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Applications of Forward Performance Processes in Dynamic Optimal Portfolio Management

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Applications of Forward Performance Processes in Dynamic Optimal Portfolio Management

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The classical optimal investment models are cast in a finite or infinite horizon setting, assuming an a priori choice of a market model (or a family of models) as well as a priori choice of a utility function of terminal wealth and/or intermediate consumption. Once these choices are made, namely, the horizon, the model and the risk preferences, stochastic optimization technique yield the maximal expected utility (value function) and the optimal policies wither through the Hamilton-Jacobi-Bellman equation in Makovian models or, more generally, via duality in semi-martingale models. A fundamental property of the solution is time-consistency, which follows from the Dynamic Programming Principle (DPP). This principle provides the intuitively pleasing interpretation of the value function as the intermediate (indirect) utility. It also states that the value function is a martingale along the optimal wealth trajectory and a super-martingale along every admissible one. These properties provide a time-consistent framework of the solutions, which "pastes"

naturally one investment period to the next.

Despite its mathematical sophistication, the classical expected utility framework cannot accommodate model revision, nor horizon flexibility nor adaptation of risk preferences, if one desires to retain time-consistency. Indeed, the classical formulation is by nature "backwards" in time and, thus, it does not allow any "forward in time" changes. For example, on-line learning, which typically occurs in a non-anticipated way, cannot be implemented in the classical setting, simply because the latter evolves backwards while the former progresses forward in time.

To alleviate some of these limitations while, at the same time, preserving the time-consistency property, Musiela and Zariphopoulou proposed an alternative criterion, the so-called forward performance process. This process satisfies the DPP forward in time, and generalizes the classical expected utility. For a large family of cases, forward performance processes have been explicitly constructed for general Ito-diffusion markets. While there has already been substantial mathematical work on this criterion, concrete applications to applied portfolio management are lacking.

In this thesis, the aim is to focus on applied aspects of the forward performance approach and build meaningful connections with practical portfolio management. The following topics are being studied.

Chapter 2 starts with providing an intuitive characterization of the underlying performance measure and the associated risk tolerance process, which are the most fundamental ingredients of the forward approach. It also provides a novel decomposition of the initial condition and, in turn, its inter-temporal preservation as the market evolves. The main steps involve a system of stochastic differential equations modeling various stochastic sensitivities and risk metrics. Chapter 3 focuses on the applications of the above results to lifecycle portfolio management. Investors are firstly classified by their individual risk preference generating measures and, in turn, mapped to different groups that are consistent with the popular practice of age-based de-leveraging. The inverse problem is also studied, namely, how to infer the individual investor-type measure from observed investment behavior.

Chapter 4 provides applications of the forward performance to the classical problem of mean-variance analysis. It examines how sequential investment periods can be "pasted together" in a time-consistent manner from one evaluation period to the next. This is done by mapping the mean-variance to a family of forward quadratic performances with appropriate stochastic and path-dependent coefficients. Quantitative comparisons with the classical approach are provided for a class of market settings, which demonstrate the superiority and flexibility of the forward approach.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Since the seminal work of Samuelson (1969) and Merton (1971), the theory of stochastic control has been extensively applied to solve problems in dynamic portfolio management in both discrete and continuous-time settings. The fundamental ingredients of these problems are the trading horizon (finite or infinite), the market model (or a family of models) and a utility functional (from terminal wealth and/or intermediate consumption). The market models can be quite complex allowing, for example, for multi-correlated stochastic factors, filtering, "hidden" processes, jumps, and other features (see, for example, Kim and Omberg (1996); Liu (2007); Watcher (2002); Brandt et al. (2005); Ait-Sahalia et al. (2009)). Such models have been also extended to accommodate Knightian uncertainty (see, for example, Maenhout (2005)).
The associated stochastic optimization problems are solved either via the Hamilton-Jacobi-Bellman equation or the duality approach for general semimartingale markets. A fundamental property of the optimal solution (the so-called value function) is that it satisfies the Dynamic Programming Principle (DPP). This universal result yields that the value function has the semi-group property across arbitrary trading times and, furthermore, that it can be interpreted as the intermediate utility function. The DPP also implies that the

value function is a martingale when complied with the optimal wealth process and a super-martingale along all admissible wealth processes.

Despite the considerable abstract mathematical advances, the expected utility approach has very little, if any, application to practical portfolio management. There are various reasons for it. From the one hand, it is very difficult to estimate the asset returns, a notoriously difficult problem that is independent of the utility framework.

From the other hand, it is difficult to provide a quantitative assessment of the utility function itself (see, for example, the old note of Black (1988)). Indeed, as argued in the latter paper, the concept of utility is to a certain extent elusive and investors prefer to express their desires and concerns in different metrics. Additional difficulties for the applicability of expected utility stem from the fact that the classical framework is, in terms of horizon, model and utility choice, static. Indeed, once the model, the horizon and the risk preferences are chosen at the initial time, no further revisions can be made if time-consistency has to be maintained. This is directly manifested in the DPP, which yields, by its nature, a backward in time construction. As a result, desirable practical features, like for example on-line learning, (non-anticipated) rolling horizons, revision of preferences and adaptation of the market model cannot be made unless they are incorporated from the beginning within a richer modeling universe. But even this modeling universe may turn out to be inaccurate, especially when long-term horizons are considered.

To remedy some of the above difficulties, an alternative criterion, the so-called forward performance process was introduced by Musiela and Zariphopoulou (see, Musiela and Zariphopoulou (2008, 2009, 2010a)). This performance process adapts to the incoming market information and provides substantial flexibility in terms of model revision, horizon choice and evolution of risk preferences, while preserving time-consistency at all times. As a matter of fact, time consistency is its fundamental property, for the forward criterion is being defined via a "rolling-type" DPP.

This thesis contributes to two distinct directions, theoretical and applied. Firstly, it provides a novel characterization and interpretation of the "forward performance measure", which is one of the main modeling ingredients in the forward approach. This measure models the initial utility datum that the investor chooses, which can be, for example, the initial value function of the classical backward problem before model revisions occur, or an estimate of the overall expected upcoming utility, and others. As it is discussed in Chapter 2, viable forward solutions are directly related to a bilateral Laplace transform that involves a specific measure. For example, for risk preferences of power type, this measure turns out to be a Dirac mass at a point related to the risk aversion coefficient. Interpreting this measure has been an open question for the forward approach.

In Chapter 2, a complete characterization of the performance measure is provided. It is shown that it corresponds to an intuitively pleasing separation of the initial wealth to distinct fractions and that to each of these components the investor assigns an "individual" risk tolerance. We argue that this separation is in accordance with the "mental account" framework postulated by behavioral economists. Once this allocation of risk and deployment of wealth are specified, the forward problem reduces to a family of smaller problems of "individual" wealth and risk tolerance. This separation also provides a novel description of the structure of the risk tolerance process itself. It is shown how it can be entirely specified component-by-component, as the market moves and new information becomes available. The mathematical problem amounts in solving a system of stochastic differential equations related to moments of underlying quantities.

In Chapter 3, these new results are, in turn, applied to fund management and, in particular, to target date funds. We use the forward approach to provide a normative framework for practical investment practices for such financial instruments. Among others, we show that, only when an investor is strictly more risk-seeking than a log-utility investor, should he agree with the "glide path" practice currently adopted by target date funds. Moreover, we look at the problem in the reversed direction as well. For an observed "glide path" strategy, we find the forward utility that generates the closest behavior. To demonstrate the application, we carry out this calibration exercise on the Vanguard target retirement 2045 fund.

In Chapter 4, the focus is on the application of the forward theory to the classical mean-variance optimization problem. While this problem has been extensively solved for the one-period and the dynamic setting, still the latter is cast within a single pre-specified investment target. Naturally, within the investment horizon, one may view this problem as a dynamic one with a family of changing targets. However, even in this setting, new information cannot be incorporated unless it is a priori incorporated in a richer model, which however may turn out very soon to be inaccurate. Furthermore, the manager may wish to paste forward (and not backwards) in time sequential investment periods, when trading horizons are dynamically adjusted. These problems have been open for quite some time.

It is not hard to see that the fundamental difficulty is to build a time-consistent model for the forward in time evolution of the process that models the risk-return trade-off coefficients. We accomplish this by first mapping each mean-variance optimization problem to a problem of maximizing a path-dependent quadratic utility. We then build a family of forward criteria that "match" these quadratic utilities and, in turn, we map this family back to a sequence of mean-variance problems. We then establish that this construction yields a sequence of risk-return trade-off coefficients, which takes into account both the new incoming information as well as the past performance of the optimal strategies. Finally, we extend these results to the case of model uncertainty by building the robust analogue of forward mean-variance optimization.

Once the theoretical results are established, we perform a quantitative comparative analysis between the classical and the forward cases. Specifically, we compare the classical scenario, where the new information is being treated by just "restarting" the mean-variance model, to the forward mean-variance setting in which the trade-off coefficients are being updated in a time-consistent manner, as described above. For the one period case, we demonstrate through a market model with serially correlated returns, that the forward mean-variance approach strictly dominates the classical one in terms of long-term Sharpe ratio. Similar results are also obtained for the dynamic case. Furthermore, we demonstrate that in a setting where the true model cannot be observed, the optimal strategy of the classical single-target multi-period setting is outperformed by the multi-period forward approach. Finally, we provide numerical examples for the case of forward mean-variance optimization in the presence of model ambiguity.

Chapter 2

The forward performance criterion

2.1 Introduction

Assume the market we trade consists of K risky assets S_1, S_2 $\ldots$, S_K, with dynamics governed by the SDE system,

$$dS_k(t) = S_k(t)\big(\mu_k(t)dt + \sum_{j=1}^{K} \sigma_{kj}(t)dW_j(t)\big),\ k = 1, 2, \ldots, K, \tag{2.1}$$

and one risk-free asset S_0, with short rate $r(t)$, i.e.,

$$dS_0(t) = r(t)S_0(t)dt. \tag{2.2}$$

The parameters $\mu_k(t)$, $\sigma_{kj}(t)$, $r(t)$ are stochastic processes adapted to the filtration generated by the Brownian motion $W(t)$.

Let $\pi_k(t)$ denote the amount of wealth invested in S_k at time t, and $\overline{X}^{\pi}(t) = e^{-\int_0^t r(s)ds}X^{\pi}(t)$ the (risk-free rate) discounted wealth process under the strategy π. Assume that $\pi_k(t)$, $k = 1, \ldots, K$ satisfies the standard assumption of being square-integrable and self-financing. Then $X^{\pi}(t)$ evolves according to the controlled process

$$d\overline{X}^{\pi}(t) = \big(\sum_{k=1}^{K} \mu_k(t)\pi_k(t)\big)dt + \sum_{k=1}^{K} \big(\pi_k(t) \sum_{j=1}^{K} \sigma_{kj}(t)dW_j(t)\big)$$

In the classical optimal portfolio problem, we prespecify an investment horizon $[0, T]$, and assume a utility function $U(x)$ which evaluates $X^\pi(T)$. Then one solves the following optimal control problem by applying backward in time the dynamic programming principle

$$\sup_{\pi \in \mathcal{A}} E[U(X^\pi(T))],$$

where $\mathcal{A}$ denotes the set of all admissible policies

$$\mathcal{A} = \{\pi : \text{ self-financing with } \pi(t) \in \mathcal{F}_t \text{ and } \mathbb{E}(\int_0^t |\sigma(s)\pi(s)|^2 ds) < \infty,\ t > 0\}.$$

While it seems a reasonable formulation at short horizons (i.e. for T small), it becomes unrealistic and impossible to implement for large T. Indeed, note that due to frequently arising model decay, the model given by (4.60) is predetermined at $t = 0$, and thus, it is not subject to change according to the theory. What if, however, at a later time $t > 0$, new information suggests that the model no longer accurately approximates the observed dynamics of the market and needs to be updated? If the fund manager ignores this new information and keeps using the same model, his portfolio strategy is no longer optimal because of model mis-specification. On the other hand, if he updates his model based on new information, then his actions at $s < t$ and $s > t$ are actually derived from two different distributional assumptions. The so call "inconsistency" issue would arise and erode his performance.

Furthermore, the assumption that the horizon T is known at $t = 0$ is also problematic. Consider, for example, the following statements: "I will keep investing until my total wealth exceeds \$1 million",. "As long as my fund keeps outperforming the market, I will stay in this business", or "I plan to get out of the market before getting hit by the next financial crisis". In all the above situations, the investors do not have a clear idea of T when investment starts.

The above issues motivate us to consider an alternative framework, that would allow for more flexibility with regards to both model and horizon revisions. For example, one could subdivide the entire horizon $[0, T]$ into small intervals, say $0 < t_1 < \ldots < t_n = T$, and solve a sequence of "shorter horizon" problems

$$\sup_{\pi_t,\ t\in[t_i,t_{i+1}]} E_{t_i}^{\mathbb{P}_i}[U_i(X^\pi(t_{i+1}))].$$

The investor then has the flexibility and ability to re-estimate model at each t_i, $i < n$, and solve the raw expected utility problem in $(t_i, t_{i+1}]$. The fundamental question is then how to properly define a sequence of appropriate utility functions $U_i(\cdot)$, such that the aforementioned "contradictory behavior" would not arise.

Musiela and Zariphopoulou (2008, 2009, 2010a) proposed such a dynamic utility theory, the so called "forward performance". In the limiting case, as $\Delta t = t_{i+1} - t_i \to 0$. We provide the definition next.

Definition 2.1.1. An $\mathcal{F}_t$-adapted process $U_t(x)$ is a forward performance process if, for $t \geq 0$ and $x \in \mathbb{R}$:

i) the mapping $x \to U_t(x)$ is strictly concave and strictly increasing,

ii) for each admissible portfolio strategy π, $\mathbb{E}[U(X_t^\pi)^+] < \infty$, and

$$\mathbb{E}[U_s(X_s^\pi)|\mathcal{F}_t] \leq U_t(X_t^\pi), \quad s \geq t,$$

iii) there exists an admissible strategy π^*, for which,

$$\mathbb{E}[U_s(X_s^{\pi^*})|\mathcal{F}_t] = U_t(X_t^{\pi^*}), \quad s \geq t.$$

Conditions ii) and iii) resemble the dynamic programming principle (DPP), in that, for any $s \geq t > 0$,

$$U_t(x) = \max_{\pi \in \mathcal{A}} \mathbb{E}[U_s(X_s^{\pi})|X_t = x]. \tag{2.3}$$

This condition, so called “self-generating”, was used in Zitkovic (2009) to provide an alternative characterization of forward preferences. Here $U_s(x)$ plays the role of a utility function at horizon s, while $U_t(x)$ plays the role of “value function”, derived from maximizing the terminal utility.

The main difference between the above construction and the classical framework is that, the latter is solved backward in time. This requires the specification of a terminal utility function and a stochastic model that describes the asset return distributions for the entire investment horizon. Here, however, the situation is reversed. The investor specifies an “initial utility” $U_0(x)$, after which the utility process as well as the optimal strategy are solved forward in time. More specifically, we are facing now the *inverse problem* posed by equation (2.3). Where $U_t(x)$ is given, and we look for $U_s(x)$ such that (2.3) is satisfied.

Musiela and Zariphopoulou (2010b) provided a concrete mathematical characterization of forward performance processes defined above. Let $\sigma(t) = \{\sigma_{ij}(t)\}$ denote the volatility matrix process, and define the market price of risk

$$\lambda(t) = (\sigma(t)^T)^+\big(\mu(t) - r(t)\mathbf{1}\big).$$

Here the matrix $(\sigma(t)^T)^+$ is the Moore-Penrose pseudo-inverse of the matrix $\sigma(t)^T$.

Assume that the utility process $U(x,t)$ admits the Ito-decomposition

$$dU(x,t) = b(x,t)dt + a(x,t) \cdot dW(t),$$

where both $b(x,t)$ and $a(x,t)$ are $\mathcal{F}_t$-progressively measurable, d-dimensional and continuously differentiable in the spatial argument processes. Then, it was shown that $U(x,t)$ is a forward performance process if it satisfies the stochastic partial differential equation (SPDE),

$$dU(x,t) = \frac{1}{2}\frac{|U_x(x,t)\lambda(t) + \sigma(t)\sigma(t)^+ a_x(x,t)|^2}{U_{xx}(x,t)}dt + a(x,t)dW(t). \tag{2.4}$$

In this thesis, we throughout assume that $a(x,t) \equiv 0$. We will call this the "zero-volatility" case. General solutions for $a(x,t) \neq 0$ are currently being investigated by several authors; see existing works Nadtochiy and Tehranchi (2015), Nadtochiy and Zariphopoulou (2014). The above equation simplifies to the following random PDE

$$U_t(x,t) = \frac{1}{2}|\lambda(t)|^2 \frac{U_x(x,t)^2}{U_{xx}(x,t)}. \tag{2.5}$$

and was studied in detail by Musiela and Zariphopoulou (2010a). It turns out that the performance process $U(x,t)$ can be represented as the compilation of a stochastic market related input

$$A(t) = \int_0^t |\lambda(s)|^2 ds, \tag{2.6}$$

which summarizes the cumulative investment opportunity up to time t, and a investor-specific differential input $u(x,t)$, which is a deterministic function of space and time, solving the PDE,

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}.$$

Then $U(x,t)$ is obtained as

$$U(x,t) = u(x, A(t)).$$

The complete construction of the function $u(x,t)$ can be found in Propositions 14, 15, 19 of Musiela and Zariphopoulou (2010a). Below, we provide for the reader's convenience the main steps of the analysis. To solve for $u(x,t)$, let $I(x,t) = u^{(-1)}(x,t)$, where $u_x^{(-1)}$ denote the space inverse of $u_x(x,t)$, and let $h(x,t) = I(e^{-x},t)$. It can be verified that h solves the (backward) heat equation,

$$h_t + \frac{1}{2}h_{xx} = 0. \tag{2.7}$$

So far, the problem of constructing zero-volatility forward performance process has been reduced to solving the backward heat equation (2.7). After the function h is obtained, $u(x,t)$ can be obtained as

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x(h^{(-1)}(x,s),s)ds + \int_0^x e^{-h^{(-1)}(z,0)}dz \tag{2.8}$$

Next, we define the risk tolerance function,

$$r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{2.9}$$

and the corresponding risk tolerance process

$$R(t) = r(X(t), A(t)),$$

with $A(t)$ as in (2.6). We are now ready for the following main result.

Proposition 2.1.2. *Let the performance process* $U(x,t) = u(x, A_t)$, *with* $A_t = \int_0^t |\lambda(s)|^2 ds$ *and* $u(x,t)$ *given by* (2.8), *and assume that* $h(x,t)$ *solves the backward heat equation* (2.7). *Then* $U(x,t)$ *is a forward performance process (in terms of definition 2.1.1), and the optimal portfolio strategy* $\pi^*(t)$ *and optimal wealth process* $X^*(t)$ *under* $U(x,t)$ *are given by,*

$$X^*(t) = h(h^{(-1)}(x,0) + A(t) + M(t), A(t)), \tag{2.10}$$

$$\begin{aligned} R^*(t) &= h_x(h^{(-1)}(X^*(t), A(t)), A(t)), \\ &= h_x(h^{(-1)}(x,0) + A(t) + M(t), A(t)) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \pi^*(t) &= R^*(t)\sigma(t)^+\lambda(t) \\ &= h_x(h^{(-1)}(x,0) + A(t) + M(t), A(t))\sigma(t)^+\lambda(t), \end{aligned} \tag{2.12}$$

where $M_t = \int_0^t \lambda(s) dW(s)$.

From the above results, one can see that all quantities may be constructed once the function h that solves (2.7) is specified. The classical results of Widder (1975) show that positive solutions of (2.7) can be represented as a bilateral transform of a positive finite Borel measure. Let us define,

$$\mathcal{B}^+(\mathbb{R}) = \left\{\nu \in \mathcal{B}(\mathbb{R}) :\ \forall B \in \mathcal{B},\, \nu(B) \geq 0 \text{ and } \int_{\mathbb{R}} e^{yx}\nu(dy) < \infty,\, x \in \mathbb{R}.\right\}$$

We reproduce Proposition 9 of Musiela and Zariphopoulou (2010a) as below

Proposition 2.1.3. *Let $\nu \in \mathcal{B}^+(\mathbb{R})$. Then the function $h(x,t)$ defined by*

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t} - 1}{y} \nu(dy) + C \tag{2.13}$$

is a strictly increasing solution to (2.7).

To demonstrate the use of results obtained so far, we provide two examples in which we show how to construct $u(x,t)$ based on specific choices of the measure ν.

Example 1. Assume that the measure ν is given by

$$\nu = \gamma\delta_{\{0\}}.$$

Here γ is a positive constant and $\delta_{\{0\}}$ is a Dirac function with point mass at 0. Then,

$$h(x,t) = \gamma x, \quad h^{(-1)}(x,t) = \frac{1}{\gamma}x.$$

By equation (2.8), we obtain the forward exponential solution

$$u(x,t) = -\frac{1}{2}\frac{1}{\gamma}\int_0^t e^{-\frac{1}{\gamma}x+\frac{s}{2}} + \int_0^x e^{-\frac{1}{\gamma}z}dz = \gamma(1 - e^{-\frac{1}{\gamma}x+\frac{t}{2}}).$$

By (2.12), the optimal strategy is given by

$$\pi^*(t) = \gamma\sigma^+(t)\lambda(t).$$

Then the forward process is given by

$$U(x,t) = u(x, A(t)) = \gamma(1 - e^{-\frac{1}{\gamma}x+\frac{1}{2}\int_0^t |\lambda(s)|^2 ds}).$$

While the above criterion and strategy resemble the analogous classical exponential quantities, there are fundamental differences between the classical and the forward cases. Indeed, the investment horizon is not prespecified. The model dynamics, $\sigma(t)$ and $\lambda(t)$, are not prechosen either, as they are taken to be arbitrary $\mathcal{F}_t$-adapted processes.

Example 2. The measure ν is given by

$$\nu = \alpha\delta_{\{\alpha\}} \quad \text{with} \quad \alpha \neq 0, 1, \;\; \delta \text{ is the Dirac function .}$$

By (2.17) and (2.8), we obtain

$$h(x,t) = e^{\alpha x - \frac{1}{2}\alpha^2 t}.$$

$$\begin{aligned} u(x,t) &= -\frac{1}{2}\int_0^t e^{-\frac{1}{2}\alpha s - \frac{1}{\alpha}\ln x + \frac{s}{2}} ds + \int_0^x e^{-\frac{1}{2}\alpha t - \frac{1}{\alpha}\ln z} dz \\ &= \frac{\alpha}{\alpha - 1}(e^{\frac{1-\alpha}{2}t} - 1)x^{1-\frac{1}{\alpha}}, \end{aligned}$$

and

$$\pi^*(t) = \alpha X^*(t)\sigma^+(t)\lambda(t).$$

The forward process is given by

$$U(x,t) = \frac{\alpha}{\alpha - 1}(e^{\frac{1-\alpha}{2}\int_0^t |\lambda(s)|^2 ds} - 1)x^{1-\frac{1}{\alpha}}.$$

Again, $U(x,t)$ and $\pi^*(t)$ resemble the classical quantities of the power case. One can similar show that the case $\nu = \delta_{\{1\}}$ corresponds to forward log utility. For more example, see Musiela and Zariphopoulou (2010a). Several authors have expanded the scope to study forward utility beyond CRRA or CARA types. Geng and Zariphopoulou (2017) showed that for a general time-monotone forward utility, the spatial and temporal limit of the relative risk tolerance function are related to the right and left boundaries of the support of ν. Also, Zariphopoulou and Zhou (2009) studied a particular class of forward utility functions, with time-dependent but asymptotically linear risk tolerance. While in the existing works, there are concrete mathematical results, the intuition backing the underlying measure ν, which is the defining element in the entire construction, has not been developed. We contribute to this next. We first provide a new interpretation of the nature of forward performance process. The optimal behavior is subsequently studied in detail in section 2.3. In section 2.4, we describe computational methods to calculate various quantities of interest for the fund manager, and we conclude in section 2.5.

2.2 Interpreting the performance generating measure

In this section we start our analysis by providing an intuitive explanation of the exact nature of forward investing, which is not immediately clear from the closed form solution given in Proposition 2.1.2 alone. We will show that a forward performance process can be thought of as a "static" combination of simpler performance processes. This representation is similar to the *mental account* framework postulated by behavioral portfolio theorists.

This new interpretation plays a fundamental role here as it will serve as the basis for our study of the relative risk tolerance process in section 2.3, as well as the time series property of the optimal allocation strategy that we propose in chapter 3.

2.2.1 Model dynamics

To simplify the notation and exposition, and present the main idea more clearly, we work for now with the log-normal dynamics in (4.60) and (2.2), with a single risky asset

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

We also assume that the interest rate is zero. As we will comment later on, the assumptions are innocuous and generalizations to multiple assets, stochastic parameters and non-zero risk-free rate are straightforward, albeit tedious.
We have seen from the last section that the utility process is determined by the a positive Borel measure ν, normally,

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t} - 1}{y} \nu(dy) + C, \tag{2.14}$$

where C is an immaterial constant. However, the forward utilities derived from the above includes many subcategories (see Proposition 9 of Musiela and Zariphopoulou (2010a) for a comprehensive classification), making it very difficult to derive any general result. We therefore restrict the scope of our analysis by assuming that the measure ν is not concentrated at or near zero (this excludes for instance, the exponential utility). For this, we introduce the following assumption. More general cases can be solved by similar arguments.

Assumption 2.2.1. The measure ν satisfies $\int_{0+}^{\infty} \frac{1}{y}\nu(dy) < \infty$, $\int_{-\infty}^{0-} \frac{1}{y}\nu(dy) < \infty$ and $\nu(\{0\}) = 0$.

Under this assumption, the function h can be rewritten as

$$
\begin{aligned}
h(x,t) &= \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t}}{y}\nu(dy) + (C - \int_{\mathbb{R}} \frac{\nu(dy)}{y}) \int_{\mathbb{R}} e^{yx-\frac{1}{2}y^2t}\delta_{\{0\}}(dy) \\
&= \int_{\mathbb{R}} e^{yx-\frac{1}{2}y^2t}\Big(\frac{\nu(dy)}{y} + (C - \int_{\mathbb{R}} \frac{\nu(dy)}{y})\delta_{\{0\}}(dy)\Big),
\end{aligned}
\tag{2.15}
$$

where $\delta_{\{0\}}$ is the Dirac delta function with point mass at 0. Next, we make the re-parameterization,

$$
\tilde{\nu}(dy) = \frac{\nu(dy)}{y} + \tilde{C}\delta_{\{0\}}(dy), \quad \tilde{C} = C - \int_{\mathbb{R}} \frac{\nu(dy)}{y}. \tag{2.16}
$$

This simplifies (2.15), yielding,

$$
h(x,t) = \int_{\mathbb{R}} e^{yx-\frac{1}{2}y^2t}\tilde{\nu}(dy).
$$

Therefore, given that ν is a positive measure, we deduce that the new measure $\tilde{\nu}$ satisfies

$$
\boldsymbol{\tilde{\nu}(dy) \geq 0} \text{ for } \boldsymbol{y > 0} \text{ and } \boldsymbol{\tilde{\nu}(dy) \leq 0} \text{ for } \boldsymbol{y < 0}.
$$

Next, we distinguish two cases since, as we will see, they will imply qualitatively different optimal portfolios.

If $\tilde{\nu}((-\infty, 0)) = 0$, then it is easy to show that, for each $t \geq 0$,

$$
\text{Range}(h(x,t)) = (\tilde{C}, \infty).
$$

In other words, the corresponding utility function $u(x,t)$ is only defined for $x > \tilde{C}$. On the other hand, if $\tilde{\nu}((-\infty, 0)) < 0$ and $\nu((0,\infty)) > 0$, then $h(x,t)$ can be written as,

$$
h(x,t) = \int_{0^+}^{\infty} e^{yx-\frac{1}{2}y^2t}\tilde{\nu}^+(dy) - \int_{0^+}^{\infty} e^{-yx-\frac{1}{2}y^2t}\tilde{\nu}^-(dy),
$$

where $\tilde{\nu}^+(dy) = \tilde{\nu}(dy)$ and $\tilde{\nu}^-(dy) = -\tilde{\nu}(-dy)$ both positive Borel measures on the positive half axis. In this case we have,

$$\text{Range}(h(x,t)) = (-\infty, +\infty).$$

For this case, $u(x,t)$ is defined for all $x \in \mathbb{R}$.

With a slight abuse of notation, we still use ν and C to denote $\tilde{\nu}$ and $\tilde{C}$ defined above. The above discussions are summarized below.

Proposition 2.2.2. *Under Assumption 2.2.1, and re-parameterizing ν (and C), the function $h(x,t)$ can be written as,*

$$h(x,t) = \int_{\mathbb{R}} e^{yx - \frac{1}{2}y^2 t} \nu(dy), \tag{2.17}$$

with ν satisfying $\nu(dy) \geq 0$ for $y > 0$ and $\nu(dy) \leq 0$ for $y < 0$.

Moreover if $C = \nu(\{0\})$, then,

(i) Range$(h) = (C, \infty)$, if $\nu((-\infty, 0)) = 0$ and $\nu((0,\infty)) > 0$,

(ii) Range$(h) = (-\infty, C)$, if $\nu((-\infty, 0)) < 0$ and $\nu((0,\infty)) = 0$,

(iii) Range$(h) = (-\infty, +\infty)$, if $\nu((-\infty, 0)) < 0$ and $\nu((0,\infty)) > 0$.

For the rest of our analysis we only consider cases (i) and (iii), since case (ii) does not correspond to a practically useful scenario.

2.2.2 The measure ν satisfies $\nu((-\infty, 0)) = 0$.

We first consider the case where ν is only supported on the non-negative half axis. To see what the measure represents, we recall example 2 where,

$$\nu = \delta_{\{\alpha\}} \quad \text{with} \quad \alpha \neq 0, 1.$$

This results in a CRRA utility with relative risk tolerance α, under which the investor allocates a constant proportion $\tilde{\pi}^* = \frac{\mu}{\sigma^2}\alpha$ of his wealth to the stock. Hence the optimal wealth solves,

$$dX^*(t) = \alpha X^*(t)(\lambda^2 dt + \lambda dW(t)).$$

Next consider the slight generalization that ν is a linear combination of two Dirac masses,

$$\nu = a_1\delta_{\{\alpha_1\}} + a_1\delta_{\{\alpha_1\}}, \tag{2.18}$$

and assume, for now, that $a_i > 0$, $\alpha_i \geq 0$. By representation (2.17), we have

$$h(x,t) = a_1 e^{\alpha_1 x - \frac{1}{2}\alpha_1^2 t} + a_2 e^{\alpha_2 x - \frac{1}{2}\alpha_2^2 t}. \tag{2.19}$$

The optimal wealth process is readily obtained following (2.10),

$$X^*(t) = a_1 e^{\alpha_1 h^{(-1)}(x,0) + (\alpha_1 - \frac{1}{2}\alpha_1^2)A(t) + \alpha_1 M(t)} + a_2 e^{\alpha_2 h^{(-1)}(x,0) + (\alpha_2 - \frac{1}{2}\alpha_2^2)A(t) + \alpha_2 M(t)},$$

where $A(t) = \int_0^t \lambda(s)^2 ds = \lambda^2 t$, $M(t) = \int_0^t \lambda(t) dW(t) = \lambda W(t)$.
We then, see that $X^*(t)$ can be written as

$$X^*(t) = X_1^*(t) + X_2^*(t),$$

with

$$X_i^*(t) = X_i(0)e^{(\alpha_i - \frac{1}{2}\alpha_i^2)A(t)+\alpha_i M(t)} \quad \text{and} \quad X_i(0) = a_i e^{\alpha_i h^{(-1)}(x,0)}$$

The processes $X_i^*(t)$, $i = 1, 2$, are then the optimal wealth process of a CRRA investor, with relative risk tolerance α_i, and initial wealth $X_i^*(0)$. Hence, we obtain the following intuitive interpretation of the optimal behavior arising from the two point measure (2.18). *At $t = 0$, it is as if the investor splits his initial wealth into $X_1^*(0)$ and $X_2^*(0)$ and puts them into two separate trading accounts. He then acts as an individual CRRA agent in account i, with risk tolerance α_i, $i = 1, 2$.*

The above analysis is easily generalized and applies to any discrete measure ν. Simply put, the optimal strategy of a forward performance process is a *static* combination of different CRRA strategies. The measure ν describes different degrees of risk tolerances in the investor's mind, as well as their respective weights. Altogether, they determine the types of CRRA subportfolios and the initial wealth allocated to each one of them. The logic holds in the reversed direction as well, in that if there are two CRRA investors with relative risk tolerances α_1 and α_2, then their optimal portfolios combined can be considered as the optimal portfolio of a single investor whose preference is described by (2.18).

The above are summarized in the following theorem.

Theorem 2.2.3 (Separation and Aggregation)**.**

(i) *Let $X^*(t)$ denote the optimal wealth process of a forward investor with* *performance generating measure ν. Assume that ν can be written as a sum of measures,*

$$\nu(dy) = \sum_{i=1}^{n} \nu_i(dy),$$

with the support of each ν_i being a subset of $[0, \infty)$. Then $X^*(t) = \sum_{i=1}^{n} X_i^*(t)$, with $X_i^*(t)$ being the optimal wealth process of an investor with initial wealth,

$$X_i(0) = \int_0^\infty e^{yh^{(-1)}(X(0),0)} \nu_i(dy),$$

and whose performance process is generated by ν_i.

(ii) *Conversely, let $X_i^*(t)$, $i = 1, 2, \ldots, n$ be the optimal wealth processes of investors with individual performance generating measures ν_i. Then, the combined wealth process,*

$$X^*(t) = \sum_{i=1}^{n} X_i^*(t)$$

is the optimal wealth process of an investor with initial wealth $X(0) = \sum_{i=1}^{n} X_i(0)$, and performance generating measure,

$$\nu(dy) = \sum_{i=1}^{n} e^{yh_i^{(-1)}(X_i(0),0)} \nu_i(dy).$$

Proof. See Appendix A.1. □

The view that the forward investor manages an individual account for each risk tolerance included in the support of ν is in striking resemblance to the behavioral portfolio theory of Shefrin and Statman (2000) and the mental accounts framework of Das et al. (2010). Therein, the authors postulated that investors often have many attitudes toward risk, and they consider their portfolios as collections of mental accounting subportfolios. The investor acts as if he cared about the risk and return of each subportfolio individually. The difference of our work lies in the choice of preference at the subportfolio level. In Das et al. (2010), the risk of each mental account is perceived as the probability of not reaching a predetermined target. In contrast, in the forward approach, the subportfolio risk is evaluated by a single CRRA utility.

2.2.3 The measure ν satisfies $\nu((-\infty, 0)) \neq 0$.

Now we consider the case when the support of ν contains both positive and negative values. In this case, the conclusions in theorem 2.2.3 still hold. However, the interpretation needs to be modified due to the "negative part" of ν. Consider the following two-point example,

$$\nu = a_1\delta_{\{\alpha_1\}} - a_2\delta_{\{-\alpha_2\}}, \tag{2.20}$$

here $a_i, \alpha_i \in \mathbb{R}^+$. By Theorem 2.2.3 we know that the investor acts as a CRRA agent for two subportfolios individually. Now, the investor actually borrows $X_2^*(0)$ dollars from the α_2 account, and invest them along with $X(0)$ into the α_1-account, which he then optimizes by maintaining $\frac{\mu}{\sigma^2}\alpha_1$ percent invested in the stock. For the second account which he owes money, he purchases stock by further borrowing, in the amount of $\frac{\mu}{\sigma^2}\alpha_2$ percent of the current debt. The strategy poses greater downside risk, since the loss from the α_2-account is potentially unlimited.

2.3 The relative risk tolerance process

We have provided a direct and intuitive interpretation of the optimal investment strategy under forward performance processes for discrete additive measures. The interpretation has yet another valuable implication. Indeed, if the forward optimal portfolios is a static combination of CRRA subportfolios, and because each subportfolio invests a different proportion in the stock, a change in stock price leads to different changes in the subportfolio wealth, which in turn changes the overall stock proportion in the combined portfolio. Therefore, as long as the collection of risk tolerance parameters (along with their weights) are known, it is possible to describe exactly how the optimal portfolio strategy $\tilde{\pi}^*(t)$ varies over time. To pursue this idea, next we derive a system of stochastic differential equations which completely describes the time-evolution of the relative risk tolerance process (which is essentially the same as $\tilde{\pi}^*$ up to a constant multiplicative factor). Such an SDE system is desirable for two reasons. Firstly, it is essential for a long term portfolio manager to understand how his wealth is time-diversified. This notion will be further explored in the next chapter. Secondly, the SDE system provides a universal computational tool which, as we will show, applies beyond the log-normal market assumption. For the subsequent analysis, we do not distinguish the two types of measure ν as we did in the previous section, since the SDE systems are the same for both cases.

From (2.12), we know that $\tilde{\pi}^*(t) = \frac{\mu}{\sigma^2}\tilde{R}^*(t)$, where $\tilde{R}^*(t)$ is the relative risk tolerance process at the optimum, defined by

$$\tilde{R}^*(t) = \frac{r(X^*(t), A(t))}{X^*(t)}.$$

Here $r(x,t)$ is the absolute risk tolerance function as defined in (2.9). We can verify from (2.8) that,

$$r(x,t) = h_x(h^{(-1)}(x,t),t).$$

More explicitly, (2.11) and (2.17) imply that,

$$\tilde{R}^*(t) = \frac{\int_{\mathbb{R}} y e^{yh^{(-1)}(x,0)+(y-\frac{1}{2}y^2)A(t)+yM(t)}\nu(dy)}{\int_{\mathbb{R}} e^{yh^{(-1)}(x,0)+(y-\frac{1}{2}y^2)A(t)+yM(t)}\nu(dy)}.$$

The above solution provides a stochastic description of $\tilde{R}^*(t)$, but the dependence of $\tilde{R}^*$ on time is difficult to analyze. For example, we do not know how does $\tilde{R}^*(t)$ changes in behavior if, for example, the dynamics of the volatility process changes. What is lacking here is an equation that describes the stochastic evolution of $\tilde{R}^*(t)$. We start the analysis by deriving the following general SDE satisfied by $\tilde{R}^*(t)$.

Proposition 2.3.1. *The relative risk tolerance process $\tilde{R}^*(t)$ satisfies the stochastic differential equation,*

$$d\tilde{R}^*(t) = \lambda R^*(t)\tilde{r}_x(X^*(t), A(t))\big(\lambda(1-\tilde{R}^*(t))dt + dW(t)\big), \tag{2.21}$$

with $\tilde{r}(x,t) = \frac{r(x,t)}{x}$ being the relative risk tolerance function.

Proof. See Appendix A.2. □

Still, equation (2.21) is not readily applicable for solving computational problems as it involves the additional state variable $\tilde{r}_x(X^*(t), A(t))$. To derive a full SDE system for $\tilde{R}^*(t)$, we need to figure out all the state variables involved. From the previous section, we have seen that $X^*(t)$ is a static combination of multiple CRRA subportfolios, and the initial ν specifies all the risk tolerance parameters along with their relative weights at time 0. At $t > 0$, as the wealth level in each subportfolio changes, the distribution of risk tolerances will be different from that at time 0. Hence, to fully characterize the investor's optimal strategy, it is necessary to describe the dynamics of the entire risk tolerance distribution. We now make the important observation that $\tilde{R}^*(t)$ is exactly the "average risk tolerance" (or the first moment) at t.

Assume that ν is given by (2.18), the two point measure. Then, at each time t, the investor allocates wealth to two CRRA subportfolios $X_1^*(t)$ and $X_2^*(t)$, with

$$\begin{cases} dX_1^*(t) = \alpha_1 X_1^*(t)(\lambda^2 dt + \lambda dW(t)), \\ dX_2^*(t) = \alpha_2 X_2^*(t)(\lambda^2 dt + \lambda dW(t)). \end{cases} \tag{2.22}$$

The amount of wealth allocated to S_t is then given by

$$\pi^*(t) = \frac{\mu}{\sigma^2}(\alpha_1 X_1^*(t) + \alpha_2 X_2^*(t)).$$

Hence, the relative risk tolerance process is given by,

$$\tilde{R}^*(t) = \frac{\sigma^2}{\mu}\frac{\pi^*(t)}{X^*(t)} = \alpha_1 \frac{X_1^*(t)}{X^*(t)} + \alpha_2 \frac{X_2^*(t)}{X^*(t)}.$$

Then, $\tilde{R}(t)$ can be interpreted as the "average α" at t, weighted by their respective proportion in the entire portfolio. Since the weights sum up to one, we can, at least informally, think of them as probabilities, and consider a hypothetical random variable, denoted by $Y(t)$, such that $Y(t)$ satisfies $\text{Prob}(Y(t) = \alpha_i) = \frac{X_i^*(t)}{X^*(t)}$, $i = 1, 2$. With this new notation, $\tilde{R}^*(t)$ can be simply considered as the mean of $Y(t)$.

To rigorously define this random variable $Y(t)$ under a general measure ν, we work as follows. Define the process

$$D(t) := h^{(-1)}(X^*(t), A(t)) = h^{(-1)}(x, 0) + M(t) + A(t).$$

Then, $\tilde{R}^*(t)$ can be written as,

$$\tilde{R}^*(t) = \frac{\int_{\mathbb{R}} y e^{yD(t) - \frac{1}{2}y^2 A(t)} \nu(dy)}{\int_{\mathbb{R}} e^{yD(t) - \frac{1}{2}y^2 A(t)} \nu(dy)}. \tag{2.23}$$

Let $\Omega^y = \text{supp}(\nu)$, and $\mathcal{N}$ the collection of all Borel subsets of Ω^y. We define the product measure $\mathbb{P}^{y,W,t}$ on the space $(\Omega \times \Omega^y, \mathcal{F}_t \otimes \mathcal{N})$,

$$\mathbb{P}^{y,W,t}(F \times N) = \int_F \Big(\int_N \mathbb{P}^{y,\omega,t}(dy) \Big) \mathbb{P}(d\omega),$$

for each $F \in \mathcal{F}_t$ and $N \in \mathcal{N}$, with the conditional density $\mathbb{P}^{y,\omega,t} = \mathbb{P}^{y,W_t}|\mathcal{F}_t$ given by,

$$\mathbb{P}^{y,\omega,t}(dy) = \frac{e^{yD(t) - \frac{1}{2}y^2 A(t)} \nu(dy)}{\int_{\mathbb{R}} e^{yD(t) - \frac{1}{2}y^2 A(t)} \nu(dy)}. \tag{2.24}$$

Let $Y(t)$ denote the random variable defined by the probability measure $\mathbb{P}^{y,W,t}$ on the product space $\Omega \times \Omega^y$. Then we see that $\tilde{R}^*(t)$ is the conditional mean,

$$\tilde{R}^*(t) = \mathbb{E}[Y(t)|\mathcal{F}_t].$$

The previous SDE of $\tilde{R}^*(t)$ can now be rewritten using the language of $Y(t)$. To this end, we first notice that,

$$r(x,t)\tilde{r}_x(x,t) = r(x,t)\frac{\partial}{\partial_x}\Big(\frac{r(x,t)}{x}\Big) = \frac{1}{x} h_{xx}(h^{(-1)}(x,t),t) - (\tilde{r}(x,t))^2.$$

Therefore,

$$R^*(t)\tilde{r}_x(X^*(t), A(t)) = \frac{\int_{\mathbb{R}} y^2 e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)}{\int_{\mathbb{R}} e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)} - (\tilde{R}^*(t))^2. \tag{2.25}$$

Note, however, that the right hand side can be written as,

$$\mathbb{E}[(Y(t))^2|\mathcal{F}_t] - (\mathbb{E}[Y(t)|\mathcal{F}_t])^2,$$

which is the conditional variance of $Y(t)$.

We summarize the above observations below.

Proposition 2.3.2. *Let $Y(t)$ be the random variable defined by the probability measure $\mathbb{P}^{y,W,t}$. Then,*

$$\tilde{R}^*(t) = \mathbb{E}[Y(t)|\mathcal{F}_t].$$

Moreover, $\tilde{R}^*(t)$ satisfies the SDE,

$$d\tilde{R}^*(t) = \lambda\, Var(Y(t)|\mathcal{F}_t)\big(\lambda(1-\tilde{R}^*(t))dt + dW(t)\big). \tag{2.26}$$

Example 1. Consider again the case, $\nu = a_1\delta_{\{\alpha_1\}} + a_2\delta_{\{\alpha_2\}}$. Then

$$\mathrm{Var}(Y|\mathcal{F}_t) = (\alpha_1 - \mathbb{E}[Y_t|\mathcal{F}_t])(\mathbb{E}[Y_t|\mathcal{F}_t] - \alpha_2).$$

Therefore by Proposition 2.3.2,

$$d\tilde{R}^*(t) = \lambda(\alpha_1 - \tilde{R}^*(t))(\tilde{R}^*(t) - \alpha_2)\big(\lambda(1-\tilde{R}^*(t))dt + dW(t)\big).$$

The above example is special in that $\tilde{R}^*(t)$ is completely described by a single SDE, since a two-outcome distribution is determined by its first moment.

If, however, $\nu = a_1\delta_{\{\alpha_1\}} + a_2\delta_{\{\alpha_2\}} + a_3\delta_{\{\alpha_3\}}$, then $\mathrm{Var}(Y|\mathcal{F}_t)$ can no longer be determined by $\tilde{R}^*(t)$ alone. In general, higher moments will be involved to describe the conditional distribution $\mathbb{P}^{y,\omega,t}$, hence a system of SDE's will be required to fully characterize the dynamics of $\tilde{R}^*(t)$. We work on this in the sequel.

To generalize Proposition 2.3.2, let $h^{(n)}(x,t)$ denote the n-th order derivative of h w.r.t. the spatial variable, and define the stochastic process $R_n(t)$,

$$R_n(t) = h^{(n)}(h^{(-1)}(X^*(t), A(t)), A(t)) = \int_{\mathbb{R}} y^n e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy). \tag{2.27}$$

The following result provides a surprising iterative connection among the processes $R_1(t)$, $R_2(t)$, ...

Lemma 2.3.3. *For $\forall\ n \geq 0$, $R_n(t)$ solves the following SDE,*

$$dR_n(t) = R_{n+1}(t)(\lambda^2 dt + \lambda dW(t)). \tag{2.28}$$

Proof. See Appendix A.2. □

Similarly, define the relative counterpart of $R_n(t)$,

$$\tilde{R}_n(t) = \frac{R_n(t)}{X^*(t)} = \frac{R_n(t)}{R_0(t)} = \frac{\int_{\mathbb{R}} y^n e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)}{\int_{\mathbb{R}} e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)}. \tag{2.29}$$

Using our previous notation of $Y(t)$, we readily deduce that, $\tilde{R}_n(t)$ coincides with the n-th moment $Y(t)$, namely,

$$\tilde{R}_n(t) = \mathbb{E}[(Y(t))^n|\mathcal{F}_t].$$

We have, trivially, the zero-th moment $\tilde{R}_0(t) = 1$. Also, by our previous discussion, the first moment is nothing other than the relative risk tolerance process $\tilde{R}^*(t)$, i.e. $\tilde{R}_1(t) = \tilde{R}^*(t)$. Therefore, the following system for $\tilde{R}_n(t)$, $n = 0, 1, 2, \ldots$, is a system that contains $\tilde{R}^*(t)$.

Proposition 2.3.4. *The process $\tilde{R}_n(t)$ defined in* (2.29) *satisfies, for $n \geq 0$, the SDE*

$$d\tilde{R}_n(t) = \lambda(\tilde{R}_{n+1}(t) - \tilde{R}^*(t)\tilde{R}_n(t))(\lambda(1 - \tilde{R}^*(t))dt + dW(t)). \tag{2.30}$$

Proof. Assertion (2.30) follows from (2.28) and Ito's lemma. □

Equation (2.30) formulates an SDE system, for $n = 0, 1, 2, \ldots$, satisfied by the moment processes, required in general to fully characterize the behavior of $\tilde{R}^*(t)$. However, when the conditional distribution $\mathbb{P}^{y,\omega,t}$ can be characterized by finitely many moments, then (2.30) can be reduced into a finite system. For tractability, we assume that the measure ν is given as a finite sum of Dirac masses,

$$\nu = a_1\delta_{\{\alpha_1\}} + a_2\delta_{\{\alpha_2\}} + \ldots + a_n\delta_{\{\alpha_n\}}. \tag{2.31}$$

Then the measure $\mathbb{P}^{y,\omega,t}$ defined in (2.24) is a discrete probability measure which has the support $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. We now show how to obtain a closed system including $\tilde{R}^*(t)$ with only finitely many equations.

Proposition 2.3.5. *Let Y be a discrete random variable with the set of outcomes $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and let $p_i = Prob\,(Y = \alpha_i)$. Denote by $Z_n = \sum_{i=1}^n p_i\alpha_i^n$*

the n-th moment of Y. The following moment equality holds,

$$Z_{n+l} = q_0 Z_l + q_1 Z_{l+1} + \dots q_{n-1} Z_{l+n-1}, \quad \textit{for} \quad l = 0, 1, \dots \tag{2.32}$$

where,

$$q_{n-k} = (-1)^{k+1} \sum_{1 \le i_1 < \dots < i_k \le n} \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_k}. \tag{2.33}$$

Proof. See Appendix A.2. □

Hence, for a discrete distribution with n outcomes, linear combinations of moments up to order $n-1$ can generate *all higher* order moments. Furthermore, the involved linear relations *do not depend* on the probabilities p_i. Applying this result to the moment SDE system corresponding to the discrete measure ν yields the system

$$\begin{cases} d\tilde{R}^*(t) & = \lambda(\tilde{R}_2(t) - (\tilde{R}^*(t))^2)(\lambda(1-\tilde{R}^*)dt + dW(t)) \\ d\tilde{R}_2(t) & = \lambda(\tilde{R}_3(t) - \tilde{R}^*(t)\tilde{R}_2(t))(\lambda(1-\tilde{R}^*)dt + dW(t)) \\ & \vdots \\ d\tilde{R}_{n-2}(t) & = \lambda(\tilde{R}_{n-1}(t) - \tilde{R}^*(t)\tilde{R}_{n-2}(t))(\lambda(1-\tilde{R}^*)dt + dW(t)) \\ d\tilde{R}_{n-1}(t) & = \lambda(\sum_{i=0}^{n-1} q_i \tilde{R}_i(t) - \tilde{R}^*(t)\tilde{R}_{n-1}(t))(\lambda(1-\tilde{R}^*)dt + dW(t)). \end{cases} \tag{2.34}$$

We summarize the above analysis below.

Proposition 2.3.6. *Let $\tilde{\boldsymbol{R}}(t) = (\tilde{R}_0(t), \tilde{R}_1(t), \dots, \tilde{R}_{n-1}(t))^T$ denote the moment vector process corresponding to the measure*

$$\nu = \sum_{i=1}^{n} a_i \delta_{\{\alpha_i\}}.$$

Then, $\tilde{\boldsymbol{R}}(t)$ *satisfies the SDE,*

$$d\tilde{\boldsymbol{R}}(t) = \lambda\big(N_\alpha - \tilde{R}^*(t)I\big)\tilde{\boldsymbol{R}}(t)(\lambda(1-\tilde{R}^*)dt + dW_t), \tag{2.35}$$

where I *is the* $n \times n$ *identity matrix, and* N_α *is defined by,*

$$N_\alpha = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ q_0 & q_1 & \dots & q_{n-2} & q_{n-1} \end{bmatrix}$$

with the q_i*'s given in* (2.33)*.*

Proof. Since $Y(t)$ is a discrete random variable with support $\{\alpha_1, \dots, \alpha_n\}$, the result follows directly from Proposition 2.3.4 and Lemma 2.3.5. □

Example 2. Let $n = 3$ and $\nu = a_1\delta_{\{\alpha_1\}} + a_2\delta_{\{\alpha_2\}} + a_3\delta_{\{\alpha_3\}}$. Then, from (2.35) we deduce that the processes $\tilde{R}^*(t)$, $\tilde{R}_2(t)$ satisfy the system,

$$\begin{aligned} d\tilde{R}^*(t) =& \lambda(\tilde{R}_2(t) - (\tilde{R}^*(t))^2)(\lambda(1-\tilde{R}^*)dt + dW(t)) \\ d\tilde{R}_2(t) =& \lambda\big((\alpha_1+\alpha_2+\alpha_3 - \tilde{R}^*(t))\tilde{R}_2(t) + \alpha_1\alpha_2\alpha_3 \\ & - (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3)\tilde{R}^*(t)\big)(\lambda(1-\tilde{R}^*)dt + dW(t)). \end{aligned} \tag{2.36}$$

Although Proposition 2.35 provides a finite closed SDE system which includes the process $\tilde{R}^*(t)$ that we want to study, SDE's such as (2.36) are still difficult to analyze. An alternative approach is to describe the dynamics of $\tilde{R}^*(t)$ through a "factor model". For the case of a discrete measure ν, we can then define the "probability process",

$$p_i(t) = \frac{\alpha_i e^{\alpha_i D(t) - \frac{1}{2}\alpha_i^2 A(t)}}{\sum_{i=1}^n e^{\alpha_i D(t) - \frac{1}{2}\alpha_i^2 A(t)}}, \tag{2.37}$$

i.e. $p_i(t) = \text{Prob}(Y(t) = \alpha_i | \mathcal{F}_t)$. The real meaning of $p_i(t)$ is the proportion of wealth the investor allocates to the CRRA subportfolio with risk tolerance α_i. By the definition of $Y(t)$,

$$\tilde{R}^*(t) = \mathbb{E}[Y(t)|\mathcal{F}_t] = \sum_{i=1}^n \alpha_i p_i(t). \tag{2.38}$$

Thus, if we can find a system of SDE's satisfied by $p_i(t)$, then we have a complete characterization of the optimal process $\tilde{R}^*(t)$.

Corollary 2.3.7. *Let $\boldsymbol{p}(t) = (p_1(t), \ldots, p_n(t))^T$ denote the vector of probability processes defined in* (2.37)*. Then, $\boldsymbol{p}(t)$ solves the SDE,*

$$d\boldsymbol{p}(t) = \lambda\big(D_\alpha - \tilde{R}^*(t)I\big)\boldsymbol{p}(t)(\lambda(1 - \tilde{R}^*)dt + dW(t)), \tag{2.39}$$

with $\tilde{R}^*(t) = \sum_{i=1}^n \alpha_i p_i(t)$, (cf. (2.38)*).*

Proof. See Appendix A.2. □

Example 2 (Con'd) We have

$$\tilde{R}^*(t) = \alpha_1 p_1(t) + \alpha_2 p_2(t) + \alpha_3 p_3(t) = (\alpha_1 - \alpha_3)p_1(t) + (\alpha_2 - \alpha_3)p_2(t) + \alpha_3,$$

with

$$\begin{cases} dp_1(t) &= \lambda p_1(t)(\alpha_1 - \tilde{R}^*(t))(\lambda(1 - \tilde{R}^*(t))dt + dW(t)) \\ dp_2(t) &= \lambda p_2(t)(\alpha_2 - \tilde{R}^*(t))(\lambda(1 - \tilde{R}^*(t))dt + dW(t)). \end{cases} \tag{2.40}$$

2.4 Performance and risk measures

For investment managers who construct portfolios based on forward performance criteria, it is essential that they are able to obtain estimates of the risk and return of their portfolios at targeted horizons. We now apply the tools developed in the previous sections to compute various quantities of interest to measure performance as well as for risk management purposes.

We recall that the optimal wealth process is given by

$$X^*(t) = \int_{\mathbb{R}} e^{(h^{(-1)}(x,0)+M(t)+A(t))y - A(t)\frac{1}{2}y^2} \nu(dy).$$

Based on the above, Musiela and Zariphopoulou (2010a) have derived the cumulative distribution for $X^*(t)$,

$$F(z) = \mathbb{P}\left(X^*(t) \leq z\right) = \mathcal{N}\left(\frac{h^{(-1)}(z, A(t)) - h^{(-1)}(x,0) - A(t)}{\sqrt{A(t)}}\right), \qquad (2.41)$$

where $\mathcal{N}(\cdot)$ is the cdf for a standard normal distribution. Thus, various risk measures can be evaluated by integrating the above distribution with an objective function $G(z)$,

$$\mathbb{E}[G(X^*(T)] = \int_{\mathbb{R}} G(z) dF(z).$$

For example, to obtain the variance we set $G(z) = z^2$. For conditional value at risk (CVaR) at the d-th percentiles, we set $G(z) = z\mathbb{1}_{\{F(z)\leq d\}}$.

However, there are several drawbacks that render the explicit approach less useful. Firstly, evaluating the distribution function requires numerically inverting the function $h(x,t)$, which can be expensive as h itself is given as an integral w.r.t. to the measure ν. More importantly, equation (2.41) is only valid under the assumption that market parameters are constant. Furthermore, there is no immediate way to generalize (2.41) if the market price of risk, λ, is a stochastic process driven by a Brownian motion correlated with $W(t)$. On the other hand, the SDE system derived in the last section does not rely on any assumption made about λ, hence it provides a more flexible computational tool. In addition to calculating $X^*(t)$, our approach also enables various calculations around the relative risk tolerance process $\tilde{R}^*(t)$, which is ignored by most, if not all previous literature in portfolio management.

2.4.1 Calculating performance measures under random parameters

Next we revert to a more general market environment with time varying parameters. The price of the risky asset follows,

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t),$$

where $\mu(t)$ and $\sigma(t)$ are stochastic processes, such that the market price of risk $\lambda(t) = \frac{\mu(t)}{\sigma(t)}$ (again the risk-free rate is set at 0) follows the SDE below

$$d\lambda(t) = b(\lambda(t))dt + a(\lambda(t))dW^{\lambda}(t).$$

Here, a, b are known Lipschtz functions of λ. The instantaneous correlation between S and λ is given by,

$$dW(t)dW^{\lambda}(t) = \rho dt,$$

The optimal wealth then follows,

$$dX^*(t) = X^*(t)\tilde{R}^*(t)\big((\lambda(t))^2dt + \lambda(t)dW(t)\big).$$

Notice that the SDE system (2.30) derived in the last section does not rely on the assumption that market is log-normal. Hence we still have, for $n = 1, 2, \ldots$,

$$d\tilde{R}_n(t) = \lambda(t)(\tilde{R}_{n+1}(t) - \tilde{R}^*(t)\tilde{R}_n(t))(\lambda(t)(1 - \tilde{R}^*(t))dt + dW(t)), \quad (2.42)$$

with $\tilde{R}_1(t) = \tilde{R}^*(t)$. We therefore have the *joint dynamics* for $X^*(t)$ and $\tilde{R}^*(t)$.

Theorem 2.4.1. *The optimal wealth process $X^*(t)$ and the relative risk tolerance process $\tilde{R}^*(t) = \tilde{R}_1(t)$ satisfy the SDE,*

$$\begin{cases} dX^*(t) = X^*(t)\tilde{R}_1(t)\big((\lambda(t))^2dt + \lambda(t)dW(t)\big) \\ d\tilde{R}_n(t) = \lambda(t)(\tilde{R}_{n+1}(t) - \tilde{R}^*(t)\tilde{R}_n(t))(\lambda(t)(1 - \tilde{R}^*(t))dt + dW(t)), \quad n = 1, 2, \ldots \\ d\lambda(t) = b(\lambda(t))dt + a(\lambda(t))dW^\lambda(t), \end{cases} \quad (2.43)$$

where $\tilde{R}_n(t)$ $n \geq 2$, are higher moment processes of the (conditional) risk tolerance distribution, defined in (2.29).

Note that if one is only interested in evaluating moments of the form $\mathbb{E}[G(X^*(T)]$, then it is more convenient to write SDE system (2.43) in terms of the absolute risk tolerance process $R^*(t)$. Indeed, recall that equation (2.27) defined the unnormalized moments $R_n(t)$, which solve the recursive SDE system,

$$dR_n(t) = R_{n+1}(t)\big((\lambda(t))^2dt + \lambda(t)dW(t)\big).$$

By definition we have that, $R_0(t) = X^*(t)$, $R_1(t) = R^*(t)$. Hence, (2.43) can be rewritten as,

$$\begin{cases} dR_n(t) = R_{n+1}(t)\big((\lambda(t))^2dt + \lambda(t)dW(t)\big), \quad n = 0, 2, \ldots \\ d\lambda(t) = b(\lambda(t))dt + a(\lambda(t))dW^\lambda(t). \end{cases} \quad (2.44)$$

In general, neither (2.43) or (2.44) is applicable for computational purposes as the number of equations is infinite. However, for the case that ν is a discrete measure (2.31), the moment relation of a discrete distribution can reduce the infinite system (2.28) to one with finite size. We have shown that the moment restriction holds for $R_n(t)$,

$$R_{n+l} = q_0 R_l + q_1 R_{l+1} + \ldots + q_{n-1} R_{l+n-1} \quad l = 0, 1, \ldots,$$

where q_i's are given in lemma (2.3.5). Hence, the system (2.44) only contains n equations. By Feynman Kac's theorem, the problem of evaluating expectations of $G(X^*(T))$ amounts to solving a system of partial differential equations.

Proposition 2.4.2. *Let $\boldsymbol{R}^*(t)$ denote the moment vector process $[R_0(t), R_1(t), \ldots, R_{n-1}(t)]^T$, and $g(\boldsymbol{r}, \lambda, t)$ denote the conditional expectation,*

$$g(\boldsymbol{r}, \lambda, t) = \mathbb{E}[G(X^*(T))|\boldsymbol{R}^*(t) = \boldsymbol{r}, \lambda(t) = \lambda].$$

Assume that ν is a discrete measure given by (2.31)*. Then, g solves the partial differential equation,*

$$\begin{aligned}
&\frac{\partial g}{\partial t} + \lambda^2 \Big(\sum_{i=0}^{n-2} r_{i+1} \frac{\partial g}{\partial r_i} + (\sum_{i=0}^{n-1} q_i r_i) \frac{\partial g}{\partial r_{n-1}} \Big) + b(\lambda) \frac{\partial g}{\partial \lambda} \\
&\quad + \frac{1}{2} \lambda^2 \Big(\sum_{i=0}^{n-2} \sum_{j=0}^{n-2} r_{i+1} r_{j+1} \frac{\partial^2 g}{\partial r_i \partial r_j} + \sum_{i=0}^{n-1} q_i r_i \sum_{i=0}^{n-2} r_{i+1} \frac{\partial^2 g}{\partial r_i \partial r_{n-1}} \\
&\quad + (\sum_{i=0}^{n-1} q_i r_i)^2 \frac{\partial^2 g}{\partial r_{n-1}^2} \Big) + \rho \lambda a(\lambda) \big(\sum_{i=0}^{n-2} r_{i+1} \frac{\partial^2 g}{\partial r_i \partial \lambda} + (\sum_{i=0}^{n-1} q_i r_i) \frac{\partial^2 g}{\partial r_{n-1} \partial \lambda} \big) \\
&\quad + \frac{1}{2} a(\lambda)^2 \frac{\partial^2 g}{\partial \lambda^2} = 0.
\end{aligned} \tag{2.45}$$

with $g(\boldsymbol{r}, \lambda, T) = G(r_0)$.

Proof. If the measure ν is given by (2.31), then (2.44) reduces to

$$d\mathbf{R}^*(t) = N_\alpha \mathbf{R}^*(t)\big(\lambda(t)^2 dt + \lambda(t) dW(t)\big),$$

where N_α is defined in Proposition 2.3.6. The rest follows from the Feynman Kac's formula. $\square$

Example Let $\nu = a_1\delta_{\{\alpha_1\}} + a_2\delta_{\{\alpha_2\}}$, then $q_0 = -\alpha_1\alpha_2$, $q_1 = \alpha_1 + \alpha_2$ and $\mathbf{R}^* = [R_0(t), R_1(t)]^T$ solves the SDE system,

$$\begin{cases} dR_0(t) = R_1(t)\big((\lambda(t))^2 dt + \lambda(t)dW(t)\big) \\ dR_1(t) = \big((\alpha_1+\alpha_2)R_1(t) - \alpha_1\alpha_2 R_0(t)\big)\big((\lambda(t))^2 dt + \lambda(t)dW(t)\big). \end{cases} \tag{2.46}$$

This generalizes Theorem 4.1 of Zariphopoulou and Zhou (2009), where the authors obtained the same SDE for the special case that $\alpha_1 = -\alpha_2$, $a_1 = -a_2$. Following (2.45), we can obtain the PDE for

$$g(r_0, r_1, \lambda, t) = \mathbb{E}[G(X^*(T)|X(t) = r_0, R_1(t) = r_1, \lambda(t) = \lambda].$$

Then, the second order parabolic PDE can be easily solved by numerical schemes such as finite-difference or finite-element. Furthermore, observe that the coefficients of (2.45) are polynomials up to second order, thus one can derive analytic approximations of $g(\mathbf{r}, \lambda, t)$ following, for example, the commutator method of Grischenko et al. (2014).

From a different perspective, asset managers also tend to use $\tilde{\pi}^*(t)$, the proportion of wealth invested in the stock, as a direct way to measure riskiness of an investment strategy. For example, the debate on the practice of target date fund management is essential on how to choose $\tilde{\pi}^*(t)$ as a (deterministic) function of time. In our case where $\tilde{\pi}^*$ is stochastic, we can calculate $\mathbb{E}[\tilde{\pi}^*(t)]$ by numerically solving the PDE derived from (2.43). A numerical example along these lines is presented in Chapter 3.

2.5 Summary

Although the theory of forward performance process was developed over a decade ago, the intuition for the structure of these stochastic risk preferences and the economic implications of the optimal investment strategies are still not well understood. Indeed, the "closed form" solution of $\tilde{\pi}^*$ derived from solving a time-reversed heat equation hardly provides any economic insight. The results in this section contribute to a better understanding of the criteria and the policies. Firstly, we show that in the absence of forward volatility, a general forward performance process coincides with a static combination of multiple (finite or infinite) CRRA preferences with different degrees of risk tolerance. Specifically the investor splits his investment into multiple subportfolios, and behaves as a CRRA optimizer within each such sub-investment problem. This feature shares many similarities with the *mental account* approach that behavior portfolio researchers have proposed.

Secondly, our work is the first that derives a complete SDE system which describes the dynamics of the distribution of the entire risk tolerance process. Furthermore, we show how to reduce the infinite system to a finite one when the risk tolerances are discretely distributed. A direct implication is that, various calculations for the optimal wealth process $X^*(t)$ and portfolio strategy $\tilde{\pi}^*(t)$, which as we show are essentially the zeroth and first moment of the risk tolerance distribution, are now placed under the universal framework of solving a particular second order parabolic equation. This approach applies even when the market opportunity set is stochastic and driven by a correlated source of randomness. As we will see in the next chapter, the SDE system also provides answer to a long debated empirical question: in a person's lifetime investment, whether the proportion of wealth invested in risky assets should increase of decrease according to the investor's age. Once more, it is not possible to answer this question by merely considering the "closed form" solution of $\tilde{\pi}^*$.

Chapter 3

Applications in lifecycle portfolio management

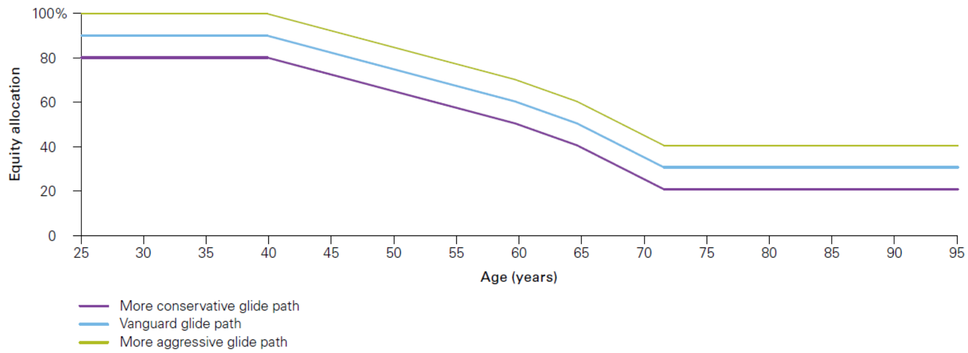
3.1 Background and literature review

In Samuelson (1969), the economist famously argued that investors' asset allocation should be time-independent, if the market return and volatility are constant over time. However, *lifecycle funds*, also called target date funds (henceforth TDF), as a segment of mutual funds which manage people's retirement savings, hold the alternative belief that the riskiness of one's portfolio should be age-based. People at younger age should invest more in high risk/high return assets. As people grow older, the need for capital preservation out weighs the need for growth. Thus, the portfolio should weigh more on less risky assets such as bonds and cash, and less on stocks.

In practice, a target date portfolio strategy is typically characterized by a "glide path", which is determined at portfolio inception and describes how the transition from stocks to bonds is carried out during the investment lifetime. Figure 3.1 illustrates a typical glide path adopted by the Vanguard target date fund. Initial equity allocation is set at 90%. At age 40, it starts to decline and eventually reaches 20% at the retirement age of 72.

We stress that the asset allocation plan prescribed by the glide path is static in nature. Once determined, the fund manager faithfully adheres to the glide

Figure 3.1: Vanguard target date fund glide paths



path without making adjustments based on the actual performance of the portfolio, non-anticipated changes in market conditions, etc.

Following the introduction of Pension Protection Act in 2006, target date retirement funds experienced exponential growth. By the end of 2015, the total asset under management had reached $763 billion. However, their practice did not live without controversy. In fact, the central idea of TDF strategies, that asset allocation should be shifted away from equities over time is heavily criticized by many. Economist Robert Shiller argued that age-based risk reduction exactly prevents people from benefiting at the right time from high returns of the stock market. In Shiller (2005), he wrote

> The lifecycle portfolio would be heavily in the stock market (in the early years) only for a relatively small amount of money, and would pull most of the portfolio out of the stock market in the very years when earnings are highest.

Shiller (2005) then showed through simulation that TDF strategies offer much lower expected returns than a 100% stock portfolio. While the latter only loses money 2% of the time, Basu and Drew (2009) argued that *contrarian strategies*, which increase equity allocations over time, actually generate far better risk-return profiles. Their simulation study showed that only when comparing the bottom 10 percentile outcomes, TDF strategies perform slightly better. Empirical evidence which support this argument was complied by Estrada (2015), who considered a comprehensive sample of 19 countries and two regions over 110 years, and discovered that contrarian strategies generally outperform all TDF strategies in terms of upside potentials, while at the same time, keeping the downside risk more limited. All the above literature suggests that the conventional wisdom seems to be misleading, in the sense that age-based risk reduction forgoes too much growth potential but does not offer enough downside protection in return. Therefore, the opposite approach, contrarian strategies, should be adopted instead.

The second type of criticism TDF strategies receive focuses on the static nature of glide paths. As argued by Basu et al. (2011), if it happens that the stock market declines right before the risk reduction kicks in, the investor will have no chance at all to recover. The authors advocated instead a dynamic portfolio strategy, a feedback glide path which only "glides" if the investor has achieved his capital growth target. From the risk budgeting perspective, Yoon (2010) also emphasized the importance of adopting a dynamic strategy, so that the term structure of risk can be properly taken into account.

Herein we develop a rigorous alternative approach for lifecycle portfolio design based on the theory of time-monotone forward performance processes. In section 2.1 we have seen that in the forward approach the notion of an investment horizon is no longer relevant, making it a reasonable theory for tackling problems in lifecycle portfolio management, which can generally be considered as infinite/flexible horizon problems. Moreover, the optimal portfolio strategies derived from the forward theory are genuinely dynamic, since the stock proportion at time t depends both on the current level of wealth and the market parameters estimated at t. Hence, the forward approach "reacts" to both changing market conditions and to realized portfolio returns.

Naturally we are interested in the implications of forward theory on the heavily debated issue. Should the proportion of wealth allocated to equities be a decreasing function of time? The answer is, provided that the investor's behavior satisfies the time-consistency condition, TDF style risk reduction is only justifiable if the investor is a high risk seeker.

As an example, assume that the stock market generates 6% return per year, with 20% volatility. It is well known that investors with log-utility would allocate $\frac{\mu}{\sigma^2} = 150\%$ of his wealth to the stock market. The forward theory implies that, only if the investor is at least as risky as the log-utility investor, should he shift allocations away from the stock market over time. This is clearly not the case for the glide path shown the in figure 3.1, as the stock proportion starts out at 90%, far below the 150% threshold, and yet the stock proportion glides downward over the years. On the other hand, the forward theory supports the argument in the literature that the contrarian strategy is the more reasonable alternative. If the investor is at least as risk-averse as the log-utility investor, his stock allocation should be a non-decreasing function of time.

The case of investors with mixed attitude toward risk is more complicated to analyze. Suppose, for example, that an investor starts out with \$10000 wealth to invest, allocates \$1000 to a high risk hedge fund with above 200% leverage and invests the rest with a conservative asset manager with leverage below 1. Then, his overall (expected) stock proportion is no longer a monotonic function of time. In fact, numerical tests suggest that it will increase in the first couple years and then will decrease for the rest of the investment lifetime. Therefore, investors of this type would not adopt either the glide path or the contrarian approaches.

In the next section, we start by illustrating through a simple example that asset allocation under a time-monotone forward performance criterion may systematically shift over time. Hence the theory has implications for the practice of mutual funds, such as lifecycle funds, which manage investors' savings for very long horizons. The result is then rigorously proved using the argument developed in the previous chapter. We also provide in section 3.3 two numerical examples that show the diversity of behaviors the forward theory is able to generate. Section 3.4 discusses the problem of preference calibration. The method is then applied to finding the forward utility implied by the Vanguard target retirement fund glide path. Section 3.5 concludes.

3.2 Time-dependent asset allocations under the time-monotone forward performance criterion

We treat the lifetime asset allocation problem as a dynamic portfolio problem with *undefined horizon*. For tractability, we assume the market consists of a log-normal stock market index $S(t)$, with,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

and a risk-free asset with zero interest rate. The investor starts with initial wealth x, and trades continuously between the two accounts. We assume that the investor optimizes the forward performance process $U(x,t) = u(x, A(t))$, derived from his initial utility $u(x,0)$ (see section 2.1 for the exact definition). As it was mentioned in the previous chapter, there are, in general, two types of appropriate forward utilities. By Proposition 2.2.2, the types can be determined by the range of the function $h(x,t)$ (which is defined by (2.17)).

Type 1: Range$(h) = (C, \infty)$, where C is the point mass the measure ν assigns to the point 0.

Type 2: Range$(h) = (-\infty, \infty)$.

The first type can be considered as forward investment with no-bankruptcy constraint since the optimal wealth can never fall below the threshold level C. In this section, we only consider investors of type 1. Equivalently, we make the following assumption on ν,

Assumption 3.2.1.

$$\text{supp}(\nu) \subset [0, \infty)$$

As an example, if ν is a single Dirac mass,

$$\nu = \delta_{\{\alpha\}}, \quad \alpha \geq 0, \tag{3.1}$$

we have seen that the corresponding utility is CRRA and the optimal portfolio weight in S, $\tilde{\pi}^*(t) = \frac{\mu}{\sigma^2}\alpha$. The optimal asset allocation in this case is time-independent. However, the assumption on ν amounts to saying that the investor's risk preferences can be described by a single risk tolerance parameter. However, this is unlikely to be realistic, as empirical evidence from experimental psychologists indicates that the same person often exhibits different degrees of tolerance to risk in his financial decision making. For example, one may be very risk averse regarding his retirement plan investment, but much less so for the proportion of wealth allocated to get richer. For such reasons, we do not make any specific assumptions on ν, other than assuming that its support does not include any subset of the negative half-axis. To the best of our knowledge, the problem of lifetime asset allocation under a general utility has not been explored in existing literature.

The measure given by (3.1) is in fact, the only case that constant allocation occurs. To see how $\tilde{\pi}^*$ changes over time under ν different from a single Dirac mass, we consider the following simple generalization,

$$\nu = a_1\delta_{\{\alpha_1\}} + a_2\delta_{\{\alpha_2\}},$$

which results in a mixture of two power utilities. As discussed in section 2.2, $\tilde{\pi}^*$ coincides with a static combination of two CRRA portfolios. At $t = 0$, the investor splits his initial wealth x into $X_1(0)$ and $X_2(0)$, with $X_1(0) +$

$X_2(0) = x$. Then, he manages the $X_i(0)$, $i = 1, 2$ investments following the CRRA strategy with risk tolerance parameters α_i, $i = 1, 2$. Hence, we have $X^*(t) = X_1(t) + X_2(t)$, with

$$dX_i(t) = \alpha_i X_i(t)(\lambda^2 dt + \lambda dW(t)).$$

The optimal stock proportion is then given by the allocation

$$\tilde{\pi}^*(t) = \frac{\mu}{\sigma^2} \frac{\alpha_1 X_1(t) + \alpha_2 X_2(t)}{X_1(t) + X_2(t)}.$$

Obviously, $\tilde{\pi}^*(t)$ is no longer constant but, rather a time-dependent weighted average of α_1, α_2. To see how the weights change, we solve $X_i(t)$ explicitly obtaining,

$$X_i(t) = X_i(0)\exp\big((\alpha_i - \frac{1}{2}\alpha_i^2)A(t) + \alpha_i M(t)\big),$$

where $A(t) = \int_0^t |\lambda(s)|^2 ds$ and $M(t) = \int_0^t \lambda(s) dW(s)$.

We see that the speed of growth of $X_i(t)$ depends on the factor $\alpha_i - \frac{1}{2}\alpha_i^2$, which is higher when α_i is closer to 1. If we assume $1 \geq \alpha_1 > \alpha_2$, then $X_1(t)$ will outgrow $X_2(t)$ and, over time, α_1 will receive higher weight in $\tilde{\pi}^*$. In other words, $\tilde{\pi}^*(t)$ has a upward shifting trend. Similarly, if $\alpha_1 > \alpha_2 \geq 1$, then $X_2(t)$ outgrows $X_1(t)$ and $\tilde{\pi}^*(t)$ shifts downwards.

As $t \to \infty$, $\frac{\sigma^2}{\mu}\tilde{\pi}^*$ should converge to the α_i that is closer to 1. Therefore, when the initial utility falls outside of the CRRA class, asset allocation decision does not stay constant, and exhibit systematic changes as time increases.

To put the above discussion on rigorous ground, first note that under the log-normal market assumption, the change in $\tilde{\pi}^*(t) = \frac{\mu}{\sigma^2}\tilde{R}^*(t)$ comes solely from the change in $\tilde{R}^*(t)$, the relative risk tolerance process. Recall, from equation (2.26) that the process $\tilde{R}^*(t)$ satisfies the SDE,

$$d\tilde{R}^*(t) = \lambda \mathrm{Var}(Y(t)|\mathcal{F}_t)\big(\lambda(1-\tilde{R}^*(t))dt + dW(t)\big), \tag{3.2}$$

where the random variable $Y(t)$, defined by the conditional probability measure $\mathbb{P}^{y,\omega,t}$ (2.24), describes the time t distribution of the risk tolerance process. If ν is different from a single Dirac mass, $Y(t)$ is a non-constant random variable and the conditional variance must be positive. We have the following results.

Theorem 3.2.2. *Let supp(ν) denote the support of ν. Then the optimal risk tolerance process $\tilde{R}^*(t)$ is a submartingale if supp(ν) $\subset (0,1]$, while it is a supermartingale if supp(ν) is bounded and supp(ν) $\subset [1,\infty)$.*

Proof. See Appendix B.1. □

The intuition that $\tilde{R}^*(t)$ has a upward/downward trend is more clearly seen when ν is a discrete measure,

$$\nu = a_1\delta_{\{\alpha_1\}} + \ldots + a_n\delta_{\{\alpha_n\}}.$$

Under this assumption, the investor's optimal portfolio consists of n CRRA subportfolios. Let $p_i(t)$ denote the proportion of wealth at t allocated to the subportfolio corresponding to risk tolerance α_i. We have shown in (2.39) that $p_i(t)$ solves the SDE,

$$dp_i(t) = \lambda p_i(t)(\alpha_i - \tilde{R}^*(t))\big(\lambda(1-\tilde{R}^*(t)) + dW(t)\big).$$

Assume $\text{supp}(\nu) \subset [0,1]$, i.e. $\alpha_i \leq 1$ for all i. Therefore at any $t > 0$, $p_i(t)$ has a positive drift if $\alpha_i > \tilde{R}^*(t)$, and a negative drift if $\alpha_i < \tilde{R}^*(t)$. In other words, proportions of those subportfolio with large risk tolerance tend to grow, while proportions of less risky subportfolios tend to decline. The net result is that, the process $\tilde{R}^*(t)$, which represents the mean of the risk tolerance distribution at t, will always trend upward.

It is then interesting to ask about the behavior of $\tilde{R}^*(t)$ as $t \to \infty$. The problem is difficult to solve in general. But in the case that ν is a discrete measure, we have definitive answers.

Theorem 3.2.3. *Let ν be a discrete measure,*

$$\nu = a_1\delta_{\{\alpha_1\}} + \ldots + a_n\delta_{\{\alpha_n\}}, \quad \textit{with} \quad \alpha_1 \geq 0, \ldots, \alpha_n \geq 0$$

Assume that for $\forall\ i \neq j$, $\alpha_i \neq \alpha_j$ and $|1-\alpha_i| \neq |1-\alpha_j|$.

(i) Let $\underline{\alpha}_i \in \{\alpha_1, \ldots, \alpha_n\}$ be such that $|1-\alpha_i| > |1-\underline{\alpha}_i|$, for any $i = 1, \ldots, n$. Then,

$$\tilde{R}^*(t) \xrightarrow{p} \underline{\alpha}_i, \ \textit{as}\ t \to \infty,$$

where $\xrightarrow{p}$ denotes convergence in probability.

(ii) Furthermore, if $\text{supp}(\nu) \subset [0,1]$ or $\text{supp}(\nu) \subset [1,\infty)$, then,

$$\tilde{R}^*(t) \xrightarrow{a.s.} \underline{\alpha}_i, \ \textit{as}\ t \to \infty,$$

Proof. See Appendix B.1. $\square$

Remark 3.2.4. The same conclusion cannot be obtained for type 2 investors, because in that case $\tilde{R}^*(t)$ is no longer bounded by the support of ν and the integrability condition does not hold.

To summarize, our study provides answers to some important questions raised by economists and investment professionals. Should equity allocation, as a proportion of total wealth, increase or decrease with the investor's age? Our findings are twofold. Firstly, deterministic (constant) allocation is only optimal for CRRA investors. That is, investors whose risk attitude can be represented by a single relative risk tolerance parameter. For all other investors however, the optimal allocations are feedback policies, which depend on the realized returns.

Secondly, the dynamic allocation policies have a systematic trend to shift upward or downward given that the investor's risk tolerance parameters are all larger or smaller than 1. Therefore, a dynamic "glide path" is only justifiable if all the investor's CRRA subportfolios are riskier than the log-utility portfolio. Practically, this is unlikely the case given that the log-utility portfolio is widely regarded as highly risky. It is more reasonable to assume that $\text{supp}(\nu) \subset [0, 1]$, which then supports the argument that the "contrarian" strategy,i.e. $\tilde{\pi}^*(t)$ is increasing in t, is more suitable for the majority of the investors.

The only case left unexamined is when $\text{supp}(\nu) \cap [0, 1) \neq \emptyset$ and $\text{supp}(\nu) \cap (1, \infty] \neq \emptyset$, i.e. the investor is partially more risk seeking while partially more risk averse than the log-utility investor. In this case, there is no definitive trend for $\tilde{\pi}^*(t)$ as $\tilde{R}^*(t)$ takes values both above and below 1. Moreover, numerical tests suggest that the average allocation will not always be monotone in time. Therefore, this is the case which does not lend support to either the glide path or the contrarian strategies.

3.3 Numerical examples

We provide in this section two concrete examples to demonstrate the systematic shift in optimal stock proportion discussed earlier. The first example revisits the simplest but non-trivial discrete case, that the measure ν is supported on two points. The second example studies the case when ν is a continuous measure. Specifically, we look at ν supported uniformly on an interval. We will see that, although the two measures have very different analytic structures, the directions of the trend in $\tilde{R}^*(t)$ are determined only by the location of the measures' support.

3.3.1 Example 1: The measure ν is given by $\nu = a_1\delta_{\{\alpha_1\}} + a_2\delta_{\{\alpha_2\}}$

We have shown in Example 1 of section 2.3 that $\tilde{R}^*(t)$ follows the SDE,

$$d\tilde{R}^*(t) = \lambda(\tilde{R}^*(t) - \alpha_1)(\alpha_2 - \tilde{R}^*(t))\big(\lambda(1 - \tilde{R}^*(t))dt + dW(t)\big). \tag{3.3}$$

Assume $\alpha_1 < \alpha_2$. The results of Theorem 3.2.2 are then apparent. The scaling factor $(\tilde{R}^*(t) - \alpha_1)(\alpha_2 - \tilde{R}^*(t))$ is always positive as $\tilde{R}^*(t) \in [\alpha_1, \alpha_2]$. If $\alpha_2 \leq 1$, then $1 - \tilde{R}^*(t) > 0$, which implies that the drift of the above SDE is positive as well. Similarly, the drift is negative if $\alpha_1 \geq 1$.

To see the temporal changes in $\tilde{R}^*$, we calculate numerically the average risk tolerance,

$$\overline{R}(T) = \mathbb{E}[\tilde{R}^*(T)].$$

To distinguish the two cases, $\alpha_1 \geq 1$ and $\alpha_2 \leq 1$, we call the former type of investor "risk averse", and call the latter "risk seeking". The terms come from the fact that a CRRA investor with risk tolerance larger than 1 is actually willing to accept the same level of expected (log) returns with additional variance.

To calculate $\overline{R}(T)$, we set $f(\tilde{r}, t) = \mathbb{E}[\tilde{R}^*(T)|\tilde{R}^*(t) = \tilde{r}]$. Then, $\overline{R}(T) = f(\tilde{r}, 0)$. By the Feynman-Kac's formula, the function $f(\tilde{r}, t)$ solves the partial differential equation,

$$\begin{cases} \frac{\partial f}{\partial t} + \lambda^2(\tilde{r} - \alpha_1)(\alpha_2 - \tilde{r})\big((1 - \tilde{r})\frac{\partial f}{\partial \tilde{r}} + \frac{1}{2}(\tilde{r} - \alpha_1)(\alpha_2 - \tilde{r})\frac{\partial^2 f}{\partial \tilde{r}^2}\big) = 0. \\ f(\tilde{r}, T) = \tilde{r} \end{cases} \tag{3.4}$$

We solve the above PDE using finite difference method. Figure 3.2 plots $\overline{R}(T)$ for both risk averse ($\alpha_1 = 0.1, \alpha_2 = 0.6$) and risk seeking ($\alpha_1 = 1.2, \alpha_2 = 3$) investors.

Assume retirement occurs 50 years after the portfolio inception. We can see that the expected relative risk tolerance of a risk-averse investor is monotonically increasing. Assuming the stock market generates an annual return of 6% with volatility 20%, the plot suggests that the investor's stock allocation starts at 67.5%, and gradually increases to (on average) 81% at the time of retirement.

However, the uncertainty around $\tilde{R}^*(t)$ does not increase monotonically in

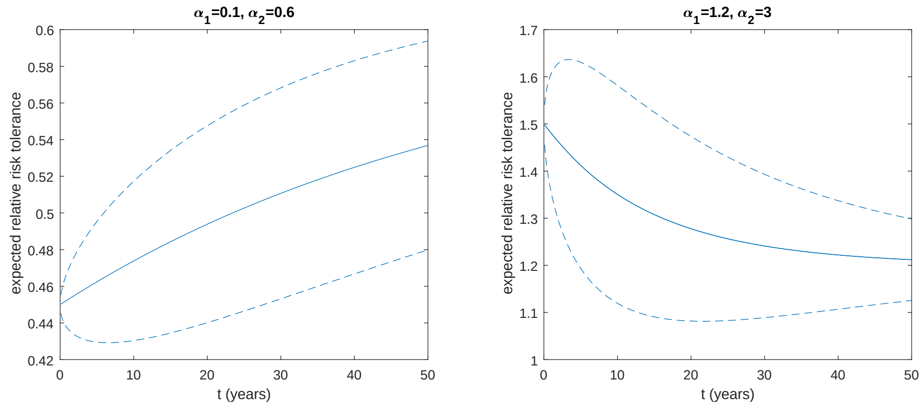


Figure 3.2: Expected relative risk tolerance as a function of time.

time. One standard deviation interval for $\tilde{\pi}^*$ at $T = 50$ is $[69\%, 90\%]$. In fact, as demonstrated in theorem 3.2.3, $\tilde{R}^*(t)$ would converge to α_1 in probability. On the other hand, the right panel shows that $\overline{R}(T)$ is decreasing in T for the risk seeking investor. Again with $\mu = 0.06, \sigma = 0.2$, $\mathbb{E}[\tilde{\pi}^*(T)]$ starts at 225% and decreases to slightly above 180% at $T = 50$. Obviously, this allocation policy is impractical as few investors are willing to maintain such large positions in stocks throughout their investment lifetimes. Therefore, under the assumption that the investor is reasonably risk averse ($\tilde{\pi}^* < 150\%$), the risk reduction behavior displayed in the right panel should not occur, otherwise the portfolio policy would violate the time-consistency condition (such as the one shown in figure 3.1).

Next we consider the case that the investor is partially risk averse and partially risk seeking, i.e. $\alpha_1 < 1 < \alpha_2$? In this case, the drift of $\tilde{R}^*(t)$ no longer maintains a constant sign for all t, as it can take values both above and below 1. Equation (3.3) yields that $\tilde{R}^*(t)$ is a mean-reverting process, with α_1 and α_2 serving as two unattainable boundaries. Therefore, unlike in the previous cases when the entire support is on a single side of 1, we do not have a definitive answer for the drift of $\tilde{R}^*(t)$. Also, in this case, numerical tests seem to suggest that the monotonicity of the average $\overline{R}(T)$ holds if the initial risk tolerance is consistent with the "average" risk tolerance. That is, if we have both $\tilde{R}^*(0) > 1$ and $\frac{1}{2}(\alpha_1+\alpha_2) > 1$ or both $\tilde{R}^*(0) < 1$ and $\frac{1}{2}(\alpha_1+\alpha_2) < 1$, then $\overline{R}(T)$ is still monotone in T. See figure 3.3 for example. However, this hypothesis will remain unjustified as a rigorous proof is quite difficult to obtain. We leave exact characterizations of this "mixed" case to future work.

Another way to gauge how fast stock proportion changes over time is through calculating the amount of time to achieve a certain level. For example, an investor who allocates 50% to the stock initially may wish to know after how many years the proportion will grow to, say, 80% for the first time. Moreover, the hitting time distribution is also necessary to measure performances of investment strategies based on "stopping rules". For example, the dynamic glide path of Basu et al. (2011) only start to decrease the stock proportion after certain return targets have been achieved.

Let $\tau_d = \inf\{t;\, \tilde{R}^*(t) > d\}$ denote the first time $\tilde{R}^*(t)$ crosses d, where $d \in (\tilde{R}^*(0), \alpha_2)$. We calculate $\mathbb{E}[\tau_d]$ below.

Proposition 3.3.1. *Let* $g(\tilde{r}) = \mathbb{E}[\tau_d \,|\, \tilde{R}^*(0) = \tilde{r}]$. *If* $\frac{1}{2}(\alpha_1 + \alpha_2) \geq 1$, *then*

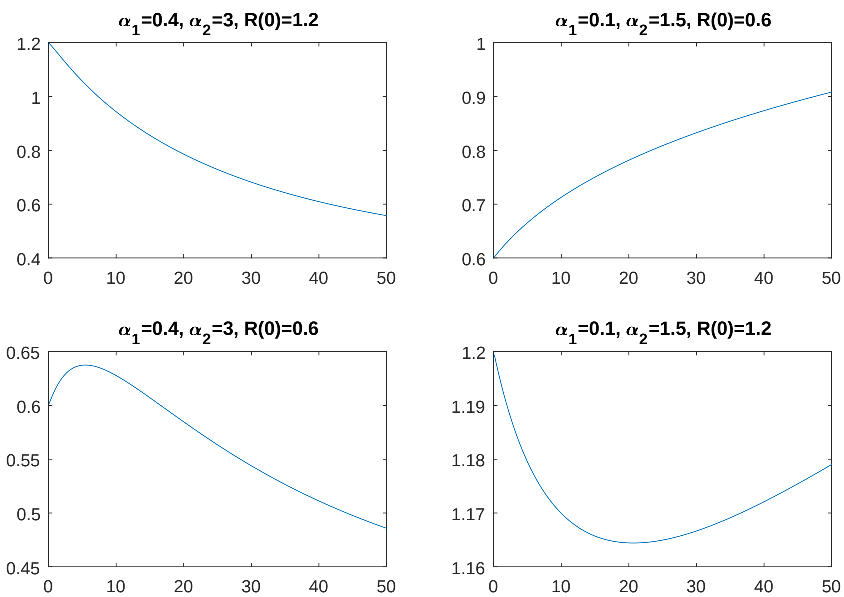


Figure 3.3: Expected relative risk tolerance when $(1-\alpha_1)(1-\alpha_2) < 0$

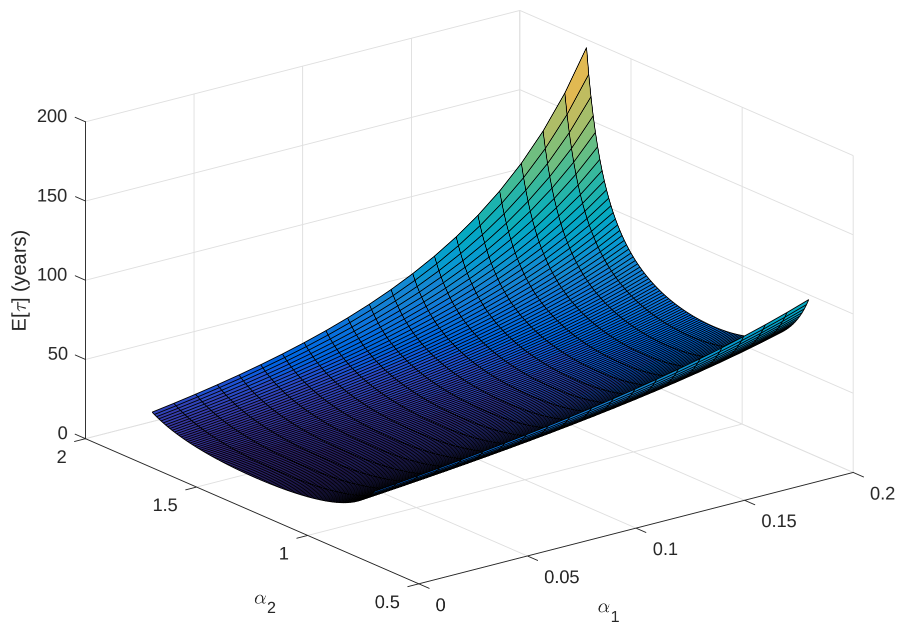


Figure 3.4: $\mathbb{E}[\tau]$ as a function of α_1 and α_2.

$g(\tilde{r}) = \infty$. *While if* $\frac{1}{2}(\alpha_1 + \alpha_2) < 1$,

$$g(\tilde{r}) = \frac{1}{\lambda^2(\alpha_1 - \alpha_2)(1 - \frac{1}{2}(\alpha_1 + \alpha_2))}\Big(\ln\big(\frac{d - \alpha_2}{\alpha_1 - d}\big) - \ln\big(\frac{\tilde{r} - \alpha_2}{\alpha_1 - \tilde{r}}\big)\Big).$$

Proof. See Appendix B.2. □

Next, we compute the average hitting time for investors for different combinations of (α_1, α_2). To make things comparable, we assume that all the investors have the same relative risk tolerance at $t = 0$, such that they all allocate $\pi(0) = 50\%$ of their wealth into the stock. Then, we calculate the average amount of time for $\pi^*(t)$ to reach 90%, as a function of α_1, α_2.

Figure 3.4 shows that, although the investors' initial risk tolerances are the same, there is great diversity among their time diversification strategies. For example, it only takes the investor, with $\alpha_1 = 0$, $\alpha_2 = 1.2$, 22 years to increase his stock proportion to 90%. While it takes the investor, with $\alpha_1 = 0.2$, $\alpha_2 = 1.7$, more than 180 years to do so. The variations in $\mathbb{E}[\tau]$ can be attributed to the several reasons. Firstly, investors with the smaller differences between the α_1, α_2 tend not to change their stock allocations too much over time (the extreme case $\alpha_1 = \alpha_2$ implies constant $\tilde{\pi}^*$). Secondly, $\pi^*(t)$ grows faster if α_2 is closer to 1, since the α_2-induced subportfolio is closer to being "growth optimal". Thirdly, somewhat counterintuitively, investors with smaller α values tend to increase $\pi^*(t)$ at a faster rate. The reason is that, to have 50% invested at $t = 0$, such investors have to allocate more wealth to the CRRA subportfolio corresponding to α_2. For these reasons, investors with polarized risk appetite - one component aims for high safety (α_1 close to 0), while the other for high growth (α_2 close to 1) - would want to increase their stock proportions more quickly as they age.

3.3.2 Example 2: $\nu \sim$ Uniform(α_1, α_2)

In this section we present one example when ν is given by a continuous distribution. Assume that the involved risk tolerances occupy an entire closed interval $[\alpha_1, \alpha_2]$, with equal weights,

$$\nu = a\mathbb{1}_{\{[\alpha_1,\alpha_2]\}}, \tag{3.5}$$

where $a > 0$, $\alpha_2 > \alpha_1 \geq 0$. The aim is to compare its optimal portfolio with that induced from the discrete counterpart,

$$\nu = a(\delta_{\{\alpha_1\}} + \delta_{\{\alpha_2\}}),$$

where *equal* weights are only assigned to the end points of the interval. It would be interesting to see how, having a smoother risk tolerance structure affects the investor's portfolio choice.

We start by studying the investor's relative risk tolerance at time zero. Hence, we need to calculate $\tilde{R}^*(0) = \tilde{r}(x, 0)$. We have

$$\tilde{r}(x, 0) = \frac{h_x(h^{(-1)}(x, 0), 0)}{x},$$

where,

$$h(x, 0) = a \int_{\alpha_1}^{\alpha_2} e^{yx} dy = \frac{a}{x}(e^{\alpha_1 x} - e^{\alpha_2 x}).$$

Figure 3.5 plots the initial risk tolerance as a function of initial wealth, for both the uniform distribution and the two point distribution under the same set of parameters. As expected, the investor with uniformly distributed risk tolerance is less sensitive to changes in wealth, therefore is less likely to dramatically increase or decrease his stock holdings following a market rally or crash.

Next, we calculate the average proportion of stocks $\mathbb{E}[\tilde{\pi}^*(t)]$ at $t > 0$, or equivalently, $\overline{R}(t) = E[\tilde{R}^*(t)]$. In the previous example, $\overline{R}(t)$ was calculated based on the SDE of $\tilde{R}^*$. The problem was converted to solving a parabolic differential

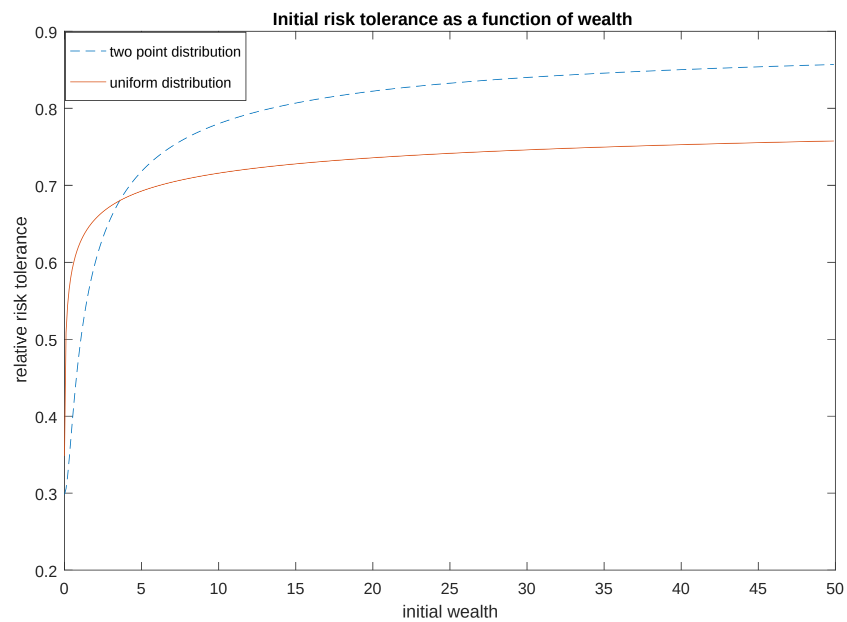


Figure 3.5: Initial risk tolerance and initial wealth

equation with polynomial coefficients. In the case of the uniform distribution however, the coefficients of the PDE are complicated functions, and it is difficult to evaluate and interpret them. A better approach is to explore the "closed form" solution of $\tilde{R}^*(t)$. Recall that in previous chapter, we used the random variable $Y(t)$ to describe the distribution of risk tolerances at t. For the reader's convenience we recall from equation (2.24) that, under a general measure ν, $Y(t)$ is defined by the conditional distribution,

$$\mathbb{P}^{y,\omega,t}(dy) = \frac{e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)}{\int_{\mathbb{R}} e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)},$$

where $D(t) = h^{(-1)}(x,0) + M(t) + A(t)$. Then, $\tilde{R}^*(t)$ is simply the conditional mean $\mathbb{E}[Y(t)|\mathcal{F}_t]$. It is easy to see that under $\nu = a\mathbb{1}_{\{[\alpha_1,\alpha_2]\}}$, $\mathbb{P}^{y,\omega,t}(dy)$ is exactly the density of a normal distribution truncated at α_1, α_2. We then have the following result.

Proposition 3.3.2. *Let the measure be of the form $\nu = a\mathbb{1}_{\{[\alpha_1,\alpha_2]\}}$. Then, the random variable $Y(t)$ admits a truncated normal distribution (conditioned on $\mathcal{F}_t$), with normal parameters $\mu_N = \frac{D(t)}{A(t)}$, $\sigma_N^2 = \frac{1}{A(t)}$, and boundary parameters α_1, α_2. Let $\theta_i = \frac{\alpha_i - \mu_N}{\sigma_N}$, $i = 1, 2$. Then,*

$$\tilde{R}^*(t) = \mathbb{E}[Y(t)|\mathcal{F}_t] = \mu_N - \frac{\phi(\theta_1) - \phi(\theta_2)}{\Phi(\theta_1) - \Phi(\theta_2)}\sigma_N,$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and cumulative distribution functions of the standard normal distribution.

Since $\tilde{R}^*(t)$ is obtained as an explicit function of the process $D(t)$, which is normally distributed for all $t > 0$, the average $\overline{R}(t)$ can be calculated using

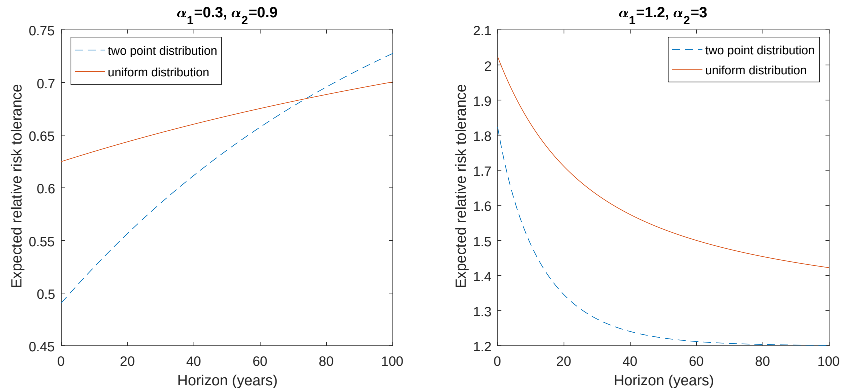


Figure 3.6: $\overline{R}$ for uniform ν.

numerical integration. Figure 3.6 plots $\overline{R}$ for both the uniform and the two point measures at various horizons. The parameters α_1 and α_2 are chosen such that the support of ν is on a single side of 1. As expected, when both α's are smaller than 1, $\overline{R}(T)$ is increasing in T for both type of investors.

However, the changes in the uniform distribution case is less dramatic. For market parameter $\mu = 0.06$, $\sigma = 0.2$, $\mathbb{E}[\tilde{\pi}^*]$ only increases from 94% to 105%, compared with the increase of 73.5% to 109% for the two-point case. Also, one can notice from the graph on the right that, $\overline{R}$ for the two point measure already converges to its theoretical limit of α_1 by the end year 100. However, the speed of convergence for the uniform distribution is much slower. It takes over 10000 years for $\overline{R}(T)$ to approach its asymptotic limit.

The above examples have demonstrated that when the CRRA assumption on the initial utility is not any more valid, a wide variety of time dependent asset allocation behaviors can be rationally justified. Moreover, the speed of change in stock allocation over time is closely related to the investor's initial distribution of risk tolerance. In particular, when advising a client with very different aims of growth for different portions of his wealth, the investment manager should more aggressively increase the equity proportion based on the client's age.

3.4 Preference calibration

To make the forward performance approach practical, one essential input is the initial utility function, or equivalently, the measure ν which describes the distribution of risk tolerance parameters. Classical methods for utility assessment usually involves asking the agent to choose between, or to offer prices for different lotteries. See for example, Farquhar (1984) for a comprehensive review. From the optimal investment perspective, He and Huang (1994) and Dybvig and Rogers (1997) discussed how to infer the utility function from known optimal portfolio strategies. More related to our work is Monin (2014), who studied utility inference under the forward investment framework, based on the investor's desired wealth distribution at a particular horizon.

In this section we propose a different approach. Instead of asking the investor to specify the probability distribution of wealth at a single fixed horizon, we ask about the desired expected returns at multiple horizons. We believe that, in the context of lifecycle investment management, target wealth to be achieved at each stage of life is best aligned with the very purpose of investment.

3.4.1 The algebraic moment problem

The problem can be described as follows. Let T_i, $i = 0, 2, \dots N$ be a series of horizons. An investor with initial wealth x specifies the series of expected total returns R_i ($R_0 = 1$) to be achieved at each individual T_i. We then look for ν that describes the investor's risk tolerance distribution, or equivalently, the measure ν that produces optimal wealth satisfying

$$\mathbb{E}[\frac{X^*(T_i)}{x}] = R_i, \quad i = 0, 1, \dots, N.$$

The implicit assumption here is that $\{R_i\}_{i=0}^N$ can indeed be generated through the above equation for some ν. We call such sequence *forward return sequence.* The treatment for $\{R_i\}_{i=0}^N$ that is not a forward return sequence will be discussed in the next section.

For simplicity, we assume that the horizons are equidistant, or $T_i = i\Delta T$. Also assume that $N = 2K - 1$ (the case when N is odd can be solved with slight modification).

We recall that $X^*(t)$ has the explicit representation

$$X^*(t) = \int_{\mathbb{R}} e^{y(h^{(-1}(x,0)+A(t)+M(t))-\frac{1}{2}y^2A(t)}\nu(dy).$$

Musiela and Zariphopoulou (2010a) has shown that in calculating $\mathbb{E}[X^*(t)]$, the expectation and integration can exchange order. Thus the total return at horizon T can be explicitly calculated as

$$\text{Ret}(T) = \frac{1}{x}\mathbb{E}[X^*(T)] = \frac{1}{x}\int_{\mathbb{R}} e^{y(h^{(-1}(x,0)+A(t))}\nu(dy).$$

If we then define the measure,

$$\nu_x(dy) := \frac{1}{x} e^{yh^{(-1}(x,0)} \nu(dy),$$

then $\mathrm{Ret}(T)$ simplifies into,

$$\mathrm{Ret}(T) = \int_{\mathbb{R}} e^{\lambda^2 yt} \nu_x(dy).$$

Obviously, to determine the investor's initial utility, it is enough to determine ν_x. Compared with ν, ν_x is easier to work with since it is already normalized and can be considered as a probability measure. Indeed,

$$\int_{\mathbb{R}} \nu_x(dy) = \int_{\mathbb{R}} \frac{1}{x} e^{yh^{(-1}(x,0)} \nu(dy) = \frac{1}{x} h(h^{(-1)}(x,0),0) = 1.$$

Recall in the previous chapter that we have defined the random variable $Y(t)$, which describes the distribution of risk tolerance at t. In fact, here ν_x coincides with the probability distribution of $Y(0)$, with $\mathrm{Ret}(T)$ being its moment generating function, i.e.

$$\mathrm{Ret}(T) = \mathbb{E}^{\nu_x}[e^{\lambda^2 Y(0) T}].$$

In order to match the target expected returns R_i at $T_i = i\Delta T$, we need then to have

$$R_i = \mathbb{E}^{\nu_x}[e^{\lambda^2 Y(0) i\Delta T}] = \mathbb{E}^{\nu_x}[Z^i], \quad i = 1, 2, \ldots, N, \tag{3.6}$$

where $Z = e^{\lambda^2 Y(0)\Delta T}$.

Equation (3.6) reduces the preference calibration problem into the so called *algebraic moment problem*, in which a finite sequence of positive numbers $\{R_i\}_{i=0}^N$

is given, and one is asked to find a positive random variable whose first N moments are exactly $\{R_i\}_{i=1}^N$. However, we note that the statement is only correct in the case range$(h) = (0, +\infty)$, since this is the case when ν_x is non-negative. If range$(h) = (-\infty, +\infty)$ or range$(h) = (C, +\infty)$, with $C < 0$, $\nu_x(dy)$ cannot be considered as a probability measure. Fortunately, when $\{R\}_{i=0}^N$ is indeed a forward return sequence, the solution method described below still applies for such cases. When $\{R_i\}_{i=0}^N$ is not a forward return sequence, we need to explicitly take into the account the fact that ν_x might be negative.

Brockett (1987) described procedures for finding the random variable Z that satisfies (3.6), which we recall below for completeness. To this end, since (3.6) gives $N = 2K - 1$ moment equations, we can only determine Z as a discrete random variable with at most K outcomes. Let $p_i = \text{Prob}\,(Z = z_i) > 0$, $i = 1, 2, \ldots, n_0$, with $n_0 \leq K$, and define the Hankel matrices

$$\Delta_k(R) = (R_{i+j})_{i,j=0}^k, \quad \Delta_k^{(1)}(R) = (R_{i+j+1})_{i,j=0}^k, \quad k = 0, 1, \ldots, K-1$$

Then we have the following theorem.

Theorem 3.4.1. *Let Z be a positive random variable which takes n_0 different values $\{z_1, z_2, \ldots, z_{n_0}\}$ with positive probability, and let R_i denote the i-th moment of Z. Then, $det(\Delta_k(R)) > 0$ and $det(\Delta_k^{(1)}(R)) > 0$ for $k \leq n_0 - 1$, $det(\Delta_k(R)) = det(\Delta_k^{(1)}(R)) = 0$ for $k \geq n_0$, where z_i, $i = 1, \ldots, n_0$ are the* *(distinct) roots of the polynomial,*

$$q(z) = \begin{vmatrix} 1 & z & z^2 & \dots & z^{n_0} \\ R_0 & R_1 & R_2 & \dots & R_{n_0} \\ R_1 & R_2 & R_3 & \dots & R_{n_0+1} \\ \vdots & \vdots & \vdots & & \vdots \\ R_{n_0-1} & R_{n_0} & R_{n_0+1} & \dots & R_{2n_0-1} \end{vmatrix}. \tag{3.7}$$

Remark 3.4.2. Although the theorem requires Z to be a random variable, the fact that z_i solve equation $q(z) = 0$ does not require ν_x to be a probability measure. Therefore, for a general measure ν_x which assigns both positive and negative measures, we can still obtain its support by solving $q(z) = 0$, provided that $\{R_i\}_{i=0}^N$ is indeed a forward return sequence. However, the Hankel determinants will no longer be non-negative.

After we solve for z_i, p_i can be obtained by solving the linear system,

$$\begin{cases} p_1 z_1 + \ldots + p_{n_0} z_{n_0} = R_1 \\ p_1 z_1^2 + \ldots + p_{n_0} z_{n_0}^2 = R_2 \\ \quad \vdots \\ p_1 z_1^{n_0} + \ldots + p_{n_0} z_{n_0}^{n_0} = R_{n_0} \end{cases} \tag{3.8}$$

In turn, if $y_i := \frac{1}{\lambda^2 \Delta T} \ln(z_i)$, then ν_x is given by,

$$\nu_x = p_1 \delta_{\{y_1\}} + \ldots + p_{n_0} \delta_{\{y_{n_0}\}}. \tag{3.9}$$

As an example, consider an investor whose expected total returns at various horizons are given in table 3.1, We then have $\Delta T = 5$, $N = 7$, $K = 4$, $n_0 \leq 4$. The Hankel determinants are shown in table 3.2. Both determinants vanish

Table 3.1: Expected Total Returns

Horizon (years)	5	10	15	20	25	30	35
Return (%)	160	250	383	580	873	1310	1962

Table 3.2: Hankel Determinants

	$\det(\Delta_k(R))$	$\det(\Delta_k^{(1)}(R))$
$k = 0$	1.00	1.61
$k = 1$	-0.08	-0.11
$k = 2$	0.00	0.00
$k = 3$	0.00	0.00

at $k = 2$ and, thus, $n_0 = 2$. Solving the quadratic polynomial (3.7), we obtain $z_1 = 1.49$, $z_2 = 0.92$. By (3.8) we have, $p_1 = 1.2$, $p_2 = -0.2$. Further, assume the market parameters are $\mu = 0.06$, $\sigma = 0.2$. Then $y_i = \frac{1}{\lambda^2 \Delta T} \ln(z_i)$ yields $y_1 = 0.5$ and $y_2 = -0.1$. Therefore, we obtain ν_x, namely,

$$\nu_x = 1.2\delta_{\{0.5\}} - 0.2\delta_{\{-0.1\}}.$$

This measure yields the function

$$h(x,t) = 1.2e^{0.5x-0.125t} - 0.2e^{-0.1x-0.005t},$$

which is of full range.

3.4.2 Forward performance approximation

Consider another example, where the expected total returns at 5, 10, 15 years are 150%, 250%, 350%. One can then verify that none of the Hankel determinants are zero. Hence we must have $n_0 = K = 2$, and $q(z) = \frac{1}{4}z^2 + \frac{1}{4}z - 1$.

The two roots are $z_1 = -2.56$, $z_2 = 1.56$. Since z_1 is negative, y_1 is no longer a real number. Therefore, we do not have a valid solution for ν_x.

The issue here is that, the expected returns provided above cannot be generated by any forward performance process. In practice, this issue will almost always arise since no one can state the returns they expect exactly according to their utility functions. It is therefore necessary that we find the measure ν_x which generates a return sequence that is closest to the sequence given. Here, we choose the l^2 norm as the objective function to minimize. In order to formally state the problem, we denote the discrete measure we wish to approximate as,

$$\nu_x = \sum_{i=1}^{n_0} p_i \delta_{\{y_i\}}.$$

As in the last section, we allow y_i and p_i to be negative so that both the full range and half range cases are included. However, we need $p_i y_i \geq 0$, since ν_x assigns positive (resp. negative) measure to positive (resp. negative) values. Finally, denote $z_i = e^{\lambda^2 y_i \Delta T}$. Then, for a given sequence of returns $\{R_i\}_{i=0}^N$, we solve the optimization problem below,

$$\begin{aligned} \underset{p,z}{\text{minimize}} \quad & L(p, z, n_0) = \sum_{j=1}^{N} (\sum_{i=1}^{n_0} p_i z_i^j - R_j)^2 \\ \text{subject to:} \quad & (z_i - 1) p_i \geq 0, \quad i = 1, 2, \ldots, n_0 \end{aligned} \tag{3.10}$$

The only question left is, how do we choose n_0, the number of elements included the support of ν_x. Obviously, larger n_0 will only decrease the approximation error, but it also renders the optimization less stable due to higher dimensionality. Therefore, we stop at the point when increasing n_0 only marginally decreases the value of L. The procedure is demonstrated in the example below.

3.4.3 Example: Vanguard Target Retirement 2045 Fund

We apply the tools developed so far to study asset allocation strategies of Vanguard target retirement fund, currently the largest lifecycle fund in AUM. In particular, we focus on the 2045 fund (VTIVX), which is designed for investors planning to leave the workforce in or within a few years of 2045. It would be interesting to find the forward performance process that best explains the fund's strategy. Figure 3.7 displays the glide path adopted by the fund. The chart shows that the fund invests in five major asset classes. To simplify things, we consider the strategy as investing in only a stock with log-normal dynamics and a bond with zero interest rate. The proportion allocated to stocks, $\tilde{\pi}(t)$, as distinguished by the grey and blue area, starts at 90%, then gradually declines following a piecewise linear function, and eventually settles at 30% after year 55. Since $\tilde{\pi}(t)$ is deterministic, under log-normal market assumption, it is straightforward to calculate the expected total returns,

$$\mathrm{Ret}(T) = e^{\mu \int_0^T \tilde{\pi}(t)dt}.$$

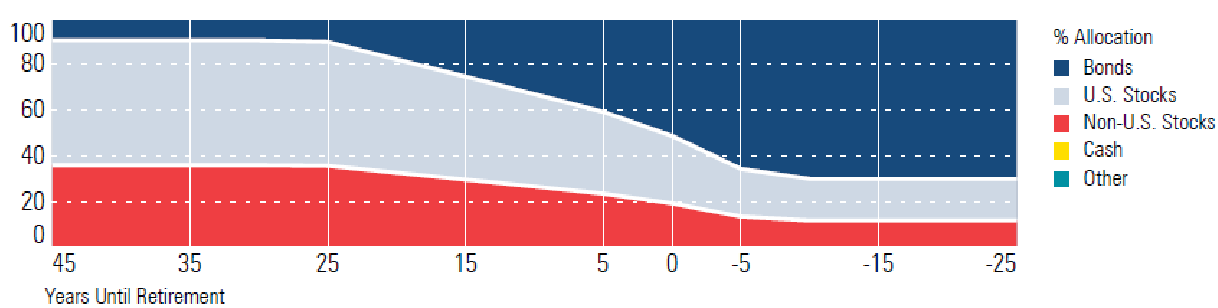


Figure 3.7: Asset allocation of Vanguard Target Retirement 2045 Fund

We apply the preference calibration tool to find the measure ν_x that produces expected returns that best matches the returns implied by the Vanguard deterministic strategy. To increase accuracy, we sample the returns every quarter ($\Delta T = 0.25$) for 70 years, which generates a sequence of 281 returns, $R_i = \text{Ret}(i\Delta T)$, $i = 0, 1, \ldots, 280$. We then solve the optimization problem (3.10) for $n_0 = 1, 2, 3, 4$. The outputs are reported in table 3.3.

Table 3.3: Estimation results for ν_x

	$n_0 = 1$	$n_0 = 2$	$n_0 = 3$	$n_0 = 4$
y_1	0.34	0.16	0.16	0.00
y_2	-	0.00	0.00	0.16
y_3	-	-	-2.88	-1.16
y_4	-	-	-	0.00
p_1	1.00	7.38	7.38	0.00
p_2	-	-6.38	-6.38	7.38
p_3	-	-	0.00	0.00
p_4	-	-	-	-6.38
L	4.00×10^3	371.49	317.49	317.49

Surprisingly, after the large drop in L, the penalty function we try to minimize, when n_0 increases from 1 to 2, there seems to be no further improvement by using a larger n_0, which suggests that a two point measure might be the best solution. Further confirming this are the distribution structures generated under $n_0 = 2, 3, 4$. All three of them show that ν_x is supported at 0.16 and 0 only, with weights 7.38 and -6.38. Therefore, we can firmly conclude the forward performance that best describes the allocation strategy of Vanguard

2045 retirement fund is generated by, the measure

$$\nu_x = 7.38\delta_{\{0.16\}} - 6.38\delta_{\{0\}}. \tag{3.11}$$

The forward optimal strategy derived from the above measure is actually quite simple. The investor with initial wealth \$1 would borrow \$6.38 from the bank, then invest the entire \$7.38 with a CRRA manager with relative risk tolerance 0.16. Although, the forward optimal strategy is stochastic, hence different from the deterministic glide path of Vanguard fund, the expected returns they produce are reasonably close (see panel A of figure 3.8).

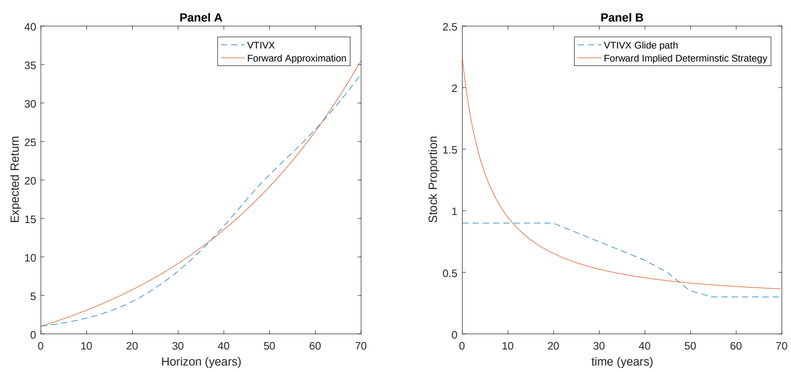


Figure 3.8: A: expected returns. B: stock proportion

A second, more illustrative approach is to compare the strategies themselves. Since $\tilde{\pi}$ for the forward performance process is stochastic, we have to introduce some kind of averaging before comparing to the deterministic glide path. Here we introduce the forward implied deterministic strategy $\tilde{\pi}^{\text{fImp}}(t)$ as the deterministic strategy that produces the same expected return function

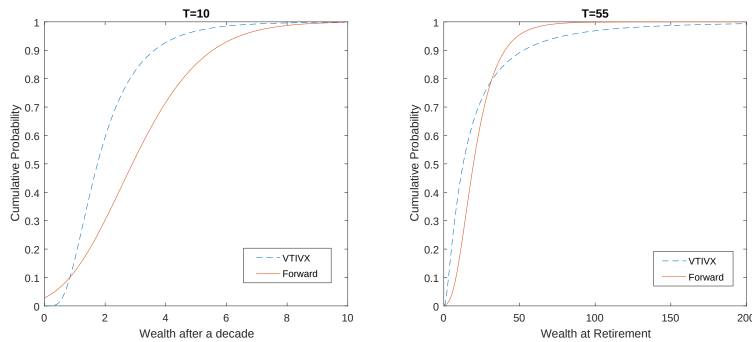


Figure 3.9: cdf for $X^v(T)$ and $X^f(T)$.

as the forward optimal strategy, i.e.

$$\tilde{\pi}^{\mathrm{fImp}}(t) = \frac{1}{\mu}\frac{\partial}{\partial t}\big(\ln(\mathbb{E}[\frac{X^*(t)}{x}])\big).$$

where $X^*(t)$ is the optimal wealth process derived from ν_x.

Panel B of figure 3.8 shows that $\tilde{\pi}^{\mathrm{fImp}}(t)$ is a much steeper "glide path". With an initial leverage higher than 200%, it rapidly declines and comes down to the same level as the VTIVX glide path around year 10, then it slowly converges to the post-retirement level, tracking the fund glide path more closely. Judging from the high leverage in the first decade, one might suspect that the dynamic forward strategy is too risky. To find out if this is case, we need to compute the risk-return profiles for both strategies. Assume for simplicity that the initial wealth is one dollar. Let $X^v(T)$ and $X^f(T)$ denote the wealth processes by following the Vanguard glide path and the forward optimal strategy. We plot the cumulative distribution functions for both wealth variables at horizons $T = 10$ and $T = 55$. As shown in figure 3.9, at year 10 the forward strategy is indeed riskier since it poses a greater downside risk, as there is a 6% chance of losing 50% or more, while for the Vanguard glide path the probability of such loss is only slightly above 1%. However, apart from the worst cases, the forward strategy does dominate in most of the other scenarios. For example, it offers a 48% chance of at least tripling the initial wealth, while under the glide path, the probability goes down to 17%. In fact, under the notion of "*almost stochastic dominance*", introduced by Leshno and Levy (2002), the forward strategy has "almost first order stochastic dominance" over the Vanguard glide path, with violation parameter $\epsilon = 0.031$ ($\epsilon \leq 0.059$ is commonly considered as acceptable).

Surprisingly, the situation reverses at the time of retirement ($T = 55$). The forward strategy is actually more conservative in that it offers a higher considerably probability of getting a decent return while forgoes some chances of exceptional returns. This can also be observed from the summary statistics in table 3.4. While the mean returns are about the same, standard deviation for the forward strategy is much lower. Among the recorded quantiles, the Vanguard glide path only outperforms at the highest decile.

Table 3.4: Retirement wealth summary statistics

	D1	Q1	Median	Q3	D10	Mean	Stdev
forward strategy	8.1	12.7	19.6	29.1	40.5	22.5	13.9
Vanguard glide path	3.2	6.1	12.8	27	52.8	23.6	24.3

3.5 Concluding remarks

In this chapter, we studied optimal investment problem under an initial, instead of terminal utility function, based on the additional assumption of time-consistency. The focus is on how the investment strategy depend on time. What separates our work from the previous ones is that we do not make specific assumptions about the functional form of investor's risk preference. Instead, we seek to find the connections between the structure of risk preferences and the way equity allocations change dynamically over time. For those preferences that generate a strictly positive optimal wealth, the proportion of wealth allocated to stock does exhibit a systematic upward/downward trend depending on whether the investor is strictly more/less risk averse than a log utility investor. Moreover, through numerical examples, we show that the speed of change in asset allocation depends heavily on the risk tolerance distribution as well. From a preference based perspective, our work provides answers to the long debated question of whether lifecycle funds should follow aged-based risk reduction schemes, or exactly the opposite, as some academics have proposed. While there seem to be no "one-size-fits-all" solutions, in that the exact path of equity allocation should be designed based on the investor's risk attitude towards different proportions of his wealth, our work does tilt toward the "contrarian" view since even at the subportfolio level, being more risk-seeking than the log-utility investor is rarely the case.

The second part of our work deals with the problem of preference calibration. We show how to infer the risk tolerance distribution (hence the initial utility) based on the investor's expected returns for different stages of life. As an application, we study the asset allocation strategy of the Vanguard target retirement 2045 fund from the forward investment perspective. We find the forward performance process that best mimics the return function (of horizon) implied from the fund's glide path. It turns out that at the year of retirement, the dynamic forward optimal portfolio generates a more balanced risk-return profile compare to the deterministic glide path. Although the glide path does outperform at the most optimistic scenarios (first decile), for all the other quantiles, the forward portfolio delivers higher returns. For future work, it will be interesting to conduct empirical tests which compare performances of deterministic glide path strategies and their forward approximation counterpart.

While here we focus on the optimal strategy itself, performance related questions are largely left unexplored. One interesting example would be the connection between the distribution of risk tolerance parameters and the probability distribution of optimal wealth at a given horizon, since the portfolio is essentially driven by all the risk tolerance moments. For the calibration part, a more general problem we did not pursue is to find the theoretical connection between a given portfolio strategy and the forward strategy closest to it. This would let us understand, for example, why the Vanguard glide path, a piecewise linear function, should correspond to a risk tolerance distribution as simple as the one given in (3.11). These questions will be explored in future works.

Chapter 4

Applications in dynamic mean-variance analysis

4.1 Introduction

Mean-variance analysis has become one of the most widely adopted portfolio construction tool since its introduction in Markowitz (1952). The idea is intuitive, a desirable portfolio allocation should achieve the highest expected return while keeping its risk as small as possible. The method is easy to implement in that one only needs to estimate the mean and variance of asset returns and the optimal portfolio weights are obtained by solving a quadratic program. As a result, most existing works are along the lines of making the single period optimal portfolio more practical and yield better out of sample performance. The focus has been on refining the statistical estimation procedure or reformulating the optimization program by imposing portfolio constraints or introduce robust optimization criteria. On a different direction, some researchers went beyond the "buy-and-hold" framework to study dynamic mean-variance problems which allow discrete or continuous trading. However, static or dynamic, all the work done so far is built essentially on the single period framework, in the sense that a single mean-variance objective function is imposed at a pre-determined horizon T, which guides every investment decision before T. If optimization needs to be carried out repeatedly in multiple periods, a common practice is to assume the same mean-variance objective function for every period. So far, to the best of our knowledge, there has been no research that takes this question to a theoretical level. To fill this gap, we propose applying the forward theory to the special context of mean-variance optimization. Compared with the ad hoc choice, the forward approach imposes time-consistency condition, such that the individual mean-variance problems are now connected in a way that improves performance in the long term. The following simple example illustrate the point.

Consider a mean-variance investor who trades between cash (with zero risk-free rate) and a stock market index. The index has an annualized expected return of 10%, with 20% volatility. The investor sets a targets return of 8% to be achieved by the end of the first year, then run the mean-variance optimization and starts investing. At $t = 1$, the investor decides to invest for one more year. However, according to his estimate, market volatility for the second year has gone up to 60% (with the same expected return). The question is, facing the volatility hike, what is the return target the investor should pursue for the second year? If he keeps seeking an 8% return, then one can compute that the two year Sharpe ratio is at 0.33. The forward theory on the other hand, insists that the choice of the second year target should take into account both the new market condition and portfolio performance in the first year, hence is a random variable realized at $t = 1$. We can calculate that, the time zero average of the second target is merely 0.74%. Therefore, the forward theory actually advises the investor to dramatically adjust his target downward in response to the market volatility increase. The result is that two-year Sharpe ratio now increases to 1.1, which is a significant improvement compared with the ad hoc choice.

In the next section, we first construct forward mean-variance in the buy-and-hold setting, i.e. the underlying single period problems are static problems. Performance comparison will be made between the forward mean-variance approach and the ad hoc approach of keeping a constant mean-variance trade-off coefficient. In particular, we show that in the entire spectrum of market auto-correlation, the forward approach always outperforms in terms of long term Sharpe ratio.

In section 4.3, we discuss the dynamic mean-variance problem with continuous-time trading. We show how the wealth target should be chosen when multiple dynamic problems are solved sequentially in time. Furthermore, we discuss the trade-off between solving a single dynamic (backward) problem with long horizon and splitting it into several forward problems of short horizons. We find that even a slight model risk suffices to justify the latter approach. Section 4.4 deals with robust mean-variance in continuous time, and its forward generalization. Section 4.5 concludes.

4.2 Forward mean-variance with discrete valuation and static trading

We consider a fund manager faces the task of managing a portfolio for twenty years, with rebalancing occurring at the beginning of each year. What approach should he choose for this problem? The theoretically best approach is to run a dynamic optimization, and solve all optimal decisions in the future using the dynamic programming principle. If we denote by $\{\omega_t\}_{t=0}^{19}$ the portfolio strategies at $t = 0, 1, \ldots, 19$, and denote X_{20} the terminal wealth, the manager solves the single optimization program,

$$\max_{\omega_0,\ldots,\omega_{19}} \mathbb{E}^{\mu,\Sigma}[X_{20}^{\omega}] - \frac{\gamma}{2}\mathrm{Var}^{\mu,\Sigma}(X_{20}^{\omega}) \tag{4.1}$$

This approach is termed by some as the *dynamic mean-variance*. When asset returns are normally distributed and independent over time, the above problem has been solved in closed form by Li and Ng (2000). Under much more general model assumptions, one may apply the linear approximation scheme of Collin-Dufresne et al. (2003). In practice, however, the manager may be aware that to actually adopt the solution he would have to assume the market parameters μ, Σ, that he estimated initially, are valid for the entire twenty year horizon. Realizing that it is unlikely the case, he might feel safer to simply optimize one year at a time. Hence, the following sequence of one-period mean-variance problems are solved instead,

$$\max_{\omega_t} \mathbb{E}_t^{\mu_t,\Sigma_t}[X_{t+1}^{\omega_t}] - \frac{\gamma_t}{2}\mathrm{Var}_t^{\mu_t,\Sigma_t}(X_{t+1}^{\omega_t}), \quad t = 0, \ldots, 19 \tag{4.2}$$

The obvious advantage is that μ_t, Σ_t can now be updated each year based on new information. Indeed, this is the framework adopted by most existing literature when empirically testing the performance of mean-variance optimal portfolios. An important question then arises: how shall one choose the sequence of mean-variance tradeoff parameters $\{\gamma_t\}$?

Before starting the discussion, it is necessary to point out that the mean-variance problem proposed by (4.2), which aims at optimizing the wealth variable, can be equivalently formulated to optimizing the variable of returns,

$$\max_{\omega_t} \mathbb{E}_t[R^e_{t+1}] - \frac{\gamma_t}{2}\text{Var}_t(R^e_{t+1}), \quad t = 0, \ldots, 19 \tag{4.3}$$

Here $R^e_{t+1} = \frac{X_{t+1}}{X_t} - R_f$ is the excess return at time t.

Obviously, if one is concerned only about optimizing in a single period, it makes no difference to pick either formulation, since the corresponding γ_t parameters only differ by a multiple of X_t. However, if one aims to define a series of mean-variance problems inter-connected through time, formulations (4.2) and (4.3) need to be treated separately as they lead to different approaches. We defer discussing the differences to the end of the section.

In the empirical mean-variance literature, the convention is to consider formulation (4.3) with the γ parameter assumed to be time-independent, i.e.

$$\gamma_t = \gamma, \quad \forall t.$$

However, such a choice is ad hoc in at least two ways. Firstly, $\gamma_t = \gamma$ does not take into consideration portfolio performances up to t, hence the sequence of mean-variance problems are virtually a naive concatenation of unrelated individual problems. While it does not affect single-period performance, multi-period performance, which depend heavily on the autocorrelation of portfolio returns, is left entirely to chance. On the other hand, from a preference perspective, the manager is indeed likely to determine the objective of the current period based on performances of the previous periods.

Secondly, a time-independent γ_t does not compare market conditions at different times. For example, if volatility estimate for this year has doubled, does this imply the same person would want to keep the γ parameter the same?

To tackle this issue, it is necessary to establish a multi-period mean-variance theory that connects forward in time the single-period problems in an economically meaningful way. Moreover, for the theory to be practically implementable, the inputs required to determine γ_t should be no more than the market parameters for the current period. Therefore, the forward approach described in previous chapters, which provides consistent, forward in time optimization, becomes a natural candidate for prescribing a reasonable *dynamic structure* on γ_t. Indeed, this idea was considered by Musiela et al. (2015), based on the view that the (time-dependent) mean-variance portfolio of (4.3) can be considered as a time-discretization of the optimal portfolio of a time-monotone forward utility. To see this, let $U(x, t) = u(x, A_t)$ denote a forward utility process. If we discretize the time dimension and solve for the optimal portfolio policy at each small time interval $[t, t + \Delta t]$, then at each t we face a single-period utility maximization problem

$$\max_{\omega_t} \mathbb{E}_t[u(X^{\omega}_{t+\Delta t}, A_t)]. \tag{4.4}$$

Considering the Taylor series expansion of $u(X^{\omega}_{t+\Delta t}, A_t)$ around the point X_t yields

$$u(X^{\omega}_{t+\Delta t}, A_t) \approx u(X_t, A_t) + u'(X_t, A_t)\Delta X_t + \frac{1}{2}u''(X_t, A_t)(\Delta X_t)^2 + o(\Delta t),$$

where $\Delta X_t = X^{\omega}_{t+\Delta t} - X_t$.

In particular, as Δt approaches 0, we can replace the difference operator by the differential operator and omit the high order terms in Δt. Then taking expectation on both sides then gives

$$\mathbb{E}_t[u(X^{\omega}_{t+\Delta t}, A_t)] \approx u(X_t, A_t) + u'(X_t, A_t)\mathbb{E}[dX_t] + \frac{1}{2}u''(X_t, A_t)\mathrm{Var}_t(dX_t).$$

Note that $\mathbb{E}_t[(dX_t)^2]$ is replaced by $\mathrm{Var}_t(dX_t)$ because they only differ by $(\mathbb{E}[dX_t])^2$, a term of order $(\Delta t)^2$.

If we use $R_t = \frac{dX_t}{X_t}$ to denote the portfolio return at $[t, t+\Delta t]$, then the above equation implies that the single-period utility maximization problem is equivalent to the mean-variance problem

$$\max_{\omega_t} \mathbb{E}_t[R_t] + \frac{1}{2}\frac{u''(X_t, A_t)X_t}{u'(X_t, A_t)}\mathrm{Var}_t(R_t). \tag{4.5}$$

Therefore we arrive at a natural choice for the mean-variance trade off parameter γ,

$$\gamma_t = -\frac{u''(X_t, A_t)X_t}{u'(X_t, A_t)} = \frac{1}{\tilde{r}(X_t, A_t)}, \tag{4.6}$$

where $\tilde{r}(x,t)$ as before denotes the relative risk tolerance function.

To the best of our knowledge, the above idea of Musiela et al. (2015) is the first one that discusses dynamic choice of γ in a multi-period mean-variance setting. The time-consistency of the forward approach thus guarantees that the sequence of mean-variance problems defined by γ in (4.6) is infinitesimally time-consistent. However, the downside is, by construction, that the approach is valid only if Δt can be considered as very small. For large Δt, the sequence of mean-variance problems become a poor approximation of the underlying forward utility problem, hence time-consistency might break down. This brings up another serious issue. This method requires that the timing of updating γ coincides with that of trading. In reality, however, it is more natural for an investor to revise his objective function less frequently than for him to trade. To address these issues, we propose in this chapter an alternative construction for γ_t. Our approach is more restrictive in that, as we will show, it only accommodates quadratic type forward performance. However our approach does not rely on Δt being small. Moreover, it is straightforward to generalize it to the case that trading and preference update can occur at separate frequencies. For the rest of the section, we limit our discussion to the case where trading and updating γ occur discretely and at same times. Starting from section 4.3, we proceed with the "discrete-continuous" case, where the investor still solves a sequence of mean-variance problems as defined in (4.2), but trades continuously within each sub-period.

4.2.1 Forward mean-variance with risk-free asset

To illustrate the main idea of our approach, we consider the simplified problem of only two trading periods, say $[0, T]$ and $[T, \tilde{T}]$. The market consists of n risky assets $S^1, S^2, \ldots, S^n$ and a risk-free asset S^0. At $t = 0$, the investor has estimated that the total returns $R_0^i = \frac{S_T^i}{S_0^i}$ follow the normal distribution

$$R_0 = [R_0^1, R_0^2, \ldots, R_0^n] \sim \mathcal{N}(\mu_0, \Sigma_0),$$

where $\mu_0 = [\mu_0^1, \mu_0^2, \ldots, \mu_0^n]'$ and Σ_0 denote the vector of expected returns and the covariance matrix. With initial risk aversion parameter γ_0, the investor solves at $t = 0$

$$\max_{\omega_0} \mathbb{E}[X_T^{\omega_0}] - \frac{\gamma_0}{2} \mathrm{Var}(X_T^{\omega_0}), \tag{4.7}$$

where $\omega_0 = [\omega_0^1, \omega_0^2, \ldots, \omega_0^n]'$ denote the proportion of wealth invested in $S^1, S^2, \ldots, S^n$. At $t = T$, the investor re-estimates the market parameters (μ_T, Σ_T), decides on a new parameter γ_T and solves the updated mean-variance problem

$$\max_{\omega_T} \mathbb{E}_T[X_{\tilde{T}}^{\omega_T}] - \frac{\gamma_T}{2} \mathrm{Var}_T(X_{\tilde{T}}^{\omega_T}). \tag{4.8}$$

We now aim to determine γ_T such that a reasonable connection between the two problems can be established. To this end, we apply the forward approach introduced in Chapter 2. Recall that in the continuous time setting, the utility process $U_t(x)$ is called a forward utility if it satisfies the martingale-supermartingale condition

$$U_s(x) = \max_{\omega} \mathbb{E}[U_t(X_t^{\omega}) | X_s = x], \quad \forall\, t > s \geq 0. \tag{4.9}$$

In our current setting where time is discretized, there is a discrepancy between the times when the utility is applied and it is determined. For example, the utility $U_0(x)$ determined at 0 is applied to evaluate portfolio payoff at $t = 1$. We now state the definition that reflects the correct measurability condition.

Definition 4.2.1. A stochastic sequence of utility functions $\{U_t(x)\}_{t=0}^{\infty}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a forward utility if the following conditions are satisfied for $t = 0, 1, \ldots$

(i) $U_t(x)$ is increasing, concave, twice continuously differentiable and satisfies the Inada condition.

(ii) $U_t(x)$ is measurable w.r.t. $\mathcal{F}_t$

(iii) For any admissible wealth process $\{X_t\}_{t=0}^{\infty}$, the utility process $\{U_t(X_{t+1})\}_{t=0}^{\infty}$ is a supermartingale. And there exists an admissible wealth process $\{X_t^*\}_{t=0}^{\infty}$ such that $\{U_t(X_{t+1}^*)\}_{t=0}^{\infty}$ is a martingale. In other words, for any $t > s > 0$ we have

$$U_{s-1}(x) = \max_{\omega} \mathbb{E}_s[U_{t-1}(X_t^{\omega})|X_s = x]. \tag{4.10}$$

The above definition however, is not directly applicable to constructing mean-variance preferences in a forward manner. The reason is that the variance operator does not enjoy the tower property of conditioning. Hence condition (iii) above would lose meaning. To circumvent this issue, we apply the well known result that for the classical, single period mean-variance, there is an equivalent quadratic utility preference which yields the same optimal portfolio.

Proposition 4.2.2. *The quadratic utility problem,*

$$\max_{\omega_0} \mathbb{E}[X_T^{\omega_0} - \frac{\delta_0}{2}(X_T^{\omega_0})^2], \tag{4.11}$$

is equivalent to the mean-variance problem (4.7) *if and only if*

$$\frac{1}{\delta_0} = \frac{1}{\gamma_0}(1 + \mu_0^e{}'\Sigma_0^{-1}\mu_0^e) + r_f X_0. \tag{4.12}$$

Here $\mu_0^e = [\mu_0^1 - r_f, \ldots, \mu_0^n - r_f]'$ *denote the vector of excess returns.*

Proof. See Appendix C.1. □

Applying the above result at each trading period $(t, t+1]$, "transforms" the sequence of mean-variance preferences to a sequence of quadratic utility preferences. This allows us to completely bypassing the trouble of sequential conditioning of the variance operator. Natually, we can now define the notion of *forward mean-variance* in terms of its forward quadratic utility counterpart.

Definition 4.2.3. A sequence of mean-variance preferences $\{\mathrm{MV}_t\}_{t=0}^{\infty}$ is called a *forward mean-variance* if the corresponding quadratic utility sequence $\{U_t\}_{t=0}^{\infty}$ (determined by (4.12)) is a forward performance satisfying Definition 4.2.1.

The construction of forward mean-variance can thus be implemented by iteratively performing the following steps:

1. At t, estimate the market parameters (μ_t, Σ_t), and apply equation (4.12) to find δ_t.

2. At $t+1$, solve the forward quadratic utility problem to determine δ_{t+1}

3. Re-estimate the market parameters $(\mu_{t+1}, \Sigma_{t+1})$, and apply equation (4.12) again to find γ_{t+1}.

Now only the second step is yet to be solved. In the previous two-period example, suppose that at $t = 0$ the first period quadratic utility $U_0(x) = x - \frac{\delta_0}{2}x^2$ has been determined through equation (4.12). Then at $t = T$, we need to solve the reversed optimization problem, in that we need to determine the quadratic utility $U_T(\cdot)$ at T, such that

$$U_0(x) = \max_{\omega_T} \mathbb{E}_T[U_T(X_{\tilde{T}}^{\omega_T})|X_T = x]. \tag{4.13}$$

Here $U_T(x)$ is a general quadratic utility of the form,

$$U_T(x) = a_T(x - \frac{\delta_T}{2}x^2) + b_T,$$

where a_T, b_T and δ_T are parameters measurable w.r.t. $\mathcal{F}_T$. We now look for the appropriate coefficient δ_T, such that equation (4.13) is satisfied.

Proposition 4.2.4. *The quadratic utilities $U_0(x) = x - \frac{\delta_0}{2}x^2$ and $U_T(x) = a_T(x - \frac{\delta_T}{2}x^2) + b_T$ satisfy equation* (4.13) *if and only if*

$$\delta_T = \frac{\delta_0}{r_f}. \tag{4.14}$$

Proof. See Appendix C.1. □

We are now ready to derive the main result following proposition 4.2.2 and 4.2.4.

Theorem 4.2.5. *A sequence of mean-variance preferences parameterized by* $\{\gamma_t\}_{t=0}^{\infty}$,

$$\mathbb{E}_t[X_{t+1}] - \frac{\gamma_t}{2} Var_t(X_{t+1}) \tag{4.15}$$

is a forward mean-variance preference if and only if for $t = 1, 2, \ldots$.

$$\frac{1}{\gamma_t} = r_f\Big(\frac{1}{\gamma_{t-1}} \frac{1 + {\mu_{t-1}^e}'(\Sigma_{t-1})^{-1}\mu_{t-1}^e}{1 + \mu_t^{e\prime}(\Sigma_t)^{-1}\mu_t^e} + \frac{r_f X_{t-1} - X_t}{1 + \mu_t^{e\prime}(\Sigma_t)^{-1}\mu_t^e}\Big). \tag{4.16}$$

Proof. See Appendix C.1. □

We can see from equation (4.16) that the way γ_t is updated now takes into consideration the market condition estimated at t, and the performance realized in the previous period. In fact, the interpretation (4.16) is more straightforward if we re-write the mean-variance problems in terms of the return variable.

$$\max_{\omega_t} \mathbb{E}_t[R_{t+1}^e] - \frac{\tilde{\gamma}_t}{2}\mathrm{Var}_t(R_{t+1}^e)$$

Under this parameterization, we obtain that $\tilde{\gamma}_t = \gamma_t X_t$. Then (4.16) implies that

$$\frac{1}{\tilde{\gamma}_t} = \frac{r_f}{R_{t+1}}\Big(\frac{1}{\tilde{\gamma}_{t-1}} \frac{1 + \theta_{t-1}^2}{1 + \theta_t^2} - \frac{R_{t+1}^e}{1 + \theta_t^2}\Big). \tag{4.17}$$

Here R_{t+1} and R_{t+1}^e are the total and excess returns realized at $t+1$, and $\theta_t := \sqrt{\mu_t^{e\prime}(\Sigma_t)^{-1}\mu_t^e}$ denotes the market *Sharpe ratio* estimated at t. Hence, as opposed to the common practice assumption that the investor maintains a constant $\tilde{\gamma}$ over time, the investor under forward mean-variance would decrease his risk appetite if he anticipates an improvement in market conditions, as indicated by the term $\frac{1+\theta_t^2}{1+\theta_{t-1}^2}$, or if he realized a poor performance in the last period. In particular, in the knife edge case that the portfolio return coincides with the risk-free rate and market condition stays unchanged, we would have $\tilde{\gamma}_{t+1} = \tilde{\gamma}_t$.

4.2.2 Multi-period performance analysis

We have mentioned in the previous section that one of the motivations of introducing forward mean-variance is to establish a (time-consistent) connection between the individual mean-variance problems solved in each period. Although the new approach has no impact on single period performances (since optimal mean-variance portfolio always achieves the highest single period Sharpe ratio, regardless of the choice for γ), performance evaluated over multiple periods will differ depending on the dynamic choice of γ_t. In this section, we conduct a comparative study, between the forward approach of setting γ and the conventional approach of keeping γ constant, under a market where the risky asset returns are serially correlated.

The reason we introduce serial correlation is twofold. First, we have seen from (4.17) that the investor will decrease risk if he did relatively well in the past. This is a type of dynamic strategy that is sensitive to return serial correlation. The second reason comes from the fact the investor compares market conditions. Again, by (4.17), γ will be set higher or lower if he estimates an increase or decrease in the market Sharpe ratio. Hence, the predictability in asset returns introduced by autocorrelation makes this consideration on γ non-trivial as well. As a result, we believe that the simple autoregressive model is a parsimonious approach that highlights the two main features of the forward mean-variance preference.

For the rest of the section, we first focus on the γ parameter itself. In particular, we quantify the impact of a market shock in a single period to the value of γ in the next period. We will see that the past performance effect and the market condition effect discussed above actually influence the future value of γ in opposite directions.

Next, we compare the long term performances measured as unconditional Sharpe ratio. It turns out that the forward investor always outperforms the conventional investor regardless of the way returns are correlated. This result is in fact related to the studies of Dybvig and Ross (1985a,b), Ferson and Siegel (2001), which we will discuss at the end of this section.

Suppose the investor solves two consecutive mean-variance problems, at $t = 0$ and $t = 1$, trading between a single risky and risk-free asset. The excess return of the risky asset is assumed to follow an AR(1) process,

$$R^e_{t+1} - \mu = \beta(R^e_t - \mu) + \epsilon_{t+1},$$

where μ denotes the long term mean, and ϵ_{t+1} is i.i.d. normal with variance σ^2. Assume also that the initial period return follows the long run stationary distribution, which can be written as $R^e_1 = \mu + \epsilon_1$, with $\epsilon_1 \sim \mathcal{N}(0, \frac{\sigma^2}{1-\beta^2})$.

Hence $R_2^e = \mu + \beta\epsilon_1 + \epsilon_2$.

We study two types of mean-variance investors, defined by $\{\gamma_t^C\}_{t=0,1}$ and $\{\gamma_t^F\}_{t=0,1}$, with $\gamma_0^C = \gamma_0^F = \gamma$, using the superscripts C and F to denote the "classical" and the "forward" approaches. At $t = 1$, γ_1^F is set according to the forward mean-variance formula (4.16), while γ_1^C is simply $\gamma\frac{X_0}{X_1}$ (i.e. $\tilde{\gamma}_1^C = \tilde{\gamma}_0^C$). The initial wealth is set at 1, without loss of generality.

For the first period $[0,1]$, the investors solve identical problems as they start with the same initial γ. Based on the above model assumptions, the market parameter estimates for the first period would be,

$$\mu_0^e = \mu \quad \text{and} \quad \sigma_0^2 = \frac{\sigma^2}{1-\beta^2}$$

The portfolio strategy at 0 is given by $\omega_0^C = \omega_0^F = \frac{1}{\gamma}\frac{\mu_0}{\sigma_0^2}$, which implies that the wealth at $t = 1$ is given by

$$X_1^C = X_1^F = \frac{1}{\gamma}\frac{\mu_0}{\sigma_0^2}R_1^e + r_f = \frac{1}{\gamma}\frac{1-\beta^2}{\sigma^2}(\mu^2 + \mu\epsilon_1) + r_f.$$

At $t = 1$, the optimal portfolio strategy is given by

$$\omega_1^* = \frac{1}{\gamma_1^i X_1}\frac{\mu_1}{\sigma_1^2}, \quad i \in \{C, F\}.$$

To study the difference in risky asset holdings at $t = 1$, we focus on the more relevant quantity $\frac{1}{\tilde{\gamma}_1} = \frac{1}{\gamma_1 X_1}$ (instead of γ_1 itself). The conventional investor would simply set his new preference as

$$\frac{1}{\tilde{\gamma}_1^C} = \frac{1}{\gamma}.$$

The forward investor, however, would first need to re-estimate market parameters

$$\mu_1 = \beta\epsilon_1 + \mu, \ \sigma_1^2 = \sigma^2,$$

and then apply equation (4.16) to find $\frac{1}{\gamma_1^F} = \frac{r_f}{\gamma}\frac{\sigma^2-(1-\beta^2)\mu\epsilon_1}{\sigma^2+(\mu+\beta\epsilon_1)^2}$, or, equivalently

$$\frac{1}{\tilde{\gamma}_1^F} = \frac{1}{\gamma_1^F X_1} = r_f \frac{(\sigma^2 - (1-\beta^2)\mu\epsilon_1)\sigma^2}{(\sigma^2 + (\mu+\beta\epsilon_1)^2)((1-\beta^2)(\mu^2+\mu\epsilon_1) + \gamma r_f \sigma^2)}. \tag{4.18}$$

To understand how does $\frac{1}{\tilde{\gamma}_1^F}$ relates to asset performance in the first period, measured by the unexpected return ϵ_1, first consider the case $\beta = 0$, that is, the returns for the two periods are i.i.d. As mentioned before, in setting γ_1^F, the forward investor would take into account both past performances and the updated estimate of the market. When $\beta = 0$, the second effect vanishes. Hence, $\frac{1}{\tilde{\gamma}_1^F}$ depends on ϵ_1 solely through the performance effect, and as it is suggested by (4.17), the dependence is negative since good performance in the past would induce the investor to reduce risk. The observation is verified in mid panel of Figure 4.1. When $\beta \neq 0$ however, the second effect is also present. If β is negative, a positive ϵ_1 not only means positive portfolio performance but also a lower expected return in the next period, as indicated by

$$\mu_1 = \mathbb{E}_1[R_2^e] = \mu + \beta\epsilon_1.$$

Therefore by (4.17), the lower estimate for θ_1 would prompt a forward investor to increase risk so that sufficient return is guaranteed. As shown in the left panel of figure 4.1, the market effect and performance effect do introduce

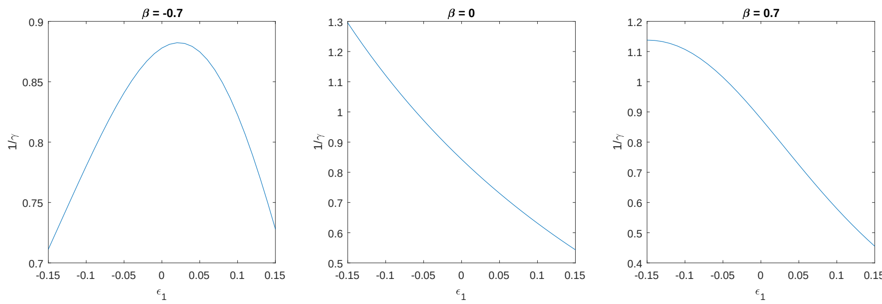


Figure 4.1: $\frac{1}{\tilde{\gamma}_1^F}$ vs. ϵ_1.

competing impacts and they interplay to dominate at different locations of the ϵ_1 domain.

Next, we analyze the long term performance measured as the two-period Sharpe ratio,

$$\theta_{0,2} := \sqrt{\frac{\mathbb{E}[X_2] - r_f^2}{\text{Var}(X_2)}}.$$

To calculate $\theta_{0,2}$, first we explicitly calculate the optimal wealth at $t = 2$,

$$X_2^C = \Big(\frac{1}{\gamma}\frac{1-\beta^2}{\sigma^2}(\mu^2 + \mu\epsilon_1) + r_f\Big)\Big(\frac{(\mu + \beta\epsilon_1)(\mu + \beta\epsilon_1 + \epsilon_2)}{\gamma\sigma^2} + r_f\Big)$$

$$X_2^F = \frac{r_f}{\gamma\sigma^2}\left((\mu + \beta\epsilon_1)(\mu + \beta\epsilon_1 + \epsilon_2)\frac{\sigma^2 - (1-\beta^2)\mu\epsilon_1}{\sigma^2 + (\mu + \beta\epsilon_1)^2} + (1-\beta^2)(\mu^2 + \mu\epsilon_1)\right) + r_f^2.$$

In turn, the moments of the terminal wealth as well as $\theta_{0,2}$ can be calculated using Monte Carlo simulation. As shown in figure 4.2, the two-period Sharpe ratio varies wildly as β changes. However, the forward investor always outperforms, irrespective of the direction and magnitude of return autocorrelation. Unlike the conventional investor, the forward investor is able to capture the high benefit of strong return predictability, realizing sizable Sharpe ratio at both ends of the β interval. Comparing the two plots in figure 4.2, the per-

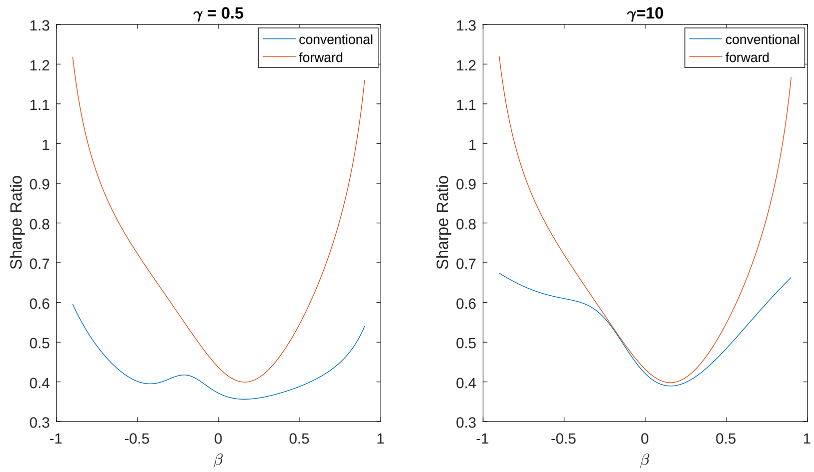


Figure 4.2: Two-period Sharpe ratio vs. AR coefficient

formance gap is even larger when both investors are more risk-seeking in the initial period.

The above difference in multi-period performance is best understood from the perspective of Hansen and Richad (1987), Dybvig and Ross (1985a,b) and Ferson and Siegel (2001), who studied performance measurement using conditioning information. The authors pointed out that, when a fund manager constructs mean-variance optimal portfolios based on more refined information sets, the portfolio might be seen as inefficient by outside investors who evaluates moments of returns based on coarser information. In fact, if we reformulate the single period problem studied in Ferson and Siegel (2001) into a two-period forward mean-variance problem, the forward optimal portfolio exactly coincides with the unconditionally optimal portfolio they derived. To show this, consider the following model for the risky asset return at $t = 1$,

$$R_1 = \mu(s) + \epsilon.$$

Here s is a random signal, which can be observed by the fund manager but cannot be observed by outside investors. The market noise ϵ, conditional on s, is normally distributed with mean 0 and variance $\sigma_\epsilon^2(s)$.

The manager who can observe the signal will optimize the conditional mean-variance preference,

$$\mathbb{E}_s[r_f + \omega(s)(R_1 - r_f)] - \frac{\gamma_s}{2}\mathrm{Var}_s(\omega(s)(R_1 - r_f)), \tag{4.19}$$

where we use $\omega(s)$ to denote the manager's portfolio based on the observed signal s. On the other hand, an "outside" investor who cannot observe s would evaluate the portfolio with the unconditional mean-variance preference,

$$\mathbb{E}[r_f + \omega(s)(R_1 - r_f)] - \frac{\gamma}{2}\mathrm{Var}(\omega(s)(R_1 - r_f)). \tag{4.20}$$

This unconditional problem was solved by Ferson and Siegel (2001). They showed that the optimal portfolio is given by

$$\omega^*(s) = \frac{1}{\gamma}\frac{\mu(s) - r_f}{(\mu(s) - r_f)^2 + \sigma_\epsilon^2(s)}. \tag{4.21}$$

However, a relevant question not explicitly stated in Ferson and Siegel (2001) is the following: in order to achieve the outside investor's unconditional objective (4.20), how should the fund manager, who only solves conditional problems, determine the trade-off parameter γ_s to be applied to problem (4.19)? If he simply set $\gamma_s = \gamma$, he would obtain the solution

$$\omega(s) = \frac{1}{\gamma_s}\frac{\mu(s) - r_f}{\sigma_\epsilon^2(s)} = \frac{1}{\gamma}\frac{\mu(s) - r_f}{\sigma_\epsilon^2(s)},$$

which is clearly different from (4.21), hence is not unconditionally optimal, as evaluated by the outside investor.

Now we look at the manager's problem from a forward mean-variance perspective. Assume that there is an actual time, denoted as t_s, $0 < t_s < 1$, at which the signal s is indeed revealed to the manager. We can then consider the manager's problem as a two-period forward mean-variance problem, defined at $[0, t_s]$ and $[t_s, 1]$. Before t_s, with the investor's objective in mind, the manager would set his initial risk-return trade-off parameter as γ. However, after he observes s at t_s, the forward theory suggests that γ be updated to γ_s according to (4.16),

$$\frac{1}{\gamma_s} = r_f\big(\frac{1}{\gamma}\frac{1+\theta_0^2}{1+\theta_{t_s}^2} + \frac{r_f X_0 - X_{t_s}}{1+\theta_{t_s}^2}\big). \tag{4.22}$$

However, $[0, t_s]$ is an "artificial" time interval, inside which no trading opportunity is available. Hence we have $\theta_0 = 0$, $r_f = 1$ and $X_{t_s} = X_0$. At t_s with s observed, the manager would estimate $\theta_{t_s}^2 = \frac{(\mu(s)-r_f)^2}{\sigma_\epsilon^2(s)}$. Equation (4.22) then becomes,

$$\frac{1}{\gamma_s} = \frac{1}{\gamma}\frac{1}{1+\frac{(\mu(s)-r_f)^2}{\sigma_\epsilon^2(s)}}. \tag{4.23}$$

Therefore, the manager's conditional portfolio is given by

$$\omega(s) = \frac{1}{\gamma_s}\frac{\mu(s)-r_f}{\sigma_\epsilon^2(s)} = \frac{1}{\gamma}\frac{1}{1+\frac{(\mu(s)-r_f)^2}{\sigma_\epsilon^2(s)}}\frac{\mu(s)-r_f}{\sigma_\epsilon^2(s)} = \frac{1}{\gamma}\frac{\mu(s)-r_f}{(\mu(s)-r_f)^2+\sigma_\epsilon^2(s)},$$

which is exactly the same as the unconditionally optimal portfolio in (4.21) derived by Ferson and Siegel (2001).

We can now understand the performance gap found in our previous example.

From the perspective of Ferson and Siegel (2001), the forward investor achieves "efficient use of conditioning information", by optimally setting γ_1 based on the observed signal, which in this case is the market shock ϵ_1. On the other hand, the conventional investor who keeps a constant γ ignores conditioning information conveyed by ϵ_1, hence cannot obtain portfolio that is efficient when evaluated based on information sets at $t < 1$. Therefore we see the difference in two-period Sharpe ratio, which by definition is an unconditional measure.

4.2.3 Forward mean-variance without risk-free asset

In the first section we considered an asset market where a risk-free asset is available. The corresponding forward mean-variance strategy can be considered as a dynamic trading strategy between the risk-free asset and the tangency portfolio (with relative weights determined by γ_t). However, there has been ample evidence showing that, in practice, the portfolio that minimizes variance alone actually achieves much higher risk adjusted return than the tangency portfolio (e.g. Jagannathan and Ma (2003)). Hence in this section we re-derive forward mean-variance requiring full investment (i.e. no risk-free asset available). The resulting optimal strategy then becomes a dynamic rebalancing strategy between the tangency portfolio and the minimum variance portfolio. In particular, the forward strategy degenerates to the minimum variance strategy if the investor estimates the same expected returns for all the risky assets.

We follow exactly the same logic as in section 4.1 Firstly, for any single pe-

riod mean-variance problem, we determine the quadratic utility problem that generates the same optimal portfolio. Then, we proceed with the multi-period setting, and derive the equation showing the time-evolution of the forward quadratic utility coefficients, under the additional portfolio weight constraint. Finally, we map the forward quadratic utility problem back to forward mean-variance problem, completing our proposed construction.
The single period mean-variance now takes the form,

$$\begin{aligned} &\max_{\omega} \mathbb{E}[X_T^{\omega}] - \frac{\gamma}{2}\text{Var}(X_T^{\omega}) \\ &\text{subject to: } \omega_1 + \omega_2 + \ldots + \omega_n = 1 \end{aligned} \tag{4.24}$$

Let λ denote the Lagrange multiplier of the weights constraint. Then, (4.24) can be rewritten into the unconstrained problem,

$$\max_{\omega} \omega\mu X_0 - \frac{\gamma}{2}(\omega\Sigma\omega')X_0^2 - \lambda\omega\boldsymbol{e}, \tag{4.25}$$

where $\boldsymbol{e}$ denote the column vector of all 1's. Then, the first order condition gives

$$\mu X_0 - \gamma\Sigma\omega' X_0^2 - \lambda\boldsymbol{e} = 0.$$

Therefore, $\boldsymbol{\omega^{*\prime} = \frac{1}{\gamma X_0^2}\Sigma^{-1}(\mu X_0 - \lambda e)}$ and $\boldsymbol{\lambda = \frac{\mu'\Sigma^{-1}e}{e'\Sigma^{-1}e}X_0 - \frac{1}{e'\Sigma^{-1}e}\gamma X_0^2}$. In other words, we have found the single period optimal portfolio

$$w_{MV}^{*}{}' = \frac{1}{\gamma X_0}\Sigma^{-1}(\mu - r_v\boldsymbol{e}) + \omega_v,$$

where $\omega_v = \frac{\Sigma^{-1}\boldsymbol{e}}{\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e}}$ and $r_v = \frac{\mu'\Sigma^{-1}\boldsymbol{e}}{\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e}}$ are the weights and expected return of the *minimum variance portfolio*.

On the other hand, we may solve the fully invested quadratic utility problem

$$
\begin{aligned}
&\max_{\omega} \ \omega\mu X_0 - \frac{\delta}{2}\omega\Theta\omega' X_0^2 \\
&\text{subject to: } \omega_1 + \omega_2 + \ldots + \omega_n = 1,
\end{aligned}
\tag{4.26}
$$

where $\Theta = \Sigma + \mu\mu'$ denote the second moment matrix. Similarly, we can derive the solution,

$$
{w_{QU}^*}' = \frac{1}{\delta X_0}\Theta^{-1}(\mu - r_u\boldsymbol{e}) + \omega_u,
$$

where ω_u and r_u denote the weights and expected return of the *minimum second moment portfolio.*

Now, our objective is to find the γ, δ correspondence such that the w_{MV}^* and w_{QU}^* are the same. The result is analogous to the one in Proposition 4.1.1.

Proposition 4.2.6. *The optimal portfolio strategies for* (4.24) *and* (4.26) *coincides if and only if*

$$
\frac{1}{\delta} = \frac{1}{\gamma}\big(1 + (\mu' - r_v\boldsymbol{e})\Sigma^{-1}(\mu - r_v\boldsymbol{e})\big) + r_v X_0. \tag{4.27}
$$

Proof. See Appendix C.1. □

Comparing Propositions 4.2.2 and 4.2.6, the results are surprisingly similar. When a risk-free asset is unavailable, the manager simply synthesizes a risky portfolio that is closest to being “risk-free”, and the expected return of this (minimum variance) portfolio is treated as the risk-free rate. However, the analogy stops here. In the next step, where we derive the updating scheme of the forward quadratic utility coefficient, we no longer have that, $\tilde{\delta} = \frac{\delta}{r_v^T}$. The reason is that our “risk-free” asset is not actually risk-free. Therefore its covariance with other risky assets will distort the upcoming quadratic utility coefficient.

Before discussing the forward quadratic performance in this setting, we first derive the value function of a one period quadratic utility problem. Recall that ${w^*_{QU}}' = \frac{1}{\delta X_0}\Theta^{-1}(\mu - r_u\boldsymbol{e}) + \omega_u$. Therefore, we easily obtain that

$$\begin{aligned} V_0 &= \mathbb{E}[w^*_{QU}R]X_0 - \frac{\delta}{2}\mathbb{E}[(w^*_{QU}R)^2]X_0^2 \\ &= w^*_{QU}\mu X_0 - \frac{\delta}{2}w^*_{QU}\Theta {w^*_{QU}}' X_0^2 \\ &= r_u\big(X_0 - \frac{\delta}{2}\frac{\omega_u\Theta\omega_u'}{r_u}X_0^2\big) + c, \end{aligned} \tag{4.28}$$

where c is a function of market parameters which does not depend on X_0. Moreover, since $w_u = \frac{\Theta^{-1}\boldsymbol{e}}{\boldsymbol{e}'\Theta^{-1}\boldsymbol{e}}$ and $r_u = \frac{\mu'\Theta^{-1}\boldsymbol{e}}{\boldsymbol{e}'\Theta^{-1}\boldsymbol{e}}$, it can be verified that

$$\frac{\omega_u\Theta\omega_u'}{r_u} = \frac{1+\mu'\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\boldsymbol{e}}.$$

Therefore, we readily see that the coefficient δ gets scaled by $\frac{1+\mu'\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\boldsymbol{e}}$. To interpret this constant, let R_v denote the realized return of the minimum variance portfolio and R_μ the return of the (tangency) mean-variance efficient portfolio with weights $\omega_\mu = \frac{\Sigma^{-1}\mu}{\boldsymbol{e}'\Sigma^{-1}\mu}$. It can be verified that,

$$\frac{1+\mu'\Sigma^{-1}\mu}{\mu'\Sigma^{-1}\boldsymbol{e}} = \frac{\mathbb{E}[R_vR_\mu]}{\mathbb{E}[R_v]}.$$

In the case that a risk-free asset does exist, both the tangency and the minimum variance portfolio become the risk-free asset itself. Thus, the above ratio degenerates to r_f.

To illustrate the time evolution of the forward quadratic coefficient, consider the two trading periods $[0,T]$ and $T,\tilde{T}]$, with market parameters μ_0, Σ_0 and μ_T, Σ_T. At $t=0$ and $t=T$, the investor optimizes the quadratic utilities $U_t(X)=x-\frac{\delta_t}{2}x^2$, $t=0,T$. We now derive the equation that δ_0, δ_T should satisfy such that $U_0(x)$ and $U_T(x)$ are time-consistent, in the sense of Definition 4.2.1.

Proposition 4.2.7. *The quadratic utility problems defined by $U_0(x)$ and $U_T(x)$ are time-consistent if and only if,*

$$\delta_T=\frac{{\mu_T}'{\Sigma_T}^{-1}\boldsymbol{e}}{1+{\mu_T}'{\Sigma_T}^{-1}\mu_T}\delta_0. \tag{4.29}$$

The proof follows directly from equation (4.28). Combining Propositions 4.2.6 and 4.2.7, we arrive at the main result.

Theorem 4.2.8. *Following the notation of Theorem 4.2.5, the sequence of coefficient $\{\gamma_t\}$ defines a forward mean-variance preference if, for $t=1,2,\ldots$,*

$$\begin{aligned}&\frac{1}{\gamma_t}\big(1+({\mu_t}'-r_t^v\boldsymbol{e}')(\Sigma_t)^{-1}(\mu_t-r_t^v\boldsymbol{e})\big)+r_t^vX_t\\=&\frac{1+{\mu_t}'(\Sigma_t)^{-1}\mu_t}{{\mu_t}'(\Sigma_t)^{-1}\boldsymbol{e}}\left(\frac{1}{\gamma_{t-1}}\big(1+({\mu_{t-1}}'-r_{t-1}^v\boldsymbol{e}')(\Sigma_{t-1})^{-1}(\mu_{t-1}-r_{t-1}^v\boldsymbol{e})\big)+r_{t-1}^vX_{t-1}\right).\end{aligned} \tag{4.30}$$

Here

$$r_t^v:=\frac{{\mu_t}'(\Sigma_t)^{-1}\boldsymbol{e}}{\boldsymbol{e}'(\Sigma_t)^{-1}\boldsymbol{e}}$$

is the expected return of the global minimum-variance portfolio at $[t,t+1]$.

Proof. By Proposition 4.2.6, the equivalent quadratic utility coefficients at time $t-1$ and t are given by,

$$\frac{1}{\delta_i}=\frac{1}{\gamma_i}\big(1+({\mu_i}'-r_i^v\boldsymbol{e}')(\Sigma_i)^{-1}(\mu_i-r_i^v\boldsymbol{e})\big)+r_i^vX^i,\quad i=t-1,t.$$

By Proposition 4.2.7, time-consistency is satisfied if

$$\delta_t = \frac{\mu_t{}'(\Sigma_t)^{-1}\boldsymbol{e}}{1+\mu_t{}'(\Sigma_t)^{-1}\mu_t}\delta_{t-1}.$$

Combining the above equations we obtain (4.30). □

Recall that we have derived before,

$$\omega_t^* = \frac{1}{\gamma_t X_t}\Sigma_t^{-1}(\mu_t - r_t^v \boldsymbol{e}) + \omega_t^v = \eta_t \omega_t^\mu + (1-\eta_t)\omega_t^v,$$

where ω_t^μ, ω_t^v are the portfolio weights of the tangency and minimum variance portfolios at t, and

$$\eta_t = \frac{\mu_t{}'(\Sigma_t)^{-1}\boldsymbol{e}}{\gamma_t X_t}.$$

Therefore, the forward mean-variance strategy is essentially a dynamic rebalancing strategy between the tangency and minimum variance portfolios, with relative weights determined by γ_t. Compared with equation (4.16), γ_t determined by (4.30) also takes into account relative market condition as well as past performance. However, here "market condition" is no longer measured as the Sharpe ratio, but as the information ratio of the tangency portfolio, with the minimum variance portfolio chosen as the benchmark asset. Indeed, it is straightforward to verify, that the information ratio, denoted by IR_t, is given by

$$\mathrm{IR}_t := \frac{\mathbb{E}_t[R_{t+1}^\mu - R_{t+1}^v]}{\sqrt{\mathrm{Var}_t(R_{t+1}^\mu - R_{t+1}^v)}} = (\mu_t{}' - r_t^v \boldsymbol{e}')(\Sigma_t)^{-1}(\mu_t - r_t^v \boldsymbol{e}).$$

Therefore, (4.30) implies that the forward investor will increase γ in the case that he estimates a higher IR_t. In particular, the case of "no information" comes when the investor cannot distinguish expected returns of the risky assets, $\mu = k\boldsymbol{e}$, for some $k \in \mathbb{R}$. Then, we have that $\omega_t^\mu = \omega_t^v$, $\omega_t^* = \omega_t^v$. The entire wealth is invested in the minimum variance portfolio. In this case, the forward mean-variance problem degenerates since we cannot find γ_t following (4.30) with $\mathrm{IR}_t = 0$.

4.3 Forward mean-variance with discrete valuation and continuous trading

The forward mean-variance framework established in the previous section introduced a special time-concatenation of single period static problems. Hence, in each trading period, the investor is only allowed to trade once at the beginning, then hold the portfolio fixed until the end of the period, when the mean-variance preference gets updated. In other words, trading and preference update happen at exactly the same times. However, in reality these are independent events, therefore an ideal multi-period investment theory should allow them to occur at separate frequencies. To address this issue, we generalize the forward framework, such that the investor is permitted to trade arbitrarily many times within each period. In fact, we will look at the limiting case and allow for continuous trading. Hence, we look to define the problem in the following form. Let $0 = t_0 < t_1 < \dots t_i < \dots$, on each time interval $[t_i, t_{i+1}]$ the investor solves the mean-variance problem,

$$\max_{\pi_s, t_i \le s \le t_{i+1}} \mathbb{E}[X^\pi(t_{i+1})] - \frac{\gamma_i}{2}\mathrm{Var}(X^\pi(t_{i+1})), \tag{4.31}$$

where $\pi(s)$, $s \in [t_i, t_{i+1}]$ is a continuous rebalancing strategy between t_i and t_{i+1}. Then we aim to apply the idea of the previous section to find the time-consistent approach to generate the sequence $\{\gamma_i\}_{i=0}^{\infty}$. We stress however, that since we allow continuous-time rebalancing, the single period problem we now face becomes the so called *dynamic mean-variance* optimization problem, which is no longer trivial to solve. We briefly review recent developments in dynamic mean-variance in section 4.3.1. In section 4.3.2, we introduce the notion of predictable forward utility, which generalizes the time-monotone forward utility in chapter 1, such that utility functions are updated only discretely while trading is continuous. Finally in section 4.3.3, we combine the notions above and establish our main theory of predictable mean-variance.

4.3.1 Dynamic mean-variance optimization within a single period

Assume that the market consists of n risky assets S_1, S_2 $\ldots$, S_n, with dynamics governed by the SDE system,

$$dS_k(t) = S_k(t)\big(\mu_k(t)dt + \sum_{j=1}^{K} \sigma_{kj}(t)dW_j(t)\big),\ k = 1, 2, \ldots, n, \tag{4.32}$$

and one risk-free asset S_0, with short rate $r(t)$, i.e.,

$$dS_0(t) = r(t)S_0(t)dt.$$

The parameters $\mu(t) = [\mu_1(t), \ldots, \mu_n(t)]'$ and $\Sigma(t) = \{\sigma_{kj}(t)\}_{k,j=1}^{n}$ are stochastic processes adapted to the filtration generated by the Brownian motion $W(t)$.

The risk-free rate $r(t)$ is assumed to be deterministic. The dynamic mean-variance problem defined at a single period $[0, T]$ aims at finding an optimal continuous-time self-financing strategy $\pi(t)$, $t \in [0, T]$, such that the terminal wealth $X^\pi(T)$ under π maximizes the mean-variance objective function,

$$\mathbb{E}[X^\pi(T)] - \frac{\gamma}{2}\mathrm{Var}((X^\pi(T))^2). \tag{4.33}$$

The above objective function is parameterized by the investor's trade-off between the mean and variance of terminal wealth. It is well known that problem (4.33) can be more intuitively formulated as minimizing the variance while achieving a target terminal wealth.

$$\begin{aligned} \underset{\pi(t)}{\text{minimize}}\ & \mathrm{Var}(X^\pi(T)), \\ \text{subject to:}\ & \mathbb{E}[X^\pi(T)] = d. \end{aligned} \tag{4.34}$$

It is then straightforward to derive a one-to-one correspondence between γ and d such that problems (4.33) and (4.34) are equivalent. To make our results comparable to those in the existing literature, in most of the analysis that follows we adopt the formulation given by (4.34). Before discussing the results, it is worth mentioning that there is one specific, well-known issue in the definition of dynamic mean-variance optimization. Unlike the static counterpart, the dynamic problem is inherently time-inconsistent, in the sense that for any $t_2 > t_1 \geq 0$, an optimal policy $\pi^*_{t_1}(s)$ derived at t_1 is no longer seen as optimal at t_2, or $\pi^*_{t_2}(s) \neq \pi^*_{t_1}(s)$ for any $s > t_2$. Hence, a policy that is "optimal" from the perspective of every point in time is virtually non-existent. Most of the work done so far circumvented this issue by only focusing on the so called pre-commitment policy $\pi_0^*(s)$, the policy that is only optimal when seen at time zero. The first study that truly deals with the issue of time-inconsistency was Basak and Chabakauri (2010). There, dynamic mean-variance optimization was formulated as an intra-personal game, which the investor plays with different copies of himself at all points in time. Hence the "optimal" policy was defined as the equilibrium policy, a policy that makes each time-copy of the investor equally happy. For further results along this line of research, see Bjork et al. (2014), Bjork et al. (2017).

The drawback with the game-theoretic approach is that the portfolio it generates is often too conservative. The comparative study of Angoshtari et al. (2015) found that because of the low risk feature, the annual certainty equivalent return it generates it less than one fifth that of the pre-commitment policy. For this reason we still adopt the pre-commitment approach for solving each single period dynamic problems.

The pre-commitment optimal solution to problem (4.34) was first provided by Bajeux-Besnainou and Portait (1998) and Li and Zhou (2000), under a log-normal market with deterministic model coefficients. Under a complete market model with random coefficients, Lim and Zhou (2002) characterized the optimal portfolio using the techniques of stochastic linear-quadratic control. Furthermore, also assuming market completeness, Bielecki et al. (2005) employed the convex duality argument to derive the efficient portfolio under no-bankruptcy constraints. Xiong and Zhou (2007) considered the case of partial information, in which the model parameters are uncertain and needs to be learned. For dynamic mean-variance with transaction cost and more general trading constraints, see Dai et al. (2010), Hu and Zhou (2005).

We now follow the idea of Bielecki et al. (2005) and describe the method for solving problem (4.34). Let $Z(t)$ denote the stochastic discount factor process. It follows easily that

$$dZ(t) = Z(t)(-r(t)dt - \lambda(t) \cdot dW(t)),$$

where $\lambda(t) = \Sigma^{-1}(t)(\mu(t) - r(t)\boldsymbol{e})$ denote the market price of risk vector. The self-financing constraint can be simply expressed as,

$$\mathbb{E}[X^\pi(T)Z(T)] = X(0).$$

Hence, we can rewrite problem (4.34) as

$$\begin{aligned} &\underset{\pi(t)}{\text{minimize}}\ \text{Var}(X^\pi(T)), \\ &\text{subject to: } \mathbb{E}[X^\pi(T)] = d, \quad \mathbb{E}[Z(T)X(T)] = X(0). \end{aligned} \tag{4.35}$$

By proposition 4.1 of Bielecki et al. (2005), there exists a pair of deterministic coefficients η_1 and η_2, such that the above is equivalent to the unconstrained optimization,

$$\underset{\pi(t)}{\text{minimize}}\ \mathbb{E}[(X^\pi(T))^2] - 2\eta_1(T)\mathbb{E}[X^\pi(T)] - 2\eta_2(T)\mathbb{E}[X(T)Z(T)], \tag{4.36}$$

From the first order condition we have

$$X^*(T) = \eta_1(T) + \eta_2(T)Z(T),$$

For some $\eta_1(T)$ and $\eta_2(T)$. Then η_1 can η_2 can be obtained by solving the system of equations

$$\begin{cases} \eta_1(T) + \eta_2(T)\mathbb{E}[Z(T)] = d, \\ \eta_1(T)\mathbb{E}[Z(T)] + \eta_2(T)\mathbb{E}[(Z(T))^2] = X(0). \end{cases} \tag{4.37}$$

Therefore,

$$\eta_1(T) = \frac{d\mathbb{E}[(Z(T))^2] - X(0)\mathbb{E}[Z(T)]}{\text{Var}(Z(T)}, \quad \eta_2(T) = \frac{X(0) - d\mathbb{E}[Z(T)]}{\text{Var}(Z(T))}.$$

We now have an explicit solution for the optimal terminal wealth,

$$X^*(T) = \frac{d\mathbb{E}[(Z(T))^2] - X(0)\mathbb{E}[Z(T)]}{\text{Var}(Z(T)} + \frac{X(0) - d\mathbb{E}[Z(T)]}{\text{Var}(Z(T))}Z(T). \quad (4.38)$$

Finally, the optimal portfolio $\pi^*(t)$ is obtained by solving the backward stochastic differential equation

$$dX^*(t) = rX^*(t)dt + (\mu(t) - r(t)\boldsymbol{e})\pi(t)'dt + \Sigma(t)\pi(t)dW(t), \quad (4.39)$$

where the terminal condition is given by (4.38). In particular, when the market parameters are deterministic, the above BSDE can be solve be in closed form, yielding

$$\pi^*(t) = -(\Sigma(t)\Sigma(t)')^{-1}(\mu(t) - r(t)\boldsymbol{e})'[X^*(t) - \phi e^{-\int_t^T r(s)ds}], \quad (4.40)$$

where

$$\phi := \frac{d - X(0)e^{\int_0^T (r(t) - |\lambda(t)|^2 dt}}{1 - e^{-\int_0^T |\lambda(t)|^2 ds}}. \quad (4.41)$$

4.3.2 Predictable forward performance process

In section 4.2.1 we have defined the discrete time forward utility process. However, the definition requires the underlying single period problems to be static, hence dynamic rebalancing is not allowed. For this reason, in this section we apply the theory of predictable utility process recently established in Angoshtari et al. (2015), where the authors assumed continuous or discrete trading under discretely updated utility functions. We recall their definition below.

Assume that an investment paradigm is defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ augmented with the filtration $(\mathcal{F})_t$, $t \geq 0$, and that the set of the admissible wealth processes is denoted by $\mathcal{A}$. (See Angoshtari et al. (2015) for further details.)

Definition 4.3.1. Let $0 = t_0 < t_1 < \ldots < t_i < \ldots$ be a sequence of points in time, and $\{\mathcal{G}\}_i$, $i = 0, 1, \ldots$ be a sub-filtration, i.e. $\mathcal{G}_i \subset \mathcal{F}_{t_i}$. A family of random functions $\{U_i(\cdot)\}_{i \geq 0}$ is a forward performance criteria if:

(i) $U_i(\cdot) \in C^2(\mathbb{R}^+)$ is $\mathcal{G}_{i-1}$-measurable, increasing, concave and satisfy the *Inada conditions*, for $i = 0, 1, \ldots$.

(ii) For any admissible wealth process $X(t)$, $t \geq 0$, the discrete process $\{U_i(X(t_i))\}_{i \geq 0}$ is a $\{\mathcal{G}_i\}$-supermartingale, namely

$$U_{i-1}(X(t_{i-1})) \geq \mathbb{E}\big[U_i(X(t_i))|\mathcal{G}_{i-1}\big], \quad \forall i = 0, 1, \ldots \quad X(t) \in \mathcal{A}. \tag{4.42}$$

(iii) There exists an admissible wealth process $\{X(t)^*\}$, such that $\{U_i(X(t_i)^*)\}$ is a $\{\mathcal{G}_i\}$-martingale, i.e.

$$U_{i-1}(X^*(t_{i-1})) \geq \mathbb{E}\big[U_i(X^*(t_i))|\mathcal{G}_{i-1}\big], \quad \forall i = 0, 1, \ldots \tag{4.43}$$

To implement the above framework, one specifies an initial utility input $U_0(x)$ at $t = 0$. The investment problem at $[t_{i-1}, t_i]$ is solved by recursively following the procedure described below.

1. At t_{i-1}, estimate a model M_i which describes the dynamics of the market at $[t_{i-1}, t_i]$.

2. Find U_i by solving the *inverse* optimization problem,

$$U_{i-1}(x) \;=\; \max_{\pi\in\mathcal{A}} \mathbb{E}^{M_i}_{t_{i-1}}[U_i(X^\pi(t_i))|X(t_{i-1}) = x]. \tag{4.44}$$

3. Solve optimal portfolio strategy π_t, $t \in [t_i, t_{i+1}]$ by,

$$\pi \;=\; \operatorname*{argmax}_{\pi\in\mathcal{A}} \mathbb{E}^{M_i}_{t_{i-1}}[U_i(X^\pi(t_i))|X_(t_{i-1}) = x]. \tag{4.45}$$

Note that the major advantage of the above framework is that model specification as well as portfolio construction can both be conducted in short, medium or long horizons. Based on the information available up to t_{i-1}, the agent has full flexibility in determining which model to adopt for the current investment period, while the utility functions constructed from equation (4.44) guarantee that portfolio strategies at different periods are *time-consistent*, which prohibits the agent from arbitrarily altering his risk attitude from period to period. As it will be illustrated in subsequent sections, maintaining a stable risk attitude over time is crucial for achieving better long run performance.

4.3.3 Predictable forward mean-variance

We are now ready to establish the theory of predictable forward mean-variance. Let $0 = t_0 < t_1 < \ldots < t_N < \ldots$ and assume that at each $[t_{i-1}, t_i]$ the investor solves the dynamic mean-variance problem (MV$_i$),

$$\begin{gathered}\min_{\pi} \operatorname{Var}_{i-1}\big(X^\pi(t_i)\big),\\ \text{subject to: } \mathbb{E}_{i-1}[X^\pi(t_i)] = d_{i-1}, \quad d_{i-1} \in \mathcal{F}_{t_{i-1}}.\end{gathered} \tag{4.46}$$

The condition $d_{i-1} \in \mathcal{F}_{t_{i-1}}$ ensures that the target wealth to be achieved at t_i is fully determined by the information available up to the beginning of the period. We now provide a construction for the sequence of target wealth $\{d_i\}_{i=0}^{\infty}$, such that the dynamic mean-variance problems defined at different trading periods are time-consistent.

Definition 4.3.2. The sequence of mean-variance preferences $\{\mathrm{MV}_i\}_{i=0}^{\infty}$ given by (4.46) is said to be a forward mean-variance preference, if there exists a predictable forward quadratic performance process $\{U_i\}_{i=0}^{\infty}$, such that MV_i and U_i are equivalent in that they have identical optimal portfolios, for all $i \geq 0$.

The definition prescribes an approach to generate mean-variance preferences forward in time, while maintaining time-consistency. Suppose that the investor has determined the target mean d_{i-1} to be achieved at t_i, in order to find an appropriate target for the period $[t_i, t_{i+1}]$, we first "transform" the mean-variance problem at t_{i-1} into its equivalent quadratic utility problem. Then we apply the definition of forward utility, and solve the inverse dynamic optimization problem (4.44) to determine the time-consistent quadratic utility to be applied for the next period. Finally, we map the quadratic utility function back to the mean-variance objective function, thus obtaining d_i.

Assume at t_{i-1} the investor has estimated the market model at $[t_{i-1}, t_i]$ to be

$$dS_k(t) = S_k(t)\big(\mu_k^{i-1}(t)dt + \sum_{j=1}^{K} \sigma_{kj}^{i-1}(t)dW_j(t)\big),\ k = 1, 2, \ldots, n,\ t \in [t_{i-1}, t_i]. \tag{4.47}$$

Denote the pricing kernel in this period by $Z(t)$, i.e.

$$Z(t) = Z(t_{i-1})\exp\big(-\int_{t_{i-1}}^{t}(r(s)+\frac{1}{2}|\lambda(s)|^2)ds-\int_{t_{i-1}}^{t}\lambda(s)\cdot dW(s)\big), \quad t \in [t_{i-1}, t_i].$$

To accomplish the first step, define the quadratic utility function to be solved at $[t_{i-1}, t_i]$ as $U_{i-1}(x) = 2\eta_{i-1}x - x^2$, with η_{i-1} being $\mathcal{F}_{t_{i-1}}$-measurable. We have the following equivalence result,

Theorem 4.3.3. *The quadratic utility problem*

$$\min_{\pi\in\mathcal{A}} \mathbb{E}_{i-1}[X^\pi(t_i)^2 - 2\eta_{i-1}X^\pi(t_i)] \tag{4.48}$$

has the same optimal portfolio as the mean-variance problem (4.46) *if and only if* η_{i-1} *is given by,*

$$\eta_{i-1} = d_{i-1}\frac{\mathbb{E}_{i-1}[Z(t_i)^2]}{Var_{i-1}(Z(t_i))} - Z(t_{i-1})\frac{X(t_{i-1})\mathbb{E}_{i-1}[Z(t_i)]}{Var_{i-1}(Z(t_i))}. \tag{4.49}$$

Proof. See Appendix C.2. □

Next, we show how to construct predictable forward quadratic performance process based on solving the inverse utility optimization problem (4.44). The following theorem provides the main result.

Theorem 4.3.4. *Define a sequence of quadratic utility* $\{U_i(x)\}_{i=0}^{\infty}$ *functions as,*

$$U_i(x) = a_i(x-\eta_i)^2 + b_i, \quad a_i < 0 \ a.s. \quad i = 1, 2, \ldots,$$

where the coefficients a_i*,* b_i *and* η_i *are* $\mathcal{F}_i$*–measurable. If the risk-free rate* $r(t)$ *is deterministic, then* $\{U_i(x)\}_{i=0}^{\infty}$ *is a predictable forward performance if and only if*

$$\eta_i = e^{\int_{t_i}^{t_{i+1}} r(s)ds}\eta_{i-1}, \quad n = 1, 2, \ldots. \tag{4.50}$$

Proof. See Appendix C.2. □

Combing the theorems 4.3.3 and 4.3.5, we obtain the main result summarized as below,

Theorem 4.3.5. *The sequence of mean-variance preferences* $\{MV_i\}_{i=0}^{\infty}$,

$$\begin{aligned} &\min_{\pi} \ Var_{i-1}\big(X^{\pi}(t_i)\big), \\ &\textit{subject to: } \mathbb{E}_{i-1}[X^{\pi}(t_i)] = d_{i-1}, \quad d_{i-1} \in \mathcal{F}_{t_{i-1}}. \end{aligned} \tag{MV_i}$$

is a predictable forward mean-variance preference if the wealth targets satisfy

$$d_i = e^{\int_{t_i}^{t_{i+1}} r(s)ds}\big[d_{i-1}\frac{1+\xi_{i-1}}{1+\xi_i} + \frac{\xi_i X(t_i) - e^{\int_{t_{i-1}}^{t_i} r(s)ds}\xi_{i-1}X(t_{i-1})}{1+\xi_i}\big], \tag{4.51}$$

where $\xi_j = \frac{\mathbb{E}_j[Z(t_{j+1})]^2}{Var_j(Z(t_{j+1}))}$, *with* $j = i-1, i$.

Proof. By equation (4.49), we can replace η_j, $j = i-1, i$ in (4.50) by $d_j\frac{\mathbb{E}_j[Z(t_{j+1})^2]}{\mathrm{Var}_j(Z(t_{j+1}))} - Z(t_j)\frac{X(t_j)\mathbb{E}_j[Z(t_{j+1})]}{\mathrm{Var}_j(Z(t_{j+1}))}$. Rearranging the terms gives us equation (4.51). □

To provide some intuition for the above results, we first note that the term ξ_j is related to the market ratio. Let SR_{i-1} denote the highest Sharpe ratio achievable at $[t_{i-1}, t_i]$. By Cvitanić et al. (2008), we have

$$SR_i = \frac{\sqrt{\mathrm{Var}_{i-1}(Z(t_i))}}{\mathbb{E}_{i-1}[Z(t_i)]} \tag{4.52}$$

Let us further set $\theta_i := \frac{SR_i^2}{1+SR_i^2}$. Define also the target gain (TG) and realized gain (RG), as the wealth gain in excess of the gain by investing in the risk-free asset only, i.e.,

$$\begin{aligned}\mathrm{TG}_{i-1} &:= d_{i-1} - e^{\int_{t_{i-1}}^{t_i} r(s)ds} X(t_{i-1}) \\ \mathrm{RG}_{i-1} &:= X(t_i) - e^{\int_{t_{i-1}}^{t_i} r(s)ds} X(t_{i-1}).\end{aligned}$$

Then equation (4.51) can be more concisely represented as

$$\mathrm{TG}_i = \frac{\theta_i}{\theta_{i-1}} \mathrm{TG}_{i-1} - \theta_i \mathrm{RG}_{i-1}. \tag{4.53}$$

Equation (4.53) suggests a straightforward interpretation of how the wealth targets get updated. At $[t_i, t_{i+1}]$, the agent would target a gain that equals the gain targeted in the last period, scaled by the relative "market condition", expressed via the coefficient $\frac{\theta_i}{\theta_{i-1}}$, and subtract the gain realized in the last period, scaled by θ_i. Therefore, a connection between the MV problems at different periods has been established. At the beginning of each period, the agent would assess whether the market has became better or worse, and respond to the changes by adjusting upward or downward the target pursued before.

In the special case when the parameters μ^i and σ^i are constant at $[t_{i-1}, t_i]$, we have

$$\begin{aligned}SR_i &= \sqrt{e^{|\lambda^i|^2 \Delta t} - 1} \\ \theta_i &= 1 - e^{-|\lambda^i|^2 \Delta t}.\end{aligned}$$

For a concrete example, assume that the investor solves two consecutive mean-variance problems at $t \in [0,1]$ and $t \in [1,2]$. Assumed also that the risk-free rate is zero in the first period, the stock market has an annualized rate of return of 10%, with 20% volatility. Then $SR_0 = 0.53$ and $\theta_0 = 0.22$. If the investor starts with wealth of, say, one dollar, and sets a target return of $d_0 = 8\%$ to be achieved by the end of the first year, then, according to equation (4.41), the optimal amount of wealth invested in the market index is given by

$$\pi^*(t) = 3.4 - 2.5X^*(t), \quad t \in [0,1].$$

At $t = 0$ in particular, 90% of the wealth will be invested in the index.

At $t = 1$, suppose that the investor estimates a turbulent market in the coming year, with volatility increases to 60%, but expected return stays at 10%. Then, $SR_1 = 0.17$ and $\theta_1 = 0.027$. The question is, how should the investor set the target to be achieved by year two? According to (4.53), we have

$$TG_1 = 0.01 - 0.027(X^*(1) - 1).$$

Since $X^*(1)$ given by equation (4.38) follows a shifted log-normal distribution, we are able to calculate the expected target gain at $t = 1$. In terms of target return, we have

$$\mathbb{E}[\frac{TG_1}{X^*(1)}] - 1 = 0.0074.$$

In fact, in response to the volatility hike, the investor dramatically lowers his target to less than one percent!

This is in sharp contrast to the assumption made in Cvitanić et al. (2008), that a multi-period mean-variance investor would maintain constant return targets. In fact, if the investor indeed chooses to maintain the 8% target return for the second period, he will only achieve a two-year Sharpe ratio of 0.3. On the contrary, the forward mean-variance investor who lowers the target achieves a Sharpe ratio of 1.1, which is a considerable improvement!

We close this section by pointing out that the forward mean-variance constructed in (4.51) and (4.53) is 'viable, in the sense that the target gain $\mathrm{TG}_n \geq 0$, $\forall\, i = 0, 1, \ldots$. If this condition fails, then the agent would target a return smaller than the risk-free rate, resulting in a portfolio that falls on the inefficient half of the mean-variance frontier. The results below ensure that this situation is excluded, provided that the initial mean-variance criteria is efficient.

Theorem 4.3.6. *If $TG_0 \geq 0$, then $TG_i \geq 0$ for $i = 0, 1, \ldots$.*

Proof. Clearly, it is enough to show $\mathrm{TG}_{i-1} \geq 0$ implies $\mathrm{TG}_i \geq 0$. First, we prove that

$$TG_{i-1} \geq 0 \Longleftrightarrow \eta_{i-1} \geq e^{\int_{t_{i-1}}^{t_i} r(s)ds} X(t_{i-1}) \ , \forall i \geq 1. \tag{4.54}$$

Rearranging equation (4.49) into

$$\mathrm{TG}_{i-1} = d_{i-1} - \frac{Z(t_{i-1})}{\mathbb{E}_{i-1}[X(t_i)]} X(t_{i-1}) = \frac{\mathrm{Var}_{i-1}(Z(t_i))}{\mathbb{E}_{i-1}[Z(t_i)^2]} \big(\eta_{i-1} - \frac{Z(t_{i-1})}{\mathbb{E}_{i-1}[X(t_i)]} X(t_{i-1})\big),$$

the equivalence follows since $\frac{\mathrm{Var}_{i-1}(Z(t_i))}{\mathbb{E}_{i-1}[Z(t_i)^2]} > 0$ and $\frac{Z(t_{i-1})}{\mathbb{E}_{i-1}[X(t_i)]} = e^{\int_{t_{i-1}}^{t_i} r(s)ds}$, when $r(t)$ is deterministic.

It is easy to derive that the optimal terminal wealth $X^*(t_i)$ has the representation,

$$X^*(t_i) = \eta_{i-1} + \big(X(t_{i-1}) - e^{-\int_{t_{i-1}}^{t_i} r(s)ds}\eta_{i-1}\big)\frac{Z(t_i)Z(t_{i-1}))}{\mathbb{E}_{i-1}[Z(t_i)^2]} \tag{4.55}$$

Since $X(t_{i-1}) - e^{-\int_{t_{i-1}}^{t_i} r(s)ds}\eta_{i-1} \leq 0$ by (4.54) and $Z(t) > 0$, we have

$$X^*(t_i) \leq \eta_{i-1}.$$

By (4.50) and the above inequality, we deduce that $\eta_i = e^{\int_{t_i}^{t_{i+1}} r(s)ds}\eta_{i-1} \geq e^{\int_{t_i}^{t_{i+1}} r(s)ds}X^*(t_i)$. The equivalence relation in (4.54) implies $\mathrm{TG}_i \geq 0$. □

4.3.4 Multi-period performance analysis under model uncertainty

A natural question is why in the first place do we choose to solve dynamic mean-variance problems period over period? Instead of solving N problems at $[t_i, t_{i+1}]$, $i = 0, 2, \ldots N-1$, shouldn't it be better to just solve a single dynamic problem defined at $[t_0, t_N]$? Indeed, if the investor is capable of knowing exactly what the market model is, running a single mean-variance optimization formulated over the entire investment lifetime would generate the best risk-adjusted return. However, uncertainties in reality more often present themselves as the "unknown-unknowns", the distributions of which cannot be modeled easily far in advance, for even with dynamic learning, a priori parametric assumption still needs to be made! Hence, the need to model and optimize in shorter horizons, to delay making assumptions until sufficient information arises.

Because of the multi-period nature of forward mean-variance, at t_{i-1}, the agent needs only to estimate a model that works at $[t_{i-1}, t_i]$, and remains agnostic about what would happen beyond t_i, therefore avoiding making incorrect assumptions when information is insufficient. However, this convenience comes at a cost. Since forward MV optimizes in short terms, it seems that only short term optimality is guaranteed. Indeed, optimal strategies derived under the forward approach may not take into account return distributions beyond the current period, hence is impossible to achieve "global", long term optimality. As a result, when model uncertainty is not significant, we expect the forward MV to underperform the classical approach in terms of long run performance measures. In this section, we present numerical examples to illustrate *the trade-off between investing myopically and investing under wrong model assumptions*. Regarding the former, we further compare short term mean-variance strategies with and without imposing the time-consistency condition. Our comparative study is closely related to that of Cvitanić et al. (2008). Therein, the authors focused on comparing long-run performances of dynamic mean-variance investors who choose to optimize a single, long horizon mean-variance objective function, and those who optimize repeatedly but over short horizons. They find that short term optimizers do suffer tremendously in terms of long term Sharpe ratio.

In our study, we add to the comparative study a third type of investors, namely the forward mean-variance investors. From now on we refer to the long term and short term investors as the *backward* and *myopic* investors, respectively, to emphasize that the former solves a single dynamic control problem backward in time, while the latter is concerned exclusively with near-term optimality. Their optimization criteria are listed blow.

1. The *backward investor* solves a single MV problem on $[0, T]$,

$$\begin{aligned} &\min_{\pi} \operatorname{Var}\big(X^{\pi}(T)\big), \\ &\text{subject to: } \mathbb{E}[X^{\pi}(T)] = d = e^{RT} X(0). \end{aligned} \tag{4.56}$$

2. The *forward investor* solves a sequence of short term MV problems,

$$\begin{aligned} &\min_{\pi} \operatorname{Var}_{i-1}\big(X^{\pi}(t_i)\big), \\ &\text{subject to: } \mathbb{E}_{i-1}[X^{\pi}(t_i)] = d_{i-1}, \quad d_{i-1} \in \mathcal{F}_{t_{i-1}}. \end{aligned} \tag{4.57}$$

   with $d_0 = e^{R\Delta t} X(0)$. $d_i, i \geq 1$ are determined by (4.51).

3. The myopic investor solves a sequence of short term MV problems, as in (4.57), with only difference being that the investment targets are given by

$$d_{i-1} = e^{R\Delta t} X(t_{i-1}).$$

In subsequent numerical analysis, we take $T = 5$, $t_i = i\Delta t$, $\Delta t = 0.25$, i.e. the forward and myopic investors would split up the five-year investment program into twenty quarter-long trading periods, while the backward investor optimizes a single 5-year mean-variance objective.

To develop a meaningful comparison between the investors, we have assumed the backward and myopic investor target the same instantaneous rate of return R. Hence, the difference in their wealth targets, namely $d_b = X(0)e^{RT}$ and $d_m^i = X(t_i)e^{R\Delta t}$, comes only from the difference in their horizons. The forward investor sets targets dynamically, but we assume that his initial target matches that of the myopic investor.

The main difference between our work and Cvitanić et al. (2008) lies in the market model assumptions. In their study, the risky asset is modeled as a diffusion with stochastic drift,

$$\begin{cases} dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t), \\ d\mu(t) = \kappa(\beta - \mu(t))dt - \sigma_\mu dW(t). \end{cases} \tag{CLW08}$$

The more important underlying assumption is that the above "true" model is fully known by the investors. However, in reality the true model can never be accurately estimated, hence it is more practically relevant to study how the investors perform when only part of model can be estimated. As a result, we propose the following changes to the model.

$$\begin{cases} dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t), \\ d\mu(t) = \kappa(\beta - \mu(t))dt + \sigma_\mu d\tilde{W}(t). \\ dW(t)d\tilde{W}(t) = \rho dt, \quad \rho \in \{-1, 1\}. \end{cases} \tag{4.58}$$

Furthermore, we assume that ρ oscillates between -1 and 1, and its distribution is not known to the investor. When $\rho = 1$, the market is said to be in *momentum*, as it suggests that a positive shock to the realized return is accompanied by a positive shock to the expected return. Similarly, when $\rho = -1$, we call the market *mean-reverting*. Note that these are polar opposite market conditions under which optimal portfolio strategies are drastically different. Hence, a wrong forecast of market state could lead to dire consequences in performance. To see this, we calculate in equation (4.59) the time zero optimal portfolios under both market states. We then obtain

$$\pi^*_{MR} = X(0)\Lambda^+(\tau,\mu(0))\frac{e^{(R-r)\tau}-1}{\Lambda^+(\tau,\mu(0))-1}\left((1+A^+(\tau)\sigma_\mu))\frac{\mu(0)}{\sigma^2}+B^+(\tau)\frac{\sigma_\mu}{\sigma}\right)$$
$$\pi^*_{MTM} = X(0)\Lambda^-(\tau,\mu(0))\frac{e^{(R-r)\tau}-1}{\Lambda^-(\tau,\mu(0))-1}\left((1-A^-(\tau)\sigma_\mu))\frac{\mu(0)}{\sigma^2}-B^-(\tau)\frac{\sigma_\mu}{\sigma}\right), \tag{4.59}$$

where $\tau = T, \Delta t$ denotes the investment horizon. The quantities Λ, A, B are functions of model parameters and τ, derived in Cvitanić et al. (2008).

We also, define the instantaneously optimal portfolio π^{ins} as the optimal portfolio of an investor who does not take into account future changes in market parameters. Then π^{ins} can be obtained by replacing $\mu(t)$ with $\mu(0)$ in equation (4.40). The difference between $\pi^*(t)$ and π^{ins} is what is usually referred to as the hedging demand, namely, the additional risky holdings of the investor to hedge future changes in the market opportunity set.

We plot the hedging demand as a function of investment horizon for both π^{MR} and π^{MTM} in Figure 4.3. We can then see that, under different market states, the hedging demand have different signs, and the discrepancy grows larger with longer horizon. Therefore, the backward investor who solves a long horizon problem would suffer the most when model mis-identification occurs. For instance, if a momentum market is estimated but a mean-reverting market occurs, instead of putting an extra (in excess of π^{ins}) 70% of wealth in the index, the backward investor would sell short S in the amount 14% of his wealth. On the other hand, the forward and myopic investors looking at one quarter ahead, will hedge by putting only 20% (as opposed to 70%) in the index, a significant under-hedging! Hence it is clear that when the exact model

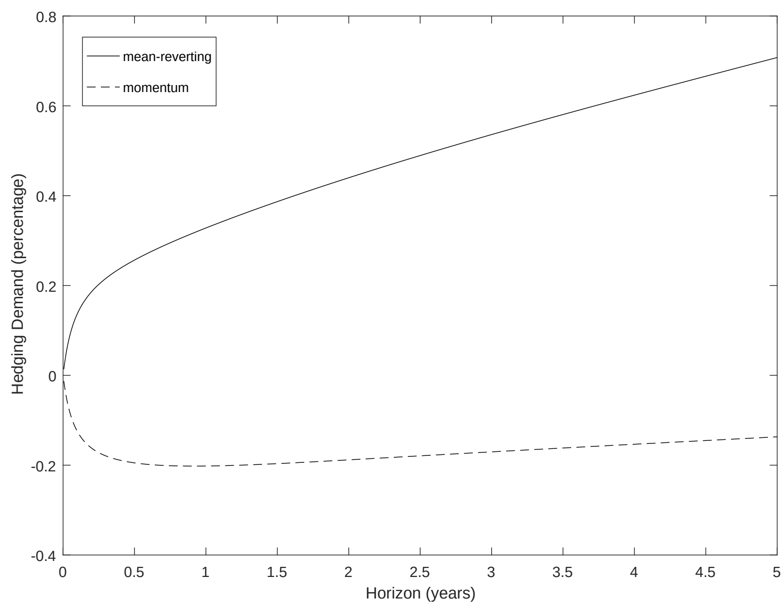


Figure 4.3: Hedging demand and horizon

cannot be known, the trade-off between optimizing in long and short horizons is essentially a complicated trade-off between sub-optimality from modeling error and from under-hedging.

To gauge the exact impact of these effects, we further assume that the true value of ρ changes after every trading period. At $[t_{i-1}, t_i]$, the market model is then given by

$$\begin{cases} dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t), \\ d\mu(t) = \kappa(\beta - \mu(t))dt + \rho_{i-1}\sigma_\mu dW(t). \quad t \in [t_{i-1}, t_i]. \end{cases} \tag{4.60}$$

where ρ_i is a stochastic process takes value in $\{-1, 1\}$ and is independent of $W(t)$. The probability structure is characterized by $(\Omega, \mathcal{F}, \mathbb{P})$, where the filtration $\mathcal{F}$ is generated by both W_t and ρ_i, i.e. $\mathcal{F}^{true} = \mathcal{F}_W \vee \mathcal{F}_\rho$. We assume that the investors observe S_t, μ_t and ρ_i but does not know the true probability structure of ρ_i. Note that the short term agents (forward and myopic) are not affected by the uncertainties in future values of ρ_i, as their optimization problems only require observing the model in the current period. The backward agent however, needs to know the probability distribution for the entire future path of ρ and therefore he has to make subjective assumptions. Since the true distribution of ρ_i is unknown, one approach for the backward investor is to treat it as Knightian uncertainty and formulate the problem as a min-max optimization. In other words, the agent chooses a portfolio strategy such that the variance is minimized if the worst possible model for B_n turns out. However, our objective here is not to look for the best possible way for the backward agent to deal with such model risk, but rather, we are interested in quantifying the relationship between model errors and portfolio performance. Therefore, we make the rather simplistic assumption that the backward investor imposes a trivial subjective probability measure on ρ_i, i.e. $\rho_j \equiv \rho_i$ for $j \geq i$.

We assume however, that the true ρ_i follows a discrete Markov process, with transition probability matrix,

$$Q = \begin{bmatrix} 1-q & q \\ q & 1-q \end{bmatrix}, \quad q \in [0,1].$$

Here q characterizes the rate at which the market switches states at the beginning of each trading period. In particular, $q = 0$ implies that the market remain at a fixed state throughout $[0, T]$. In this case, the backward agent makes no modeling errors, hence he outperforms the other two since his optimization is based on the entire investment horizon.

Figure 4.4 plots 5-year mean-standard deviation frontiers, for $q = 1,\ 0.4\,, 0.05\,, 0$. In all cases except the last one ($q = 0$) the forward frontier dominates. As q decreases, the performance gap shrinks as the backward agent is less prone to modeling error. Notice also that the backward and forward frontiers are both straight lines, suggesting that the Sharpe ratio is a target-independent performance measure. Figure 4.5 plots 5-year Sharpe ratio for all three types of investors. The backward investor outperforms only for q close to 0. As q grows, his Sharpe ratio rapidly declines and is soon dominated by the forward, after $q > 0.02$. Therefore, running a classical dynamic optimization throughout the entire horizon is no longer optimal if model estimation is error-prone. Here a 2% chance of error lowers the long run Sharpe ratio by as much as 23%, making the short term optimization with forward preference a better alternative. Furthermore, note that, while both investors optimize at short

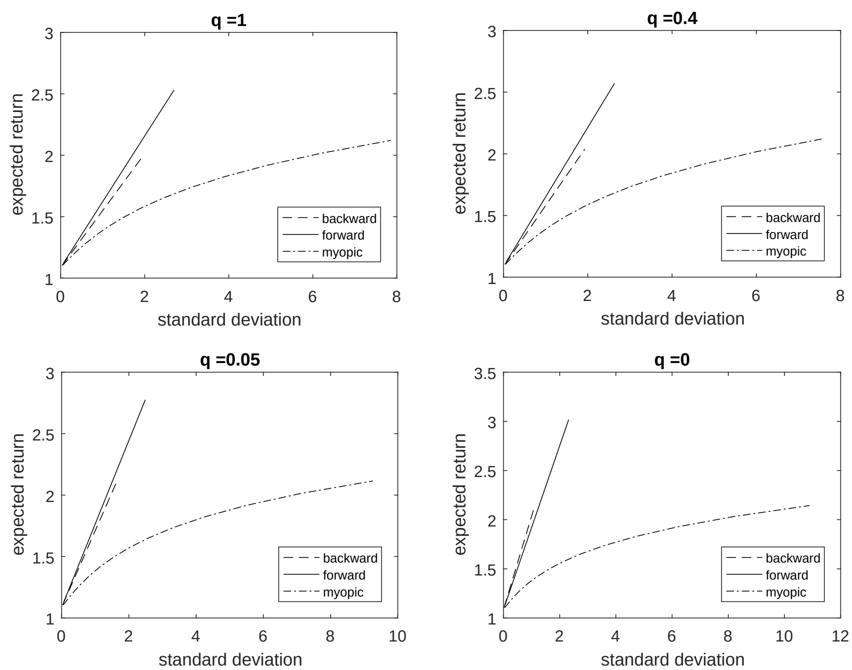


Figure 4.4: mean-standard deviation frontier.

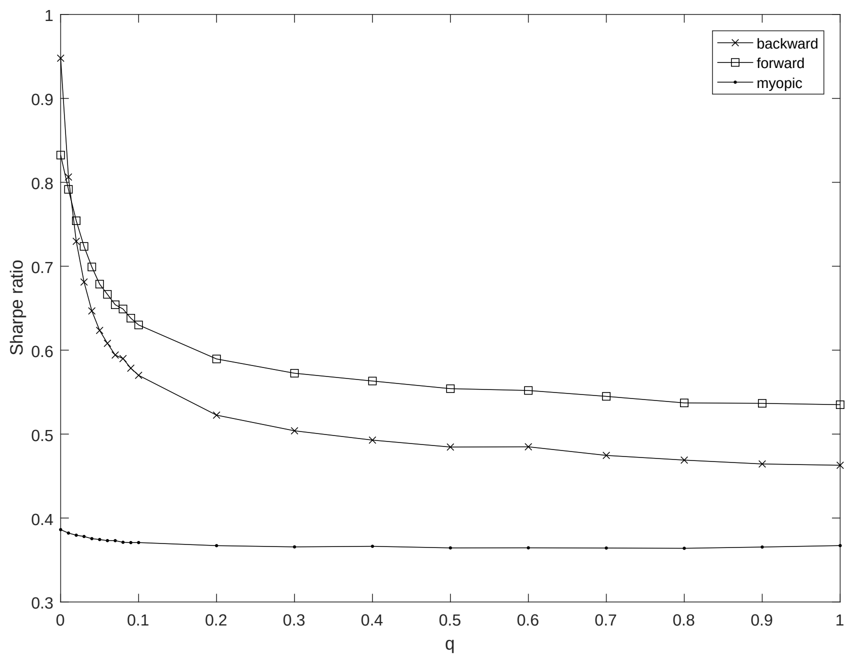


Figure 4.5: 5-year Sharpe ratios as a function of q.

horizons, the forward outperforms the myopic one substantially. As mentioned before, the myopic investor targets the same rate of return in each period ($d_{i-1} = e^{R\Delta t}X(t_{i-1})$), irrespective of the market condition. But because the market switches between the mean-reverting and momentum states, with the former being less volatile in general, a constant target return actually implies a swinging risk attitude. Figure 4.5 shows that inconsistent strategies over time significantly damages the performance in the long term. The forward mean-variance therefore indicates that target expected return in each period should be chosen such that time-consistency throughout $[0, T]$ is guaranteed.

4.3.5 Conflicting objectives: measuring performance at different horizons

The horizon issue arises often in the practice of investment management. On the one hand, fund managers are more likely to focus on short term performance as it is more closely related to their compensations. On the other hand, clients may expect long term stable growth of their investments, hence tend to evaluate fund performance using long run measures.

Cvitanić et al. (2008) discussed the implication of using Sharpe ratio to measure performances of fund managers aiming at different horizons. Under the assumption that asset returns are i.i.d. and mean-reverting, the authors showed that in both cases managers optimizing at short horizons obtain much lower long term Sharpe ratios. In fact, the converse is also true if the model parameters are stochastic. A mean-variance strategy set out to maximize five-year Sharpe ratio will not achieve the highest one-year Sharpe ratio. Hence, maximizing Sharpe ratios at different horizons are conflicting objectives and cannot be accomplished by any single mean-variance optimal strategy. That being said, it does not mean that a better trade-off cannot be pursued.

In this section we will show that, even for the managers who optimize in short horizons, if he chooses his (short term) preferences such that time-consistency in the long term is guaranteed, he will not suffer much long run Sharpe ratio loss. In particular, when the return distribution is i.i.d., there will be no Sharpe ratio loss at all.

We start our discussion with a log-normal market environment, assume that S_t follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{4.61}$$

where μ and σ are constant parameters known to the manager at time zero. Cvitanić et al. (2008) considered a horizon $[0, T]$ and a short-term manager who solves a series of mean-variance problems at $[t_i, t_{i+1}]$, with $0 = t_0 < t_1 < \ldots < t_N = T$, namely, he solves

$$\begin{aligned} &\min_{\pi} \text{Var}_{i-1}\big(X^{\pi}(t_n)\big), \\ &\text{subject to: } \mathbb{E}_{i-1}[X^{\pi}(t_i)] = d_{i-1}^S = e^{R\Delta t} X(t_{i-1}), \end{aligned} \tag{4.62}$$

with R fixed. In other words, the short-term manager is assumed to target the same expected return in each period. On the other hand, the long term manager solves

$$\begin{aligned} &\min_{\pi} \text{Var}\big(X^{\pi}(T)\big), \\ &\text{subject to: } \mathbb{E}[X^{\pi}(T)] = d^L = e^{RT} X_0, \end{aligned} \tag{4.63}$$

Cvitanić et al. (2008) reported that solving (4.62) instead of (4.63) incurs significant loss in long-term Sharpe ratio. For example, at 5-year horizon with $\Delta t = 0.25$ years and $R = 14\%$, the loss is at 68%! However, since the model parameters are assumed to be deterministic, there should be no hedging demand for market parameter risk. Therefore, optimizing at shorter horizons should not necessarily mean worse long-run performance. Here we argue that the issue with the short-term manager is rather due to the way he chooses his targets, which causes the portfolio strategies across different periods to be *time-inconsistent*. Indeed, the following proposition indicates the improvement if the short term manager adopts instead the forward MV framework,

Proposition 4.3.7. *The forward mean-variance problem with initial target wealth* d_0^F *is equivalent to the long term mean-variance problem* (4.63) *with target wealth*

$$d^L = e^{(r-\lambda^2)(N-1)\Delta t}\Big(\frac{e^{\lambda^2 T}-1}{e^{\lambda^2\Delta t}-1}d_0^F - \frac{e^{\lambda^2(N-1)\Delta t}-1}{e^{\lambda^2\Delta t}-1}e^{r\Delta t}X_0\Big). \tag{4.64}$$

Hence, the forward manager achieves the same Sharpe ratio as the long term manager,

$$SR^F = SR^L = \sqrt{e^{\lambda^2 T}-1}.$$

One may now wonder how does the forward manager perform at different horizons when the market parameters are stochastic. To see this, we consider the mean-reversion and momentum market studied in section 3, i.e. for $t \in [0,T]$, $S(t)$ follows

$$\begin{cases} dS(t) = \mu(t)S(t)dt + \sigma S(t)dW(t), \\ d\mu(t) = \kappa(\beta - \mu(t))dt + \rho\sigma_\mu dW(t). \end{cases} \tag{4.65}$$

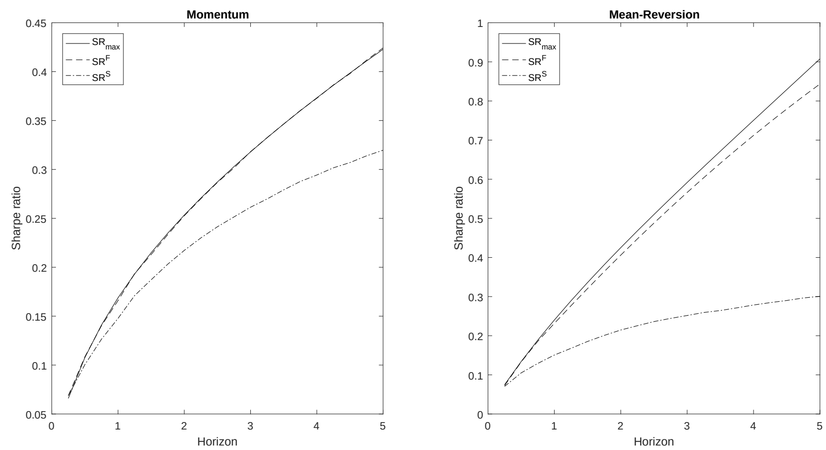


Figure 4.6: Sharpe ratio and investment horizon

To simplify the analysis, we assume here that ρ is constant (1 or -1) throughout the entire horizon, and is known to the investor at time zero.

Let $SR_{\max}(t)$ denote the maximal Sharpe ratio achievable at horizon $[0, t]$, namely,

$$SR_{\max}(t) = \max_{\pi \in \mathcal{A}} \frac{\mathbb{E}[X^{\pi}(t)] - e^{rt} X_0}{\sqrt{\mathrm{Var} X^{\pi}(t)}}. \tag{4.66}$$

It is well known that mean-variance optimal strategies achieve the highest Sharpe ratio at the horizon which the mean-variance objective is defined. Therefore, we have

$$SR^F(\Delta t) = SR^S(\Delta t) = SR_{\max}(\Delta t),\ SR^L(T) = SR_{\max}(T).$$

The question is then how do $SR^F(t)$ and $SR^S(t)$ compare with $SR_{\max}(t)$, for $t > \Delta t$.

Figure 4.6 plots the Sharpe ratios for the forward and short term managers, measured at horizons ranging from 0.25 to 5 years. As expected, both SR^F and SR^S attain the maximal value at t=0.25. However, divergence from $SR_{\max}$ grows substantial as the measurement horizon increases. At 5-year horizon, SR^S is only a third of $SR_{\max}$ in the mean-reversion case. Cvitanić et al. (2008) reported even more dramatic difference (due to difference in parameter choice), that $SR_{\max}$ is more than 50 times higher than SR^S.

However, the difference between SR^F and $SR_{\max}$ is not nearly as dramatic. In the momentum case, the curves almost overlap at all horizons, implying little or no Sharpe ratio loss at all. In the mean-reversion case, SR^F traces $SR_{\max}$ closely and the loss at 5-year horizon is less than 8%. The result is quite surprising in that the forward manager's portfolio strategy is derived based entirely on short term preferences. Hence, any long term objective is not sought after by the manager. However, by merely choosing the targets such that time-consistency is maintained, the manager achieves much better long term performance without looking at the future asset return distributions at all!

4.4 Robust forward mean-variance

The classical mean-variance approach to portfolio selection suffers from a major shortcoming, in that the associated optimal portfolios are often sensitive to changes in the input parameters of the problem. The inability of estimating the parameters in accuracy often result in unrealistic risky positions as well as large turn over ratios with periodic readjustments of the input estimates. To dampen the sensitivity to input parameters, the first, Bayesian approach is to assign a prior probability distribution to input parameters and explicitly incorporate this distribution into mean-variance optimization; see for example, Bawa et al. (1979) and Jorion (1986).

An alternative view of the parameter uncertainty, called Knightian uncertainty (following Knight (1921)) or ambiguity, recognizes that the probability structure cannot be modeled. Therefore, a set of possible parameter values are instead specified.

The robust mean-variance approach is built on this view. It aims to structure the portfolio such that it optimizes performance should the worse case parameter from the specified set occurs. The robust problem has been studied by a number of authors in the past decade. For example, Goldfarb and Iyengar (2003) reformulated this problem as a second order convex-cone program. Tütüncü and Koenig (2004) discussed numerical algorithms under different choices of the parameter uncertainty set. Garlappi et al. (2007) provided an interpretation of robust optimal portfolio, as a shrinkage of the mean-variance portfolio towards either the risk-free asset or the minimum variance portfolio. Boyle et al. (2010) linked ambiguity to investor's familiarity toward assets, and characterized its asset allocation implications.

In this section, we provide a generalization to the forward mean-variance framework established earlier, such that it incorporates robustness considerations in each trading period. Our main contribution is the proof that, the robust forward mean-variance problem is equivalent to the non-robust problem under the worst case parameter. In other words, switching the order of minimization (over the parameter set) and maximization (over the portfolio set) does not change the solution of the forward mean-variance problem. To achieve this, we first study robust mean-variance with continuous time trading in a single horizon. Next, we discuss how the forward machinery can be applied to generate subsequent mean-variance objectives when market parameter is ambiguous. A simulation exercise is also provided to demonstrate the steps for implementing our theory in practical portfolio management.

4.4.1 Continuous time robust mean-variance in a single period

Although the robust mean-variance optimization problem has been widely studied, most of the existing literature focused on the static, buy-and-hold setting. A framework which allows continuous trading has not been established. In this section, we fill this gap.

We assume the investment universe consists of a risk-free asset and n risky securities with the following dynamics

$$dS_k(t) = S_k(t)\big(\mu_k(t)dt + \sum_{j=1}^{n} \sigma_{kj}(t)dW_j(t)\big),\ k = 1, 2, \ldots, n. \tag{4.67}$$

Here $\mu(t)$ is $\mathcal{F}_t$-measurable, $\mathbb{R}^n$-valued stochastic process. According to Merton (1980), the expected returns of the market are much harder to estimate than the variance. Therefore we will assume the volatility matrix $\{\sigma_{kj}(t)\}$ can be estimated with perfect accuracy, while the return parameter $\mu(t)$ is only known to lie within a certain set, denote by $C(t)$, which we assume to be a convex bounded closed subset of $\mathbb{R}^n$. The risk-free rate r is set to be constant, and the vector $r\mathbf{1}$ is excluded from $C(t)$, $\forall t \in [0,T]$. To retain tractability, we make the additional assumption that both the set function $C(t)$ and volatility matrix $\{\sigma_{kj}(t)\}$ are deterministic.

We denote by $\mathbb{Q}^\mu$ the probability measure under which the assets have drift μ. To simplify notations, when μ is used as a superscript, it will be identical to $\mathbb{Q}^\mu$ (e.g. $\mathbb{E}^\mu[\cdot]$ and $W^\mu(t)$ are the same as $\mathbb{E}^{\mathbb{Q}^\mu}[\cdot]$ and $W^{\mathbb{Q}^\mu}(t)$).

We define $Z^\mu(t)$, $t \in [0,T]$ to be the (risk-free rate discounted) density process $e^{-rt}\frac{dQ^0}{dQ^\mu}|_{\mathcal{F}_t}$. Where Q^0 is the unique risk neutral measure. Thus, Z^μ solves under Q^μ the SDE,

$$dZ^\mu(t) = -rZ^\mu(t)dt - Z^\mu(t)\lambda^\mu \cdot dW^\mu(t),$$

where $\lambda^\mu(t) := (\mu(t) - r\mathbf{1}) \cdot \Sigma(t)^{-1}$ is the market price of risk vector associated with Q^μ. By definition, under $\mathbb{Q}^\mu$ we have

$$\mathbb{E}^\mu[X(T)Z^\mu] = X_0, \text{ for } \forall \mu \in C \text{ and } X(T) \in \mathcal{A}_T, \tag{4.68}$$

where $\mathcal{A}_T$ is the set of "terminal wealth" generated by admissible portfolio strategies. Before defining robust mean-variance optimization, we need to make one additional assumption (which are often implicitly made in existing literature) to clarify what are considered as "elements" in the parameter set. For example, if the investor thinks the expected return can be anywhere from 3% to 6%, then only deterministic values within this range are considered as valid candidate parameters, while random variables whose values fall into range are excluded.

Assumption: The subjective return parameter μ is deterministic.

Under this assumption, the applicability of our method is a bit limited. Our robust solution cannot protect against, for example, measures that are state dependent. The reason for the assumption is following, notice that in the continuous time setting, there are two channels through which μ can affect the variance of the optimal terminal wealth. Firstly, roughly speaking, a higher μ makes the terminal wealth more volatile, due to the effect of return compounding. Secondly, if μ is random, variances in μ itself would manifest into variance of terminal wealth. Without restricting μ to be deterministic, the second effect would render our subsequent search for a "worst case" measure a daunting task, and it is unclear whether the equivalence relation between quadratic utility and mean-variance problems remain valid. For these reasons, we choose to leave the study of more general cases for future research.
We denote by $\tilde{C}$ the set of deterministic subjective return parameters, i.e.

$$\tilde{C} := \{\mu : \ \mu(t) \in C(t) \text{ for } \forall\, t \text{ and } \mu(t) \text{ is deminterministic in } t\}.$$

Then, the robust mean-variance is defined as follows:

Definition 4.4.1. A mean-variance investor, who is averse to uncertainty in the return parameters solves the problem,

$$\begin{aligned} &\min_{X(T)\in\mathcal{A}_T} \max_{\mu\in\tilde{C}} \mathrm{Var}^{\mu}(X(T)), \\ &\text{subject to: } \mathbb{E}^{\mu}[X(T)] \geq d, \ \forall \mu \in \tilde{C}. \end{aligned} \tag{4.69}$$

where d is the investor's target level of terminal wealth.

To solve the above problem, we make use of the techniques commonly seen in robust utility preferences. In most cases, if the mean-variance objective function is replaced with an expected utility functional,

$$\max_{X(T)\in\mathcal{A}_T}\ \min_{\mu\in C}\ \mathbb{E}^{\mu}[U(X(T))],$$

then the robust problem can be reduced to a standard utility problem under a fixed measure $\mathbb{Q}^{\hat{\mu}}$ (at least for the case where model parameters are deterministic). It is called by some the "least favorable" measure, in the sense that the market opportunity set under this measure is minimized. Equivalently, the associated drift parameter $\hat{\mu}$ minimizes the Sharpe ratio of the *market portfolio*. We provide a straightforward definition of $\hat{\mu}$ under our deterministic model assumption. A more general and abstract definition can be found in Schied (2004).

Definition 4.4.2. Define the market portfolio under $\mathbb{Q}^{\mu}$ as the self financing portfolio which replicates at time T the (negative) stochastic discount factor $-Z^{\mu}(T)$, and denote by SR^{μ}_m the market portfolio's Sharpe ratio. The least favorable return parameter is defined by

$$\hat{\mu} = \operatorname*{argmin}_{\mu\in C}\ SR^{\mu}_m.$$

By a straightforward calculation, we can verify that $\hat{\mu}$ minimizes the norm of the market price of risk vector. Computationally, it can be solved by solving, at each time t, the quadratic program,

$$\hat{\mu}(t) = \operatorname*{argmin}_{\mu\in C(t)} |\lambda^{\mu}(t)|^2 = \operatorname*{argmin}_{\mu(t)\in C(t)} (\mu(t)-r\mathbf{1})\cdot(\Sigma(t)'\Sigma(t))^{-1}\cdot(\mu(t)-r\mathbf{1})' \quad (4.70)$$

Thus, the conventional wisdom states that

$$\max_{X(T)\in\mathcal{A}_T}\ \min_{\mu\in C}\ \mathbb{E}^{\mu}[U(X(T))] \Longleftrightarrow \max_{X(T)\in\mathcal{A}_T}\ \mathbb{E}^{\hat{\mu}}[U(X(T)]$$

We will show that the same conclusion holds with robust mean-variance optimization. In the proofs that follow, we will make extensive use of the following lemma.

Lemma 4.4.3. *The following three inequalities hold, for any $\mu \in \tilde{C}$:*

1. $\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)] \leq \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)], \quad \mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)^2] \leq \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2].$
2. $\mathbb{E}^{\mu}[Z^{\mu}(T)^2] \geq \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2], \quad Var^{\mu}(Z^{\mu}(T)) \geq Var^{\hat{\mu}}(Z^{\hat{\mu}}(T)).$
3. *If μ is deterministic, then* $Var^{\mu}(Z^{\hat{\mu}}(T)) \leq Var^{\hat{\mu}}(Z^{\hat{\mu}}(T)).$

For all the above, equality holds if and only if $\mu = \hat{\mu}$.

Proof. See the Appendix C.3. □

Next, we state the main result of this section, which states that robust mean-variance problem can be reduced to its standard version under a fixed subjective measure $\mathbb{Q}^{\hat{\mu}}$. The theorem below shows that $\hat{\mu}$ is exactly the "least favorable" measure given by (4.70).

Theorem 4.4.4. *Let $\hat{\mu}$ be given by* (4.70)*. The robust problem* (4.69) *has the same optimal solution as the standard mean-variance problem under the measure $\mathbb{Q}^{\hat{\mu}}$,*

$$\begin{aligned} &\min_{X(T)\in\mathcal{A}_T} \; Var^{\hat{\mu}}(X(T)), \\ &\text{subject to: } \mathbb{E}^{\hat{\mu}}[X(T)] \geq d. \end{aligned} \tag{4.71}$$

provided that $d \geq X_0 e^{rT}$.

Proof. See Appendix C.3. □

Remark 4.4.5. We recognize that problem (4.69) is not the only robust generalization of the standard mean-variance problem. As is commonly known, the standard mean-variance problem can also be formulated as maximizing a single objective function $F(\mathbb{E}[X(T)], \text{Var}(X(T))$, where F is increasing in the first argument and decreasing in the second. We call this the Lagrangian formulation when F is linear. In this setting, a natural generalization to account for uncertainty in μ is

$$\max_{X(T)\in\mathcal{A}_T} \min_{\mu\in C} \mathbb{E}^{\mu}[X(T)] - \gamma \text{Var}^{\mu}(X(T)), \tag{4.72}$$

the parameter $\gamma > 0$ expresses the investor's desired risk-return trade-off. One thing worth noticing from above is that the deterministic assumption on alternative drift parameters is no longer needed. Thus, the minimization over μ in (4.72) is now taken over the entire parameter set C, instead of the subset $\tilde{C}$ which contains only the μ's that are deterministic, as it was done for problem (4.69). As we show in the next result, the solution to (4.72) again reduces to the one of a standard mean-variance problem under the "least favorable" measure $\mathbb{Q}^{\hat{\mu}}$.

Theorem 4.4.6. *The robust Lagrangian problem* (4.72) *has the same solution as the standard Lagrangian problem under the fixed measure* $\mathbb{Q}^{\hat{\mu}}$

$$\max_{X(T)\in\mathcal{A}_T} \mathbb{E}^{\hat{\mu}}[X(T)] - \gamma\, Var^{\hat{\mu}}(X(T)) \tag{4.73}$$

Proof. See Appendix C.3. □

Recall that in the previous sections, the forward mean-variance is defined based on the fact that each single period mean-variance problem has its equivalent quadratic utility counterpart. Then, the theory of predictable forward performance can be applied to generate sequential quadratic utilities with time-consistency guarantee. In the robust optimization domain, our strategy of updating mean-variance objectives forward in time mimics that of the previous section. Therefore, we now study the closely related robust quadratic utility problem and show that the equivalence result still exists under the robust context. To this end, we define the robust quadratic utility problem as

$$\max_{X(T)\in\mathcal{A}_T} \min_{\mu\in C} \mathbb{E}^{\mu}[\eta X(T) - X(T)^2]. \tag{4.74}$$

As before, we will show that the above problem can be reduced to the non-robust problem under the fixed measure $\mathbb{Q}^{\hat{\mu}}$. We do this by exploiting the fact that the optimal terminal wealth for a quadratic utility is linear in the stochastic discount factor.

Theorem 4.4.7. *The robust quadratic utility problem*

$$\max_{X(T)\in\mathcal{A}_T} \min_{\mu\in C} \mathbb{E}^{\mu}[\eta X(T) - X(T)^2] \tag{4.75}$$

is equivalent to the standard utility problem under the fixed measure $\mathbb{Q}^{\hat{\mu}}$,

$$\max_{X(T)\in\mathcal{A}_T} \mathbb{E}^{\hat{\mu}}[\eta X(T) - X(T)^2]$$

Proof. See Appendix C.3. □

Theorems 4.4.7 and 4.4.4 establish connections between the robust quadratic utility, robust mean-variance and their respective standard versions without robustness components. We also recall that the standard quadratic utility and mean-variance problems have been shown to be equivalent by theorem 4.3.3. These results can be summarized into the following diagram,

$$\text{RQU} \overset{\text{Theorem 4.4.7}}{\Longleftrightarrow} \text{QU}(\hat{\mu}) \overset{\text{Theorem 4.3.3}}{\Longleftrightarrow} \text{MV}(\hat{\mu}) \overset{\text{Theorem 4.4.4}}{\Longleftrightarrow} \text{RMV}.$$

We can see from above an immediate implication, in that the two robust problems are also equivalent.

Corollary 4.4.8. *The robust mean-variance problem* (4.69) *parameterized by target mean wealth* d*,* $d \geq x_0 e^{rT}$*, is equivalent to the robust quadratic utility problem* (4.75) *parameterized by* η*, with* d *and* η *related by,*

$$\frac{\eta}{2} = \frac{d\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2] - X_0 E^{\hat{\mu}}[Z^{\hat{\mu}}(T)]}{Var^{\hat{\mu}}(Z^{\hat{\mu}}(T))}. \tag{4.76}$$

The optimal terminal wealth $X^*(T)$ *under both preferences can be represented as*

$$X^*(T) = \frac{d\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2] - X_0 E^{\hat{\mu}}[Z^{\hat{\mu}}(T)]}{Var^{\hat{\mu}}(Z^{\hat{\mu}}(T))} - \frac{de^{-rT} - X_0}{Var^{\hat{\mu}}(Z(T))} Z^{\hat{\mu}}(T).$$

Remark 4.4.9. Notice that the η parameter in (4.76) depends only on the least favorable measure $\mathbb{Q}^{\hat{\mu}}$, instead of on a "to-be-determined" subjective measure $\mathbb{Q}^{\mu}$. Alternatively, the above equivalence can be established with a quadratic utility with measure-dependent coefficients. In that case $\hat{\mu}$ would be replaced by μ in (4.76), and η would be given by

$$\frac{\eta}{2} = \frac{\eta^{\mu}}{2} = \frac{d\mathbb{E}^{\mu}[Z^{\mu}(T)^2] - X_0 E^{\mu}[Z^{\mu}(T)]}{\text{Var}^{\mu}(Z^{\mu}(T))}.$$

One can solve the robust utility problem using the measure-dependent utility function $U^{\mu}(X(T)) := \eta^{\mu}X(T) - X(T)^2$. We will then have the same solution as using the fixed utility function $U^{\hat{\mu}}(X(T)) = \eta^{\hat{\mu}}X(T)^2 - X(T)$. We choose to state corollary 4.4.8 with the fixed utility as we want to obtain a utility preference where aversion to market risk and to model estimation risk be separated.

4.4.2 Multi-period robust mean-variance under the forward approach

Having solved the robust mean-variance and quadratic problems in a single period, we are ready to apply the established procedure for generating robust mean-variance forward in time. First, we introduce the appropriate multi-period problem.

Definition 4.4.10 (multi-period robust mean-variance)**.** Let $0 = T_0 < T_1 < \ldots < T_N = T$. A multi-period robust mean-variance preference is a sequence of robust mean-variance preferences $\{RMV_n\}_{n=1}^{N}$, imposed at $\{T_n\}_{n=1}^{N}$. Within each period $[T_{n-1}, T_n]$, the investor solves the problem:

$$\begin{aligned} &\min_{X(T_n)\in\mathcal{A}_{T_n}} \max_{\mu\in\tilde{C}} \mathrm{Var}^{\mu}_{n-1}(X(T_n)), \\ &\text{subject to: } \mathbb{E}^{\mu}_{n-1}[X(T_n)] \geq d_n, \ \forall \mu \in \tilde{C}, \end{aligned} \tag{4.77}$$

where d_n is the desired target level at the end of the time interval $[T_{n-1}, T_n]$. In section 4.3, we have shown that the d_n's, instead of being specified period by period in an *ad hoc* manner, can be generated endogenously, in a way that guarantees inter-temporal consistency of the optimal investment strategy. As we discussed, the key point was that mean-variance at each period is replaced by an equivalent quadratic utility, which can then be extended forward in time using the existing forward approach. Here, we apply the same idea to the robust preferences. The rest of the section will be devoted to establishing a multi-period robust mean-variance forward criterion which is predictable and time-consistent. We will call it a *robust forward mean-variance* preference.

We start with extending the definition of predictable forward performance to take robustness into account.

Definition 4.4.11. A sequence of random functions $\{U_n\}_{n=1}^N$ imposed at $\{T_n\}_{n=1}^N$ is a *robust predictable forward performance* if

(i) $U_n(\cdot)$ is measurable w.r.t $\mathcal{F}_{T_{n-1}}$, for $n \in \{1, 2, \ldots, N\}$.

(ii) For any admissible wealth process $X(t)$, $t \in [0, T]$, the following holds, for $n \in \{1, 2, \ldots, N\}$,

$$U_{n-1}(X(T_{n-1})) \geq \min_{\mu \in \tilde{C}} \mathbb{E}^{\mu}[U_n(X(T_n))|\mathcal{F}_{T_{n-1}}] \tag{4.78}$$

(iii) There exists an admissible wealth process $X^*(t)$, $t \in [0, T]$, such that, for $\forall\, t \in \{1, 2, \ldots, N\}$,

$$U_{n-1}(X^*(T_{n-1})) = \min_{\mu \in \tilde{C}} \mathbb{E}^{\mu}[U_n(X^*(T_n))|\mathcal{F}_{T_{n-1}}] \tag{4.79}$$

In the case of quadratic utility functions $U_n(x) = a_n x^2 + b_n x + c$, $n = 1, 2, \ldots, N$, the lemma below provides a necessary conditions for $\{U_n\}_{n=1}^N$ to be a robust predictable forward performance.

Lemma 4.4.12. *Let* $\eta_n = -\frac{b_n}{a_n}$. *If the family of quadratic utility functions* $\{U_n\}_{n=1}^N$ *is a robust predictable forward utility with* $\frac{\eta_1}{2} \geq X_0 e^{rT}$, *then it must be that*

$$\eta_n = \eta_{n-1} e^{r(T_n - T_{n-1})} \tag{4.80}$$

Proof. See Appendix C.3. □

We have established in Corollary 4.4.8 the equivalence of robust mean-variance and quadratic utility preferences in a single-period setting. This result along with Definition 4.4.11 yield the definition of robust forward mean-variance preferences.

Definition 4.4.13. The multi-perod robust mean-variance preference $\{RMV_n\}_{n=1}^N$ defined in 4.4.10 is a *robust forward mean-variance preference* if,

(i) d_n is measurable with respect to $\mathcal{F}_{T_{n-1}}$, for any $n = 1, 2, \ldots, N$.

(ii) There exists a sequence of quadratic utility problems $\{U_n\}_{n=1}^N$, defined over time intervals $\{[T_{n-1}, T_n]\}_{n=1}^N$, such that (a) U_n and RMV_n imply the same optimal portfolio strategy at $[T_{n-1}, T_n]$ and (b) The family of (random) quadratic functions $\{U_1(\cdot), \ldots, U_N(\cdot)\}$ is a robust predictable forward preference in the sense of Definition 4.4.11.

We are now ready to characterize the conditions under which $\{RMV_n\}_{n=1}^N$ is a robust forward mean-variance preference. In particular, we seek a dependence relation of the n-th period wealth target d_n on the wealth target at the previous period. By Corollary 4.4.8, RMV_n and U_n being equivalent requires that d_n and η_n be related by the following identity,

$$\frac{\eta_n}{2} = \frac{d\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2] - X(T_{n-1})\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T_n)]}{\mathrm{Var}^{\hat{\mu}}(Z^{\hat{\mu}}(T_n))}. \tag{4.81}$$

On the other hand, by Lemma 4.4.12 we also have that $\boldsymbol{\eta_n = \eta_{n-1} e^{r(T_n - T_{n-1})}}$. Hence after eliminating the η-variable, it is straightforward to solve from (4.81) the equation relates d_n and d_{n-1}.

Theorem 4.4.14. *The sequence of robust mean-variance preference* $\{RMV_n\}_{n=1}^N$ *defined in 4.4.10 is a robust forward mean-variance preference if the target wealth levels satisfy the recursive relation:*

$$d_n = e^{r(T_n - T_{n-1})}\big[d_{n-1}\frac{1+\xi^{\hat{\mu}}_{n-2}}{1+\xi^{\hat{\mu}}_{n-1}} + \frac{\xi^{\hat{\mu}}_{n-1}X(T_{n-1}) - e^{r(T_{n-1}-T_{n-2})}\xi^{\hat{\mu}}_{n-2}X(T_{n-2})}{1+\xi^{\hat{\mu}}_{n-1}}\big], \tag{4.82}$$

where $\xi^{\hat{\mu}}_{n-1} = \frac{\mathbb{E}^{\hat{\mu}}_{n-1}[Z^{\hat{\mu}}(T_n)]^2}{Var^{\hat{\mu}}_{n-1}(Z^{\hat{\mu}}(T_n))}$.

Recall that in section 4.3, when robustness issue was not considered, we have derived a similar formula (e.q. (4.51)) for d_n. The only difference in (4.82) is that μ in (4.51) is replaced by $\hat{\mu}$. Therefore, the robust forward mean-variance can be considered as a standard forward mean-variance preference under the least favorable measure $\mathbb{Q}^{\hat{\mu}}$.

4.4.3 A simulation exercise

We now conduct performance comparison between the robust forward mean-variance investor and the robust backward mean-variance investor. We assume as before that the expected rate of return cannot be estimated accurately, and both types of investors exhibit aversion to this parameter uncertainty. However, we do allow learning to occur in our model. As more data arrives, the investors adopt a Bayesian updating procedure to refine their previous estimates. Our setup differs from other Bayesian learning methods in dealing with parameter uncertainty in that we do not take averages over the probability distribution of parameters (ambiguity neutral). Instead, for a given level of ambiguity aversion, the estimated parameter distribution is used only to specify the set C of rival parameters.

More specifically, C is modeled as follows. Suppose that at time T_i we have estimated that, the expected rate of return over the interval $[T_i, T_{i+1}]$, denoted by μ_i, has posterior mean $\overline{\mu}_i$ and variance $\sigma^2_{\mu_i}$, then the uncertainty set for μ_i is given by

$$C(i) := \{\mu_i : \frac{|\mu_i - \overline{\mu}_i|}{\sigma_{\mu_i}} \leq \alpha\}. \tag{4.83}$$

Here, α specifies how much estimation error the investor intends to be protected against, reflecting his level of ambiguity aversion. As we have seen in the previous sections, all robust problems can eventually be reduced to their corresponding standard problems under the "least favorable" parameter, defined as,

$$\hat{\mu}_i := \operatorname*{argmin}_{\mu_i \in C(i)} |\mu_i - r|, \tag{4.84}$$

with r being the risk-free rate. Therefore, the learning procedure in our model aims at deceasing the size of the uncertainty set C, and increasing the value of $\hat{\mu}$.

The difference in the performances of the forward and backward investors arises for two reasons. Firstly, the forward investor only needs to estimate expected returns for the next period. In contrast, the backward investor, who tries to solve a problem backwards in time, is forced to estimate an entire path of expected returns for the remaining trading horizon. For returns at more distant future, he faces considerable parameter uncertainty and the "least favorable" return $\hat{\mu}$ will be extremely low. He then has to take on excessive risky assets holdings in order to achieve his return target.

Secondly, as the backward investor moves into the next period, he needs to re-estimate parameters and re-optimize his portfolio, making his decisions time-inconsistent. The same problem however, does not exist for the forward investor.

For the rest of the section, we will obtain numerical results for each type of investor's performance, and the aforementioned performance difference will be quantified.

The entire investment horizon $[0, T]$ is divided into N equal length sub-intervals, $\{[T_i, T_i + 1]\}_{i=0}^{N-1}$, $T_{i+1} - T_i = \Delta T$. Trading takes place continuously throughout $[0, T]$, but parameter estimations will only occur at the beginning of each interval $[T_i, T_{i+1}]$. Assume that the market consists of one risky asset $S(t)$ and one risk-free bond $B(t)$. Inside each $[T_i, T_{i+1}]$, the price of the risky asset follows a geometric Brownian motion with constant (but unknown) return parameter μ_i, i.e.

$$dS(t)/S(t) = \mu_i dt + \sigma dW(t). \quad t \in [T_i, T_{i+1}]$$

We assume that μ_i evolves discretely, period over period, following an AR(1) process. For $0 < \beta < 1$, we have,

$$\mu_{i+1} = (1-\beta)\overline{\mu} + \beta\mu_i + \nu_{i+1}, \tag{4.85}$$

where $\overline{\mu}$ is the unconditional mean rate of return for asset S, and $\{\nu_i\}_{i=0}^{N-1}$ denotes normally distributed i.i.d. innovations, with zero mean and variance σ_ν^2. To generate mean-reversion in realized returns, we further assume that ν_{i+1} is correlated with the Brownian increment accumulated during the i-th period, with correlation $\rho < 0$. Then, the shocks have the covariance matrix,

$$\text{Cov}(\nu_{i+1}, \sigma\Delta W_{i+1}) = \begin{bmatrix} \sigma_\nu^2 & \rho\sqrt{\Delta T}\sigma\sigma_\nu \\ \rho\sqrt{T}\sigma\sigma_\nu & \sigma^2\Delta T \end{bmatrix}$$

The above AR(1) model assumption for μ_i is consistent with the Pastor and Stambaugh (2012) and Campbell and Viceira (2002). Its continuous time analog, the Ornstein-Ulenbeck process, is also commonly employed to model mean-reverting expected returns in the continuous time portfolio literature, (see, for example, Kim and Omberg (1996), Watcher (2002)).

Throughout the discussion below, we will assume that the return rates $\{\mu_i\}_{i=0}^{N-1}$ are the only parameters with uncertainty. The investor has perfect knowledge and can correctly specify all the other parameters.

4.4.3.1 The Forward Problem

As we previously discussed, the forward investor by his nature, only solves a robust mean-variance problem one period ahead of him. Thus at T_i, the problem he faces is defined over $[T_i, T_{i+1}]$. To him, the only relevant parameter information is the current period expected return rate μ_i, which he cannot observe and has to make a subjective choice (denoted by $\hat{\mu}_i$) before investing. After $\hat{\mu}_i$ is determined, the investor then solves the standard mean-variance problem,

$$\begin{aligned} &\min_{\pi \in \mathcal{A}} \mathrm{Var}_i^{\mathbb{Q}^{\hat{\mu}_i}}(X^\pi(T_{i+1})) \\ &\text{subject to: } \mathbb{E}_i^{\mathbb{Q}^{\hat{\mu}_i}}[X^\pi(T_{i+1})] \geq d_{i+1}. \end{aligned} \tag{4.86}$$

We have shown in section 4.4.1 that, being a robust mean-variance investor, he will choose $\hat{\mu}_i$ as the worst case parameter from a set $C(i)$ of rival parameters, where by equation (4.83) $C(i)$ is derived from his estimation of the distribution of μ_i. Thus, the problem of specifying $\hat{\mu}_i$ reduces to a statistical inference problem for the unobservable variable μ_i. We discuss this problem next.

We will be using Bayesian inference as the main tool to learn about the distribution of μ_i. We assume that at the initial time t=0, the investor has a Gaussian prior on the return rate μ_0 for the first period $[T_0, T_1]$. We explain below in an iterative manner how the prior distribution of μ_i can be combined with the signals observed at $[T_i, T_{i+1}]$, in a way that at T_{i+1} the investor is able to make a prediction on the distribution of μ_{i+1}.

If at time T_i, the investor has arrived at a prior belief over μ_i:

$$\mu_i \ \sim \ N(\overline{\mu}_i, \sigma^2_{\mu_i}), \tag{4.87}$$

At each $t \in [T_i, T_{i+1}]$, the investor observes the realized instantaneous return on the risky asset, $\frac{dS(t)}{S(t)}$, based on which the prior estimate of the expected return μ_i will be refined. Since $\frac{dS(t)}{S(t)} \sim$ i.i.d. $N(\mu_i dt, \sigma\sqrt{dt})$, each observation

$\frac{dS(t)}{S(t)}$ should contain the same amount of information, and should receive the same weight as they appear in the posterior distribution of μ_i. As a result, the learning problem in the i-th period can be based solely on the average realized return over the period $[T_i, T_{i+1}]$ instead of the entire path. Denote it by b_{i+1}, then by definition,

$$b_{i+1} = \frac{1}{\Delta T} \int_{T_i}^{T_{i+1}} \frac{dS(t)}{S(t)}, \tag{4.88}$$

The dynamics of the stock implies that

$$b_{i+1} = \mu_i + \sigma \frac{\Delta W_{i+1}}{\Delta T}.$$

On the other hand, the next period expected return μ_{i+1} is predicted by the AR(1) process,

$$\mu_{i+1} = (1-\beta)\overline{\mu} + \beta\mu_i + \nu_{i+1}.$$

Therefore, conditional on a fixed value of μ_i, we can deduce the joint distribution of (b_{i+1}, μ_{i+1}),

$$\left[\begin{array}{c} b_{i+1} \\ \mu_{i+1} \end{array}\right]\Bigg|_{\mu_i} \sim N\left(\left[\begin{array}{c} \mu_i \\ (1-\beta)\overline{\mu} + \beta\mu_i \end{array}\right], \left[\begin{array}{cc} \frac{\sigma^2}{\Delta T} & \rho\frac{\sigma\sigma_\nu}{\sqrt{\Delta T}} \\ \rho\frac{\sigma\sigma_\nu}{\sqrt{\Delta T}} & \sigma_\nu^2 \end{array}\right]\right)$$

By time T_{i+1}, the average realized return b_{i+1} has been observed by the investor. This new information helps in two ways. Firstly, the observed returns reveal information about the true expected return parameter μ_i. According to Bayes' rule,

$$P(\mu_i|b_{i+1}) = \frac{P(b_{i+1}|\mu_i)P(\mu_i)}{P(b_{i+1})}.$$

where $P(\mu_i|b_{i+1})$ denotes the posterior distribution.

Secondly, because the shocks to b_{i+1} and μ_{i+1} are correlated, the investor is able to make inference on μ_{i+1} based on the b_{i+1} he has already observed, using that

$$P(\mu_{i+1}|b_{i+1}, \mu_i) = \frac{P(\mu_{i+1}, b_{i+1}|\mu_i)}{P(b_{i+1}|\mu_i)}.$$

Finally, in order for the estimation be conditioned only on the observed data (and not on the unknown μ_i), we average μ_i out by integrating the above conditional density over the posterior distribution of μ_i,

$$\begin{aligned} P(\mu_{i+1}|b_{i+1}) &= \int_{-\infty}^{+\infty} P(\mu_{i+1}|b_{i+1}, \mu_i)P(\mu_i|b_{i+1})d\mu_i \\ &= \int_{-\infty}^{+\infty} \frac{P(b_{i+1}|\mu_i)P(\mu_i)}{P(b_{i+1})} \frac{P(\mu_{i+1}, b_{i+1}|\mu_i)}{P(b_{i+1}|\mu_i)} d\mu_i \\ &= \frac{\int_{-\infty}^{+\infty} P(b_{i+1}, \mu_{i+1}|\mu_i)P(\mu_i)d\mu_i}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(b_{i+1}, \mu_{i+1}|\mu_i)P(\mu_i)d\mu_i d\mu_{i+1}} \end{aligned} \tag{4.89}$$

Combing the above with the two density functions, $P(b_{i+1}, \mu_{i+1}|\mu_i)$ and $P(\mu_i)$, we obtain,

$$\mu_{i+1}|b_{i+1} \;\sim\; N(\overline{\mu}_{i+1}, \sigma^2_{\mu_{i+1}}), \tag{4.90}$$

where

$$\begin{cases} \overline{\mu}_{i+1} = (1-\beta)\overline{\mu} + \beta\overline{\mu}_i + (b_{i+1} - \overline{\mu}_i)\frac{\sigma\sigma_\nu\rho\sqrt{\Delta T} + \beta\sigma^2_{\mu_i}\Delta T}{\sigma^2 + \sigma^2_{\mu_i}\Delta T} \\ \sigma^2_{\mu_{i+1}} = \sigma^2_\nu + \beta^2\sigma^2_{\mu_i} - \frac{(\sigma\sigma_\nu\rho + \beta\sigma^2_{\mu_i}\sqrt{\Delta T})^2}{\sigma^2 + \sigma^2_{\mu_i}\Delta T} \end{cases} \tag{4.91}$$

Equation (4.90) gives the investor's prior belief about μ_{i+1}.

We are now ready to describe the steps for solving the robust forward mean-variance problem.

(i). Suppose at time T_i the investor has estimated $\mu_i \sim N(\overline{\mu}_i, \sigma^2_{\mu_i})$ and has determined his target d_{i+1}. He picks the subjective parameter $\hat{\mu}_i$ according to (4.84), and then solves the standard mean-variance problem (4.86).

(ii). At T_{i+1}, the investor follows the steps described in this section and estimates the distribution for μ_{i+1} from (4.90), and determines $\hat{\mu}_{i+1}$ accordingly.

(iii). With $\hat{\mu}_i$ and $\hat{\mu}_{i+1}$ known to the investor, he is able to determine his new target d^F_{i+2} to be imposed at T_{i+2}. By Theorem 4.4.14, we have

$$d^F_{i+2} = e^{r\Delta T}\big[d^F_{i+1}\frac{1+\xi_i^{\hat{\mu}_i}}{1+\xi_{i+1}^{\hat{\mu}_{i+1}}} + \frac{\xi_{i+1}^{\hat{\mu}_{i+1}}X(T_{i+1}) - e^{r\Delta T}\xi_i^{\hat{\mu}_i}X(T_i)}{1+\xi_{i+1}^{\hat{\mu}_{i+1}}}\big]$$

where $\xi_j^{\hat{\mu}_j} = \frac{\mathbb{E}_j^{\hat{\mu}_j}[Z^{\hat{\mu}_j}(T_{j+1})]^2}{\mathrm{Var}_j^{\hat{\mu}_j}(Z^{\hat{\mu}_j}(T_{j+1}))}$, $j = i, i+1$.

Then, he goes back to step (i) and solves a standard mean-variance problem with target d_{i+2} and under the revised parameter $\hat{\mu}_{i+1}$.

4.4.3.2 The Backward Problem

Unlike the forward investor, the backward investor solves a single mean-variance problem defined over the entire horizon $[0, T]$. At time 0, in order to determine the optimization program defined over $[0, T]$, the investor first has to specify the expected rate of return $\hat{\mu}_i$ for each future period $[T_i, T_{i+1}]$, $i = 0, 1, \ldots, N-1$. He does this by estimating the probability distribution of μ_i, and then choose the $\hat{\mu}_i$ from the set $C(i)$ induced by this distribution, all of these being based only on information available at $t = 0$.

We will be using double-script notation $\hat{\mu}_i^j$ to denote the investor's estimate for expected return at period $[T_i, T_{i+1}]$, based on information available up to time T_j. Thus, the time 0 problem can be written as,

$$\begin{aligned} &\min_{X(T)\in\mathcal{A}_T} \ \mathrm{Var}^{\hat{\mu}^0}(X(T)), \\ &\text{subject to: } \mathbb{E}^{\hat{\mu}^0}[X(T)] \geq d^B. \end{aligned} \tag{4.92}$$

where $\hat{\mu} := [\hat{\mu}_0^0, \hat{\mu}_1^0, \ldots, \hat{\mu}_{N-1}^0]$.

Note that after the investor reaches T_1, he has observed the path of returns $\frac{dS(t)}{S(t)}$, $t \in [T_0, T_1]$. Using this new information, the investor will update his parameters into $\hat{\mu}_i^1$, $i = 1, 2, \ldots, N-1$. Because $\hat{\mu}_i^1$ is different from $\hat{\mu}_i^0$, the investor would have to resolve his mean-variance problem using the updated parameter set. In general, re-optimization occurs whenever the investor reaches a new period.

The next theorem provides the investor's optimal policy at T_i.

Theorem 4.4.15. *Assume that at T_i, the investor has estimated return parameters to be $\hat{\mu}^i = [\hat{\mu}_0^i, \hat{\mu}_1^i, \ldots, \ldots, \hat{\mu}_{N-1}^i]$. Then, the optimal amount of wealth to be invested in the risky asset at t, $t \in [T_i, T_{i+1}]$, is given by,*

$$\pi^B(t) = \frac{\hat{\mu}_i^i - r}{\sigma^2}\big(\eta_i^B e^{r(T_{i+1}-t)} - X^B(t)\big). \tag{4.93}$$

At T_{i+1}, the investor's optimal wealth is given by

$$X^B(T_{i+1}) = \eta_i^B + \phi_i^B \frac{Z^{\hat{\mu}^i}(T_{i+1})}{Z^{\hat{\mu}^i}(T_i)}, \tag{4.94}$$

where

$$\begin{aligned} \eta_i^B &= \frac{d e^{\Delta T(\sum_{j=0}^{N-1}(\hat{\lambda}_j^i)^2)} - X_0 e^{rT}}{e^{\Delta T(\sum_{j=0}^{N-1}(\hat{\lambda}_j^i)^2)} - 1} e^{-r(T-T_{i+1})} \\ \phi_i^B &= e^{-((\hat{\lambda}_i^i)^2 - 2r)\Delta T}\big(X^B(T_i) - \eta e^{-r(T-T_i)}\big). \end{aligned} \tag{4.95}$$

and $\hat{\lambda}_j^i = \frac{\hat{\mu}_j^i - r}{\sigma}$, for $j = 0, 1, \ldots, N-1$.

Therefore the only problem now left is parameter estimation. Note that since the entire path of parameters is needed by (4.93), the investor is forced to make multi-period return forecast. We describe his parameter estimation procedure next.

Assume at T_i the investor already has a known Gaussian prior distribution for μ_i, with mean $\overline{\mu}_i$ and variance $\sigma^2_{\mu_i}$. The conditional distribution for k-period ahead return $P(\mu_{i+k}|\mu_i)$ can be derived from the AR(1) dynamics (4.85) followed by μ, namely,

$$\mu_{i+k}|\mu_i \;\sim\; N\Big((1-\beta^k)\overline{\mu} + \beta^k\mu_i,\, \frac{1-\beta^{2k}}{1-\beta^2}\sigma^2_\nu\Big).$$

Integrating over the prior distribution of μ_i given by (4.87), we obtain the k-period ahead forecast

$$\mu_{i+k} \;\sim\; N(\overline{\mu}_{i+k},\, \sigma_{\mu_{i+k}}), \tag{4.96}$$

with

$$\begin{aligned} \overline{\mu}_{i+k} &= (1-\beta^k)\overline{\mu} + \beta^k\overline{\mu}_i \\ \sigma_{\mu_{i+k}} &= \frac{1-\beta^{2k}}{1-\beta^2}\sigma^2_\nu + \beta^{2k}\sigma^2_{\mu_i}. \end{aligned} \tag{4.97}$$

The above distribution of μ_{i+k} then generates a sequence of uncertainty sets

$$C(i+k) = \{\mu_{i+k} : \; \frac{|\mu_{i+k} - \overline{\mu}_{i+k}|}{\sigma_{\mu_{i+k}}} \le \alpha\}, \text{ for } k = 0, 1, \ldots, N-i-1,$$

from which we can determine the investor's subjective parameters by solving

$$\hat{\mu}^i_{i+k} := \operatorname*{argmin}_{\mu_{i+k}\in C(i+k)} |\mu_{i+k} - r|, \quad k = 1, 2, \ldots, N-i-1 \tag{4.98}$$

The i-th period optimization problem is then solved by equation (4.94), with parameters provided by (4.98).

We can already see from the above that parameter uncertainty has a considerable impact on the backward investor's performance. As shown by equation (4.97), $\sigma^2_{\mu_{i+k}}$, the uncertainty for future parameter μ_{i+k} increases with k. In particular, if β is close to 1, that is the shocks to expected return are persistent, a relatively small variance in ν will be aggregated into a large variance k periods in the future, making the investor's current estimate very inaccurate. As a result, the investor's current parameter $\hat{\mu}^i_{i+k}$ may be very different from $\hat{\mu}^{i+k}_{i+k}$, the parameter he will choose at T_{i+k}. We will see from the numerical results how this time-inconsistency in parameter specification would hurt the investor's long term performance.

The uncertainty in future parameter value has yet another impact on the investor's portfolio, entering through the ambiguity aversion. Note that a large $\sigma^2_{\mu_{i+k}}$ will increase the size of the parameter set $C(i+k)$, which in turn lowers the value of the "worst case" parameter $\hat{\mu}^i_{i+k}$. Facing huge uncertainty in future returns, the investor has to presume that future returns are very low, so as to insure his portfolio against undesirable parameter realizations. By equation (4.93), the amount of risky investment at T_i depends positively on the parameter η^B_i, which by (4.95) can be written as,

$$\eta^B_i = de^{-r(T-T_{i+1})} + \frac{d - X_0 e^{rT}}{e^{\Delta T(\sum_{j=0}^{N-1}(\hat{\lambda}^i_j)^2)} - 1} e^{-r(T-T_{i+1})}.$$

Thus low values $\hat{\mu}^i_{i+k}$ increase the value of η^B_i which in turn increases risky investment. Because the ambiguity averse investor sees the future as too uncertain, he would rather achieve his wealth target early by taking large risks in the current period!

Before ending this section, we summarize the steps the backward investor takes, to formulate and implement his multi-period mean-variance optimization.

(i). Suppose at time T_i the prior distribution of μ_i is known, the investor determines the path $\{\hat{\mu}_{i+k}\}_{k=1}^{N-i-1}$ from (4.96)–(4.98).

(ii). At $[T_i, T_i + 1]$ the investor solves a regular dynamic mean-variance problem defined over $[T_i, T]$, under the parameter $\hat{\mu}$ obtained in step (i). At T_{i+1}, he obtains the terminal wealth $X^*(T_{i+1})$ given by (4.94).

(iii). At T_{i+1}, the investor refines his estimation for μ_{i+1}, based on the sample average return observed during $[T_i, T_{i+1}]$. This leads to the posterior distribution given by (4.90), which is then served as the prior distribution of μ_{i+1} for the next trading period.

4.4.3.3 Simulation

We now conduct a simulation study on the forward and backward mean-variance problems discussed in previous sections. Recall that the joint dynamics of per-period average realized return, $b_{i+1} = \frac{1}{\Delta T}\int_{T_i}^{T_{i+1}} \frac{dS(t)}{S(t)}$ and the expected return parameter μ_i can be described by the system,

$$\begin{cases} b_{i+1} = \mu_i + \sigma \frac{\Delta W_{i+1}}{\Delta T}, \\ \mu_{i+1} = (1-\beta)\overline{\mu} + \beta\mu_i + \nu_{i+1}, \end{cases} \tag{4.99}$$

for $i = 0, 1, \ldots, N-1$.

Here we take risky asset S to be the U.S. stock market index. We assume the following parameter values to describe the discrete dynamics of the expected return, $\overline{\mu} = 0.15$, $\beta = 0.72$, $\sigma_\nu = 0.09$.. Additionally, we set the correlation between the two shocks at $\rho = -0.7$, the volatility of unexpected return at $\sigma = 0.4$, and the risk-free rate at $r = 0.02$. The prior distribution μ_0 is assumed to have a mean $\overline{\mu}_0 = 0.2$ and variance $\sigma^2_{\mu_0} = 0.01$. All parameter values are annualized, meaning that these values are valid under the assumption that ΔT equals one year. When we choose to work with ΔT different from one year, β and σ_ν need to be rescaled to match the period length. (For example, if the investors update parameters every $\frac{1}{k}$ year, we change the value of $\beta = 0.72$ into $\beta = 0.72^{1/k}$, and $\sigma_\nu = 0.09$ into $\sigma_\nu = 0.09\sqrt{\frac{1-\beta^2}{1-\beta^{2k}}}$).

To obtain probability distributions for terminal wealth $X^F(T)$ and $X^B(T)$, we first simulate 1,000,000 paths of $\{b_i\}$ and $\{\mu_i\}$. Under each path simulated, we compute the terminal wealth $X(T)$ by calculating recursively the end-of-period wealth $X(T_i)$, for $i = 1, 2, \dots, N$.

For the backward investor, the following equation was derived in Theorem 4.4.15. Recall that

$$X^B(T_{i+1}) = \eta_i^B + \phi_i^B \frac{Z^{\hat{\mu}}(T_{i+1})}{Z^{\hat{\mu}}(T_i)}. \tag{4.100}$$

where the coefficients η_i^B and ϕ_i^B are given in equation (4.95).

Similarly for the forward investor, the fact that he solves a standard mean-variance problem on $[T_i, T_{i+1}]$ implies

$$X^F(T_{i+1}) = \eta_i^F + \phi_i^F \frac{Z^{\hat{\mu}}(T_{i+1})}{Z^{\hat{\mu}}(T_i)}, \tag{4.101}$$

where,

$$\begin{aligned} \eta_i^F &= \frac{d_{i+1} e^{\hat{\lambda}_i^2 \Delta T} - X^F(T_i) e^{r\Delta T}}{e^{\hat{\lambda}_i^2 \Delta T} - 1} \\ \phi_i^F &= -\frac{d_{i+1} e^{r\Delta T} - X^F(T_i) e^{2r\Delta T}}{e^{\hat{\lambda}_i^2 \Delta T} - 1} \end{aligned} \tag{4.102}$$

Now, the necessary input are the coefficients η_i, ϕ_i and the random variable $\frac{Z^{\hat{\mu}}(T_{i+1})}{Z^{\hat{\mu}}(T_i)}$. To obtain the former for the backward investor, we find at each T_i the subjective return parameters $\hat{\mu}_i$ following the procedures described in section 4.4.3.2, then η and ϕ are computed from equations and (4.95). Similarly, for the forward investor, parameter estimation follows section 4.4.3.1, and then the coefficients are calculated by (4.102). For computing $\frac{Z^{\hat{\mu}}(T_{i+1})}{Z^{\hat{\mu}}(T_i)}$, we recall that $Z^{\hat{\mu}}$ is the density process of the investor's subjective measure $\mathbb{Q}^{\hat{\mu}}$ with respect to the risk neutral measure $\mathbb{Q}^0$. Then, by Girsanov's theorem, under the true probability measure $\mathbb{Q}^{\mu}$, $Z^{\hat{\mu}}(t)$ solves in $[T_i, T_{i+1}]$ the SDE

$$dZ^{\hat{\mu}}(t) = \big(- r - \lambda^{\hat{\mu}_i}(\lambda^{\mu_i} - \lambda^{\hat{\mu}_i})\big) Z^{\hat{\mu}}(t)dt - \lambda^{\hat{\mu}_i} Z^{\hat{\mu}}(t)dW(t).$$

Here $\lambda^{\mu_i} = \frac{\mu_i - r}{\sigma}$, $\lambda^{\hat{\mu}_i} = \frac{\hat{\mu}_i - r}{\sigma}$, with μ_i being the true parameter and $\hat{\mu}_i$ the investor's estimate. Then $Z^{\hat{\mu}}(T_{i+1})$ can be solved explicitly,

$$\begin{aligned} \frac{Z^{\hat{\mu}}(T_{i+1})}{Z^{\hat{\mu}}(T_i)} &= \exp\Bigg((-r - \frac{1}{2}(\lambda^{\hat{\mu}_i})^2)\Delta T - \lambda^{\hat{\mu}_i}(\lambda^{\mu_i}\Delta T + \Delta W_{i+1}) \Bigg). \\ &= \exp\Bigg((-r - \frac{1}{2}(\lambda^{\hat{\mu}_i})^2)\Delta T - \lambda^{\hat{\mu}_i}\frac{\Delta T}{\sigma}(b_{i+1} - r) \Bigg) \end{aligned} \tag{4.103}$$

where the last step follows from the relation $b_{i+1} = \mu_i + \sigma\frac{\Delta W_{i+1}}{\Delta T}$. Therefore, after we draw a random sample of b_{i+1} at T_{i+1}, $\frac{Z^{\hat{\mu}}(T_{i+1})}{Z^{\hat{\mu}}(T_i)}$ is explicitly computed using the above equation. This concludes the last step of our simulation procedure.

Next, we compare the long term performances of forward and backward investors. Therefore, instead of measuring performance period over period, we should only focus on the long term Sharpe ratio,

$$SR = \frac{\mathbb{E}[X(T)] - e^{rT}X_0}{\sqrt{\text{Var}(X(T))}}.$$

Here the moments are conditional only on the information set at time zero. As the forward preference suggests, the investor only needs to specify his target mean d_1^F for the first period. The model will then combine information of market and the investor's past performances to endogenously generate all remaining targets for periods that follows. On the contrary, the backward investor specifies a fixed target d^B at the end of the horizon T. It is reasonable to imagine that, the two approaches will have greater differences under longer horizons. Therefore, we first look at how does the horizon affect performances of the two investors differently.

The horizon effect

We calculate the Sharpe ratios for both the forward and backward investors for horizons ranging from one to ten years. We take $\Delta T = 0.25$, that is we assume the investors re-estimate parameters every quarter (at the same time the forward investor will update his preference). To also get a sense of how ambiguity aversion affects performance, the calculations are done separately under the assumptions $\alpha = 0$ and $\alpha = 1$. When estimating the return parameters, investors with $\alpha = 0$ do not take any action against parameter uncertainty, and simply take the mean of the prior distribution as their estimate. In this sense, they are "ambiguity neutral". The $\alpha = 0$ case is shown figure 4.7 panel A. We can see that both the forward and backward investors' Sharpe ratios increase with time, suggesting that the returns of their portfolios increases in horizon at a faster rate than the risk. This is a consequence of return mean-reversion as a result of the negative correlation between the two shocks. Also, notice that the forward investor almost always outperforms the backward investor at all horizons. Although the difference is unnoticeable at shorter horizons (for T=1, the forward investor's Sharpe ratio is only 1.7 percent higher), it is highly significant at 10 years horizon, where the forward investor's Sharpe ratio is 38 percent higher! This is quite intuitive, for the backward investor's preference is placed at the terminal time, so his optimization problem requires knowledge of all the returns parameters in the entire horizon, which can only be estimated based on insufficient information currently available. The forward investor on the other hand dynamically sets preferences one period ahead, his optimization problem requires only the parameter for the next period, which is much easier to estimate. Because the forward investor optimizes based on more accurate parameter estimations, it is expected that he would achieve a better performance.

The plot in panel B assumes that $\alpha = 1$. The Sharpe ratios now are uniformly lower in comparison to the ones in panel A, and grow with the horizon at a smaller rate, for the investors would take in this case larger risks. Indeed, driven by ambiguity aversion, the investors would make more conservative estimates on expected returns. Thus they have to make larger investments in the risky asset in order to achieve the same wealth target. However, this effect has different impact on forward and backward investors. As parameter uncertainty faced by the backward investor is significantly larger, aversion to this uncertainty would render his portfolio excessively risky. Therefore, panel B shows a more striking difference in the performances of the two types of investors. For example, at $T = 10$, forward investor's Sharpe ratio is 100 percent higher.

The effect of learning

We examine how does learning affects the performances of forward and backward investors? Recall that in our setup, although the stock prices are observed continuously, the investors are only allowed to incorporate this information at the end of each period T_i. That is, parameters are updated every ΔT years. If we fix the horizon at T but increases the number of periods N, we then allow the investors to update their parameters at a higher frequency. This will obviously benefit the forward investor as the signals he observes can now be incorporated earlier in his parameter/preference update. The backward investor would benefit by the same effect but to a lesser extent, since more frequent change of subjective parameters also has the negative effect of introducing a higher degree of time-inconsistency. It is not yet clear which effect plays the more significant role.

In the numerical study that follows, we fix the horizon at $T = 10$ years, and let N vary from 1 to 20. In Figure 4.8, we plot the investors' Sharpe ratio as function of N. Again, in all cases expect $N = 1$, the forward investor outperforms the backward investor, and the performance gap widens as N increases, implying that forward investor receives a greater benefit from additional learning.

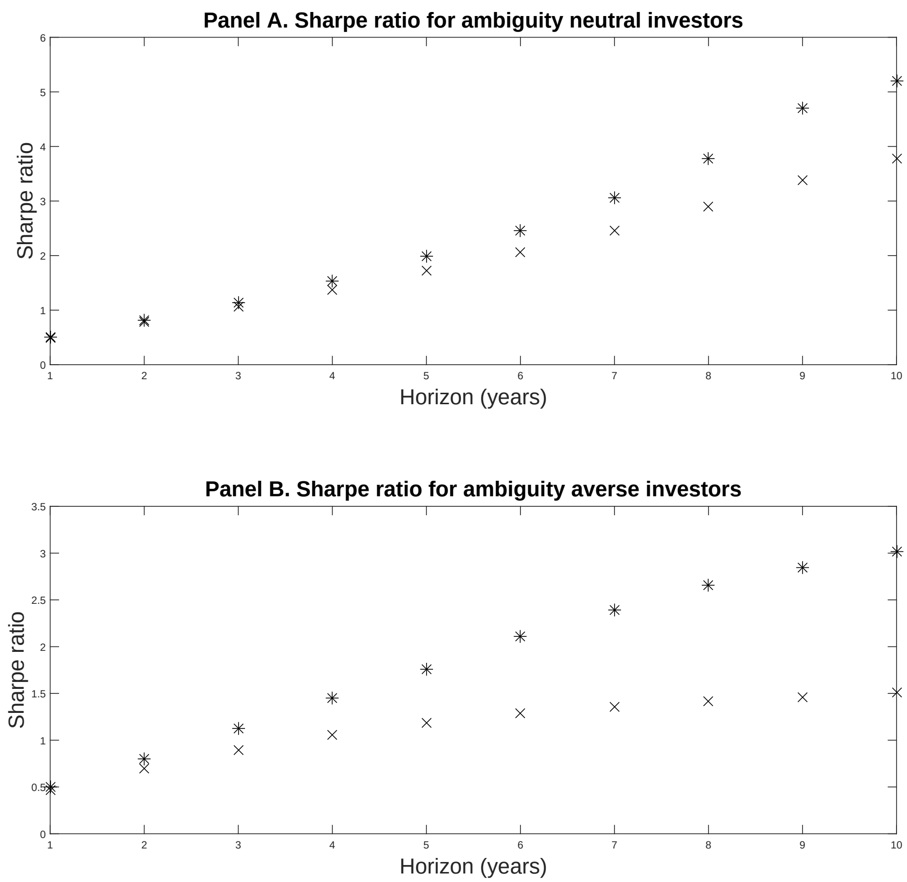


Figure 4.7: Sharpe ratio and investment horizon.

The case $N = 1$ is special in that it is the only case when the forward investor does not update his preference in between $[0, T]$. As a result, the forward and backward investors have identical Sharpe ratio performances. $N = 1$ is also the only case when learning is completely ignored. We can see that there is a sizable upward jump in Sharpe ratio when we just increase N from one to two, which provides evidence that the cost of ignoring learning is substantial, and is consistent with the findings in Xia (2001).

We also observe that the backward investor's Sharpe ratio is only increasing in N when $\alpha = 0$. When $\alpha = 1$, the Sharpe ratio first increases when $N \leq 3$, and after that it gradually decreases. As it has been discussed in section 4.4.3.2, in the presence of ambiguity aversion, the backward investor's choice for subjective parameter has a greater variation over time (compare to the ambiguity neutral case), that is, for a fixed period $[T_i, T_{i+1}]$, the investor would choose very different $\hat{\mu}_i$ at different times, introducing a higher degree of time-inconsistency in his portfolio decision. As the marginal benefit of learning decreases with N, the time-inconsistency effect would eventually outweigh the benefit of additional learning. That is why we see the Sharpe ratio drop at $N \geq 3$.

4.5 Conclusions

In the practice of portfolio management, investment decisions are almost never made based on a single-period optimization approach. Regardless

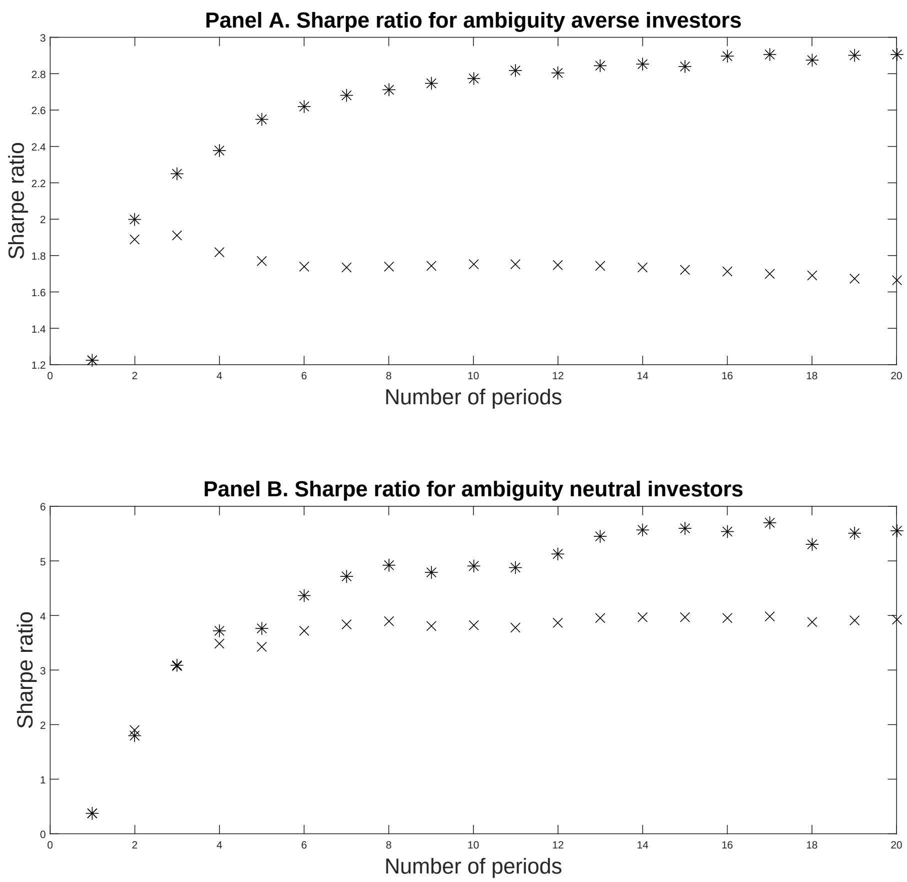


Figure 4.8: Sharpe ratio vs. number of investment periods.

of the actual investment lifetime, managers are more likely to construct portfolios based on optimizing objective functions placed at much shorter horizons, and repeat this effort sequentially and forward in time. Therefore, an inevitable decision is to specify a sequence of objective functions which guide investment decisions in each period. So far, to the best of our knowledge, there has been little work that takes this issue to a theoretical level. In mean-variance optimization in particular, the wealth targets in each period are chosen in ad hoc ways that ignore both the changing market conditions and realized performances.

We fill this crucial gap by proposing the forward approach in constructing multi-period mean-variance investment criteria, imposing that portfolio decisions should be *consistent* over time.

Our numerical examples under both discrete and continuous trading demonstrate that, maintaining time-consistency when generating multi-period investment criteria, leads to much higher long term Sharpe ratio, compared to other multi-period mean-variance approaches that ignore it.

An alternative to the multi-period approach is to formulate the investment problem as a single period, long horizon dynamic mean-variance optimization. While it does generate the best long run performance under the assumption that the investor can precisely estimates all the uncertainties, the quality of this approach quickly deteriorates even under a slight possibility of estimation error. As shown in section 4.3.4, a 2% error rate is already enough to make the multi-period forward mean-variance formulation more attractive.

Even in an idealized situation where model estimation is not a problem, the forward approach is still relevant in that it strikes a good balance when performance is measured both at long term and short term. Contrary to what Cvitanić et al. (2008) claims that managers who seek to maximize short term performance suffer greatly in the long term, the forward approach, while allowing managers to focus primarily on short term performances, generates far better long term Sharpe ratios that are only slightly below the managers who target at long term.

There are several lines of research that are left unexplored herein. Firstly, since the forward approach grants the investor full flexibility to his model estimation based on new information, it would be desirable to combine forward framework with existing statistical model selection techniques. For example, one might have a number of factors that have the potential to predict asset returns. At the beginning of each period, data analysis techniques may be applied to determine the best factor combination, as well as factor loading. Another example is the Black-Litterman model (see Black and Litterman (1991)), which allows the investor to express his subjective views about the asset returns in each trading period and to combine them quantitatively with the equilibrium model in order to obtain more refined predictive distributions. Based on these models, the forward theory can then generate the target wealth to be pursued in the next period.

A potential issue with this idea, however, is that if the investor's estimates or views are very wrong, the wealth targets and the resulting optimal portfolios obtained from the forward theory will not be able to guarantee time-consistency. A possible alternative is the non-parametric approach of Brandt (1999) or the semi-parametric approach of Ait-Sahalia and Brandt (2001). In order to avoid model specification errors, in these studies the authors suggest using the generalized method of moments to directly solve optimal portfolio weights through solving the empirical Euler equation. This bypasses the modeling step altogether. In fact, one can go one step further to solve not just for the optimal portfolio, but also for the forward target in a single step. Hence, the obtained forward mean-variance preference depend only on historical data and not model specification. These are ongoing research topics which will be addressed in future work.

Appendices

Appendix A

Proofs of Theorems for Chapter 2

A.1

Proof of Theorem 2.2.3

Proof. (i) By (2.10), we have

$$X_i^*(t) = \int_0^\infty e^{y h_i^{(-1)}(X_i(0),0)+(y-\frac{1}{2}y^2)A(t)+yM(t)}\nu_i(dy).$$

Moreover, $X_i(0) = \int_0^\infty e^{y h^{(-1)}(X(0),0)}\nu_i(dy)$, which implies that,

$$h^{(-1)}(X(0),0) = h_i^{(-1)}(X_i(0),0).$$

Hence,

$$\begin{aligned}
\sum_{i=1}^n X_i^*(t) &= \sum_{i=1}^n \int_0^\infty e^{y h_i^{(-1)}(X_i(0),0)+(y-\frac{1}{2}y^2)A(t)+yM(t)}\nu_i(dy), \\
&= \sum_{i=1}^n \int_0^\infty e^{y h^{(-1)}(X(0),0)+(y-\frac{1}{2}y^2)A(t)+yM(t)}\nu_i(dy), \\
&= \int_0^\infty e^{y h^{(-1)}(X(0),0)+(y-\frac{1}{2}y^2)A(t)+yM(t)}\sum_{i=1}^n \nu_i(dy), \\
&= \int_0^\infty e^{y h^{(-1)}(X(0),0)+(y-\frac{1}{2}y^2)A(t)+yM(t)}\nu(dy) = X^*(t).
\end{aligned} \tag{A.1}$$

(ii) The definitions of ν and $X(0)$ yield that $h^{(-1)}(X(0),0) = 0$. The rest of proof is similar to (i). □

A.2

Proof of Proposition 2.3.1

Proof. In Musiela and Zariphopoulou (2010a), it was shown that the risk tolerance function $r(x,t)$ solves

$$r_t + \frac{1}{2}r^2 r_{xx} = 0. \tag{A.2}$$

Therefore the function $\tilde{r}(x,t)$ solves,

$$\tilde{r}_t + x\tilde{r}^2\tilde{r}_x + \frac{1}{2}x^2\tilde{r}^2\tilde{r}_{xx} = 0.$$

Recall that the optimal wealth process is given by

$$dX^*(t) = \tilde{R}^*(t)X^*(t)(\lambda^2 dt + \lambda dW(t)). \tag{A.3}$$

Applying Ito's lemma formula yields that $\tilde{R}^*(t)$ needs to satisfy

$$\begin{aligned}
d\tilde{R}^*(t) =& d\tilde{r}(X^*(t), A(t)) \\
=& \tilde{r}_t(X^*(t), A(t))dA(t) + \tilde{r}_x(X^*(t), A(t))dX^*(t) + \frac{1}{2}\tilde{r}_{xx}(X^*(t), A(t))(dX^*(t))^2 \\
=& \lambda^2\bigg(\tilde{r}_t(X^*(t), A(t)) + \tilde{r}_x(X^*(t), A(t))\tilde{R}^*(t)X^*(t) \\
& + \frac{1}{2}\tilde{r}_{xx}(X^*(t), A(t))(\tilde{R}^*(t)X^*(t))^2\bigg)dt + \lambda\tilde{r}_x(X^*(t), A(t))\tilde{R}^*(t)X^*(t)dW(t) \\
=& \lambda R^*(t)\tilde{r}_x(X^*(t), A(t))\big(\lambda(1 - \tilde{R}^*(t))dt + dW(t)\big),
\end{aligned} \tag{A.4}$$

where we used (A.2). □

Proof of Lemma 2.3.3

Proof. By definition, $R_0(t) = X^*(t)$, $R_1(t) = R^*(t)$. Thus, equation (2.28) is automatically satisfied at $n = 0$, for $X^*(t)$ and $R^*(t)$ are related by the SDE,

$$dX^*(t) = R^*(t)(\lambda^2 dt + \lambda dW(t)).$$

If we write the stochastic processes in their integral forms, the above equation implies,

$$d\Big(\int_{\mathbb{R}} e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)\Big) = \int_{\mathbb{R}} ye^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)(\lambda^2 dt + \lambda dW(t)). \quad \text{(A.5)}$$

Note that the equality above is satisfied for any measure ν characterized in Proposition 2.2.2. Hence, if we replace $\nu(dy)$ by $\tilde{\nu}(dy) = y^n\nu(dy)$, the equation still holds,

$$d\Big(\int_{\mathbb{R}} y^n e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)\Big) = \int_{\mathbb{R}} y^{n+1}e^{yD(t)-\frac{1}{2}y^2A(t)}\nu(dy)(\lambda^2 dt + \lambda dW(t)).$$

This is precisely SDE (2.28) by the definition of $R_n(t)$. □

Proof of Proposition 2.3.5

Proof. We first show that (2.32) holds when the shift index l equals 0. Consider the $n \times (n+1)$ matrix ,

$$M_\alpha = \begin{bmatrix} 1 & \alpha_1 & \dots & \alpha_1^n \\ 1 & \alpha_2 & \dots & \alpha_2^n \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_n & \dots & \alpha_n^n \end{bmatrix}$$

It is then easy to show that M_α has rank n, since for $\forall\ i \neq j$, $\alpha_i \neq \alpha_j$. Thus, there exists a non-zero vector $\mathbf{q} = (q_0, q_1, \ldots, q_n)^T$ such that,

$$M_\alpha \mathbf{q} = 0.$$

Without loss of generality we can set $q_n = 1$. Therefore, for $i = 1, 2, \ldots, n$, we have

$$q_0 + q_1\alpha_i + q_2\alpha_i^2 + \ldots + q_{n-1}\alpha_i^{n-1} + \alpha_i^n = 0.$$

In other words, α_1,...,α_n are n distinct roots of the polynomial,

$$x^n + q_{n-1}x^{n-1} + \ldots + q_1 x + q_0.$$

Hence, for $k = 1, \ldots, n-1$, we have

$$q_{n-k} = (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq n} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}.$$

Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{Z} = (Z_0, Z_1, \ldots, Z_n)$ denote the probability and moment vectors. Then,

$$\mathbf{Z} = \mathbf{p}M_\alpha.$$

Since $\mathbf{p}M_\alpha\mathbf{q} = 0$, we have $\mathbf{Z}\mathbf{q} = 0$, and thus, the moment equality (2.32) holds at $l = 0$. For the case $l > 0$, first notice that the relation $M_\alpha\mathbf{q} = 0$ implies

$$\mathbf{p}D_\alpha^l M_\alpha \mathbf{q} = 0, \tag{A.6}$$

where D_α is the diagonal matrix with α_i's on the main diagonal. Because the i-th row of $D_\alpha^l M_\alpha$ equals $(\alpha_i^l, \alpha_i^{l+1}, \ldots, \alpha_i^{l+n})$, we have

$$\mathbf{p}D_\alpha^l M_\alpha = (Z_l, Z_{l+1}, \ldots, Z_{l+n}).$$

Therefore equation (A.6) coincides with (2.32). □

Proof of Corollary 2.3.7

Proof. For the matrix M_α defined in the proof of lemma 2.3.5, one can easily verify that,

$$\tilde{\mathbf{R}}(t) = M_\alpha^T \mathbf{p}(t).$$

Moreover, the column vectors of M_α^T give the eigenvectors of N_α with eigenvalues $\alpha_1, \ldots, \alpha_n$. Hence,

$$N_\alpha = M_\alpha^T D_\alpha (M_\alpha^T)^{-1}.$$

Multiplying equation (2.35) by $(M_\alpha^T)^{-1}$ yields

$$d(M_\alpha^T)^{-1}\tilde{\mathbf{R}}(t) = \lambda\big((M_\alpha^T)^{-1}N_\alpha - \tilde{R}^*(t)(M_\alpha^T)^{-1}\big)\tilde{\mathbf{R}}(t)(\lambda(1-\tilde{R}^*)dt + dW(t)),$$

and, in turn,

$$d\mathbf{p}(t) = \lambda\big(D_\alpha - \tilde{R}^*(t)I\big)\mathbf{p}(t)(\lambda(1-\tilde{R}^*)dt + dW(t)).$$

□

Appendix B

Proofs of Theorems for Chapter 3

B.1

Proof for Theorem 3.2.2

Proof. The drift term of $\tilde{R}^*(t)$ is,

$$\lambda^2 \mathrm{Var}(Y(t)|\mathcal{F}_t)(1 - \tilde{R}^*).$$

Clearly, under the above assumption we have $\mathrm{Var}(Y(t)|\mathcal{F}_t) \geq 0$, for all $t \geq 0$. To see the sign of $(1 - \tilde{R}^*(t))$, recall the explicit solution of $\tilde{R}^*(t)$,

$$\tilde{R}^*(t) = \frac{\int_0^\infty y e^{yD(t) - \frac{1}{2}y^2 A(t)} \nu(dy)}{\int_0^\infty e^{yD(t) - \frac{1}{2}y^2 A(t)} \nu(dy)}.$$

Therefore,

$$\inf \{y|\, y \in \mathrm{supp}(\nu)\} < \tilde{R}^*(t) < \sup \{y|\, y \in \mathrm{supp}(\nu)\}.$$

If $\mathrm{supp}(\nu) \subset [0, 1]$, we have $\tilde{R}^*(t) < 1$, for all t. Hence, $\tilde{R}^*(t)$ has non-negative drift. Since $\tilde{R}^*(t)$ is also bounded, and therefore integrable at all t, it must be a submartingale. The supermartingale case can be proved similarly. □

Proof for Theorem 3.2.3

Proof. (i) Without loss of generality, assume α's are arranged such that $|1-\alpha_1| < |1-\alpha_2| < \ldots < |1-\alpha_n|$. Let $g_i = \alpha_i - \frac{1}{2}\alpha_i^2$. Then, $g_1 > g_2 > \ldots g_n$. $\tilde{\pi}$ can be rewritten as,

$$\tilde{R}^*(t) = \frac{a_1\alpha_1 + \sum_{i=2}^n a_i\alpha_i e^{(\alpha_i-\alpha_1)(h^{(-1)}(x)+M(t))+(g_i-g_1)A(t)}}{a_1 + \sum_{i=2}^n a_i e^{(\alpha_i-\alpha_1)(h^{(-1)}(x)+M(t))+(g_i-g_1)A(t)}}.$$

For any $c \in \mathbb{R}$, if $\alpha_i > \alpha_1$, we have

$$\begin{aligned} &\text{Prob}\left((\alpha_i-\alpha_1)(h^{(-1)}(x)+M(t))+(g_i-g_1)A(t) > c\right) \\ =&\text{Prob}\left(M(t) > \frac{c}{\alpha_i-\alpha_1} - h^{(-1)}(x) - \frac{g_i-g_1}{\alpha_i-\alpha_1}\lambda^2 t\right) \\ =&1-\mathcal{N}\Big(\frac{1}{\lambda\sqrt{t}}\big(\frac{c}{\alpha_i-\alpha_1} - h^{(-1)}(x)\big) - \frac{g_i-g_1}{\alpha_i-\alpha_1}\lambda\sqrt{t}\Big). \end{aligned} \tag{B.1}$$

Here $\mathcal{N}(\cdot)$ is the cumulative density function of a standard normal distribution. Since $g_i - g_1 < 0$, the above probability goes to 0 as $t \to \infty$. It is easy to show that the case $\alpha_i < \alpha_1$ leads to the same result. Therefore, for any $i = 2, 3, \ldots, n$ we have,

$$\operatorname*{plim}_{t\to\infty} e^{(\alpha_i-\alpha_1)(h^{(-1)}(x)+M(t))+(g_i-g_1)A(t)} = 0.$$

The result immediately follows.

(ii) If $\text{supp}(\nu) \subset [0,1]$, we assume without loss of generality that $1 \geq \alpha_1 > \alpha_2 > \ldots > \alpha_n$. We have shown that when the entire support of ν is on the left side of 1, then $\tilde{R}^*(t)$ is a submartingale bounded above by α_1. Therefore, by the martingale convergence theorem, there exists a finite limit $\tilde{R}^*_\infty$, such that

$$\tilde{R}^*(t) \overset{a.s.}{\to} \tilde{R}^*_\infty.$$

We have $\tilde{R}^*_\infty \leq \alpha_1$ a.s.

On the other hand, by Fatou's lemma,

$$\mathbb{E}[\tilde{R}^*_\infty] \geq \limsup_{t\to\infty} \mathbb{E}[\tilde{R}^*(t)].$$

We have shown in part (i) that $\operatorname*{plim}_{t\to\infty} \tilde{R}^*(t) = \alpha_1$. Since $\tilde{R}^*(t)$ is also bounded, we must have,

$$\lim_{t\to\infty} \mathbb{E}[\tilde{R}^*(t)] = \alpha_1.$$

Therefore, $\mathbb{E}[\tilde{R}^*_\infty] \geq \alpha_1$, which implies that $\tilde{R}^*_\infty = \alpha_1$, a.s.. The proof is similar when $\text{supp}(\nu) \subset [1, \infty)$. □

Proof for Theorem 3.3.1

Proof. Since $\tilde{R}^*(t)$ solves (3.3), classical results (see for example øksendal (1985)) yield that $g(r)$ solves the following Poisson problem,

$$\begin{cases} \lambda^2(\tilde{r}-\alpha_1)(\alpha_2-\tilde{r})\big((1-\tilde{r})g' + \frac{1}{2}(\tilde{r}-\alpha_1)(\alpha_2-\tilde{r})g''\big) = -1 \\ g(d) = 0. \end{cases} \tag{B.2}$$

One can first solve the ODE under the additional boundary condition $g(d_1) = 0$ and then take $d_1 \to \alpha_2$. The ODE can be solved through multiplying (B.2) by the function,

$$\Big(\frac{\tilde{r}-\alpha_1}{d-\alpha_1}\Big)^{2\frac{1-\alpha_1}{\alpha_2-\alpha_1}}\Big(\frac{\tilde{r}-\alpha_2}{d-\alpha_2}\Big)^{2\frac{\alpha_2-1}{\alpha_2-\alpha_1}},$$

and then integrate. The calculations are elementary but tedious, and are hence omitted.

□

Appendix C

Proofs of Theorems for Chapter 4

C.1

Proof for Proposition 4.2.2

Proof. The wealth at T takes the form $X_T = (\omega_0(R_0 - r_f) + r_f)X_0$, and $\mathbb{E}[X_T] = (\omega_0\mu_0^e + r_f)X_0$, $\mathrm{Var}(X_T) = (\omega_0\Sigma_0\omega_0')X_0^2$. Therefore, the mean-variance problem can be explicitly written as

$$\max_{\omega} \ (\omega\mu_0^e + r_f)X_0 - \frac{\gamma_0}{2}(\omega_0\Sigma_0\omega_0')X_0^2 \tag{C.1}$$

The first order condition implies that $\mu_0^e X_0 - \gamma_0\Sigma_0\omega_0' X_0^2 = 0$, and thus

$$\omega_{MV}^{*}{}' = \frac{1}{\gamma_0 X_0}\Sigma_0^{-1}\mu_0^e \tag{C.2}$$

On the other hand, let $\Theta_0 = \Sigma_0 + \mu_0^e\mu_0^e{}'$ denote the second moment matrix of excess returns. Then, the quadratic utility problem has the explicit form

$$\max_{\omega_0} \ (\omega_0\mu_0^e + r_f)X_0 - \frac{\delta_0}{2}(\omega_0\Theta_0\omega_0' + 2r_f\omega_0\mu_0^e + r_f^2)X_0^2 \tag{C.3}$$

Take first order condition we obtain,

$$\omega_{QU}^{*}{}' = (\frac{1}{\delta_0 X_0} - r_f)\Theta_0^{-1}\mu_0^e. \tag{C.4}$$

It then follows from (C.2) and (C.4) that equivalence holds if and only if

$$\frac{1}{\gamma_0 X_0}\Sigma_0^{-1}\mu_0^e = (\frac{1}{\delta_0 X_0} - r_f)\Theta_0^{-1}\mu_0^e.$$

It is straightforward to verify that,

$$\Theta_0^{-1}\mu_0^e = \frac{1}{1+\mu_0^{e\prime}\Sigma_0^{-1}\mu_0^e}\Sigma_0^{-1}\mu_0^e. \tag{C.5}$$

Combining the above two equations we obtain (4.12). □

Proof for Proposition 4.2.4

Proof. For any $U_T(x)$ in the above form, we can calculate the value function $V_T(\cdot)$ at $t = T$ as,

$$\begin{aligned} V_T(X_T) &= a_T \mathbb{E}_T[X_{\tilde{T}}^{\omega^*} - \frac{\delta_T}{2}(X_{\tilde{T}}^{\omega^*})^2] + b_T \\ &= a_T\big(r_f(1-c)(X_T - \frac{\delta_T r_f}{2}X_T^2) + \frac{c}{2\delta_T}\big) + b_T, \end{aligned} \tag{C.6}$$

where $c = \mu_T^e{}'\Theta_T^{-1}\mu_T^e$. Identity (C.5) implies that $c = \frac{\mu_T^e{}'\Sigma_T^{-1}\mu_T^e}{1+\mu_T^e{}'\Sigma_T^{-1}\mu_T^e}$. Therefore,

$$V_T(x) = \frac{r_f a_T}{1+\mu_T^e{}'\Sigma_T^{-1}\mu_T^e}(x - \frac{\delta_T r_f}{2}x^2) + \frac{a_T}{2\delta_T}\frac{\mu_T^e{}'\Sigma_T^{-1}\mu_T^e}{1+\mu_T^e{}'\Sigma_T^{-1}\mu_T^e} + b_T. \tag{C.7}$$

In order to have $V_T(x) = U_0(x)$, we need

$$\begin{cases} \delta_T = \frac{\delta_0}{r_f}, \\ a_T = \frac{1+\mu_T^e{}'\Sigma_T^{-1}\mu_T^e}{r_f} \\ b_T = -\frac{1}{2\delta_0}(\mu_T^e{}'\Sigma_T^{-1}\mu_T^e), \end{cases} \tag{C.8}$$

and we easily conclude. □

Proof for Theorem 4.2.5

Proof. Let δ_{t-1} and δ_t be the coefficients that define the equivalent quadratic utility problems at $t-1$ and t. Then, by (4.12),

$$\frac{1}{\delta_i} = \frac{1}{\gamma_i}(1 + \mu_i^{e\prime}(\Sigma_i)^{-1}\mu_i^e) + r_f X_i, \;\; i = t-1, t.$$

Since by (4.14) the δ coefficients need to satisfy $\delta_t = \frac{\delta_{t-1}}{r_f}$, we need

$$\frac{1}{\gamma_t}(1 + \mu_t^{e\prime}(\Sigma_t)^{-1}\mu_t^e) + r_f X_t = r_f\big(\frac{1}{\gamma_{t-1}}(1 + \mu_{t-1}^e{}'(\Sigma_{t-1})^{-1}\mu_{t-1}^e) + r_f X_{t-1}\big)$$

Rearranging this equation we obtain (4.16). □

Proof for Proposition 4.2.6

Proof. By definition, $w_u = \frac{\Theta^{-1}\boldsymbol{e}}{\boldsymbol{e}'\Theta^{-1}\boldsymbol{e}}$ and $r_u = \frac{\mu'\Theta^{-1}\boldsymbol{e}}{\boldsymbol{e}'\Theta^{-1}\boldsymbol{e}}$. We can verify the following matrix identity,

$$\Theta^{-1}\mu = \frac{1}{1 + \mu'\Sigma^{-1}\mu}\Sigma^{-1}\mu$$
$$\Theta^{-1}\boldsymbol{e} = \Sigma^{-1}\boldsymbol{e} - \frac{\mu'\Sigma^{-1}\boldsymbol{e}}{1 + \mu'\Sigma^{-1}\mu}\Sigma^{-1}\mu.$$

Therefore we notice that both w_{MV}^* and w_{QU}^* are linear combinations of $\Sigma^{-1}\mu$ and $\Sigma^{-1}\boldsymbol{e}$, and they are identical if and only the weights on $\Sigma^{-1}\mu$ coincide. Hence, we need that

$$\frac{1}{\gamma X_0} = \frac{1}{\delta X_0}\big(\frac{1}{1 + \mu'\Sigma^{-1}\mu} + r_u\frac{\mu'\Sigma^{-1}\boldsymbol{e}}{1 + \mu'\Sigma^{-1}\mu}\big) - \frac{1}{\boldsymbol{e}'\Theta^{-1}\boldsymbol{e}}\frac{\mu'\Sigma^{-1}\boldsymbol{e}}{1 + \mu'\Sigma^{-1}\mu}.$$

Simplifying the equation yields

$$\begin{aligned}\frac{1}{\delta} &= \frac{1}{\gamma}(1 + \mu'\Sigma^{-1}\mu - \frac{(\mu'\Sigma^{-1}\boldsymbol{e})^2}{\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e}}) + \frac{\mu'\Sigma^{-1}\boldsymbol{e}}{\boldsymbol{e}'\Sigma^{-1}\boldsymbol{e}}X_0 \\ &= \frac{1}{\gamma}\big(1 + (\mu' - r_v\boldsymbol{e})\Sigma^{-1}(\mu - r_v\boldsymbol{e})\big) + r_v X_0.\end{aligned}$$

□

C.2

Proof for Theorem 4.3.3

Proof. By proposition 4.1 of Bielecki et al. (2005), problem (4.3.3) under the wealth constraint $\mathbb{E}_{i-1}[X(t_i)Z(t_i)] = X(t_{i-1})Z(t_{i-1})$ is equivalent to the unconstrained problem,

$$\min_{\pi \in \mathbb{A}} \mathbb{E}[X^\pi(t_i)^2 - 2\eta_{i-1}X^\pi(t_i) - 2\tilde{\eta}X(t_i)Z(t_i)], \tag{C.9}$$

for some $\tilde{\eta}$ measurable w.r.t. $\mathcal{F}_{t_{i-1}}$. Therefore, the first order condition implies that, the optimal terminal wealth has the following form,

$$X^*(t_i) = \eta_{i-1} + \tilde{\eta}Z(t_i).$$

On the other hand, recall that in section 3.3.1 we have derived the optimal terminal wealth for the mean-variance problem

$$\begin{aligned} X^*(t_i) =& \frac{d_{i-1}\mathbb{E}_{i-1}[(Z(t_i))^2] - X(t_{i-1})Z(t_{i-1})\mathbb{E}[Z(t_i)]}{\mathrm{Var}(Z(t_i))} + \\ & \frac{X(t_{i-1})Z(t_{i-1}) - d_{i-1}\mathbb{E}_{i-1}[Z(t_i)]}{\mathrm{Var}_{i-1}(Z(t_i))} Z(t_i). \end{aligned} \tag{C.10}$$

The optimal wealth then coincide if and only if,

$$\begin{cases} \eta_{i-1} = \frac{d_{i-1}\mathbb{E}_{i-1}[(Z(t_i))^2] - X(t_{i-1})Z(t_{i-1})\mathbb{E}[Z(t_i)]}{\mathrm{Var}(Z(t_i))}, \\ \tilde{\eta} = \frac{X(t_{i-1})Z(t_{i-1}) - d_{i-1}\mathbb{E}_{i-1}[Z(t_i)]}{\mathrm{Var}_{i-1}(Z(t_i))}. \end{cases} \tag{C.11}$$

which proves η_{i-1} is given by (4.49). □

Proof for Theorem 4.3.4

Proof. By definition, $\{U_i(x)\}_{i=0}^{\infty}$ is a predictable forward performance if it is constructed iteratively through the time-reversed HJB equation,

$$U_{i-1}(x) = \max_{\pi \in \mathcal{A}} \mathbb{E}_i[U_i(X^{\pi}(t_{i+1}))|X(t_i) = x]. \tag{C.12}$$

To calculate the "value function" on the right hand side, note from the proof of Proposition 4.3.3 that the wealth variable that maximizes $\mathbb{E}_i[U_i(x)]$ has the form,

$$X^*(t_{i+1}) = \eta_i + \tilde{\eta} Z(t_{i+1}).$$

The wealth constraint $\mathbb{E}_i[X(t_{i+1})Z(t_{i+1})] = X(t_i)Z(t_i)$ then implies that

$$\tilde{\eta} = \frac{X(t_i)Z(t_i) - \eta_i \mathbb{E}_i[Z_{t_{i+1}}]}{\mathbb{E}_i[Z(t_{i+1})^2]}.$$

Therefore,

$$\begin{aligned} &\max_{\pi \in \mathcal{A}} \mathbb{E}_i[U_i(X^{\pi}(t_{i+1}))|X(t_i) = x] \\ =&\mathbb{E}_i[a_i(X^*(t_{i+1}) - \eta_i)^2 + b_i|X_i = x] \\ =&a_i\Big(\frac{Z(t_i)x - \eta_i \mathbb{E}_i[Z_{t_{i+1}}]}{\mathbb{E}_i[Z(t_{i+1})^2]}\Big)^2 \mathbb{E}_i[Z(t_{i+1})^2] + b_i. \end{aligned} \tag{C.13}$$

In order for the value function to match the quadratic utility $U_{i-1}(x) = a_{i-1}(x - \eta_{i-1})^2 + b_{i-1}$, the η-coefficients need to satisfy

$$\eta_{i-1} = \frac{\mathbb{E}_i[Z(t_{i+1})]}{Z(t_i)} \eta_i.$$

In particular, when the risk-free rate is deterministic, we have

$$\frac{\mathbb{E}_i[Z(t_{i+1})]}{Z(t_i)} = e^{-\int_{t_i}^{t_{i+1}} r(s)ds}.$$

□

C.3

Proof of Lemma 4.4.3

Proof. (1). By definition, $Z^{\hat{\mu}}(t)$ has the following dynamics under $\mathbb{Q}^{\hat{\mu}}$,

$$dZ^{\hat{\mu}}(t) = -rZ^{\hat{\mu}}(t)dt - Z^{\hat{\mu}}(t)\lambda^{\hat{\mu}} \cdot dW^{\hat{\mu}}(t).$$

By Girsanov's theorem, $Z^{\hat{\mu}}(t)$ should solve under $\mathbb{Q}^{\mu}$ the SDE,

$$dZ^{\hat{\mu}}(t) = \big(-r - \lambda^{\hat{\mu}}(t) \cdot (\lambda^{\mu}(t) - \lambda^{\hat{\mu}}(t))' \big) Z^{\hat{\mu}}(t)dt - Z^{\hat{\mu}}(t)\lambda^{\hat{\mu}}(t) \cdot dW^{\mu}(t).$$

We can then prove that $\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)] \le \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)]$ by showing $Z^{\hat{\mu}}$ under Q^{μ} has a smaller drift. In other words, it would be sufficient to prove

$$\lambda^{\hat{\mu}}(t) \cdot (\lambda^{\mu}(t) - \lambda^{\hat{\mu}}(t))' \ge 0, \text{ a.s., and for } \forall\, t \in [0, T].$$

To show this let us first define the function

$$f(\theta) = |\theta\lambda^{\mu} + (1-\theta)\lambda^{\hat{\mu}}|^2, \, \theta \in [0, 1].$$

Let $D(t)$ denote the set of market price of risk vectors: $\{\lambda(t) = (\mu(t) - r\mathbf{1})\Sigma(t)^{-1} : \mu(t) \in C(t)\}$. Clearly, $D(t)$ is convex and closed since it is obtained as an affine transformation of $C(t)$. Therefore, the convex combination $\theta\lambda^{\mu}(t) + (1-\theta)\lambda^{\hat{\mu}}(t) \in D(t)$, for $\forall\, \theta \in [0, 1]$. By definition, $\lambda^{\hat{\mu}}(t)$ achieves the smallest norm in $D(t)$, implying that the vector $\theta\lambda^{\mu}(t) + (1-\theta)\lambda^{\hat{\mu}}(t)$ should attain minimum norm at $\theta = 0$, or equivalently, $f(\theta)$ is minimized at $\theta = 0$. After rearranging the terms we get (time variable t is omitted) that

$$f(\theta) = |\lambda^{\mu} - \lambda^{\hat{\mu}}|^2\theta^2 + 2\lambda^{\hat{\mu}} \cdot (\lambda^{\mu} - \lambda^{\hat{\mu}})'\theta + |\lambda^{\hat{\mu}}|^2.$$

The only way that $f(\theta)$, defined over $[0,1]$, attains minimum at $\theta = 0$ is when the first order coefficient is non-negative. We thus get $\lambda^{\hat{\mu}} \cdot (\lambda^{\mu} - \lambda^{\hat{\mu}})' \geq 0$. This proves $\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)] \geq \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)]$. $\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)^2] \leq \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2]$ can be proved in a similar fashion.

(2). By Itŏ's lemma, after differentiating $Z^{\mu}(t)^2$ under $\mathbb{Q}^{\mu}$:

$$dZ^{\mu}(t)^2 = Z^{\mu}(t)^2\big((|\lambda^{\mu}(t)|^2 - 2r)dt - 2\lambda^{\mu}(t) \cdot dW^{\mu}(t)\big).$$

Taking expectations on both sides and apply the fact that $|\lambda^{\mu}(t)|^2 \geq |\lambda^{\hat{\mu}}(t)|^2$, $\mathbb{Q}^{\mu}$-a.s. we get

$$d\mathbb{E}^{\mu}[Z^{\mu}(t)^2] \geq \mathbb{E}[Z^{\mu}(t)^2(|\lambda^{\hat{\mu}}(t)|^2 - 2r)dt]$$

. Notice that the deterministic assumption on $C(t)$ and $\Sigma(t)$ implies that $\hat{\mu}(t)$, and hence $\lambda^{\hat{\mu}}(t)$ is deterministic. The above is therefore reduced to

$$d\mathbb{E}^{\mu}[Z^{\mu}(t)^2] \geq (|\lambda^{\hat{\mu}}(t)|^2 - 2r)\mathbb{E}[Z^{\mu}(t)^2]dt.$$

By Gronwall's inequality,

$$\mathbb{E}^{\mu}[Z^{\mu}(t)^2] \geq e^{\int_0^T (|\lambda^{\hat{\mu}}(t)|^2 - 2r)\, dt} = \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(t)^2].$$

$\mathrm{Var}^{\mu}(Z^{\mu}(T)) \geq \mathrm{Var}^{\hat{\mu}}(Z^{\hat{\mu}}(T))$ follows immediately since $\mathbb{E}^{\mu}[Z^{\mu}(T)] = \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)] = e^{-rT}$.

(3). Notice that if we also assume $\lambda^{\mu}(t)$ to be deterministic, then the dynamics of $Z^{\hat{\mu}}(t)$ under both $\mathbb{Q}^{\mu}$ and $\mathbb{Q}^{\hat{\mu}}$ would have deterministic coefficients. This enables us to directly compute the variances. We can then verify that

$$\mathrm{Var}^{\mu}(Z^{\hat{\mu}}(T)) = e^{-2\int_0^T \lambda^{\hat{\mu}}(t)\cdot(\lambda^{\mu}(t)-\lambda^{\hat{\mu}}(t))'dt}\mathrm{Var}^{\hat{\mu}}(Z^{\hat{\mu}}(T)),$$

and $\mathrm{Var}^{\mu}(Z^{\hat{\mu}}(T)) \leq \mathrm{Var}^{\hat{\mu}}(Z^{\hat{\mu}}(T))$ follows again from the inequality $\lambda^{\hat{\mu}}(t)\cdot(\lambda^{\mu}(t)-\lambda^{\hat{\mu}}(t))' \geq 0$. □

Proof of Theorem 4.4.4

Proof. Denote by $RV(\cdot)$ the robust variance functional

$$RV(X(T)) := \max_{\mu\in\tilde{C}} \mathrm{Var}^{\mu}(X(T)).$$

It would be enough to show that $X^{\hat{\mu}}(T)$, the optimal terminal wealth of the standard mean-variance problem (4.73), satisfies the robust wealth target constraint,

$$\mathbb{E}^{\mu}[X^{\hat{\mu}}(T)] \geq d, \ \forall \mu \in \tilde{C}, \tag{C.14}$$

and minimizes the robust variance functional, i.e.,

$$X^{\hat{\mu}}(T) = \operatorname*{argmin}_{X(T)\in\mathcal{A}_T} RV(X(T)). \tag{C.15}$$

To show (C.14), we work as follows. By definition we have $\mathbb{E}^{\hat{\mu}}[X^{\hat{\mu}}(T)] \geq d$. Also from the proof of Theorem 4.3.3, we know that the terminal wealth $X^{\hat{\mu}}(T)$ is of the form

$$X^{\hat{\mu}}(T) = \frac{\eta}{2} - \phi Z^{\hat{\mu}}(T),$$

where η and ϕ are $\mathcal{F}_0$-measurable, depending on $\hat{\mu}$. Under the assumption that $d \geq X_0 e^{rT}$, we deduce that $\phi = \frac{de^{-rT}-X_0}{\mathrm{Var}^{\hat{\mu}}(Z(T))}$ should be non-negative. Applying part (1) of lemma 4.4.3 yields

$$\mathbb{E}^{\mu}[X^{\hat{\mu}}(T)] = \frac{\eta}{2} - \phi\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)] \geq \frac{\eta}{2} - \phi\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)] = \mathbb{E}^{\hat{\mu}}[X^{\hat{\mu}}(T)] \geq d,$$

which proves (C.14).

To prove (C.15), we first note that for $\forall\, X(T) \in \mathcal{A}_T$, $RV(X(T)) \geq \max_{\mu\in\tilde{C}} \mathrm{Var}^{\mu}(X^{\mu}(T))$. Here, again, $X^{\mu}(T)$ denotes the optimal terminal wealth of the standard mean-variance problem under subjective measure $\mathbb{Q}^{\mu}$. Straightforward calculation then shows that

$$\mathrm{Var}^{\mu}(X^{\mu}(T)) = \frac{(dE^{-rT} - X_0)^2}{\mathrm{Var}^{\mu}(Z^{\mu}(T))}.$$

By Lemma 4.4.3 part (2), the above is maximized at $\mu = \hat{\mu}$. We thus obtain the lower bound

$$RV(X(T)) \geq \frac{(dE^{-rT} - X_0)^2}{\mathrm{Var}^{\hat{\mu}}(Z^{\hat{\mu}}(T))}.$$

The calculations below shows that the above lower bound is attained at $X^{\hat{\mu}}(T)$,

$$\begin{aligned} RV(X^{\hat{\mu}}(T)) &= \max_{\mu\in\tilde{C}} \mathrm{Var}^{\mu}(X^{\hat{\mu}}(T)) \\ &= \max_{\mu\in\tilde{C}} \frac{(dE^{-rT} - X_0)^2}{\left(\mathrm{Var}^{\hat{\mu}}(Z^{\hat{\mu}}(T))\right)^2} \mathrm{Var}^{\mu}(Z^{\hat{\mu}}(T)) \\ &= \frac{(dE^{-rT} - X_0)^2}{\mathrm{Var}^{\hat{\mu}}(Z^{\hat{\mu}}(T))}, \end{aligned} \tag{C.16}$$

where the last equality follows from part (3) of Lemma 4.4.3. Thus, so far we have shown that the terminal wealth $X^{\hat{\mu}}(T)$ minimizes the robust variance functional function, while satisfying the target wealth constraint under any $\mathbb{Q}^{\mu}$. We easily deduce that $X^{\hat{\mu}}(T)$ is the solution to the robust mean-variance problem. □

Proof of Theorem 4.4.6

Proof. Let $RL(X(T)) := \min_{\mu \in C} \big(\mathbb{E}^{\mu}[X(T)] - \gamma \mathrm{Var}^{\mu}(X(T))\big)$ denote the robust Lagrangian functional. The theorem states that $X^{\hat{\mu}}(T) = \operatorname*{argmax}_{X(T) \in \mathcal{A}_T} RL(X(T))$. To prove this, we will first derive an upper bound for $RL(X(T))$, and then show that $RL(X^{\hat{\mu}}(T))$ attains this upper bound.

For an arbitrary but fixed $\mu \in C$, let $X^{\mu}(T)$ denote the optimal terminal wealth for the standard problem under $\mathbb{Q}^{\mu}$,

$$\max_{X(T) \in \mathcal{A}_T} \big(\mathbb{E}^{\mu}[X(T)] - \gamma \mathrm{Var}^{\mu}(X(T))\big). \tag{C.17}$$

Applying the embedding technique of Li and Zhou (2000), $X^{\mu}(T)$ can be shown to have the representation,

$$X^{\mu}(T) = \frac{x_0}{\mathbb{E}^{\mu}[Z^{\mu}(T)]} + \frac{1}{2\gamma}\frac{\mathbb{E}^{\mu}[Z^{\mu}(T)^2]}{\mathbb{E}^{\mu}[Z^{\mu}(T)]^2} - \frac{1}{2\gamma}\frac{Z^{\mu}(T)}{\mathbb{E}^{\mu}[Z^{\mu}(T)]} \tag{C.18}$$

Thus, by the optimality assumption of X^{μ}, we have

$$\begin{aligned} RL(X(T)) &= \min_{\mu \in C} \mathbb{E}^{\mu}[X(T)] - \gamma \mathrm{Var}^{\mu}(X(T)) \\ &\le \min_{\mu \in C} \mathbb{E}^{\mu}[X^{\mu}(T)] - \gamma \mathrm{Var}^{\mu}(X^{\mu}(T)) \\ &= \min_{\mu \in C} \frac{x_0}{\mathbb{E}^{\mu}[Z^{\mu}(T)]} + \frac{1}{4\gamma}\Big(\frac{\mathbb{E}^{\mu}(Z^{\mu}(T)^2)}{(\mathbb{E}^{\mu}[Z^{\mu}(T)])^2} - 1\Big). \qquad \text{(by (C.18))} \end{aligned} \tag{C.19}$$

Because $\mathbb{E}^{\mu}[Z^{\mu}(T)] = \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)] = e^{-rT}$, and by part (2) of Lemma 4.4.3 we have $\mathbb{E}^{\mu}[Z^{\mu}(T)^2] \ge \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2]$. In other words, the right hand side of (C.19) is minimized at $\hat{\mu}$. Therefore,

$$RL(X(T)) \le \frac{x_0}{\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)]} + \frac{1}{4\gamma}\Big(\frac{\mathbb{E}^{\hat{\mu}}(Z^{\hat{\mu}}(T)^2)}{(\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)])^2} - 1\Big). \tag{C.20}$$

Next we prove that the above upper bound is attained by the robust Lagrangian functional RL at $X^{\hat{\mu}}(T)$. Straightforward calculations give

$$\begin{aligned} RL(X^{\hat{\mu}}(T)) &= \min_{\mu\in C} \big(\mathbb{E}^{\mu}[X^{\hat{\mu}}(T)] - \gamma \mathrm{Var}^{\mu}(X^{\hat{\mu}}(T))\big) \\ &= \min_{\mu\in C} \Big(\frac{x_0}{\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)]} + \frac{1}{2\gamma}\frac{\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2]}{\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)]^2} - \frac{1}{4\gamma} \\ &\quad \cdot \frac{1}{(\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)])^2}(\mathrm{Var}^{\mu}(Z^{\hat{\mu}}(T)) + 2\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)]\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)])\Big). \end{aligned} \tag{C.21}$$

We can ignore the first two terms as they are independent of the minimization argument μ. For the third term, we rewrite

$$\begin{aligned} &\mathrm{Var}^{\mu}(Z^{\hat{\mu}}(T)) + 2\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)]\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)] \\ =&\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)^2] - (\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)] - \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)])^2 + (\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)])^2. \end{aligned} \tag{C.22}$$

By part (1) of Lemma 4.4.3, $\mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)^2] \leq \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2]$. Thus, the first and second terms of the above expression are both maximized at $\mu = \hat{\mu}$, and therefore the minimum on the RHS of (C.21) is achieved at $\mu = \hat{\mu}$. We obtain from (C.21),

$$RL(X^{\hat{\mu}}(T)) = \frac{x_0}{\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)]} + \frac{1}{4\gamma}\Big(\frac{\mathbb{E}^{\hat{\mu}}(Z^{\hat{\mu}}(T)^2)}{(\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)])^2} - 1\Big).$$

Comparing the above with equation (C.20), we have shown that $X^{\hat{\mu}}(T)$ attains the upper bound of the robust Lagrangian functional, and we easily conclude.

□

Proof of Theorem 4.4.7

Proof. We first let $X^{\mu}(T)$ denote the optimal terminal wealth that maximizes under $\mathbb{Q}^{\mu}$ the expected utility $QU^{\mu}(X(T)) = \mathbb{E}^{\mu}[\eta X(T) - X(T)^2]$, and let

$RQU(\cdot)$ denote the robust utility functional

$$RQU(X(T)) = \min_{\mu \in C} \left(\mathbb{E}^{\mu}[\eta X(T) - X(T)^2]\right).$$

The theorem then states that

$$X^{\hat{\mu}}(T) = \underset{X(T)\in\mathcal{A}_T}{\operatorname{argmax}} RQU(X(T)).$$

We prove this in two steps. First, we derive an upper bound of the robust utility functional, and then we show that the upper bound is attained at $X^{\hat{\mu}}(T)$.

By definition, $X^{\mu}(T)$ is the optimal terminal wealth for the standard utility problems under $\mathbb{Q}^{\hat{\mu}}$, for which we have the solutions:

$$X^{\mu}(T) = \frac{\eta}{2} - \frac{\phi}{2} Z^{\mu}(T), \text{ with } \phi = \frac{\eta e^{-rT} - 2X_0}{\mathbb{E}^{\mu}[Z^{\mu}(T)^2]}. \tag{C.23}$$

This in turn gives the utility at the optimum

$$QU^{\mu}(X^{\mu}(T)) = \frac{\eta^2}{4} - \frac{(\eta e^{-rT} - 2X_0)^2}{4\mathbb{E}^{\mu}[Z^{\mu}(T)^2]}. \tag{C.24}$$

Next, we consider the robust problem. We have by the definition of $X^{\mu}(T)$ that, for any admissible terminal wealth $X(T)$, the inequality $QU^{\mu}(X(T)) \leq QU^{\mu}(X^{\mu}(T))$ holds, for $\forall\, \mu \in C$. This implies

$$RQU(X(T)) = \min_{\mu \in C}\ QU^{\mu}(X(T)) \leq \min_{\mu \in C}\ QU^{\mu}(X^{\mu}(T)).$$

By (C.24), we can see that the right hand side above is minimized exactly when $\mathbb{E}^{\mu}[Z^{\mu}(T)^2]$ is minimized. By Lemma 4.4.3 part (2), this minimum is achieved at the "least favorable" measure $\hat{\mu}$. This gives an upper bound for $RQU(X(T))$,

$$RQU(X(T)) \le QU^{\hat{\mu}}(X^{\hat{\mu}}(T)) = \frac{\eta^2}{4} - \frac{(\eta e^{-rT} - 2X_0)^2}{\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2]}.$$

It remains to show that $RQU(X^{\hat{\mu}}(T)) = QU^{\hat{\mu}}(X^{\hat{\mu}}(T))$. Substituting the representation (C.23) of $X^{\mu}(T)$ into the above, the problem reduces to proving

$$\max_{\mu \in C} \mathbb{E}^{\mu}[Z^{\hat{\mu}}(T)^2] = \mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T)^2],$$

which however follows directly from part (1) of Lemma 4.4.3. □

Proof of Theorem 4.4.12

Proof. Part (ii) of definition 4.4.11 implies the restriction on U_n,

$$U_{n-1}(X(T_{n-1})) = \max_{X(T_n) \in \mathcal{A}_{T_n}} \min_{\mu \in \tilde{C}} \mathbb{E}^{\mu}[U_n(X(T_n))|\mathcal{F}_{T_{n-1}}].$$

By Theorem 4.4.7, the robust utility problem on the right hand side is equivalent to the standard utility problem under the "least favorable" measure $\hat{\mu}$. The equation can thus be rewritten as,

$$U_{n-1}(X(T_{n-1})) = \max_{X(T_n) \in \mathcal{A}_{T_n}} \mathbb{E}^{\hat{\mu}}[U_n(X(T_n))|\mathcal{F}_{T_{n-1}}].$$

In other words, $\{U_n\}_{n=1}^N$ is a predictable forward performance under the fixed measure $\mathbb{Q}^{\hat{\mu}}$. Applying theorem 4.3.4 gives,

$$\eta_n = \eta_{n-1} \frac{Z^{\hat{\mu}}(T_{n-1})}{\mathbb{E}^{\hat{\mu}}[Z^{\hat{\mu}}(T_n)]}.$$

Under the assumption that the interest rate is constant, the above reduces to,

$$\eta_n = \eta_{n-1} e^{r(T_n - T_{n-1})},$$

and we conclude. □

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