag{5.10}$$

Expression (5.6) is then simply the truncation to $|\gamma| = 21$ of (cf. (4.7))

$$\Pi^{(0,0)}(\gamma_3,\gamma_4)\ \Pi^{(0,1)}(\gamma_1+\gamma_3,\gamma_2+\gamma_4)\ \Pi^{(0,0)}(\gamma_1,\gamma_2+\gamma_4)\ \Pi^{(1,0)}(\gamma_1,\gamma_2) \tag{5.11}$$

Similarly, for (5.9) we have

$$\begin{aligned}
\Pi^{(0,0)}(\gamma_3,\gamma_4)\ \Pi^{(0,1)}(\gamma_1+\gamma_3,\gamma_2+\gamma_4)\ \Pi^{(0,1)}(\gamma_1,\gamma_2+\gamma_4)\\
\Xi(2\gamma_1+\gamma_2,\gamma_2+\gamma_4)\ \Pi^{(2,0)}(\gamma_1,\gamma_2)
\end{aligned} \tag{5.12}$$

where $\Xi(2\gamma_1+\gamma_2, \gamma_2+\gamma_4)$ represents the contribution from the full $\mathcal{C}_3(2\gamma_1+\gamma_2, \gamma_2+\gamma_4)$ cohort, which we now analyze in greater detail.

[5] We adopt the following conventions: a product of noncommutative factors $\prod_{k\nearrow a}^{b}$ indicates that values of k increase from left to right between a and b, while $\prod_{k\searrow a}^{b}$ denotes decreasing values of k from left to right.

5.3 Exponential growth of the BPS degeneracies

We now focus on the part of BPS spectrum within the cohort $\mathcal{C}_3(\gamma_2+\gamma_4, 2\gamma_1+\gamma_2)$. Let

$$\mathcal{K}_{(a,b)} \equiv \mathcal{K}_{a(2\gamma_1+\gamma_2)+b(\gamma_2+\gamma_4)}, \qquad a,b\in\mathbb{Z}, \tag{5.13}$$

then, up to $a+b=15$, $\Xi(2\gamma_1+\gamma_2, \gamma_2+\gamma_4)$ reads

$$\begin{aligned}
&\mathcal{K}_{(1,0)}\mathcal{K}_{(3,1)}\mathcal{K}_{(8,3)}\mathcal{K}^{-6}_{(10,4)}\mathcal{K}^{3}_{(5,2)}\mathcal{K}^{13}_{(7,3)}\mathcal{K}^{68}_{(9,4)}\mathcal{K}^{465}_{(10,5)}\mathcal{K}^{-84}_{(8,4)}\mathcal{K}^{18}_{(6,3)}\\
&\quad\mathcal{K}^{-6}_{(4,2)}\mathcal{K}^{3}_{(2,1)}\mathcal{K}^{2530}_{(9,5)}\mathcal{K}^{399}_{(7,4)}\mathcal{K}^{68}_{(5,3)}\mathcal{K}^{4242}_{(8,5)}\mathcal{K}^{34227}_{(9,6)}\mathcal{K}^{-478}_{(6,4)}\mathcal{K}^{13}_{(3,2)}\mathcal{K}^{4242}_{(7,5)}\\
&\quad\quad\mathcal{K}^{-32050}_{(8,6)}\mathcal{K}^{68}_{(4,3)}\mathcal{K}^{399}_{(5,4)}\mathcal{K}^{2530}_{(6,5)}\mathcal{K}^{16965}_{(7,6)}\mathcal{K}^{118668}_{(8,7)}\mathcal{K}^{18123}_{(7,7)}\mathcal{K}^{-2808}_{(6,6)}\mathcal{K}^{465}_{(5,5)}\\
&\quad\quad\quad\mathcal{K}^{-84}_{(4,4)}\mathcal{K}^{18}_{(3,3)}\mathcal{K}^{-6}_{(2,2)}\mathcal{K}^{3}_{(1,1)}\mathcal{K}^{118668}_{(7,8)}\mathcal{K}^{16965}_{(6,7)}\mathcal{K}^{2530}_{(5,6)}\mathcal{K}^{399}_{(4,5)}\mathcal{K}^{-32050}_{(6,8)}\mathcal{K}^{68}_{(3,4)}\\
&\quad\quad\quad\quad\mathcal{K}^{4242}_{(5,7)}\mathcal{K}^{34227}_{(6,9)}\mathcal{K}^{-478}_{(4,6)}\mathcal{K}^{13}_{(2,3)}\mathcal{K}^{4242}_{(5,8)}\mathcal{K}^{68}_{(3,5)}\mathcal{K}^{399}_{(4,7)}\mathcal{K}^{2530}_{(5,9)}\mathcal{K}^{465}_{(5,10)}\mathcal{K}^{-84}_{(4,8)}\\
&\quad\quad\quad\quad\quad\mathcal{K}^{18}_{(3,6)}\mathcal{K}^{-6}_{(2,4)}\mathcal{K}^{3}_{(1,2)}\mathcal{K}^{68}_{(4,9)}\mathcal{K}^{13}_{(3,7)}\mathcal{K}^{-6}_{(4,10)}\mathcal{K}^{3}_{(2,5)}\mathcal{K}_{(3,8)}\mathcal{K}_{(1,3)}\mathcal{K}_{(0,1)}
\end{aligned} \tag{5.14}$$

The BPS degeneracies appearing in (5.14) look rather *wild* at first sight. One way of looking at them is to consider sequences of charges (a_0+na, b_0+nb) approaching different "slopes" a/b for $n\to\infty$, and study the asymptotics of Ω for large n. As illustrated in figure 5.1, the BPS index grows exponentially with n, the asymptotic exponential behavior depends entirely on a/b and not on a_0, b_0.

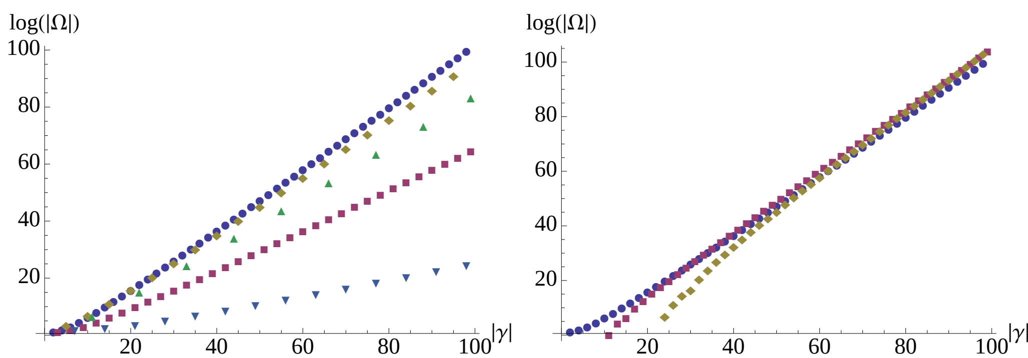


Figure 5.1: Left: values of $\log\Omega(an,bn)$ for several slopes a/b: 1 (circles), 3/2 (diamonds), 7/4 (up-triangles), 2 (squares), 5/2 (down-triangles). Right: sequences of type (a_0+an, b_0+bn) have the same asymptotics; here we show $a=b=1$ with $a_0-b_0=0,5,10$.

Chapter Six: Relation to quivers

In addition to spectral networks, one alternative route to the BPS spectrum is the dual description in terms of quiver quantum mechanics [15, 16, 6]. The problem of counting BPS states gets mapped into that of counting cohomology classes of moduli spaces of quiver representations. These classes are organized into Lefschetz multiplets, which correspond to the $\mathfrak{so}(3)$ multiplets. The PSC $\Omega(\gamma, u; y)$ is then given by the Poincaré polynomial associated to a certain quiver representation.

The basic observation here is that an isolated wall-crossing of hypermultiplets with charges γ, γ' such that $\langle \gamma, \gamma' \rangle = m$ will produce the spectrum of the Kronecker m-quiver in the wild stability region.

6.1 Derivation of the Kronecker m-quivers from the strong coupling regime

Here we briefly describe how the quiver description fits in our study of the BPS spectrum of this theory. We start in the strong coupling chamber: we choose a half-plane as shown in the first frame of figure 6.1, the corresponding BPS quiver is shown in the second frame of the same figure. As we move along the path (4.3), we come to the situation shown in the third frame of figure 6.1: three MS walls have been crossed, and the corresponding $m = 2$ cohorts are indicated (this corresponds to the situation shown in the fifth frame of figure 4.5 above.). Note that no walls of the second kind[1] have been crossed, hence the same BPS quiver is still valid. Now, while keeping the moduli fixed, we rotate the half-plane clockwise inducing a mutation on the quiver, as shown in the first two frames of figure 6.2. We then proceed a little further along our path on $\mathcal{B}$, until we cross the wall $MS(\gamma_1, \gamma_2 + \gamma_4)$, again this does not involve crossing walls of the second kind, and the same quiver is still valid. The charge disposition and cohorts are shown in the third frame of figure 6.2.

[1] In the physics literature, a wall of the second kind is, roughly speaking, the locus on the moduli space where the central charge of a populated state *exits* the half-plane associated with the quiver under study. When this happens, the quiver description changes by a mutation, for more details, see [6].

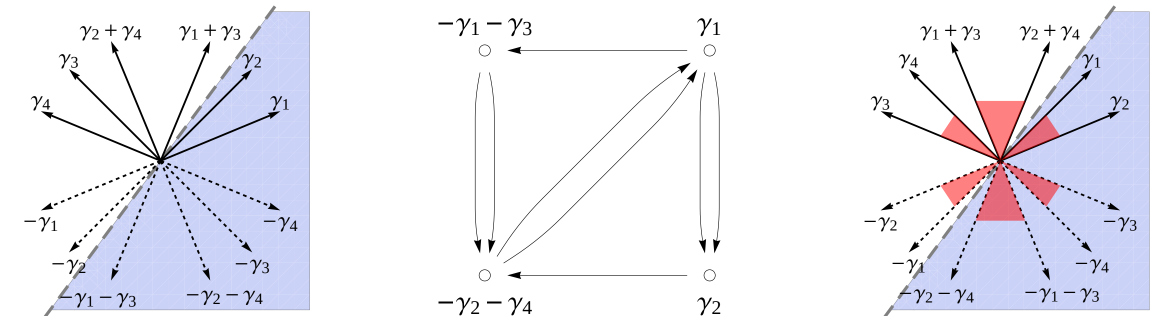


Figure 6.1: Left: the disposition of charges and choice of half plane in the strong coupling chamber. The depiction of the central charges is schematic. Center: the quiver at strong coupling. Right: central charges and cohorts after crossing the first three MS walls along our path.

Finally, we rotate the half-plane counterclockwise, as shown in figure 6.3, inducing

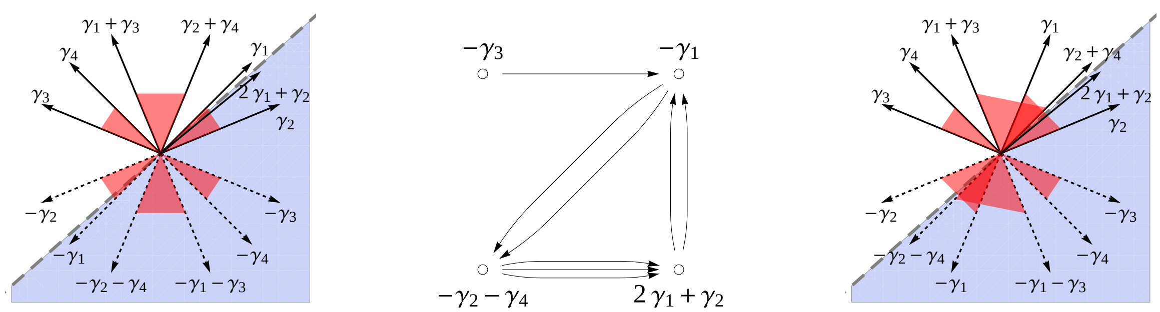


Figure 6.2: Left: a clockwise rotation of the half-plane past the ray Z_{γ_1}. Center: the corresponding BPS quiver. Right: after proceeding further on $\mathcal{B}$ we cross $MS(\gamma_1, \gamma_2+\gamma_4)$

an inverse mutation on the node $-\gamma_2 - \gamma_4$, which results in the desired BPS quiver.

The two lower nodes of the quiver we just obtained manifestly exhibit the 3-Kronecker quiver involved in wild wall-crossing as a subquiver. In particular, it offers a convenient starting point for studying stable quiver representations on both sides of $MS(\gamma_2 + \gamma_4, 2\gamma_1 + \gamma_2)$: states with charge $a(\gamma_2 + \gamma_4) + b(2\gamma_1 + \gamma_2)$ correspond to particularly simple dimension vectors, in which the two upper nodes decouple leaving the pure 3-Kronecker quiver. We will not pursue the stability analysis in this thesis, let us stress however that, since we have been working with stability parameters constrained by special geometry on the Coulomb branch (as opposed to working in $\mathbb{C}^4$), it should be possible to perform such analysis on both sides of the above-mentioned

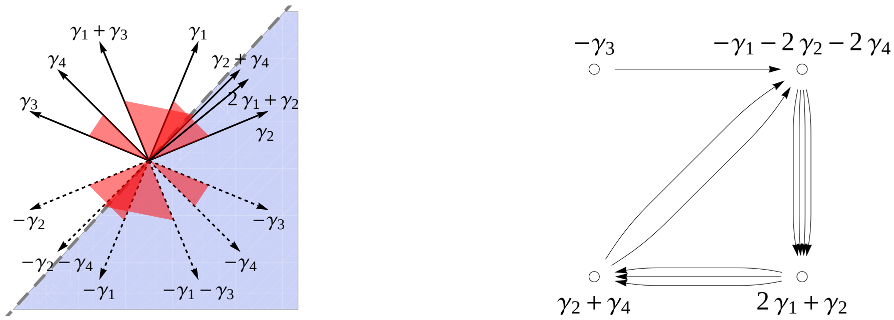


Figure 6.3: Left: a counterclockwise rotation past $Z_{\gamma_2+\gamma_4}$. Right: the corresponding BPS quiver.

MS wall, thus recovering the related wild degeneracies.

The above construction generalizes easily to higher m. Consider indeed the situation in frame three of Figure 6.2: here one could rotate the half-plane clockwise up until crossing the ray of $\gamma^{(j+1,j)} := (j+1)\gamma_1 + j\gamma_2$, resulting in a sequence of mutations leading to the quiver of Figure 6.4. Then, without crossing walls of the second kind, one can move on $\mathcal{B}$ on a continuation of our path, as discussed in Section 4.4, until getting past $MS((j+1)\gamma_1 + j\gamma_2, \gamma_2 + \gamma_4)$, the same quiver description still holds.

At this point, a counterclockwise rotation of the half-plane, corresponding to an

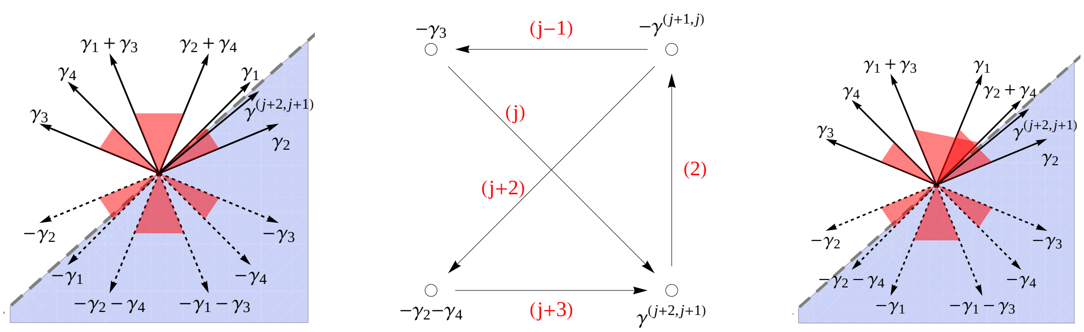


Figure 6.4: Left: a clockwise rotation of the half-plane past the ray $Z_{(j+1)\gamma_1+j\gamma_2}$. Center: the corresponding BPS quiver, arrow multiplicities are indicated in red. Right: after proceeding further on $\mathcal{B}$ we cross $MS((j+1)\gamma_1 + j\gamma_2, \gamma_2 + \gamma_4)$

inverse mutation on $-\gamma_2 - \gamma_4$ yields the quiver given in Figure 6.5. Again the two lower nodes exhibit the Kronecker subquiver of interest.

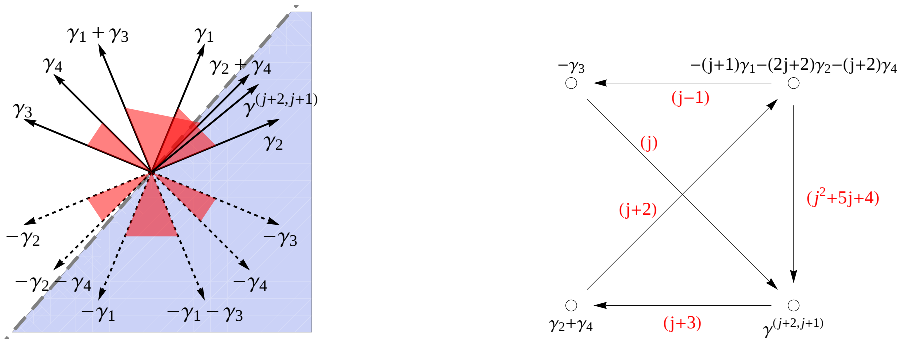


Figure 6.5: Left: a counter-clockwise rotation of the half-plane past the ray $Z_{\gamma_2+\gamma_4}$. Right: the corresponding BPS quiver, with arrow multiplicities indicated in red.

6.2 A nontrivial symmetry of BPS degeneracies

One very nice application of the quiver approach is that it reveals an intriguing symmetry of BPS degeneracies which would be very hard to discover using spectral networks.

Our previous analysis of the $\mathcal{C}_3$ spectrum has focused on sequences of states $(na + a_0)\gamma_1 + (nb + b_0)\gamma_2$ with fixed slope a/b as $n \to \infty$. In this section we will instead consider sequences of states with the same BPS index.

In full generality, given two hypermultiplets with charges γ, γ' such that $\langle \gamma, \gamma' \rangle = m > 0$, we know already from the semi-primitive WCF that, across the wall $MS(\gamma, \gamma')$, a new hypermultiplet of charge $\gamma + m\gamma'$ will be a stable boundstate. The constituents γ, γ', as well as their CPT conjugates will also be stable. Now, note that $\langle -\gamma', \gamma + m\gamma' \rangle = m$, moreover we have the following relation between stability parameters

$$\mathrm{sign}\left(\mathrm{Im}\frac{Z_\gamma}{Z_{\gamma'}}\right) \equiv \mathrm{sign}\left(\mathrm{Im}\frac{Z_{-\gamma'}}{Z_{\gamma+m\gamma'}}\right). \tag{6.1}$$

Thus, any boundstate of γ, γ' can *equivalently* be described as a boundstate of $-\gamma', \gamma + m\gamma'$. Such change of *simple roots* for the $K(m)$ quiver simply corresponds[2] to a change of duality frame by

$$g_m = \begin{pmatrix} 0 & 1 \\ -1 & m \end{pmatrix} \quad \in Sp(2, \mathbb{Z}) \tag{6.2}$$

in a basis where γ, γ' are represented by column vectors $(1, 0), (0, 1)$ respectively. That is, there is a mutation of the quiver corresponding to the change of basis g_m.

[2]In the mathematics literature this correspondence is a known isomorphism among Kronecker moduli spaces, see for example [56], Remark 3.2.

Since this is detectable by the semiprimitive wall crossing formula there should be a halo interpretation, to which we return in Section §8, Remark 4.

The above is essentially an observation of [56] and it immediately implies some remarkable identities for BPS indices. The group

$$\mathcal{R} = \langle h, h' | h^2 = 1, {h'}^2 = 1 \rangle \; = \; \mathbb{Z}_2 \star \mathbb{Z}_2 \tag{6.3}$$

has an action on $\mathbb{Z}\gamma \oplus \mathbb{Z}\gamma'$ by

$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad h' = \begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}, \quad g_m = h\, h', \tag{6.4}$$

then the BPS indices must have the symmetry:

$$\Omega(g \cdot \gamma) = \Omega(\gamma), \quad \forall g \in \mathcal{R}. \tag{6.5}$$

In other words, the spectrum can be organized into orbits of $\mathcal{R}$.

Remarks

- The identity (6.5) extends to the protected spin character

$$\Omega(g \cdot \gamma; y) = \Omega(\gamma; y). \tag{6.6}$$

- Consider for example $m = 3$, we call the *slope* of (a, b) the ratio a/b. The eigenvalues of g_3 are

$$\xi_\pm = \frac{3 \pm \sqrt{5}}{2} \tag{6.7}$$

  corresponding to the slopes limiting the cone of dense states of Fig. 6.6. All g_3 orbits are confined to lie either inside or outside of the cone, and asymptote to the limiting rays.

- The only orbits falling outside of the cone are those of "pure" hypermultiplets i.e. states with $\Omega = 1$. All the other orbits are contained within the cone.

- In the pure $SU(2)$ theory the limiting rays of the g_2 cone collapse into a single ray, which coincides with the slope of the gauge boson. In that context, the g_2 action has an interpretation in terms of a half-turn around the strong coupling chamber, combined with the residual R-symmetry, in a similar spirit to the approach of [22]. One can check that, in a suitable duality frame g_2 is a square root of the monodromy at infinity, up to an overall factor.

For the $m = 1$ Kronecker quiver, the g_1 action simply recovers the whole spectrum.

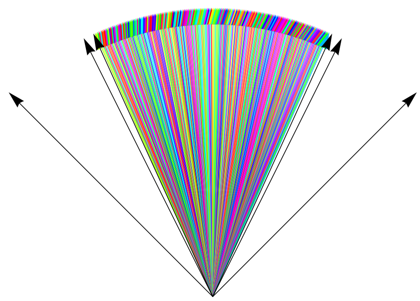


Figure 6.6: Schematic picture of BPS states charges for 3-Kronecker quiver. A dense cone is bounded by rays of slopes $a/b = (3 \pm \sqrt{5})/2$. Only hypermultiplets fall out of the cone.

Chapter Seven: Physical estimates and expectations

7.1 An apparent paradox

In this section we first present a physical argument which seems to lead to a very general bound on the behavior of the BPS index in any supersymmetric field theory. The purported bound, however, is explicitly violated by the "wild" degeneracies we have just found in the pure $SU(3)$ theory. Thus, naïvely, there is a paradox. We first explain the paradox in more detail, and then explain how this paradox is resolved.[1]

At very large energy our effective theory should approach a UV conformal fixed point. So consider a d-dimensional CFT put in a box of volume V and heated up to temperature T. Since we have only two dimensionful parameters and we assume the energy and the entropy of the system to be extensive quantities, simple dimensional analysis is enough to predict their form up to dimensionless constants (which will depend on the theory):

$$E(T, V) = \alpha V T^d,$$
$$S(T, V) = \beta V T^{d-1}.$$

Eliminating the temperature dependence we derive the scaling of the entropy with the energy:

$$S(E, V) = \kappa V^{1/d} E^{(d-1)/d}. \tag{7.1}$$

This provides an estimate for the behavior of the number of microstates of energy E supported in a volume V, and gives the correct asymptotic dependence for $E \to \infty$.

In order to excite massive states we can increase the temperature, thus taking into account heavier BPS states. The BPS index, being a signed sum over the states in the theory, cannot exceed the overall number of states. [2] Thus we come to the

[1] We thank T. Banks and S. Shenker for crucial remarks on this matter.

[2] In fact, the data for the Kronecker m-quiver suggest that in this case all the summands contributing to the BPS index have the same sign, so the BPS index actually counts the number of states up to an overall sign.

following chain of inequalities (here we take $d = 4$ and set $E = |Z_\gamma|$):

$$|\Omega(\gamma)| = \left|\mathrm{Tr}_{\mathfrak{h}^\gamma}(-1)^{2J_3}\right| \le \frac{1}{4}\,\mathrm{Tr}_{\mathcal{H}_{BPS,\gamma}}\,1 \le \frac{1}{4}\,\mathrm{Tr}_{\mathcal{H},E}\,1 = \frac{1}{4}e^{S(E)} \sim e^{\kappa V^{\frac{1}{4}}E^{\frac{3}{4}}}, \tag{7.2}$$

where the last estimate assumes large E. Thus the observed behavior $\log|\Omega(\gamma)| \sim E$ for large γ in the pure $SU(3)$ theory seems to give a contradiction.

The resolution of this paradox comes from taking into account the fact that our bound applies only to the theory in a finite volume. If the size of BPS states becomes large enough and they do not fit into the box of finite volume, then they do not contribute to the naïve counting of degrees of freedom. So we should instead consider a "truncated BPS index" $\check{\Omega}_V$, counting only the states which fit into a box of size V; we should expect this index to satisfy the inequality

$$|\check{\Omega}_V| = |\,\mathrm{Tr}_{\mathcal{H}_{BPS,M=|E|,R\le V^{\frac{1}{3}}}}(-1)^{2J_3}| \lesssim e^{\kappa V^{\frac{1}{4}}E^{\frac{3}{4}}} \tag{7.3}$$

with R the average size of a BPS state.

The rest of this section is devoted to arguing that the above scenario is indeed correct. We will use the semiclassical picture of BPS states given by the Denef equations, reviewed in Section 7.2, to give a lower bound for the average size of the semiclassical BPS states. The resolution of the paradox is spelled out in some more detail in Section 7.3. We give some supporting evidence for the validity of the use of the Denef equations for describing the exponentially large number of BPS states in Section 7.4.

7.2 Denef equations

In order to estimate the size of the BPS states arising in the theory, we refer to the interpretation [16] of those BPS states that arise from wall-crossing as multi-centered solutions similar to those arising in $\mathcal{N} = 2$ supergravity [15]. We assume Denef's multicentered picture has a good $\alpha' \to 0$ limit and can be applied to field theory. Suppose we have a set of elementary BPS states with charges $\{\gamma_A\}_{A=1}^n$ placed at corresponding points $\{r_A\}_{A=1}^n$ of $\mathbb{R}^3$. This configuration is again BPS only if the following set of equations is satisfied:

$$\sum_{\substack{B=1\\B\neq A}}^{n} \frac{\langle\gamma_A,\gamma_B\rangle}{|r_A - r_B|} = 2\mathrm{Im}(e^{-i\vartheta}Z_{\gamma_A}), \tag{7.4}$$

where $\vartheta = \arg \sum_{A=1}^{n} Z_{\gamma_A}$.

Now let us consider a BPS state of total charge $M\gamma_1 + N\gamma_2$, with $\langle \gamma_1, \gamma_2 \rangle = m$. Let us, *for the moment*, suppose that the dominant contribution to the entropy comes from a boundstate of M elementary constituents of charge γ_1 and N elementary constituents of charge γ_2.

In the case where the charges are of the form

$$\{\underbrace{\gamma_1, \dots, \gamma_1}_{M}, \underbrace{\gamma_2, \dots, \gamma_2}_{N}\} \tag{7.5}$$

the equations simplify to

$$\begin{aligned} \sum_{a=1}^{N} \frac{m}{r_{ia}} &= \kappa_1 := 2\mathrm{Im}(e^{-i\vartheta} Z_{\gamma_1}), \qquad 1 \le i \le M \\ \sum_{i=1}^{M} \frac{-m}{r_{ia}} &= \kappa_2 := 2\mathrm{Im}(e^{-i\vartheta} Z_{\gamma_2}), \qquad 1 \le a \le N \end{aligned} \tag{7.6}$$

We can view the index a as running over "electrons" and i over "magnetic monopoles," in an appropriate duality frame.

Now we are interested in the size of the boundstate. Therefore we consider the sum over the first equation in (7.6). (Doing the analogous sum over the second equation produces an equivalent result.) The result is that

$$\sum_{i,a} \frac{1}{r_{ia}} = \frac{NM}{|MZ_{\gamma_1} + NZ_{\gamma_2}|} \left(\frac{2\mathrm{Im}(\bar{Z}_{\gamma_2} Z_{\gamma_1})}{m} \right) \tag{7.7}$$

We can rewrite this equation nicely in terms of the *harmonic average* of the distances r_{ia}:

$$\langle r_{ia} \rangle_h = \left(\frac{m}{2\mathrm{Im}(\bar{Z}_{\gamma_2} Z_{\gamma_1})} \right) |MZ_{\gamma_1} + NZ_{\gamma_2}|. \tag{7.8}$$

On the other hand, we can use the well-known inequality that the harmonic average is a lower bound for the ordinary average, $\langle r_{ia} \rangle_h \le \langle r_{ia} \rangle$, to conclude that

$$\left(\frac{m}{2\mathrm{Im}(\bar{Z}_{\gamma_2} Z_{\gamma_1})} \right) |MZ_{\gamma_1} + NZ_{\gamma_2}| \le \langle r_{ia} \rangle. \tag{7.9}$$

Equation (7.9) is a key result. It shows that if N or M goes to infinity then the size of the average BPS molecule grows linearly with N or M, respectively.

We have shown that boundstates of total charge $M\gamma_1 + N\gamma_2$ with constituents (7.5) become large when $N, M \to \infty$. However, other partitions of N, M can and do lead to BPS boundstates. In general, given a pair of partitions

$$M = \sum_{j=1}^{M} l_j j, \quad N = \sum_{k=1}^{N} s_k k \tag{7.10}$$

there can be other boundstates where there are l_j centers of charge $j\gamma_1$ and s_k centers of charge $k\gamma_2$. In order to deal with these cases, let us introduce, for any set of charges $\{\gamma_A\}$, the moduli space $\mathcal{M}(\{\gamma_A\})$ of solutions to the Denef equations (7.4). If there are n centers it is a subspace of $\mathbb{R}^{3n}$. Note that the moduli space for charges

$$\{\underbrace{\gamma_1, \ldots, \gamma_1}_{l_1}, \underbrace{2\gamma_1, \ldots, 2\gamma_1}_{l_2}, \ldots, \underbrace{\gamma_2, \ldots, \gamma_2}_{s_1}, \underbrace{2\gamma_2, \ldots, 2\gamma_2}_{s_2}, \ldots\} \tag{7.11}$$

is in fact a subspace of the moduli space for (7.5), where certain collections of centers r_i and r_a have (separately) collided. Nevertheless, the identity (7.8) applies uniformly throughout the moduli space and hence applies to all possible partitions. As an extreme example, the moduli space $\mathcal{M}(\{M\gamma_1, N\gamma_2\}) \cong \mathbb{R}^3 \times S^2$, where the $\mathbb{R}^3$ is the center of mass and the S^2 has a radius

$$R_{12} = \left(\frac{m}{2\mathrm{Im}(\bar{Z}_{\gamma_2} Z_{\gamma_1})}\right) |M Z_{\gamma_1} + N Z_{\gamma_2}|. \tag{7.12}$$

In any case, we can conclude that for any partition of charges such as (7.11) the average BPS state has a size bounded below by a linear expression in N and M. We call these large semiclassical BPS states *BPS giants*.

7.3 Resolution and Revised Bound

The giant BPS states resolve the paradox explained in Section 7.1 above. Indeed we can adapt the bound (7.2) by adjusting the volume of the box V in such a way that states of mass E fit in a volume $V_E := R_E^3 := E^3$. From our estimate of the sizes of BPS molecules we know that the average size indeed scales linearly with E. Therefore the new bound is

$$\log |\Omega(E)| \sim \alpha E \lesssim \kappa E^{3/4} V_E^{1/4} \sim \kappa' E^{3/2} \tag{7.13}$$

and is indeed satisfied.

In equation (7.13) κ' is a dimensionful constant, it scales as $\kappa' \sim (\text{length})^{\frac{3}{2}}$. Let us give a physical interpretation for this scale. If we consider a sequence of charges $N(a\gamma_1 + b\gamma_2)$, with $N \to \infty$ and $\gamma_p := a\gamma_1 + b\gamma_2$ primitive, then the size of an average BPS molecule behaves as $R \sim r_0 N$, where r_0 is the size of a state with charge γ_p. The energy behaves as $E = |Z_0|N$, where Z_0 is a central charge of the state with charge γ_p. Thus we can give a formula accounting for the scaling dimension of κ' in (7.13) by using

$$V_E = R_E^3 \sim (r_0 N)^3 \sim (r_0 E/|Z_0|)^3 \tag{7.14}$$

to deduce

$$\begin{aligned} E^{3/4} V_E^{1/4} &\sim \left(\frac{r_0}{|Z_0|}\right)^{3/4} E^{3/2}, \\ \Rightarrow \quad \kappa' &\sim \left(\frac{r_0}{|Z_0|}\right)^{3/4}. \end{aligned} \tag{7.15}$$

We remark that the length scale $(r_0/|Z_0|)^{1/2}$ is a function of the moduli, since both r_0 and Z_0 are functions of the moduli.

7.4 Discussion of validity of the semiclassical picture

In this section we will address the question of how reliable the semiclassical approximation is. We will review some supporting evidence for the reliability of the semiclassical pictures based on the relation of an exact result for BPS degeneracies Ω to certain symplectic volumes.

Let us recall the symplectic structure on Denef moduli space $\mathcal{M}(\{\gamma_A\})$. Overall translation acts on this space and the reduced space $\overline{\mathcal{M}}(\{\gamma_A\}) = \mathcal{M}(\{\gamma_A\})/\mathbb{R}^3$ is generically $2n-2$ dimensional. Moreover, the reduced space admits a symplectic form [10]:

$$\omega = \frac{1}{4}\sum_{i<j} \langle \gamma_i, \gamma_j \rangle \frac{\epsilon_{abc} dr_{ij}^a \wedge dr_{ij}^b r_{ij}^c}{r_{ij}^3}. \tag{7.16}$$

In the semiclassical approximation we identify a subspace of the space of BPS states with a set of BPS field configurations. We expect that the dimension of a subspace corresponding to a charge decomposition can be estimated, in the semiclassical approximation, by the symplectic volume

$$\mathrm{Vol}(\{\gamma_A\}) := \frac{1}{(n-1)!}\int_{\overline{\mathcal{M}}}\left(\frac{\omega}{2\pi}\right)^{n-1}. \tag{7.17}$$

where n is the number of centers.

Now, thanks to a result of Manschot, Pioline, and Sen [44, 43], in the example of the m-Kronecker quiver the protected spin character in the wild chamber can in fact be expressed exactly as a sum over two partitions (7.10) so that

$$\Omega(M\gamma_1+N\gamma_2;y) = \\ = \sum_{\{l_j\},\{s_k\}} g_{\mathrm{ref}}(\{l_j\},\{s_k\};y)\prod_{j,k}\frac{1}{l_j!j^{l_j}s_k!k^{s_k}}\left(\frac{y-y^{-1}}{y^j-y^{-j}}\right)^{l_j}\left(\frac{y-y^{-1}}{y^k-y^{-k}}\right)^{s_k} \tag{7.18}$$

where g_{ref} refers to an equivariant Dirac index on the space of solutions to Denef's equations with distinguishable centers described by charge partitions $\{l_j\}$, $\{s_k\}$. If we specialize to the index at $y=1$ [3] then g_{ref} has a very nice interpretation as the symplectic volume (7.17) of the moduli space of solutions to Denef's equations (up to a sign):

$$\Omega(M\gamma_1+N\gamma_2) = \sum_{\{l_j\},\{s_k\}}(-1)^{mMN+1-\sum_j l_j-\sum_k s_k}\mathrm{Vol}(\{l_j\},\{s_k\})\prod_{j,k}\frac{1}{l_j!j^{2l_j}s_k!k^{2s_k}}. \tag{7.19}$$

where $\mathrm{Vol}(\{l_j\},\{s_k\})$ is (7.17) for the charges (7.11).

We will take this relation of the exact number of BPS states to symplectic volumes as sufficient evidence for the validity of our resolution. There are, however, some further interesting aspects of this formula which we will comment on in the following Sections 7.4 and 7.4 below.

A toy example: the Hall halo

A very nice exactly solvable example of BPS configurations is provided by the Hall halo of [15]. Consider a configuration of N electric particles and a single magnetic monopole of charge m. This corresponds to the case $(M,N)=(1,N)$ in the notation

[3] In the conventions of [44] we take $y\to 1$ rather than $y\to -1$ to get the index.

above. In this case the equations (7.6) imply that the N electric particles all lie on a single sphere centered on the magnetic particle and of radius:

$$R_{12} = \left(\frac{m}{2\mathrm{Im}(\bar{Z}_{\gamma_2} Z_{\gamma_1})} \right) |Z_{\gamma_1} + N Z_{\gamma_2}|. \tag{7.20}$$

Now, in this case Denef argued that to get the spin character we can just apply the usual quantum mechanics of Landau levels on a sphere with a magnetic monopole inside. Counting the ground states gives the corresponding protected spin character [54]

$$\Omega(y) = (-y)^{-(m-N)N} \frac{\prod_{j=1}^{m} (1 - y^{2j})}{\prod_{j=1}^{N} (1 - y^{2j}) \prod_{j=1}^{m-N} (1 - y^{2j})}, \tag{7.21}$$

in perfect agreement with Reineke's general formula (see eq. (5.3) of [15]).

There are two interesting lessons we can draw from (7.21):

1. First, naive physical intuition suggests that the large size of BPS states is due to large angular momentum. This example shows that in fact this is not necessarily the case. In this case the size of the configuration is given by formula (7.20). Nevertheless this configuration contains representations of many different spins.

2. Second, we can derive the number of states in a multiplet by taking $y \to -1$. Then $\Omega = \frac{m!}{N!(m-N)!}$. In the limit $N \ll m$ the number of allowed states is much greater than the number of populated states, so quantum statistics does not play an important role, and the semiclassical approximation should work. Indeed,

$$\Omega = \frac{m!}{N!(m-N)!} \underset{N \ll m}{\sim} \frac{m^N}{N!} + \cdots \tag{7.22}$$

   This confirms the semiclassical expectation that the number of states should be counted by the symplectic volume since the volume is proportional to m^N. Note however that, for fixed N the binomial coefficient is really a polynomial in m and (7.22) is only the leading term at large m. Since $1/m$ plays the role of $\hbar$ we can interpret the subleading terms as quantum corrections to the naive semiclassical reasoning.

Estimating the contribution of the maximal partition

Let us consider the contribution to the BPS degeneracy of the maximal partition (7.5) in the formula (7.19). The symplectic volume for this partition is

$$\mathrm{Vol}((N,M),\kappa_1,\kappa_2,m) := \frac{1}{(N+M-1)!}\int_{\overline{\mathcal{M}}}\left(\frac{\omega}{2\pi}\right)^{N+M-1} \tag{7.23}$$

where we used the fact that there are $n = N+M$ centers. We would like to estimate this volume when N, M become large.

Rescaling both $\kappa_{1,2}$ in (7.6) by $\lambda \in \mathbb{R}$ together with $r_{ij} \mapsto r_{ij}\lambda^{-1}$ shows that solutions for rescaled values of $\kappa_{1,2}$ are obtained by simply rescaling the distances. Therefore the ratio

$$H((N,M),\kappa_1/\kappa_2) := \mathrm{Vol}((N,M),\kappa_1,\kappa_2,m)/m^{N+M-1}$$

only depends on the ratio κ_1/κ_2 and on N, M. For simplicity, let us specialize to $M = N-1$. in the limit $N \to \infty$ we have

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\log\left(\mathrm{Vol}((N-1,N),\kappa_1,\kappa_2,m)\right)\\ &\sim \log m^2 + F\left(\frac{\kappa_1}{\kappa_2}\right).\end{aligned}$$

Note that the second piece is independent of m.

There are two important lessons we can draw from this computation:

1. This behavior nicely coincides with the Weist coefficient, but only in the large m limit when:

   $$C_{1,1}(m) \sim \log m^2 + \mathcal{O}(m^{-1}) \tag{7.24}$$

   The fact that we must take $m \to \infty$ is not terribly surprising in view of the Hall halo example discussed above.

2. It is interesting to note that at finite values of m the maximal partition does *not* fully account for the exponential growth coefficient, even in the large charge regime. Indeed, as pointed out in [56] we should take into account many other partitions to derive even the leading asymptotic behavior of the BPS index. One important (and subtle) aspect of (7.19) is that the different symplectic volumes are weighted with *signs*. This might imply some subtlety in applying

the semiclassical pictures we have used, and should be understood better. In the meantime, as we discuss further in Remark 5 of Section 8: in the formula (7.19), considering the case where the BPS ray lies in the dense cone, there can be striking cancelations between volumes of different partitions.

Chapter Eight: Open Problems

In conclusion we would like to mention a few open problems and questions raised by the current work.

1. It is natural to guess that wild degeneracies will be a common feature among higher rank theories of class S. Strictly speaking, the only examples we have given are for gauge group $SU(3)$, but we fully expect that the phenomenon will persist for $SU(K)$ with $K > 3$. This is strongly suggested by the quiver analysis of Section 6, but a fully rigorous proof would require that one demonstrate that the path exhibited in the moduli space of stability parameters of the the Fiol quiver, which leads to wild wall crossing for $K > 3$, actually can be chosen in the moduli space of physical stability parameters. (While not fully mathematically rigorous, a compelling physical argument that this is indeed the case is that we could consider a hierarchy of symmetry breaking where $SU(K)$ is much more strongly broken to $SU(3) \times U(1)^{K-3}$ than the $SU(3)$ is broken to $U(1)^2$.)

2. Another open problem along similar lines is how the presence of, say, matter multiplets affects the existence of wild degeneracies.

3. It should be noted that the explicit point on the Coulomb branch illustrated in Figure 3.5 is in fact different from the region explored in Section 4.2. Nevertheless, using the techniques of Appendix B we have checked that the same crucial algebraic equation (3.2) governing the street factors of herds indeed appears in the spectral networks that arise in this region. These networks are very similar to but not quite the same as the m-herds. One might ask for a succinct test to see whether a degenerate spectral network leads to m-wild degeneracies.

4. It would be nice to understand better the physics of the curious invariance of the BPS degeneracies under the transformation by the g_m matrix discussed in Section 6.2 above. To the extent that the relation to quivers is physical, a physical understanding is indeed provided by the arguments in Section 6. However, we would like to suggest an alternative interpretation using the halo picture of BPS states. If we consider a core particle γ with halo particles of charge

γ' then the replacement of the hypermultiplet of charge γ for the hypermultiplet $\gamma + m\gamma'$ is simply flipping the Fermi sea of the halo Fock space. (See, e.g. Section 3.5 of [7] for a similar transformation.) Perhaps then a physical derivation of the symmetry could proceed by using Fermi flips to establish such a symmetry for framed BPS states and then using recursion relations between framed and unframed BPS states to deduce it for general degeneracies. This symmetry also raises the interesting possibility that the mutation method for determining BPS degeneracies can be extended to higher spin states.

5. The g_m symmetry of Kronecker quivers makes a surprising prediction about two well-known formulae: Reineke's formula for Poincaré polynomials of quiver varieties [49], and the Manschot-Pioline-Sen wall-crossing formula [44, 43]. These formulae involve sums over certain partitions. For certain charges, there is rather extensive cancelation between terms in these formulae implied by the g_m symmetry of the BPS degeneracies. Since the individual terms in the sum in the MPS formulae have a simple geometrical interpretation [43] the g_m symmetry together with the MPS formula imply nontrivial identities on equivariant Dirac indices. For a simple and dramatic example we can choose $m = 3$ and note that that $(1, 1)$ has a very simple PSC, but

$$(g_3)^k \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_{2k-1} \\ F_{2k+1} \end{pmatrix} \tag{8.1}$$

(where F_n is the n^{th} Fibonacci number) involves arbitrarily large charges. Clearly there are many terms in the MPS formula (7.19) and, as we just said, their coefficients have a beautiful geometrical interpretation as equivariant indices of Dirac operators on the Denef moduli spaces. So the identity[1]

$$\Omega((F_{2k-1}, F_{2k+1}); y) = \Omega((1, 1); y) = [3]_y \tag{8.2}$$

is a very remarkable set of identities for these indices. It would be interesting to understand better these identities (and their analogues for $m > 3$) from a geometrical point of view.

6. Returning to the key algebraic equation (3.2), a natural question is whether there is a physical interpretation of the other roots of this equation. We expect

[1]We use the notation $[n]_y := \frac{y^n - y^{-n}}{y - y^{-1}}$

that there will be. For example, choose a small path $\wp$ crossing a c-street in an m-herd. The corresponding supersymmetric interface has a vev when wrapped on the circle in $\mathbb{R}^3 \times S^1$ given by

$$\langle L_\zeta(\wp)\rangle_m = \begin{pmatrix} q(m,\zeta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q(m,\zeta) \end{pmatrix} \tag{8.3}$$

where m is a point in Hitchin moduli space $\mathcal{M}$, $\zeta \in \mathbb{C}^*$ has phase $\arg\zeta = \arg Z(\gamma+\gamma')$, and $q(m,\zeta) = Q(c)|_{X_{\gamma_c}\to\mathcal{Y}_{\gamma_c}}$, where $\mathcal{Y}_{\gamma_c}$ is a function on the twistor space of the Hitchin moduli space $\mathcal{M}$ constructed in [25, 27]. It therefore makes sense to ask about the physical meaning of the analytic behavior of $\langle L_\zeta(\wp)\rangle$, and this might well involve the other roots of (3.2). Exploring this point further is beyond the scope of this thesis.

7. A closely related point to the previous one is that the exponential growth of Ω for certain charges implies a similar exponential growth for μ and therefore for $\overline{\underline{\Omega}}$. We expect this will have important implications for the construction of hyperkahler metrics of associated Hitchin systems proposed in [25] and for the definition of the nonabelianization map of [27, 28]. Again, we leave this important point for future work.

Part II

Algebraicity and Asymptotics

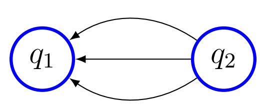
Chapter Nine: Main Results Part I: A short-lived exercise in succinctness

In this part of the thesis, we concern ourselves with some results relating to a property we will call *algebraicity*: suitably defined generating series of Donaldson-Thomas Invariants, BPS-indices, or even just plain Euler-characteristics of (stable) quiver moduli– a priori defined as formal series– are secretly algebraic functions over $\mathbb{Q}$. Kontsevich and Soibelman have an understanding of algebraicity in the context of DT-invariants [38] associated to a large class of 3CY categories; however they claim that their proof uses rather indirect methods: explicit algebraic relations are not produced.

On the other hand, in a large class of supersymmetric field theories called *theories of class* $S[A]$, there exists algorithmic machinery for computing generating series of BPS indices: *spectral networks*. Using this machinery, algebraicity can be seen directly, and one can algorithmically construct explicit algebraic relations. Roughly speaking, a spectral network is a directed, decorated graph associated a $\mathbb{Z}$-family of BPS states; algebraic relations for generating series of the associated BPS-indices follows by a system of algebraic relations determined by the edges and vertices of this graph.

The main results of this part are threefold:

(A): Algebraic equations that produce DT-invariants for the 3-Kronecker quiver:



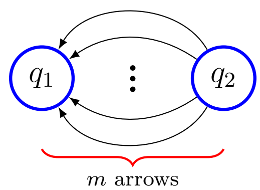
associated to the family of collinear dimension vectors $(3nq_1 + 2nq_2)_{n=1}^{\infty}$;

(B): As a corollary of (A) and a functional equation due to Reineke: an algebraic equation determining associated Euler-characteristics of stable quiver moduli;

(C): Explicit constraints on the asymptotics of BPS-indices/DT-invariants (or m-Kronecker Euler characteristics) due to algebraicity: "*Either there are finitely* *many BPS indices or exponentially many*". Formulae are produced that allow for the extraction of asymptotics from algebraic equations; these formulae are applied to the algebraic equations of (A) and (B).

9.1 Main Results Part II: The Return of the Main Results

The main examples of this thesis center around spectral networks that compute DT-invariants associated to representations of the m-Kronecker quiver K_m shown below.



Specifically, let $d(a, b, m)$ denote the DT-invariants $\Omega^{\mathrm{DT}}(aq_1 + bq_2; Z)$ defined with respect to a stability condition $Z : \mathbb{Z}[q_1, q_2] \to \mathbb{C}$ such that $\arg(Z(q_1)) < \arg(Z(q_2))$. Let us begin by recalling some results about the "slope-1" invariants $d(n, n, m)$: If we encode these invariants via the formal series

$$\mathsf{T}_1 := \prod_{n=1}^{\infty} (1 - (-1)^n z^n)^{nd(n,n,m)}$$

then there is an algebraic relation satisfied by T_1; specifically, $\mathsf{T}_1 = P^m$ where P is the unique solution to the algebraic relation

$$0 := P - zP^{(m-1)^2} - 1. \tag{9.1}$$

which is a formal series in z with constant coefficient 1. The algebraic relation (9.1) was first postulated in the mathematics literature by Kontsevich and Soibelman. Later it was proven by Reineke using a functional equation that relates DT-invariants for the m-Kronecker and Euler-characteristics of stable moduli: using known results about Euler characteristics of the moduli space of m-Kronecekr stable representations for dimension vectors $n(q_1 + q_2)$, the functional equation is equivalent to the algebraic relation (3.2).

On the other hand, in Part I, the algebraic relation (9.1) was independently produced by utilizing spectral networks: specifically, a family of spectral networks called m-herds. Moreover, using BPS quiver techniques, it was argued that these spectral networks are, indeed, computing the DT-invariants associated to the m-Kronecker quiver.

In this part of the thesis our main example is concerned with the 3-Kronecker DT-invariants $d(3n, 2n, 3)$; by encoding them into the generating series

$$\mathsf{T}_{3/2} = \prod_{n=1}^{\infty} (1 - (-1)^n z^n)^{nd(3n,2n,3)} \in \mathbb{Z}[\![z]\!]$$

we show, using a spectral network we call the $(3, 2|3)$-herd – a generalization of the m-herd family– that $\mathsf{T}_{3/2}$ is an algebraic function over $\mathbb{Q}$. Specifically, $\mathsf{T}_{3/2} = MVW$ for three formal series $M, V, W \in \mathbb{Z}[\![z]\!]$ that satisfy the algebraic relations

$$\begin{aligned} M &= 1 + zM^4 \left\{ (1+V)(1+V-W)^2 [V^2(1+W) - 1]^3 \right\}, \\ 0 &= (-1+V)(1+V)^2 + (1+V^3)W - V(M+V)W^2, \\ 0 &= V\left(V^2 - 1\right) - \left[M(V+1) + V(V-2) - 1\right] W. \end{aligned} \tag{9.2}$$

With a bit of elimination theory, one can, moreover determine a 39th degree polynomial $\mathcal{F}_{3/2} \in (\mathbb{Z}[z])[t]$ (see (12.9)), that is irreducible as a polynomial in $\mathbb{C}[z, t]$, and has T as the unique root with series representation of the form $1 + \sum_{n=1}^{\infty} t_n z^n$.

9.2 Organization of the Second Part

The second part of this thesis is organized in the manner listed below.

1. 11: A brief review of some relevant results in Part I.

2. 12 Continuation of the work in Part I to generalizations of the m-herd networks. This section culminates in the statement of the result 9.2 (derived in Appendix H), the algebraic equation (12.9) satisfied by $\mathsf{T}_{3/2}$, and the resulting analysis of the corresponding BPS-index asymptotics.

3. 13: A review Reineke's functional equation that relates DT-invariants and Euler characteristics of stable moduli for the m-Kronecker quiver.

4. 14: An in-depth discussion of how spectral networks lead to algebraicity. The precise statement of algebraicity is included in Claim 14.0.4 and Corollary 14.0.5.

5. 15 The derivation of how algebraicity of generating series lead to exponential asymptotics.

When reading this part of the thesis, the following picture of interrelationships between BPS-indices, DT-invariants, and Euler-characteristics may be helpful. Corresponding references for these relationships can be found throughout the main body of the thesis. Dotted arrows indicate partially-defined correspondences (with the conditions defining the domain of definition indicated); the squigglyness of an arrow is negatively correlated with the author's (subjective and biased) notion of rigour. Reineke's arrow is magenta coloured to stress that the relationship is not a simple equality of invariants, but a relationship between generating functions.

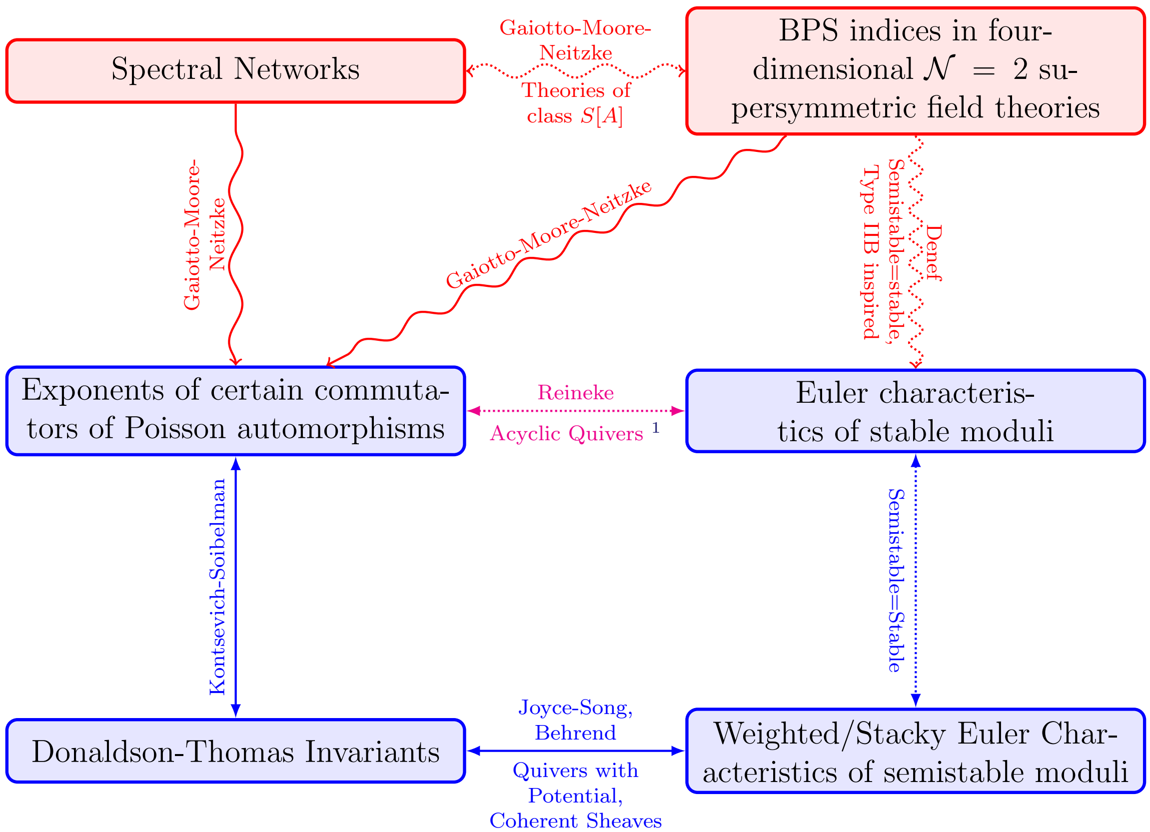


Figure 9.1: Interrelationships between BPS-indices, DT-invariants, and Euler-characteristics of Stable Moduli.

Chapter Ten: Global Notation

If $F \in \mathbb{C}[\![z]\!]$ is a formal series in the variable z

$$F = \sum_{n=0}^{\infty} f_n z^n$$

we will frequently use the notation

$$[z^n]F := f_n.$$

The term *generating series* G for a sequence of rational numbers $(\beta_n)_{n=1}^{\infty} \subset \mathbb{Q}$ (which are expected to be integers in all of our examples) will mean a formal series

$$\mathsf{G} = 1 + \sum_{n=1}^{\infty} g_n x^n \in \mathbb{Q}[\![x]\!]$$

such that the sequence β_n appears in the "Euler-product factorization" of G:

$$\mathsf{G} = \prod_{n=1}^{\infty} (1 - x^n)^{n\beta_n}. \tag{10.1}$$

For any formal series with constant coefficient 1, such a factorization exists and is unique. Indeed, the β_n can be extracted from G by taking the logarithm of both sides of (10.1) and applying Möbius inversion:

$$\beta_n = -\frac{1}{n^2} \sum_{k|n} k\mu\left(\frac{n}{k}\right)\left([x^k]\log(\mathsf{G})\right), \tag{10.2}$$

where $\mu : \mathbb{Z}_{>0} \to \{0, 1, -1\}$ is the Möbius-mu function, $\log(\mathsf{G}) \in \mathbb{Q}[\![x]\!]$ is defined by composing the Taylor series expansion of $\log(1 + x)$ around $x = 0$ with $\mathsf{G} - 1$, and $[x^k]\log(\mathsf{G})$ is the coefficient of x^k in the formal series $\log(\mathsf{G})$; in terms of derivatives,

$$[x^d]\log(\mathsf{G}) = \frac{1}{d!}\left[\frac{d^n}{dx^d}\log(\mathsf{G})\right]_{x=0}.$$

Note that *any* formal series in $\mathbb{Q}[\![z]\!]$ with constant coefficient 1 is a generating series for a sequence of rational numbers defined by (10.2).

Remark In §15.2 we will also allow generating functions $\mathsf{G} \in \overline{\mathbb{Q}}[\![x]\!]$ of algebraic numbers $(\beta_n)_{n=1}^{\infty} \subset \overline{\mathbb{Q}}$.

In practice the generating functions of interest will be series in a distinguished variable we will denote by z. Two important generating functions will be denoted by symbols (in sans-serif font) that we make an effort to protect.

- The generating series for DT-invariants or BPS indices will be denoted by T; it will be a generating series in the variable $x = \widetilde{z} = \pm z$ (where the sign depends on the context).
- The generating series for Euler characteristics of stable quiver moduli will be denoted by E; it will be a generating series in the variable $x = z$.

Other formal series in z, that are closely related to generating functions, will also appear throughout with protected symbols:

- the symbol P will denote the formal series that controls various sorts of generating functions (street-factors, soliton-generating functions, and the generating series for the corresponding BPS indices) related to m-herds;
- the symbols M, V, and W will denote the formal series that control various sorts of generating functions related to $(3, 2|3)$-herds (introduced below).

The m-Kronecker quiver will be denoted by the quiver with vertices q_1 and q_2 and m arrows from q_2 to q_1. In its physical embodiment, the m-Kronecker quiver will arise as a sub-quiver of the BPS quiver, and the vertices q_1, q_2 will be identified with distinguished "charges" γ_1 and γ_2 respectively. DT-invariants associated to the m-Kronecker quiver (equipped with a non-trivial stability condition) with dimension vector $(a, b) \in \mathbb{Z}^2$ will be denoted in two ways:

1. $d(a, b, m)$ if we are referring to results purely from the mathematical literature.
2. $\Omega\,(a\gamma_1 + b\gamma_2)$ if we are interpreting the DT-invariants as BPS indices in some four-dimensional $\mathcal{N} = 2$ field theory. The parameter m will be clear from context.

Chapter Eleven: A Brief Review of m-Herds

We first roughly sketch the wall-crossing motivation behind the definition of an m-herd, as precisely defined in Part I. For the specifics of spectral networks the reader is referred to [27, 28].

11.1 Setting, Terminology, and Conventions

We begin our journey in the realm of theories of class $S[A_{K-1}]$ for $K \geq 3$. For concreteness we take our canonical example to be pure $SU(3)$ SYM which is given by the theory $S[A_2, C, D]$ where $C = \mathbb{CP}^1$, and D is a specific pair of defect operators located at 0 and ∞ (providing irregular punctures). The discussion in this thesis does not refer to the details of the punctures so nothing is lost by imagining C as the cylinder $S^1 \times \mathbb{R}$. We now recall some essential notation and terminology.

Definitions

1. The *Coulomb branch* $\mathcal{B}$ of $S[A_{K-1}, C, D]$ is the set of tuples $(\phi_2, \cdots, \phi_K)$ of holomorphic r-differentials ϕ_r with singularities at $\mathfrak{s}_1, \cdots, \mathfrak{s}_n \in C$ prescribed by the defect operators D.

2. Let $u = (\phi_2, \cdots, \phi_r) \in \mathcal{B}$ and denote the holomorphic cotangent bundle of C as $\mathcal{T}^*C$. Then the *spectral cover* (a.k.a Seiberg-Witten curve) is a K-sheeted branched cover $\pi_u : \Sigma_u \to C$, where Σ_u is the subvariety

$$\Sigma_u := \{\lambda \in \mathcal{T}^*C : \lambda^K + \sum_{r=2}^{K} \phi_r \lambda^{K-r} = 0\} \subset \mathcal{T}^*C, \tag{11.1}$$

and the projection π_u is the restriction of the standard projection $\mathcal{T}^*C \to C$.

3. On $\mathcal{B}$ there may be (complex) codimension-1 loci where a cycle of Σ_u degenerates. Let $\mathcal{B}^* = \mathcal{B} - \{\text{degeneration loci}\}$. Then $\widehat{\Gamma} \to \mathcal{B}^*$ is the local system of charge lattices. In the theories of type A_{K-1}, the fibre $\widehat{\Gamma}_u$ is a sublattice of $H_1(\Sigma_u; \mathbb{Z})$, but for our purposes, nothing will be lost if we take $\widehat{\Gamma}_u$ to be the full

lattice:

$$\widehat{\Gamma}_u := H_1(\Sigma_u; \mathbb{Z}).$$

For each $u \in \mathcal{B}^*$, $\widehat{\Gamma}_u$ is equipped with the skew-symmetric pairing on homology: $\langle \cdot, \cdot \rangle_u : \widehat{\Gamma}_u^{\otimes 2} \to \mathbb{Z}$.

4. $Z \in \widehat{\Gamma}^* \otimes_{\mathbb{Z}} \mathbb{C}$ is the central charge function.[1]

Notation Unless otherwise noted, from this point on we will work over fixed $u \in \mathcal{B}^*$ and use the streamlined notation $\Sigma := \Sigma_u$ and $\Gamma := \widehat{\Gamma}_u$.

Definition

1. Denoting the unit tangent bundle to Σ via $UT\Sigma$, and defining H to be the homology class represented by a 1-chain that wraps once around some fibre of $UT\Sigma \to \Sigma$, then

$$\widetilde{\Gamma} := H_1(UT\Sigma; \mathbb{Z})/(2H);$$

   it is a $\mathbb{Z}/2\mathbb{Z}$-extension of the charge lattice Γ. We abuse notation and denote the image of H in the quotient by H again.

2. $\widetilde{(\cdot)} : \Gamma \to \widetilde{\Gamma}$, taking γ to $\widetilde{\gamma}$, is the *standard lift* defined in App. D.

3. Let L be a finitely generated monoid, then $\mathbb{Z}[\![L]\!]$ is the ring whose underlying set is given by formal expressions of the form

$$\sum_{\alpha \in L} c_\alpha X_\alpha$$

   such that

   a) $c_\alpha \in \mathbb{Z}$, $\forall \alpha \in L$;

   b) $\{\alpha \in \Gamma : c_\alpha \neq 0\}$ is well-ordered[2] with respect to a total order such that for any $l \in L$ we have $l < nl$ if $n > 1$. In particular, if L is a

[1] It is given fibrewise via period integrals on Σ_u: $Z_u : \gamma \mapsto \int_\gamma \lambda_u$, where λ_u is the pullback of the tautological 1-form on $\mathcal{T}^*C$ to Σ_u (λ_u is the Seiberg-Witten differential).

[2] This condition ensures there is a least-element among the c_α: a necessary condition in order to define the product of two such formal series. This is similar to the reason why it only makes sense to take products of Laurent series (series infinite in only one direction) as opposed to series that are infinite in "two directions".

finitely generated submonoid of $\widetilde{\Gamma}$ and we are equipped with a central charge function $Z : \Gamma \to \mathbb{C}$, we have in mind a total order induced by (a possibly small perturbation of) the mass function

$$L \hookrightarrow \widetilde{\Gamma} \overset{\text{mod } H}{\longrightarrow} \Gamma \overset{|Z|}{\to} \mathbb{R}_{\geq 0}.$$

The ring structure is given (with identity $1 = X_0$) by the sum and product structure:

$$\sum_{\alpha \in L} c_\alpha X_\alpha + \sum_{\beta \in L} d_\beta X_\beta = \sum_{\alpha \in L} (c_\alpha + d_\alpha) X_\alpha;$$
$$\left(\sum_{\alpha \in L} c_\alpha X_\alpha\right) \left(\sum_{\beta \in L} d_\beta X_\beta\right) = \sum_{\alpha+\beta=\delta} c_\alpha d_\beta X_\delta.$$

$\mathbb{Z}[\![L]\!]$ contains the group-ring $\mathbb{Z}[L]$ as all such formal series with only finitely many non-vanishing coefficients.

Remark 11.1.1. *We will mostly be concerned with $\mathbb{Z}[\![L]\!]$ when $L \leq \widetilde{\Gamma}$ is a one-dimensional sub-lattice. In such a situation, then L has two possible generators (differing by a sign) and $\mathbb{Z}[\![L]\!]$ has two corresponding identifications, each corresponding to a choice of generator, with formal series in a single variable.*

Terminology We will refer to WKB spectral networks as $\mathcal{W}$-networks. When referring to the WKB network at a particular choice of $u \in \mathcal{B}^*$ and $\vartheta \in S^1$, we call the network $\mathcal{W}_\vartheta$ (the point $u \in \mathcal{B}^*$ will be clear from context). Abstract spectral networks (which do not necessarily have any WKB realization) will be referred to as either "spectral networks" or simply "networks".

Definition

1. Recall to each street p of a spectral network, we associate a formal series $Q(p) \in \mathbb{Z}[\![\widetilde{\Gamma}]\!]$ called a *street-factor*. The constant term (the coefficient in front of $X_0 = 1$) of a street-factor is always 1.

2. A street p is *two-way* if $Q(p) \neq 1$. A spectral network is *degenerate* if it contains a two-way street.

3. The collection of two-way streets of a spectral network is its *degenerate skeleton*.

Notes

- The spectral networks of interest in this thesis hypothetically correspond to BPS states and, hence, are all degenerate. Furthermore, all networks will have the property that (up to a choice of sign) there exists a unique primitive $\gamma_c \in \Gamma$ such that every street-factor $Q(p)$ is an element of the subring $\mathbb{Z}[\![\widetilde{\Gamma}_c]\!] \leq \mathbb{Z}[\![\widetilde{\Gamma}]\!]$ generated by $\widetilde{\Gamma}_c := \mathbb{Z}\widetilde{\gamma}_c$. Off of walls of marginal stability in $\mathcal{B}$, every degenerate $\mathcal{W}$-network satisfies this property.

- Only three sheets of Σ will be relevant for any network that we will consider; following the conventions of Part I, in some local coordinate chart we will trivialize the spectral cover and denote two-way streets of type 12 by red lines, streets of type 13 by blue lines, and streets of type 13 by fuchsia lines.

- Some network diagrams will be drawn with only the two-way streets. Following the argument in App. B.3, one may decorate any such network with an arbitrarily complicated "background" of one-way streets without modifying any computations relevant to the generating functions attached to the two-way streets. This is a particularly convenient observation as any such degenerate network that happens to be realized as a $\mathcal{W}$-network will usually have such complicated backgrounds.

- Some diagrams in this part of the thesis that also appear in Part I are mirror images of their original versions in Part I (e.g. red streets run northwest instead of northeast and blue streets run northeast instead of northwest). As a result, the statements about m-herds are can be superficially translated via the map Part I $\rightarrow$ "this part of the thesis" given by $\gamma' \mapsto \gamma_1$ and $\gamma \mapsto \gamma_2$.

Definition

1. We will say a formal series $\mathsf{T} \in \mathbb{Z}[\![\widetilde{\Gamma}_c]\!] \cong \mathbb{Z}[\![X_{\widetilde{\gamma}_c}]\!]$ *generates* $\Omega(n\gamma_c)$ (alternatively: T is the *generating series for* $\{\Omega(n\gamma_c)\}_{n=1}^{\infty}$) if

$$\mathsf{T} = \prod_{n=1}^{\infty} (1 - X_{n\widetilde{\gamma}_c})^{n\Omega(n\gamma_c)}. \tag{11.2}$$

For notational convenience (when $\widetilde{\gamma}_c$ is known from context), it will be useful to define the variable

$$\widetilde{z} := X_{\widetilde{\gamma}_c}; \tag{11.3}$$

so that we may simply write,

$$\mathsf{T} = \prod_{n=1}^{\infty} (1 - \widetilde{z}^n)^{n\Omega(n\gamma_c)}.$$

2. If $u \in \mathcal{B}$ is an m-wild point, and $\gamma_1,\, \gamma_2 \in \widehat{\Gamma}_u$ are the[3] two hypermultiplet charges such that $\langle \gamma_1, \gamma_2 \rangle = m \in \mathbb{Z}_{>0}$, then the generating series for the BPS indices $(\Omega\,[n\,(a\gamma_1 + b\gamma_2)])_{n=1}^{\infty}$ will be denoted by $\mathsf{T}_{a/b}$.

Note that by taking the logarithm of both sides of (11.2), applying Möbius inversion, and (for simplicity of notation) defining $\widetilde{z} := X_{\widetilde{\gamma}_c}$, one finds:

$$\Omega(n\gamma_c) = -\frac{1}{n^2} \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{1}{(d-1)!} \left[\frac{d^n}{d\widetilde{z}^n} \log\left(\mathsf{T}\right)\right]_{\widetilde{z}=0}, \tag{11.4}$$

where $\mu : \mathbb{Z}_{>0} \to \{0, 1, -1\}$ is the Möbius-mu function.

Remark 11.1.2. *When the BPS indices $\Omega(n\gamma_c)$ are computed at an m-wild point, and are expected to coincide with DT-invariants associated to the m-Kronecker quiver, it is useful to compare our definition of generating series in (11.2) to the generating functions appearing in the work of Reineke [47]. Indeed, if $\gamma_c = a\gamma_1 + b\gamma_2$ as argued in Appendix I, it is natural to define the variable*

$$z := (-1)^{mab+a+b} X_{\widetilde{a\gamma_1 + b\gamma_2}}. \tag{11.5}$$

Thus,

$$\begin{aligned} \mathsf{T}_{a/b} &= \prod_{n=1}^{\infty} (1 - (-1)^{n(mab+a+b)} z^n)^{n\Omega(n\gamma_c)} \in \mathbb{Z}[\![z]\!] \\ &= \prod_{n=1}^{\infty} (1 - \left((-1)^N z\right)^n)^{n\Omega(n\gamma_c)}, \end{aligned}$$

where $N := mab - a^2 - b^2$. It follows that $\mathsf{T}_{a/b}$ coincides precisely with the generating series "$G_\mu(t)$" of [47, §6] with $\mu = a/b$ and $t = z$.

[3]The use of "the" here may suggest that there is only one m-Kronecker BPS subquiver, with non-trivial stability condition, at $u \in \mathcal{B}$. This may not be the case, but implicitly we will always restrict our attention to a single m-Kronecker BPS subquiver (corresponding to a particular choice of γ_1 and γ_2).

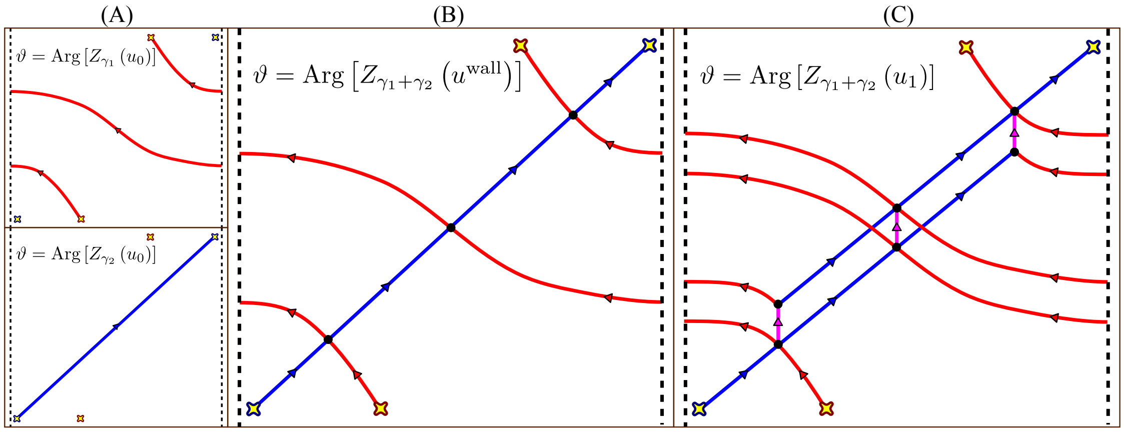


Figure 11.1: Snapshots of the family of hypothetical $\mathcal{W}$-networks described in §11.2 that depict a wall-crossing of two hypermultiplets with charges γ_1 and γ_2 such that $\langle\gamma_1, \gamma_2\rangle = 3$. Streets of type 12 are shown in red, 23 in blue, and 13 in fuchsia; only two-way streets are depicted. Arrows denote street orientations according to the conventions set in Part I; yellow crosses denote branch points. The black dotted lines are identified to form the cylinder. (A): The two hypermultiplet networks at a point $u^0 = U(t = 0)$ just "before" the wall of marginal stability. (B): The $\mathcal{W}_\vartheta$-network at the point $u^{\mathrm{wall}} = u(t_*)$ on the wall of marginal stability and at phase $\vartheta = \arg\left[Z_{\gamma_1}(u^{\mathrm{wall}})\right] = \arg\left[Z_{\gamma_2}(u^{\mathrm{wall}})\right] = \arg\left[Z_{\gamma_1+\gamma_2}(u^{\mathrm{wall}})\right]$. (C): Slightly "after" the wall at a point $u_1 = U(t = 1)$, a BPS bound state of charge $\gamma_1 + \gamma_2$ is born and a two-way street of type 13 "grows" as one proceeds away from the wall.

11.2 Generating m-herds via Wall-Crossing

Suppose there exists a point $u_0 \in \mathcal{B}^*$ and a path $U : [0, 1] \to \mathcal{B}^*$ beginning at u_0 ($U(0) = u_0$) satisfying the following:

1. At u_0 there are two BPS states with respective charges $\gamma_1, \gamma_2 \in \Gamma_{u(0)}$, intersection pairing $\langle\gamma_1, \gamma_2\rangle = m \geq 1$, and BPS indices $\Omega(\gamma_i) = 1$ for $i = 1, 2$ (i.e. BPS hypermultiplets).

2. U crosses a wall of marginal stability for this pair of hypermultiplets at some time $t_* \in (0, 1)$.

3. For $t \in (t_*, 1]$, u does not cross any other walls of marginal stability involving the bound state of charge $\gamma_1 + \gamma_2$ (produced by the wall-crossing at $U(t_*)$).

Suppose further that the degenerate networks corresponding to the hypermultiplets at u_0 (i.e. the degenerate $\mathcal{W}_{\vartheta_i}$ networks at phases $\vartheta_i = \arg(Z_{\gamma_i})$ for $i = 1, 2$) appear as simple saddle-connections. Then the quasi-imaginative reader may be able to visualise a process by which m-herds are generated. Indeed, let $\gamma_i(t)$ denote the parallel transport of $\gamma_i(t), i = 1, 2$ along the path U from $U(0)$ to $U(t)$; further, for $t \in [0, 1]$ define

$$\vartheta_i(t) = \begin{cases} \arg\left(Z_{\gamma_i(t)}\right) & \text{for } t \leq t_w \\ \arg\left(Z_{\gamma_1(t)+\gamma_2(t)}\right) & \text{for } t > t_w \end{cases}, \; i = 1, 2.$$

Then $\{\vartheta_1(t)\}_{t\in[0,1]}$ and $\{\vartheta_2(t)\}_{t\in[0,1]}$ define two families of phases, equal to the central charge phases of the BPS states of charges γ_1 and γ_2, that are distinct for $t \in [0, t_*)$ but are equal (to the central-charge phase of the bound state $\gamma_1 + \gamma_2$) for $t \in [t_*, 1]$. Then looking at the family of networks $\{\mathcal{W}_{\vartheta_i(t)}\}_{t\in[0,1]}$ we should see two distinct saddle connections for $t \in [0, t_*)$ that combine into a single network on the wall of marginal stability at $t = t_*$ (c.f. Fig. 11.2). This network, which supports the linearly independent charges $\gamma_1(t_*)$ and $\gamma_2(t_*)$, should just be the superposition of two saddle connections with m distinct, transverse intersections. Now the claim is that as t continues to increase, the network at t_* will resolve via the growth of a new two-way street growing from each of the intersection points. Specifically, if we choose a local trivialization of our cover around each intersection point, and let the intersecting saddle connections be of types 12 and 23, then a new two-way street of type 13 should grow from the intersection. A visualization of this process is shown in Fig. 11.1 for the case $m = 3$.

As it turns out, these hypothetical degenerate networks can be formed by glueing together copies of the local-patch "building-block" of Fig. 11.3, named a *horse*. Roughly speaking, a $\mathcal{W}$-network is called an *m-herd* if its sub-network consisting of only two-way streets is given by glueing together m horses in a chain, and then coupling the two end horses to branch points.[4] The building-block construction provides the appropriate machinery for understanding the BPS spectrum associated to any m-herd.

Proposition 11.2.1 (c.f. Prop. 3.1.1 of Part I)**.** *Let N be an m-herd, then the street-factors $Q(p)$ for all two-way streets p on N are given in terms of powers of a*

[4] After this coupling, the six-way street equations constrain some of the streets on a horse to be only one-way.

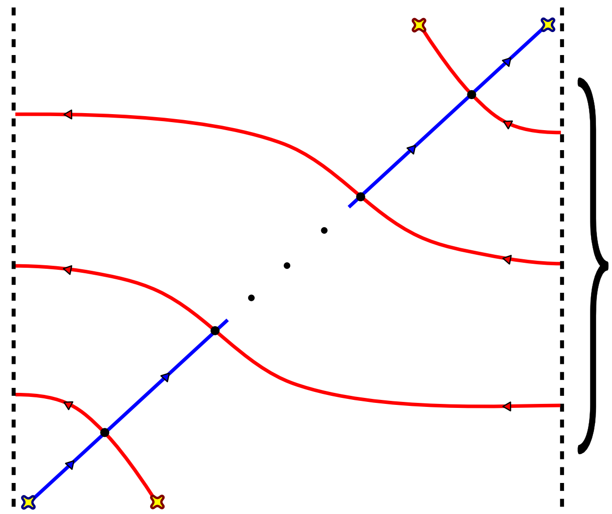


Figure 11.2: Two saddle connections appearing simultaneously on a wall of marginal stability. The dotted lines are identified to form the cylinder. Letting γ_1 be the charge supported by the red saddle connection and γ_2 the charge supported by the blue saddle connection, then we have the intersection pairing $\langle\gamma_1, \gamma_2\rangle = m$ (using the orientation on the cylinder induced by the standard orientation on the plane).

single generating series $P \in \mathbb{Z}[\![z]\!]$ *satisfying the functional equation*

$$P = 1 + zP^{(m-1)^2}, \tag{11.6}$$

where $z = (-1)^m X_{\widetilde{\gamma_c}}$ *with* $\gamma_c = \gamma_1 + \gamma_2$ *for some* γ_1, $\gamma_2 \in \Gamma = H_1(\Sigma; \mathbb{Z})$ *such that* $\langle\gamma_1, \gamma_2\rangle = m$. *Furthermore, the series* P^m *generates the BPS indices* $\Omega(n\gamma_c)$.

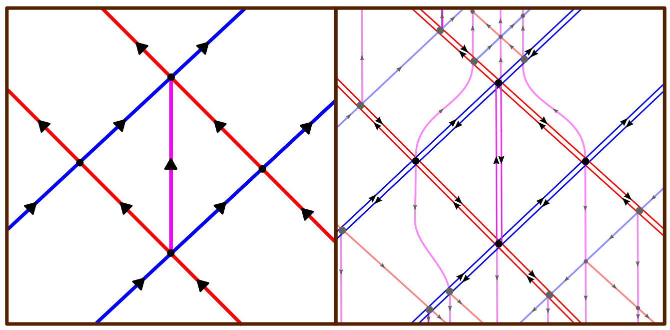


Figure 11.3: *Left Frame*: The building-block for m-herds, named a *horse*. The solid streets depicted are capable of being two-way. The sheets of the cover $\Sigma \to C$ are (locally) labelled from 1 to $K \geq 3$. Red streets are of type 12, blue streets are of type 23, and fuchsia streets are of type 13. We choose an orientation for this diagram such that all streets "flow up". *Right Frame*: A horse as it may appear in an actual spectral network. One-way streets shown as partially transparent and two-way streets are resolved using the "British resolution" (c.f [27]). One can imagine horses with increasingly intricate "backgrounds" of one-way streets.

Chapter Twelve: Seeking Non-diagonal Herds

Now we pursue a natural question: what are the degenerate networks associated to slope $a/b \neq 1$ BPS states; particularly, we are interested in slopes a/b such that $\arg(Z_{a\gamma_1+b\gamma_2})$ lies inside of the densely populated arc of phases, i.e. the arc of phases corresponding to integers a and b satisfying (see App. G.2):

$$\frac{m-\sqrt{m^2-4}}{2} < \frac{a}{b} < \frac{m+\sqrt{m^2+4}}{2}. \tag{12.1}$$

Galakhov-Longhi-Moore have a constructive definition of candidates for such slope-a/b networks [30, App. D]; however, we will not use their construction– instead, in order to aid in our definitions, we will proceed by looking at actual $\mathcal{W}$-networks associated to an m-wild point.

12.1 A 3-wild Point

Let us turn our attention to pure $SU(3)$ SYM; as mentioned in §11.1 this is just $S[A_2, C, D]$ where $C = \mathbb{P}^1$ and D denotes a particular set of defect operators at 0 and ∞. Let z denote the coordinate on $\mathbb{P}^1\backslash\{0,\infty\}$, given by the restriction of the standard coordinate patch on $\mathbb{P}^1\backslash\{\infty\} \cong \mathbb{C}$ to $\mathbb{P}^1\backslash\{0,\infty\}$. A point on the Coulomb branch $\mathcal{B}$ is a pair (ϕ_2, ϕ_3) of a quadratic (meromorphic) differential ϕ_2 and cubic (meromorphic) differential ϕ_3 on $\mathbb{P}^1$ with singularities at 0 and ∞ fixed by our defect operators. Explicitly, there is an identification $\mathbb{C}^2 \xrightarrow{\sim} \mathcal{B}$ sending the point $(u_2, u_3) \in \mathbb{C}^2$ to the pair (ϕ_2, ϕ_3) defined by:

$$\phi_2 = \frac{u_2 dz^{\otimes 2}}{z^2},$$
$$\phi_3 = \left(\frac{\Lambda}{z^2} + \frac{u_3}{z^3} + \frac{\Lambda}{z^4}\right) dz^{\otimes 3}.$$

Using this identification, we will abuse notation and say $u = (u_2, u_3)$ is a point in $\mathcal{B}$. The spectral cover (Seiberg-Witten curve) at the point $u = (u_2, u_3) \in \mathcal{B}$ of the pure $SU(3)$, $\mathcal{N} = 2$ theory with dynamical scale Λ is given by

$$\Sigma_u = \left\{\lambda \in \mathcal{T}^*C : \lambda^3 + \lambda\left(\frac{u_2}{z^2}\right)(dz)^{\otimes 2} + \left(\frac{\Lambda}{z^2} + \frac{u_3}{z^3} + \frac{\Lambda}{z^4}\right)(dz)^{\otimes 3} = 0\right\};$$

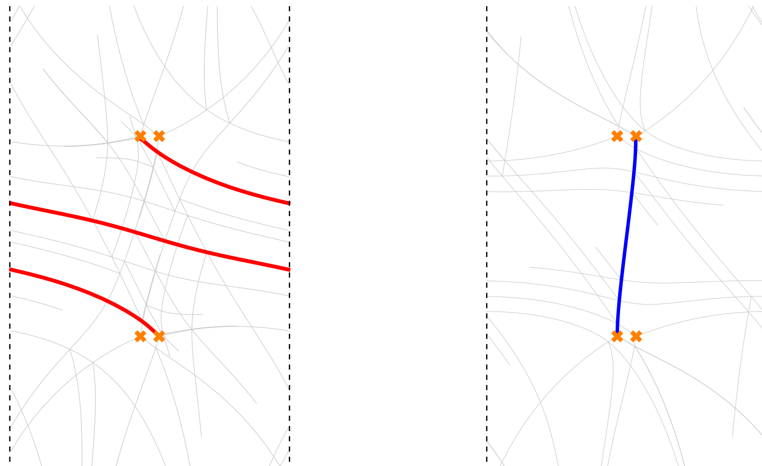


Figure 12.1: Degenerate saddle-connection spectral networks at the point u^w (defined in §12.1). Streets that become two-way at a critical phase are thickened and highlighted in colour; in each figure, the dotted lines are identified in order to form a cylinder. Both networks are produced at mass-cutoff 50 in $\Lambda = 1$ units. *Left*: $\mathcal{W}$-network computed close to the critical phase $\arg(Z_{\gamma_1})$. *Right*: $\mathcal{W}$-network computed close to the critical phase $\arg(Z_{\gamma_2})$.

as usual we will work in units where $\Lambda = 1$. Our focus will be on the point $u^w \in \mathcal{B}$ given by $(u_2^w, u_3^w) = \left(3, \frac{95}{10}\right)$ (where the superscript w is for "wild") where numerical observations indicate that the 3-wild BPS spectrum presents itself.

Claim u^w is a 3-wild point.

To support this claim, we note that at this point there are two BPS states with linearly independent charges $\gamma_1, \gamma_2 \in \Gamma = \hat{\Gamma}_u$ and central charges

$$\begin{aligned} Z_{\gamma_1} &= 20.980 - 40.148i, \\ Z_{\gamma_2} &= 7.244 - 9.083i. \end{aligned} \tag{12.2}$$

The existence of such states is established by observing the occurrence of two saddle-connection $\mathcal{W}$-networks at the phases $\arg(Z_{\gamma_1})$ and $\arg(Z_{\gamma_2})$, both depicted in Fig. 12.1; from this figure it can be deduced that $\langle \gamma_1, \gamma_2 \rangle = 3$. Moreover, the fact that both BPS states are represented by saddle-connections implies that these states are BPS-hypermultiplets.[1] This makes it reasonable to conjecture that there exists a BPS

[1] One can verify this claim directly by computing the protected spin-character [30] for a saddle-connection, or indirectly by noting that a saddle connection corresponds to a special Lagrangian 3-

quiver at the point u^w such that γ_1 and γ_2 label two of its nodes; under this conjecture the 3-Kronecker quiver is a subquiver of a BPS quiver at u^w. Furthermore, the central charge phases (12.2) impose a wild stability condition on this subquiver (c.f. Appendices F.2-F.3). Indeed, for any θ such that $e^{-i\theta}Z_{\gamma_1}$ and $e^{-i\theta}Z_{\gamma_2}$ lie in the half-space $bbH \subset \mathbb{C}$, we have

$$\arg\left(e^{-i\theta}Z_{\gamma_1}\right) > \arg\left(e^{-i\theta}Z_{\gamma_2}\right)$$

as elements of $(0,\pi)$. So, via Prop. F.3.1 and the related discussion in Apps. F and F.4, u^w is indeed a 3-wild point.

The $\mathcal{W}_\vartheta$-network at the phase $\vartheta = \arg\left(Z_{\gamma_1+\gamma_2}\right)$ is a 3-herd. Now our interest lies in discovering the form of the degenerate networks inside of the densely populated arc of phases: i.e. networks associated to BPS states of charges $a\gamma_1 + b\gamma_2$ for a, b coprime and satisfying (12.1). The process of drawing such networks becomes progressively more computationally taxing as a and b grow: for slope-a/b, in order to see the production of two way streets, the mass cutoff of the $\mathcal{W}$-network must be at least the mass of the predicted BPS state: $M_{a/b} := |aZ_{\gamma_1} + bZ_{\gamma_2}|$. As a result, in this thesis we will restrict our attention to results obtained by studying the two "low mass" cases: slopes 1/2 and 2/3 (at $m = 3$). As mentioned in the introduction, the network at slope 2/3 produces novel functional equations for the BPS spectrum. On the other hand, the network at slope 1/2, and its generalization to slope $1/(m-1)$ networks in the m-wild spectrum, produces the same spectrum as the slope 1 (m-herd) situation (and, in fact, produces essentially the same algebraic equations as for the m-herd); nevertheless it merits discussion as it may lead to future insight toward the general structure for networks at general slopes.

For the time-being it is helpful to fix some terminology when referring to these "off-diagonal" slope networks. As generalizations of m-herds, plenty of names come to mind to a sufficiently mischievous author. However, in a rare triumph of science over good-fun, rather than parsing through the full spectrum of farm animals, we settle on a more descriptive terminology.

Terminology [2] Let a and b be coprime integers. Then by an $(a, b|m)$-herd we will

sphere in the type IIB origin of theories of class S. Such 3-spheres correspond to BPS hypermultiplets in the field theory limit.

[2] In [30, App. D], the authors use the terminology "m-(a, b)-herd" instead of what we would call an $(a, b|m)$ herd. The author rationalizes his notation due to a deep-rooted derision of multiple hyphens.

mean a degenerate spectral network such that:

1. If $a = b = 1$, it is an m-herd;

2. One can continuously deform its degenerate skeleton into two saddle-connections with m-intersection points: if (in a trivialization of the spectral cover) the branch points are of type ij and jk, respectively, then in the limit that one shrinks the degenerate streets of type ik to length zero, one recovers two saddle-connections that intersect m-times (c.f. Fig. 11.2);

3. The BPS indices $\Omega(n\gamma_c)$ coincide with the DT-invariants $d(na, nb, m)$ (for all $n \in \mathbb{Z}_{\geq 1}$) of the Kronecker m-quiver equipped with a non-trivial stability condition;

4. Its degenerate skeleton is constructed by glueing together m basic building-blocks.

The last condition is what prevents this terminology from being a definition: we have not provided what the building-blocks should be for general (a, b). This condition is motivated by our experiences with m-herds, in particular the expectation that– as in the case for m-herds– the BPS spectrum of an $(a, b|m)$-herd can (hopefully) be calculated inductively for all m.

Speaking more vagely: an $(a, b|m)$-herd is a type of degenerate spectral network (with sufficiently simple skeleton) that can appear as a $\mathcal{W}_\vartheta$ network at phase $\vartheta = \arg(Z_{a\gamma_1 + b\gamma_2})$ in an m-wild region (i.e. γ_1 and γ_2 are two hypermultiplet charges with $\langle \gamma_1, \gamma_2 \rangle = m$).

Remark Not all $\mathcal{W}_\vartheta$ networks for $\vartheta = \arg(Z_{\gamma_1+\gamma_2})$ at an m-wild point may be m-herds. Indeed, during the work on [31], examples of spectral networks supporting the charge $\gamma_1 + \gamma_2$ in an m-wild region, but were *not* m-herds were found by D. Galakhov and P. Longhi. Hence, one should not expect that all spectral networks producing the DT-invariants $d(a, b, m)$ should be $(a, b|m)$-herds.

In this thesis, we will give definitions for $(m-1, 1|m), (1, m-1|m), (2, 3|3)$ and $(3, 2|3)$-herds. As mentioned in the beginning of this section, in [30, App. D], the authors provide an excellent candidate for a rigorous definition of the degenerate skeleton of an $(a, b|m)$-herd for general (a, b); in that paper, the building-blocks are referred to as "fat-horses". The reader should be warned, however, that our method of construction for $(m-1,1|m)$ and $(1,m-1|m)$-herds use different building-blocks than the "fat-horses", but the resulting degenerate skeletons appear to be the same.

Remark In all $(a,b|m)$-herds considered, the BPS indices $\Omega_m(n(a\gamma_1+b\gamma_2))$ generated by an $(a,b|m)$-herd are identical to $\Omega_m(n(b\gamma_1+a\gamma_2))$. As described in G, for the m-Kronecker quiver K_m this is a consequence of the "transposition" autofunctor acting on the category of representations of K_m. At the level of all the spectral networks in this thesis, this symmetry is immediate consequence of the fact that the $(a,b|m)$-herd degenerate skeleton is the mirror image of the $(b,a|m)$-herd degenerate skeleton. The networks of [30] also appear to have this mirror-image property.

12.2 $(1,m-1|m)$-herds (for $m\geq 2$)

First, we turn our attention toward the $(1,2|3)$-herd, realized as a $\mathcal{W}_\vartheta$-network at the point $u^w\in\mathcal{B}$ described above in §12.1 and with central charge phase $\vartheta=\arg(Z_{\gamma_1+2\gamma_2})$. A simplified version of this network, with only the two-way streets shown, is depicted in Fig. 12.2. Just as for the m-herds, Fig. 12.2 suggests a rather simple generalization for diagrams corresponding to any $m\geq 2$. Indeed, one can check that the diagram in Fig. 12.2 is reproduced by glueing together three copies of the building-block[3] shown in the left panel of Fig. 12.3, and coupling the end building blocks to four distinct branch points in the appropriate way.[4]

Definition Let $m\geq 2$. A network is an $(m-1,1|m)$-herd (respectively $(1,m-1|m)$-herd) if:

1. Its two-way streets agree with the diagram formed by glueing m-copies of the building-block in the left panel (resp. right panel) of Fig. 12.3 in the following

[3]This building block looks roughly like a horse (see Fig. 11.3) tipped 45 degrees: the key difference between the two diagrams is that a horse has only one "secondary" street (fuchsia street), while the building block shown in Fig. 12.3 has two secondary streets.

[4]In this procedure, just as with horses, some of the streets in the building block are constrained to be one-way streets after coupling to the branch points.

manner:

$$\begin{aligned} a_1^{(l)} &= a_2^{(l-1)} \\ \overline{a}_2^{(l)} &= \overline{a}_1^{(l-1)} \\ b_1^{(l)} &= \overline{b}_2^{(l-1)} \\ b_2^{(l)} &= \overline{b}_1^{(l-1)}, \end{aligned}$$

for every $l \in \{2, \cdots, m\}$.

2. Each of the two-way streets labelled by $a_1^{(1)}$, $b_2^{(1)}$, $\overline{a}_1^{(m)}$, and $\overline{b}_2^{(m)}$ are connected to distinct branch points.

3. A version of the *No Holes* condition[5], described in App. B.3, with the horse replaced by the building-block in the left (resp. right) panel of Fig. 12.3.

The name $(m-1, 1|m)$-herd suggests that these diagrams are related to the slope $(m-1)/1$ BPS states. Indeed, the proofs and analyses of Apps. B - C can all be appropriately modified with either one of the building-blocks of Fig. 12.3 replacing the horse (Fig. 11.3); with these simple modifications, we arrive at the following.

Proposition 12.2.1. *The BPS indices* $\{\Omega(n\gamma_c)\}_{n=1}^{\infty}$ *for the* $(m-1, 1|m)$*-herds (resp.* $(1, m-1|m)$*-herds) are generated by a formal series* P *satisfying the algebraic equation*

$$P = 1 + zP^{(m-1)^2} \tag{12.3}$$

with $z := -X_{\widetilde{\gamma_c}}$ *for some* $\gamma_c \in \Gamma$*. Furthermore, there exist charges* $\gamma_1, \gamma_2 \in \Gamma$ *such that* $\langle\gamma_1, \gamma_2\rangle = m$ *and* $\gamma_c = (m-1)\gamma_1 + \gamma_2$ *(resp.* $\gamma_c = \gamma_1 + (m-1)\gamma_2$*).*

The reason for the negative sign appearing in $z = -X_{\widetilde{\gamma_c}}$ is explained in Appendix I.

Note that, modulo the identification of the formal variable z with an element of $\mathbb{Z}[\![\widetilde{\Gamma}]\!]$, the algebraic equation (12.3) is identical to the algebraic equation (11.6)

[5] This is a technical, yet important, condition that allows one to define an $(m-1, m|m)$-herd (resp. $(m, m-1|m)$-herd) on a general curve C and insure that all street factors are formal series in a single variable $X_{\widetilde{\gamma}}$ for a primitive $\gamma \in \Gamma$, unique up to a choice of sign (c.f. Rmk. 11.1.1 in §11.1). Without this condition, street factors may be formal series in a collection of formal variables, associated to linearly independent, primitive, elements of Γ.

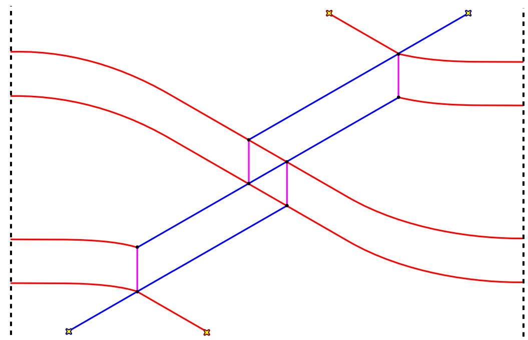


Figure 12.2: Two-way streets of a $(2,1|3)$-herd on the cylinder.

for m-herds. In particular, the BPS indices $\{\Omega(n\gamma_c)\}_{n=1}^{\infty}$ are identical to the m-herd situation. This should be expected under the assumption of equivalence of BPS indices $\Omega\left[n(a\gamma_1 + b\gamma_2)\right]$ for $(a,b|m)$-herds with the DT-invariants $d(a,b,m)$ associated to the m-Kronecker quiver equipped with dimension vector (a,b). In particular, the observation that $\Omega\left[n(m\gamma_1 + (m-1)\gamma_2)\right]$ is the same as $\Omega\left[n(\gamma_1+\gamma_2)\right]$ is expected from the more general equivalence of m-Kronecker DT-invariants

$$d(a,b,m) = d(ma-b,b,m).$$

As described in Appendix G, this equivalence follows as a consequence of "reflection" endofunctor [55] on the m-Kronecker quiver representation category – which takes representations with dimension vector (a,b) to representations with dimension vector $(ma-b,b,m)$. In the physics, this equivalence is expected to manifest itself as a monodromy of the local system $\hat{\Gamma} \to \mathcal{B}^*$ (c.f. §6.2 and Appendix G).

12.3 The $(2,3|3)$ and $(3,2|3)$-herds

To achieve a spectral network with the potential to produce a more interesting BPS spectrum, we turn our attention toward the $\mathcal{W}$-network associated to bound states of charges $\{2n\gamma_1 + 3n\gamma_2\}_{n=1}^{\infty}$: this is the network $\mathcal{W}_\vartheta$ at phase $\vartheta = \arg\left(Z_{2\gamma_1+3\gamma_2}\right)$ lying in the densely populated arc. At the point $u^w \in \mathcal{B}$, the bound state of charge $\gamma_c = 2\gamma_1 + 3\gamma_2$ has central charge $Z_{\gamma_c} = 63.692 - 107.545i$ and mass $M = |Z_{\gamma_c}| = 124.99$. The corresponding $\mathcal{W}_\vartheta$ network at $\vartheta = \arg(Z_{\gamma_c})$ is shown in Fig. 12.4 with the relevant

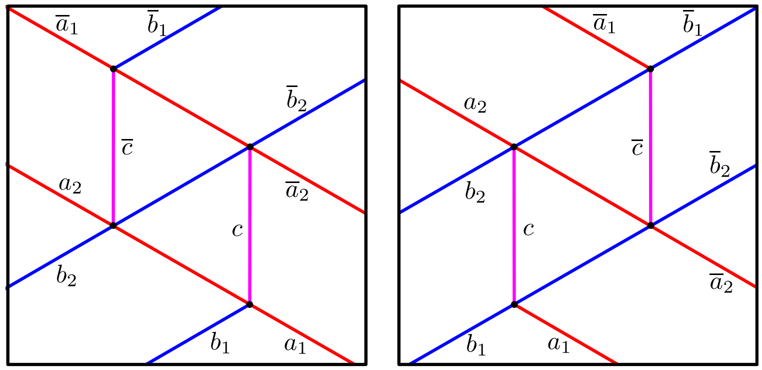


Figure 12.3: Left: Building-block for an $(m-1,1|m)$-herd. Right: Building-block for a $(1,m-1|m)$-herd (which is the mirror image of the building-block in the left panel).

two-way streets highlighted using our colour-coding convention.[6] A simplified picture, showing only two-way streets, is shown in Fig. 12.6.

One may hope that the process of solving for the street-factors reduces to a simple recursive/inductive building-block procedure as with the m-herds and $(m-1,1|m)$-herds. Indeed, there is an obvious candidate for a building-block, shown in Fig. 12.7. However, the presence of the two coupled joints of valence-six (where all streets are two-way) leads to a set of highly-coupled non-linear equations with no simple explicit solution for the incoming soliton data in terms of the outgoing soliton data;[7] so, we reluctantly abandon this technique.

Instead, we will pursue a more ad-hoc approach specific to the case $m=3$. First, we recall the "abelian" six-way junction rules: the equations that impose conditions on the street-factors at a general type of joint (where six streets, all possibly two-way,

[6]In practice, one way to determine the two-way streets by perturbing the phase of interest slightly, and carefully (read painfully) watching for near collisions (that become *actual* collisions for the unperturbed phase) of anti-parallel streets as the mass-cutoff is increased. This technique also helps rule out the myriad of one-way streets that run close and parallel to one another.

[7]Perhaps, a more clever reader may be able to find such a solution.

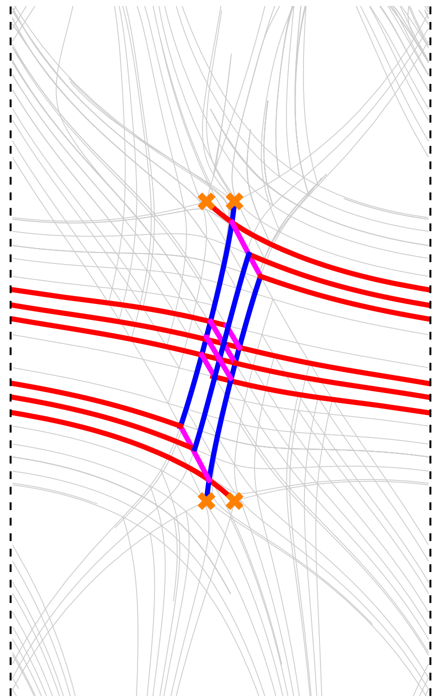


Figure 12.4: $\mathcal{W}$-network demonstrating a $(2,3|3)$-herd at the point u^w. The network is produced at mass cutoff 140 close to the critical phase $\arg(Z_{2\gamma_1+3\gamma_2})$. Streets that become two-way at the critical phase are thickened and highlighted in colour; in each figure, the dotted lines are identified in order to form a cylinder.

meet).[8] Referring to the labelled streets of such a joint in Fig. 12.5, we have:

$$\begin{aligned} 1 &= Q\left(q_{12}\right) Q\left(q_{13}\right) Q\left(p_{12}\right)^{-1} Q\left(p_{13}\right)^{-1} \\ 1 &= Q\left(q_{23}\right) Q\left(q_{12}\right) Q\left(p_{23}\right)^{-1} Q\left(p_{12}\right)^{-1} \\ 1 &= Q\left(q_{13}\right) Q\left(q_{23}\right) Q\left(p_{13}\right)^{-1} Q\left(p_{23}\right)^{-1} . \end{aligned} \tag{12.4}$$

Any two of these equations are algebraically independent – however, by taking the inverse of both sides of one equation, and multiplying by another, produces the third. Hence, the abelian rules can only eliminate two effective street-factor degrees of freedom from each joint. Nevertheless they are quite powerful in reducing the number of unknown street-factors. From these equations, along with the 180-degree rotation symmetry of the diagram, it is a simple exercise to show that all street factors

[8] These equations can be derived either from the "non-abelian" six-way junction rules for soliton generating functions, or directly using the homotopy invariance techniques of [27] (which is how the non-abelian rules are derived).

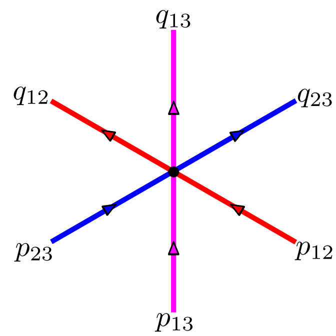


Figure 12.5: A "six-way junction": joint where six (possibly) two-way streets meet.

can be expressed as Laurent monomials in terms of the three street-factors, labelled M, V, and W, shown next to their corresponding streets in Fig. 12.6; in particular, the street factors attached to the streets extending from the branch points are given as powers of $\mathsf{T}_{3/2} := MVW$.

Claim

1. All street factors are power series in the variable $z := -X_{\widetilde{\gamma}_c}$, where $\gamma_c = 3\gamma_1 + 2\gamma_2$ with γ_1 and γ_2 the charges represented by the lifts of the saddle-connections of Fig. 11.2 to the spectral cover (i.e. the hypothetical hypermultiplet charges before their wall-crossing).

2. $MVW \in \mathbb{Z}[\![z]\!]$ is the generating series $\mathsf{T}_{3/2}$ for the BPS indices $\{\Omega(n\gamma_c)\}_{n=1}^{\infty}$ in the sense of (11.2).

To argue the first claim we recall that a $(3, 2|3)$-herd appears as a $\mathcal{W}$-network off of a wall of marginal stability; thus, all generating functions are formal series in the formal variable $X_{\widetilde{\gamma}_c}$ assigned to the standard lift $\widetilde{\gamma}_c \in \widetilde{\Gamma}$ of a unique (up to sign) charge $\gamma_c \in \Gamma$. (For reasons expressed in Appendix I, it is more convenient express the street factors as formal series in the formal variable $z = -X_{\widetilde{\gamma}_c}$). Using the six-way junction equations, an order-by-order computation of street factors is consistent with this fact and shows that $\gamma_c = 3\gamma_1 + 2\gamma_2$. This shows the first claim.

To argue the second claim we recall some essential features of the machinery that produces the BPS indices from a spectral network. Let N be a spectral network, $\mathrm{str}(N)$ be the set of streets of N, and $\ell : \mathrm{str}(N) \to C_1(\Sigma; \mathbb{Z})$ the map describing the “lift” of each street to a 1-chain on Σ. Assume each street-factor is a formal series in some variable $X_{\widetilde{\gamma_c}}$ with integer coefficients. Now for each $p \in \mathrm{str}(N)$ define a sequence of integers $(\alpha_n(p))_{n=1}^{\infty} \subset \mathbb{Z}$ by a factorization of the street-factor $Q(p)$:

$$Q(p) = \prod_{n=1}^{\infty} \left(1 - (X_{\widetilde{\gamma_c}})^n\right)^{\alpha_n(p)}, \tag{12.5}$$

then we define a collection of 1-chains $\{L(n\gamma_c)\}_{n=1}^{\infty} \subset C_1(\Sigma; \mathbb{Z})$ via

$$L(n\gamma_c) := \sum_{p \in \mathrm{str}(N)} \alpha_n(p)\ell(p). \tag{12.6}$$

It can be shown that $L(n\gamma_c)$ is closed and its homology class is a multiple of $n\gamma_c$ – in fact, the BPS index is related to the homology class $[L(n\gamma_c)] \in H_1(\Sigma; \mathbb{Z})$ via

$$[L(n\gamma_c)] = n\gamma_c\Omega(n\gamma_c). \tag{12.7}$$

Turning back to the case of the $(3, 2|3)$-herd, define c_n as the exponents in the following expansion

$$\begin{aligned} MVW &= \prod_{n=1}^{\infty} \left(1 - (X_{\widetilde{\gamma_c}})^n\right)^{c_n} \\ &= \prod_{n=1}^{\infty} \left(1 - (-1)^n z^n\right)^{c_n}. \end{aligned}$$

Note that any 1-chain supported on the lift of the spectral network and representing the homology class γ_1 must project down to a 1-chain passing through the branch points of type 12 (emanating from the red-streets in Fig. 12.6); similarly, any 1-chain on the lift of the spectral network representing the homology class γ_2 must project down to a 1-chain passing through the branch points of type 23 (emanating from the blue-streets in Fig. 12.6). Hence, any 1-chain representing the cohomology class $\gamma_c = 3\gamma_1 + 2\gamma_2$ must project down to a path passing three times through the branch points of type 12, and two times through the branch points of type 13. With this observation and the fact that the street-factors attached to the red and blue streets emanating from branch points are given by $(MVW)^3$ and $(MVW)^2$ respectively, then from (12.6) and (12.7) it follows that

$$c_n = n\Omega(n\gamma_c);$$

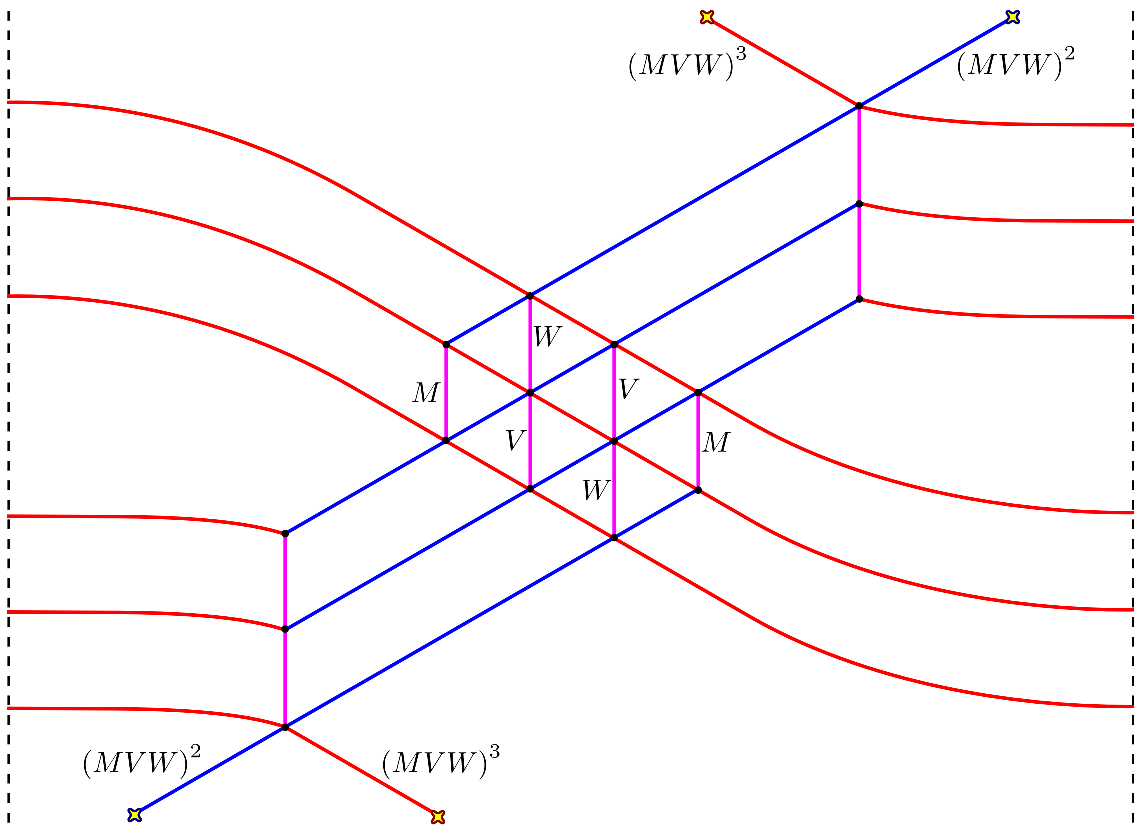


Figure 12.6: Two-way streets of a $(3, 2|3)$-herd.

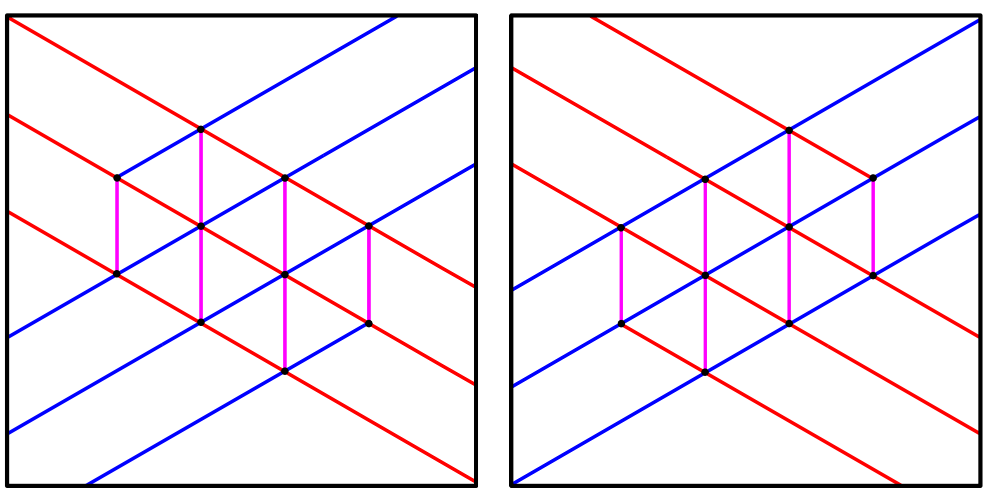


Figure 12.7: Candidate building-blocks for “higher” networks.

the second claim follows.

As sketched in the appendix, using the six-way junction equations, along with some order-by-order numerical calculations, one can derive the following system of algebraic relations:

$$\begin{aligned} M &= 1 + zM^4\left\{(1+V)(1+V-W)^2[V^2(1+W)-1]^3\right\} \\ 0 &= (-1+V)(1+V)^2 + (1+V^3)W - V(M+V)W^2 \\ 0 &= V\left(V^2-1\right) - \left[M(V+1)+V(V-2)-1\right]W. \end{aligned} \tag{9.2}$$

Of course, there is a plethora of ways of rewriting (9.2), the above is a result of what the author found the most pleasing. In other words, if we define three polynomials[9] in $(\mathbb{Z}[z])\,[m,v,w]$

$$\begin{aligned} \mathcal{G} &= 1 + zm^4\left\{(1+v)(1+v-w)^2[v^2(1+w)-1]^3\right\} - m \\ \mathcal{M} &= (-1+v)(1+v)^2 + (1+v^3)w - v(m+v)w^2 \\ \mathcal{N} &= v\left(v^2-1\right) - \left[m(v+1)+v(v-2)-1\right]w \end{aligned}$$

then $(m,v,w) = (M,V,W)$ must provide a simultaneous root of these three polynomials. There are, in fact, forty-two[10] simultaneous root tuples; only one of which can be identified as the tuple of street-factors (M,V,W). Indeed, any tuple of simultaneous roots that has the interpretation as a street-factor must be a tuple of formal power series, each with constant coefficient 1. One can check that there exists precisely one such simultaneous root tuple; we can produce it by substituting such a formal power series ansatz (e.g. $M = 1 + \sum_{n=1}^{\infty} m_n z^n$) into (9.2) and solving order-by-order in z. The first few terms are given by,

$$\begin{aligned} M &= 1 + 2z + 146z^2 + 15824z^3 + 2025066z^4 + 284232734z^5 + 42316425168z^6 + \mathcal{O}(z^7), \\ V &= 1 + 4z + 324z^2 + 36224z^3 + 4704404z^4 + 665965148z^5 + 99703601696z^6 + \mathcal{O}(z^7), \\ W &= 1 + 7z + 514z^2 + 55685z^3 + 7121694z^4 + 999071727z^5 + 148683258448z^6 + \mathcal{O}(z^7). \end{aligned} \tag{12.8}$$

Hence,

$$\begin{aligned} \mathsf{T}_{3/2} = 1 &+ 13z + 1034z^2 + 115395z^3 + 14986974z^4 + 2122315501z^5 + 317853709072z^6 \\ &+ \mathcal{O}(z^7). \end{aligned}$$

[9]polynomials in the variables m, v, and w and coefficients in $\mathbb{Z}[z]$.

[10]Making a tempting relation to the meaning of life, the universe, and everything.

Using (11.4), the first few BPS-indices are

$$\left(\Omega\left[n\left(3\gamma_1+2\gamma_2\right)\right]\right)_{n=1}^{6}=(13,-478,34227,-3279848,367873950,-45602813070)$$

The first three values of this sequence are consistent with BPS-indices obtained from the `HiggsBranchFormula` function[11] in the Mathematica package `CoulombHiggs.m` [42], based on formulae developed by Reineke [49] and Manschot-Pioline-Sen [44, 43, 41].

Using an expansion of $\mathsf{T}_{3/2}$ out to the coefficient multiplying z^K, we can reliably extract the first K BPS-indices from (11.4). This was performed for $K = 1360$ and the resulting BPS-indices can be found in a file included with the arXiv preprint version of this part of the thesis (to appear).

In the next section we discuss the large n asymptotics of BPS-indices using an algebraic equation satisfied by $\mathsf{T}_{3/2}$.

12.4 Asymptotics of BPS-indices associated to dimension vectors $(3n, 2n)$

With a bit of elimination theory, the algebraic relations (H.13) and the relation $\mathsf{T}_{3/2} = MVW$ can be reduced to a single algebraic relation between z and $\mathsf{T} := \mathsf{T}_{3/2}$:

$$\begin{aligned}0=&-(1+z)+\mathsf{T}(4-5z)+\mathsf{T}^2(-6+z)+\mathsf{T}^3(4+21z)+\\&\mathsf{T}^4(-1-34z)-\mathsf{T}^5(7z)+\mathsf{T}^6\left(76z+z^2\right)+\mathsf{T}^7\left(-64z-13z^2\right)+\\&\mathsf{T}^8\left(6z-114z^2\right)+\mathsf{T}^9\left(7z-80z^2\right)+\mathsf{T}^{10}\left(6z^2\right)+\mathsf{T}^{11}\left(119z^2\right)+\\&\mathsf{T}^{12}\left(53z^2+z^3\right)+\mathsf{T}^{13}\left(-55z^2+44z^3\right)+\mathsf{T}^{14}\left(-21z^2-38z^3\right)+\mathsf{T}^{15}\left(77z^3\right)-\\&\mathsf{T}^{16}\left(382z^3\right)+\mathsf{T}^{17}\left(270z^3\right)+\mathsf{T}^{18}\left(80z^3-z^4\right)+\mathsf{T}^{19}\left(35z^3+7z^4\right)+\\&\mathsf{T}^{20}\left(39z^4\right)-\mathsf{T}^{21}\left(367z^4\right)-\mathsf{T}^{22}\left(173z^4\right)-\mathsf{T}^{23}\left(30z^4\right)-\\&\mathsf{T}^{24}\left(35z^4\right)+\mathsf{T}^{25}\left(3z^5\right)-\mathsf{T}^{26}\left(17z^5\right)-\mathsf{T}^{27}\left(77z^5\right)-\\&\mathsf{T}^{28}\left(14z^5\right)+\mathsf{T}^{29}\left(21z^5\right)-\mathsf{T}^{32}\left(3z^6\right)+\mathsf{T}^{33}\left(9z^6\right)-\\&\mathsf{T}^{34}\left(7z^6\right)+\mathsf{T}^{39}z^7.\end{aligned}\tag{12.9}$$

[11] As mentioned in the documentation for `CoulombHiggs.m`, the `HiggsBranchFormula` function is based on Reineke's work in [49]. Values of (11.4) were not checked with this package for $n \geq 4$ due to the large computation-time required.

With this algebraic relation, using the techniques developed in §15 we can determine an explicit form for the $n \to \infty$ asymptotics of $\Omega\left[n\left(3\gamma_1 + 2\gamma_2\right)\right]$. Indeed, using Corollary 15.2.10 and (12.9) we have the $n \to \infty$ asymptotics[12]

$$\Omega\left[n\left(3\gamma_1 + 2\gamma_2\right)\right] \sim C(-1)^{n+1} n^{-5/2} \rho^{-n} + \mathcal{O}(n^{-7/2}) \tag{12.10}$$

where

$$\rho \approx 0.005134$$
$$C \approx 0.075084.$$

The derivation of these asymptotics is elaborated on in Example 2 at the end of §15.2. More precisely, ρ is an algebraic number given by the smallest-magnitude root of the 10th degree polynomial (15.30).

Note that, defining the sequence,

$$R_n := 1 - \frac{\Omega\left[n\left(3\gamma_1 + 2\gamma_2\right)\right]}{C(-1)^{n+1} n^{-5/2} \rho^{-n}} \tag{12.11}$$

the asymptotics (12.10) are equivalent to the statement that $R_n \in \mathcal{O}(n^{-1})$. A plot of R_n– displaying the $\mathcal{O}(n^{-1})$ behaviour– is shown in Fig. 12.8 for $1 \leq n \leq 1360$. In fact, for the range of n shown in this plot, $0 < R_n \leq cn^{-1}$ for any constant $c \geq 0.12$.

[12]Written in a different form, the "exponential part" of the asymptotics can be expressed as $\rho^{-n} = e^{Kn}$ for $K = -\log(\rho) \approx 5.27187$.

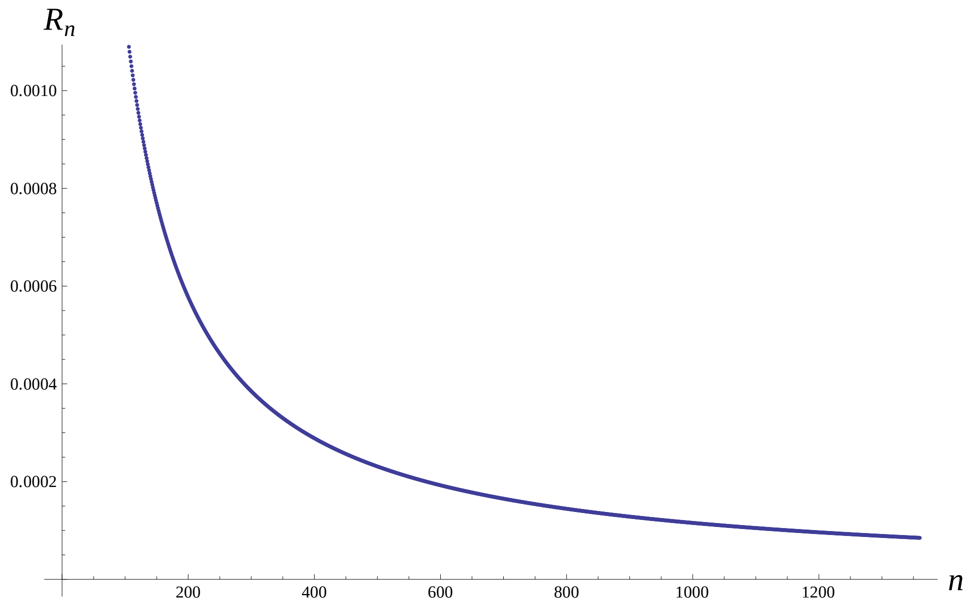
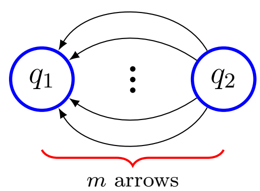


Figure 12.8: Plot of R_n (defined in (12.11)) vs. n for $1 \leq n \leq 1360$.

Chapter Thirteen: Euler Characteristics of Stable Moduli and their Asymptotics

13.1 Setup

Fix $m \geq 1$ an integer, then m-Kronecker quiver will be represented as the quiver with two vertices q_1, q_2 and m arrows between them.



As described in [48, §5.1], and appendix F, there is a single *non-trivial* stability condition that one can put on representations of this quiver. Fix this non-trivial condition, define $\Lambda := \mathbb{Z}\langle q_1, q_2\rangle$, and denote the corresponding moduli space of stable representations with dimension vector $\alpha q_1^* + \beta q_2^* \in \Lambda^*$ by $\mathcal{M}_{\mathrm{s}}^m(\alpha, \beta)$ – it is a smooth, complex, quasi-projective variety.

Now, fix $(a, b) \in \mathbb{Z}^2$ coprime. We are interested in studying the sequence of Euler characteristics

$$\chi_n := \chi\left(\mathcal{M}_{\mathrm{s}}^m(na, nb)\right)$$

and DT-invariants

$$d_n := d(na, nb, m).$$

To do so, we encapsulate these sequences' generating functions in such a way that they arise as Euler-product factorizations. For the Euler characteristics we define,

$$\mathsf{E}_{a/b} := \prod_{n=1}^{\infty}(1 - z^n)^{-n\chi_n} \in \mathbb{Z}[\![z]\!]$$

and for the DT-invariants:

$$\mathsf{T}_{a/b} := \prod_{n=1}^{\infty}\left(1 - \left((-1)^N z\right)\right)^{nd_n} \in \mathbb{Z}[\![z]\!] \tag{13.1}$$

where

$$N := mab - a^2 - b^2 = \dim\left(\mathcal{M}_{\mathrm{s}}^{(a,b)}\right) - 1$$

Because (a, b) is fixed, without fear of confusion, we will write $\mathsf{T} = \mathsf{T}_{a/b}$ and $\mathsf{E} = \mathsf{E}_{a/b}$ for the remainder of this section. The reason for the sign $(-1)^N$ in (13.1) is at first sight an unnecessary complication, unless the formal variable z has some sort of meaning in the appropriate context. Indeed, secretly z is leading a double-life as a generator of the group-ring $K_0(\mathsf{Rep}_{\mathbb{C}}(K_m)) \cong \mathbb{Z}[\Lambda]$– but we will not expose z for what it is, as we will not need to use this fact in any meaningful way.

Of course, we have intentionally confused the notation used for the generating series of BPS indices $\Omega\left[n\left(a\gamma_1 + b\gamma_2\right)\right]$ at an m-wild point, and the generating series for the $d(na, nb, m)$: although BPS indices are a priori defined via super-traces over Hilbert spaces, when the theory has a BPS quiver, strong physical evidence indicates that the BPS indices are precisely the DT-invariants associated to the quiver (where the stability condition is derived from the central-charge function, and a quiver-potential may be present). Hence, if m-Kronecker quiver arises as a BPS subquiver at a point $u \in \mathcal{B}$, the nodes q_1 and q_2 can be identified with two charges γ_1 and γ_2 in $\Gamma = \widehat{\Gamma}_u$ such that $\langle\gamma_1, \gamma_2\rangle = m$; in particular, the sub-lattice $\mathbb{Z}\langle\gamma_1, \gamma_2\rangle$ of the full charge lattice Γ can be identified with Λ.

On the other hand, the existence of a physical role for Euler characteristics (of stable moduli) associated to non-primitive dimension vectors is an interesting open question; nevertheless there appears to be a relationship between DT-invariants and Euler characteristics of stable-moduli.

13.2 Reineke's Functional Equation and Algebraicity for Stable Kronecker Moduli

The key element of Reineke's proof of the integrality of the DT-invariants $d(a, b, m)$ is the functional equation [47] relating Euler characteristics and DT-invariants:

$$\mathsf{T} = \mathsf{E} \circ \left(z\mathsf{T}^N\right) \tag{13.2}$$

where $\circ$ indicates composition of formal series. As written, (13.2) is a functional equation with a "recursive flavour"; however, it is equivalent to a statement about compositional inverses of formal series. To see this we will reverse-engineer some of Reineke's work, beginning by recalling some facts about composition of formal series.

First, recall that formal power series $\mathbb{Z}[\![z]\!]$ form a ring under addition $(+)$ and multiplication $(\cdot)$ that is, furthermore, equipped with a partially-defined composition $(\circ)$ operation (analogous the composition of functions) given by substitution of formal series. Specifically, letting $(z) \subset \mathbb{Z}[\![z]\!]$ denote the ideal of formal series with vanishing constant coefficient, composition is a map

$$\circ : \mathbb{Z}[\![z]\!] \times (z) \to \mathbb{Z}[\![z]\!]$$

satisfying (for A, B and C taken in the proper domain of definition for $\circ$):

- $0 \circ C = 0$ and $A \circ 0 = 0$;
- $(A + B) \circ C = (A \circ C) + (B \circ C)$;
- $(A \cdot B) \circ C = (A \circ C) \cdot (B \circ C)$;
- $(A \circ B) \circ C = A \circ (B \circ C)$;
- $A \circ z = A$, and $z \circ C = C$.

It is defined by the property that, for any $g \in (z)$, the map $(\cdot) \circ g : \mathbb{Z}[\![z]\!] \to \mathbb{Z}[\![z]\!]$ is the unique morphism of rings such that $(\cdot) \circ g : z \mapsto g$; in other words, $f \circ g$ is the formal series defined by substituting every instance of z in f with g. Note that, in order for such a substitution to give a well-defined formal power series for general $f \in \mathbb{Z}[\![z]\!]$, the series g must have zero constant coefficient[1]– which is why we only pre-compose by formal series in (z).

Now, the restricted composition $\circ : (z)^{\times 2} \to (z)$ equips (z) with the structure of a composition ring with compositional identity given by $z \in (z)$– so it makes sense to ask whether or not a formal series in $S \in (z)$ has a compositional inverse.[2] It can be checked that such an inverse exists if and only if $[z^1]S = \pm 1$ (more generally, $[z^1]S$ should be a unit in the the coefficient ring). Every such S can be decomposed as

[1] Otherwise the resulting series $f \circ g$ would have constant coefficient given by an infinite sum of elements of whatever ring we are working over.

[2] Every left inverse must be a right inverse and vice-versa.

$S = zF$ for some $F \in \mathbb{Z}[\![z]\!]$ with $[z^0]F = \pm 1$ and the existence of a compositional inverse is the statement that there exists $G \in (z)$ satisfying

$$(zF) \circ G = z.$$

We can manipulate this equation into a form with a more "recursive flavour": because F has constant coefficient in the ring of units of $\mathbb{Z}$, then F admits multiplicative inverse $F^{-1} \in \mathbb{Z}[\![z]\!]$. Using this fact, it follows that $(zF) \circ G = z$ is equivalent to the functional equation

$$G = z \cdot \left(F^{-1} \circ G\right).$$

With this observation in mind, we are ready to rewrite Reineke's functional equation. Indeed, define the compositionally invertible series

$$D := z\mathsf{T}^N;$$

then (13.2) can be written as

$$D = z \cdot \left(\mathsf{E}^N \circ D\right);$$

so via our discussion above, (13.2) is equivalent to the statement that $D = z\mathsf{T}^N$ is the compositional inverse of $z\mathsf{E}^{-N}$. We repeat this fact in the proof of the theorem below.

Theorem 13.2.1.

1. *If T obeys the relation $f(z, \mathsf{T}) = 0$ in $\mathbb{Z}[\![z]\!]$ for some $f \in \mathbb{Z}[z, t]$, then E obeys the relation $f(z\mathsf{E}^{-N}, \mathsf{E}) = 0$.*

2. *If E obeys the relation $f(z, \mathsf{E}) = 0$ in $\mathbb{Z}[\![z]\!]$ for some $f \in \mathbb{Z}[z, e]$, then T obeys the relation $f(\mathsf{T}, z\mathsf{T}^N) = 0$.*

Proof. For the first statement; taking $D = z\mathsf{T}^N$ as above, we have

$$\begin{aligned} f(z\mathsf{E}^{-N}, \mathsf{E}) \circ D &= f\left[D \cdot (\mathsf{E} \circ D)^{-N}, \mathsf{E} \circ D\right] \\ &= f\left(z\mathsf{T}^N \cdot (\mathsf{T})^{-N}, \mathsf{T}\right) \\ &= f(z, \mathsf{T}) \\ &= 0. \end{aligned}$$

where on the second line we used Reineke's functional equation (13.2). As D is compositionally invertible, then $f(w\mathsf{E}^{-N}, \mathsf{E}) = 0$. The second statement follows by a similar computation. □

This results in the obvious corollary.

Corollary 13.2.2. E *is algebraic over* $\mathbb{Q}(z)$ *if and only if* T *is algebraic over* $\mathbb{Q}(z)$.

Moreover, if we know the polynomial relation for either T or E, then the proposition provides us with an explicit polynomial relation: e.g. if $f(z, \mathsf{T}) = 0$ for some two-variable polynomial f, then a polynomial relation $g(z, \mathsf{E}) = 0$ for E is given by the polynomial $g(z, e) = e^d f(ze^{-N}, e) \in \mathbb{Z}[z, e]$ for sufficiently large d.

Notes

- Reineke's functional equation generalizes to any finite quiver (with no relations) for certain stability conditions [14, Second proof of Thm 7.29]. The corresponding statement about algebraicity follows.

- The interpretation of Reineke's functional equation as a statement that the two functions

$$z\mathsf{T}^N = \prod_{n=1}^{\infty} \left(1 - \left((-1)^N z\right)^n\right)^{Nnd_n}$$

and

$$z\mathsf{E}^{-N} = \prod_{n=1}^{\infty} (1 - z^n)^{Nn\chi_n}$$

are compositional inverses seems to suggest that we should have a priori defined the generating functions for DT-invariants and Euler characteristics (at least for the m-Kronecker quiver) directly in terms of $D = z\mathsf{T}^N$ and $C := z\mathsf{E}^{-N}$ to make Reineke's functional equation seem more symmetric; doing so would make the statement of algebraicity also symmetric: if there exists a $g \in \mathbb{Z}[z, d]$ such that $g(z, D) = 0$, then by composing this relation with C we have $g(C, z) = 0$. Thus D is algebraic if and only if C is algebraic.

Example Let $a = b = 1$ so that $N = m-2$. Defining $f_m(z,t) := -1+t(1-zt^{(m-2)})^m$ (see (14.3)), then $f(z, \mathsf{T}_1) = 0$; hence, by Thm. (13.2.1), (using the fact that $N = m-2$ when $a = b = 1$) we have a degree 1 polynomial satisfied by $\mathsf{E} = \mathsf{E}_1$:

$$0 = -1 + \mathsf{E}_1(1 - z\mathsf{E}_1^{-(m-2)}\mathsf{E}_1^{(m-2)})^m = -1 + \mathsf{E}_1(1-z)^m$$

Hence $\mathsf{E}_1 = (1-z)^{-m}$ corresponding to the Euler characteristics $\chi_1 = -m$ and $\chi_n = 0$ for all $n > 1$. This is effectively a reverse-engineering of Reineke's proof of the algebraic relation satisfied by T_1. From the point of view of spectral networks, however, it is easiest to start from the algebraic relation for T_1 to derive the algebraic relation (and corresponding Euler characteristics) for E_1. Indeed, if one accepts that the BPS generating series for m-herds is the generating series T_1 for DT-invariants $d(n,n,m)$, then the spectral-network derivation of the relation $f_m(z,\mathsf{T}_1) = 0$ in Part I, along with the discussion above, constitutes an independent physical derivation of the Euler characteristics for stable quiver moduli with dimension vectors $(an, bn) = (n, n)$.

In §13.4 we compute the Euler characteristics in the case $(a,b) = (3,2)$ using the algebraic relation (12.9) for $\mathsf{T}_{3/2}$, which was derived through spectral network techniques applied to the $(3,2|3)$-herd.

13.3 Determining Euler characteristics from DT-invariants and Vice-Versa

Using (13.2), Reineke uses Lagrange-inversion to write down a rather explicit formula expressing DT-invariants in terms of a weighted sum over Euler characteristics of stable moduli (see Lemmata 5.2, 5.3 and Prop. 5.4 of [47]):

$$k^2 d_k = \sum_{l|k} \mu\left(\frac{k}{l}\right)(-1)^l \sum_{r=1}^{l} \left(\sum_{(\lambda_1,\cdots\lambda_r)\in\mathcal{P}(l,r)} (-1)^{\lambda_1} \prod_{j=1}^{r} \binom{jN\chi_j}{\lambda_j - \lambda_{j+1}} \right)$$

where

$$\mathcal{P}(l,r) := \{\text{Partitions of } l \text{ of length } r\} = \left\{ (\lambda_1, \cdots, \lambda_r) \in \mathbb{Z}^r : \sum_{j=1}^{r} \lambda_j = l \right\}.$$

On the other hand, by composing (13.2) with $z\mathsf{E}^{-N}$, we have (immediate from our discussion about compositional inverses)

$$\mathsf{E} = \mathsf{T} \circ \left(z\mathsf{E}^{-N}\right)$$

so we can apply the same Lagrange-inversion procedure to derive an explicit formula expressing Euler characteristics in terms of a weighted-sum over DT-invariants:

$$k^2\chi_k = \sum_{l|k} \mu\left(\frac{k}{l}\right)(-1)^l \sum_{r=1}^{l}\left(\sum_{(\lambda_1,\cdots\lambda_r)\in\mathcal{P}(l,r)} (-1)^{\lambda_1}\prod_{j=1}^{r}\binom{-jNd_j}{\lambda_j-\lambda_{j+1}}\right).$$

An alternative method, which is quite easy to implement into computer algebra software, is provided by the following. Suppose we are given the BPS generating series $\mathsf{T} = \mathsf{T}_{a/b}$; then the corresponding Euler characteristics can be extracted via the following recursive procedure:

$$\chi_k = \frac{1}{k}[z^k]\left\{\frac{\mathsf{T}}{\prod_{l=0}^{k-1}\left[1-(z\mathsf{T}^N)^l\right)]^{-l\chi_l}}\right\}, \tag{13.3}$$

which can be derived from (13.2). If one knows the BPS generating series accurately up to the coefficient multiplying z^n, then we can reliably extract the first n Euler characteristics in this manner. Similarly, if one were provided with the Euler characteristic generating series $\mathsf{E} = \mathsf{E}_{a/b}$, then

$$d_k = \frac{(-1)^{Nk}}{k}[z^k]\left\{\frac{\mathsf{E}}{\prod_{l=0}^{k-1}\left[1-(z\mathsf{E}^{-N})^l\right)]^{ld_l}}\right\}$$

Remark 13.3.1. *Note that from (13.3), we have*

$$\chi_1 = [z]\mathsf{T} = (-1)^{N+1}d(a,b,m) = (-1)^{\dim\mathcal{M}_{\mathrm{s}}^{(a,b)}}d(a,b,m).$$

Indeed, for primitive dimension vectors (i.e. coprime pairs (a,b)), all semi-stable representations are stable and the DT-invariants $d(a,b,m)$ are weighted Euler characteristics of the smooth complex projective variety $\mathcal{M}_{\mathrm{s}}^m(a,b) = \mathcal{M}_{\mathrm{ss}}^m(a,b)$ parameterizing stable representations; in this situation the weighting function is just an overall sign: (see [14, §4]):

$$d(a,b,m) = (-1)^{\dim\mathcal{M}_{\mathrm{s}}^{(a,b)}}\chi\left[\mathcal{M}_{\mathrm{s}}^m(a,b)\right] = (-1)^{N+1}\chi\left[\mathcal{M}_{\mathrm{s}}^m(a,b)\right].$$

On the other hand, for non-primitive dimension vectors, one should not expect that $|d(ka, kb, m)| = \chi_k$. *Indeed, from the perspective of Joyce-Song/Behrend, the DT-invariant* [3] $d(ka, kb, m)$, *for* $k > 1$, *is a weighted Euler characteristic (with non-trivial weighting function: see[14, Equation 7.56]) of the (typically singular) projective variety* $\mathcal{M}^{m}_{\mathrm{ss}}(ka, kb)$ *that parametrizes polystable representations.*

13.4 Euler Characteristics of Stable Moduli with Dimension vectors $(3n, 2n)$

In this section we specialize to the case $(a, b) = (3, 2)$. By applying Thm. 13.2.1 to the conjectured algebraic relation (12.9) for the generating series $\mathsf{T} = \mathsf{T}_{3/2}$, we find that $\mathsf{E} = \mathsf{E}_{3/2}$ must obey the algebraic relation:

$$\begin{aligned}
0 = & -z + \mathsf{E}(-5z + z^2) + \mathsf{E}^2(z - 13z^2 + z^3)+ \\
& \mathsf{E}^3(21z - 114z^2 + 44z^3 - z^4)+ \\
& \mathsf{E}^4(-34z - 80z^2 - 38z^3 + 7z^4)+ \\
& \mathsf{E}^5(-1 - 7z + 6z^2 + 77z^3 + 39z^4 + 3z^5)+ \\
& \mathsf{E}^6(4 + 76z + 119z^2 - 382z^3 - 367z^4 - 17z^5)+ \\
& \mathsf{E}^7(-6 - 64z + 53z^2 + 270z^3 - 173z^4 - 77z^5 - 3z^6)+ \\
& \mathsf{E}^8(4 + 6z - 55z^2 + 80z^3 - 30z^4 - 14z^5 + 9z^6)+ \\
& \mathsf{E}^9(-1 + 7z - 21z^2 + 35z^3 - 35z^4 + 21z^5 - 7z^6 + z^7).
\end{aligned} \tag{13.4}$$

Remark Note that, while (12.9) defines T as a root of a 39th degree polynomial (with coefficients in $\mathbb{Z}[z]$, the relation (13.4) defines E as a root of a 9th degree polynomial; in this sense, the relation satisfied by E is much easier than that satisfied by T.

By substituting in a formal power series (with constant coefficient 1) ansatz for E, one can determine the coefficients order by order. The first few values of the expansion are,

$$\mathsf{E} = 1 + 13z + 189z^2 + 1645z^3 - 6611z^4 - 429139z^5 + \mathcal{O}(z^6).$$

[3] In the notation of [14], the DT-invariants we are interested in are called "BPS-invariants" and denoted by the symbol $\hat{DT}$; the DT-invariants denoted $\bar{DT}$ are related to the coefficients of generating functions for DT-invariants.

If we determine this expansion out to the term multiplying z^K, we can reliably determine the first K Euler characteristics (of stable moduli). Section 13.3 describes various other procedures for extracting the Euler characteristics χ_n given the DT-invariant generating series T. The first few Euler characteristics are given by

$$(\chi_n)_{n=1}^{8} = (13, 49, -28, -5277, -50540, 546995, 19975988, 103632164)\,.$$

For leisure reading, the first 1360 χ_n– corresponding to dimension-vectors $(3n, 2n)$– can be found in the file `Euler1360.csv` included with the arXiv preprint version of this part of the thesis (to appear).[4]

As shown in §15.2, one can determine the $n \to \infty$ asymptotics of the χ_n directly from the algebraic relation (13.4). In particular, via Corollary 15.2.9, we find

$$\chi_n = n^{-5/2}\left[C\rho^{-n} + \overline{C}\,(\overline{\rho})^{-n}\right] + \mathcal{O}\left(n^{-7/2}|\rho|^{-n}\right) \tag{13.5}$$

where

$$\rho \approx 0.0151352 + 0.0373931i$$

and its complex conjugate $\overline{\rho}$ are roots of the polynomial

$$\begin{aligned} &-84375 - 86320988z + 1929101336z^2 - 60820581960z^3 \\ &+ 219678053524z^4 - 230881412176z^5 + 38310552576z^6 \\ &- 946356992z^7 + 4917248z^8 \end{aligned} \tag{13.6}$$

(which arises as a factor of the discriminant of (13.4) thought of as a polynomial in with coefficients in $\mathbb{Z}[z]$), and

$$C \approx -0.077407 + 0.200149i.$$

A numerical study of the first 1360 such Euler characteristics indicates a very strong fit to these asymptotics. Indeed, note that we may rewrite (13.5) as

$$\chi_n = 2|C|\Theta(n)n^{-5/2}|\rho|^{-n} + \mathcal{O}\left(n^{-7/2}|\rho|^{-n}\right) \tag{13.7}$$

where

$$\Theta(n) := \cos\left[n\arg(\rho) - \arg(C)\right]. \tag{13.8}$$

[4]These were extracted using (13.3) and the solution $\mathsf{T}_{3/2}$ expanded out to the coefficient multiplying z^{1360}.

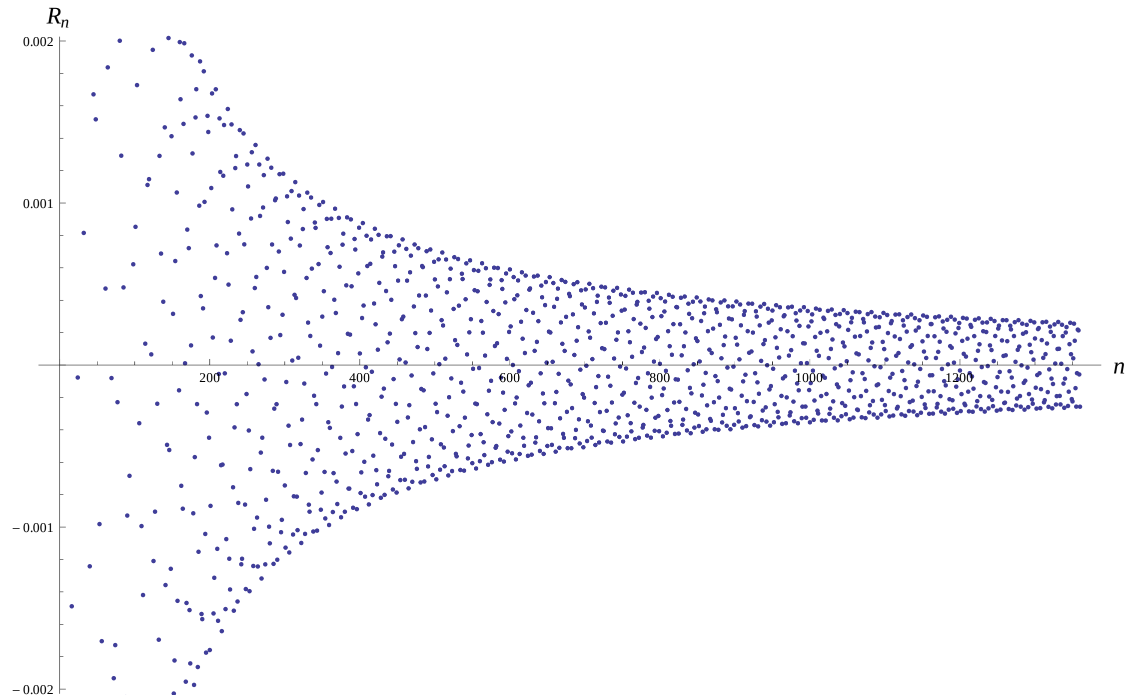


Figure 13.1: Plot of R_n (defined in (13.9)) for $1 \le n \le 1360$.

Now, defining

$$R_n := \frac{\chi_n}{2|C|n^{-5/2}|\rho|^{-n}} - \Theta(n) \tag{13.9}$$

then via (13.5) we have $R_n \in \mathcal{O}(n^{-1})$. In Fig. 13.1, the first 1360 values of the sequence R_n are plotted; visual inspection (and a bit of curve-fitting) suggests that R_n is snugly bounded by the curves $\pm cn^{-1}$, where $c \approx .36$.

Some Numerology

The scattered appearance of Fig. 13.2 is expected due to the sampling of the cosine function $\cos[n \arg(\rho) - \arg(C)]$ at integer points n: if $\arg(\rho)$ is an irrational multiple of 2π, then this sampling will never exhibit any exact periodicity in the variable n. Indeed, using the polynomial (13.6), a calculation (assisted by computer-algebra software) using resultants shows that the minimal polynomial of $\rho/|\rho|$ is not a cyclotomic polynomial. As a result, $\rho/|\rho|$ is not a root of unity; hence, $\arg(\rho)$ must be an irrational multiple of 2π.

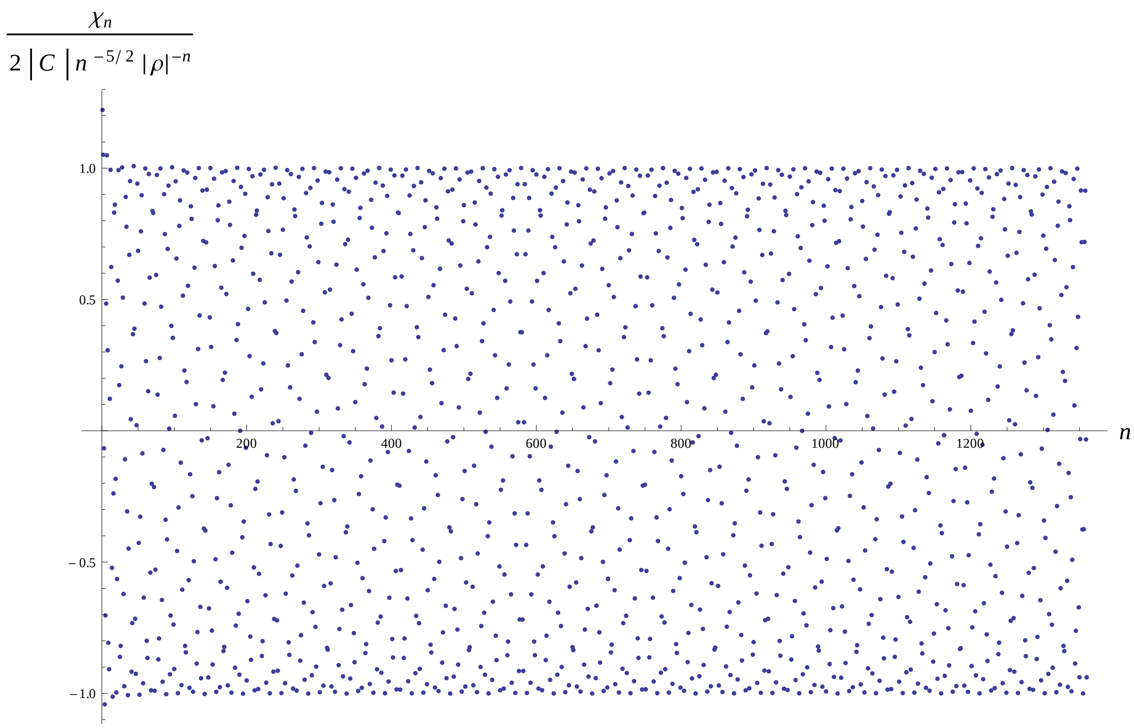


Figure 13.2: Plot of $\chi_n/(2|C|n^{-5/2}|\rho|^{-n})$ for $1 \leq n \leq 1360$

Nevertheless, among the chaos there seems to be some regularity: it is possible to recognize patterns in Figs. 13.2 13.1. In particular, a visually-striking pattern reveals itself if one defines the sequence

$$\mathcal{X}_n := \frac{\chi_n}{2|C|n^{-5/2}|\rho|^{-n}},$$

(which quickly approaches $\Theta(n)$ defined in (13.8)) and studies the sixteen subsequences defined by residue classes of n mod 16:

$$(\mathcal{X}_n : n \bmod 16 \equiv l)_{n=0}^{\infty} = (\mathcal{X}_{16k-l})_{k=0}^{\infty}\,,\ l = 0, \cdots, 15. \tag{13.10}$$

Each such subsequence appears to be sampled from a sinusoidal function with period $97/2$ in the index k. The reason for this is due to the close proximity of $\arg(\rho^{16})$ to $4\pi/97$; specifically

$$\left|\frac{16\arg(\rho)}{2\pi} - \frac{2}{97}\right| \mod 1 < 2.41 \times 10^{-6}$$

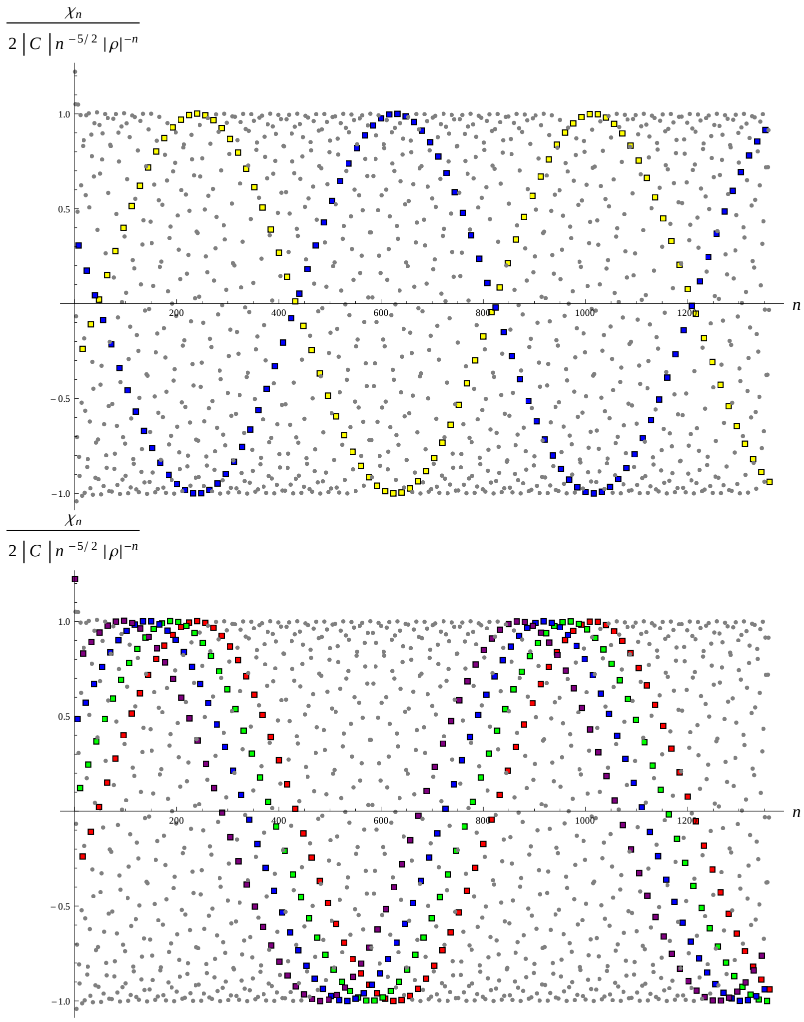


Figure 13.3: Plot of $\mathcal{X}_n = \chi_n/(2|C|n^{-5/2}|\rho|^{-n})$ for $1 \leq n \leq 1360$ with some approximately periodic subsequences highlighted. **Top**: Subsequence $(\mathcal{X}_n : n \mod 16 \equiv 0)$ highlighted with yellow squares; subsequence $(\mathcal{X}_n : n \mod 16 \equiv 8)$ highlighted with blue squares. **Bottom**: subsequences $(\mathcal{X}_n : n \mod 16 \equiv a)$ highlighted with red $(a = 0)$, green $(a = 5)$, blue $(a = 10)$, and purple $(a = 15)$ squares.

which leads to the approximate 97/2 periodicity in the index k over intervals of size $< 10^6$. Plots highlighting some of the subsequences (13.10) (indexed by n) are shown in Fig. 13.3.

An aside on the Weist-Douglas Asymptotics

Exponential asymptotics of DT-invariants and Euler characteristics associated to *collinear* dimension vectors are reminiscent of the conjecture that appears in the work of Thorston Weist [56, §6] regarding asymptotics of Euler characteristics/DT-invariants associated to sequences of *primitive* dimension-vectors.

Define the set of pairs of coprime integers (equivalent to the set of primitive dimension vectors for the m-Kronecker quiver)

$$C := \{(a,b) \in \mathbb{Z}_{\geq 0} : \gcd(a,b) = 1\}$$

and consider the function $\alpha : C \to \mathbb{R}_{\geq 0}$ given by

$$\alpha : (a,b) \mapsto \frac{\log |\chi(\mathcal{M}_{\mathrm{s}}^{m}(a,b)|}{b}.$$

Note that, because we are restricting our attention to primitive dimension vectors, DT-invariants are (up to a sign) the same as Euler characteristics: i.e. for $(a,b) \in C$ we have

$$\chi(\mathcal{M}_{\mathrm{s}}^{m}(a,b)) = (-1)^{\dim[\mathcal{M}_{\mathrm{s}}^{m}(a,b)]} = (-1)^{N+1} d(a,b,m).$$

so, equivalently,

$$\alpha : (a,b) \mapsto \frac{\log |d(a,b,m)|}{b}.$$

The Weist-Douglas conjecture concerns the behaviour of $\alpha(a,b)$ for (a,b) such that $a+b$ is sufficiently large and (see Appendix G.2):

$$m_{-} := \frac{m - \sqrt{m^2-4}}{2} \leq \frac{a}{b} \leq \frac{m + \sqrt{m^2+4}}{2} =: m_{+}.$$

In particular, the conjecture states that, for such dimension vectors, $\alpha(a,b)$ is well-approximated by a continuous function that depends only on the ratio a/b. We state the precise form of the conjecture below (taken essentially verbatim from [55, Conj. 6.1]).

Theorem 13.4.1 (Conjecture of M. Douglas; Precise Formulation due to T. Weist)**.** *There exists a continuous function $f : [m_{-}, m_{+}] \to \mathbb{R}$ such that the following holds: for all $r \in [m_{-}, m_{+}]$ and all $\epsilon > 0$ there exists a $\delta > 0$ and an $n \in \mathbb{Z}_{\geq 0}$ such that for all $(a,b) \in C$ with $|r - a/b| < \delta$, and $|a+b| > n$ we have*

$$|f(r) - \alpha(a,b)| < \epsilon$$

If we assume the conjecture, then along any sequence $(a_n, b_n)_{n=1}^{\infty} \subset C$ such that $\lim_{n\to\infty} b_n/a_n \in [m_-, m_+]$ and $a_n + b_n$ is strictly increasing, then $\lim_{n\to\infty} \alpha(a_n, b_n)$ exists and is given by

$$\lim_{n\to\infty} \alpha(a_n, b_n) = f(r).$$

An explicit form for $f(r)$ can be given.

Theorem 13.4.2 (T. Weist)**.** *If the conjecture is true then*

$$f(r) = \frac{W_m}{\sqrt{m-2}} \sqrt{r(m-r)-1};$$

where

$$W_m := (m-1)^2 \log\left[(m-1)^2\right] - m(m-2)\log\left[m(m-2)\right].$$

Of particular interest to us are sequences whose ratio a_n/b_n limits to a rational number in (m_-, m_+), i.e. sequences limiting to an imaginary Schur root (c.f. §G.2). Given a fixed $r \in \mathbb{Q}_{>0}$, Weist provides a method for determining an explicit subsequence $(a_n, b_n)_n \subset C$ such that $\lim_{n\to\infty} a_n/b_n = r$.

Proposition 13.4.3. *Let $(a, b) \in C$. Then there is a unique pair of positive integers (a_0, b_0) (determined using the techniques in [55, §4]) such that*

1. $\gcd(a_0 + na, b_0 + nb) = 1$ *for all* $n \in \mathbb{Z}_{\geq 0}$*;*
2. $a_0 \leq a$ *and* $b_0 \leq b$*.*

Via the proposition, for every $r = a/b \in (m_-, m_+)$, we can find a subsequence $(a_n^*, b_n^*)_n \subset C$ such that $\lim_{n\to\infty} a_n^*/b_n^* = r$ by defining

$$(a_n^*, b_n^*) := (a_0 + na, b_0 + nb).$$

According to Conj. 13.4.1 and Thm. 13.4.2, we have

$$\lim_{n\to\infty} \frac{1}{n} \log|\chi(\mathcal{M}_{\mathrm{s}}^m(a_n^*, b_n^*))| = W_m \sqrt{\frac{mab - a^2 - b^2}{m-2}}.$$

Now, let us focus on the sequence whose slope approaches $r = 1$ (i.e. choose $a = b = 1$); in this case we have $(a_n^*, b_n^*) = (n, n+1)$ and

$$\lim_{n\to\infty} \frac{1}{n} \log|\chi(\mathcal{M}_{\mathrm{s}}^m(n, n+1))| = W_m.$$

Strangely enough, these asymptotics agree with the asymptototics for DT-invariants associated to collinear dimension vectors (n, n); that is, via (3.22) (rederived in Example 1 at the end of §15.2)), we have

$$\lim_{n\to\infty} \frac{1}{n} \log |d(n, n, m)| = W_m$$
$$= \lim_{n\to\infty} \frac{1}{n} \log |\chi(\mathcal{M}_{\mathrm{s}}^m(n, n+1))|$$
$$= \lim_{n\to\infty} \frac{1}{n} \log |d(n, n+1, m)|.$$

On the other hand, $\chi(\mathcal{M}_{\mathrm{s}}^m(n, n)) = 0$ for $n > 1$.

This coincidence of asymptotics does not hold for the sequence approaching slope $r = 2/3$ when $m = 3$. Indeed, in this case $(a_n^*, b_n^*) = (1 + 2n, 2 + 3n)$ and

$$\lim_{n\to\infty} \frac{1}{n} \log |d(1 + 2n, 2 + 3n, 3)| = \sqrt{5}\,[4\log(4) - 3\log(3)] \approx 5.02698.$$

On the other hand, as a corollary of (12.10) and $d(2n, 3n, 3) = d(3n, 2n, 3)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log |d(2n, 3n, 3)| \approx -\log(0.005134) \approx 5.27187.$$

In the spirit of comparisons, one can also extract the exponential growth-rate[5] of Euler characteristics of stable moduli using (13.7):

$$\limsup_{n\to\infty} \frac{1}{n} \log \left|\chi\left(\mathcal{M}_{\mathrm{s}}^3(2n, 3n)\right)\right| \approx -\log(0.040340) \approx 3.21041.$$

[5]The function $\Theta(n)$ in (13.7) requires us to use the limit superior versus an actual limit when defining the exponential-growth rate: because $\arg(\rho)$ is an irrational multiple of π (see §13.4), then for any fixed N the sequence $(\Theta(n))_{n=N}^{\infty}$ densely fills the interval $[-1, 1]$.

Chapter Fourteen: Algebraic Generating Series and Spectral Networks: General Features

The appearance of algebraic equations for BPS indices seems to be a feature of spectral networks: in all degenerate spectral-networks so far encountered, the resulting equations for soliton-generating functions around joints (e.g. the six-way street equations) and branch points generate a system of algebraic equations that completely determine the corresponding street-factors. Indeed, in App. D it is argued that this is a property for "sufficiently nice" spectral networks: the corresponding street-factors are determined by a collection of algebraic equations. In this section we will give a more precise statement of algebraicity for spectral networks. Our key statement is Claim 14.0.4, which requires a slightly weaker version of the condition on the spectral network compared to the condition[1] in App. D. We will not prove this claim here, but defer the reader to future work: an honest proof requires a development of careful definitions for spectral networks; in particular, to make sense of the product of soliton generating functions– which are supposed to be formal series in non-commutative variables– one must carefully define the appropriate non-commutative ring of formal series as the completion of a filtered ring. Such a development deserves separate treatment that will lead us too far astray.

In this section, by spectral network we mean something either satisfying a generalization[2] of the conditions D1-C4 in [27, §9.1] or a $\mathcal{W}$-network. Recall that, to each street of the spectral network we associate two quantities we call *soliton generating functions* (see §2.2);[3] they are elements of a non-commutative ring of formal series (which, as mentioned above, takes some care to properly define). From a polynomial

[1] In App. D, if $\mathcal{N}$ is a spectral network subordinate to $\pi : \Sigma \to C$, the argument required the existence of a compact set $K \subset C$ such that all two-way streets are contained in K, and only finitely many streets (one-way or two-way) are contained in K. The condition we state here weakens this to only require finitely many two-way streets; this weakening may be important in the context of $(a, b|m)$-herds.

[2] Namely, we allow for spectral networks with no singular points and relax the condition on C1 that any compact subset of C' intersects only finitely many segments (allowing for spectral networks with densely defined streets).

[3] In Part I and in the appendices, these generating functions are denoted by the symbols Υ_p and Δ_p or, in [27], as τ and ν.

combination of soliton generating functions on the street p of a spectral network, we can derive *street factors* $Q(p)$ (see §2.2);)– these latter quantities are elements of a *commutative* ring of formal series, and are directly related to the generating function for BPS indices (see the proof of Cor. 14.0.5).

We should emphasize that the soliton generating functions and street-factors are not taken as data in the definition of a spectral network, but are derived directly from the data of spectral networks. Indeed, from the data, we can derive a system of relations that must be obeyed by soliton generating functions; let us temporarily dub these as the *vertex relations*.[4] We claim that, the vertex relations can be used to define the soliton generating functions: interpreting the relations as equations in non-commutative variables, there is a unique solution in a particular non-commutative ring of formal series.

Now let us specialize our attention to sufficiently "generic" degenerate spectral networks: these are spectral networks that support a single charge (for $\mathcal{W}$ networks, these are degenerate networks that are off any walls of marginal stability). To define what this means, recall that the data of any spectral network $\mathcal{N}$ defines a directed graph $\mathrm{Lift}(\mathcal{N})$ with embedding into Σ (see §2.2 and App. C.7.2 of [31]).

Definition We will say a spectral network $\mathcal{N}$ *supports a single charge* γ_c if there exists a non-constant closed path on $\mathrm{Lift}(\mathcal{N})$ (whose orientation is consistent with each edge) and, moreover, the image of each such closed path under the embedding $\mathrm{Lift}(\mathcal{N}) \hookrightarrow \Sigma$ has homology class contained in $\mathbb{Z}_{>0}\gamma_c \leq H_1(\Sigma;\mathbb{Z})$.

By the discussion above, using the definition of the soliton generating functions as the unique solution to the vertex relations (in a particular non-commutative ring of formal series), we state the following.

Claim Suppose that $\mathcal{N}$ is a spectral network supporting a single charge γ_c; define $\widetilde{z} := X_{\widetilde{\gamma}_c}$. Then, for any street p of $\mathcal{N}$, we can use the vertex relations to define $Q(p)$ as an element of the ring $\mathbb{Z}[\![\widetilde{z}]\!]$ of formal series; moreover, $Q(p)$ has constant coefficient 1.

[4]Alternatively, as suggested by Greg Moore, these should be called "non-abelian Kirchoff rules" due to their alternative interpretation as conservation laws for solitons of certain types coming in or out of each vertex.

Now we may give a statement of algebraicity in the context of spectral networks: under some mild assumptions, the (a priori) formal series $Q(p)$ are algebraic functions over $\mathbb{Q}$. The precise statement is the following.

Claim 14.0.4. *Let $\mathcal{N}$ be a spectral network subordinate to the branched cover $\Sigma \to C$, such that*

1. *$\mathcal{N}$ supports a single charge $\gamma_c \in H_1(\Sigma; \mathbb{Z})$;*
2. *Its collection of strictly two-way streets*

$$\mathrm{degstr}(\mathcal{N}) := \{p \in \mathrm{str}(\mathcal{N}) : Q(p) \neq 1\}$$

is a finite set.

Then, letting $\widetilde{z} = X_{\widetilde{\gamma}_c}$, for each $p \in \mathrm{degstr}(\mathcal{N})$ there exists a $\mathcal{F}_p \in (\mathbb{Q}[\widetilde{z}])[x]$ (a polynomial with coefficients in $\mathbb{Q}[\widetilde{z}]$) such that

$$0 = \mathcal{F}_p [Q(p)] . \tag{14.1}$$

Although we are deferring the proof of this claim until future work, it should be mentioned that the proof is of a constructive nature: the polynomial relations $\mathcal{F}_p$ of the claim can be extracted through applications of elimination theory to a system of polynomial equations given produced by “abelianizing” the vertex relations. In other words, from a spectral network we can (in principle) algorithmically extract explicit algebraic relations obeyed by the $Q(p)$.

Remark Although no examples are known, in principle it may be possible for a degenerate $\mathcal{W}$-network to violate the first condition. However, for the network to correspond to a BPS state of finite mass, the total length of any closed curve supported on the lift of the two-way streets to the spectral cover (defined using the holomorphic differential on the spectral cover) must remain finite.

The claim induces the following corollary.

Corollary 14.0.5. *Let T be the BPS generating series associated to any spectral network satisfying the conditions of Claim 14.0.4, then there exists a polynomial $\mathcal{R} \in (\mathbb{Z}[z])[t]$ such that*

$$0 = \mathcal{R}(\mathsf{T}) .$$

Proof. The BPS generating series (c.f (11.2)) T is a monomial in the street-factors $Q(p)$, $p \in \mathrm{degstr}(\mathcal{N})$: let

- $\mathcal{I}(\cdot,\cdot) : C_1(\Sigma;\mathbb{Z})^{\otimes 2} \to \mathbb{Z}$ be the skew-symmetric form on $C_1(\Sigma;\mathbb{Z})$ defined by linearizing the oriented-intersection pairing of paths on Σ;
- $g_c^\vee \in C_1(\Sigma;\mathbb{Z})$ be a *closed* 1-chain such that its corresponding homology class $\gamma_c^\vee := [g_c^\vee] \in H_1(\Sigma;\mathbb{Z})$ satisfies

$$\langle \gamma_c^\vee, \gamma_c \rangle = 1$$

  under the usual intersection pairing on homology $\langle \cdot,\cdot \rangle : H_1(\Sigma;\mathbb{Z})^{\otimes 2} \to \mathbb{Z}$;
- $\ell(p) \in C_1(\Sigma;\mathbb{Z})$ be the lift of p to a 1-chain on Σ;

then, via (12.5)-(12.7), the BPS generating series is given by

$$\mathsf{T} = \prod_{p \in \mathrm{str}(\mathcal{N})} Q(p)^{\mathcal{I}(g_c^\vee, \ell(p))},$$

where this latter product has finite support by assumption: $Q(p) \neq 1$ if and only if p is two-way. As any polynomial combination of elements algebraic over some field is again algebraic, then T is algebraic over $\mathbb{Q}(z)$: by successive applications of resultants to the polynomials $\mathcal{F}_p \in (\mathbb{Z}[z])[x]$, one can produce a polynomial $\mathcal{R} \in (\mathbb{Z}[z])[t]$ such that

$$0 = \mathcal{R}(\mathsf{T}).$$

□

Examples

1. m**-herds**: Let m be an integer ≥ 1. One of the results of [31] was that all generating functions (the soliton generating functions, the street-factors, and the BPS generating series) associated to the m-herd can be expressed in terms of powers of a formal series P satisfying the relation

$$0 = P - zP^{(m-1)^2} - 1;$$

that is, if we define

$$\mathcal{H}_k := p - zp^k - 1 \in (\mathbb{Z}[z])\,[p] \tag{14.2}$$

then all street-factors and the BPS generating series are given by powers of the root P of $\mathcal{H}_{(m-1)^2}$ and, hence, must satisfy algebraic relations themselves (which may be computed explicitly by use of resultants). For example, it was shown in [31] that the BPS generating series for an m-herd is given in terms of P as

$$\mathsf{T}_{(1|m)} = P^m;$$

one can verify that $\mathsf{T}_{(1|m)}$ arises as a root of the polynomial[5]

$$\mathcal{F}_{(1|m)} = -1 + t(1 - zt^{(m-2)})^m. \tag{14.3}$$

2. **The $(3,2|3)$-herd**: All street-factors of the $(3,2|3)$-herd can be written as Laurent-polynomials in the quantities M, V, W satisfying the system of algebraic relations (9.2). Using elimination theory, from this system of relations, one can deduce univariate polynomials $\mathcal{F}_V$, $\mathcal{F}_W$, $\mathcal{F}_M$ with coefficients in $\mathbb{Z}[z]$ such that

$$\begin{aligned} 0 &= \mathcal{F}_M(V) \\ 0 &= \mathcal{F}_V(W) \\ 0 &= \mathcal{F}_W(M); \end{aligned}$$

moreover, the roots of any one of $\mathcal{F}_M$, $\mathcal{F}_V$, $\mathcal{F}_W$ are in one-to-one correspondence tuples of algebraic functions (m, v, w) obeying

$$\begin{aligned} 0 &= 1 + zm^4\left\{(1+v)(1+v-w)^2[v^2(1+w)-1]^3\right\} - m, \\ 0 &= (-1+v)(1+v)^2 + (1+v^3)W - v(m+v)w^2, \\ 0 &= v\left(v^2-1\right) - [m(v+1) + v(v-2) - 1]\,w, \end{aligned} \tag{14.4}$$

Now, (14.4) has forty-two tuples $(m, v, w) \in \overline{\mathbb{Q}(z)}^3$ of solutions, three of which are constant solutions (elements of $\overline{\mathbb{Q}}^3$). As already mentioned in §12.4, elimination theory provides us with the algebraic relation (12.9) involving only z

[5]By substituting $t = p^m$ into (14.3), one can prove that $\mathcal{H}_{(m-1)^2}(p) = 0$ if and only if $\mathcal{F}_{(1|m)}(p^m) = 0$ (the "if" direction requiring a tiny bit more thought than the "only if" direction).

and the BPS generating series $\mathsf{T}_{3/2} = MVW$. In other words, $\mathsf{T}_{3/2}$ is a root of the degree 39 polynomial $\mathcal{F}_{(3/2|3)} \in (\mathbb{Z}[z])\,[t]$ listed below:

$$\begin{aligned}
\mathcal{F}_{(3/2|3)} := -(1+z) + t(4-5z) + t^2(-6+z) + t^3(4+21z)+ \\
t^4(-1-34z) - t^5(7z) + t^6(76z+z^2) + t^7(-64z-13z^2)+ \\
t^8(6z-114z^2) + t^9(7z-80z^2) + t^{10}(6z^2)+ \\
t^{11}(119z^2) + t^{12}(53z^2+z^3) + t^{13}(-55z^2+44z^3)+ \\
t^{14}(-21z^2-38z^3) + t^{15}(77z^3) - t^{16}(382z^3)+ \\
t^{17}(270z^3) + t^{18}(80z^3-z^4) + t^{19}(35z^3+7z^4)+ \\
t^{20}(39z^4) - t^{21}(367z^4) - t^{22}(173z^4)- \\
t^{23}(30z^4) - t^{24}(35z^4) + t^{25}(3z^5) - t^{26}(17z^5)- \\
t^{27}(77z^5) - t^{28}(14z^5) + t^{29}(21z^5) - t^{32}(3z^6)+ \\
t^{33}(9z^6) - t^{34}(7z^6) + t^{39}z^7.
\end{aligned} \tag{14.5}$$

This polynomial enjoys the following properties:

- it is irreducible as a polynomial in $\mathbb{C}[z,t]$;
- each of its roots is in one-to-one correspondence with a triple (m, v, w) of non-constant solutions to (14.4).

As a consequence of the latter property and the discussion at the end of §12.3, the BPS generating series $\mathsf{T}_{3/2} = MVW$ is the unique solution given by a formal power series with constant coefficient 1.[6]

In §15 we go about exploring the general properties of the algebraic curves associated to these algebraic equations. In the process, we show that we can always find a factor of $\mathcal{R} \in (\mathbb{Z}[z])\,[y]$ (c.f. Cor. 14.0.5) that is irreducible over $\mathbb{C}(z)$ and, furthermore, still has coefficients in $\mathbb{Z}[z]$.

[6] An alternate derivation of this statement is provided in Example 2 at the end of §15.1.

Chapter Fifteen: Algebraic Generating Functions and Asymptotics

15.1 Some Recollections on Algebraic Curves and Algebraic Equations

We first present a geometric reinterpretation of the (algebraic) functional equations of Claim 14.0.4 that will prove useful in the following sections. In essence, the following is no more than a review of classical algebraic/complex geometry and most of the techniques here can be found scattered throughout elementary references (e.g. [37]; nevertheless, the hope is that some readers (apart from the author) will find it convenient to have these observations in one place.

Notation

1. $\overline{\mathscr{K}}$ will denote the algebraic closure of a field $\mathscr{K}$.

2. Let R be a ring, then $\mathsf{taut}_R : (R[z])\,[t] \to R[z,t]$ will denote the tautological map; it is an isomorphism of $R[z]$-modules. Elements of $(R[z])\,[t]$ will be denoted by calligraphic letters and their images under taut_R will be denoted by their respective Roman-typeface analogues.

3. In the following two sections: T will denote a general algebraic function, free from a particular context (as opposed to the BPS generating series T that is indicated with sans-serif font). The symbol z will denote a formal variable, which can be interpreted free from any particular context; however, when we specialize to examples coming from $(a,b|m)$-herds, z will specifically denote the variable defined in (11.5) (and explained in Appendix I). We will occasionally abuse notation and write statements like "$z \in \mathbb{C}$" when we mean the evaluation of the variable z at some value in $\mathbb{C}$.

Algebraic Curves

Suppose we wish to study solutions T to the algebraic functional equation

$$\mathcal{R}(T) = 0 \tag{15.1}$$

for some $\mathcal{R} \in (\mathbb{Z}[z])\,[t]$, i.e. a polynomial with coefficients in $\mathbb{Z}[z]$. Equivalently, we wish to study roots of $\mathcal{R}$, which splits in $\overline{\mathbb{Q}(z)}[t]$: if $c = \deg_{\mathbb{Z}[z]} \mathcal{R}$, then

$$\mathcal{R} = A\,(t - T_1) \cdots (t - T_c)$$

for some $A \in \overline{\mathbb{Q}(z)}$ and $T_l \in \overline{\mathbb{Q}(z)}$, $l = 1, \cdots, c$. Being geometrically minded, we will study the collection of roots $\{T_l\}_{l=1}^{c}$ by analysing complex affine/projective (typically singular) algebraic curve(s) associated to $\mathcal{R}$; when we pass to the analytic world, the roots can be interpreted as local sections of a branched cover defined by this curve.

Before proceeding further, we make an observation that will prove useful for analysing the behaviour of T at "poles" (points on the z-plane where T becomes infinite).[1]

Remark 15.1.1. *The space of algebraic functions $\overline{\mathbb{Q}(z)}$ is a field; in particular, if $T \not\equiv 0$ is algebraic, then $1/T$ must be algebraic. In fact, if T is a root of (15.1), then*

$$0 = \widehat{\mathcal{R}}\left(\frac{1}{T}\right)$$

where $\widehat{\mathcal{R}} \in \mathbb{Z}[z, y]$ is defined by

$$\widehat{\mathcal{R}}(y) = y^e \mathcal{R}\left(\frac{1}{y}\right).$$

Note that the operation $\widehat{(\cdot)} : \mathcal{R} \mapsto \widehat{\mathcal{R}}$ is an involution and commutes with factorizations; in particular $\widehat{\mathcal{R}}$ is irreducible (over some chosen field) iff $\mathcal{R}$ is irreducible.

Let $r = \mathsf{taut}_{\mathbb{Z}}\mathcal{R} \in \mathbb{Z}[z, t]$ be the polynomial in two variables defined by $\mathcal{R}$. The zero locus of r defines an affine curve $V_r \subset \mathbb{C}^2$; to simplify the following discussion, we focus on any given irreducible component of this curve: suppose r factors as a polynomial in $\mathbb{C}[z, t]$ as

$$r = f_1^{m_1} \cdots f_k^{m_n} \tag{15.2}$$

[1] We place "poles" in quotes as the sort of singularities encountered may also manifest themselves as branch points: e.g. the point $z = 0$ of the function $z^{-1/2}$.

for irreducible $f_l \in \mathbb{C}[z,t]$ and multiplicities $m_1, \cdots, m_n \in \mathbb{Z}_{>0}$, then each V_{f_i}, $i = 1, \cdots k$ is an irreducible complex algebraic curve.

Remark 15.1.2. *As $r \in \mathbb{Q}[z,t]$, then the irreducible factors $f_l \in \mathbb{C}[z,t]$ in (15.2) actually have $\overline{\mathbb{Q}}$ coefficients: $f_l \in \overline{\mathbb{Q}}[z,t]$.*[2]

Let f be one of the irreducible factors f_l. Regarding our choice of f in practice, note that the roots of $\mathcal{F}$ are a subset of the roots of $\mathcal{R}$; so if our interest lies in studying a particular root T_* of $\mathcal{R}$, then we choose f to be an irreducible factor such that

$$\mathcal{F} := \mathsf{taut}_{\overline{\mathbb{Q}}}^{-1} f \in \left(\overline{\mathbb{Q}}[z]\right)[t]$$

has T_* as a root. We make one further remark that will come into use later.

Remark $\mathcal{F}$ is irreducible in $(\mathbb{C}(z))[t]$ by virtue of the irreducibility of f in $\mathbb{C}[z,t]$ in combination with Gauss' Lemma. Hence, $\mathcal{F}$ has a root in rational functions $\mathbb{C}(z)$ if and only if $\deg_{\overline{\mathbb{Q}}[z]} \mathcal{F} = 1$.

Our ulterior motive behind these statements are, of course, that we wish to choose f such that $\mathcal{F}$ has a root $T \in \mathbb{Z}[\![z]\!] \cap \overline{\mathbb{Q}(z)}$ that enjoys an interpretation either as a street-factor, or the generating series for a collection of BPS indices. If we choose f in such a way, we can use the following gem of a lemma (attributed to Eisenstein [23]) that allows us – without loss of generality – to assume that f has coefficients in $\mathbb{Z}$.

Lemma 15.1.1 (Eisenstein)**.** *Let $T \in \mathbb{Q}[\![z]\!] \cap \overline{\mathbb{C}(z)}$ be non-zero (i.e. an algebraic function over $\mathbb{C}$ that can be represented as a formal series with rational coefficients); suppose $f \in \mathbb{C}[z,t]$ is a non-zero polynomial such that $f(z,T(z)) \equiv 0$, then:*

1. *there exists a non-zero $g \in \mathbb{Z}[z,t]$ such that $g(z,T(z)) = 0$ and the degree of g as a polynomial in t (resp. z) is less than or equal to the degree of f as a polynomial in t (resp. z);*

[2] Indeed, the factorization (15.2) induces a factorization $\mathcal{R} = \mathcal{F}_1 \cdots \mathcal{F}_n$ where each factor $\mathcal{F}_l \in (\mathbb{C}[z])[t]$ is the image of f_l under the obvious tautological map. The algebraic closure $\overline{\mathbb{Q}(z)}$ contains the splitting field of $\mathcal{R}$; so $\mathcal{F}_l \in \overline{\mathbb{Q}(z)}[t]$ for all l. On the other hand, $f_l \in \mathbb{C}[z,t]$ means that $\mathcal{F}_l$ must have coefficients in $\overline{\mathbb{Q}(z)} \cap \mathbb{C}[z] = \overline{\mathbb{Q}}[z]$. The equality in the previous sentence (which completes the proof), follows by showing that any element of $\overline{\mathbb{Q}(z)}$ can have zeros only at algebraic numbers.

2. *if f is irreducible in $\mathbb{C}[z,t]$, then there exists a constant $c \in \mathbb{C}$ such that $f = cg$.*

Proof. Assuming the first statement, the second statement is immediate from the first part along with properties of minimal polynomials. To prove the first statement, we fill out the details of the proof in [23, p. VII.37] . Let $A \subset \mathbb{C}$ be the set of coefficients of f and $V = \mathbb{Q}A$ the (finite-dimensional) $\mathbb{Q}$-vector space generated by A. Choose a basis $B \subset \mathbb{C}$ (of $\mathbb{Q}$-linearly independent elements) for V, then f can be written as

$$f = \sum_{b \in B} b f_b,$$

where $f_b \in \mathbb{Q}[z,t]$ for every $b \in B$ and at least one f_b is non-zero. If there is only one non-zero f_b then we are done; so assume that there are at least two non-zero f_b. Because $f(z, T(z)) = 0$, we have

$$0 = \sum_{b \in B} b f_b(z, T(z)); \tag{15.3}$$

the right hand side is a formal power series in z. Now, if none of the $f_b(z, T(z))$ are identically zero, then there exists a k such that, extracting the coefficient of z^k on the right hand side of (15.3), produces a $\mathbb{Q}$-linear relation between at least two elements of B – a contradiction. □

In summary, the discussion above shows us that if we only care about roots of (15.1) that arise as generating functions for a combinatorial/ "counting" problem (e.g. generating functions that count solitons/BPS states, or closed curves on a spectral network), then we can always reduce to another algebraic equation with coefficients in $\mathbb{Z}$ which, moreover is irreducible over $\mathbb{C}[z,t]$ (a.k.a *absolutely irreducible*). Hence, we can study the geometry of the (complex) curve associated to the "counting" problem root, while maintaining $\mathbb{Z}$-coefficients.

Remark Because the results of the following sections are quite general and do not rely on $\mathbb{Z}$ coefficients, we will work in complete generality and assume – via Remark 15.1.2– that we have chosen $f \in \overline{\mathbb{Q}}[z,t]$, irreducible in $\mathbb{C}[z,t]$.

Now, the curve V_f is equipped with two maps to copies of $\mathbb{C}$ via restrictions of the standard coordinate projections of $\mathbb{C}^2$ to $\mathbb{C}$:

$$
\begin{array}{ccc}
& V_f := \{(z,t) \in \mathbb{C}^2 : f(z,t) = 0\} \subset \mathbb{C}^2 & \\
& \swarrow^{\mathrm{pr}_1|_{V_f} = \phi}_{z \,\leftarrow\!\shortmid\, (z,t)} \qquad \searrow^{\tau = \mathrm{pr}_2|_{V_f}}_{(z,t) \mapsto t} & \\
\mathbb{C} & & \mathbb{C}
\end{array}
\tag{15.4}
$$

Letting $e := \deg_{\overline{\mathbb{Q}}[z]} \mathcal{F}$, then ϕ is a degree e (algebraic) branched cover: there exists a Zariski closed set (the branching locus) $\mathcal{B}$ such that, defining $V_f' := V_f \backslash \mathcal{B}$, the restriction $\phi|_{V_f'} : V_f' \to \mathbb{C} \backslash \mathcal{B}$ is a degree e cover.

For simplicity, first suppose that $\mathcal{F}$ is monic; then the branching locus (coinciding with the critical values of ϕ) is given precisely by the values of z_* where the polynomial $f|_{z=z_*} \in \overline{\mathbb{Q}}[t]$ has a repeated root: hence $\mathcal{B}$ is just the zero locus Δ_0 of the discriminant polynomial $\mathrm{Disc}(\mathcal{F}) \in \overline{\mathbb{Q}}[z]$:

$$\Delta_0 := \{z \in \mathbb{C} : \mathrm{Disc}(\mathcal{F}) = 0\} \subset \overline{\mathbb{Q}}; \tag{15.5}$$

the corresponding set of critical points of ϕ is given by the *ramification locus*:

$$\mathcal{R}_0 := \left\{ (z_*, t_*) \in V_f : \left.\frac{\partial f}{\partial t}\right|_{(z_*,t_*)} = 0 \right\}.$$

We can define the ramification index $\nu_p \in \mathbb{Z}_{>0}$ of a point $p = (z_*, t_*) \in V_f$ directly in terms of derivatives of f:

$$\nu_p := \min \left\{ k \in \mathbb{Z}_{>0} : \left.\frac{\partial^k f}{\partial t^k}\right|_{(z_*,t_*)} \neq 0 \right\} + 1; \tag{15.6}$$

geometrically ν_p is the number of sheets of the cover $V_f \to \mathbb{C}$ that collide at the point p. Of course, $\nu_p > 1$ if and only if $p \in \mathcal{R}_0$.

Now let us consider the more general situation – the situation encountered in our examples – where $\mathcal{F}$ is not monic. The resulting story is not the typical "textbook" picture for affine algebraic branched covers; in particular, the curve may have infinite branches at finite values of z. To see this, note that f can be written in the form

$$f(z,t) = a_e(z)t^e + a_{e-1}(z)t^{e-1} + \cdots a_0(z),$$

where $a_l \in \overline{\mathbb{Q}}[z]$; $l = 0, \cdots, e$. Then, for generic $z_* \in \mathbb{C}$, the degree of the polynomial $f|_{z=z_*} \in \overline{\mathbb{Q}}[t]$ is just e; however, this degree drops precisely for z_* in the set of isolated points

$$\mathcal{P} := \{z \in \mathbb{C} : a_e(z) = 0\} \subset \overline{\mathbb{Q}}. \tag{15.7}$$

From the geometric point of view, branches of $\phi_f : V_f \to \mathbb{C}$ "go off to infinity" as we approach points in $\mathcal{P}$; if the degree of f at $z_* \in \mathcal{P}$ is $r < e$, then precisely $e - r$ branches diverge in this manner. Thus, the branching locus[3] for the affine curve associated to a general irreducible f is given by

$$\mathcal{B} = \Delta_0 \cup \mathcal{P}.$$

Moreover, from the topological perspective (using the analytic topology), it still makes sense to talk about ramification around a chosen point in $\mathcal{P}$ by analysing the monodromy corresponding to lifts of small loops encircling the point. One way to see the ramification algebraically is to eliminate the whole infinite branch problem by passing to the projectivized curve, as we will do shortly; however, we can also see any ramification from the perspective of affine curves. Indeed, let $\widehat{\mathcal{F}}$ be defined according to the operation in Rmk. 15.1.1; then the zero locus of $\widehat{f} = \mathsf{taut}_{\overline{\mathbb{Q}}}^{-1}\widehat{\mathcal{F}}$ defines an affine curve $V_{\widehat{f}}$, birationally equivalent to V_f. Note that,

$$\widehat{f}(z,y) = a_0(z)y^e + a_1(z)y^{e-1} + \cdots a_e(z);$$

so if the t-degree of f at z_* is $r \leq e$, then $a_l(z_*) = 0$ for $l \geq r+1$; thus,

$$\widehat{f}(z_*,y) = y^{e-r}\left[a_0(z_*)y^r + \cdots + a_r(z_*)\right].$$

Hence, there is a ramification point of the "z-projection"

$$\begin{array}{rccl} \phi_{\widehat{f}}: & V_{\widehat{f}} & \longrightarrow & \mathbb{C} \\ & (z,y) & \longmapsto & z \end{array}$$

at the point $p = (z_*, 0) \in V_{\widehat{f}} \subset \mathbb{C}^2$ with index $\nu_p = e - r + 1$.

Remark It is a useful fact that the discriminant locus Δ_0 defined in (15.5) detects the presence of ramification of ϕ_f at points in $\mathcal{P}$; however, strictly speaking, there is no corresponding ramification point to speak of on the affine curve V_f (although there will be on its projectivization).

Similarly, the branched cover $\phi_{\widehat{f}}$ also has infinite (possibly ramified) branches over any point $z_* \in \mathbb{C}$ such that $(z_*, 0) \in V_f$, i.e. for any z_* in the set

$$\mathcal{Z} := \{z \in \mathbb{C} : a_0(z) = 0\} \subset \overline{\mathbb{Q}}. \tag{15.8}$$

[3] The name "branching locus" may be misleading in this context: there may be no monodromy around a particular point in $\mathcal{P}$; this problem will be eliminated from from the projective perspective.

Remark The sets $\mathcal{R}$, Δ, $\mathcal{P}$, and $\mathcal{Z}$ will play an important role in §15.2, during the analysis of asymptotics of sequences (e.g. of DT-invariants/Euler characteristics) encoded in algebraic generating functions.

Returning to the diagram (15.4), the map $\tau \in \mathbb{C}[V_f] = \mathbb{C}[z,t]/(f) \subset \mathbb{C}(V_f) = (\mathbb{C}(z))\,[t]/(\mathcal{F})$ can be thought of as a root of $\mathcal{F}$ in the field $(\mathbb{C}(z))\,[t]/(\mathcal{F})$ (where τ is just the class defined by t). In other words, there is a solution of (15.1) that is a regular algebraic function on a branched cover of the "z-plane" $\mathbb{C}$; the attentive reader will realize nothing deep in this statement: from the analytic point of view τ is the analytic continuation of any root of $\mathcal{F}$ to a function on V_f.

To understand the behaviour of ϕ and τ near various sorts of "infinity", it is convenient to consider the projectivized curve $\overline{V_f}$ formed from the zero locus of the homogenization $f_h \in \overline{\mathbb{Q}}[z,t,w]$ of $f \in \overline{\mathbb{Q}}[z,t]$, defined below.

$$\overline{V_f} = \{[z:t:w] \in \mathbb{P}^2 : f_h(z,t,w) = 0\}$$

$$\mathbb{P}^1 \xleftarrow[\;[z:w] \leftarrow\!\shortmid [z:t:w]\;]{\overline{\phi}} \overline{V_f} \xrightarrow[\;[z:t:w] \mapsto [t:w]\;]{\overline{\tau}} \mathbb{P}^1$$

Restricting $\overline{V_f}$ to the affine coordinate patch $U_{w\neq 0} = \{[z:t:w] \in \mathbb{P}^2 : w \neq 0\}$, we recover (15.4). Indeed, the rational function $\overline{\tau} \in \mathbb{C}(\overline{V_f}) \cong \mathbb{C}(V_f)$ is an extension of $\tau \in \mathbb{C}[V_{\mathcal{F}}]$ (i.e. $\overline{\tau}|_{U_{w\neq 0}} = \tau$), and the the map $\overline{\phi}$ is a degree e branched cover of $\mathbb{P}^1$ that extends ϕ.

Technical Note Of course, $\overline{\phi}$ is everywhere defined only if $[0:1:0] \notin \overline{V_f}$, and $\overline{\tau}$ is everywhere defined only if $[1:0:0] \notin \overline{V_f}$. Annoyingly, our examples include such points, but luckily the point $[1:0:0]$ will not cause any philosophical or practical qualms: we just eliminate it from the domain of $\overline{\tau}$ and consider $\overline{\tau}$ as a birational map $\overline{V_f} \dashrightarrow \mathbb{P}^1$. On the other hand, the point $[0:1:0]$ is a technical nuisance as it is convenient to have $\overline{\phi}$ defined everywhere; however, it can be removed by application of a projective transformation (an element of $\mathrm{PGL}_2(\mathbb{C})$) on $\mathbb{P}^2$ in order to change the embedding $\overline{V}_f \to \mathbb{P}^2$ before performing the coordinate projections that define $\overline{\phi}$ and $\overline{\tau}$. Because we will eventually wish to write down expressions for sections of the cover $\overline{\phi}$ in terms of the coordinate z, the transformation should not mix t and z. In examples, applying one of the two $\mathrm{PGL}_2(\mathbb{Z})$ transformations:

$$[z:t:w] \mapsto [z':t':w'] = [z+w:t:w]$$

or

$$[z:t:w] \mapsto [z':t':w'] = [z:w:t]$$

will eliminate the annoyance points. From the affine perspective (restricting to the coordinate patch where $w \neq 0$), the first map is just the shift $z \mapsto z+1$ and the second map is the birational map

$$\begin{aligned} V_f &\dashrightarrow V_{\widehat{f}} \\ (z,t) &\mapsto (z, 1/t). \end{aligned}$$

Without further discussion of this issue, we will proceed as if this coordinate transformation has been performed (if necessary); and assume that z, t, and w represent the transformed coordinates.

For later convenience, we note that $\overline{\phi}$ is ramified over the points

$$\mathcal{R} := \left\{ [z:t:w] \in \overline{V_f} : \left.\frac{\partial f_h}{\partial t}\right|_{(z,t,w)} = 0 \right\};$$

Moreover, the image $\overline{\phi}(\mathcal{R})$ (also referred to as branch points) is encoded in the zero locus of the discriminant $\mathrm{Disc}\,(\mathcal{F}_h) \in \overline{\mathbb{Q}}[z,w]$ of $\mathcal{F}_h := \mathrm{taut}_{\mathbb{Q}[w]} f_h \in \left(\overline{\mathbb{Q}}[z,w]\right)[t]$:

$$\begin{aligned} \Delta &:= \overline{\phi}(\mathcal{R}) \\ &= \left\{ [z:w] \in \mathbb{P}^1 : \mathrm{Disc}(\mathcal{F}_h)(z,w) = 0 \right\}. \end{aligned}$$

Restricting to the standard coordinate patch $U_{w\neq 0}$ (and implicitly using the isomorphism $\mathbb{C}^2 \to U_{w\neq 0}$), then $\mathcal{R}$ and Δ restrict to $\mathcal{R}_0$ and Δ_0 respectively.

Examples

1: m-**herds**: Recall that all generating functions (e.g. soliton-generating functions, street factors, and the BPS generating series) associated to an m-herd are given by powers of a root P of the polynomial $\mathcal{H}_{(m-1)^2} \in (\mathbb{Z}[z])\,[p]$, where

$$\mathcal{H}_k := p - zp^k - 1 \in (\mathbb{Z}[z])\,[p]. \tag{14.2}$$

The discriminant of this polynomial is given as [51, §2.7, pg. 41]

$$\mathrm{Disc}(\mathcal{H}_k) = (-1)^{k(k-1)/2} z^{k-2} \left(-(k-1)^{k-1} + k^k z\right).$$

which has two roots, one at $z = 0$, and the other at

$$z_* = (k-1)^{k-1} k^{-k}.$$

The root of the discriminant at $z = 0$ corresponds to ramification point of ramification degree $k-1$ (present on the projective curve, but not the affine curve): the degree of (14.2) drops at the point $z = 0 \in \mathcal{P}$. To study ramification around this point, we can either pass to the projective curve or study the birationally equivalent curve defined by

$$\widehat{\mathcal{H}_k} = y^{k-1} - y^k - z.$$

2: $(3, 2|3)$**-herd**: Using computer algebra software (e.g. *Mathematica* or *Magma*), one can check that the polynomial $f_{(3/2|3)} = \mathsf{taut}_{\mathbb{Z}}^{-1} \mathcal{F}_{(3/2|3)} \in \mathbb{Z}[z, t]$ is irreducible; the associated affine curve V_f is a degree 39 branched cover of "z-plane" $\mathbb{C}$.

The discriminant $\mathrm{Disc}(\mathcal{F}_{(3/2|3)}) \in \mathbb{Z}[z]$ is a degree 306 polynomial that factorizes as a product

$$\mathrm{Disc}(\mathcal{F}_{(3/2|3)}) = -387420489 \, [d_1(z)]^{256} \, [d_2(z)] \, [d_3(z)]^2 \, ; \tag{15.9}$$

where the d_i; $i = 1, 2, 3$; are all distinct irreducible polynomials; specifically $d_1 = z$, d_2 is a degree 10 polynomial,[4] and d_3 is a polynomial of order 20. Thus, there are 31 distinct points in Δ_0

The branch point $z = 0$ is of particular interest as this is the point around which we expect to find a series solution for the BPS generating series. The degree of f drops at the point $z = 0$ where 35 roots become infinite; there are two associated ramification points above $z = 0$: one at infinity (present on the projective curve, or after performing the birational transformation $t \mapsto 1/t = y$), and one at the point $(z_*, t_*) = (0, 1)$.

The point $(z_*, t_*) = (0, 1)$ is a singular point where four branches of the curve V_f meet transversely; however, only one of these four branches extends as an

[4] The expression for d_2 is explicitly shown below in (15.30); the root of least magnitude of this polynomial will play a role in the asymptotics of the associated BPS indices.

analytic function in the variable z– it is this unique analytic extension that represents the BPS generating series. Monodromy around the point $z = 0$ induces a permutation that is the product of a 3-cycle that permutes the three roots that do not analytically extend, a 35-cycle that permutes the roots that become infinite, and a 1-cycle that preserves the BPS generating series root.

Analytic Viewpoint

The above discussion gives us a solution τ of (15.1) that is a regular function on a branched cover of the z-plane $\mathbb{C}$. However, one of our goals was to seek out the roots $T_l \in \overline{\mathbb{Q}(z)}$ of (15.1) that are functions of the coordinate z; geometrically this is equivalent to seeking (local) sections of the branched cover $\phi : V_f \to \mathbb{C}$. When $e > 1$, such sections cannot be birational maps (rational functions of z) as their existence would violate irreducibility of $\mathcal{F}$ over $\mathbb{C}(z)$. However, passing over into the holomorphic world, on the complement of the finite set $\Delta \cup \mathcal{P}$, local holomorphic sections are guaranteed by the analytic implicit function theorem. In fact, all of the above could have been said in analytic language.

Notation Let X be an analytic space, then $\mathcal{O}_X$ will denote the sheaf of analytic functions on X and $\mathcal{M}_X$ will denote the sheaf of meromorphic functions on X.

Passing to the world of analytic spaces:

- The projective variety $\overline{V_f}$ is re-interpreted as an analytic space; we will denote this corresponding analytic space via $\mathcal{V}_f \subset \mathbb{P}^2$.
- $\overline{\tau} \in \mathbb{C}(\overline{V_f}) \hookrightarrow \mathcal{M}_{\mathcal{V}_f}(\mathcal{V}_f)$ is a meromorphic function on $\mathcal{V}_f$; it restricts to the analytic function $\tau \in \mathbb{C}[V_f] \hookrightarrow \mathcal{O}_{\mathcal{V}_f}(\mathcal{V}_f \cap U_{w\neq 0})$ where $U_{w\neq 0} := \{[z : t : w] \in \mathbb{P}^2 : w \neq 0\}$.
- The (algebraic) branched cover $\overline{\phi}$ is re-interpreted as a branched cover of analytic spaces $\varphi : \mathcal{V}_f \to \mathbb{P}^1$.

If the domain of φ is restricted to $\mathcal{V}_f' := \mathcal{V}_f \backslash \varphi^{-1}(\Delta)$, we have a holomorphic degree e cover of Riemann surfaces:

$$\varphi_{\text{cov}} := \varphi|_{\mathcal{V}_f'} : \mathcal{V}_f' \to \mathbb{P}^1 \backslash \Delta.$$

Then for any simply-connected open set $U \subset \mathbb{P}^1 \backslash \Delta$, the pre-image $\varphi_{\text{cov}}^{-1}(U)$ is a disjoint union of e open sets; in fact, for any component V of $\varphi_{\text{cov}}^{-1}(U)$ we can use the analytic implicit function theorem, along with properties of analytic continuation, to show that there is a unique holomorphic section $s_V : U \to V \subset \mathcal{V}'_f$; by precomposing this section with the map $\overline{\tau}$ we get a *meromorphic* function (a holomorphic map to $\mathbb{P}^1$) on U:

$$T_V := \overline{\tau} \circ s_V : U \to \mathbb{P}^1.$$

Moreover, this meromorphic function[5] is a root of $\mathcal{F}$ in the field $\mathcal{M}_{\mathbb{P}^1\backslash\Delta}(U)$; in fact, $\mathcal{F}$ splits over $\mathcal{M}_{\mathbb{P}^1\backslash\Delta}(U)$ as there are e distinct roots corresponding to the e components of $\varphi_{\text{cov}}^{-1}(U)$.

15.2 Asymptotics and Algebraicity

Let G be a generating series for the sequence of rational numbers $(\beta_n)_{n=1}^{\infty}$, i.e.:

$$\mathsf{G} := \prod_{n=1}^{\infty} (1 - (\mathtt{s} z)^n)^{n\beta_n} \in \mathbb{Q}[\![z]\!], \tag{15.10}$$

where $\mathtt{s} \in \{+1, -1\}$ is some fixed sign depending on the context under consideration; of course, we have two cases in mind.

1. $\mathsf{G} = \mathsf{T}$ is the generating series for BPS-indices/DT-invariants: $\beta_n = \Omega(n\gamma_c)$ (where γ_c is some primitive charge) γ_c and $\mathtt{s}z = X_{\widetilde{\gamma}_c} =: \widetilde{z}$. The particular value of $\mathtt{s} \in \{\pm 1\}$ depends on the definition of z suited to the problem at hand; however, for $(a, b|m)$-herds/the m-Kronecker quiver we choose $\mathtt{s} = (-1)^{mab-a^2-b^2}$ according to the definition of z given in (15.8) and Appendix I.

2. $\mathsf{G} = \mathsf{E}$ is the generating series for the Euler characteristics of stable moduli for the Kronecker m-quiver: $\beta_n = -\chi(\mathcal{M}_{\text{s}}^m(an, bn))$ where $(a, b) \in \mathbb{Z}_{>0}^2$ is a pair of coprime integers, $\mathcal{M}_{\text{s}}^m(an, bn)$ is the moduli space of stable representations of the m-Kronecker quiver with dimension vector (an, bn) (with respect to the non-trivial stability condition), and χ denotes the Euler characteristic. In this

[5] It is not hard to see that the poles of any such T_V are precisely at points in $\mathcal{P}$ with ramification index 1, i.e. points in $\mathcal{P} \cap (\mathbb{P}^1 \backslash \Delta)$.

case the choice $\mathsf{s} = 1$ is forced upon us via Reineke's functional equation (13.2) if we wish to use the same variable z used in the DT-invariant generating series T (for which we choose the signs $\mathsf{s} = (-1)^{mab-a^2-b^2}$).

If G is an algebraic function, then the discussion in §15.1 presents a geometric picture that aids in the classification of the $n \to \infty$ behaviour of the β_n via the study of local sections of the branched cover φ. In particular holomorphic techniques allow us to extract asymptotics of the β_n through a study of the possible zeros, poles, and branch point singularities of the local section corresponding to the G; the fact that G is, moreover, algebraic places further restrictions on the asymptotic behaviour.

In the following section we will derive the most general possible form for the asymptotics of the β_n relying only on the assumption that G is an algebraic function over $\mathbb{Q}$. As a warmup, we will first determine the asymptotics of the coefficients $\{t_n\}_{n=0}^{\infty}$ in the series expansion

$$T = \sum_{n=0}^{\infty} t_n (z - z_0)^n$$

of an arbitrary algebraic function T, holomorphic around the point z_0 (for simplicity we will eventually take $z_0 = 0$);[6] the asymptotics of the β_n follow a closely related story. The techniques used in this section are drawn heavily from the rather powerful book [23] by Flajolet and Sedgewick. Before beginning, we introduce some terminology.

Terminology In the subsequent discussion: a *singularity* is one of the following:

(P): A pole of the function (where the function can be interpreted as a meromorphic function), e.g. the point 0 in $h(\zeta) = \zeta^{-k}$ for some $k \in \mathbb{Z}$;

($B_{>0}$): A "finite" branch point: the singularity is due to a failure of coordinates and the function can be continued as a holomorphic function on some finite-degree branched cover of U, e.g. the point $\zeta = 0$ in ζ^{α} for some $\alpha \in \mathbb{Q}_{>0} \backslash \mathbb{Z}_{>0}$;

($B_{<0}$): An "infinite" branch point: the singularity is due to a failure of coordinates and the function can be continued as a meromorphic function on some finite-degree cover of U, e.g. the point $\zeta = 0$ in $h(\zeta) = \zeta^{-\alpha}$ for some $\alpha \in \mathbb{Q}_{>0} \backslash \mathbb{Z}_{>0}$;

[6] When this algebraic function is the BPS generating series, these coefficients can be interpreted as counts of Halo-BPS particles [26][§3.4].

($B_{\log}$): A logarithmic branch point: the singularity is due to a failure of coordinates and the function can be continued as a meromorphic function on some infinite-degree branched cover of U, e.g. the point $\zeta = 0$ in $h(\zeta) = \log(\zeta)$.

Singularities of type (P), ($B_{>0}$), or ($B_{<0}$) are the only possibilities for an algebraic function; a singularity of type ($B_{\log}$) can occur as the logarithm of a meromorphic function (specifically $\log(h)$ for some meromorphic function h will have a logarithmic singularity at any zero or pole of h). Furthermore, taking the logarithm of an algebraic function converts all zeros, singularities of type (P), and singularities of type ($B_{<0}$) into logarithmic branch points.

Asymptotics of Algebraic Series Coefficients

First, we begin by noting that local sections of the cover $\varphi_{\text{cov}} : \mathcal{V}_f' \to \mathbb{P}^1 \backslash \Delta$, defined only on open sets $U \subset \mathbb{P}^1 \backslash \Delta$, are guaranteed to be analytically continued to regions including points in Δ as long as the image of the section does not hit a ramification point. Thus, if we wish to study maximal analytic extensions of sections– a necessary study in order to understand type of singularities local sections may encounter– we should include non-ramification points of $\mathcal{V}_f$ in our study as well. Indeed, on the complement of ramification points, φ defines a holomorphic map of Riemann surfaces (non-singular analytic curves)

$$\varphi_{\text{sub}} := \varphi|_{\mathcal{V}_f \backslash \mathcal{R}} : \mathcal{V}_f \backslash \mathcal{R} \to \mathbb{P}^1.$$

Even though this is generally not a covering space map, it remains a submersion; hence, we can still apply the analytic inverse function theorem to speak of sections.

Lemma 15.2.1. *Denote the disk of radius r, centred at z_0, via*

$$D_{z_0}(r) := \{z \in \mathbb{C} : |z - z_0| < r\};$$

Let $s_0 \in \varphi_{\text{sub}}^{-1}(z_0)$ for some $z_0 \in \mathbb{C} \cap \varphi_{\text{sub}}(\mathcal{V}_f)$; if the component of $\varphi^{-1}(D_{z_0}(r))$ containing s_0 does not contain any point in $\mathcal{R}$, then there is a unique holomorphic section $s : D_{z_0}(r) \to \mathcal{V}_f$ of φ_{sub} such that $s(z_0) = s_0$.

Proof. For sufficiently small $\epsilon > 0$, the analytic implicit function theorem guarantees the existence of a unique holomorphic section $s : D_{z_0}(\epsilon) \to \mathcal{V}_f$ such that $s(z_0) = s_0$; we can analytically continue s to any disk $D_{z_0}(r)$ such that the component of $\varphi^{-1}(D_{z_0}(r))$ containing s_0 does not contain any ramification points. $\square$

Our true interests lie in finding roots of the polynomial $\mathcal{F}$, thought of as functions of the coordinate z; as described via the discussion in §15.1, these are given by pushing holomorphic sections of $\mathcal{V}_f \to \mathbb{P}^1$ forward via $\overline{\tau}$. Doing so, we obtain a meromorphic function, i.e. a holomorphic map $T := \overline{\tau} \circ s : D_{z_0}(r) \to \mathbb{P}^1$; the poles of this function correspond precisely to the location of the poles of $\overline{\tau}$

$$\widehat{\mathcal{P}} := \overline{\tau}^{-1}([1:0]) \subset \mathcal{V}_f$$

(note that $\varphi(\widehat{\mathcal{P}}) = \mathcal{P}$). Hence, we have the following corollary.

Corollary 15.2.2. *Let s be defined as in the previous lemma; if the component of $\varphi^{-1}(D_{z_0}(r))$ containing s_0 does not contain any point in $\mathcal{R} \cup \widehat{\mathcal{P}}$, then $T := \overline{\tau} \circ s : D_{z_0}(r) \to \mathbb{C}$ is a holomorphic function.*

With this in mind, suppose T is a root of $\mathcal{F}$ that is holomorphic on a disk centred at $z_0 \in \mathbb{C}$. Then, to speak of the *maximal* analytic continuation to such a disk, we define the following.

Definition Let h be a holomorphic function on a disk centered at $z_0 \in \mathbb{C}$, then

$$\mathtt{R}_h := \sup \{ r \in \mathbb{R}_{>0} \cup \{\infty\} : h \text{ can be analytically continued to } D_{z_0}(r) \} .$$

On $D_{z_0}(\mathtt{R}_T)$, we may write T as a convergent series

$$T(z) = \tau(s_0) + \sum_{n=1}^{\infty} t_n (z - z_0)^n, \quad z \in D_{z_0}(\mathtt{R}_T). \tag{15.11}$$

Assume that $\mathtt{R}_T < \infty$; then, heuristically speaking, we can predict the asymptotic behaviour of the coefficients t_n in the large n limit by noticing that the failure of the series representation of T to converge on a disk of radius larger than $\mathtt{R}_T$ is due to the growth of $|t_n|$ as fast as $|\mathtt{R}_T|^{-n}$; in fact, we can even derive the precise asymptotics of t_n through a closer analysis of its behaviour near the singular points of T (a collection of points in in $\mathcal{R} \cup \widehat{\mathcal{P}}$) that obstruct analytic continuation to a larger disk.

Remark 15.2.1. *If $\mathtt{R}_T < \infty$, then via Cor. 15.2.2 the failure to holomorphically extend T to a larger disk is due to a (non-empty) subset of points in $\varphi^{-1}\left[\partial \overline{D_{z_0}(\mathtt{R}_T)}\right] \cap \left(\mathcal{R} \cup \widehat{\mathcal{P}}\right)$.*

Definition Let h be a holomorphic function on some disk centred at z_0, then the set of *dominant singularities* of h is

$$\mathsf{sing}_{z_0}(h) := \left\{ z \in \partial\overline{D_{z_0}(\mathtt{R}_h)} : \begin{array}{c} h \text{ does not extend as a holomorphic function} \\ \text{to any open set containing } z \end{array} \right\}.$$

Remark/Notation For the algebraic function T, we can lift each dominant singularity uniquely to a point on the curve $\mathcal{V}_f$. Denote this set of lifts via $\mathsf{Sing}_{z_0}(T)$.

Remark

1. By Rmk. 15.2.1,

$$\mathsf{sing}_{z_0}(T) \subseteq \Delta \cup \mathcal{P}$$

   Moreover, under the map $\iota : \mathbb{C}^2 \cup \{\infty\} \hookrightarrow \mathbb{P}^2$ which sends $(z,t) \mapsto [z:t:1]$ for $t \neq \infty$ and $(z,\infty) \mapsto [z:1:0]$, we have

$$\iota\left(\mathsf{Sing}_{z_0}(T)\right) \subseteq \mathcal{R} \cup \widehat{\mathcal{P}} \subset \mathcal{V}_f$$

   In particular, $\mathsf{sing}_{z_0}(T)$ (and $\mathsf{Sing}_{z_0}(T)$) are finite sets (of the same order) as the set $\mathcal{R} \cup \widehat{\mathcal{P}}$ is finite.

2. It is helpful to keep in mind that $T(z)$ becomes infinite as z approaches dominant singularities in $\mathcal{P}$, while it remains finite as z approaches dominant singularities in $\mathcal{R} \setminus (\mathcal{R} \cap \mathcal{P})$.

Unfortunately, when $\mathcal{V}_f$ has singular points (in particular singular points projecting, via φ to the affine patch containing 0), (smooth) complex geometric techniques are insufficient for finding $\mathtt{R}_T$ and the location of dominant singularities. Indeed, let $\mathcal{W} \subset \mathcal{V}_f$ be the component of $\varphi^{-1}(D_{z_0}(r))$ containing the point s_0; assume that we have chosen r large enough such that the there is a point $p \in \mathcal{R} \cap \partial\overline{\mathcal{W}}$. If p is a smooth point of $\mathcal{V}_f$, and $\nu_p \geq 1$ is its ramification-index, then the monodromy associated to small loops containing the point p induces a cyclic permutation of order ν_p on the components of

$$\varphi^{-1}(D_{\varphi(p)}(\epsilon)) = \bigsqcup_{i=1}^{\nu_p} L_i,$$

where $\epsilon > 0$ is sufficiently small– i.e. a cyclic permutation on the set of all φ-preimages of small disks surrounding $\varphi(p)$. The result is that p is a branch-point singularity and we cannot extend s holomorphically to a larger disk. On the other hand, if p is a *singular* point– in particular, a normal-crossing singular point– then it may happen that one of the pre-images $L = L_j$, for some $j \in \{1, \cdots, \nu_p\}$, is fixed by the monodromy associated to a small loop around p; if $L \cap \mathcal{W} \neq \emptyset$, then we can analytically continue s to $\mathcal{W} \cup L$ – i.e. analytically continue past the singular point p. Furthermore, as long as p is not an element of $\mathscr{P}$, then $T = \overline{\tau} \circ s$ can also be analytically continued. In order to detect such continuations beyond singular points of V_f, it is best to take a more algebraic viewpoint and use the Puiseux expansion of the root T. The key idea is that the behaviour of T near a singularity is encoded in its Puiseux expansion around the singularity. To recall this expansion, we paraphrase the statement of Thm. VII.7 of [23, §VII.7].

Theorem 15.2.3 ((Newton-)Puiseux Expansion, C.f. Theorem VII.7 of [23])**.** *Let $T \in \overline{\mathbb{Q}(z)}$ be an algebraic function; let $z_0 \in \mathbb{C}$, then there exists an expansion of the form*

$$T(z) = \sum_{l \geq l_0} c_l (z - z_0)^{l/\kappa_{z_0}} \tag{15.12}$$

where $l_0 \in \mathbb{Z}$, $\kappa_{z_0} \geq 1$ is an integer,[7] *and $\{c_l\}_{l=l_0}^{\infty} \subset \mathbb{C}$. Moreover, for sufficiently small $r > 0$, then for any $\theta \in (0, 2\pi]$ and $\vartheta \in [0, 2\pi)$, this expansion gives a well-defined analytic function on a neighbourhood of the form*

$$\begin{aligned} W_{z_0}(r, \theta, \vartheta) &:= \{z \in \mathbb{C}\backslash\{z_0\} : |z - z_0| < r \textit{ and } \operatorname{Arg}(z - z_0) \notin [\vartheta + \theta, \vartheta - \theta]\} \\ &= e^{i\vartheta}\{z \in \mathbb{C}\backslash\{0\} : |z| < r \textit{ and } \operatorname{Arg}(z) \notin [\theta, -\theta]\} + z_0 \end{aligned} \tag{15.13}$$

(i.e. an indented disk given by a disk neighbourhood of the point z_0, minus a closed wedge with width 2θ and bisector the line at angle ϑ).

The Puiseux expansion for T is a Laurent series in the fractional power $(z-z_0)^{1/\kappa_{z_0}}$; when it is not possible to choose $\kappa_{z_0} = 1$, then such an expansion cannot define a meromorphic function on any disk containing z_0 (but does on a finite branched cover

[7] Note that κ_{z_0} depends on the choice of root T, not just the point $z_0 \in \mathbb{C}$; so the truly pedantic should adopt a notation that suggests this – however, there should be no confusion due to our notation for the following discussion.

of such a disk) due to the existence of monodromy around small loops that encircle z_0. However, it is holomorphic after removing a small wedge from the disk containing z_0 and choosing one of the κ_z branches of $(z-z_0)^{1/\kappa_{z_0}}$. Each choice of branch corresponds to a distinct choice of root around z_0 and the monodromy around z_0 induces a cyclic permutation of these κ_z roots. When it is possible to choose $\kappa_{z_0} = 1$, then there is no monodromy around z_0 (i.e. no branch point singularity) and the expansion (15.12) defines a meromorphic function on a sufficiently small disk $D_{z_0}(r) \supset W_{z_0}(r, \theta, \vartheta)$; furthermore, if $l_0 \geq 0$, then the series defines a holomorphic function on $D_{z_0}(r)$.

In order to eliminate the various choice ambiguities that arise when fractional powers arise in the Puiseux expansion, we fix a convention.

Convention When considering expressions of the form

$$\left(1 - \frac{z}{\rho}\right)^{\gamma}$$

for $\gamma \in \mathbb{Q}\backslash\mathbb{Z}$, we will always choose the unique branch that is defined for z in the domain

$$\mathbb{C}\backslash \left\{ z : 1 - \frac{z}{\rho} \in \mathbb{R}_{<0} \right\},$$

and is *positive* when $\rho^{-1}z \in [0, 1)$.

With this choice of convention, expanding T around a point $\rho \in \mathsf{sing}_{z_0}(T)$, we have

$$T(z) = \sum_{l \geq l_0} c_l \left(1 - \frac{z}{\rho}\right)^{l/\kappa_\rho} \tag{15.14}$$

where the collection of constants $\{c_l\}_{l=l_0}^{\infty} \subset \mathbb{C}$ are unambiguously determined when fixing the convention above, and T converges on any "wedge-shaped" region $W_\rho(r, \theta, \vartheta)$ (c.f. (15.13)) for sufficiently small r.

Remark To calculate l_0, κ_ρ, and the coefficients c_l, one can use a suitable Newton-polygon associated to either the polynomial $f \in \mathbb{Z}[z, t]$ (if $\rho \in \mathcal{P}$), or the polynomial $\widehat{f} \in \mathbb{Z}[z, y]$ (if $\rho \in \mathcal{P}$) (see e.g. [37, Ch. 7.2]). Indeed, assume $\rho \notin \mathcal{P}$ and let

$p = (\rho, \alpha) = (\rho, T(\rho)) \in \mathsf{Sing}_{z_0}(T)$, then we can expand f in a series about the point $p = (\rho, \alpha) \in \mathsf{Sing}_{z_0}(T) \hookrightarrow \mathbb{C}^2$

$$f(z,t) = \sum_{n,m=0}^{\infty} f_{n,m}(p)(z-\rho)^m(t-\alpha)^n$$

where

$$f_{n,m}(p) := \frac{1}{n!m!}\left(\frac{\partial^{n+m} f}{\partial z^n \partial t^m}\right)\bigg|_{(z,t)=(\rho,\alpha)} \in \overline{\mathbb{Q}}; \tag{15.15}$$

then the possibilities for the values κ_ρ are given by the inverse slopes of the leftmost convex envelope of the set $\{(n,m) \in \mathbb{Z} : f_{n,m}(p) \neq 0\}$; for each choice of κ_ρ, upon substituting in the expression

$$T = \sum_{l \geq l_0} c_l \left(1 - \frac{z}{\rho}\right)^{l/\kappa_\rho} \tag{15.16}$$

into f, the condition that all coefficients of a particular order must vanish determines l_0 and imposes polynomial conditions on the c_l that can be solved order-by-order in k. (Note that, because these polynomials have coefficients in $\overline{\mathbb{Q}}$, then $c_l \in \overline{\mathbb{Q}}$.) When $\rho \in \mathcal{P}$ one can repeat the same procedure to determine the Newton-Puiseux expansion of T^{-1} (and, hence, T) by expanding $\widehat{f} \in \overline{\mathbb{Q}}[z,y]$ around the point $p = (\rho, 0) \in \mathsf{Sing}_{z_0}(T)$.

It is important to observe that the expansion (15.16) must contain at least one non-vanishing c_l associated to an l such that l/κ_ρ is not a non-negative integer– otherwise we can analytically continue T to an open set containing $\rho \in \mathrm{sing}_{z_0}(T)$, which contradicts the definition of $\mathsf{sing}_{z_0}(T)$.

Definition Let T be an algebraic function and (15.14) an expansion of T around a point $\rho \in \mathsf{sing}_{z_0}(T)$; then define

$$\begin{aligned} \ell_\rho(T) &:= \min\left\{ l : c_l \neq 0 \text{ and } \frac{l}{\kappa_\rho} \notin \mathbb{Z}_{\geq 0} \right\}, \\ C_\rho(T) &:= c_{\ell(\rho)}, \\ \sigma_\rho(T) &:= \frac{\ell_\rho}{\kappa_\rho}; \end{aligned} \tag{15.17}$$

In practice we will be working with respect to a fixed function T; so we will simplify our notation by suppressing appearances of T, e.g. just writing C_ρ and σ_ρ.

Using the definition above, as $z \to \rho$

$$T(z) = E_\rho(z) + C_\rho \left(1 - \frac{z}{\rho}\right)^{\sigma_\rho} + c_0 + \mathcal{O}\left[\left(1 - \frac{z}{\rho}\right)^{\sigma_\rho + 1/\kappa_\rho}\right], \tag{15.18}$$

where

$$E_\rho(z) = \sum_{\{k : k \neq 0 \text{ and } k < \sigma_\rho\}} c_k (z-\rho)^k$$

is a (possibly identically zero) polynomial function of z such that $E_\rho(z) \to 0$ as $z \to \rho$.

Remark 15.2.2. *In the special case that the point $p \in \mathsf{Sing}_{z_0}(T)$ is a smooth point of the curve $\mathcal{V}_f$ (i.e. $f_{1,0} := f_{1,0}(p) \neq 0$), then κ_ρ is equal to the ramification index $\nu := \nu_p$ (defined by equation (15.6)) at the point p and*

$$\ell_\rho = \begin{cases} +1, & \text{if } \rho \notin \mathcal{P} \\ -1, & \text{if } \rho \in \mathcal{P} \end{cases};$$

$$C_\rho = \begin{cases} \omega_\nu \left(\rho \frac{f_{1,0}}{f_{0,\nu}}\right)^{1/\nu}, & \text{if } \rho \notin \mathcal{P} \\ \omega_\nu \left(\rho \frac{\widehat{f}_{1,0}}{\widehat{f}_{0,\nu}}\right)^{-1/\nu}, & \text{if } \rho \in \mathcal{P} \end{cases};$$

$$\sigma_\rho = \begin{cases} +\frac{1}{\nu_\rho}, & \text{if } \rho \notin \mathcal{P} \\ -\frac{1}{\nu_\rho}, & \text{if } \rho \in \mathcal{P} \end{cases}.$$

where ω_ν is a suitable choice of ν^{th} root of unity. However, if p is a singular point of $\mathcal{V}_f$, then (the smallest choice for) κ_ρ in the expansion for T is $\leq \nu_p$; its precise value depends on the choice of section T, not just the point p (different sections passing through the same point p may have different smallest values for κ_ρ).

Now, one can show that these local expansions completely determine the asymptotics of the coefficients t_n in (15.11). To see why this is possible, we make the following remark.

Remark 15.2.3. *We have $W_\rho(r, \theta, \vartheta) \cap D_{z_0}(\mathtt{R}_T) \neq \emptyset$ so it makes sense to compare the expansions (15.12) around any singular point ρ with the expansion (15.11) around z_0; furthermore, T can be analytically continued to the region*

$$D_{z_0}(\mathtt{R}_T) \cup \bigcup_{\rho \in \mathsf{sing}_{z_0}(T)} W_\rho(r, \theta, \arg(\rho)),$$

for some choice of sufficiently small $r > 0$.

Remark As we are ultimately interested in expansions about $z_0 = 0$, it is in our best interests to simplify our notation by imposing $z_0 = 0$ in the rest of discussion. The formulae for non-zero z_0 can be derived by a simple translation of coordinates.

In the case that there is only one dominant singularity $\rho \in \mathsf{sing}_0(T)$, one can pass from the Puiseux expansion around ρ to the expansion around 0 by using the series

$$\left(1 - \frac{z}{\rho}\right)^\gamma = \sum_{n=0}^{\infty} \binom{\gamma}{n} (-1)^n \rho^{-n} z^n,$$

which converges on $D_0(\rho)$. using Stirling's asymptotics, to leading order in n we have

$$[z^n] \left(1 - \frac{z}{r}\right)^\gamma = \left(\frac{n^{-1-1/\gamma}}{\Gamma(-\gamma)}\right) \rho^{-n} + \mathcal{O}(n^{-2-1/\gamma} \rho^{-n}).$$

In the case of a single dominant singularity, it is possible to show that the $n \to \infty$ asymptotics of the coefficients t_n can be extracted by applying the above expansion to the expression (15.18), ignoring the big-$\mathcal{O}$-terms of (15.18); this yields the $n \to \infty$ asymptotics

$$t_n \sim \left(\frac{C_\rho}{\Gamma(-\sigma_\rho)}\right) n^{-1-\sigma_\rho} \rho^{-n}.$$

In the case that there are multiple dominant singularities, the correct asymptotics are given by summing up individual contributions from each dominant singularity. The following theorem expresses this more general situation, including the subleading asymptotics. The full proof, which is mainly an application of Cauchy's integral formula, can be found in Theorem VI.5 of [23].

Theorem 15.2.4. *If h is an analytic function on $D_0(\mathtt{r})$ such that*

- *h has a finite number of dominant singularities $\{\rho_1, \cdots, \rho_r\} \subset \partial\overline{D_0(\mathtt{r})}$;*
- *$h(z)$ is analytic on a region of the form specified in Rmk. 15.2.3;*
- *there exist functions*

  $$\Phi_1, \cdots, \Phi_r, \mathcal{E} \in \left\{(1-z)^a \log(1-z)^b : a \in \mathbb{C}, b \in \mathbb{Z}\right\}$$

  such that, for each $i = 1, \cdots, r$,

  $$h(z) = \Phi_i\left(\frac{z}{\rho_i}\right) + \mathcal{O}\left[\mathcal{E}\left(\frac{z}{\rho_i}\right)\right]$$

  as $z \to \rho_i$;

then

$$[z^n]h(z) = \sum_{i=1}^{r} \rho_i^{-n}\left([z^n]\Phi_i\right) + \mathcal{O}\left[\mathtt{r}^{-1}\left([z^n]\mathcal{E}\right)\right].$$

The following is a corollary of the above theorem in the context of the algebraic functions under study.[8]

Theorem 15.2.5. *Let*

$$\begin{aligned} \sigma_* &:= \min\{\sigma_\rho : \rho \in \mathrm{sing}_0(T)\}, \\ \sigma_{\mathrm{sub}} &:= \min\left\{\sigma_\rho + \frac{1}{\kappa_\rho} : \rho \in \mathrm{sing}_0(T) \text{ and } \sigma_\rho + \frac{1}{\kappa_\rho} \notin \mathbb{Z}_{>0}\right\}, \\ \mathsf{leadsing}_0(T) &:= \{\rho \in \mathsf{sing}_0(T) : \sigma_\rho = \sigma_*\}. \end{aligned} \tag{15.19}$$

Then

$$t_n = \left(\frac{n^{-1-\sigma_*}}{\Gamma(-\sigma_*)}\right) \sum_{\rho \in \mathsf{leadsing}_0(T)} C_\rho \rho^{-n} + \mathcal{O}\left(n^{-1-\sigma_{\mathrm{sub}}} \mathtt{R}_T^{-n}\right). \tag{15.20}$$

Proof. From the discussion above, for any $\rho \in \mathsf{sing}_0(T)$, as $z \to \rho$ we have

$$T(z) = C_\rho\left(1 - \frac{z}{\rho}\right)^{\sigma_\rho} + \mathrm{Poly}_\rho(z) + \mathcal{O}\left[\left(1 - \frac{z}{\rho}\right)^{\sigma_{\mathrm{sub}}}\right],$$

where $\mathrm{Poly}_\rho(z)$ is some polynomial in z of degree $< \sigma_{\mathrm{sub}}$. Then from Thm. 15.2.4,

$$t_n = \sum_{\rho \in \mathsf{sing}_0(T)} \left(\frac{C_\rho}{\Gamma(-\sigma_\rho)}\right) n^{-1-\sigma_\rho} \rho^{-n} + \mathcal{O}\left(n^{-1-\sigma_{\mathrm{sub}}} \mathtt{R}_T^{-n}\right).$$

Note that this expression may have some redundancies: some terms in the summation may be in the same big-$\mathcal{O}$ class as the unspecified subleading terms; absorbing such redundant terms into the unspecified big-$\mathcal{O}$ terms, we arrive at (15.20). □

Remark 15.2.4. *Assume that we can choose $z_0 = 0$, and f has real coefficients. Then the fact that f has real coefficients ensures that all non-real dominant singularities of T come in pairs with their complex conjugates. Moreover, if T is a root such that $T(x) \in \mathbb{R}$ for all $x \in \mathbb{R} \cap D_0(\mathtt{R}_T)$, then $T(\overline{z}) = \overline{T(z)}$ for every $z \in D_0(\mathtt{R}_T)$. As a corollary of this latter fact, the fact that the Puiseux expansions around $z = \rho$*

[8] For a more refined statement of the structure of the subleading asymptotics for algebraic functions, see [23, Thm. VII.8.].

and $z = \overline{\rho}$ are holomorphic on a common domain, and the fact that $\overline{(1-\rho^{-1}z)^{\gamma}} = (1-\overline{\rho}^{-1}\overline{z})^{\gamma}$, it follows that $C_{\overline{\rho}} = \overline{C_{\rho}}$; hence, all (non-real) terms in (15.20) come in complex-conjugate pairs and we have,

$$t_n = \left(\frac{1}{\Gamma(-\sigma_*)}\right) \operatorname{Osc}(n) n^{-1-\sigma_*} \mathtt{R}_T^{-n} + \mathcal{O}\left(n^{-1-\sigma_{\text{sub}}} \mathtt{R}_T^{-n}\right)$$

where

$$\operatorname{Osc}(n) := \sum_{\rho \in \mathsf{leadsing}_0(T)} |C_\rho| \cos\left[n \arg(\rho) - \arg\left(C_\rho\right)\right].$$

Asymptotics of Euler-Product Exponents from Algebraicity

Let $\mathsf{G} \in \mathbb{Q}[\![z]\!]$ be a generating series for a sequence of rational numbers. If G is algebraic, then it must be the series representation (Puiseux-expansion) of a holomorphic function T around $z = 0$ such that $T(0) = 1$. Alternatively, if T is an algebraic function that:

1. is holomorphic on a disk containing $z = 0$,
2. satisfies $T(0) = 1$,

then T can has a series representation $T(z) = 1 + \sum_{n=1}^{\infty} t_n z^n \in \overline{\mathbb{Q}}[\![z]\!]$. Note that any such series admits an Euler-product factorization: for choice of a fixed $\mathtt{s} \in \{+1, -1\}$, we may write

$$T(z) = \prod_{n=1}^{\infty} (1 - (\mathtt{s}z)^n)^{n\beta_n}$$

where the sequence of algebraic numbers $(\beta_n)_{n=1}^{\infty}$ is defined by (10.2). In other words, T can be thought of as a generating series for a sequence of algebraic numbers.

Of course, our interests lies in the case where the (β_n) are rational (or better yet, integers). However, in the spirit of generality of the results in this section, we do not impose any rationality condition and allow $\mathsf{G} \in \overline{\mathbb{Q}}[\![z]\!]$ to denote a generating series of *algebraic* numbers.

The following lemma hints that, if we wish to study asymptotics of the β_n, then we should really be studying the coefficient asymptotics of $\log(\mathsf{G})$ expanded around $z = 0$.

Lemma 15.2.6. *Let* $\mathsf{G} \in \overline{\mathbb{Q}}[\![z]\!]$ *be the generating series for* $(\beta_n)_{n=1}^{\infty}$*; suppose* G *is algebraic over* $\mathbb{Q}(z)$ *and define*

$$\mathtt{R} = \min\{\mathtt{R}_{\mathsf{G}}, \mathtt{R}_{1/\mathsf{G}}\}.$$

(A): If $\mathtt{R} \leq 1$*, then for any* $0 < r < \mathtt{R}$

$$\beta_n = -\frac{1}{n}\left([z^n]\log\left(\mathsf{G}\right)\right) + \mathcal{O}\left(d(n)r^{-n/2}\right), \tag{15.21}$$

where $d(n)$ *is the number of divisors of* n *that are less than* n*.*

(B): If $\mathtt{R} > 1$*, then* $\beta_n \in \mathcal{O}\left(n^{-2}\right)$*.*

Proof. From (11.4) we have

$$\begin{aligned}\beta_n &= -\frac{1}{n^2}\sum_{k|n} k\mu\left(\frac{n}{k}\right)\mathtt{s}^k\left([z^k]\log\left(\mathsf{G}\right)\right)\\ &= -\frac{\mathtt{s}^n}{n}\left([z^n]\log\left(\mathsf{G}\right)\right) - \underbrace{\frac{1}{n^2}\sum_{\substack{k|n\\ k<n}} k\mu\left(\frac{n}{k}\right)\mathtt{s}^k\left([z^k]\log\left(\mathsf{G}\right)\right)}_{R(n)}\end{aligned}$$

Next, note that G is defines a holomorphic function on $D_0(\mathtt{R}_{\mathsf{G}})$ with $\mathsf{G}(0) = 1$; hence, the composite function $\log(\mathsf{G})$ is also holomorphic on any disk $D_0(r) \subset D_0(\mathtt{R}_{\mathsf{G}})$ such that $D_0(r)$ contains no zeros of G, i.e. any sub-disk where both G and $1/\mathsf{G}$ are holomorphic. The maximal such sub-disk has radius $\mathtt{R} = \min\{\mathtt{R}_{\mathsf{G}}, \mathtt{R}_{1/\mathsf{G}}\}$. Now, it is a corollary of Cauchy's integral formula that if $h : D_{z_0}(r) \to \mathbb{C}$ is a holomorphic function on $D_{z_0}(r)$, then $[z^n]h \leq r^{-n}\left(\sup_{r' \in D_{z_0}(r')} h\right)$ for any $0 < r' < r$. Applying this to our situation: for any $0 < r < \mathtt{R}$,

$$\left|[z^k]\log\left(\mathsf{G}\right)\right| \leq Cr^{-k} \tag{15.22}$$

where $0 < C_r < \infty$ is given by the supremum of $|\log(\mathsf{G})|$ over the circle of radius r.

1. **Case** (A). First note that

$$|R(n)| \leq \frac{1}{n}\sum_{\substack{k|n\\ k<n}}\left|[z^k]\log\left(\mathsf{G}\right)\right|.$$

Using the fact that $r < \mathtt{R} < 1$, the bound (15.22), and the fact that the next largest divisor of n (other than n itself) is $\geq n/2$, then

$$|R(n)| \leq C_r d(n) r^{-n/2},$$

where $d(n)$ is the number of divisors of n that are $< n$. This verifies (15.21).

2. **Case** (B). Choose r such that $1 < r < \mathtt{R}$. In this case

$$\begin{aligned} |R(n)| &\leq \frac{1}{n^2} \sum_{k=1}^{n} k\mathtt{R}^{-k} \\ &\leq \frac{1}{(r-1)^2} \left[\frac{r}{n^2} - \frac{r^{1-n}}{n^2} - \frac{r^{1-n}}{n} + \frac{r^{-n}}{n} \right] \end{aligned}$$

so $R(n) \in \mathcal{O}(n^{-2})$ as $n \to \infty$. Furthermore, by (15.22) $[z^n] \log(\mathsf{G}) \in \mathcal{O}(r^{-n})$ as $n \to \infty$; hence, $\beta(n) \in \mathcal{O}(n^{-2})$.

□

Remark 15.2.5. *One can use the rather crude estimate $d(n) \leq n$ to rewrite* (15.21) *as*

$$\beta_n = -\frac{\mathtt{s}^n}{n}\left([z^n] \log(\mathsf{G})\right) + \mathcal{O}\left(n r^{-n/2}\right)$$

however, there are various improvements on this bound. As an example of one improvement (see [8, §13.10]): for any $\epsilon > 0$, there exists a C_ϵ such that

$$d(n) \leq C_\epsilon n^\epsilon$$

for all $n \geq 1$; with this estimate, the subleading terms of (15.21) *are in $\mathcal{O}(n^\epsilon r^{-n/2})$.*

We are most interested in the case that $\mathsf{G} \in \mathbb{Z}[\![z]\!]$; indeed, if $(\beta_n)_{n=1}^{\infty} \subset \mathbb{Z}$ – which must be the case for BPS indices[9] or Euler characteristics– it follows that $\mathsf{G} \in \mathbb{Z}[\![z]\!]$.

Proposition 15.2.7. *If $\mathsf{G} \in \mathbb{Z}[\![z]\!]$, then $\mathtt{R} \leq 1$.*

[9] A BPS index can be expressed as the trace of an integer-valued operator over a finite dimensional vector space: see (F.8)).

Proof. Assume that $\mathtt{R} > 1$; then if G has integer coefficients, it must have only finitely many as (15.20) guarantees that $|g_n|$ becomes arbitrarily small as $n \to \infty$. Hence, G must be a polynomial with integer coefficients and

$$\mathtt{R} = \min\{|\omega| : \omega \in \mathbb{C} \text{ and } \mathsf{G}(\omega) = 0\}.$$

Factorizing G over $\mathbb{C}$ we have $\mathsf{G} = K(z - \omega_1) \cdots (z - \omega_n)$ for some collection of roots $\omega_i \in \overline{\mathbb{Q}}$, $i = 1, \cdots, n$. Integrality of the coefficients of G ensures that $K \in \mathbb{Z}$, but the product of the roots of T must satisfy

$$\omega_1 \cdots \omega_n = (-1)^n \frac{\mathsf{G}(0)}{K} = (-1)^n \frac{1}{K};$$

so at least one root must have magnitude ≤ 1, a contradiction. □

Hence, we will restrict our attention to the situation where $\mathtt{R} \leq 1$. For the interested reader, in §15.2 we will revisit the case that $\mathtt{R} > 1$ (which can only occur for generating series with some β_n non-integral).

Our next step is to mimic the singularity analysis of the previous section with the function $\log(\mathsf{G})$; although $\log(\mathsf{G})$ is no longer an algebraic function,[10] its singularities are closely related to the singularities (and zeros) of G.

Remark Denote the set of dominant singularities of $\log(\mathsf{G})$, defined with respect to its analytic continuation to disks centred about $z = 0$, by $\mathsf{sing}_0(\log(\mathsf{G}))$. This set of singularities can be described in terms of the singularities and zeros of G:

1. $\overline{D_0(\mathtt{R}_\mathsf{G})} \cap \mathcal{Z} = \emptyset$, i.e. there are no zeros of G in the closed-disk $\overline{D_0(\mathtt{R}_\mathsf{G})}$. Then $\mathtt{R} = \mathtt{R}_\mathsf{G} = \mathtt{R}_{1/\mathsf{G}}$ and we have

$$\mathsf{sing}_0(\log(\mathsf{G})) = \mathsf{sing}_0(\mathsf{G}) = \mathsf{sing}_0(1/\mathsf{G}).$$

2. $\overline{D_0(\mathtt{R}_\mathsf{G})} \cap \mathcal{Z} \neq \emptyset$, i.e. there is a zero of G in the closed disk $\overline{D_0(\mathtt{R}_\mathsf{G})}$. We separate this case into two further subcases:

[10]Let T be an algebraic function such that $T \not\equiv 0$ and $T \not\equiv 1$, then it follows as a corollary of the Lindemann-Weierstrass theorem that $\log(T)$ is transcendental over $\mathbb{Q}(z)$. Indeed, suppose $T \not\equiv 0, 1$ is an algebraic function – i.e. T satisfies $r(z, T(z)) = 0$ for some $r \in \mathbb{Q}[z, t]$– then evaluating z at an algebraic number $z_* \in \overline{\mathbb{Q}}$ we must have $T(z_*) \in \overline{\mathbb{Q}}$. Furthermore, we can choose z_* such that $T(z_*) \neq 0, 1$. Now, by Lindemann-Weierstrass $\log(T(z_*)) \notin \overline{\mathbb{Q}}$; hence, $\log(T)$ cannot be an algebraic function.

a) There are no zeros of G in the interior $D_0(\mathtt{R}_\mathsf{G})$ of $\overline{D_0(\mathtt{R}_\mathsf{G})}$: then $\mathtt{R} = \mathtt{R}_\mathsf{G} = \mathtt{R}_{1/\mathsf{G}}$ and $\mathsf{sing}_0(\log(\mathsf{G}))$ is given by adjoining the zeros on the boundary to the set $\text{sing}_0(\log(\mathsf{G}))$. More pedantically

$$\mathsf{sing}_0(\log(\mathsf{G})) = \mathsf{sing}_0(\mathsf{G}) \cup \mathsf{sing}_0(1/\mathsf{G}).$$

b) There is a zero of G in the interior of $\overline{D_0(\mathtt{R}_\mathsf{G})}$: then $\mathtt{R} = \mathtt{R}_{1/\mathsf{G}} < \mathtt{R}_\mathsf{G}$ and

$$\mathsf{sing}_0(\log(\mathsf{G})) = \mathsf{sing}_0(1/\mathsf{G}).$$

As the proof of Thm. 15.2.8 will show, the dominant singularities of $\log(\mathsf{G})$ will be a "finite" branch point if it lies in the complement of $\mathcal{Z} \cup \mathcal{P}$, and it will be a logarithmic branch point if it lies in $\mathcal{Z} \cup \mathcal{P}$.

Because singularities in $\mathcal{Z} \cup \mathcal{P}$ (i.e. singularities arising as zeros and poles of G) will play a different role than finite branch points, it is helpful to introduce some notation implicitly defined in the following remark.

Remark 15.2.6. *Let $\rho \in \text{sing}_0(\log(\mathsf{G})) \cap (\mathcal{Z} \cup \mathcal{P})$; then the Puiseux expansion of G around ρ will take the form*

$$\mathsf{G}(z) = (1 - \rho^{-1}z)^{m_\rho} g\left[\left(1 - \rho^{-1}z\right)^{1/\kappa_\rho}\right]$$

where g is a holomorphic function on the unit disk centred about 0 such that $g(0) \neq 0$ and

- *$m_\rho > 0$ if $\rho \in \mathcal{Z}$ (i.e. is a zero of G),*
- *$m_\rho < 0$ if $\rho \in \mathcal{P}$ (i.e. is a zero of $1/\mathsf{G}$).*

Note that if $m_\rho \in \mathbb{Q} \backslash \mathbb{Z}_{>0}$ then $m_\rho = \sigma_\rho$; otherwise, if $m_\rho \in \mathbb{Z}_{>0}$ (i.e. ρ is an unramified zero of G), then $m_\rho < \sigma_\rho$.

Equipped with Lemma 15.2.6, we state the following classification theorem.

Theorem 15.2.8. *Suppose $\mathsf{G} \in \overline{\mathbb{Q}}[\![z]\!]$ is an algebraic series generating the sequence $(\beta_n)_{n=1}^{\infty} \subset \overline{\mathbb{Q}}$, and such that $\mathtt{R} \leq 1$.*

1. *If* $\overline{D_0(\mathtt{R}_\mathsf{G})} \cap (\mathcal{Z} \cup \mathcal{P}) = \emptyset$*– i.e.* G *has no zeros or poles on* $D_0(\mathtt{R}_\mathsf{G})$*– then* $\mathtt{R} = \mathtt{R}_\mathsf{G}$*,* $\sigma_* > 0$*, and*

$$\beta_n = \left(\frac{\mathtt{s}^n}{\Gamma(-\sigma_*)}\right) n^{-2-\sigma_*} \sum_{\rho \in \mathsf{leadsing}_0(\mathsf{G})} \left(\mathsf{G}(\rho) C_\rho\right) \rho^{-n} + \mathcal{O}\left(n^{-2-\sigma_{\text{sub}}} \mathtt{R}^{-n}\right), \quad (15.23)$$

as $n \to \infty$*; where* σ_**,* $\sigma_{\text{sub}} > \sigma_*$*, and* $\mathsf{leadsing}_0(\mathsf{G})$ *are defined in* (15.19)*. Furthermore, if* G *is a series with real-coefficients,*

$$\beta_n = \left(\frac{\mathtt{s}^n}{\Gamma(-\sigma_*)}\right) \mathrm{Osc}(n) n^{-2-\sigma_*} \mathtt{R}^{-n} + \mathcal{O}\left(n^{-2-\sigma_{\text{sub}}} \mathtt{R}^{-n}\right) \quad (15.24)$$

as $n \to \infty$*; where*

$$\mathrm{Osc}(n) := \sum_{\rho \in \mathsf{leadsing}_0(\mathsf{G})} |\mathsf{G}(\rho) C_\rho| \cos\left[n \arg(\rho) - \arg\left(\mathsf{G}(\rho) C_\rho\right)\right].$$

2. *If* $\overline{D_0(\mathtt{R}_\mathsf{G})} \cap (\mathcal{Z} \cup \mathcal{P}) \neq \emptyset$*– i.e.* G *has a zero or a pole on* $\overline{D_0(\mathtt{R}_\mathsf{G})}$*– then*

$$\beta_n = \mathtt{s}^n n^{-2} \left(\sum_{\rho \in \mathrm{sing}_0(\log(\mathsf{G})) \cap (\mathcal{Z} \cup \mathcal{P})} m_\rho \rho^{-n} \right) + \mathcal{O}(n^{-2-1/\kappa_*} \mathtt{R}^{-n}); \quad (15.25)$$

where m_ρ *is defined in Rmk.* 15.2.6 *and*

$$\kappa_* := \max\{\kappa_\rho : \rho \in \mathcal{Z} \cup \mathcal{P}\} \geq 1.$$

Furthermore, if G *is a series with real-coefficients,*

$$\beta_n = \mathtt{s}^n \mathrm{Osc}(n) n^{-2} \mathtt{R}^{-n} + \mathcal{O}(n^{-2-1/\kappa_*} \mathtt{R}^{-n}) \quad (15.26)$$

as $n \to \infty$*; where*

$$\mathrm{Osc}(n) := \sum_{\rho \in \mathrm{sing}_0(\log(\mathsf{G})) \cap (\mathcal{Z} \cup \mathcal{P})} m_\rho \cos\left[n \arg(\rho)\right].$$

Proof. To prove the theorem, we first analyze the leading order asymptotics of $\log(\mathsf{G})$ and then apply Lem. 15.2.6. To begin, we divide the dominant singularities of $\log(\mathsf{G})$ into two cases:

(i): $\rho \in \mathrm{sing}_0(\log(\mathsf{G})) \cap (\mathcal{Z} \cup \mathcal{P})$, i.e. ρ is a zero of G or $1/\mathsf{G}$;

(ii): $\rho \notin \mathrm{sing}_0(\log(\mathsf{G})) \cap (\mathcal{Z} \cup \mathcal{P})$, i.e. $\mathsf{G}(\rho) \neq 0$ and ρ is a "finite" branch point of G in the sense of ($B_{>0}$).

For simplicity of notation, throughout we define

$$\zeta := (1 - \rho^{-1} z)$$

where the value of ρ under consideration will be clear from context.

Case (i)

Assume that $\rho \in \mathrm{sing}_0(\log(\mathsf{G}))$ is a zero of G or $1/\mathsf{G}$, then $\exists m_\rho \in \mathbb{Q}_{\neq 0}$ such that the Puisuex expansion of G around ρ can be written as

$$\mathsf{G}(z) = \zeta^{m_\rho} g\left[\zeta^{1/\kappa_\rho}\right]$$

where g is a holomorphic function on the unit disk centred about 0 such that $g(0) \neq 0$, m_ρ is a positive rational number, and $\kappa_\rho \in \mathbb{Z}_{\geq 1}$. Hence,

$$\begin{aligned}
\log(\mathsf{G}) &= \log\left(\zeta^{m_\rho}\right) + \log\left\{g\left[\zeta^{1/\kappa_\rho}\right]\right\} \\
&= m_\rho \log(\zeta) + \log\left\{g(0) + \mathcal{O}\left[\zeta^{1/\kappa_\rho}\right]\right\} \\
&= m_\rho \log\left(\zeta\right) + \log(-g(0)) + \mathcal{O}\left[\zeta^{1/\kappa_\rho}\right].
\end{aligned}$$

Case (ii)

Next we move on to case (ii). Let $\alpha := \mathsf{G}(\rho) \neq 0$ so that $(\rho, \alpha) \in \mathrm{Sing}_0(\mathsf{G}) \subset \mathcal{R}_0$. Then, on the appropriate open region, we may write

$$\mathsf{G}(z) = \zeta^{\sigma_\rho} g\left[\zeta^{1/\kappa_\rho}\right] + \alpha + E_\rho(z).$$

where g is a holomorphic function on the unit disk such that $g(0) = C_\rho \neq 0$, σ_ρ is a *positive* rational number, and $E_\rho(z)$ is a polynomial in z of degree $< \sigma_\rho$. Taking the logarithm of both sides, we have

$$\log\left[\mathsf{G}(z)\right] = \log(\alpha) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \alpha^{-m} \left[\zeta^{\sigma_\rho} g\left(\zeta^{1/\kappa_\rho}\right) + E_\rho(z)\right]^m;$$

Hence, as $z \to \rho$,

$$\log\left[\mathsf{G}(z)\right] = -\alpha^{-1} g(0) \zeta^{\sigma_\rho} + \mathrm{Poly}_\rho(z) + \mathcal{O}\left[\zeta^{\sigma_\rho + 1/\kappa_\rho}\right].$$

for some polynomial $\mathrm{Poly}_\rho(z)$ of degree $\leq \sigma_\rho$.

Our analyses of cases (i) and (ii) show that, as $z \to \rho$,

$$\log(\mathsf{G}) = \begin{cases} m_\rho \log\left(1-\rho^{-1}z\right), & \text{if } \rho \in \mathcal{Z} \cup \mathcal{P} \\ -\alpha^{-1} C_\rho \left(1-\rho^{-1}z\right)^{\sigma_\rho}, & \text{if } \rho \notin \mathcal{Z} \cup \mathcal{P} \end{cases} + \mathrm{Poly}'_\rho(z) + \mathcal{O}\left[\left(1-\rho^{-1}z\right)^{D}\right]$$

where

- $D = \min\left[\{\sigma_\rho + \frac{1}{\kappa_\rho} : \rho \notin \mathcal{Z} \cup \mathcal{P} \text{ and } \sigma_\rho + \frac{1}{\kappa_\rho} \notin \mathbb{Z}_{>0}\} \cup \{\frac{1}{\kappa_\rho} : \rho \in \mathcal{Z} \cup \mathcal{P}\}\right]$
- $\mathrm{Poly}'_\rho(z)$ is a polynomial of degree $< D$;

The asymptotics of $[z^n]\log(\mathsf{G})$ follow after applying Thm. 15.2.4 to the above discussion, eliminating redundant terms that are absorbed into subleading asymptotics. Equations (15.23) and (15.25) follow by application of Lemma 15.2.6, which transfers the asymptotics of $[z^n]\log(\mathsf{G})$ to the asymptotics of β_n. A few more words are in order for this last statement: it remains to show that the subleading asymptotics of (15.21), which live in $\mathcal{O}(d(n)r^{-n/2})$, can be safely absorbed into the subleading asymptotics of $n^{-1}\left([z^n]\log(\mathsf{G})\right)$, which are in $\mathcal{O}(n^{\beta}\mathtt{R}^{-n})$ for some $\beta \in \mathbb{R}$. There are two cases at hand.

(A): **When** $\mathtt{R} < 1$. We may choose an r in (15.21) such that $0 < r < \mathtt{R}$ and $r^{1/2} > \mathtt{R}$ (e.g. $r = \mathtt{R}^{3/2}$); the choice of such an r ensures that $\mathcal{O}(n^{\alpha}r^{-n/2}) \subset \mathcal{O}(n^{\beta}\mathtt{R}^{-n})$ for any α, $\beta \in \mathbb{R}$. So, by Rmk. 15.2.5, we have $\mathcal{O}(d(n)r^{-n/2}) \subset \mathcal{O}(n^{\beta}\mathtt{R}^{-n})$ for any $\beta \in \mathbb{R}$ and we are done.

(B): **When** $\mathtt{R} = 1$. The situation is slightly more subtle: the subleading asymptotics of $[z^n]\log(\mathsf{G})$ are decreasing (being in $\mathcal{O}(n^{-\beta})$ for some $\beta > 0$) while $d(n)r^{-n/2}$ grows exponentially for any $r < \mathtt{R} = 1$; hence, Lem. 15.2.6 appears useless at first sight. However, we may salvage the situation by utilizing the following trick that takes us back to the $\mathtt{R} < 1$ situation. First, let $a > 1$ be any integer and define the series

$$\widetilde{\mathsf{G}} = 1 + \sum_{n=1}^{\infty} g_n a^n z^n;$$

as a is an integer, then $\widetilde{\mathsf{G}} \in \overline{\mathbb{Q}}[\![z]\!]$. Moreover, $\widetilde{\mathsf{G}}$ is an algebraic series over $\mathbb{Q}$ (if $f(z,\mathsf{G}) = 0$ for some $f \in \mathbb{Q}[z,t]$ then $f(az,\widetilde{\mathsf{G}}) = 0$). Now, because $\widetilde{\mathsf{G}}$

has constant coefficient 1, it admits an Euler product expansion that defines a sequence of algebraic numbers $(\widetilde{\beta}_n)_n$ (defined via (10.2)):

$$\widetilde{\mathsf{G}} = \prod_{n=1}^{\infty} \left(1 - (\mathsf{s}z)^k\right)^{n\widetilde{\beta}_n}.$$

Next, observe that $\rho \in \operatorname{sing}_0(\log(\mathsf{G}))$ if and only if $a^{-1}\rho \in \operatorname{sing}_0(\log(\widetilde{\mathsf{G}}))$; in particular, $\widetilde{\mathtt{R}} = \min(\mathtt{R}_{\widetilde{\mathsf{G}}}, \mathtt{R}_{\widetilde{\mathsf{G}}^{-1}}) = a^{-1}\mathtt{R} < 1$. Hence, by the argument in (A), Thm. 15.2.8 holds with G, β, and $\mathtt{R}$ replaced with their twiddled-counterparts $\widetilde{\mathsf{G}}$, $\widetilde{\beta}_n$, and $\widetilde{\mathtt{R}}$. To reduce to the asymptotics for β_n, we observe that $[z^n]\log\left(\widetilde{G}\right) = a^n[z^n]\log(G)$; so from (10.2), we have

$$\widetilde{\beta}_n = a^n \beta_n.$$

Moreover, from our observations, when passing back to all un-twiddled quantities, an overall multiplicative factor of a^n may be extracted from the right hand side of the twiddled analogues of (15.23) or (15.25). Cancelling factors of a^n, the un-twiddled (15.23) and (15.25) remain valid.

Lastly, if G has coefficients in $\mathbb{R}$, we may rewrite (15.23) (respectively (15.25)) as (15.24) (respectively (15.26)) by using the reality of the coefficients of f (c.f. Remark 15.2.4). □

Remark As shown in §15.2, if we restrict to $\mathsf{G} \in \mathbb{Q}[\![z]\!]$, we may drop the condition that $\mathtt{R} \leq 1$ from the statement of Thm. 15.2.8.

The following explicit expression follows immediately when combining Thm. 15.2.8 with Prop. 15.2.7 and Rmk. 15.2.2.

Corollary 15.2.9. *Let*

1. $\mathsf{G} \in \overline{\mathbb{Q}}[\![z]\!]$ *be a generating series for* $(\beta_n)_{n=1}^{\infty} \subset \overline{\mathbb{Q}}$, *algebraic over* $\mathbb{Q}(z)$;
2. $f \in \overline{\mathbb{Q}}[z,t]$ *be the (unique up to a constant) absolutely irreducible polynomial such that* $f(z, \mathsf{G}) = 0$;
3. $\operatorname{sing}_0(\log(\mathsf{G})) = \{\rho_1, \cdots, \rho_k\}$ *be the set of dominant singularities of* $\log(\mathsf{G})$;

If $\mathtt{R} \leq 1$, $p_i := (\rho_i, \mathsf{G}(\rho_i))$ *is a smooth ramification point of* $\phi : V_f \to \mathbb{C}$ *for all* $i \in \{1, \cdots, k\}$, *and the ramification indices of each* p_i *are all equal:* $\nu := \nu_{p_1} = \nu_{p_2} = \cdots = \nu_{p_n}$, *then as* $n \to \infty$

$$\beta_n = -\left[\frac{\mathtt{s}^n n^{-2-1/\nu}}{\Gamma\left(-\frac{1}{\nu}\right)}\right] \sum_{i=1}^{k} \frac{C_{\rho_i}}{\mathsf{G}(\rho_i)} \rho_i^{-n} + \mathcal{O}\left(n^{-2-\sigma_{\text{sub}}} \mathtt{R}^{-n}\right) \tag{15.27}$$

where

$$\sigma_{\text{sub}} = \begin{cases} 3/2, & \textit{if } \nu = 2 \\ 2/\nu, & \textit{if } \nu > 2 \end{cases}$$

and the constants C_{ρ_i} *are defined in* (15.17) *(via the Newton-Puiseux expansion of* G *around* ρ_i*): explicitly, via Rmk. 15.2.2, there exist a collection of* ν*th roots of unity* $\{\omega_{\nu,i}\}_{i=1}^{k}$ *(satisfying the condition that* $\omega_i = \overline{\omega_j}$ *if* ρ_i *and* ρ_j *are conjugate) such that*

$$C_{\rho_i} = \omega_{\nu,i} \left[\rho_i \left(\frac{f_{1,0}(p_i)}{f_{0,\nu}(p_i)}\right)\right]^{1/\nu}$$

where $f_{n,m}(p_i)$ *is defined via* (15.15). *Moreover, if* $\mathsf{G} \in \mathbb{Q}[\![z]\!]$, *then the quantity in the radical is a rational number and* $\omega_{\nu,i} \in \{\pm 1\}$.

Remark Notice that we have been careful to not use the symbol "$\sim$"; the reason for this is the following. If there are multiple dominant singularities, at least one (along with its complex conjugate) of which is complex, then $\mathrm{Osc}(n)$ is a non-constant "oscillatory" function. The result is that, as $n \to \infty$

$$\frac{\beta_n}{\mathtt{s}^n n^{-l} \mathrm{Osc}(n) \mathtt{R}^{-n}} = 1 + \mathcal{O}\left(n^{-s} \mathrm{Osc}(n)^{-1}\right);$$

where $l > 0$ and $s > l$ are rational numbers, depending on the specific example at hand, that are specified by Thm. 15.2.8. The big-$\mathcal{O}$-terms, however, are not guaranteed to vanish (or even have a well-defined limit) as $n \to \infty$. For example, in the case of two complex dominant singularities, we would have

$$\mathrm{Osc}(n) = \cos\left[n \arg(\rho) - \phi\right]$$

for some $\phi \in [0, 2\pi)$. When $\arg(\rho)$ is an irrational multiple of π, then the set $\{\mathrm{Osc}(n)\}_{n=1}^{\infty}$ densely fills the interval $[-1, 1]$. In particular the set $\{\mathrm{Osc}(n)\}_{n=1}^{\infty}$ contains infinitely many elements of magnitude $< \epsilon$ for any fixed ϵ; so for any fixed

$M > 0$, then $\{\text{Osc}(n)^{-1}\}_{n=1}^{\infty}$ contains infinitely many elements of magnitude $> M$– it follows that $n^{-s}\text{Osc}(n)^{-1}$ has no well-defined limit as $n \to \infty$.

However, returning to the general scenario, note that, because $\text{Osc}(n)$ is a bounded function we always have that

$$\limsup_{n\to\infty} \frac{\beta_n}{\mathtt{s}^n n^{-l}\mathtt{R}^{-n}} = \limsup_{n\to\infty} \text{Osc}(n) = C_1$$
$$\liminf_{n\to\infty} \frac{\beta_n}{\mathtt{s}^n n^{-l}\mathtt{R}^{-n}} = \liminf_{n\to\infty} \text{Osc}(n) = C_2$$

for some constants C_1, $C_2 \in \mathbb{R}$ (at least one of which is non-vanishing).

In the simple case of a single dominant singularity then $\text{Osc}(n) \equiv C$ for some non-zero $C \in \mathbb{R}$; hence, we are justified in writing

$$\beta_n \sim \mathtt{s}^n C n^{-l}\mathtt{R}^{-n}.$$

Corollary 15.2.10. *Let* G *and* f *be defined as in Corollary 15.2.9. If* $\log(\mathsf{G})$ *has a single dominant singularity* ρ *such that* $\rho \notin \mathcal{Z} \cup \mathcal{P}$ *and* $(\rho, \mathsf{G}(\rho))$ *is a smooth ramification point of* $\phi : V_f \to \mathbb{C}$ *with ramification index* ν*, then as* $n \to \infty$

$$\beta_n \sim \omega_v \left(\rho \frac{f_{1,0}(p)}{f_{0,\nu}(p)}\right)^{1/\nu} \left[\frac{\mathtt{s}^n}{\mathsf{G}(\rho)\Gamma\left(-\frac{1}{\nu}\right)}\right] n^{-2-1/\nu}\rho^{-n} \tag{15.28}$$

where $f_{n,m}(p)$ *is defined via (15.15),* ω_v *is a* ν*th root of unity and the big-*$\mathcal{O}$ *class of the subleading-asymptotics is provided by the previous corollary.*

One can determine the overall multiplicative-constant ω_v by an appropriate Puiseux expansion around ρ. However, an observation for the truly lazy is the following.

Remark If $\mathsf{G} \in \mathbb{Q}[\![z]\!]$, then $\omega_v \in \{\pm 1\}$. Furthermore, if the coefficients of G are non-negative, then $\omega_v = -1$.

Example

1: **The m-herd**. One can study the m-herd BPS asymptotics by studying the BPS generating series T, given as a root of

$$\mathcal{F}_{(3,2|3)} = -1 + t(t - zt^{(m-2)})^m \in (\mathbb{Z}[z])\,[t];$$

however, as with our previous m-herd examples, we will proceed indirectly with the slightly simpler polynomial

$$\mathcal{H}_k = p - zp^k - 1 \in (\mathbb{Z}[z])\,[p]$$

Letting P be the unique root of this equation with $P(0) = 1$, we define the sequence $(c_n)_{n=1}^{\infty}$ via the Euler-product factorization

$$P = \prod_{n=1}^{\infty} (1 - (\mathtt{s}z)^n)^{nc_n}\,;$$

noting that, when $k = (m-1)^2$ and choosing $\mathtt{s} = (-1)^m$, the m-herd BPS generating series is given by

$$\mathsf{T} = P^m$$

hence,

$$\Omega(n\gamma_c) = mc_n. \tag{15.29}$$

Letting $k := (m-1)^2$, there is a single dominant singularity located at

$$\rho = k^{-k}(k-1)^{k-1}$$

with

$$P(\rho) = \frac{k}{k-1} = \frac{(m-1)^2}{m(m-2)}.$$

The corresponding ramification point $p = (\rho, P(\rho))$ is a smooth point of ramification index $\nu_p = 2$. Working through the details of (15.28), we have

$$c_n \sim -\frac{\mathtt{s}^n}{k}\sqrt{\frac{k}{2\pi(k-1)}}\, n^{-5/2}\left(k^k(k-1)^{-(k-1)}\right)^n.$$

Setting $k = (m-1)^2$, $\mathtt{s} = (-1)^m$ and using (15.29), we have

$$\Omega(n\gamma_c) \sim (-1)^{mn+1}\left(\frac{1}{m-1}\sqrt{\frac{m}{2\pi(m-2)}}\right) n^{-5/2}\left[(m-1)^{(m-1)}m(m-2)^{-m(m-2)}\right]^n;$$

which agrees with the asymptotics of [31, Prop. 3.4], proved via a procedure using the explicit series expansion of P around $z = 0$.

2: **The** $(3,2|3)$**-herd**. The BPS generating series T is given by the root of a 39th degree (absolutely irreducible) polynomial (c.f. (12.9)) defined by the unique *analytic* section of the associated algebraic curve, passing through the point $(0,1) \in \mathbb{C}^2$ (the point $(0,1)$ is a singular point of the algebraic curve corresponding to the intersection of two branches; only one branch defines an analytic section at $z = 0$). There is a single dominant singularity of T at $\rho \approx 0.005134$ that arises as the root of smallest magnitude of the degree 10 polynomial $d_2(z)$ (a factor of the discriminant polynomial of $\mathcal{F}_{(3/2|3)}$, c.f. (15.9)); explicitly,

$$\begin{aligned} d_2(z) = {} & 745322705120504762121096956080349336843937635421752929687 5 \\ & - 2322891406807452663970230316950021070808611526489257812500000z \\ & + 15311010666500137752425638769424063408504645586658916178560000 0z^2 \\ & - 7817871138183350523859861706656575827739367498904849637993525248z^3 \\ & + 22164287894637678029471180210381050956647457432719525515987779665 92z^4 \\ & - 1291818332171307829988878009469101678790621386359656804725186009497 6z^5 \\ & + 540128615752419031066854948701538543743667536496461974382147010560 00z^6 \\ & + 2067823170060188178302247072833500754014100750420991514751809880064z^7 \\ & - 7316813062372500911991434039240197319506245663152468018174361 6z^8 \\ & - 692626494138074646657585699931857043089076077167902720z^9 \\ & + 1066279825830391566189364544685962795501486 08z^{10}. \end{aligned} \tag{15.30}$$

It can be checked that $(\rho, \mathsf{T}(\rho))$ (where $\mathsf{T}(\rho) \approx 1.203051$) is a smooth point with ramification index 2. Applying (15.28), we arrive at the asymptotics stated in (12.10), which we restate here for convenience:

$$\Omega\left(n\gamma_c\right) \sim (-1)^{n+1} C n^{-5/2} \rho^{-n},$$

where $C \approx 0.075084$.

2: 3**-Kronecker Quiver Euler characteristics**: The generating series $\mathsf{E} = \mathsf{E}_{3/2}$ is a root of a 9th degree (absolutely irreducible) polynomial (c.f. (13.4)) $\mathcal{E} \in (\mathbb{Z}[z])\,[e]$. It has two dominant singularities at two complex conjugate roots of the polynomial (13.6), a factor of the discriminant polynomial $\mathrm{Disc}_{\mathbb{Z}[e]}(\mathcal{E}) \in \mathbb{Z}[z]$. Explicitly these dominant singularities are given by

$$\rho \approx 0.0151352 + 0.0373931i$$

and its complex conjugate. The corresponding points $(\rho, \mathsf{E}(\rho))$ and $(\overline{\rho}, \mathsf{E}(\overline{\rho}) = \overline{\mathsf{E}(\rho)})$, where

$$\mathsf{E}(\rho) \approx 0.6590388 + 0.7452078i$$

are non-singular points of ramification index 2. Applying (15.27), we arrive at the asymptotics stated in (13.5).

The physical consequences of the results in this section, when combined with Claim 14.0.4, can be summarized in a single statement.

Physical Corollary 15.2.11. *Let γ_c be a primitive charge of a BPS state in a theory of class $S[A_{K-1}]$ such that:*

- *the state occurs at a point on the Coulomb branch that is off of any walls of marginal stability,*
- *and it is represented by a spectral network with finitely many two-way streets;*

then the generating series for the BPS indices $\{\Omega(n\gamma_c)\}_{n=1}^{\infty} \subset \mathbb{Z}$ is algebraic. Furthermore, $\Omega(n\gamma_c)$ grows asymptotically as

$$\Omega(n\gamma_c) = \mathrm{Osc}(n) n^{-2-\alpha} \mathtt{R}^{-n} + \mathcal{O}\left(n^{-2-\alpha-\epsilon} \mathtt{R}^{-n}\right)$$

as $n \to \infty$; where $\mathtt{R} \in \overline{\mathbb{Q}} \cap (0, 1]$, $\alpha \in \mathbb{Q}_{\geq 0}$, $\epsilon \in \mathbb{Q}_{>0}$, and $\mathrm{Osc}(n)$ is a bounded "oscillatory" function. Furthermore, because $(\Omega(n\gamma_c))_n \subset \mathbb{Z}$ then either there are finitely many BPS indices ($\mathtt{R} = 1$), or there are infinitely many ($\mathtt{R} < 1$) with asymptotic growth of the form given above.

Algebraic Generating Series with Rational Coefficients

Suppose we were interested in studying the asymptotics of a series of rational numbers $(\beta_n)_{n=1}^{\infty}$ with algebraic generating series $\mathsf{G} \in \mathbb{Q}[\![z]\!]$ that does not necessarily lie in $\mathbb{Z}[\![z]\!]$. Such a scenario cannot occur for the generating functions of m-Kronecker Euler characteristics or BPS-index generating functions: in both scenarios $(\beta_n)_{n=1}^{\infty} \subset \mathbb{Z}$; however, this case may be of interest in the study of DT-invariants that do not necessarily satisfy an integrality condition (and, hence, cannot arise as BPS indices). If the radius of convergence $\mathtt{R}$ of $\log(\mathsf{G})$ is less than or equal to 1, then we can simply apply Thm. 15.2.8; however, it is possible that $\mathtt{R} > 1$ (a situation that cannot occur for series in $\mathbb{Z}[\![z]\!]$ by Prop. 15.2.7). In this latter situation, Lemma 15.2.6 indicates that the associated β_n must shrink: $\beta_n \in \mathcal{O}(n^{-2})$ as $n \to \infty$. However, as we will see, this is a rather crude estimate and, in fact, the formulae of Thm. 15.2.8 continue to hold for $\mathsf{G} \in \mathbb{Q}[\![z]\!]$. The reason for this magic is due to the following Lemma.

Lemma 15.2.12 (Eisenstein, Heine)**.** *Let $\sum_{n=0}^{\infty} t_n z^n \in \mathbb{Q}[\![z]\!]$ be algebraic over $\mathbb{Q}(z)$, then there exists an integer a (an Eisenstein constant) such that $a^n t_n$, $n \geq 1$ are all integers.*

Proof. See [19, pg. 327] which attributes the statement to Eisenstein (1852) and the proof to Heine (1854); part of the proof is identical to the proof of Lem. 15.1.1. □

Next we apply a trick, already seen in the proof of Thm. 15.2.8, but for convenience we will repeat here. First, let a denote an Eisenstein constant of $\mathsf{G} = 1 + \sum_{n=1}^{\infty} g_n z^n$, and define the formal series

$$\widetilde{\mathsf{G}} = 1 + \sum_{n=1}^{\infty} g_n a^n z^n$$

by definition of a, then $\widetilde{\mathsf{G}} \in \mathbb{Z}[\![z]\!]$. Moreover, $\widetilde{\mathsf{G}}$ is algebraic (if $f(z, \mathsf{G}) = 0$ for some $f \in \mathbb{Z}[z,t]$ then $f(az, \widetilde{\mathsf{G}}) = 0$). Now, because $\widetilde{\mathsf{G}}$ has constant coefficient 1, it admits an Euler-product expansion that defines a sequence of rational numbers $\left(\widetilde{\beta}_n\right)_{n=1}^{\infty}$ (defined via (10.2))

$$\widetilde{\mathsf{G}} = \prod_{n=1}^{\infty} \left(1 - (\mathtt{s}z)^k\right)^{n\widetilde{\beta}_n}.$$

Note that we can apply Thm. 15.2.8 to extract the $n \to \infty$ asymptotics of $\widetilde{\beta}_n$: because $\widetilde{\mathsf{G}} \in \mathbb{Z}[\![z]\!]$, by Prop. 15.2.7

$$\widetilde{\mathtt{R}} := \min\left\{\mathtt{R}_{\widetilde{\mathsf{G}}}, \mathtt{R}_{1/\widetilde{\mathsf{G}}}\right\} \leq 1.$$

Now, because $\widetilde{\mathsf{G}}$ is just given by the composition of G with a rescaling of the variable z, we can translate the asymptotic results which express the asymptotics of $\widetilde{\beta}_n$ in terms of asymptotics of the β_n; indeed, it is easy to see that:[11]

$$\operatorname{sing}_0(\widetilde{\mathsf{G}}) = \{a^{-1}\rho : \rho \in \operatorname{sing}_0(\mathsf{G})\}; \tag{15.31}$$

[11] As an aside: note that because $\widetilde{\mathtt{R}} \leq 1$ then we have the bound $\mathtt{R} \leq |a|$; hence, for any algebraic series G with constant coefficient 1 and Eisenstein constant $n \in \mathbb{Z}_{>0}$, then $\log(\mathsf{G})$ must have radius of convergence $\leq n$.

in particular,

$$\widetilde{\mathtt{R}} = |a|^{-1}\mathtt{R}; \tag{15.32}$$

and from 10.2 along with the fact that $[z^n]\log\left(\widetilde{G}\right) = a^n[z^n]\log(G)$, it is immediate that

$$\widetilde{\beta}_n = a^n\beta_n. \tag{15.33}$$

Corollary 15.2.13. *Theorem 15.2.8 holds for any algebraic* $\mathsf{G} \in \mathbb{Q}[\![z]\!]$ *(without the condition* $\mathtt{R} \leq 1$*); moreover, Corollaries 15.2.9 and 15.2.10 hold for any algebraic* $\mathsf{G} \in \mathbb{Q}[\![z]\!]$*.*

Proof. As mentioned, Thm. 15.2.8 holds with all G, β, and $\mathtt{R}$ replaced with their twiddled-counterparts $\widetilde{\mathsf{G}}$, $\widetilde{\beta}$, and $\widetilde{\mathtt{R}}$. With (15.31) - (15.33), one can check that the expressions (15.23) and (15.25) remain valid when passing back to the un-twiddled quantities. □

Part III

Appendices

Appendix A: The Six-Way Junction

For reference, we present some basic conditions on soliton generating functions as enforced by the homotopy invariance of the framed 2D-4D generating functions $F(\wp, \vartheta)$. First, using the convention described in Section 2.2, we assign every two-way street an orientation. If the network in question is degenerate, we resolve all two-way streets into "constituent one-way streets" using the *British resolution*: let p be a two-way street; using the orientation on p, we resolve p into two one-way streets running in opposite directions, infinitesimally displaced from one another, and such that the street pointing along the orientation of p is to the *left* of the street running against the orientation. If p is a two-way street of type ij (i.e. composed of coincident streets of type ij and type ji), then (after resolving) the street on the left is of type ij and the street on the right is of type ji.

Just as with Kirchoff's circuit laws it is most convenient to express our equations locally around each joint (or branch point). Hence, rather than expressing them in terms of the street-dependent Υ/Δ notation introduced in (2.13)-(2.14), we will temporarily adopt a joint-dependent notation.

Definition Let $v \in C$ be a joint or branch point, then τ_{ij} will denote the soliton generating function attached to a constituent one-way street of type ij running *out* of v, and ν_{ij} will denote the soliton generating function attached to a constituent one-way street of type ij running *into* v.

In a full spectral network, the joint dependent τ, ν notation can become redundant; so we will eventually revert back to the Υ/Δ notation in Appendix B.

To define products of soliton generating functions properly we introduce the following.

Definition Let η be a formal variable that acts on each formal variable X_a in the homology path algebra via

$$\eta X_a = X_{a^{\mathrm{tw}}},$$

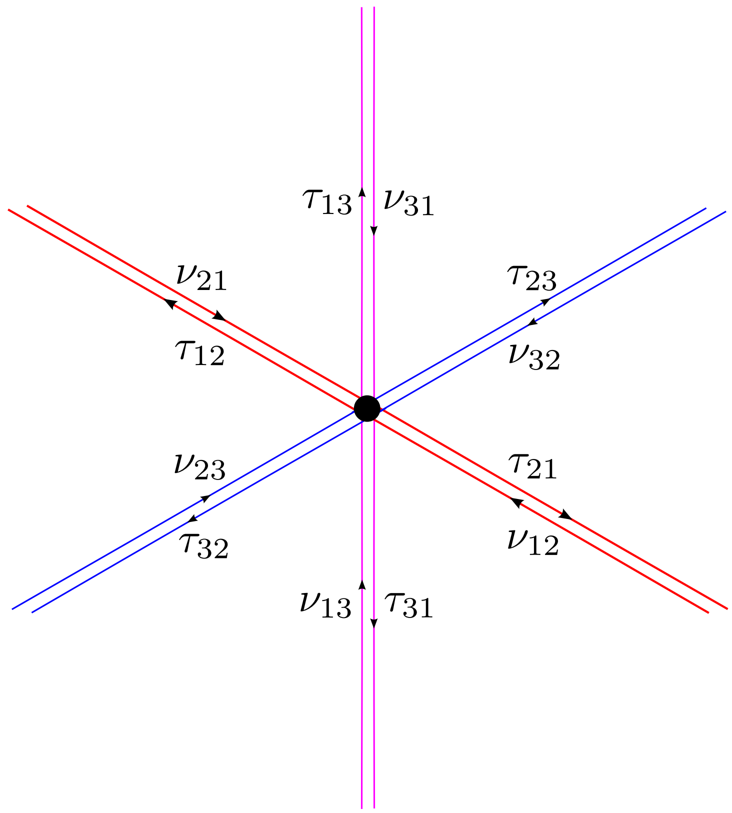


Figure A.1: A six-way junction. Two-way streets are resolved into one-way constituent streets using the British resolution. Streets of type 12 are red, type 23 are blue, and type 13 are fuchsia. A soliton generating function attached to a (one-way constituent) street is shown adjacent to its respective street. Subscripts on the soliton generating functions are ordered pairs $ij \in \{1, 2, 3\}^2$ denoting the type of solitons that the generating function "counts".

where, at the level of 1-chains, a^{tw} is the 1-chain produced by inserting a half-twist along the circle fiber of $\widetilde{\Sigma} \to \Sigma$ at some point[1] along a.

Remark It is immediate that $\forall G \in \mathcal{A}$

$$\eta^2 G = X_H G = -G.$$

We now consider a general type of joint, that can occur for a spectral network subordinate to a branched cover with $K \geq 3$ sheets, where six (possibly two-way) streets meet. The situation is shown in Fig. A.1: the (relevant) sheets of the cover are labeled from 1 to 3, and the soliton generating functions attached to a constituent one-way street (under the British resolution of all possible two-way streets) are shown adjacent to their corresponding sheet. Using homotopy invariance of $F(\wp, \vartheta)$, one arrives at the six-way junction equations:[2]

$$\begin{aligned} \tau_{12} &= \nu_{12} + \eta \tau_{13} \nu_{32}, \quad \tau_{21} = \nu_{21} + \eta^{-1} \nu_{23} \tau_{31}, \\ \tau_{23} &= \nu_{23} + \eta \tau_{21} \nu_{13}, \quad \tau_{32} = \nu_{32} + \eta^{-1} \nu_{31} \tau_{12}, \\ \tau_{31} &= \nu_{31} + \eta \tau_{32} \nu_{21}, \quad \tau_{13} = \nu_{13} + \eta^{-1} \nu_{12} \tau_{23}. \end{aligned} \tag{A.1}$$

At a branch point of type ij, we will assume that there is at most one two-way street, of type ij, emanating from the branch point; on this two-way street we will take

$$\tau_{ij} = X_{a_{ij}}$$

where a_{ij} is the charge of a simpleton.[3] As described at the end of Section 2.2, fixing a point z near the branch point, such a simpleton is represented by a path which runs from the lift of z on sheet i to the lift of z on sheet j. In [27] one can find a more general rule accommodating the situation of three two-way streets emanating from the branch point; however, we will not need this generalized rule for m-herds.

[1] Up to homotopy (rel endpoints) the insertion point does not matter; hence, it is irrelevant for relative homology.

[2] In [27] these equations were erroneously written without the factors η, η^{-1}.

[3] The coefficient of $\mu(a_{ij}) = 1$ in front of $X_{a_{ij}}$ is a result of the soliton input data (2.12).

Appendix B: m-Herds in Detail

B.1 Notational Definitions

We will consider four distinct branch points of a branched cover $\Sigma \to C$ of degree $K \geq 3$. On any local region on C', where the cover may be trivialized, only three sheets will be relevant and we will label the relevant sheets from 1 to 3. Label the branch points from 1 to 4 such that branch points 1 and 3 are branch points of type 12, while branch points 2 and 4 are branch points of type 23. For each branch point $i \in \{1, \cdots, 4\}$ we will choose a simpleton (cf. the end of Section 2.2) s_i with endpoints on distinct lifts of some $z_i \in C'$ close to the ith branch point. s_1 and s_2 will be simpletons of type 12 and 23, respectively, while s_3 and s_4 will be of type 21 and 32, respectively. We denote the charges of these simpletons by

$$\begin{aligned} a_* &= [s_1] \in \Gamma_{12}(z_1, z_1) \\ b_* &= [s_2] \in \Gamma_{23}(z_2, z_2) \\ \overline{a}_* &= [s_3] \in \Gamma_{21}(z_3, z_3) \\ \overline{b}_* &= [s_4] \in \Gamma_{32}(z_4, z_4). \end{aligned} \tag{B.1}$$

More often, however, computations are performed in the "$\mathbb{Z}/2\mathbb{Z}$-extended" sets $\widetilde{\Gamma}(\widetilde{z}, -\widetilde{z})$ for $\widetilde{z} \in \widetilde{C'}$ where we define

$$\begin{aligned} a &= [\widehat{s_1}] \in \widetilde{\Gamma}_{12}(\widetilde{z}_1, -\widetilde{z}_1) \\ b &= [\widehat{s_2}] \in \widetilde{\Gamma}_{23}(\widetilde{z}_2, -\widetilde{z}_2) \\ \overline{a} &= [\widehat{s_3}] \in \widetilde{\Gamma}_{21}(\widetilde{z}_3, -\widetilde{z}_3) \\ \overline{b} &= [\widehat{s_4}] \in \widetilde{\Gamma}_{32}(\widetilde{z}_4, -\widetilde{z}_4). \end{aligned} \tag{B.2}$$

where $\widehat{(\cdot)}$ denotes the *tangent framing lift* (first discussed in Section 2.2) and the $\widetilde{z}_i \in \widetilde{C}$ are the unit tangent vectors at the starting points of the tangent framing lifts.

In a slight abuse of notation, horse streets[1] (which may be two-way), will be denoted by decorated latin letters: a_i, $\overline{a}_i$ are streets of type 12, b_i, $\overline{b}_i$ are of type 23, and c is of type 13. The subscripts, denoted by $i \in \{1, 2, 3\}$, denote which street is in

[1] See the definition in Section 3.1.

question and the use of overlines are just a notational exploit of the duality operation described below in B.2.

Furthermore, in contrast with the "joint-dependent" τ, ν notation of Appendix A, we will (more naturally) denote soliton generating functions[2] "streetwise." The point $z \in p \subset C$ in the definition of soliton generating functions will be dropped for notational convenience. As mentioned in a remark at the end of Section 2.2: for any $z, z' \in p$ the generating functions $\Upsilon_z(p)$ and $\Upsilon_{z'}(p)$ are related by parallel transport (similarly for $\Delta_z(p)$ and $\Delta_{z'}(p)$).

For the sake of readability, we will modify our notation slightly from Section 2.2 and write streets as subscripts.

Definition Let p be a street, the generating function of solitons on p which *agree* with the orientation of p is denoted Υ_p, the generating function of solitons which *disagree* with the orientation of p is denoted Δ_p. In all figures in this thesis streets are oriented in an upward direction (upsilon is for "up" and delta is for "down").

We now wish to associate the street factor (a generating function) Q_p to each street p. To do so, it is convenient to pass through the definition of a closely related auxiliary function.

Definition

For each street p, we define the function

$$\mathcal{Q}_p := 1 + \Upsilon_p \Delta_p \in \mathcal{A}_C \tag{B.3}$$

To produce a formal series in the X_γ, $\gamma \in \Gamma$, we use the "basepoint-forgetting" closure map.

Definition

$$Q_p := \mathrm{cl}\,[\mathcal{Q}_p] \in \mathbb{Z}[\![\widetilde{\Gamma}]\!].$$

We now make some important technical remarks about the use of $\mathcal{Q}_p$ vs. Q_p.

[2]We refer to Section 2.2 for the detailed definitions of generating functions and formal variables.

Remarks If p is a street of type ij, then $\mathcal{Q}_p$ is a formal series in formal variables over Γ_{ii}. In particular, let $a \in \Gamma_{kl}$, then this means

$$\mathcal{Q}_p X_a = \begin{cases} 0 & \text{if } k \neq i \\ Q_p X_a = X_a Q_p & \text{if } k = i \end{cases},$$

$$X_a \mathcal{Q}_p = \begin{cases} 0 & \text{if } l \neq i \\ X_a Q_p = Q_p X_a & \text{if } l = i \end{cases}.$$

Hence, if the (left or right) action of $\mathcal{Q}_p$ on a soliton function of type kl is nonvanishing, then it can be replaced with the (commutative) action of Q_p. In the following derivations, the action of $\mathcal{Q}_p$ happens to be always nonvanishing; hence, it will almost always be replaced by Q_p, except in cases where we resist such replacements for the sake of precision and pedagogy.

Terminology Occassionally we will use the term *spectral data* to refer to the collection of soliton generating functions, street factors, and the functions $\mathcal{Q}_p$ supported on a particular collection of streets.

B.2 Duality

As an oriented graph embedded in a disk, Fig. 3.3 is invariant under an involution given by rotating the diagram 180 degrees, and reversing all orientations; we denote this involution on streets p via an overline

$$p \mapsto \overline{p}, \tag{B.4}$$

for $p \in \{a_i, \overline{a_i}, b_i, \overline{b_i}, c : i = 1, 2, 3\}$. As the terminology suggests, this involution satisfies $\overline{\overline{p}} = p$ for every street p and $\overline{c} = c$. We claim that this geometric involution actually induces a duality operation on all spectral data, i.e. generating functions. In particular, on any equations involving soliton generating functions, the replacements

$$\begin{aligned} \Upsilon_p &\leftrightarrow \Delta_{\overline{p}} \\ \eta &\leftrightarrow \eta^{-1}, \end{aligned} \tag{B.5}$$

with all products taken in reverse order, will also yield a valid equation. This claim can be verified by brute-force checking. Note, in particular, applying the duality operation to the definition of $\mathcal{Q}_p$ in (B.3) will yield $\mathcal{Q}_{\overline{p}}$.

B.3 The Horse as a Machine

Recall the definition of a horse is given as a condition on the subset of two-way streets of a spectral network in an open disk region (see Section 3.1). For convenience we restate the definition.

Definitions

1. A *horse street* $p \in \{a_1, a_2, a_3, b_1, b_2, b_3, c, \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{b_1}, \overline{b_2}, \overline{b_3}\}$ is one of the streets of Fig. 3.3 (left frame).

2. Let N be a spectral network (subordinate to some branched cover $\Sigma \to C$) and $U \subset C'$ be an open disk region. Then $U \cap N$ is a horse if a subset of its streets can be identified with Fig. 3.3 in a way such that

   a) Every two-way street is a horse street.

   b) There is always a two-way street identified with the street labeled c.

It may happen, however, that on a horse there are "background" non-horse streets that cannot be identified with those of Fig. 3.3; by definition, these are one-way streets. The following claim ensures that the computation of soliton generating functions on the streets of a horse are independent of the details of the non-horse streets.

Claim The equations for soliton generating functions on horse streets, induced by (A.1), close on themselves. I.e., the equations for the soliton generating functions on a given horse street can be written entirely in terms of the soliton generating functions on horse streets.

Proof. Let us temporarily denote the four joints in Fig. 3.3 (left frame) as *horse joints*. We split non-horse streets into two classes:

(A) Streets that have no endpoints on a horse joint.

(B) Streets that have a single endpoint on a horse joint.

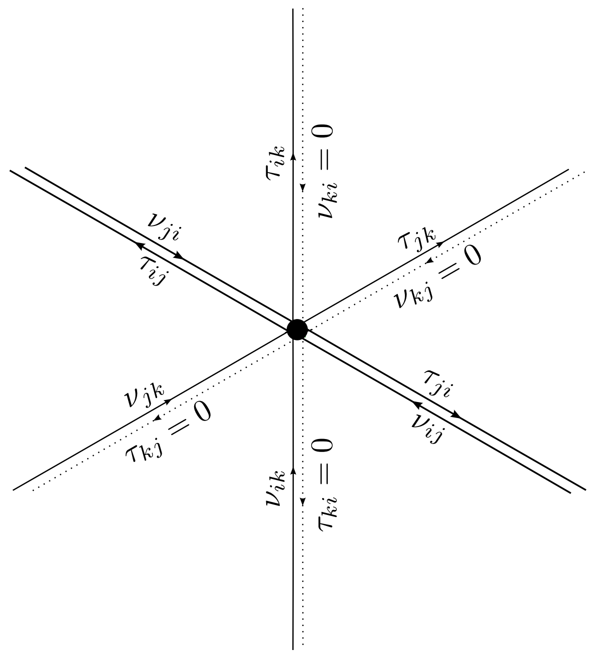


Figure B.1: The most general type of joint where non-horse streets of class (A) meets a horse street (which may be two-way). As in Fig. A.1, streets are resolved into one-way constituents using the British resolution. Soliton generating functions vanish on the dotted streets. The labels i, j, k are a permutation of the sheets $1, 2, 3$.

Let us first consider streets of class (A). The claim is trivial for (A)-streets that do not intersect a horse street. Thus, we turn our attention to a joint where an (A)-street meets a horse street. The most general picture of such a joint[3] is depicted in Fig. B.1. In this figure: i, j, k label any permutation of the sheets $1, 2, 3$, the streets of type jk and ki label background one-way streets, and the streets of type ij compose the the horse street (after being split into two streets by the joint). The soliton generating functions on the horse street are (in the "joint-wise" notation of Section A) τ_{ji}, ν_{ij}, τ_{ij}, and ν_{ji}. The claim (for (A)-streets) is then equivalent to the statement that $\tau_{ij} = \nu_{ij}$, $\tau_{ji} = \nu_{ji}$; we will show this is the case. Indeed, by the six-way joint

[3]By the "most general picture" we mean a six-way junction equipped with the weakest possible constraints on incoming soliton degeneracy functions, compatible with the condition that only the streets of type ij (for some fixed pair ij) are two-way. Using (A.1), one finds that the most general picture is Fig. B.1.

equations (A.1):

$$\tau_{ij} = \nu_{ij} + \begin{cases} \eta\tau_{ik}\nu_{kj}, & \text{if } ij \in \{12,\, 23,\, 31\} \\ \eta^{-1}\nu_{ik}\tau_{kj}, & \text{if } ij \in \{21,\, 32,\, 13\} \end{cases}$$

$$\tau_{ji} = \nu_{ji} + \begin{cases} \eta^{-1}\nu_{jk}\tau_{ki}, & \text{if } ij \in \{12,\, 23,\, 31\} \\ \eta\tau_{jk}\nu_{ki}, & \text{if } ij \in \{21,\, 32,\, 13\} \end{cases}$$

but $\nu_{kj} = 0$, $\tau_{kj} = 0$, $\nu_{ki} = 0$, and $\tau_{ki} = 0$. Hence,

$$\tau_{ij} = \nu_{ij}$$
$$\tau_{ji} = \nu_{ji}.$$

Now, via inspection of Fig. 3.3, streets of class (B) are of type 13. If a (B)-street meets a horse street at a non-horse joint, then we apply the same argument used for (A)-streets to see that the (equations for) soliton generating functions on horse streets do not depend on the (B)-street soliton generating function. Thus, we focus our attention on the horse joint.

If a (B)-street meets a horse joint, then (A.1) requires the equations for soliton generating functions, on the horse streets meeting the joint, to depend on the soliton generating function of the (B)-street. We will show that the soliton generating function on the (B) street can be rewritten in terms of generating functions on the horse streets. First, note that if a (B)-street meets the horse joint where a_1 and b_1 meet, or the joint where $\overline{a}_1$ and $\overline{b}_1$ meet, then it must be outgoing with respect to the horse joint. Indeed, the constraint that c is two-way requires the presence of outgoing streets of type 13 at the horse joints meeting c; if the (B)-street were incoming, it would combine with one of these outgoing streets to form a two-way street, violating the horse condition. Without loss of generality, assume the (B) street meets the horse joint where a_1 and b_1 meet; denote the soliton generating function on the (B)-street by $\Upsilon_{(B)}$. Then, using (A.1), it follows that $\Upsilon_{(B)} = \eta^{-1}\Upsilon_{a_1}\Upsilon_{b_1}$; so its soliton generating function is a function of the soliton generating functions on horse streets.

If a (B)-street meets one of the other two horse joints (where b_3 and $\overline{a}_3$ meet or where a_3 and $\overline{b}_3$ meet), then there are two situations: the horse streets at the horse joint are both two-way, or only one of the horse streets at the horse joint is two-way. The former situation is equivalent to the situation where the (B)-street meets the horse joint where a_1 and b_1 meet. To resolve the latter situation we repeat the same argument used for (A)-streets. □

We divide the soliton generating functions supported on horse streets into elements of three subspaces: incoming data, outgoing data, and internal data.

Incoming data

Incoming data is defined as the spectral data which flows into the internal joints of the horse and is supported on the external streets. Here, the space of such data is composed of four soliton generating functions and their duals:

$$\text{Incoming-Data} = \left\{ \begin{pmatrix} \Upsilon_{a_1}, & \Upsilon_{b_1}, & \Delta_{a_3}, & \Delta_{b_3}, \\ \Delta_{\overline{a_1}}, & \Delta_{\overline{b_1}}, & \Upsilon_{\overline{a_3}}, & \Upsilon_{\overline{b_3}} \end{pmatrix} \in \mathcal{A}_S^{\times 8} \right\}. \tag{B.6}$$

It will prove useful to subdivide this space of data further into generating functions of solitons that agree with the orientation of the diagram, Incoming-Data$^+$, and those that disagree, Incoming-Data$^-$:

$$\begin{aligned} \text{Incoming-Data}^+ &= \left\{ \left(\Upsilon_{a_1}, \Upsilon_{b_1}, \Upsilon_{\overline{a_3}}, \Upsilon_{\overline{b_3}} \right) \in \mathcal{A}_S^{\times 4} \right\}, \\ \text{Incoming-Data}^- &= \left\{ \left(\Delta_{\overline{a_1}}, \Delta_{\overline{b_1}}, \Delta_{a_3}, \Delta_{b_3} \right) \in \mathcal{A}_S^{\times 4} \right\}. \end{aligned}$$

Outgoing data

Similarly, outgoing data is defined as the spectral data which flows out of the internal joints and is supported on external streets. This consists of the space of soliton generating functions,

$$\text{Outgoing-Data} = \left\{ \begin{pmatrix} \Delta_{a_1}, & \Delta_{b_1}, & \Upsilon_{a_3}, & \Upsilon_{b_3}, \\ \Upsilon_{\overline{a_1}}, & \Upsilon_{\overline{b_1}}, & \Delta_{\overline{a_3}}, & \Delta_{\overline{b_3}} \end{pmatrix} \in \mathcal{A}_S^{\times 8} \right\}. \tag{B.7}$$

As with the incoming data, we can similarly subdivide this data into generating functions of solitons that agree or disagree with the overall orientation:

$$\begin{aligned} \text{Outgoing-Data}^+ &= \left\{ \left(\Upsilon_{\overline{a_1}}, \Upsilon_{\overline{b_1}}, \Upsilon_{a_3}, \Upsilon_{b_3} \right) \in \mathcal{A}_S^{\times 4} \right\}, \\ \text{Outgoing-Data}^- &= \left\{ \left(\Delta_{a_1}, \Delta_{b_1}, \Delta_{\overline{a_3}}, \Delta_{\overline{b_3}} \right) \in \mathcal{A}_S^{\times 4} \right\}. \end{aligned} \tag{B.8}$$

Internal/Bound data

The internal data of the diagram is composed of the ten soliton generating functions defined on the internal streets a_2, b_2, $\overline{a_2}$, $\overline{b_2}$:

$$\text{Internal-Data} = \left\{ \begin{pmatrix} \Upsilon_{a_2}, & \Upsilon_{b_2}, & \Upsilon_{\overline{a_2}}, & \Upsilon_{\overline{b_2}}, & \Upsilon_{c}, \\ \Delta_{\overline{a_2}}, & \Delta_{\overline{b_2}}, & \Delta_{a_2}, & \Delta_{b_2}, & \Delta_{c}. \end{pmatrix} \in \mathcal{A}_S^{\times 10} \right\} \tag{B.9}$$

However, as far as the results of this thesis are concerned, all that is relevant are the street factors Q_p, for p an internal street, which are derived from the soliton generating functions above:

$$\text{Internal-Data} \rightsquigarrow \left\{ \left(\; Q_{a_2}, \quad Q_{b_2}, \quad Q_{\overline{a_2}}, \quad Q_{\overline{b_2}}, \quad Q_c \; \right) \in \mathbb{Z}[\![\widetilde{\Gamma}]\!]^{\times 5} \right\}. \tag{B.10}$$

We then view a horse as a scattering-matrix machine that eats incoming solitons and spits out outgoing solitons + "bound"/internal solitons:

$$\text{Horse} : \text{Incoming-Data} \to \text{Outgoing-Data} \times \text{Internal-Data},$$

or in other words, we can determine Outgoing-Data and Internal-Data as a function of Incoming-Data; to do so we utilize the six-way junction equations (A.1) to give[4]

$$\begin{aligned}
\Upsilon_{a_2} &= \Upsilon_{a_1} + \eta \Upsilon_c \Delta_{b_2} \\
\Upsilon_{a_3} &= \Upsilon_{a_2} + \eta \left(\eta^{-1} \Upsilon_{a_2} \Upsilon_{\overline{b_2}} \right) \Delta_{\overline{b_2}} \\
&= \Upsilon_{a_2} Q_{\overline{b_2}} \\
\Upsilon_{b_2} &= \Upsilon_{b_1} \\
\Upsilon_{b_3} &= \Upsilon_{b_2} \\
\Upsilon_c &= \eta^{-1} \Upsilon_{a_1} \Upsilon_{b_1} \\
\Delta_{a_1} &= \Delta_{a_2} + \eta^{-1} \Upsilon_{b_1} \left(\Delta_c + \eta \Delta_{b_1} \Delta_{a_2} \right) \\
&= Q_{b_1} \Delta_{a_2} + \eta^{-1} \Upsilon_{b_1} \Delta_c \\
\Delta_{a_2} &= \Delta_{a_3} + \eta^{-1} \Upsilon_{\overline{b_3}} \left(\eta \Delta_{\overline{b_3}} \Delta_{a_3} \right) \\
&= Q_{\overline{b_3}} \Delta_{\overline{a_3}} \\
\Delta_{b_1} &= \Delta_{b_2} + \eta^{-1} \Delta_c \Upsilon_{a_2} \\
\Delta_{b_2} &= \Delta_{b_3}.
\end{aligned} \tag{B.11}$$

By applying the duality operation of Section B.2 to each equation above, we produce the rest of the six-way junction equations.

We wish to solve for the outgoing and internal (blue) quantities in terms of the incoming (red) quantities.

[4]When using the six-way junction equations on the four relevant joints of a horse, pictured in the left panel of Fig. 3.3, one must take into account one-way streets of type 13 that flow out of these joints. However, as shown in the proof of the claim of Section B.3, the soliton generating functions on these one-way streets can be written in terms of soliton generating functions on the horse streets.

Outgoing Soliton Generating Functions

Starting from a_1 and moving counter-clockwise around the edge of Fig. 3.3, we have

$$\begin{aligned}
\Delta_{a_1} &= \left(1+\Upsilon_{b_1}\Delta_{\overline{b_1}}\Delta_{\overline{a_1}}\Upsilon_{a_1}\right)\left(1+\Upsilon_{b_1}\Delta_{b_3}\right)\left(1+\Upsilon_{\overline{b_3}}\Delta_{\overline{b_1}}\right)\Delta_{a_3}+\Upsilon_{b_1}\Delta_{\overline{b_1}}\Delta_{\overline{a_1}} \\
\Delta_{b_1} &= \Delta_{b_3}+\Delta_{\overline{b_1}}\Delta_{\overline{a_1}}\Upsilon_{a_1}\left(1+\Upsilon_{b_1}\Delta_{b_3}\right) \\
\Delta_{\overline{b_3}} &= \Delta_{\overline{b_1}} \\
\Upsilon_{a_3} &= \Upsilon_{a_1}\left(1+\Upsilon_{b_1}\Delta_{b_3}\right)\left(1+\Upsilon_{\overline{b_3}}\Delta_{\overline{b_1}}\right) \\
\Upsilon_{\overline{a_1}} &= \Upsilon_{\overline{a_3}}\left(1+\Delta_{\overline{a_1}}\Upsilon_{a_1}\Upsilon_{b_1}\Delta_{\overline{b_1}}\right)\left(1+\Upsilon_{\overline{b_3}}\Delta_{\overline{b_1}}\right)\left(1+\Upsilon_{b_1}\Delta_{b_3}\right)+\Upsilon_{a_1}\Upsilon_{b_1}\Delta_{\overline{b_1}} \\
\Upsilon_{\overline{b_1}} &= \Upsilon_{\overline{b_3}}+\left(1+\Upsilon_{\overline{b_3}}\Delta_{\overline{b_1}}\right)\Delta_{\overline{a_1}}\Upsilon_{a_1}\Upsilon_{b_1} \\
\Upsilon_{b_3} &= \Upsilon_{b_1} \\
\Delta_{\overline{a_3}} &= \left(1+\Upsilon_{\overline{b_3}}\Delta_{\overline{b_1}}\right)\left(1+\Upsilon_{b_1}\Delta_{b_3}\right)\Delta_{\overline{a_1}}.
\end{aligned}$$

Outgoing Street Factors

We remark that all outgoing street factors can be expressed in terms of the internal street factors. Hence, starting from a_1 and moving counter-clockwise around the edge of the diagram, we have

$$\begin{aligned}
Q_{a_1} &= Q_c Q_{a_2} \\
Q_{b_1} &= Q_c Q_{b_2} \\
Q_{\overline{b_3}} &= Q_{\overline{b_2}} \\
Q_{a_3} &= Q_{a_2} \\
Q_{\overline{a_1}} &= Q_c Q_{\overline{a_2}} \\
Q_{\overline{b_1}} &= Q_c Q_{\overline{b_2}} \\
Q_{b_3} &= Q_{b_2} \\
Q_{\overline{a_3}} &= Q_{\overline{a_2}}.
\end{aligned}$$

Internal Street Factors

We now state the internal street factors in terms of the incoming soliton generating functions. These equations follow from (B.11) and are:

$$\mathcal{Q}_c = 1 + \Upsilon_{a_1}\Upsilon_{b_1}\Delta_{\overline{b_1}}\Delta_{\overline{a_1}}$$
$$\mathcal{Q}_{a_2} = 1 + \Upsilon_{a_1}Q_{b_2}Q_{\overline{b_2}}\Delta_{a_3}$$
$$\mathcal{Q}_{\overline{b_2}} = 1 + \Upsilon_{\overline{b_3}}\Delta_{\overline{b_1}}$$
$$\mathcal{Q}_{\overline{a_2}} = 1 + \Upsilon_{\overline{a_3}}Q_{b_2}Q_{\overline{b_2}}\Delta_{\overline{a_1}}$$
$$\mathcal{Q}_{b_2} = 1 + \Upsilon_{b_1}\Delta_{b_3}.$$

By applying the closure map cl one produces the corresponding Q_p functions.

Remark We note that in all the equations of sections B.3 - B.3 there is an almost magical cancellation of the half-twists η; this cancellation will ultimately ensure that the coefficients of the degeneracy generating functions Q_p (as polynomials in some formal variable $X_{\widehat{\gamma_c}}$, yet to be identified) are all positive.

Special Cases

We now cite two important special cases of incoming data for a horse.

Definitions

1. A lower-sourced horse is a horse along with exactly "two-sources from below," i.e. it is a horse restricted to the subset of Incoming-Data where a point in Incoming-Data$^+$ is specified:

$$\text{Incoming-Data}^+_{\text{LSH}} = \left\{ \begin{pmatrix} \Upsilon_{a_1} = X_a \\ \Upsilon_{b_1} = X_b \\ \Upsilon_{\overline{a_3}} = 0 \\ \Upsilon_{\overline{b_3}} = 0. \end{pmatrix} \right\} \subset \text{Incoming-Data}^+. \tag{B.12}$$

2. An upper-sourced horse is dual to a lower-sourced horse, i.e. it is a horse restricted to the subset of Incoming-Data where a point in Incoming-Data$^-$ is specified:

$$\text{Incoming-Data}^-_{\text{USH}} = \left\{ \begin{pmatrix} \Delta_{\overline{a_1}} = X_{\overline{a}} \\ \Delta_{\overline{b_1}} = X_{\overline{b}} \\ \Delta_{a_3} = 0 \\ \Delta_{b_3} = 0. \end{pmatrix} \right\} \subset \text{Incoming-Data}^-. \tag{B.13}$$

Remark Inserting the lower-sourced horse conditions into the equations of Section B.3, the most important of the resulting equations are

$$Q_{\overline{a_2}} = Q_{\overline{b_2}} = 1; \tag{B.14}$$

which, furthermore, via (B.3) require

$$Q_{\overline{a_3}} = Q_{\overline{b_3}} = 1. \tag{B.15}$$

The upper-sourced horse conditions yield the dual equations,

$$Q_{a_2} = Q_{a_3} = Q_{b_2} = Q_{b_3} = 1. \tag{B.16}$$

With this technology, we can define an m-herd on an arbitrary oriented real surface C as a collection of m-horses glued together using the relations (3.1), beginning with a lower-sourced horse coming from a pair of branch points, and ending with an upper-sourced horse near another pair of branch points (which, for the purposes of this thesis, we will take to be disjoint from the lower-sourced branch points).

Definition Let N be a spectral network subordinate to some branched cover $\Sigma \to C$ and let $H \subset N$ be the set of two-way streets of N. Then N is an m-herd if the following conditions are satisfied.

Horses: There exists a collection of open embedded disks $\{U_l\}_{l=1}^{m} \subset C'$ forming a covering of H, with $U_l \cap U_k \neq \emptyset$ iff $l = k \pm 1$, and each $N \cap U_l$ is:

- a lower-sourced horse if $l = 1$,
- a horse if $1 < l < m$,
- an upper-sourced horse if $l = m$.

Gluing: Each horse satisfies particular gluing conditions: let $p^{(l)}$ denote a horse street[5] on $N \cap U_l$. Then, for $l = 2, \cdots, m-1$, we have the conditions

$$\begin{aligned} a_1^{(l)} &= a_3^{(l-1)} \\ b_1^{(l)} &= b_3^{(l-1)} \\ \overline{a_1}^{(l)} &= \overline{a_3}^{(l+1)} \\ \overline{b_1}^{(l)} &= \overline{b_3}^{(l+1)}. \end{aligned} \tag{3.1}$$

[5]Using our previous naming convention: $p \in \{a_1, a_2, a_3, b_1, b_2, b_3, c, \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{b_1}, \overline{b_2}, \overline{b_3}\}$.

No Holes: For $l = 1, \cdots, m-1$, the oriented loops traced out by the words

- $\left(\overline{a}_2^{(l)}\right)\left(\overline{b}_1^{(l)}\right)\left(a_2^{(l+1)}\right)^{-1}\left(b_3^{(l)}\right)^{-1}$,
- $\left(\overline{b}_2^{(l)}\right)\left(\overline{a}_1^{(l)}\right)\left(b_2^{(l+1)}\right)^{-1}\left(a_3^{(l)}\right)^{-1}$

are each the oriented boundary of (separate) disks on C' (see Fig. B.2).

Remarks

- Note that a 1-herd is the spectral network for a saddle: indeed, via the above definition it consists of a single horse which is both lower and upper-sourced. The picture of a saddle is formed by viewing only the two-way streets remaining after "removing" the horse streets constrained to be one-way according to (B.14) - (B.16).

- Let Incoming-Data$^{\pm}(l)$ (Outgoing-Data$^{\pm}(l)$) be the domain of incoming (range of outgoing) data associated to the lth horse of an m-herd. Via the definition, Incoming-Data$^{+}(1)$ and Incoming-Data$^{+}(m)$ are specified by the lower sourced horse conditions (B.12) and upper-sourced horse conditions (B.13) respectively:

$$\begin{aligned} \text{Incoming-Data}^{+}(1) &= \left\{ \begin{pmatrix} \Upsilon_{a_1}^{(1)} = X_a \\ \Upsilon_{b_1}^{(1)} = X_b \\ \Upsilon_{\overline{a_3}}^{(1)} = 0 \\ \Upsilon_{\overline{b_3}}^{(1)} = 0. \end{pmatrix} \right\}, \\ \text{Incoming-Data}^{-}(m) &= \left\{ \begin{pmatrix} \Delta_{\overline{a_1}}^{(m)} = X_{\overline{a}} \\ \Delta_{\overline{b_1}}^{(m)} = X_{\overline{b}} \\ \Delta_{a_3}^{(m)} = 0 \\ \Delta_{b_3}^{(m)} = 0. \end{pmatrix} \right\}. \end{aligned} \tag{B.17}$$

Further, for $l = 2, \cdots, m-1$, the gluing conditions (3.1) force[6]

$$\begin{aligned} \text{Incoming-Data}^{+}(l) &= \text{Outgoing-Data}^{+}(l-1), \\ \text{Incoming-Data}^{-}(l) &= \text{Outgoing-Data}^{-}(l+1). \end{aligned} \tag{B.18}$$

In fact, as we will discover, all spectral data on an m-herd can be determined recursively from (B.18) using the initial conditions (B.17).

[6]We have omitted the parallel transport map (on the RHS of (B.18)), detailed in Section B.4, that transports spectral data on the $(l-1)$th horse to the lth horse.

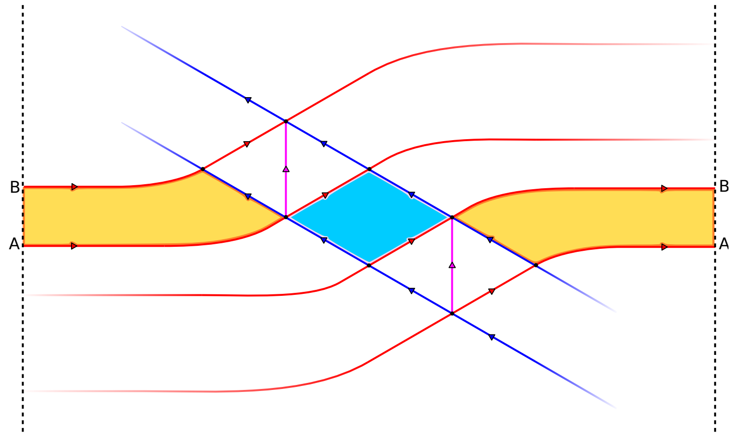


Figure B.2: A picture of two horses (cf. Fig. 3.3) glued together using the *Gluing* conditions and satisfying the *No Holes* condition; the dotted lines are identified, and we assign the "horse-indices" l and $l+1$ to the bottom and top horses respectively. The aqua-blue region is a disk with boundary traced out by the word $\left(\overline{a}_2^{(l)}\right)\left(\overline{b}_1^{(l)}\right)\left(a_2^{(l+1)}\right)^{-1}\left(b_3^{(l)}\right)^{-1}$; the yellow region is a disk with boundary traced out by the word $\left(\overline{b}_2^{(l)}\right)\left(\overline{a}_1^{(l)}\right)\left(b_2^{(l+1)}\right)^{-1}\left(a_3^{(l)}\right)^{-1}$. Two examples for which the *No Holes* condition fails can be pictured by either inserting a puncture, or connect summing with a torus (inserting a "handle"), inside of the colored regions.

- The technical *No Holes* condition excludes cases where there are "holes" between adjacent streets when gluing together horses. This condition is essential for our proof of Prop. 3.3.1, as such holes create obstructions to auxiliary streets introduced in the proof. Furthermore, the *No Holes* condition is utilized in Prop. 3.1.1 in order to produce an explicit expression for the charge $\widehat{\gamma}_c$ (defined in (B.45)) that appears in the formal variable z, but the condition is not necessary to derive the algebraic equation (3.2).[7]

[7]In particular, the *No Holes* condition is used in the definition of the parallel transport maps $\rho_*^{(l,l\pm1)}$ of Section B.4. One could use a more general notation for parallel transport in a situation without the *No Holes* condition and the proof of the algebraic equation would follow similarly, although, the final expression for z would be modified.

B.4 A Global Interlude

The following is a technical subsection dedicated to a proper definition of the symbols $\rho_*^{(k,l)}$ that appear throughout the proof of Prop. 3.1.1. Readers who wish to avoid this technical detour may skip this section and interpret the symbols $\rho_*^{(l,l\pm1)}$ as parallel transport maps along an appropriate path, from the lth horse to the $(l \pm 1)$th horse, along the graph of the m-herd living on C'; further, the $R^{(k,l)}$ can be replaced by parallel transport maps from the lth horse to the kth horse.

First, we will define the local system of soliton charges over $\widetilde{C'}$.

Definition Let $\mathfrak{s} : \bigcup_{\widetilde{z}\in\widetilde{C'}} \widetilde{\Gamma}(\widetilde{z}, -\widetilde{z}) \to \widetilde{C'}$ be the projection map with fibers $\mathfrak{s}^{-1}(\widetilde{z}) = \widetilde{\Gamma}(\widetilde{z}, -\widetilde{z})$.

Remark $\mathfrak{s}$ defines a local system of $\widetilde{\Gamma}$-sets (a locally constant sheaf of $\widetilde{\Gamma}$-sets) over $\widetilde{C'}$, when equipped with a parallel transport map defined by a lifted version of the parallel transport of solitons (2.6). More explicitly, for any path $\ell : [0,1] \to \widetilde{C'}$, the parallel transport map $\widetilde{P}_\ell : \widetilde{\Gamma}\left(\ell(0), -\ell(0)\right) \to \widetilde{\Gamma}\left(\ell(1), -\ell(1)\right)$ is given by

$$\widetilde{P}_\ell s = (s' + [\ell\{j\}] - [\ell\{i\}]) \bmod 2H,\ s \in \widetilde{\Gamma}_{ij}(\ell(0), -\ell(0)). \tag{B.19}$$

where

- s' is a lift of s to a relative homology cycle[8] on $\widetilde{\Sigma}$,
- $\ell\{n\}$ is the lift of ℓ to a path on $\widetilde{\Sigma}$ given by lifting $\ell(0)$ to sheet n,
- $[\ell\{n\}]$ is the relative homology class of $\ell\{n\}$
- $(\cdot) \bmod 2H : G(\ell(1), -\ell(1)) \to \widetilde{\Gamma}(\ell(1), -\ell(1))$ is the quotient map (where the subset of relative homology classes $G(\ell(1), -\ell(1))$ is defined in (2.7)).

By construction, $\widetilde{P}_\ell$ only depends on the homotopy class of ℓ (rel endpoints).

Let $\xi : \widetilde{C'} \to C'$ be the unit tangent bundle projection map (previously denoted $\xi^{C'}$). We now make an important observation.

[8]Recall from (2.8): $\widetilde{\Gamma}(\widetilde{z}, -\widetilde{z})$ is defined as a quotient of the the subset $G(\widetilde{z}, -\widetilde{z})$ (consisting of relative homology classes on $\widetilde{\Sigma}$).

Observation The monodromy of $\widetilde{P}_\ell$ around any loop that wraps the circle fibers of ξ is trivial. I.e., let $z \in C'$ and choose $\ell : S^1 \to (\xi)^{-1}(z) \subset \widetilde{C'}$ to be a closed loop supported on the circle fiber $\xi^{-1}(z)$, then the monodromy $\widetilde{P}_\ell$ is the identity map.

Proof. The proof is immediate: if ℓ is such a loop, then for any sheet n, we have $\mathrm{cl}([\ell\{n\}]) = H$; the result follows from (B.19). □

Definition Let S be any topological space; $\pi_1(S; z_1, z_2)$ is the set of homotopy (rel endpoints) classes of paths $p : [0,1] \to S$ with $p(0) = z_1$ and $p(1) = z_2$.

Corollary B.4.1. *Let $\ell : [0,1] \to \widetilde{C'}$ be a path. Then $\widetilde{P}_\ell : \widetilde{\Gamma}(\ell(0), -\ell(0)) \to \widetilde{\Gamma}(\ell(1), -\ell(1))$ is completely specified by the homotopy class (rel endpoints) of the projected path $\xi \circ \ell : [0,1] \to C'$.*

In particular, given $q \in \pi_1(C', z_1, z_2)$ along with lifts $\widetilde{z}_1 \in \xi^{-1}(z_1)$, $\widetilde{z}_2 \in \xi^{-1}(z_2)$, we may associate a parallel transport map $\widetilde{P}_\ell : \widetilde{\Gamma}(\widetilde{z}_1, -\widetilde{z}_1) \to \widetilde{\Gamma}(\widetilde{z}_2, -\widetilde{z}_2)$ where $\ell : [0,1] \to \widetilde{C'}$ is a lift of any path representative of the class q such that $\ell(0) = \widetilde{z}_1$, $\ell(1) = \widetilde{z}_2$. By the corollary this association $(q, \widetilde{z}_1, \widetilde{z}_2) \rightsquigarrow \widetilde{P}_\ell$ is well-defined.

Definition Let $q \in \pi_1(C'; z_1, z_2)$ and $\widetilde{z}_1 \in \xi^{-1}(z_1)$, $\widetilde{z}_2 \in \xi^{-1}(z_2)$, then $\widetilde{P}_{(q,\widetilde{z}_1,\widetilde{z}_2)} : \widetilde{\Gamma}(\widetilde{z}_1, -\widetilde{z}_1) \to \widetilde{\Gamma}(\widetilde{z}_2, -\widetilde{z}_2)$ is the unique parallel transport map assigned to $(q, \widetilde{z}_1, \widetilde{z}_2)$.

To simplify matters of computation, without ignoring global issues, we will develop a notation, suitable to combinatorics, for parallel transport on an m-herd. As each horse is embedded in a contractible region of C, it suffices to keep track of parallel transport of paths *between the horses* of an m-herd: our notation need not keep track of parallel transport between points in an individual horse as suggested by the following remark.

Remark Let $\{U_l\}_{l=1}^m$ be an open cover of disks (on C') satisfying the *Horses* condition for an m-herd, then all paths running between points $z_1, z_2 \in U_l$ and contained within U_l are homotopic (rel endpoints). Thus, by Cor. B.4.1, for each pair of points $\widetilde{z}_1 \in \xi^{-1}(z_1)$, $\widetilde{z}_2 \in \xi^{-1}(z_2)$, there is a unique parallel transport map assigned to all paths running from $\widetilde{z}_1$ to $\widetilde{z}_2$ and contained in $\xi^{-1}(U_l)$.

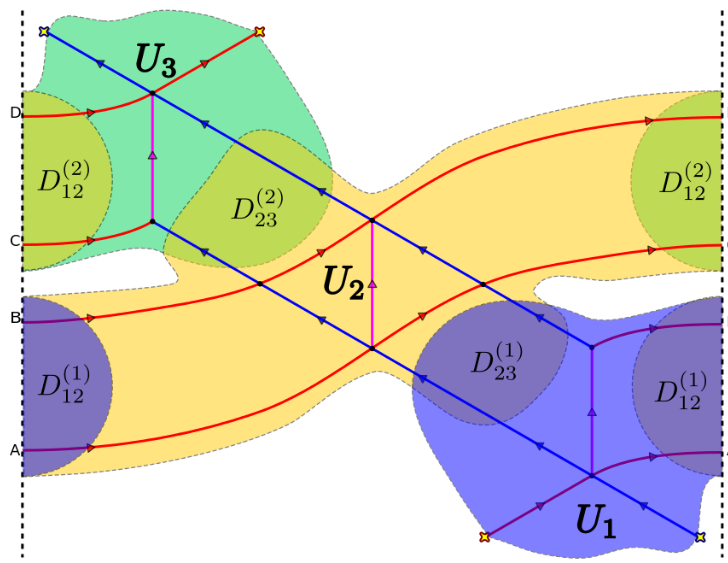


Figure B.3: A 3-herd, on $C = \mathbb{R} \times S^1$, equipped with a (two-way street) cover $\{U_l\}_{l=1}^3$ satisfying the *Horses* condition along with (B.20)-(B.21).

Now, let us turn our attention to parallel transport of paths running between horses; in particular, paths contained in $U_l \cup U_{l+1}$ for some $l = 1, \cdots, m-1$. First, note that each non-vanishing intersection $U_l \cap U_{l+1}$, $l = 1, \cdots, m-1$, will consist of some number of disconnected disks. However, on an m-herd, the *No Holes* condition allows us to modify our cover such that $U_l \cap U_{l+1}$ contains exactly *two* components:

$$U_l \cap U_{l+1} = D_{12}^{(l)} \sqcup D_{23}^{(l)}, \tag{B.20}$$

where the $D_{ij}^{(l)}$ are disks such that for $l = 1, \cdots, m-1$

$$\begin{aligned} &\left(\overline{a}_1^{(l)} = \overline{a}_3^{(l+1)}\right) \cap (U_l \cap U_{l+1}) \subset D_{12}^{(l)}, \ \left(a_3^{(l)} = a_1^{(l+1)}\right) \cap (U_l \cap U_{l+1}) \subset D_{12}^{(l)} \\ &\left(\overline{b}_1^{(l)} = \overline{b}_3^{(l+1)}\right) \cap (U_l \cap U_{l+1}) \subset D_{23}^{(l)}, \ \left(b_3^{(l)} = b_1^{(l+1)}\right) \cap (U_l \cap U_{l+1}) \subset D_{23}^{(l)}; \end{aligned} \tag{B.21}$$

an example of such a cover is shown in Fig. B.3 for the case of a 3-herd on $\mathbb{C} = \mathbb{R} \times S^1$. Thus, fixing a pair of points $\widetilde{z}_1 \in \xi^{-1}(U_l)$, $\widetilde{z}_2 \in \xi^{-1}(U_{l+1})$, our interest lies in two homotopy classes (rel endpoints) of paths that run from $\widetilde{z}_1$ to $\widetilde{z}_2$, and are contained in $U_l \cup U_{l+1}$. In particular, denoting these two classes by $q_{12}, q_{23} \in \pi_1(U_l \cup U_{l+1}; z_1, z_2)$,

1. q_{12} has a path representative given by a simple curve running from $\widetilde{z}_1$ to $\widetilde{z}_2$ and passing through $D_{12}^{(l)}$ (but not $D_{23}^{(l)}$) exactly once,

2. q_{23} has a path representative given by a simple curve running from $\widetilde{z}_1$ to $\widetilde{z}_2$ and passing through $D_{23}^{(l)}$ (but not $D_{12}^{(l)}$) exactly once.

Definition Let $z_1 \in U_l$, $z_2 \in U_{l+1}$, and take q_{ij} ($ij \in \{12, 23\}$) to be the homotopy classes described above. Then, for a choice of lifts $\widetilde{z}_1 \in \xi^{-1}(z_1)$ and $\widetilde{z}_2 \in \xi^{-1}(z_2)$,

$$\begin{aligned} \rho_{ij}^{(l,l+1)}(\widetilde{z}_1, \widetilde{z}_2) &:= \widetilde{P}_{(q_{ij}, \widetilde{z}_1, \widetilde{z}_2)} : \widetilde{\Gamma}(\widetilde{z}_1, -\widetilde{z}_1) \to \widetilde{\Gamma}(\widetilde{z}_2, -\widetilde{z}_2), \\ \rho_{ij}^{(l+1,l)}(\widetilde{z}_1, \widetilde{z}_2) &:= \widetilde{P}_{(q_{ij}^{-1}, \widetilde{z}_2, \widetilde{z}_1)} : \widetilde{\Gamma}(\widetilde{z}_2, -\widetilde{z}_2) \to \widetilde{\Gamma}(\widetilde{z}_1, -\widetilde{z}_1) = \left[\rho_{ij}^{(l,l+1)}(\widetilde{z}_1, \widetilde{z}_2)\right]^{-1}. \end{aligned} \tag{B.22}$$

Notation In the following computations we will just write $\rho_{ij}^{(l,l+1)}$, dropping the explicit dependence on the endpoints $\widetilde{z}_1 \in \xi^{-1}(U_l)$ and $\widetilde{z}_2 \in \xi^{-1}(U_{l+1})$; this notation will be sufficiently unambiguous for our purposes. Indeed, let $\widetilde{w}_1 \in \xi^{-1}(U_l)$, $\widetilde{w}_2 \in \xi^{-1}(U_{l+1})$ be another choice of endpoints with projections $w_i = \xi(\widetilde{w}_i)$, $i = 1, 2$; then, by a remark above, $\exists!$ homotopy classes $q_1 \in \pi_1(U_l; w_1, z_1)$ and $q_2 \in \pi_1(U_{l+1}, z_2, w_2)$ such that

$$\rho_{ij}^{(l,l+1)}(\widetilde{w}_1, \widetilde{w}_2) = \widetilde{P}_{(q_2, \widetilde{z}_2, \widetilde{w}_2)} \left(\rho_{ij}^{(l,l+1)}(\widetilde{z}_1, \widetilde{z}_2)\right) \widetilde{P}_{(q_1, \widetilde{w}_1, \widetilde{z}_1)},\ ij \in \{12, 23\}.$$

Now, on an m-herd, (B.21) indicates that only solitons of type 12 or 21 will be transported via $\rho_{12}^{(l,l+1)}$, and only solitons of type 23 or 32 will be transported via $\rho_{23}^{(l,l+1)}$. With this in mind, for the sake of readability, it will prove convenient to make further notation-simplifying definitions.

Definitions

1. Let $\widetilde{z} \in \xi^{-1}(U_l)$, then

$$\rho_*^{(l,l+1)} a := \begin{cases} \rho_{12}^{(l,l+1)} a & \text{if } a \in \widetilde{\Gamma}_{12}(\widetilde{z}, -\widetilde{z}) \cup \widetilde{\Gamma}_{21}(\widetilde{z}, -\widetilde{z}) \\ \rho_{23}^{(l,l+1)} a & \text{if } a \in \widetilde{\Gamma}_{23}(\widetilde{z}, -\widetilde{z}) \cup \widetilde{\Gamma}_{32}(\widetilde{z}, -\widetilde{z}) \end{cases}, \tag{B.23}$$

and $\rho_*^{(l-1,l)} := \left(\rho_*^{(l-1,l)}\right)^{-1}$.

2.

$$R^{(k,n)} := \begin{cases} \rho_*^{(n-1,n)} \cdots \rho_*^{(k+1,k+2)} \rho_*^{(k,k+1)} & \text{if } k < n \\ \rho_*^{(n+1,n)} \cdots \rho_*^{(k-1,k-2)} \rho_*^{(k,k-1)} & \text{if } n < k. \end{cases} \tag{B.24}$$

Remarks

1. The $\rho_*^{(l,k)}$ extend their action to formal variables X_a via

$$\rho_*^{(l,k)} X_a = X_{\rho_*^{(l,k)} a}.$$

2. $R^{(k,n)}$ is a parallel transport map, on the local system $\mathfrak{s}$, from the kth horse to the nth horse associated to a path that passes through each l-horse between k and n exactly once. If $R^{(k,n)}$ acts on a soliton of charge 12 or 21, this path passes through the sets $D_{12}^{(l)}$ (but never $D_{23}^{(l)}$) for $\min\{k,n\} < l < \max\{k,n\}$; if $R^{(k,n)}$ acts on a soliton of charge 23 or 32 the path passes through the sets $D_{23}^{(l)}$ (but never $D_{12}^{(l)}$) for $\min\{k,n\} < l < \max\{k,n\}$.

We make one final observation that will be of use in Section B.7.

Remark Let $\mathfrak{r} : \bigcup_{z\in C} \Gamma(z,z) \to C'$ be the projection map with $\mathfrak{r}^{-1}(z) = \Gamma(z,z)$; this forms a local system over C' when equipped with the parallel transport map

$$P_q s_* = s_* + [q\{j\}] - [q\{i\}],\ s_* \in \Gamma_{ij}(q(0), q(0)). \tag{B.25}$$

The parallel transport on $\mathfrak{r}$ is compatible with the parallel transport (B.19) on the local system $\mathfrak{s}$ in the sense that for any $q \in \pi_1(C', z_1, z_2)$ and $\widetilde{z}_1 \in \xi^{-1}(z_1)$, $\widetilde{z}_2 \in \xi^{-1}(z_2)$, we have

$$\xi_*^{\Sigma}\left(\widetilde{P}_{(q,\widetilde{z}_1,\widetilde{z}_2)} s\right) = P_q\left(\xi_*^{\Sigma} s\right), \tag{B.26}$$

where, recall, $\xi^{\Sigma} : \widetilde{\Sigma} \to \Sigma$ is the unit tangent bundle projection map.

Now, we may define the analog of the parallel transport operators $R^{(k,n)}$ for $\mathfrak{r}$.

Definition Let $s_* \in \bigsqcup_{ij\in\{12,21,23,32\}} \Gamma_{ij}(z,z)$ for some $z \in U_k$, then

$$R_{\mathfrak{r}}^{(k,n)} s_* := \xi_*^{\Sigma} R^{(k,n)} s, \tag{B.27}$$

where $s \in \bigsqcup_{ij\in\{12,21,23,32\}} \widetilde{\Gamma}_{ij}(\widetilde{z}, -\widetilde{z})$ is any lift of s_* (i.e. $s_* = \xi_*^{\Sigma} s$).

(B.26) ensures that (B.27) is a well-defined (lift-independent) statement.

B.5 Identifications of Generating Functions

Using the notation developed in Section B.4, we can express (B.18) explicitly as

$$\begin{aligned}
\Upsilon^{(l)}_{a_1} &= \rho^{(l-1,l)}_{*}\Upsilon^{(l-1)}_{a_3}, & \Delta^{(l)}_{\overline{a_1}} &= \rho^{(l+1,l)}_{*}\Delta^{(l+1)}_{\overline{a_3}},\\
\Upsilon^{(l)}_{b_1} &= \rho^{(l-1,l)}_{*}\Upsilon^{(l-1)}_{b_3}, & \Delta^{(l)}_{\overline{b_1}} &= \rho^{(l+1,l)}_{*}\Delta^{(l+1)}_{\overline{b_3}},\\
\Upsilon^{(l)}_{\overline{a_3}} &= \rho^{(l-1,l)}_{*}\Upsilon^{(l-1)}_{\overline{a_1}}, & \Delta^{(l)}_{a_3} &= \rho^{(l+1,l)}_{*}\Delta^{(l+1)}_{a_1},\\
\Upsilon^{(l)}_{\overline{b_3}} &= \rho^{(l-1,l)}_{*}\Upsilon^{(l-1)}_{\overline{b_1}}, & \Delta^{(l)}_{b_3} &= \rho^{(l+1,l)}_{*}\Delta^{(l+1)}_{b_1}.
\end{aligned} \tag{B.28}$$

In particular,

$$\begin{aligned}
Q^{(l)}_{a_1} &= Q^{(l-1)}_{a_3}, & Q^{(l)}_{\overline{a_1}} &= Q^{(l+1)}_{\overline{a_3}},\\
Q^{(l)}_{b_1} &= Q^{(l-1)}_{b_3}, & Q^{(l)}_{\overline{b_1}} &= Q^{(l+1)}_{\overline{b_3}}.
\end{aligned} \tag{B.29}$$

B.6 Proof of Proposition 3.1.1

Proof of Equations (3.3)

Using the recursion relations (*B*.29), in conjunction with the equations listed in Sections B.3 and B.3, we first solve for the internal street factors $Q^{(l)}_{a_2}$, $Q^{(l)}_{\overline{a_2}}$, $Q^{(l)}_{b_2}$, $Q^{(l)}_{\overline{b_2}}$ in terms of street factors on the lower/upper-sourced horses at $l = 1$ or $l = m$. As we noticed in Section B.3, all other street factors can be written in terms of the internal ones.

Now, via (B.29), and the equations of Section (B.3),

$$\begin{aligned}
Q^{(l)}_{a_2} &= Q^{(l)}_{a_3}\\
&= Q^{(l+1)}_{a_1}\\
&= Q^{(l+1)}_{c}Q^{(l+1)}_{a_2}.
\end{aligned} \tag{B.30}$$

Similarly, we find

$$Q^{(l)}_{\overline{a_2}} = Q^{(l-1)}_{c}Q^{(l-1)}_{\overline{a_2}} \tag{B.31}$$

$$Q^{(l)}_{b_2} = Q^{(l+1)}_{c}Q^{(l+1)}_{b_2} \tag{B.32}$$

$$Q^{(l)}_{\overline{b_2}} = Q^{(l-1)}_{c}Q^{(l-1)}_{\overline{b_2}}. \tag{B.33}$$

This leads us to the following.

Lemma B.6.1. *For $l = 1, \ldots, m$, we have*

$$Q_{a_2}^{(l)} = Q_{b_2}^{(l)} = \prod_{r=l+1}^{m+1} Q_c^{(r)} \tag{B.34}$$

$$Q_{\overline{a_2}}^{(l)} = Q_{\overline{b_2}}^{(l)} = \prod_{r=0}^{l-1} Q_c^{(r)}. \tag{B.35}$$

with the convention that $Q_c^{(m+1)} = Q_c^{(0)} = 1$.

Proof. From the upper-sourced horse conditions (B.13) we have

$$Q_{a_2}^{(m)} = Q_{b_2}^{(m)} = 1; \tag{B.16}$$

so (B.34) follows via (B.30) and (B.32). Similarly, from the lower-sourced horse conditions (B.12) we have

$$Q_{\overline{a_2}}^{(1)} = Q_{\overline{b_2}}^{(1)} = 1; \tag{B.14}$$

so (B.35) follows via (B.31) and (B.33). □

To reduce (B.34) - (B.35) further, we must compute some soliton generating functions.

Computing $\Upsilon_{b_1}^{(l)}$

Via (B.28)

$$\begin{aligned} \Upsilon_{b_1}^{(l)} &= \rho_*^{(l-1,l)} \Upsilon_{b_3}^{(l-1)} \\ &= \rho_*^{(l-1,l)} \Upsilon_{b_1}^{(l-1)}. \end{aligned} \tag{B.36}$$

Thus, propagating the lower sourced horse conditions (B.12) through this recursion relation,

$$\Upsilon_{b_1}^{(l)} = \left(\prod_{r=1}^{l} \rho_*^{(r-1,r)} \right) X_b \tag{B.37}$$

$$= R^{(1,l)} X_b. \tag{B.38}$$

Computing $\Delta^{(l)}_{\overline{b_1}}$

The idea is dual to above; indeed

$$\begin{aligned} \Delta^{(l)}_{\overline{b_1}} &= \rho_*^{(l+1,l)} \Delta^{(l+1)}_{\overline{b_3}} \\ &= \rho_*^{(l+1,l)} \Delta^{(l+1)}_{\overline{b_1}}. \end{aligned} \tag{B.39}$$

Using the upper-sourced horse conditions (B.13),

$$\begin{aligned} \Delta^{(l)}_{\overline{b_1}} &= \left(\prod_{r=l}^{m} \rho_*^{(r+1,r)} \right) X_{\overline{b}} \\ &= R^{(m,l)} X_{\overline{b}}. \end{aligned} \tag{B.40}$$

Computing $\Upsilon^{(l)}_{a_1}$

Via (B.28) and the equation for Υ_{a_3} in Section B.3,

$$\begin{aligned} \Upsilon^{(l)}_{a_1} &= \rho_*^{(l-1,l)} \Upsilon^{(l-1)}_{a_3} \\ &= \rho_*^{(l-1,l)} \Upsilon^{(l-1)}_{a_1} \left(1 + \Upsilon^{(l-1)}_{\overline{b_3}} \Delta^{(l-1)}_{\overline{b_1}}\right) \left(1 + \Upsilon^{(l-1)}_{b_1} \Delta^{(l-1)}_{b_3}\right) \\ &= \rho_*^{(l-1,l)} \Upsilon^{(l-1)}_{a_1} Q^{(l-1)}_{\overline{b_2}} Q^{(l-1)}_{b_2}. \end{aligned} \tag{B.41}$$

Using the lower-sourced horse conditions (B.12),

$$\begin{aligned} \Upsilon^{(l)}_{a_1} &= \left(\prod_{r=1}^{l} \rho_*^{(r-1,r)} X_a \right) \left(\prod_{r=1}^{l} \rho_*^{(r-1,r)} Q^{(r-1)}_{\overline{b_2}} Q^{(r-1)}_{b_2} \right) \\ &= R^{(1,l)} X_a \left(\prod_{r=0}^{l-1} Q^{(r)}_{\overline{b_2}} Q^{(r)}_{b_2} \right). \end{aligned} \tag{B.42}$$

Computing $\Delta^{(l)}_{\overline{a_1}}$

Again, the computation is dual to that for $\Upsilon^{(l)}_{a_1}$,

$$\begin{aligned} \Delta^{(l)}_{\overline{a_1}} &= \rho_*^{(l+1,l)} \Delta^{(l+1)}_{\overline{a_3}} \\ &= \rho_*^{(l+1,l)} \left[\left(1 + \Upsilon^{(l+1)}_{\overline{b_3}} \Delta^{(l+1)}_{\overline{b_1}}\right) \left(1 + \Upsilon^{(l+1)}_{b_1} \Delta^{(l+1)}_{b_3}\right) \Delta^{(l+1)}_{\overline{a_1}} \right] \\ &= \rho_*^{(l+1,l)} Q^{(l+1)}_{\overline{b_2}} Q^{(l+1)}_{b_2} \Delta^{(l+1)}_{\overline{a_1}}. \end{aligned} \tag{B.43}$$

So, using the upper-sourced horse conditions (B.13),

$$\Delta_{\overline{a_1}}^{(l)} = \left(\prod_{r=l}^{m} \rho_*^{(r+1,l)} Q_{\overline{b_2}}^{(r+1)} Q_{b_2}^{(r+1)}\right) X_a$$
$$= \left(\prod_{r=l+1}^{m+1} Q_{\overline{b_2}}^{(r)} Q_{b_2}^{(r)}\right) R^{(m,l)} X_a. \tag{B.44}$$

These computations lead us to the following key lemma that allows all street factors Q_p to be reduced to powers of a single function.

Lemma B.6.2.

$$\mathcal{Q}_c^{(l)} = \mathcal{Q}_c^{(1)},\ \forall l = 1, \cdots, m.$$

Proof. Recall (B.36), (B.39), (B.41), and (B.43)

$$\Upsilon_{b_1}^{(l)} = \rho_*^{(l-1,l)} \Upsilon_{b_1}^{(l-1)}$$
$$\Delta_{\overline{b_1}}^{(l)} = \rho_*^{(l+1,l)} \Delta_{\overline{b_1}}^{(l+1)}$$
$$\Upsilon_{a_1}^{(l)} = \rho_*^{(l-1,l)} Q_{\overline{b_2}}^{(l-1)} Q_{b_2}^{(l-1)} \Upsilon_{a_1}^{(l-1)}$$
$$\Delta_{\overline{a_1}}^{(l)} = \rho_*^{(l+1,l)} Q_{\overline{b_2}}^{(l+1)} Q_{b_2}^{(l+1)} \Delta_{\overline{a_1}}^{(l+1)};$$

we can rewrite the equations for $\Delta_{\overline{b_1}}^{(l)}$ and $\Delta_{\overline{a_1}}^{(l)}$ as

$$\Delta_{\overline{b_1}}^{(l)} = \rho_*^{(l-1,l)} \Delta_{\overline{b_1}}^{(l-1)}$$
$$\Delta_{\overline{a_1}}^{(l)} = \rho_*^{(l-1,l)} \frac{\Delta_{\overline{a_1}}^{(l-1)}}{Q_{\overline{b_2}}^{(l)} Q_{b_2}^{(l)}}.$$

Using the equation for $\mathcal{Q}_c$ in Section B.3

$$\begin{aligned}\mathcal{Q}_c^{(l)} &= 1 + \Upsilon_{a_1}^{(l)} \Upsilon_{b_1}^{(l)} \Delta_{\overline{b_1}}^{(l)} \Delta_{\overline{a_1}}^{(l)} \\ &= 1 + \left(\rho_*^{(l-1,l)} Q_{\overline{b_2}}^{(l-1)} Q_{b_2}^{(l-1)} \Upsilon_{a_1}^{(l-1)}\right) \left(\rho_*^{(l-1,l)} \Upsilon_{b_1}^{(l-1)}\right) \left(\rho_*^{(l-1,l)} \Delta_{\overline{b_1}}^{(l-1)}\right) \left(\rho_*^{(l-1,l)} \frac{\Delta_{\overline{a_1}}^{(l-1)}}{Q_{\overline{b_2}}^{(l)} Q_{b_2}^{(l)}}\right) \\ &= 1 + \left(\mathcal{Q}_c^{(l-1)} - 1\right) \left(\frac{Q_{\overline{b_2}}^{(l-1)} Q_{b_2}^{(l-1)}}{Q_{\overline{b_2}}^{(l)} Q_{b_2}^{(l)}}\right);\end{aligned}$$

where, on the last line, the cancellation of the $\rho_*^{(l-1,l)}$ (parallel transport) actions[9] can be seen by working through its definition in equations (B.19), (B.22), and (B.23). Applying the closure map we obtain

$$Q_c^{(l)} = 1 + \left(Q_c^{(l-1)} - 1\right) \left(\frac{Q_{\overline{b_2}}^{(l-1)} Q_{b_2}^{(l-1)}}{Q_{\overline{b_2}}^{(l)} Q_{b_2}^{(l)}} \right).$$

Using (B.34) and (B.35), then

$$\begin{aligned} Q_c^{(l)} &= 1 + \left(Q_c^{(l-1)} - 1\right) \left(\frac{\prod_{r \neq l-1} Q_c^{(r)}}{\prod_{r \neq l} Q_c^{(r)}} \right) \\ &= 1 + \left(Q_c^{(l-1)} - 1\right) \frac{Q_c^{(l)}}{Q_c^{(l-1)}} \\ &= 1 + Q_c^{(l)} - \frac{Q_c^{(l)}}{Q_c^{(l-1)}}. \end{aligned}$$

Hence,

$$Q_c^{(l)} = Q_c^{(l-1)}, \, l = 2, \cdots, m.$$

□

The above proposition motivates the following simplified notation.

Definition

$$P_m := Q_c^{(1)}.$$

Now, when lemmata B.6.1 and B.6.2 are combined, we have

Corollary B.6.3.

$$\begin{aligned} Q_{a_2}^{(l)} &= Q_{b_2}^{(l)} = (P_m)^{m-l} \\ Q_{\overline{a_2}}^{(l)} &= Q_{\overline{b_2}}^{(l)} = (P_m)^{l-1}. \end{aligned}$$

[9]This is consistent with the fact that, according to (B.19), parallel transport acts trivially on charges of type ii.

The above corollary, combined with the equations of Section B.3, is enough to express the remainder of the street factors in terms of P_m,

$$
\begin{aligned}
(P_m)^{m-l} &= Q^{(l)}_{a_3} = Q^{(l)}_{b_3} \\
(P_m)^{l-1} &= Q^{(l)}_{\overline{a_3}} = Q^{(l)}_{\overline{b_3}} \\
(P_m)^{m-l+1} &= Q^{(l)}_{a_1} = Q^{(l)}_{b_1} \\
(P_m)^{l} &= Q^{(l)}_{\overline{a_1}} = Q^{(l)}_{\overline{b_1}}.
\end{aligned}
$$

This completes the proof of (3.3) in Prop. 3.1.

Proof of the Algebraic Equation (3.2)

Via the equation for $\mathcal{Q}_c$ in Section B.3 along with (B.38)-(B.42),

$$
\begin{aligned}
\mathcal{Q}^{(l)}_c &= 1 + \Upsilon^{(l)}_{a_1}\Upsilon^{(l)}_{b_1}\Delta^{(l)}_{\overline{b_1}}\Delta^{(l)}_{\overline{a_1}} \\
&= 1 + \left[\left(\prod_{r=0}^{l-1} Q^{(r)}_{\overline{b_2}} Q^{(r)}_{b_2}\right) R^{(1,l)} X_a\right]\left[R^{(1,l)} X_b\right]\left[R^{(m,l)} X_{\overline{b}}\right]\left[\left(\prod_{r=l+1}^{m+1} Q^{(r)}_{\overline{b_2}} Q^{(r)}_{b_2}\right) R^{(m,l)} X_a\right] \\
&= 1 + \left(\prod_{r\neq l} Q^{(r)}_{\overline{b_2}} Q^{(r)}_{b_2}\right)\left(R^{(1,l)} X_a\right)\left(R^{(1,l)} X_b\right)\left(R^{(m,l)} X_{\overline{b}}\right)\left(R^{(m,l)} X_a\right) \\
&= 1 + (P_m)^{(m-1)^2}\left(R^{(1,l)} X_a\right)\left(R^{(1,l)} X_b\right)\left(R^{(m,l)} X_{\overline{b}}\right)\left(R^{(m,l)} X_{\overline{a}}\right);
\end{aligned}
$$

where, on the last line we utilized Corollary B.6.3.

Remark We note that,

$$R^{(1,l)}a + R^{(1,l)}b + R^{(m,l)}\overline{b} + R^{(m,l)}\overline{a}$$

represents a soliton charge of type 11 on the open set $\xi^{-1}(U_l) \subset \widetilde{C'}$. Thus, we may apply the map cl to this expression to produce an element of $\widetilde{\Gamma}$.

This leads us to the following definition.

Definition We define

$$\widehat{\gamma}_c : = \mathrm{cl}\left[R^{(1,l)}a + R^{(1,l)}b + R^{(m,l)}\overline{b} + R^{(m,l)}\overline{a}\right] \in \widetilde{\Gamma} \tag{B.45}$$

and corresponding formal variable

$$z := X_{\widehat{\gamma}_c}. \tag{B.46}$$

(We will show below that, in fact, (B.45) does not depend on l; thus, this definition is sensible.)

With the above definitions we have

$$Q_c^{(l)} = 1 + zP_m^{(m-1)^2}$$

hence, by Lemma B.6.2, P_m satisfies the algebraic equation

$$P_m = 1 + zP_m^{(m-1)^2}. \tag{3.2}$$

This completes the proof of the algebraic equation in Prop. 3.1.1.

Remark As we will show in Section B.7, $\widehat{\gamma}_c$ is the sum of two tangent framing lifts of simple closed curves with corresponding homology classes γ, $\gamma' \in \Gamma$. In fact, we will show that (B.46) can be rewritten in the form stated in Prop. 3.1.1: $z = (-1)^m X_{\widetilde{\gamma}+\widetilde{\gamma}'}$, where $\widetilde{(\cdot)} : \Gamma \to \widetilde{\Gamma}$ is defined in Section (2.2) and discussed further in Section D.

B.7 Proof of the Decomposition of $\widehat{\gamma}_c$

We begin with an example (which may be skipped for the more general proof below).[10]

Example: $\widehat{\gamma}_c$ for m-herds on the cylinder

We consider generalizations (to arbitrary m) of the herds shown in Fig. 3.4 for $m = 1, \cdots, 4$. Assume we are equipped with a branched 3-cover of the cylinder $C = S^1 \times \mathbb{R}$ with four branch points. Now, consider an m-herd such that it is contained in a presentation of the cylinder as an identification space of $[0,1] \times \mathbb{R}$: the streets of type 23 lie entirely in the interior of $(0,1) \times \mathbb{R}$, while the streets of type 12 involved in the identifications (3.1) pass through the identified boundary. First, each of the charges a, $b, \overline{a}$, $\overline{b}$ can be thought of as flat sections of the local system $\mathfrak{s} : \bigcup_{\widetilde{z} \in \widetilde{C'}} \widetilde{\Gamma}(\widetilde{z}, -\widetilde{z}) \to \widetilde{C'}$, locally defined around their respective branch points. The two-way streets are contained within the open set $U := \bigcup_{l=1}^m U_l$, which is homeomorphic to $S^1 \times I$ for $I \cong (0,1)$ an open interval. Let $U^c \cong (0,1)^2$ be the open set formed by removing the vertical line[11] $(\{0\} \times \mathbb{R}) \cap U \sim (\{1\} \times \mathbb{R}) \cap U$ from U. $\mathfrak{s}$ is trivial over the open set

[10] The following sections rely on the ideas of Section B.4.

[11] Here $\sim$ denotes the identification of the boundary of $[0,1] \times \mathbb{R}$ to form the cylinder. The removed vertical line is given by the (identified) dotted lines in Fig. 3.4.

$\xi^{-1}(U^c) \cong (0,1)^2 \times S^1$ in $\widetilde{C'}$; so, we can extend a, $b, \overline{a}$, $\overline{b}$ to flat sections over all of $\xi^{-1}(U^c)$.

Now, let $q_{\text{cyl}} : [0,1] \to U \subset C'$ denote a loop winding once around the S^1 direction of U, and oriented such that the upper-sourced horse branch points sit to its "left," while the lower-sourced horse branch points sit to its "right"; $\widehat{q}_{\text{cyl}} : [0,1] \to \widetilde{C'}$ will denote the tangent framing lift of q_{cyl}.

Working through the definition of the parallel transport maps $R^{(k,l)}$ in (B.19), (B.22)-(B.24), we have

$$\widehat{\gamma}_c = \text{cl}\left(a + b + \overline{b} + \overline{a}\right) + (m-1)\left([\hat{q}_{\text{cyc}}\{2\}] - [\hat{q}_{\text{cyc}}\{1\}]\right);$$

the expression in the closure map is defined by evaluating the sections a, $b, \overline{a}$, $\overline{b}$ at some point $\widetilde{z} \in \xi^{-1}(U^c \cap U_l)$ and taking their sum to define an element in $\widetilde{\Gamma}_{11}(\widetilde{z}, -\widetilde{z})$.

Observe that we can decompose $\widehat{\gamma}_c$ as $\widehat{\gamma}_c = \widehat{\gamma} + \widehat{\gamma}'$ where,

$$\begin{aligned}\widehat{\gamma} &= \text{cl}\left(b + \overline{b}\right)\\ \widehat{\gamma}' &= \text{cl}\left(a + \overline{a}\right) + (m-1)\left([\hat{q}_{\text{cyc}}\{2\}] - [\hat{q}_{\text{cyc}}\{1\}]\right).\end{aligned}$$

Now, note that we can realize $\widehat{\gamma}$ the tangent framing lift of a simple closed curve on Σ. Indeed, consider an auxiliary street of type 23, realized as a straight line on $U \cup \{\text{branch pts.}\}$, running between the two branch points of type 23 (beginning at the branch point emitting the charge b and ending at the branch point emitting the charge $\overline{b}$). The lift of this street to Σ is a simple closed curve; the tangent framing lift is a representative of $\text{cl}\left(b + \overline{b}\right)$. Similarly, we can realize $\text{cl}\left(a + \overline{a}\right)$ with the tangent framing lift of a simple closed curve ℓ_a on $U \cup \{\text{branch pts.}\}$ and so $\widehat{\gamma}'$ can be realized as a modification of ℓ_a by smoothly "detouring" along the lifts (to sheets 1 and 2) of a curve that winds $m-1$ times around the S^1 direction of U. The resulting curve is the tangent framing lift of a simple closed curve. Furthermore, project these simple-closed curves to Σ; then letting γ and γ' be the homology classes of our projections, with their representative curves it is clear that $\langle \gamma, \gamma' \rangle = m$.

Now, using different techniques, let us proceed on with the general proof of the decomposition $\widehat{\gamma}_c = \widehat{\gamma} + \widehat{\gamma}'$, described in the example above, for an m-herd on a general oriented curve C.

General Proof

Let $\xi^{\Sigma} : \widetilde{\Sigma} \to \Sigma$ be the unit tangent bundle projection.

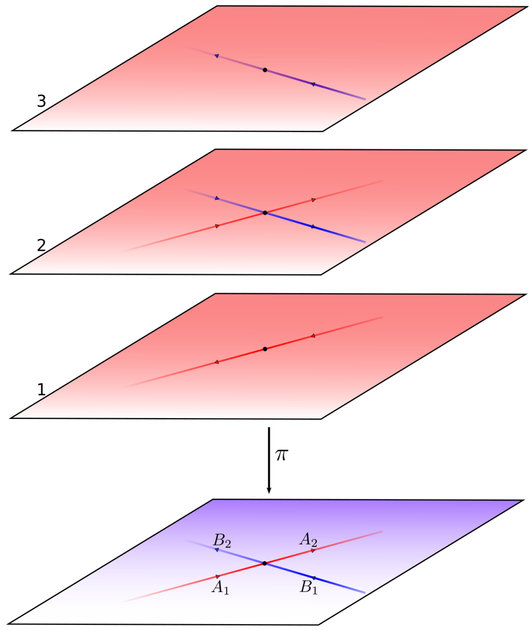


Figure B.4: A pony and its lift to Σ. Red streets are of type 12, blue streets are of type 23.

Definition

$$\gamma_c := \xi^{\Sigma}_{*}\widehat{\gamma}_c \in \Gamma.$$

To derive an explicit expression for γ_c in terms of simpleton charges (B.1) in $\bigcup_{z\in C}\Gamma(z,z)$, we "pushforward" the expression (B.45) via ξ^{Σ}. From the definitions (B.1), (B.2), and (B.45) it follows that

$$\gamma_c = \mathrm{cl}\left[R^{(1,l)}_{\mathfrak{r}}a_* + R^{(1,l)}_{\mathfrak{r}}b_* + R^{(m,l)}_{\mathfrak{r}}\overline{b}_* + R^{(m,l)}_{\mathfrak{r}}\overline{a}_*\right] \tag{B.47}$$

where $R^{(k,n)}_{\mathfrak{r}}$ are the "pushforward" of the parallel transport operators $R^{(k,n)}$ defined in (B.27).

We will construct a decomposition $\gamma_c = \gamma + \gamma'$ with $\langle\gamma,\gamma'\rangle = m$ roughly by shrinking the $c^{(l)}$ streets of the herd to points. To be precise, we introduce some definitions.

Definitions

1. A pony is a partial spectral network as shown in Fig. B.4. Upper and lower-sourced ponies are defined similar to upper and lower sourced horses.

2. The string of ponies S_m associated to an m-herd H_m is the spectral network constructed by placing

   a) A lower sourced pony on U_1

   b) Ponies on U_l, $1 < l < m$

   c) An upper-sourced pony on U_m,

   where the U_m are a good open cover satisfying the *Horses* condition for H_m, and forcing the identifications

$$A_1^{(l+1)} = A_2^{(l)}$$
$$B_1^{(l+1)} = B_2^{(l)}$$

   on each $U_l \cap U_{l+1}$.

Remarks

1. S_m is only defined up to homotopy on each disk U_l.

2. The interpretation of S_m as a spectral network is overkill for our discussion and we introduce it as such mainly for notational convenience: all that will be necessary is the graph of the lift $\mathrm{Lift}(S_m) \subset \Sigma$. However, in the wall-crossing interpretation of m-herds discussed in Section 3.1, the spectral network S_m is expected to appear on the wall of marginal stability where two hypermultiplets of intersection number m have coincident central charge phase. In fact, the procedure of deforming such a picture is what motivated the construction of m-herds.

Definition Let $p^{(l)}$ be a street of type ij, then $\boldsymbol{p}^{(l)} \in C_1(\Sigma;\mathbb{Z})$ is the 1-chain on Σ representing the lift[12] of $p^{(l)}$ as a street of type ij (using the orientation discussed in Section 2.2).

[12] If $p^{(l)}$ connects two joints, this lift has two components. If $p^{(l)}$ connects a joint to a branch point of type ij, then the two components combine to form a connected 1-chain between sheets i and j.

If we define,

$$\gamma = \left[\sum_{l=1}^{m}\left(\boldsymbol{B}_1^{(l)} + \boldsymbol{B}_2^{(l)}\right)\right] \in H_1(\Sigma;\mathbb{Z})$$
$$\gamma' = \left[\sum_{l=1}^{m}\left(\boldsymbol{A}_1^{(l)} + \boldsymbol{A}_2^{(l)}\right)\right] \in H_1(\Sigma;\mathbb{Z}),$$

then, as shown in Fig. B.4, γ and γ' intersect once in each $\pi^{-1}(U_l)$, $l = 1, \cdots m$; hence,

$$\langle \gamma, \gamma' \rangle = m.$$

Now

- $\sum_{l=1}^{m-1}\left(\boldsymbol{A}_1^{(l)} + \boldsymbol{A}_2^{(l)}\right)$ is a 1-chain representative of the parallel transported charge $R_{\mathfrak{r}}^{(1,m-1)} a_*$.
- $\sum_{l=1}^{m-1}\left(\boldsymbol{B}_1^{(l)} + \boldsymbol{B}_2^{(l)}\right)$ is a 1-chain representative of $R_{\mathfrak{r}}^{(1,m-1)} b_*$.
- $\boldsymbol{A}_1^{(m)} + \boldsymbol{A}_2^{(m)}$ is a 1-chain representative of $\overline{a}_*$.
- $\boldsymbol{B}_1^{(m)} + \boldsymbol{B}_2^{(m)}$ is a 1-chain representative of $\overline{b}_*$.

Hence,

$$\gamma_c = \gamma + \gamma'.$$

Now, each of the 1-chains $\boldsymbol{A}_i^{(l)}$, $\boldsymbol{B}_i^{(l)}$ have well-defined tangent framing lifts $\widehat{\boldsymbol{A}}_i^{(l)}$, $\widehat{\boldsymbol{B}}_i^{(l)}$ when thought of as oriented paths on $\mathrm{Lift}(S_m) \subset \Sigma$. Similarly, γ and γ' have obvious representative curves on $\mathrm{Lift}(S_m)$ that allow us to produce tangent framing lifts $\widehat{\gamma}$, $\widehat{\gamma}'$. In fact,

$$\widehat{\gamma} = \left[\sum_{l=1}^{m}\left(\widehat{\boldsymbol{B}}_1^{(l)} + \widehat{\boldsymbol{B}}_2^{(l)}\right)\right] \in H_1(\widetilde{\Sigma};\mathbb{Z})$$
$$\widehat{\gamma}' = \left[\sum_{l=1}^{m}\left(\widehat{\boldsymbol{A}}_1^{(l)} + \widehat{\boldsymbol{A}}_2^{(l)}\right)\right] \in H_1(\widetilde{\Sigma};\mathbb{Z}).$$

Via similar arguments to above, along with the definition of $\widehat{\gamma}_c$ in (B.45), we have

$$\widehat{\gamma}_c = \widehat{\gamma} + \widehat{\gamma}'.$$

Alternatively, we can lift $\gamma_c = \gamma + \gamma'$ using the map $\widetilde{(\cdot)} : \Gamma \to \widetilde{\Gamma}$ defined in (D.1) of Appendix D. Indeed, as the curves representing γ and γ' intersect m times, we have

$$\widetilde{\gamma}_c = \widehat{\gamma} + \widehat{\gamma}' + mH = \widehat{\gamma}_c + mH.$$

Thus,

$$z = X_{\widehat{\gamma}_c} = (-1)^m X_{\widetilde{\gamma}_c}.$$

B.8 Proof of Proposition 3.3.1

We wish to compute the homology class of the 1-chain $L(n\gamma_c)$. First we introduce a few notational definitions that differ slightly from the main body of the thesis.

Definition $\boldsymbol{p}^{(l,r)} \in C_1(\Sigma;\mathbb{Z})$ is the component of $\boldsymbol{p}^{(l)} \in C_1(\Sigma;\mathbb{Z})$ on the rth sheet. If $p^{(l)}$ is a street of type ij, then

$$\boldsymbol{p}^{(l,r)} = \begin{cases} +\left(\text{1-chain representing the lift of } p^{(l)} \text{ to the } r\text{th sheet}\right), & \text{if } r = j \\ -\left(\text{1-chain representing the lift of } p^{(l)} \text{ to the } r\text{th sheet}\right), & \text{if } r = i \\ 0 & \text{otherwise} \end{cases}.$$

Now,

$$\begin{aligned} L(n\gamma_c) &= \sum_{l=1}^{m} \sum_{p^{(l)}} \alpha_n(p,l) \boldsymbol{p}^{(l)} \\ &= \alpha_n \sum_{l=1}^{m} \Big\{ \boldsymbol{c}^{(l)} + (m-l)\left(\boldsymbol{a}_{\mathbf{2}}^{(l)} + \boldsymbol{a}_{\mathbf{3}}^{(l)} + \boldsymbol{b}_{\mathbf{2}}^{(l)} + \boldsymbol{b}_{\mathbf{3}}^{(l)}\right) \\ &+ (l-1)\left(\overline{\boldsymbol{a}_{\mathbf{2}}}^{(l)} + \overline{\boldsymbol{a}_{\mathbf{3}}}^{(l)} + \overline{\boldsymbol{b}_{\mathbf{2}}}^{(l)} + \overline{\boldsymbol{b}_{\mathbf{3}}}^{(l)}\right) + \\ &+ (m-l+1)\left(\boldsymbol{a}_{\mathbf{1}}^{(l)} + \boldsymbol{b}_{\mathbf{1}}^{(l)}\right) + l\left(\overline{\boldsymbol{a}_{\mathbf{1}}}^{(l)} + \overline{\boldsymbol{b}_{\mathbf{1}}}^{(l)}\right) \Big\}. \end{aligned} \tag{3.19}$$

after using the results of Prop. 3.1.1 and the definition of α_n given in equation (3.18).

For the sake of readability we introduce some simplifying notation.

Notational Definition We denote,

$$\begin{aligned} \boldsymbol{a}_{12} &:= \boldsymbol{a}_1^{(l)} + \boldsymbol{a}_2^{(l)} \\ \boldsymbol{a}_{23} &:= \boldsymbol{a}_2^{(l)} + \boldsymbol{a}_3^{(l)} \\ \boldsymbol{a}_{123} &:= \boldsymbol{a}_1^{(l)} + \boldsymbol{a}_2^{(l)} + \boldsymbol{a}_3^{(l)}; \end{aligned}$$

and similarly, for $\overline{\boldsymbol{a}}_i$, $\boldsymbol{b}_i$, and $\overline{\boldsymbol{b}}_i$.

Using this notation, we can rewrite our sum in slightly more illuminating form,

$$L(n\gamma_c) = \alpha_n \sum_{l=1}^{m} \Big\{ (m-l)\left(\boldsymbol{a}_{123}^{(l)} + \boldsymbol{b}_{123}^{(l)}\right) + l\left(\overline{\boldsymbol{a}}_{123}^{(l)} + \overline{\boldsymbol{b}}_{123}^{(l)}\right)$$
$$+\boldsymbol{a}_{\mathbf{1}}^{(l)} + \boldsymbol{b}_{\mathbf{1}}^{(l)} + \boldsymbol{c}^{(l)} - \overline{\boldsymbol{a}_{\mathbf{23}}}^{(l)} - \overline{\boldsymbol{b}_{\mathbf{23}}}^{(l)} \Big\}.$$

This form suggests we should try to find a homological equivalence taking the terms multiplying the factor l, to the terms multiplying the factor $(m-l)$. We introduce extra 1-chains to aid in our computation. To define them, it is helpful to think of them as lifts of auxiliary streets. However, the interpretation as lifts of streets on C is only a notational tool: these streets are not part of any spectral network.

Definition Let $\{U_l\}_{l=1}^{m}$ be an open covering satisfying the *Horses* condition for an m-herd. On each horse we define auxiliary streets as in Fig. B.5: $e_1^{(l)}, e_2^{(l)} \subset U_l$ of type 12, and $f_1^{(l)}$, $f_2^{(l)} \subset U_l$, of type 23 ; such that,

(*C1*):

$$e_1^{(l+1)} = -e_2^{(l)}$$
$$f_2^{(l+1)} = -f_1^{(l)},$$

where "$-$" indicates orientation reversal.

(*C2*): $e_1^{(1)}$ and $f_2^{(1)}$ end on the branch points of type 12 and 23 (respectively) of the lower-sourced horse, while $e_2^{(m)}$ and $f_1^{(m)}$ end on the branch points of type 12 and 23 (respectively) of the upper-sourced horse.

Remark The *No Holes* condition removes any obstruction to condition (*C1*).

The 1-chains that will aid in our proof are the lifts of the auxiliary streets.

Remark Keeping with the (previously defined) convention for lifts of streets, there are 1-chains (on Σ) $\boldsymbol{e}_1^{(l)}$, $\boldsymbol{e}_2^{(l)}$, $\boldsymbol{f}_1^{(l)}$, and $\boldsymbol{f}_2^{(l)}$ (also depicted in Fig. B.5). It follows that, via (*C1*),

$$\begin{aligned} \boldsymbol{e}_1^{(l+1)} &= -\boldsymbol{e}_2^{(l)} \\ \boldsymbol{f}_2^{(l+1)} &= -\boldsymbol{f}_1^{(l)}. \end{aligned} \tag{B.48}$$

for $l = 1, \cdots, m-1$.

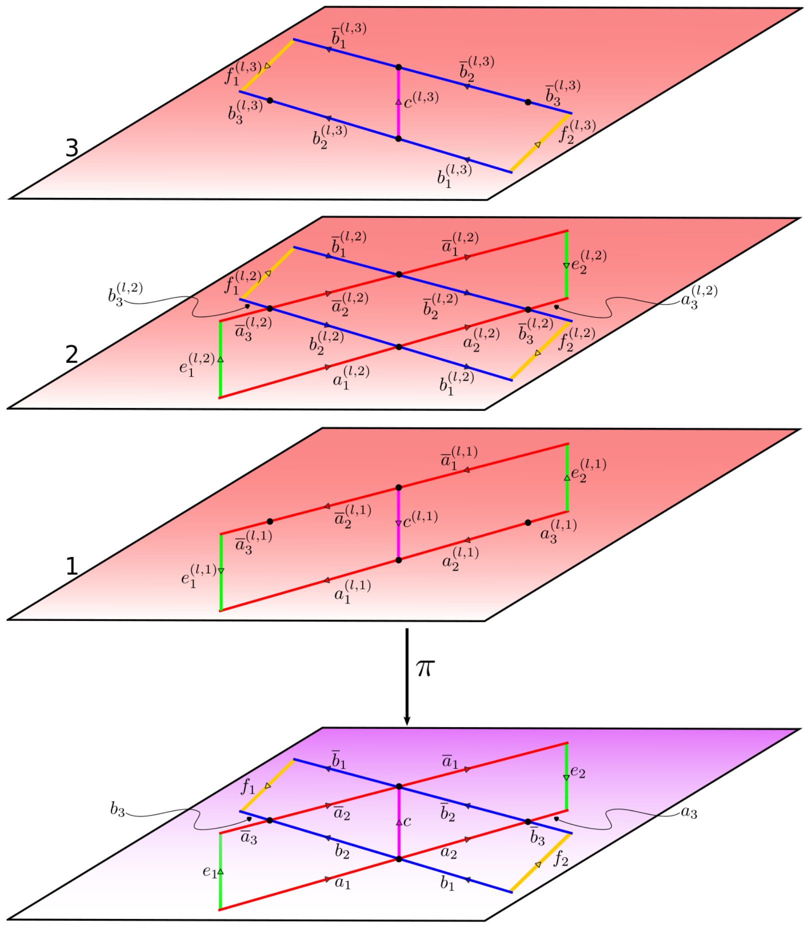


Figure B.5: Lift of a horse with extra 1-chains, pictured here as the lift of some auxiliary streets on C. For the sake of readability, the "horse label" (l) is suppressed on the base C.

Lemma B.8.1. *Let $\sim$ denote homological equivalence. Then for each $l = 1, \cdots, m$: on the first (locally defined) sheet,*

$$0 \sim \overline{\boldsymbol{a}}_{23}^{(l,1)} + \boldsymbol{e}_1^{(l,1)} - \boldsymbol{a}_1^{(l,1)} - \boldsymbol{c}^{(l,1)} \tag{B.49}$$

$$0 \sim \overline{\boldsymbol{a}}_1^{(l,1)} + \boldsymbol{c}^{(l,1)} - \boldsymbol{a}_{23}^{(l,1)} + \boldsymbol{e}_2^{(l,1)}. \tag{B.50}$$

On the second sheet,

$$0 \sim \overline{\boldsymbol{a}}_{123}^{(l,2)} + \boldsymbol{e}_2^{(l,2)} - \boldsymbol{a}_{123}^{(l,2)} + \boldsymbol{e}_1^{(l,2)} \tag{B.51}$$
$$0 \sim \overline{\boldsymbol{b}}_{123}^{(l,2)} + \boldsymbol{f}_2^{(l,2)} - \boldsymbol{b}_{123}^{(l,2)} + \boldsymbol{f}_1^{(l,2)} \tag{B.52}$$
$$0 \sim \overline{\boldsymbol{a}}_{3}^{(l,2)} + \boldsymbol{b}_2^{(l,2)} - \boldsymbol{a}_{1}^{(l,2)} + \boldsymbol{e}_1^{(l,2)} \tag{B.53}$$
$$0 \sim \boldsymbol{a}_{2}^{(l,2)} + \overline{\boldsymbol{b}}_3^{(l,2)} + \boldsymbol{f}_{2}^{(l,2)} - \boldsymbol{b}_1^{(l,2)} \tag{B.54}$$
$$0 \sim \boldsymbol{b}_{2}^{(l,2)} + \boldsymbol{a}_2^{(l,2)} - \overline{\boldsymbol{b}}_{2}^{(l,2)} - \overline{\boldsymbol{a}}_2^{(l,2)}. \tag{B.55}$$

On the third sheet,

$$0 \sim \boldsymbol{b}_{1}^{(l,3)} + \boldsymbol{c}^{(l,3)} - \overline{\boldsymbol{b}}_{23}^{(l,3)} - \boldsymbol{f}_2^{(l,3)} \tag{B.56}$$
$$0 \sim \overline{\boldsymbol{b}}_{1}^{(l,3)} + \boldsymbol{f}_1^{(l,3)} - \boldsymbol{b}_{23}^{(l,3)} + \boldsymbol{c}^{(l,3)}. \tag{B.57}$$

Proof. The lemma follows by inspection of Fig. B.5. Each of the listed sum of 1-chains is the boundary of an oriented disk. □

In particular, it follows from the lemma that

$$\overline{\boldsymbol{a}}_{123}^{(l)} \sim \boldsymbol{a}_{123}^{(l)} - \boldsymbol{e}_1^{(l)} - \boldsymbol{e}_2^{(l)}$$
$$\overline{\boldsymbol{b}}_{123}^{(l)} \sim \boldsymbol{b}_{123}^{(l)} - \boldsymbol{f}_1^{(l)} - \boldsymbol{f}_2^{(l)}.$$

Hence,

$$\begin{aligned} L(n\gamma_c) &\sim \alpha_n \sum_{l=1}^{m} \left\{ (m-l) \left(\boldsymbol{a}_{123}^{(l)} + \boldsymbol{b}_{123}^{(l)} \right) + l \left(\boldsymbol{a}_{123}^{(l)} + \boldsymbol{b}_{123}^{(l)} \right) \right\} + \alpha_n R_1 + \alpha_n R_2 \\ &\sim m\alpha_n \sum_{l=1}^{m} \left(\boldsymbol{a}_{123}^{(l)} + \boldsymbol{b}_{123}^{(l)} \right) + \alpha_n R_1 + \alpha_n R_2 \end{aligned} \tag{B.58}$$

where

$$R_1 = -\sum_{l=1}^{m} l \left\{ \boldsymbol{e}_1^{(l)} + \boldsymbol{e}_2^{(l)} + \boldsymbol{f}_1^{(l)} + \boldsymbol{f}_2^{(l)} \right\}$$
$$R_2 = \sum_{l=1}^{m} \left\{ \boldsymbol{a}_{\boldsymbol{1}}^{(l)} + \boldsymbol{b}_{\boldsymbol{1}}^{(l)} + \boldsymbol{c}^{(l)} - \overline{\boldsymbol{a}_{\boldsymbol{23}}}^{(l)} - \overline{\boldsymbol{b}_{\boldsymbol{23}}}^{(l)} \right\}.$$

Using (B.48), the first of these sums can be simplified,

$$
\begin{aligned}
R_1 &= -\sum_{l=1}^{m} l\left(\boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}\right) - \sum_{l=1}^{m} l\left(\boldsymbol{e}_2^{(l)} + \boldsymbol{f}_1^{(l)}\right) \\
&= -\sum_{l=1}^{m} l\left(\boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}\right) + \sum_{l=1}^{m-1} l\left(\boldsymbol{e}_1^{(l+1)} + \boldsymbol{f}_2^{(l+1)}\right) - m\left(\boldsymbol{e}_2^{(m)} + \boldsymbol{f}_1^{(m)}\right) \\
&= -\sum_{l=1}^{m} l\left(\boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}\right) + \sum_{l=2}^{m} (l-1)\left(\boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}\right) - m\left(\boldsymbol{e}_2^{(m)} + \boldsymbol{f}_1^{(m)}\right) \\
&= -\left(\boldsymbol{e}_1^{(1)} + \boldsymbol{f}_2^{(1)}\right) - m\left(\boldsymbol{e}_2^{(m)} + \boldsymbol{f}_1^{(m)}\right) - \sum_{l=2}^{m}\left(\boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}\right) \\
&= -m\left(\boldsymbol{e}_2^{(m)} + \boldsymbol{f}_1^{(m)}\right) - \sum_{l=1}^{m}\left(\boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}\right).
\end{aligned}
\tag{B.59}
$$

To reduce R_2, we use the following lemma.

Lemma B.8.2.

$$
\boldsymbol{a}_{\boldsymbol{1}}^{(l)} + \boldsymbol{b}_{\boldsymbol{1}}^{(l)} + \boldsymbol{c}^{(l)} - \overline{\boldsymbol{a}_{\boldsymbol{23}}}^{(l)} - \overline{\boldsymbol{b}_{\boldsymbol{23}}}^{(l)} \sim \boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}.
$$

Proof. On sheet 1,

$$
\boldsymbol{a}_1^{(l,1)} + \boldsymbol{b}_1^{(l,1)} + \boldsymbol{c}^{(l,1)} - \overline{\boldsymbol{a}}_{23}^{(l,1)} - \overline{\boldsymbol{b}}_{23}^{(l,1)} = \boldsymbol{a}_1^{(l,1)} + \boldsymbol{c}^{(l,1)} - \overline{\boldsymbol{a}}_{23}^{(l,1)}.
$$

Using (B.49),

$$
\begin{aligned}
&\sim \boldsymbol{a}_1^{(l,1)} + \boldsymbol{c}^{(l,1)} + \left(\boldsymbol{e}_1^{(l,1)} - \boldsymbol{a}_1^{(l,1)} - \boldsymbol{c}^{(l,1)}\right) \\
&\sim \boldsymbol{e}_1^{(l,1)}.
\end{aligned}
$$

Similarly, on sheet 3, using (B.56) appropriately,

$$
\begin{aligned}
\boldsymbol{a}_1^{(l,3)} + \boldsymbol{b}_1^{(l,3)} + \boldsymbol{c}^{(l,3)} - \overline{\boldsymbol{a}}_{23}^{(l,3)} - \overline{\boldsymbol{b}}_{23}^{(l,1)} &= \boldsymbol{b}_1^{(l,3)} + \boldsymbol{c}^{(l,3)} - \overline{\boldsymbol{b}}_{23}^{(l,3)} \\
&\sim \boldsymbol{b}_1^{(l,3)} + \boldsymbol{c}^{(l,3)} + \left(\boldsymbol{f}_2^{(l,3)} - \boldsymbol{b}_1^{(l,3)} - \boldsymbol{c}^{(l,3)}\right) \\
&\sim \boldsymbol{f}_2^{(l,3)}.
\end{aligned}
$$

On sheet 2

$$
\boldsymbol{a}_1^{(l,2)} + \boldsymbol{b}_1^{(l,2)} + \boldsymbol{c}^{(l,2)} - \overline{\boldsymbol{a}}_{23}^{(l,2)} - \overline{\boldsymbol{b}}_{23}^{(l,2)} = \boldsymbol{a}_1^{(l,2)} + \boldsymbol{b}_1^{(l,2)} - \overline{\boldsymbol{a}}_{23}^{(l,2)} - \overline{\boldsymbol{b}}_{23}^{(l,2)}.
$$

Now, via (B.53) and (B.54)

$$\boldsymbol{a}_1^{(l,2)} \sim \overline{\boldsymbol{a}}_3^{(l,2)} + \boldsymbol{b}_2^{(l,2)} + \boldsymbol{e}_1^{(l,2)}$$
$$\boldsymbol{b}_1^{(l,2)} \sim \boldsymbol{a}_2^{(l,2)} + \overline{\boldsymbol{b}}_3^{(l,2)} + \boldsymbol{f}_2^{(l,2)}.$$

Hence,

$$\begin{aligned}
\boldsymbol{a}_1^{(l,2)} + \boldsymbol{b}_1^{(l,2)} + \boldsymbol{c}^{(l,2)} - \overline{\boldsymbol{a}}_{23}^{(l,2)} - \overline{\boldsymbol{b}}_{23}^{(l,2)} &\sim \left(\overline{\boldsymbol{a}}_3^{(l,2)} + \boldsymbol{b}_2^{(l,2)} + \boldsymbol{e}_1^{(l,2)}\right) \\
&\quad + \left(\boldsymbol{a}_2^{(l,2)} + \overline{\boldsymbol{b}}_3^{(l,2)} + \boldsymbol{f}_2^{(l,2)}\right) - \overline{\boldsymbol{a}}_{23}^{(l,2)} - \overline{\boldsymbol{b}}_{23}^{(l,2)} \\
&\sim \boldsymbol{b}_2^{(l,2)} + \boldsymbol{a}_2^{(l,2)} - \overline{\boldsymbol{b}}_2^{(l,2)} - \overline{\boldsymbol{a}}_2^{(l,2)} + \boldsymbol{e}_1^{(l,2)} + \boldsymbol{f}_2^{(l,2)} \\
&\sim \boldsymbol{e}_1^{(l,2)} + \boldsymbol{f}_2^{(l,2)},
\end{aligned}$$

where the last reduction is due to (B.55). □

Thus,

$$R_2 \sim \sum_{l=1}^{m} \left(\boldsymbol{e}_1^{(l)} + \boldsymbol{f}_2^{(l)}\right);$$

so, with (B.59), we have

$$R_1 + R_2 = -m\left(\boldsymbol{e}_2^{(m)} + \boldsymbol{f}_1^{(m)}\right).$$

Substituting this result into (B.58),

$$L(n\gamma) \sim m\alpha_n \sum_{l=1}^{m-1} \left(\boldsymbol{a}_{123}^{(l)} + \boldsymbol{b}_{123}^{(l)}\right) + m\alpha_n \left[\left(\boldsymbol{a}_{123}^{(m)} + \boldsymbol{b}_{123}^{(m)}\right) - \left(\boldsymbol{e}_2^{(m)} + \boldsymbol{f}_1^{(m)}\right)\right].$$

After inspecting Fig. B.5, by deforming slightly on the mth horse we can convince ourselves this is precisely a 1-chain representing γ_c.

To make this claim precise, let $\boldsymbol{q}$ be a 1-chain on Σ such that $\partial \boldsymbol{q} \subset \pi^{-1}(z)$ for some $z \in C'$, and define $[\boldsymbol{q}]_R$ as the corresponding equivalence class in $\bigcup_{z \in C'} \Gamma(z, z)$. Then, for any $k = 1, \cdots, m$

$$R_{\mathfrak{r}}(1,k)a_* = \left[\sum_{l=1}^{k} \boldsymbol{a}_{123}\right]_R$$
$$R_{\mathfrak{r}}(1,k)b_* = \left[\sum_{l=1}^{k} \boldsymbol{b}_{123}\right]_R.$$

Furthermore, by parallel transporting the endpoints of $\overline{a}$ and $\overline{b}$ along an appropriate path contained in the mth horse[13]

$$\begin{aligned}\overline{a}_* &= \left[\left(\boldsymbol{a}_{123}^{(m)} - \boldsymbol{e}_2^{(m)}\right)\right]_R \\ \overline{b}_* &= \left[\left(\boldsymbol{b}_{123}^{(m)} - \boldsymbol{f}_1^{(m)}\right)\right]_R .\end{aligned}$$

Thus,

$$[L(n\gamma_c)]_R = m\alpha_n \left[R_{\mathfrak{r}}^{(1,k)} a_* + R_{\mathfrak{r}}^{(1,k)} b_* + \overline{a}_* + \overline{b}_*\right]_R .$$

Applying the closure map to both sides, by (B.47) the proposition holds:

$$[L(n\gamma_c)] = m\alpha_n\gamma_c \in H_1(\Sigma;\mathbb{Z}).$$

B.9 Table of m-herd BPS indices $\Omega(n\gamma_c)$, for low values of n and m

	n						
	1	2	3	4	5	6	7
$m = 1$	1	0	0	0	0	0	0
$m = 2$	-2	0	0	0	0	0	0
$m = 3$	3	-6	18	-84	465	-2808	18123
$m = 4$	-4	-16	-144	-1632	-21720	-318816	-5018328
$m = 5$	5	-40	600	-12400	300500	-8047440	231045220
$m = 6$	-6	-72	-1800	-58800	-2251500	-95312880	-4325917260
$m = 7$	7	-126	4410	-208740	11579925	-710338104	46716068007

Table B.1: Values of $\Omega(n\gamma_c)$ for low n and m

[13] As per our notation motivated in Section B.4, we do not write this parallel transport map explictly.

Appendix C: Proof of Proposition 3.3.3

Define the sequence

$$b_l := \binom{(m-1)^2 l}{l};$$

we will show

$$\lim_{n\to\infty} \frac{\Omega(n\gamma_c)}{(-1)^{mn+1}\left(\frac{m}{(m-1)^2n^2}\right)b_n} = 1. \tag{C.1}$$

Indeed, from (3.21),

$$\frac{\Omega(n\gamma_c)}{(-1)^{mn+1}\left(\frac{m}{(m-1)^2n^2}\right)b_n} = 1 + \overbrace{\sum_{\substack{d|n\\ d<n}}(-1)^{m(n+d)}\mu\left(\frac{n}{d}\right)\left(\frac{b_d}{b_n}\right)}^{R(n)},$$

but

$$|R(n)| \le \sum_{\substack{d|n\\ d<n}} \frac{b_d}{b_n}.$$

Now, from the bounds

$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} < n! \le n^{n+\frac{1}{2}}e^{1-n}$$

it follows that

$$\frac{b_d}{b_n} < \left(\frac{e}{\sqrt{2\pi}}\right)^3\left(\frac{n}{d}\right)^{1/2}e^{c_m(d-n)},$$

where c_m is the constant defined in (3.23). Hence,

$$|R(n)| < \left(\frac{e}{\sqrt{2\pi}}\right)^3\left(n^{1/2}e^{-c_m n}\right)\sum_{\substack{d|n\\ d<n}} d^{-1/2}e^{c_m d}.$$

Now, the next largest divisor of n, other than n itself, is $\le n/2$. Using this fact, the observation that $d^{-1/2}e^{c_m d}$ is a monotonically increasing function of d, and the crude bound that number of divisors of n is $\le n$, we have

$$\sum_{\substack{d|n\\ d<n}} d^{-1/2}e^{c_m d} \le n\left(\left(\frac{n}{2}\right)^{-1/2}e^{c_m n/2}\right) = \sqrt{2n}e^{c_m n/2};$$

so

$$|R(n)| < \sqrt{2}\left(\frac{e}{\sqrt{2\pi}}\right)^3 n e^{-c_m n/2},$$

which vanishes as $n \to \infty$, verifying (C.1). In other words, the $n \to \infty$ asymptotics of $\Omega(n\gamma_c)$ are given by the asymptotics of the largest term b_n of (3.21) inside the sum over divisors:

$$\Omega(n\gamma_c) \sim (-1)^{mn+1}\left(\frac{m}{(m-1)^2}\right) n^{-2} b_n.$$

Equation (3.22) follows by using Stirling's asymptotics on the binomial coefficient b_n: as $n \to \infty$,

$$b_n \sim \frac{1}{\sqrt{2\pi}}\left(\frac{m-1}{\sqrt{m(m-2)}}\right) n^{-1/2} e^{c_m n}.$$

Appendix D: The Standard Lift and A sign rule

In this appendix we discuss a subtle point about signs which was not treated correctly in the first version of [27].

The issue concerns the proper way of extracting 4D BPS degeneracy information from the generating functions $Q(p)$ defined in (2.2). What we want to do is factorize $Q(p)$ as we wrote in (2.15), but to do so, we need a way of choosing the lifts $\tilde{\gamma} \in \tilde{\Gamma}$ of classes $\gamma \in \Gamma$.

We propose the following rule. First, represent γ as a sum of k smooth closed curves β_m on Σ. Each such curve has a canonical lift $\hat{\beta}_m$ to $\tilde{\Sigma}$ just given by the tangent framing. Then we define *the standard lift* of γ as

$$\tilde{\gamma} = \sum_{m=1}^{k} (\hat{\beta}_m + H) + \sum_{m \leq n} \#(\beta_m \cap \beta_n) H. \tag{D.1}$$

We need to check that $\tilde{\gamma}$ so defined is independent of the choice of how we represent γ as a union of β_m. First we check that $\tilde{\gamma}$ is stable under creation/deletion of a null-homologous loop. If β denotes such a loop then $\hat{\beta} = H$ modulo $2H$ (indeed, suppose β bounds a subsurface S; S admits a vector field extending $\hat{\beta}$, with $\chi(S)$ signed zeroes in the interior; this vector field gives a 2-chain on $\tilde{\Sigma}$ which shows $\hat{\beta}$ is homologous on $\tilde{\Sigma}$ to $\chi(S)H$; but $\chi(S)$ is odd since S has a single boundary component.) Thus the extra term $\hat{\beta} + H$ added to $\tilde{\gamma}$ is zero modulo $2H$. Next we check $\tilde{\gamma}$ is stable under resolution of an intersection: indeed this changes $\sum_{m \leq n} \#(\beta_m \cap \beta_n)$ by -1, and changes k by ± 1, while not changing $\sum \hat{\beta}_m$; it thus changes $\tilde{\gamma}$ by either 0 or $-2H$, which is in either case trivial mod $2H$. Finally we note that any representation of γ as a union of smooth closed curves can be related to any other by repeated application of these two operations and their inverses. It follows that $\tilde{\gamma}$ is indeed well defined.

Moreover, this rule has the following property:

$$\tilde{\gamma} + \tilde{\gamma}' = \widetilde{\gamma + \gamma'} + \langle \gamma, \gamma' \rangle H. \tag{D.2}$$

It follows that the corresponding formal variables

$$Y_\gamma = X_{\tilde{\gamma}} \tag{D.3}$$

obey the twisted product rule

$$Y_\gamma Y_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} Y_{\gamma+\gamma'}. \tag{D.4}$$

In turn it follows (using the arguments of [26, 27]) that, if we use this particular lifting rule to extract the 4D BPS degeneracies, all the wall-crossing relations (and in particular the KSWCF for the pure 4D degeneracies) will come out as they should.

Appendix E: Spectral networks and algebraic equations

It has been noted by Kontsevich that the generating functions of Donaldson-Thomas invariants are often solutions of algebraic equations. The equation (1.2) is one example. This equation determines the BPS degeneracies $\Omega(n\gamma_c)$ corresponding to an m-cohort. As we have seen in this thesis, this equation can be derived from a close analysis of the spectral network corresponding to an m-herd.

While finding the precise equation (1.2) involved some hard work, the bare fact that the BPS generating function obeys *some* algebraic equation is not so mysterious. Indeed, this seems to be a general phenomenon, which we expect to occur for *any* theory of class S. Let us briefly explain why.

The junction equations (A.1) involve variables ν and τ attached to each street of the network. These variables lie *a priori* in the noncommutative algebra $\mathcal{A}_S$. However, one can replace them by variables lying in the commutative algebra $\mathcal{A}_C$ simply by choosing local trivializations of the torsors $\tilde{\Gamma}(\tilde{z}, -\tilde{z})$; indeed such a trivialization gives an embedding of $\mathcal{A}_S$ into the algebra of $K \times K$ matrices over $\mathcal{A}_C$; taking the individual matrix components then gives equations where all of the variables lie in $\mathcal{A}_C$. These equations alone do not quite determine ν and τ — there are not quite enough of them. However, once one supplements them with the "branch point" equations from [27] (which are also algebraic), one then has one equation for each variable.

In principle the spectral network may involve infinitely many streets and joints, so at this stage we may have an infinite set of algebraic equations in an infinite number of variables. However, in all examples we have considered, only finitely many of these equations are relevant for determining any particular BPS generating function. Indeed, in these examples the set of "two-way streets" is always supported in some compact set K obtained by deleting small discs around punctures on C; the intersection $\mathcal{W} \cap K$ only involves finitely many streets; and there are no streets which enter K from outside. It seems likely that these properties hold for *all* spectral networks, although we have not proven it. In any case, taking these properties for granted, it follows that the finitely many variables ν and τ attached to the finitely many streets in $\mathcal{W} \cap K$ are indeed determined by a finite set of algebraic equations.

The functions $Q(p)$ in turn are algebraic combinations of the ν and τ, as are the BPS generating functions $\prod_p Q(p)^{\langle \bar{a}, p_\Sigma \rangle}$. Thus we expect that the BPS generating functions in any theory of class S always satisfy algebraic equations, which gives a natural explanation of Kontsevich's observation, at least in those theories.

Appendix F: Some Generalities and Recollections on Quiver Representations

F.1 Slope-stability and (Semi)-Stable Moduli

We recall the basics of quiver representations and associated moduli spaces. Most of this material is drawn from [48] and [36]. Let $Q = (Q_0, Q_1)$ be a quiver specified by a finite set of vertices Q_0 and a finite set of arrows Q_1. In the following sections we will progressively specialize to the cases where Q is acyclic (possesses no oriented cycles) and where Q is the Kronecker m-quiver.

Definition

1. A *representation* $V = ((V_i)_{i \in Q_0}, (V_\alpha)_{\alpha \in Q_1})$ of Q is a collection of finite-dimensional complex vector spaces V_i, $i \in Q_0$ indexed by vertices, and a collection of $\mathbb{C}$-linear homomorphisms $V_\alpha : V_i \to V_j$ indexed by arrows $\alpha : i \to j \in Q_1$.

2. A *subrepresentation* W of V is a representation $((W_i)_{i \in Q_0}, (W_\alpha)_{\alpha \in Q_1})$ such that $W_i \leq V_i$ for each $i \in Q_0$ and W_α is given by the respective restrictions of the V_α for each $\alpha \in Q_1$.

3. A morphism $L : V \to W$ between representations of Q is a collection of maps $(L_i : V_i \to W_i)_{i \in Q_0}$ satisfying the obvious commutative diagrams: $L_j V_\alpha = W_\alpha L_i$ for each arrow $\alpha : i \to j$.

Remark/Definition Representations of Q, as defined above, form an abelian category $\mathsf{Rep}_{\mathbb{C}}(Q)$.

Define $\Lambda := \mathbb{Z}Q_0$ as the free abelian group generated by Q_0 and $\Lambda^+ := \mathbb{Z}_{\geq 0}Q_0$ the set of possible "dimension vectors". For each representation V we will denote its dimension vector via $\underline{\dim}(W) = \sum_{i \in Q_0} \dim_{\mathbb{C}}(V_i) i \in \Lambda^+$. Now, fix a functional $\Theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ (specified by assigning each vertex an integer weight), then we define its corresponding *slope function* $\mu_\Theta : \Lambda^+ \to \mathbb{Q}$ as

$$\mu_\Theta(d) = \frac{\Theta(d)}{\dim(d)} \tag{F.1}$$

where $\dim : \Lambda^+ \to \mathbb{Z}$ is given by $\dim(d) := \sum_{i \in Q_0} d_i$ for $d = \sum_{i \in Q_0} d_i i$.

Definition

1. A representation V of Q is Θ-*semistable* if for every non-zero subrepresentation W of V we have $\mu_\Theta(\underline{\dim}(W)) \le \mu_\Theta(\underline{\dim}(V))$.

2. A representation V of Q is Θ-*stable* if for every non-zero proper subrepresentation W of V we have $\mu_\Theta(\underline{\dim}(W)) < \mu_\Theta(\underline{\dim}(V))$.

3. A representation V of is Θ-*polystable* if it can be decomposed as a direct sum of Θ-stable representations $(V_i)_{i \in I}$ such that $\mu_\Theta(\underline{\dim}(V_i)) = \mu_\Theta(\underline{\dim}(V_j))$ for any $i, j \in I$.

Because any map between Θ-semistable representations has kernel, cokernel and image another Θ-stable representation, we have the following.[36]

Remark The full subcategory of Θ-semistable representations in $\mathsf{Rep}_{\mathbb{C}}(Q)$ forms an abelian subcategory $\mathsf{Rep}_{\mathbb{C}}^{\Theta}(Q)$; the simple (semisimple) objects of this category are the Θ-stable (polystable) representations.

Any object V in $\mathsf{Rep}_{\mathbb{C}}^{\Theta}(Q)$ admits a composition series[1], i.e. there exists a sequence of subobjects

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

such that V_i/V_{i-1} is a simple object (stable representation) for $2 \le i \le n$; a composition series is unique up to permutation and isomorphism of composition factors. Two objects are said to be *S-equivalent* if they have equivalent composition series (that is same composition length and composition factors up to permutation and isomorphism); as its name suggests, the notion of S-equivalence defines an equivalence relation.

Fixing a dimension vector $d \in \mathbb{Z}_{>0}Q_0$, one can then ask about the set of S-equivalence classes of semistable representations. By a theorem of King [36], using the stability condition Θ, this set can be given the structure of:

[1]Arising as a Jordan-Hölder filtration when thought of as a module of the quiver path algebra.

1. A (possibly singular) complex variety $\mathcal{M}^Q_{\mathrm{ss}}(d;\Theta)$ by taking a projective GIT quotient of the affine space $\mathfrak{R}_Q := \bigoplus_{i\to j\in Q_1} \mathrm{Hom}\left(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}\right)$: indeed, $\mathfrak{R}_Q$ is equipped with an action of $G = \mathbb{P}\left(\prod_{i\in Q_0} \mathrm{GL}_{d_i}\mathbb{C}\right)$ induced by conjugation; for any character $\phi : G \to \mathbb{C}^\times$, let $\mathbb{C}[\mathfrak{R}_Q]^{G,\phi}$ denote the ring of functions on $\mathfrak{R}_Q$ that satisfy $f(g\cdot x) = \phi(x)f(g)$ for any $g \in G$ and $x \in \mathfrak{R}_Q$; then, we define

   $$\mathcal{M}^Q_{\mathrm{ss}}(d;\Theta) := \mathrm{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[\mathfrak{R}_Q]^{G,\chi^n_\Theta}\right)$$

   where χ_Θ is the character defined by the descent of the character

   $$\prod_{i\in Q_0} \mathrm{GL}_{d_i}\mathbb{C} \longrightarrow \mathbb{C}^\times$$
   $$(g_i)_{i\in Q_0} \longmapsto \prod_{i\in Q_0} \det(g_i)^{\Theta(d)-\dim(d)\Theta(i)}$$

   to G.

2. A smooth Kähler orbifold by taking a Kähler quotient.

When Q is acyclic, the GIT quotient has a classical algebro-geometric interpretation.

Remark F.1.1. *If Q is acyclic, then $\mathbb{C}[\mathfrak{R}_Q]^G = \mathbb{C}$; so it follows that $\mathcal{M}^Q_{\mathrm{ss}}(d;\Theta)$ is a (possibly singular) complex projective variety.*

Remark F.1.2. *We do not describe the Kähler construction explicitly; however, in the case that Q is an acyclic quiver (so that the GIT quotient is a projective variety); the analytificiation of the GIT construction is biholomorphic to the Kähler construction away from any singular points of the Kähler quotient; moreover, both spaces are always homeomorphic as topological spaces.*

Remark Clearly every S-equivalence class contains a polystable representative, unique up to isomorphism. Hence, the space $\mathcal{M}^Q_{\mathrm{ss}}(d;\Theta)$ parametrizes polystable representations of dimension vector d up to isomorphism.

The set of *stable* representations up to isomorphism has the structure of a *smooth* open subvariety $\mathcal{M}^Q_{\mathrm{s}}(d;\Theta)$ of $\mathcal{M}^Q_{\mathrm{ss}}(d;\Theta)$. These spaces are indeed "moduli spaces" as they are solutions to a moduli problem given by a functor from $\mathbb{C}$-schemes to Set.

Remark F.1.3. $\mathcal{M}^Q_{\rm ss}(d;\Theta)$ *is a coarse moduli space for families of* Θ*-semistable modules (parametrized by* $\mathbb{C}$*-schemes) of dimension vector* d *(up to* S*-equivalence). When* d *is a primitive dimension vector then* $\mathcal{M}^Q_{\rm s}(d;\Theta) = \mathcal{M}^Q_{\rm ss}(d;\Theta)$ *and, moreover,* $\mathcal{M}^Q_{\rm s}(d;\Theta)$ *is equipped with a universal family making it into a fine moduli space.*[2]

Definition We define two equivalence relations on stability conditions in the following manner:

1. Given a fixed dimension vector d, two stability conditions Θ_1 and Θ_2 are *d-equivalent* if a representation of dimension vector d is Θ_1-(semi)stable if and only if it is Θ_2-(semi)stable.

2. Two stability conditions are *stability-equivalent* if they are d-equivalent for all dimension vectors d.

By definition, if Θ_1 and Θ_2 are d-equivalent, then the moduli spaces $\mathcal{M}^Q_{\rm ss}(d;\Theta_1)$ and $\mathcal{M}^Q_{\rm ss}(d;\Theta_2)$ are the same as subsets of (S-equivalence classes of) representations. But Rmk. F.1.3 provides a stronger equivalence: $\mathcal{M}^Q_{\rm ss}(d;\Theta_1)$ and $\mathcal{M}^Q_{\rm ss}(d;\Theta_2)$ are coarse-moduli spaces for the same moduli problem; hence, they must be isomorphic as varieties. It follows that, at the level of constructing moduli spaces, one need only be concerned with equivalence classes of a stability conditions.

Remark F.1.4. *There are two operations on functionals* $\Theta \in Hom_{\mathbb{Z}}(\Lambda,\mathbb{Z})$ *that do not change the stability-equivalence classes of (semi)stable representations:*

- *Rescaling:* $\Theta \mapsto n\Theta$ *for* $n \in \mathbb{Z}_{>0}$
- *Translating by* dim*:* $\Theta \mapsto \Theta + n\,\mathrm{dim}$ *for* $n \in \mathbb{Z}$.

We can generalize the notion of stability to allow Θ to be a real-valued functional, i.e. we can choose $\Theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{R})$. The corresponding slope function μ_Θ is then a real-valued map $\Lambda^+ \to \mathbb{R}$ and the definition of (semi)stability remains the same with respect to this slope function. The space $\mathcal{M}^Q_{\rm ss}(d;\Theta)$ can then be defined as a solution to the moduli problem of Rmk F.1.3; if such a solution exists, then $\mathcal{M}^Q_{\rm ss}(d;\Theta)$ is

[2]Some readers would correctly argue that the failure to always be a fine moduli space for non-primitive dimension vectors is an indication we should be working with moduli stacks; indeed,one always has a fine moduli (Artin) stack (even for non-primitive dimension vectors). However, we do not need to explicitly use the language of stacks in our discussion.

unique up to unique isomorphism. For existence, we may try to construct $\mathcal{M}^Q_{\mathrm{ss}}(d;\Theta)$ via a GIT quotient; however, this requires $\mathbb{Z}$-valued Θ. For the m-Kronecker quiver, one need not worry about this condition: section F.2 below shows that every real-valued stability functional on the m-Kronecker quiver is stability-equivalent to an integer-valued stability functional. However, for those concerned with generalities, the following shows that we can always replace an $\mathbb{R}$-valued functional with a d-equivalent $\mathbb{Z}$-valued functional; hence, we may always use GIT to construct $\mathcal{M}^Q_{\mathrm{ss}}(d,\Theta)$ as a projective variety.

Lemma F.1.1. *Let $d \in \Lambda^+$ be a fixed dimension vector and $\Theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{R})$, sufficiently generic in the sense that*

$$\Theta(k) = \frac{\Theta(d)}{\dim(d)} \dim(k)$$

if and only if $d = nk$ for some $n \in \mathbb{Z}_{>0}$. Then Θ is d-equivalent to a stability functional in $\mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$*.*

Proof. Fix any $\epsilon > 0$, then we can find $\phi_v \in \mathbb{Q}$ such that

$$|\Theta(q) - \phi_q| < \epsilon$$

for all $q \in Q_0$. Define

$$\Phi = \sum_{q \in Q_0} \phi_q q^* \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Q});$$

then,

$$|\mu_\Theta(k) - \mu_{\Phi'}(k)| < \epsilon.$$

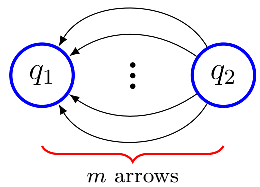
for any $k \in \Lambda$. Now equip Λ^+ with the poset structure defined by: $k \leq d$ if and only if $k_q \leq d_q$ for all $q \in Q_0$. Clearly, for a fixed d, there are only finitely many k such that $k \leq d$; hence, if we choose $\epsilon > 0$ such that

$$\epsilon < \frac{1}{2} \min \left\{ |\mu_\Theta(k) - \mu_\Theta(d)| : k \leq d \text{ and } d \neq nk \text{ for any } n \in \mathbb{Z}_{>0} \right\},$$

then Θ and Φ are d-equivalent. Letting λ denote the largest denominator of the ϕ_q, $q \in Q_0$, we have $\lambda\Phi \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$ and, moreover, $\lambda\Phi$ is stability-equivalent to Φ. ∎

F.2 Slope-Stability Conditions on the m-Kronecker Quiver

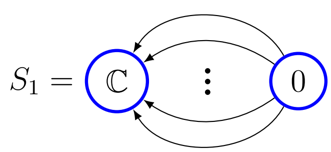
Recall the m-Kronecker quiver is defined as the quiver $Q = K_m$ with two vertices $\{q_1, q_2\}$ and $m \geq 1$ parallel arrows $\{\alpha_l : q_2 \to q_1\}_{l=1}^m$ from q_2 to q_1.

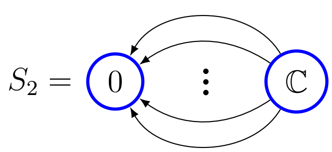


Via the operations in Remark F.1.4, any stability functional Θ on the m-Kronecker quiver (real or integer-valued) can be reduced to one of three possibilities:

1. $\Theta_1 \equiv 0$;
2. $\Theta_2 = q_1^* : eq_1 + dq_2 \mapsto e$;
3. $\Theta_3 = q_2^* : eq_1 + dq_2 \mapsto d$;

that induces the same slope-stability condition as Θ. As described in [48, §5.1], Θ_3 is the only choice that leads to an interesting notion of (semi-)stable representations. Indeed, consider the representations:





For all three stability conditions, S_1 and S_2 are stable representations; for Θ_1 and Θ_2, these are the *only* stable representations. For Θ_1, all representations are semi-stable; for Θ_2, the only semi-stable representations are isomorphic to $S_1^{\oplus r}$ or $S_2^{\oplus r}$ for some $r > 0$. We will return to the classification of semi-stable representations for Θ_3 in §G.2.

Terminology A *wild* stability condition on an m-Kronecker quiver is a stability condition that is stability-equivalent to Θ_3.

F.3 Stability Conditions on $\mathsf{Rep}_{\mathbb{C}}(Q)$

Perhaps closer to the spirit of the notion of stability that appears in the context of physics are stability conditions on the abelian category[3] $\mathsf{Rep}_{\mathbb{C}}(Q)$. First, we recall the notion of a stability condition on an abelian category [36, 11].

Definition Let A be an abelian category and $K_0(\mathsf{A})$ its Grothendieck group. A stability function on A is a group homomorphism $Z : K_0(\mathsf{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \text{object}(\mathsf{A})$,

$$Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0} = \{re^{i\theta} : \ r \in \mathbb{R}_{>0} \text{ and } \theta \in (0, \pi]\} \subset \mathbb{C} \tag{F.2}$$

The appropriate notion of (semi-)stability is then extracted from the "phase" $\arg[Z]$ thought of as a function $\text{ob}(\mathsf{A}) \to (0, \pi]$.

Definition An object $E \in \text{ob}(\mathsf{A})$ is *Z-semistable* if for every non-zero proper sub-object F of E we have

$$\arg[Z(F)] \leq \arg[Z(E)]\,. \tag{F.3}$$

E is *Z-stable* if strict equality holds in (F.3) for all proper subobjects F.

Our interest lies in the case $\mathsf{A} = \mathsf{Rep}_{\mathbb{C}}(Q)$. For simplicity we will take Q to be an *acyclic* quiver, then the map that takes each representation to its dimension vector, $\underline{\dim} : K_0(\mathsf{Rep}_{\mathbb{C}}(Q)) \to \mathbb{Z}Q_0 =: \Lambda$, is an isomorphism. Indeed, for Q acyclic, $K_0(\mathsf{Rep}_{\mathbb{C}}(Q))$ is freely generated by the (isomorphism classes of) one-dimensional representations supported at a single vertex, and the zero map associated to each arrow.

[3] Even more natural are Bridgeland stability conditions on the triangulated category $D^b\mathsf{Rep}_{\mathbb{C}}(Q)$; however, we will only be concerned with the subspace of stability conditions after choosing the t-structure whose heart is the abelian subcategory $\mathsf{Rep}_{\mathbb{C}}(Q) \hookrightarrow D^b\mathsf{Rep}_{\mathbb{C}}(Q)$ via its obvious embedding into chain complexes supported in the zeroth degree.

We can equivalently state the notion of stability in terms of an associated slope-function $\mu_Z : \Lambda^+ \to \mathbb{R} \cup \{\infty\}$ defined as

$$\mu_Z(d) = -\frac{\mathrm{Re}\,[Z(d)]}{\mathrm{Im}\,[Z(d)]} \tag{F.4}$$

$$= -\cot\{\arg[Z(d)]\}\,. \tag{F.5}$$

Then $\mu_Z(F) \leq \mu_Z(E)$ if and only if $\arg[Z(F)] \leq \arg[Z(E)]$ (where $\arg[Z]$ is thought of as a function taking values in $(0, \pi]$).

Remark F.3.1. *Note that, if we are given* $\Theta \in \mathrm{Hom}(\Lambda, \mathbb{Z})$, *then we can define* $Z \in \mathrm{Hom}(\Lambda, \mathbb{C})$ *via*

$$Z = -\Theta + i\dim \tag{F.6}$$

because dim *is positive, then the image of Z lies in $\mathbb{Z} + \mathbb{Z}_{>0}i \subset \mathbb{H}$; so this is a valid stability condition. In this case, the function* (F.5) *is precisely the slope function μ_Θ defined in* (F.1).

In the context of the m-Kronecker quiver, the following proposition shows that a stability condition on $\mathsf{Rep}_{\mathbb{C}}(K_m)$ is equivalent to one of the three slope-stability conditions in §F.2.

Proposition F.3.1. *Let K_m denote the m-Kronecker quiver (as discussed in §F.2) with vertices q_1 and q_2 and m arrows from q_2 to q_1; $\Lambda := \mathbb{Z}\langle q_1, q_2\rangle \cong K_0(\mathsf{Rep}_{\mathbb{C}}(K_m))$. If $Z : \Lambda \to \mathbb{C}$ is a stability function on K_m sufficiently generic in the sense that $Z(q_1)$ and $Z(q_2)$ are contained in $\mathbb{H}$, then*

1. *If $\arg[Z(q_1)] = \arg[Z(q_2)]$ then Z induces a stability condition stability-equivalent to the slope-stability condition induced by Θ_1;*
2. *If $\arg[Z(q_1)] > \arg[Z(q_2)]$ then Z induces a stability condition stability-equivalent to the slope-stability condition induced by Θ_2;*
3. *If $\arg[Z(q_1)] < \arg[Z(q_2)]$ then Z induces a stability condition stability-equivalent to the slope-stability condition induced by Θ_3.*

Proof. The proof of this statement is an exercise in linear algebra and relies on the fact that the Kronecker m-quiver only has two vertices. We begin by showing that

Z induces the same notion of stability as a stability condition of the form (F.6). By taking the real and imaginary parts of Z we get two group homomorphisms from Λ to $\mathbb{R}$. Indeed,

$$\mathrm{Im}(Z) = aq_1^* + bq_2^* \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R})$$

for some $a, b \in \mathbb{R}_{>0}$. We claim that there is a positive-rescaling of the lattice Λ, thought of as embedded inside of $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, such that the induced pullback on Z takes the form (F.6). Indeed, let $D \in \mathrm{Aut}_{\mathbb{R}} [\Lambda \otimes_{\mathbb{Z}} \mathbb{R}]$ given by

$$D = aq_1 \otimes q_1^* + bq_2 \otimes q_2^*$$

then it follows that

$$\mathrm{Im}(Z) = \dim \circ D$$

Hence, defining

$$Z' := \mathrm{Re}(Z) \circ D^{-1} + i \dim \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$$

we may write

$$Z = Z' \circ D.$$

Now we claim that Z' induces the same stability condition as Z. To see this, we think of Z and Z' as valued in the real two-dimensional vector space $\mathbb{R} \oplus i\mathbb{R}$; then

$$Z' \circ D = \widetilde{D} Z'$$

where $\widetilde{D}$ is a (real) automorphism of the image $I = \mathrm{span}_{\mathbb{R}} \langle Z'(q_1), Z'(q_2) \rangle \leq \mathbb{R} \oplus i\mathbb{R}$ of Z', defined via

$$\widetilde{D} = aZ'(q_1) \otimes Z'(q_1)^* + bZ'(q_2) \otimes Z'(q_2)^* \in \mathrm{Hom}_{\mathbb{R}}(I, I).$$

But $\widetilde{D}$ is an orientation-preserving map: when I is one-dimensional, $\widetilde{D}$ is a positive rescaling of I; when I is two-dimensional, $\widetilde{D}$ is a diagonal matrix with positive entries when expressed in terms of the basis $\{Z'(q_1), Z'(q_{2)}\}$; hence, $Z = \widetilde{D} Z'$ and Z' induce the same notion of stability.

Next, by Remark F.3.1, we have that Z induces the same notion of stability as the slope-stability defined by the linear functional, $\Theta = -\mathrm{Re}(Z) \circ D^{-1}$; the statement of the proposition holds by taking Θ into one of the standard forms of §F.2 via Remark F.1.4. □

F.4 The BPS Quiver

Let us put the above technology into the context of four-dimensional $\mathcal{N} = 2$ field theories admitting a Coulomb branch $\mathcal{B}$ – a more complete review can be found in [6, 5]. First, begin by fixing a point $u \in \mathcal{B}$; let $\widehat{\Gamma}_u$ be the charge lattice at the point u, $\langle \cdot, \cdot \rangle_u : \widehat{\Gamma}_u^{\otimes 2} \to \mathbb{C}$ the associated anti-symmetric pairing, and $\mathcal{Z}_u : \widehat{\Gamma}_u \to \mathbb{C}$ the central charge function.

For notational convenience, first define the subspace of *occupied* BPS charges

$$\mathtt{Occ} := \{\gamma \in \widehat{\Gamma}_u : \text{There exists a BPS state of charge } \gamma\}.$$

In order to define a BPS quiver, we require the existence of $\vartheta \in (\mathbb{R}/\mathbb{Z}) \setminus \arg[\mathcal{Z}_u(\mathtt{Occ})]$ such that:

(C1): There exists a finite set of linearly-independent charges $\mathfrak{b} \subset \widehat{\Gamma}_u$ such that $\mathcal{Z}_u^{-1}\left(e^{i\vartheta}\mathbb{H}\right) \subset \mathbb{Z}_{\geq 0}\mathfrak{b}$, i.e. any charge $\gamma \in \mathtt{Occ}$ such that $Z(\gamma) \in e^{i\vartheta}\mathbb{H}$ is a positive linear combination of elements of $\mathfrak{b}$;

(C2): Each $\gamma \in \mathfrak{b}$ is the charge of a hypermultiplet.

Condition (C1) is a purely linear/convex algebraic question about the set $\mathtt{Occ}$. It is instructive to construct a hypothetical counterexample of a set $\mathtt{Occ} \subset \mathbb{Z}^2$ and a map $Z : \mathbb{Z}^2 \to \mathbb{C}$ such that the $\mathfrak{b}$ of condition (C1) does not exist for any choice of $\vartheta \in S^1 \setminus \arg[\mathcal{Z}_u(\mathtt{Occ})]$.

Remark F.4.1. *If there exists a ϑ satisfying (C1), then the corresponding $\mathfrak{b}$ is unique. This follows from the fact that any invertible matrix with non-negative integer entries whose inverse also has non-negative entries must necessarily be a permutation matrix.*

Thus, if such a $\mathfrak{b}$ exists for a given element of $(\mathbb{R}/(2\pi\mathbb{Z})) \setminus \arg[\mathcal{Z}_u(\mathtt{Occ})]$, one knows that it is unique and need only check that (C2) holds.

Assuming ϑ satisfies (C1) and (C2), then the *BPS quiver* $Q = (Q_0, Q_1)$ at the point $u \in \mathcal{B}$ (and associated to ϑ) is the quiver with:

1. vertices $Q_0 = \mathfrak{b}$;

2. $\langle \gamma_1, \gamma_2 \rangle$ arrows from $\gamma_2 \in \mathfrak{b}$ to $\gamma_1 \in \mathfrak{b}$ if $\langle \gamma_1, \gamma_2 \rangle > 0$.

. Moreover, Q is equipped with a stability condition determined via the central charge function:

1. The lattice $\Lambda = \mathbb{Z}Q_0$, equipped with the canonical basis spanned by the vertices, is identified with the charge (sub)lattice $\mathbb{Z}\mathscr{b} \leq \widehat{\Gamma}_u$, equipped with the basis $\mathscr{b}$;

2. The stability function $Z : \Lambda \to \mathbb{C}$ defined in (F.2) is the central charge function $e^{-i\vartheta}\mathcal{Z}_u : \Gamma \cong \Lambda \to \mathbb{C}$.

A few remarks are in order.

Remarks

1. Note that (with Rmk. F.4.1 in mind), the quiver Q only depends on a choice of $\vartheta \in S^1 \backslash \arg[\mathcal{Z}_u(\mathtt{Occ})]$ satisfying (C1) and (C2). If there are several such ϑ, then the corresponding quivers are not necessarily the same; however, they may be related by mutation [5].

2. The existence and construction of a quiver Q (when it exists) depends on a priori knowledge about properties of the full set $\mathtt{Occ}$ of BPS states. However, it is not necessary to understand the full content of the 1-particle BPS subspaces associated to every element of $\mathtt{Occ}$: one needs only check (C1) and (C2). Once these are verified (or assumed via conjecture), then all information about the BPS spectrum for non-basis elements can be computed by looking at the corresponding quiver moduli spaces (see §F.5). In particular, all of the BPS-indices are given by computing DT-invariants for the associated quiver.

Terminology Let $\mathcal{B}$ be the Coulomb branch for a four-dimensional $\mathcal{N} = 2$ theory. We will say a point $u \in \mathcal{B}$ is an *m-wild point* ($m \geq 3$), if

1. there exists a BPS quiver Q, associated to some $\vartheta \in \mathbb{R}/(2\pi\mathbb{Z})$, such that Q contains the m-Kronecker quiver as a full subquiver K.

2. the central charge function $\mathcal{Z}_u : \Gamma \to \mathbb{C}$ determines a wild stability condition on K: suppose that the nodes of K are labelled by γ_1 and γ_2 with m arrows from γ_2 to γ_1, then $\mathcal{Z}$ equips K with a stability condition stability-equivalent to Θ_3 if and only if

$$\arg\left(e^{-i\vartheta}\mathcal{Z}_{\gamma_1}\right) < \arg\left(e^{-i\vartheta}\mathcal{Z}_{\gamma_2}\right) \tag{F.7}$$

as elements of $(0, \pi)$.

Remark (F.7) is satisfied if

$$\arg\left(e^{-i\theta}\mathcal{Z}_{\gamma_1}\right) < \arg\left(e^{-i\theta}\mathcal{Z}_{\gamma_2}\right)$$

for *any* $\theta \in \mathbb{R}/\mathbb{Z}$ such that $e^{-i\theta}\mathcal{Z}_{\gamma_1}$ and $e^{-i\theta}\mathcal{Z}_{\gamma_2}$ lie in $\mathbb{H}$. Thus, one can check if a point is m-wild only knowing the existence of an m-Kronecker subquiver of a BPS quiver Q, without knowing the precise (possibly family of) ϑ required to produce Q.

F.5 BPS States and Quiver Moduli

In order to define the BPS index, we first fix a rest-frame (a spacelike slice) and then focus our attention on a subspace of the 1-particle BPS states at rest: $\mathcal{H}^{\text{rest}}_{\text{BPS}}(u)$; this space is naturally graded by the charge lattice at u:

$$\mathcal{H}^{\text{rest}}_{\text{BPS}}(u) = \bigoplus_{\gamma\in\widehat{\Gamma}_u} \mathcal{H}^{\text{rest}}_{\text{BPS}}(\gamma; u).$$

Furthermore, each direct summand is a finite dimensional representation of the massive little-algebra, the even part of which contains the spatial rotation subalgebra $\mathfrak{r} \cong \mathfrak{so}(3)$. It is a fact (see [45, §4.2.3] and [57, §II]) that there is always a factorization as little-algebra representations:

$$\mathcal{H}^{\text{rest}}_{\text{BPS}}(\gamma; u) \cong \rho_{\text{hh}} \otimes \mathfrak{h}(\gamma; u)$$

where ρ_{hh} is the half-hypermultiplet representation. Using this factorization, the BPS index $\Omega(\gamma; u)$ can be written as

$$\Omega(\gamma; u) = \text{Tr}_{\,\mathfrak{h}(\gamma;u)}(-1)^{2J_3} \tag{F.8}$$

where, J_3 is a generator of a Cartan subalgebra of $\mathfrak{r}$.

Now, assume that Q is an acyclic (full) subquiver of a BPS quiver at the point $u \in \mathcal{B}$. In order to define the moduli space(s) of (semi)stable representations with respect to the stability condition $Z : \mathbb{Z}Q_0 \to \mathbb{C}$ (determined from the central charge– see §F.4), we replace Z with a slope-stability condition. If Q is the Kronecker quiver, then Prop. F.3.1 automatically provides us with a slope-stability condition that provides an equivalent notion of stability as Z. For more general acyclic quivers this can be done in the following manner: fix $\gamma \in \mathbb{Z}_{>0}Q_0$ and define $\theta_{\gamma,Z} \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{Z}Q_0, \mathbb{R})$ by

$$\theta_{\gamma,Z} : k \mapsto \mathrm{Im}\left(\frac{Z(k)}{Z(\gamma)}\right);$$

then one can verify that a representation V of dimension $\underline{\dim}(V) = \gamma$ is $\theta_{\gamma,Z}$-(semi)stable if and only if it is Z-(semi)stable. Hence, we may define

$$\begin{array}{rcl} \mathcal{M}^Q_{\mathrm{ss}}(\gamma; Z) & := & \mathcal{M}^Q_{\mathrm{ss}}(\gamma; \theta_{\gamma,Z}) \\ & & \cup\!\mid \\ \mathcal{M}^Q_{\mathrm{s}}(\gamma; Z) & := & \mathcal{M}^Q_{\mathrm{s}}(\gamma; \theta_{\gamma,Z}) \end{array}$$

where $\mathcal{M}^Q_{\mathrm{ss}}(\gamma; \theta_{\gamma,Z})$ is constructed via GIT quotient.

Now BPS indices are conjectured to be computed via quiver DT-invariants.

1. Generalized DT-invariants in the sense of Kontesevich and Soibelman: A specialization of the motivic DT-invariants of $D^b\mathsf{Rep}_{\mathbb{C}}(Q)$ equipped with the Bridgeland stability condition defined by the heart $\mathsf{Rep}_{\mathbb{C}}(Q) \hookrightarrow \mathbb{C}$ and

$$Z : K_0(\mathsf{Rep}_{\mathbb{C}}(Q)) \cong \mathbb{Z}Q_0 \to \mathbb{C}$$

   .

2. DT-invariants in the sense of Joyce-Song-Behrend: As weighted Euler characteristics of a suitable moduli space/stack of semistable representations.

Via Rmk. F.1.1, $\mathcal{M}^Q_{\mathrm{ss}}(\gamma; Z)$ is a projective variety and so its analytification is equipped with the pullback of the Fubini-Study metric given a choice of embedding into a projective space.

If $\gamma \in \mathbb{Z}_{\geq 0}Q_0 \leq \widehat{\Gamma}_u$ is a primitive dimension vector, then $\mathcal{M}^Q_{\mathrm{s}}(\gamma; Z) = \mathcal{M}^Q_{\mathrm{ss}}(\gamma; Z)$ is a smooth projective variety; hence, as an analytic space $\mathcal{M}^Q_{\mathrm{s}}(\gamma; Z)$ is a Kähler manifold. Now, as shown in [15], the space $\mathfrak{h}(\gamma; u)$ is precisely the space of BPS states for supersymmetric quantum mechanics of a point particle travelling on the configuration space $\mathcal{M}^Q_{\mathrm{s}}(\gamma; Z)$ (c.f. [33, §10.4.3]). In this situation, the $\mathfrak{r} \cong \mathfrak{su}(2)$-representation $\mathfrak{h}(\gamma; u)$ can be identified with the ring of harmonic forms on $\mathcal{M}^Q_{\mathrm{s}}(\gamma; Z)$, equipped with the Lefschetz $\mathfrak{su}(2)$-action [15, §4.3] – i.e. as $\mathfrak{r}$-representations:

$$\mathfrak{h}(\gamma; u) \cong H^*_{dR}(\mathcal{M}^Q_{\mathrm{s}}(\gamma; Z); \mathbb{C}),$$

where we have identified the ring of harmonic forms with the de-Rham cohomology ring. Now, in the Lefschetz representation, the generator of a Cartan subalgebra J_3 is

$$J_3 = \frac{1}{2}\left(\deg - \dim\left[\mathcal{M}_{\mathrm{s}}^{Q}(\gamma; Z)\right]\right) \mathbb{1}_{H^*_{dR}(\mathcal{M}_{\mathrm{s}}^{Q}(\gamma;Z);\mathbb{C})}$$

where $\deg : H^*_{dR}(\mathcal{M}_{\mathrm{s}}^{Q}(\gamma; Z); \mathbb{C}) \to \mathbb{Z}$ maps a cohomology class to its degree. As a result, the BPS index (F.8) is (up to a sign) the same as an Euler characteristic:

$$\Omega(\gamma; u) = (-1)^{\dim\left[\mathcal{M}_{\mathrm{s}}^{Q}(\gamma;Z)\right]} \chi\left(\mathcal{M}_{\mathrm{s}}^{Q}(\gamma; Z)\right).$$

For non-primitive γ, the moduli space $\mathcal{M}_{\mathrm{s}}^{Q}(\gamma; Z)$ is properly contained in the (possibly singular) projective variety $\mathcal{M}_{\mathrm{ss}}^{Q}(\gamma; Z)$ and the argument above is no longer valid. However, the BPS-index may always be computed in terms of quiver DT-invariants (which reduce to Euler characteristics of stable moduli in the case of primitive dimension vectors).

Appendix G: Useful properties of m-wild spectra

Recall that, as mentioned in the introduction, m-herds can appear as the spectral network representing slope-1 BPS states at an m-wild point on the Coulomb branch. In order to study generalizations of m-herds associated to other states in the m-wild spectrum, we provide the following recollections for some properties of the m-wild spectrum, all of which can be found in [56], [55], and [20].

G.1 Equivalences between moduli spaces

Definition For notational convenience, letting K_m denote the Kronecker m-quiver,

$$\mathcal{M}_{\mathrm{s}}^{m}(a,b) := \mathcal{M}_{\mathrm{s}}^{K_m}(aq_1 + bq_2; \Theta_3)$$

Proposition G.1.1 (c.f. Prop. 4.4 of [55])**.** *There exists isomorphisms of varieties*[1]*:*

$$\mathcal{M}_{\mathrm{s}}^{m}(a,b) \cong \mathcal{M}_{\mathrm{s}}^{m}(b,a) \tag{G.1}$$

and

$$\mathcal{M}_{\mathrm{s}}^{m}(a,b) \cong \mathcal{M}_{\mathrm{s}}^{m}(mb-a,b). \tag{G.2}$$

Hence, we have a corresponding equivalence between Euler characteristics of stable moduli; using Reineke's functional equation, this also implies a corresponding equivalence between DT-invariants.[2]

Corollary G.1.2. *We have the equivalence of DT-invariants*

$$d(a,b,m) = d(b,a,m)$$
$$d(a,b,m) = d(mb-a,b,m).$$

[1] This equivalence of moduli spaces holds for any choice of stability condition.

[2] The proof of Prop. G.1.1, sketched below, is a corollary of the existence of two autofunctors $\mathsf{Rep}_{\mathbb{C}}^{\Theta}(K_m) \to \mathsf{Rep}_{\mathbb{C}}^{\Theta}(K_m)$; from this the equivalence of DT-invariants can be seen more directly.

Hence, at an m-wild point on the Coulomb branch, if we let γ_1 and γ_2 be BPS hypermultiplet charges that satisfy (F.7) and generate a Kronecker m-subquiver of the BPS-quiver, then we have

$$\begin{aligned} \Omega\left(a\gamma_1 + b\gamma_2\right) &= \Omega\left(b\gamma_1 + a\gamma_2\right) \\ \Omega\left(a\gamma_1 + b\gamma_2\right) &= \Omega\left((mb-a)\gamma_1 + b\gamma_2\right). \end{aligned} \tag{G.3}$$

The proof of (G.1) follows by application of the transposition functor that takes representations of a quiver Q to representations of the quiver Q^{op} given by reversing all arrows of Q: each vector space is taken to its dual and each morphism $f : V_{q_i} \to V_{q_j}$ is taken to its induced action on dual-spaces: $f^*; V_{q_j}^* \to V_{q_i}^*$. When Q is the Kronecker m-quiver, then Q^{op} is the m-Kronecker quiver again with a relabelling of its vertices: $q_1^{\text{op}} = q_2$ and $q_2^{\text{op}} = q_1$; thus, $\Theta_3 = q_2^*$ (semi)-stable representations of Q are taken to $(q_2^{\text{op}})^*$ (semi)stable representations for Q^{op}.

The proof of (G.2) is a consequence of the application of another type of functor: Weist's "reflection functor", between quiver representation categories (c.f. [55, Thm. 2.6] and [9]). In the case of the m-Kronecker quiver, this functor maps representations of the m-Kronecker quiver to itself.

From a physical perspective, the equivalences (G.3) are expected to arise as a corollary of the existence of a $\mathbb{Z}/(2\mathbb{Z}) * \mathbb{Z}/(2\mathbb{Z})$ subgroup of monodromy transformations on the local system $\widehat{\Gamma} \to \mathcal{B}^*$ §6.2. In particular, let us fix an m-wild point $u \in \mathcal{B}^*$ and focus our attention on the rank 2-sublattice of $\Gamma = \widehat{\Gamma}_u$ spanned by the primitive hypermultiplet charges γ_1 and γ_2; then, in the basis (γ_1, γ_2) we expect to see a $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ monodromy subgroup generated by the $\mathrm{Sp}\,(2;\mathbb{Z})$ matrices (where $*$ denotes the free-product of groups):

$$\begin{aligned} T &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ R &= \begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

As the BPS spectrum must be invariant under all such monodromies, then the equivalences (G.3) follow.

G.2 Schur Roots and The Dense Arc

Let us now return to the case of a general acyclic quiver Q; we begin with the question: "*Given some slope-stability condition* $\Theta \in \mathrm{Hom}(\Lambda, \mathbb{Z})$*, for which dimension vectors are there non-trivial stable moduli?*". A partial answer is given by the Schur roots.

Definition [3] A dimension vector $d \in \mathbb{Z}_{\geq 0}Q$ is a *Schur root* if $d = \underline{\dim}(V)$ for some $V \in \mathsf{Rep}_{\mathbb{C}}(Q)$ such that $\mathrm{End}_{\mathsf{Rep}_{\mathbb{C}}(Q)}(V) \cong \mathbb{C}$. Moreover, d is a *real* Schur root if V is unique up to isomorphism and is an *imaginary* Schur root if there exists infinitely many non-isomorphic choices for V.

Typically Schur roots associated to a particular quiver are defined in terms of the positive roots of a Kac-Moody algebra associated to Q; defined in this manner, Schur roots can be calculated combinatorially. However, the definition above is equivalent to the usual definition by a theorem of Kac [34].

The following is a result of King.

Proposition G.2.1 ([36])**.** *A dimension vector d is a Schur root if and only if there exists a stability condition $\Theta \in \mathrm{Hom}(\Lambda, \mathbb{R})$ such that $\mathcal{M}_{\mathrm{s}}^{Q}(d; \Theta) \neq \emptyset$.*

We return to the m-Kronecker quiver and ask for its Schur roots. For a concise description of the following results see [17] or the notes [21]; these references expand upon the original reference [35, Page 159, Example (c)]. For the remainder of this section we assume $m \geq 3$.

Define a "generalized Fibbonaci" sequence $(a_k)_{k=0}^{\infty}$ via $a_k = ma_{k-1} - a_{k-2}$ with $a_0 = 0$, $a_1 = 1$; this has the closed-form solution:

$$a_k = \frac{1}{2^k\sqrt{m^2-4}}\left[\left(m + \sqrt{m^2-4}\right)^k - \left(m - \sqrt{m^2-4}\right)^k\right].$$

Then the real Schur roots are given by

$$\Delta_m^{\mathrm{Re}} := \{a_k q_1 + a_{k+1} q_2 : k \in \mathbb{Z}_{\geq 0}\} \cup \{a_{k+1} q_1 + a_k q_2 : k \in \mathbb{Z}_{\geq 0}\}$$

It is an exercise in induction to show that $\gcd(a_k, a_{k+1}) = 1$ for all $k \geq 0$; hence, all real Schur roots are primitive dimension vectors. On the other hand, the imaginary

[3] This definition was taken from the notes [21].

Schur roots need not be primitive, and are given by

$$\Delta_m^{\mathrm{Im}} := \left\{ aq_1 + bq_2 \in \mathbb{Z}^2_{>0} : \frac{a}{b} \in [m_-, m_+] \right\}.$$

where

$$m_\pm := \frac{1}{2}\left(m \pm \sqrt{m^2 - 4} \right).$$

Now, via Prop. G.2.1 every Schur root must correspond to a stable representation with respect to *some* stability condition. As the previous sections have shown, however, there are only three choices of stability conditions for the m-Kronecker quiver up to equivalence; from the discussion at the end of §F.2, we have the following.

Theorem G.2.2. *All Schur roots are the dimension vector of a stable representation with respect to the stability condition* Θ_3*.*

As a consequence of our definition for a real Schur root, we have the following corollary.

Corollary G.2.3. *The moduli space* $\mathcal{M}_{\mathrm{s}}^m(a,b)$ *is a point if* $(a,b) \in \Delta_m^{\mathrm{Re}}$

This yields the immediate physical corollary (see §F.5): "real Schur roots correspond to BPS hypermultiplets at m-wild points", or more precisely:

Corollary G.2.4 (Physical Corollary)**.** *Let* u *be an* m*-wild point with charges* $\gamma_1,\ \gamma_2 \in \widehat{\Gamma}_u$ *such that* $\langle \gamma_1, \gamma_2 \rangle = m \geq 3$ *are nodes of an* m*-Kronecker BPS subquiver satisfying* (F.7)*. If* $a\gamma_1 + b\gamma_2 \in \Delta_m^{\mathrm{Re}}$*, then the subspace* $\mathcal{H}_{\mathrm{BPS}}^{\mathrm{rest}}(a\gamma_1 + b\gamma_2)$ *is a half-hypermultiplet representation.*

More generally, one can calculate the dimension of $\mathcal{M}_{\mathrm{s}}^m(a,b)$ for both real and imaginary Schur roots:

$$\dim(\mathcal{M}_{\mathrm{s}}^m(a,b)) = mab - a^2 - b^2 + 1$$

for any $(a,b) \in \Delta_m^{\mathrm{Re}} \cup \Delta_m^{\mathrm{Im}}$. It follows that $\dim(\mathcal{M}_{\mathrm{s}}^m(a,b)) > 1$ for any a and b satisfying $a/b \in (m_-, m_+)$, i.e. any imaginary Schur root.

One may be curious about the central-charge phases of BPS states – particularly when computing spectral networks. Note that for any stability condition $Z : \Lambda \to \mathbb{C}$, the phase function

$$\begin{aligned} \arg[Z] \quad : \quad & \Lambda^+ \backslash \{0\} \quad \longrightarrow \quad [0,\pi) \\ : \quad & aq_1 + bq_2 \quad \longmapsto \quad \arg[Z(aq_1+bq_2)] \end{aligned}$$

depends only on the ratio[4] a/b. Hence, if we extend Z linearly to a map $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to [0,\pi)$ we may define a function

$$\Phi : \mathbb{R}_{>0} \to [0,\pi)$$

via

$$r \mapsto \arg[Z(\alpha q_1 + \beta q_2)],$$

for any $\alpha, \beta \in \mathbb{R}_{\geq 0}$ such that $\alpha/\beta = r$. With this bit of notation, the following corollary of Thm. G.2.2 describes the structure of the set of central-charge phases at an m-wild point.

Corollary G.2.5. *(see also [20, Corollary 3.20]) Let $Z : \Lambda \to \mathbb{C}$ be a stability condition such that* $\arg[Z(q_1)] < \arg[Z(q_2)]$, *then*

$$\arg\left[Z\left(\Delta_m^{\mathrm{Re}} \cup \Delta_m^{\mathrm{Im}}\right)\right] = F_- \cup D \cup F_+ \tag{G.4}$$

where

- $D = \arg\left[Z\left(\Delta_m^{\mathrm{Im}}\right)\right] = \Phi\left([m_-, m_+] \cap \mathbb{Q}\right)$
- *and F_- (resp. F_+) is a set consisting of a sequence of points (monotonically) converging to $\Phi(m_-)$ (resp. $\Phi(m_+)$).*[5]

Because the set D is dense in the arc $\Phi([m_-, m_+])$ we will refer to it as the *dense arc*. A qualitative depiction of the set (G.4) is shown in Fig. 6.6 for $m = 3$ (where each point on the dense arc defines a ray).

[4] In the appendix we will refrain from referring to this as the "slope" to prevent confusion with the slope-function in slope-stability conditions.

[5] The limit points $\Phi(m_-)$ and $\Phi(m_+)$ are not contained in $F_- \cup D \cup F_+$.

Appendix H: Derivation of (9.2)

In the following we provide a sketch of the methods used to obtain (9.2), which we restate here (with each individual equation numbered) for referential convenience.

$$M = 1 + zM^4 \left\{(1+V)(1+V-W)^2[V^2(1+W)-1]^3\right\} \tag{H.1}$$

$$0 = (-1+V)(1+V)^2 + (1+V^3)W - V(M+V)W^2 \tag{H.2}$$

$$0 = V\left(V^2-1\right) - [M(V+1)+V(V-2)-1]\,W. \tag{H.3}$$

Unless noted otherwise all notation will be drawn from Part I.

Notational/Colouring Remark In the following appendices, soliton generating functions (which are elements of a noncommutative ring) will be coloured red, street factors (which are elements of a commutative ring) blue, and "abelian" parts (defined below) violet. This is not to be confused with the colour coding notation of App. B.3.

The reader is warned that these derivations are ad-hoc, and rely on numerical order-by-order observations to establish particular identities. Nevertheless we present them here in the spirit of full transparency, and hopes that some reader may improve upon the technique.

Definition We will write $\widetilde{\Gamma_{ij}}(p)$ to denote the set of collection relative homology classes ("charges") representing solitons of type ij that propagate along the street p.

Remark In Part I the notation $\widetilde{\Gamma}_{ij}(\widetilde{x}, -\widetilde{x})$ was used to denote the set of soliton charges of type ij over some *point* $x \in p$ (and with $\widetilde{x} \in UT\Sigma$ a lift of x to the unit tangent bundle of Σ, using the orientation on p). Dropping the explicit point x from the notation is only be a mild abuse as all such sets attached to various points on p are related by parallel transport along p.

First, we recall a major part of the spectral network machinery: to each (two-way) street p of type $ij \in \{12, 23, 13\}$ we attach two formal series: $\Upsilon(p) \in \mathbb{Z}[\![\widetilde{\Gamma}_{ij}(p)]\!]$, $\Delta(p) \in \mathbb{Z}[\![\widetilde{\Gamma}_{ji}(p)]\!]$, called *soliton generating functions*, where $\mathbb{Z}[\![\widetilde{\Gamma}_{ij}(p)]\!]$ and $\mathbb{Z}[\![\widetilde{\Gamma}_{ji}(p)]\!]$ are certain (mildly) non-commutative rings. Less formally speaking, the terms of these formal series are formal variables of the form X_a for $a \in \widetilde{\Gamma}_{ij}$; furthermore, one can order the terms of soliton generating functions according to the mass of each term (see the proof of the claim below).

Now, assume our spectral network has the property that for every branch point of type ij, there is at most one two-way street with an endpoint on that branchpoint.[1] From a spectral network we develop a collection of equations on the soliton generating functions:

1. At each joint of a spectral network, write down the six-way junction equations (c.f. App. A).
2. For each two-way street p emanating from a branch point of type ij (according to our assumptions there is at most one),

As we are often only interested in generating functions on the two-way streets, we use the observation of App. B.3: the resulting equations on two-way streets form a closed system of equations, i.e. there is a way in which we can ignore all one-way streets[2]

For each pair of soliton generating functions $\Upsilon(p)$ and $\Delta(p)$ we make the following claim:

Claim H.0.6. *Let $\mathcal{N}$ be a spectral network supporting integer multiples of a single BPS charge γ_c and let $p \in \mathrm{str}(\mathcal{N})$ be a street of type ij, then there exists a unique decomposition*

$$\Upsilon(p) = X_s U(p) \in \mathbb{Z}[\![\widetilde{\Gamma}_{ij}(p)]\!] \tag{H.4}$$

$$\Delta(p) = X_{\overline{s}} D(p) \in \mathbb{Z}[\![\widetilde{\Gamma}_{ji}(p)]\!] \tag{H.5}$$

where

- *$s \in \widetilde{\Gamma}_{ij}(p)$ and $\overline{s} \in \widetilde{\Gamma}_{ji}(p)$ are two pairs of solitons.*
- *$U(p), D(p) \in \mathbb{Z}[\![\widetilde{\Gamma}_c]\!] = \mathbb{Z}[\![z]\!]$ are formal series with non-vanishing degree zero terms..*

[1] m-herds and all of their generalizations in this thesis satisfy this property

[2] In particular, the two-way "skeleton networks" shown throughout this thesis–which are not full spectral networks but only a specification of their two-way streets– suffice to understand the BPS state counts.

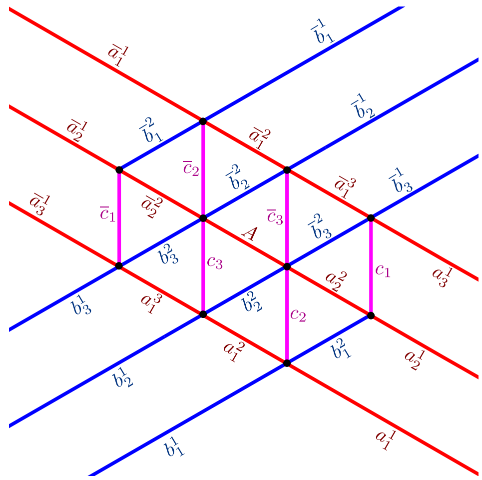


Figure H.1: The central building-block of a $(3, 2|3)$-herd with streets labeled. For the computations in App. H this diagram fits into the central region of Fig. 12.6. The labelling is chosen in a way that reflects the 180 degree rotational symmetry of the full $(3, 2|3)$-herd (which can also be seen on the central building block).

Proof. To show existence, first note that $\widetilde{\Gamma}_{ij}(\widetilde{x}, -\widetilde{x})$, for any point $x \in p$, is a $\widetilde{\Gamma}$-torsor; so, in particular we can make sense of differences $s - s' \in \widetilde{\Gamma}$ for any $s, s' \in \widetilde{\Gamma}_{ij}(\widetilde{x}, -\widetilde{x})$. In particular, as N supports only integer multiples of a single primitive charge $\widetilde{\gamma}_c$, any two solitons supported on p must differ by a multiple of γ_c. Let S denote the (countable) set of solitons that appear in the formal variables in the expansion of $\Upsilon(p)$. We now impose a total ordering on S: $s < s'$ if the mass of the soliton s is less than or equal to the mass of s'; this is a total order as $\mathrm{Mass}(s) < \mathrm{Mass}(s + \widetilde{\gamma}_c)$; furthermore, as the mass is always strictly positive, then by the well-ordering principle there exists a smallest element $s_* \in S$. Now, as any soliton in S can be obtained by adding a multiple of $\widetilde{\gamma}_c$, it follows that there exists $U(p) \in \mathbb{Z}[\![\widetilde{\Gamma}_c]\!]$ such that $\Upsilon(p) = X_{s_*}U(p)$. Furthermore, by construction the degree zero term of $U(p)$, must be non-vanishing (otherwise $s_* \notin S$).

Uniqueness is immediate from the condition of non-vanishing degree zero terms: if there exists two decomposition $\Upsilon(p) = X_s U(p) = X_{s'} U'(p)$, extracting degree zero terms from $U(p)$ and $U'(p)$ requires $X_s = X_{s'}$; hence, $U(p) = U'(p)$ must also follow. $\square$

Remark $U(p), D(p) \in \mathbb{Z}[\![z]\!]$ are invertible as elements of the ring $\mathbb{Q}[\![z]\!]$: an element of $\mathbb{Q}[\![z]\!]$ is invertible if and only if it has non-vanishing degree zero term. This observation is computationally convenient.

Terminology We will occasionally refer to the commutative formal series $U(p)$ and $D(p)$ of (H.5) as *abelian parts*.

In practice, this decomposition allows one to transform equations in the (mildly) non-commutative soliton-generating functions $\Upsilon(p)$, $\Delta(p)$ into equations in the commutative (and invertible) formal series $U(p)$ and $D(p)$.

Remark Note that in the decomposition (H.5), s and $\overline{s}$ must be solitons such that, $\mathrm{cl}\,[s+\overline{s}] = \pm n\widetilde{\gamma}_c$ under the closure map $\mathrm{cl} : \bigcup_i \widetilde{\Gamma}_{ii} \mapsto \widetilde{\Gamma}$

We will denote particular street factors via

$$\begin{aligned} M &:= Q\,(c_1) = Q\,(\overline{c}_1) \\ V &:= Q\,(c_3) = Q\,(\overline{c}_3) \\ W &:= Q\,(c_2) = Q\,(\overline{c}_2)\,. \end{aligned} \tag{H.6}$$

where the second equalities follow via the 180 degree rotation symmetry of the diagram.

In the following, a and c represent the simpleton charges of types 12 and 21 emitted from the red streets at the top and bottom (respectively) of the red streets of Fig. 12.2. Similarly, b and d represent the simpleton charges of type 23 and 32. We will make two rather abusive, but notationally convenient omissions:

1. We will explicitly omit the various parallel transport maps applied to the a, b, c, and d;

2. We will explicitly omit applications of the closure map $\mathrm{cl} : \bigcup_i \widetilde{\Gamma}_{ii} \to \widetilde{\Gamma}$.

In practice these omissions simplify notation greatly without an unrecoverable loss of rigour. Indeed, for the first type of omission: applying the six-way street equations appropriately ensures that the appropriate parallel transport maps have been applied whenever one has a product of two formal variables in soliton-charges $X_s X_{s'}$; hence, there exists a well-defined manner in which this product is non-vanishing[3] and can be

[3]Technically speaking, according to the groupoid addition rule, at least one of $s + s'$ and $s' + s$ is ill-defined (assuming s and s' are not both charges of type ii), i.e. one of $X_{s+s'}$ and $X_{s'+s}$ vanishes; if one keeps track of the correct order in the six-way street equations, one will not have such non-vanishing sums.

written as $X_{s+s'}$. With this in mind, we can freely identify factors of z in equations via:

$$z = X_{3a+2b+3c+2d}.$$

Note that, letting $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ be the charges defined[4] in App. I,

$$\widehat{\gamma}_1 = a + b$$
$$\widehat{\gamma}_2 = c + d$$

assuming the appropriate parallel transport maps have been applied to the right hand side. Thus,

$$z = X_{3\widehat{\gamma}_1 + 2\widehat{\gamma}_2}.$$

H.1 Numerical Observations

We first remark that, given the decomposition of soliton generating functions as stated in the claim, if we define $z := X_{\widehat{\gamma}_c}$ we must have

$$Q(p) = 1 + z^k U(p) D(p).$$

for some $k \in \mathbb{Z}$. For the $(3, 2|3)$-herd, it can be checked that for any two-way street p

$$\Upsilon(p)\, \Delta(p) = zU(p)D(p).$$

i.e. given $U(p)$ and $D(p)$ we can recover the street-factors via

$$Q(p) = 1 + zU(p)D(p). \tag{H.7}$$

Now, for m-herds, all soliton generating functions were of the form $X_s P^k$ for some soliton charge s, street-factor P, and positive integer k. Now, it would be a wonderful situation if something analogous held for $(3, 2|3)$-herds: if $U(p)$ and $D(p)$ could be entirely expressed as monomials in terms of the street factors M, V, and W defined in Fig. 12.6. A bit of numerical exploration shows that this dream fails, but in a mild

[4] Please be aware of the following notational conflict between this section and App. I: in the former the letters a, b denote soliton charges, in the latter a and b are integers.

way. Namely, the $U(p)$ and $D(p)$ turn out to be rational functions in the three street factors M, V, $W \in \mathbb{Z}[\![z]\!]$, the simplest of which are monomials in M, V, and W.

Order-by-order exploration suggest the following identities (a partial list among the $29 \times 2 = 58$ soliton-generating functions associated to the 29 two-way streets of Fig. H.1).

$$\begin{array}{ll}
\Upsilon(A) = X_{2a+b+c+d}U(A) & \Delta(A) = X_{a+b+2c+d}U(A) \\
\Upsilon(a_1^1) = X_a MW & \Delta(\overline{a}_1^1) = X_c MW \\
\Upsilon(a_2^1) = X_{2a+c+b+d}Y & \Delta(\overline{a}_1^3) = X_c M^2 VW \\
\Upsilon(a_2^2) = X_{2a+b+c+d}Y & \Upsilon(\overline{a}_2^1) = X_{2a+b+c+d}U(b_2^2) \\
\Delta(a_2^2) = X_{a+b+2c+d}D(a_2^2) & \Delta\left(\overline{b}_2^2\right) = X_{2a+b+2c+2d}U(b_2^2) \\
\Upsilon(b_1^1) = X_b & \Upsilon\left(\overline{b}_3^2\right) = X_{2a+2b+2c+d}U\left(\overline{b}_3^2\right) \\
\Upsilon(b_2^2) = X_{2a+2b+2c+d}U(b_2^2) & \Delta\left(\overline{b}_3^2\right) = X_{a+c+d}(MW)^2 V \\
\Delta(b_2^2) = X_{a+c+d}D(b_2^2) & \Upsilon(\overline{c}_3) = \eta^{-1}X_{a+b}U(\overline{c}_3) \\
\Delta(b_2^1) = X_{a+c+d}Y & \Delta(\overline{c}_3) = \eta X_{2a+b+3c+2d}D(\overline{c}_3) \\
\Upsilon(c_2) = \eta^{-1}X_{a+b}MW & \\
\Delta(c_2) = \eta X_{2a+b+3c+2d}D(c_2) & .
\end{array} \tag{H.8}$$

Where $Y \in \mathbb{Z}[\![z]\!]$ is a series whose first few terms are given by

$$Y = 2 + 72z + 6438z^2 + 752640z^3 + 100221708z^4 + 14415715416z^5 + \mathcal{O}(z^6).$$

Eventually we will find that Y can be expressed as a rational function in M, V and W.

Observation Let p be any street in Fig. H.1, i.e. $p \in \{a_i^j, \overline{a}_i^j, b_i^j, \overline{b}_i^j\}_{i,j=1}^3 \cup \{c_i, \overline{c}_i\}_{i=1}^3 \cup \{A\}$. With the convention that $\overline{\overline{p}} = p$ and $\overline{A} = A$, then performing a 180-degree rotation of the $(3,2|3)$-herd, we have a map $p \mapsto \overline{p}$; furthermore, the $(3,2|3)$-herd's symmetry under this rotation implies that, under this rotation we have an involution that maps soliton-generating functions to soliton-generating functions:

$$\Upsilon(p) \rightsquigarrow \Delta(\overline{p})$$
$$\Delta(p) \rightsquigarrow \Upsilon(\overline{p}),$$

which is induced by the replacements

$$a \mapsto c, \quad c \mapsto a$$
$$b \mapsto d, \quad d \mapsto b$$

and

$$\eta \mapsto \eta^{-1},$$

while fixing the abelian part of the generating series. This is reflected in some of the numerical observations above (e.g. $\Upsilon\left(a_1^1\right) = X_a MW$ and $\Delta\left(\overline{a}_1^1\right) = X_c MW$).

In the following section we show that we may recover the algebraic relations (9.2) from: the collection of results in (H.8), the observation above, the six-way junction equations, the abelian six-way junction equations[5] (12.4), and the relation between abelian parts and street factors (H.7).

H.2 Manipulations

We now develop a functional equation through applications of the six-way junction equations, combined with the numerical observations of the previous section. First, one can show through the six-way junction equations.

$$\begin{aligned}
&\Upsilon\left(b_1^2\right) = \Upsilon\left(b_1^1\right) && \text{(H.9)}\\
&\Upsilon\left(b_2^2\right) = \Upsilon\left(b_2^1\right) \\
&\Delta\left(b_1^2\right) = Q\left(\overline{b}_2^2\right) \times \\
&\quad \times \frac{\Delta\left(\overline{a}_1^1\right)\Delta\left(\overline{b}_3^1\right)\Upsilon\left(a_2^1\right)\left[1 + \Delta\left(\overline{a}_2^1\right)\Delta\left(\overline{b}_1^1\right)\Upsilon\left(a_1^1\right)\Upsilon\left(b_3^1\right)Q\left(b_2^2\right)\right]}{1 - \Delta\left(\overline{a}_1^1\right)\Delta\left(\overline{a}_2^1\right)\Delta\left(\overline{b}_1^1\right)\Delta\left(\overline{b}_3^1\right)\Upsilon\left(a_1^1\right)\Upsilon\left(a_2^1\right)\Upsilon\left(b_3^1\right)\Upsilon\left(b_1^1\right)Q\left(b_2^2\right)Q\left(\overline{b}_2^2\right)}. && \text{(H.10)}
\end{aligned}$$

Defining,

$$\Pi := \Delta\left(\overline{a}_1^1\right)\Delta\left(\overline{a}_2^1\right)\Delta\left(\overline{b}_1^1\right)\Delta\left(\overline{b}_3^1\right)\Upsilon\left(a_1^1\right)\Upsilon\left(a_2^1\right)\Upsilon\left(b_3^1\right)\Upsilon\left(b_1^1\right),$$

[5] Which follow as a corollary of the six-way junction equations.

then (H.10) implies (using (H.9) and $Q(p) = 1 + \Upsilon(p)\,\Delta(p)$ for any street p)

$$Q\left(b_1^2\right) = 1 + \frac{Q\left(\bar{b}_2^2\right)\Upsilon\left(b_1^2\right)\Delta\left(\bar{a}_1^1\right)\Delta\left(\bar{b}_3^1\right)\Upsilon\left(a_2^1\right)\left[1 + \Delta\left(\bar{a}_2^1\right)\Delta\left(\bar{b}_1^1\right)\Upsilon\left(a_1^1\right)\Upsilon\left(b_3^1\right)Q\left(b_2^2\right)\right]}{1 - \Delta\left(\bar{a}_1^1\right)\Delta\left(\bar{a}_2^1\right)\Delta\left(\bar{b}_1^1\right)\Delta\left(\bar{b}_3^1\right)\Upsilon\left(a_1^1\right)\Upsilon\left(a_2^1\right)\Upsilon\left(b_3^1\right)\Upsilon\left(b_1^1\right)Q\left(b_2^2\right)Q\left(\bar{b}_2^2\right)}$$

$$= 1 + \frac{Q\left(\bar{b}_2^2\right)\Upsilon\left(b_1^1\right)\Delta\left(\bar{a}_1^1\right)\Delta\left(\bar{b}_3^1\right)\Upsilon\left(a_2^1\right) + \Pi Q\left(b_2^2\right)Q\left(\bar{b}_2^2\right)}{1 - \Pi Q\left(b_2^2\right)Q\left(\bar{b}_2^2\right)}$$

$$= \frac{1 + Q\left(\bar{b}_2^2\right)\Upsilon\left(b_1^1\right)\Delta\left(\bar{b}_3^1\right)\Upsilon\left(a_2^1\right)\Delta\left(\bar{a}_1^1\right)}{1 - \Pi Q\left(b_2^2\right)Q\left(\bar{b}_2^2\right)} \tag{H.11}$$

$$\tag{H.12}$$

From the abelian rules (12.4) and the identifications (H.6), we have

$$Q\left(b_1^2\right) = M$$
$$Q\left(b_2^2\right) = V = Q\left(\bar{b}_2^2\right).$$

Furthermore, using the numerical results of (H.8), we find

$$\Pi = \left[z(MW)^3VY\right]^2$$

and

$$\Upsilon\left(b_1^1\right)\Delta\left(\bar{b}_3^1\right)\Upsilon\left(a_2^1\right)\Delta\left(\bar{a}_1^1\right) = z(MW)^3VY.$$

With these identities, (H.11) reduces to:

$$M = \left[1 - z(MW)^3V^2Y\right]^{-1},$$

or in a slightly different form:

$$M = 1 + zM^4W^3V^2Y \tag{H.13}$$

which is highly reminiscent of (H.1) (i.e. the top equation of (9.2)). Indeed, as mentioned previously, further analysis will show Y can be written as a rational function in M, V, and W; so (H.13) will, indeed, reduce to (H.1).

By applying appropriate six-way street equations, and identifying factors of $X_{3a+2b+3c+2d}$ with z, one can obtain the following nine equations:

$$D\left(b_2^2\right) = (MW)^2V + zD\left(\bar{c}_3\right)U\left(A\right) \tag{H.14}$$

$$D\left(a_2^2\right) = U\left(A\right) + zU\left(b_2^2\right)D\left(c_2\right) \tag{H.15}$$

$$D\left(c_2\right) = D\left(\bar{c}_3\right) + D\left(b_2^2\right)U\left(A\right) \tag{H.16}$$

$$U\left(\bar{c}_3\right) = MW + zU\left(\bar{b}_3^2\right)Y \tag{H.17}$$

$$U\left(\bar{b}_3^2\right) = U\left(b_2^2\right) + MWD\left(a_2^2\right) \tag{H.18}$$

$$U\left(A\right) = Y + (MW)^2VU\left(\bar{c}_3\right) \tag{H.19}$$

$$U\left(b_2^2\right) = D\left(a_2^2\right)M + M^4V^2W^3 \tag{H.20}$$

$$Y = D\left(b_2^2\right) + U\left(\bar{c}_3\right)M^2VW \tag{H.21}$$

$$D\left(\bar{c}_3\right) = M^2WU\left(b_2^2\right). \tag{H.22}$$

Through various manipulations of these nine equations we can express Y in terms of M, V, and W, and derive (H.1) and (H.2)(the bottom two equations of (9.2)), i.e. the equations not containing the variable z, but force further constraints on the relationships between M, V, and W. In the spirit of unattractive transparency, we present a derivation of one such constraint equation (which, by all means, may not be the most efficient way of deriving such an equation).

First, we play with (H.14):

$$\begin{aligned}
D\left(b_2^2\right) &= (MW)^2V + zD\left(\bar{c}_3\right)U\left(A\right) \\
&= (MW)^2V + zD\left(\bar{c}_3\right)\left\{Y + (MW)^2VU\left(\bar{c}_3\right)\right\} \\
&= (MW)^2V + zD\left(\bar{c}_3\right)Y + z(MW)^2VD\left(\bar{c}_3\right)U\left(\bar{c}_3\right) \\
&= (MW)^2V\left[1 + zD\left(\bar{c}_3\right)U\left(\bar{c}_3\right)\right] + zD\left(\bar{c}_3\right)Y \\
&= (MW)^2VQ\left(\bar{c}_3\right) + zD\left(\bar{c}_3\right)Y \\
&= (MVW)^2 + zD\left(\bar{c}_3\right)Y.
\end{aligned}$$

Thus,

$$\begin{aligned} Q\left(b_2^2\right) &= 1 + zU\left(b_2^2\right)D\left(b_2^2\right) \\ &= 1 + \frac{zD\left(\bar{c}_3\right)}{M^2W}\left[Y - U\left(\bar{c}_3\right)M^2VW\right] \\ &= 1 + \frac{zD\left(\bar{c}_3\right)Y}{M^2W} - zD\left(\bar{c}_3\right)U\left(\bar{c}_3\right)V \\ V &= 1 + \frac{zD\left(\bar{c}_3\right)Y}{M^2W} - \left(V-1\right)V; \end{aligned}$$

so

$$zD\left(\bar{c}_3\right)Y = M^2W\left(V^2 - 1\right). \tag{H.23}$$

This yields,

$$D\left(b_2^2\right) = \left(MVW\right)^2 + M^2W\left(V^2-1\right). \tag{H.24}$$

Using $D\left(\bar{b}_3^2\right) = (MW)^2V$ (c.f. (H.8)), $Q\left(\bar{b}_3^2\right) = W$, and (H.17)

$$\begin{aligned} W &= 1 + z(MW)^2VU\left(\bar{b}_3^2\right) \\ &= 1 + z(MW)^2V\left[\frac{U\left(\bar{c}_3\right) - MW}{zY}\right] \\ &= 1 + (MW)^2V\left[\frac{U\left(\bar{c}_3\right) - MW}{Y}\right]; \end{aligned}$$

we rewrite this as

$$Y = \frac{(MW)^2V\left[U\left(\bar{c}_3\right) - MW\right]}{W-1}. \tag{H.25}$$

Hence, (H.21) requires, along with (H.24),

$$(MVW)^2 + M^2W(V^2-1) + U\left(\bar{c}_3\right)M^2VW = \frac{(MW)^2V\left[U\left(\bar{c}_3\right) - MW\right]}{W-1};$$

solving for $U\left(\bar{c}_3\right)$, we have,

$$U\left(\bar{c}_3\right) = \frac{1 - V^2 - W + MVW^2 + V^2W^2}{V}. \tag{H.26}$$

Substituting (H.26) back into (H.25), and simplifying we have

$$\begin{aligned} Y &= M^2W^2\left[V^2 + V\left(M+V\right)W - 1\right] \\ &= (MVW)^2 + M^3VW^3 + M^2V^2W^3 - M^2W^2. \end{aligned} \tag{H.27}$$

Finally, from the identity

$$V = Q(\bar{c}_3) = 1 + (zD(\bar{c}_3))\,U(\bar{c}_3)$$

and the results (H.23), (H.26), and (H.27), we have

$$V = 1 + \left(\frac{1 - V^2 - W + MVW^2 + V^2W^2}{V}\right) \frac{M^2W(V^2-1)}{(MVW)^2 + M^3V;W^3 + M^2V^2W^3 - M^2W^2}$$

this simplifies to

$$0 = (-1+V)\left[(-1+V)(1+V)^2 + (1+V^3)W - V(M+V)W^2\right].$$

But as $V \neq 1$, then we must have

$$0 = (-1+V)(1+V)^2 + (1+V^3)W - V(M+V)W^2; \tag{H.28}$$

this is (H.2).

Now, solving (H.28) for M, and substituting this expression into (H.27) gives

$$Y = \frac{(1+V)(1+V-W)^2\left(V^2(1+W)-1\right)^3}{V^2W^3}. \tag{H.29}$$

Combining this with (H.13), we arrive at (H.1):

$$M = 1 + zM^4\left\{(1+V)(1+V-W)^2[V^2(1+W)-1]^3\right\}.$$

To derive the second interesting relation between M, V, and W we study $Q(c_2)$, $Q(a_2^2)$, or $Q(b_2^1)$, among various other generating functions. In particular, we can write

$$\begin{aligned} V &= Q\left(a_2^2\right) \\ &= 1 + xY D\left(a_2^2\right) \\ &= 1 + \left(\frac{M^2W(V^2-1)}{D(\bar{c}_3)}\right) D\left(a_2^2\right) \\ &= 1 + \frac{(V^2-1)D\left(\overline{a_2^2}\right)}{U(b_2^2)} \end{aligned}$$

or

$$\begin{aligned} W &= Q(c_2) \\ &= 1 + zMW D(\bar{c}_2) \\ &= 1 + MW\left[\frac{D(a_2^2) - U(A)}{U(b_2^2)}\right]. \end{aligned}$$

Using the finalized expressions above, the relation (H.28) (which can be used to eliminate M), and the condition that $V \neq 1$, and the assistance of Mathematica, we consistently arrive at the relation (H.3):

$$0 = (-1 + V)(1 + V)^3 + (1 + V)(1 + V^2)W - V(1 + 3V)W^2. \tag{H.30}$$

Appendix I: Signs in the definition of z

We now elaborate on the reasons for the sign $(-1)^{mab-a^2-b^2}$ that appears in the definition of the variable z for $(a,b|m)$-herds; the reasoning follows along the lines of the reason for the sign $(-1)^m$ in the definition of the formal variable in in the m-herd (c.f. Prop. 3.1.1). Indeed, as in the proof for m-herds, we expect that the generating functions are explicitly given as series in the formal variable $z = X_{m\widehat{\gamma}_1+(m-1)\widehat{\gamma}_2}$; where $\widehat{\gamma}_1$, $\widehat{\gamma}_2 \in \widetilde{\Gamma}$ are defined in the following manner:

1. Starting from an $(a,b|m)$-herd, shrink all fuchsia streets (streets of type 13) to points; the resulting diagram should appear as the superposition of two saddle connections with transverse intersections at m points (c.f. Fig. 11.2).

2. Each saddle-connection lifts to a closed, oriented loop on the spectral cover Σ; let α denote the loop whose homology class is γ_1 (using the red-blue colour-coding used throughout this thesis: the lift of the blue saddle connection), and β the lift of the loop whose homology class is γ_2 (the lift of the red saddle-connection).

3. The loops α and β have natural tangent-framing lifts to the unit tangent bundle $UT\Sigma$; denote them by $\widehat{\alpha}$ and $\widehat{\beta}$ respectively. Then by taking the homology classes of these tangent framing lifts, we define $\widehat{\gamma}_1 := [\widehat{\alpha}] \in \widetilde{\Gamma}$ and $\widehat{\gamma}_2 := [\widehat{\beta}] \in \widetilde{\Gamma}$. The explicit loops $\widehat{\alpha}$ and $\widehat{\beta}$ depend on the choice of shrinking procedure in step 1 – however, it is clear that these loops are unique up to homotopy. Hence, $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are well-defined homology classes.

Now, let $a, b \in \mathbb{Z}$, then the homology class $a\gamma_1 + b\gamma_2$ can be represented by the sum of a copies of $\widehat{\alpha}$ and b copies of $\widehat{\beta}$; thus, from (D.1), it follows that

$$(a\widetilde{\gamma_1 + b}\gamma_2) = a\widehat{\gamma}_1 + b\widehat{\gamma}_2 + (a + b + mab)H$$

Hence,

$$z := X_{a\widehat{\gamma}_1+b\widehat{\gamma}_2} = (-1)^{mab+a+b} X_{(a\widetilde{\gamma_1+b}\gamma_2)}.$$

Because $mab + a + b$ has the same parity as $mab - a^2 - b^2$, we may write (if we so wish)

$$z = (-1)^{mab-a^2-b^2} X_{(a\widetilde{\gamma_1 + b}\gamma_2)}.$$

Appendix J: Large Unwieldy Polynomials

For the reader's convenience, we rewrite the polynomial determining the series V and the polynomial determining $\mathsf{T}_{3/2}$ in a form that allows for a simple copy and paste into *Mathematica* (or other computer-algebra software). First, the absolutely irreducible polynomial (in two-variables) with root (as a polynomial with coefficients in $\mathbb{Z}[z]$) the series V is stated below.

```
PolyV[z_, v_] := -z -
   z^2 + (-11*z - 20*z^2)*v + (-24*z - 170*z^2)*v^2 + (-1 + 177*z -
      761*z^2)*v^3 + (-14 + 1053*z - 1620*z^2)*v^4 + (-75 + 1285*z +
      549*z^2)*v^5 + (-180 - 3866*z + 11973*z^2)*v^6 + (-135 -
      10062*z + 24543*z^2)*v^7 + (162 + 7192*z -
      5700*z^2)*v^8 + (243 + 35757*z - 102404*z^2)*v^9 + (-11615*z -
      129829*z^2)*v^10 + (-97765*z + 124394*z^2)*v^11 + (4563*z +
      462849*z^2)*v^12 + (180593*z + 198599*z^2)*v^13 + (8205*z -
      711637*z^2)*v^14 + (-215927*z - 932750*z^2)*v^15 + (27968*z +
      378160*z^2)*v^16 + (184698*z + 1536505*z^2)*v^17 + (-81597*z +
      527710*z^2)*v^18 + (-100800*z - 1365795*z^2)*v^19 + (74492*z -
      1276293*z^2)*v^20 + (10798*z + 539545*z^2)*v^21 + (-28991*z +
      1270220*z^2)*v^22 + (11257*z + 200132*z^2)*v^23 + (-1475*z -
      718560*z^2)*v^24 - 408046*z^2*v^25 + 204359*z^2*v^26 +
   259484*z^2*v^27 + 10647*z^2*v^28 - 87687*z^2*v^29 -
   31977*z^2*v^30 + 13506*z^2*v^31 + 11541*z^2*v^32 + 725*z^2*v^33 -
   1718*z^2*v^34 - 578*z^2*v^35 + 33*z^2*v^36 + 58*z^2*v^37 +
   13*z^2*v^38 + z^2*v^39
```

The absolutely irreducible polynomial (in two-variables) with root (as a polynomial with coefficients in $\mathbb{Z}[z]$) $\mathsf{T}_{3/2}$ is stated below.

```
PolyT[z_,t_] := -(1 + z) + t*(4 - 5*z) +  t^2*(-6 + z) + t^3*(4 + 21*z) +
t^4*(-1 - 34*z) - t^5*(7*z) + t^6*(76*z + z^2) + t^7*(-64*z - 13*z^2) +
t^8*(6*z - 114*z^2) + t^9*(7*z - 80*z^2) + t^(10)*(6*z^2) +
```

```
  t^(11)*(119*z^2) + t^(12)*(53*z^2 + z^3) + t^(13)*(-55*z^2 + 44*z^3) +
  t^(14)*(-21*z^2 - 38*z^3) + t^(15)*(77*z^3) - t^(16)*(382*z^3) +
  t^(17)*(270*z^3) + t^(18)*(80*z^3 - z^4) + t^(19)*(35*z^3 + 7*z^4) +
  t^(20)*(39*z^4) - t^(21)*(367*z^4) - t^(22)*(173*z^4) -
t^(23)*(30*z^4) - t^(24)*(35*z^4) +  t^(25)*(3*z^5) - t^(26)*(17*z^5) -
t^(27)*(77*z^5) - t^(28)*(14*z^5) + t^(29)*(21*z^5) - t^(32)*(3*z^6) +
t^(33)*(9*z^6) - t^(34)*(7*z^6) + t^(39)*z^7
```

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