Transmission Resonances Induced by Time-Periodic Driving of a Quantum Well with a V-Shaped Bottom

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Abstract

How the scattering dynamics of a quantum system is affected by an application of a time-periodic driving field, such as coherent electromagnetic radiation, has been a subject of increasing importance in the past three decades. Time-periodic fields can profoundly alter the dynamics of matter in ways that are relevant to the design of semiconductor structures, such as quantum dots and superlattices, that have possible applications to quantum computation and quantum information processing.

This thesis is a theoretical treatment of a localized one-dimensional quantum system that is subject to an external driving field, such as an oscillating electric field of a laser or of some microwave radiation, that oscillates periodically in time. When not subject to the driving field, the system is characterized by a potential energy function that is a finite one-dimensional square well (i.e. a 1D quantum potential well with finite depth and a flat bottom. Also, the zero level of potential energy is set such that the potential in the well's exterior is zero). To this well, we specifically introduce a driving field that causes the well's bottom to bend from a flat shape to a V-shape, such that the potential energy as a function of position is held fixed at the endpoints of the well's region of space and forms the spike of the letter "V" at the midpoint of the well. Our chosen external field makes the V-shaped bottom perpetually bend up and down, causing the potential energy in the region of the well to vary periodically in time.

In this study, we analyze how a plane wave that propagates toward the region of our well and is with a fixed incident energy not only scatters into outgoing plane waves with the same energy — propagating in both directions of one-dimensional space — but is also induced by the driving field to have nonzero probabilities of transition to infinitely many other states with different energies. Due to the time-periodicity of our potential, an incoming wave-particle (i.e. a matter wave) of a given energy can only access energies that differ by integer multiples of $\hbar\omega$ from its original energy, where ω is the angular frequency of the oscillation of the potential. In the case that the time-periodic driving is due to an electromagnetic field oscillating at an angular frequency ω, we can explain the transitions in which a particle can only gain or lose an amount of energy that is an integer multiple of $\hbar\omega$ as follows: The electromagnetic field induces the particle to absorb or emit photons, each carrying a quantum of energy equal to an integer multiple of $\hbar\omega$.

The analysis to be revealed comprises three main achievements, all of which would help one to accomplish suitable computational precision, accuracy, and efficiency when making a prediction about a scattering phenomenon in our chosen system. The first achievement is that my research supervisor and I managed to solve the Schrödinger equation for this system analytically (i.e. exactly), so for a choice of four intervals that together constitute one-dimensional space, we were able to find the actual space of (complex-valued) explicit solutions to the equation in every interval separately. Within that space of solutions, if we only consider functions that are continuously differentiable, square-integrable, and somewhere nonzero (that is, nonzero solutions for which their partial derivatives with respect to spatial variables exist and are continuous everywhere and the squares of their absolute values are integrable over all space), then we obtain the set of all possible states that a particle can access in our quantum system. By accurately finding the space of solutions to the Schrödinger equation and by impos-

ing the requirement that a wavefunction must be continuously differentiable, we were able to deduce into what superposition of states a given incoming plane wave must scatter into in order to form a state that is possible to include in a physically allowed superposition of states.

Accounting for the fact that a physically acceptable solution must be at least continuously differentiable is directly related to the second success of the study, which is to derive a formula enabling me to find the scattering matrix (or S-matrix), a mathematical construct that relates the incoming plane waves to the states of definite energy outside the well into which they scatter. Despite the complexity of the solutions, I managed to exploit the reflection symmetry of the system about the center of the well and other simplifying properties to come up with expressions for four matrix blocks that constitute a matrix that contains the S-matrix, such that all four expressions involve the same eight matrices.

This in turn led to the third achievement, which was that I set up an efficient method for using computational software (in my case, I used Wolfram Mathematica 11.0) to find the elements of the S-matrix. Such a method was suitably fast at generating high-quality graphs of moduli squared of some matrix elements as functions of incident "energy" (actually, quasienergy, as we shall see later). Those graphs revealed the probability for an incoming wave (with a fixed energy and a unit probability current) to both transmit through the region of the well and to transition to a state, such as an outgoing wave or a negative-energy state, of some fixed energy. The energy of the new state did not necessarily have to be equal to the incident energy.

Given these computational freedoms, I created a demo of my method by constructing these transmission graphs for a specific set of parameters expressed in Hartree atomic units: well width of 2, particle mass $\mu = 1$, amplitude $U_0 = 0.5$ and angular frequency $\omega = 4$ of the oscillation of the V-shaped bottom, and unperturbed well depth $V_0 = 10$. When I compared the different graphs of some of the combined transmission and transition probabilities provided by the elements of the S-matrix, I noticed two incident energies for which transmission resonances occur. Next, I exploited those resonances to determine some energies of quasibound states, which are states in which a particle's probability is localized inside the region of the well for a finite amount of time. (In contrast, bound states have their probability localized for an infinite amount of time). For the version of our system without the external field (i.e. with $U_0 = 0$), the bound state energies for the first and second excited states are -6.7791 and -3.0542 Hartrees, respectively, while for the driven system, I found that the corresponding energies of quasibound states were -6.7788 and -3.0534 Hartrees. The fact that these energies of quasibound states for the perturbed system are quite close to those of the unperturbed system is an indication that the chosen oscillation strength $U_0 = 0.5$ Hartrees is weak enough to preserve many of the general properties of the unperturbed system.

1 Background

1.1 An Overview of the Field of Research

Since the 1990s, there has been a growing interest among physicists in determining how the application of a time-periodic field, such as coherent time-periodic radiation (like a laser) or an alternating current's (AC) electromagnetic field, controls an atomic-scale or a mesoscopic[1] system [10, 16]. Time-periodic fields can introduce fundamental changes to the dynamics of matter that are relevant to various aspects of technological innovation, such as $Al_xGa_{1-x}As/GaAs$ quantum dots [2, 7, 13], quantum diodes [4], superlattices [11, 12], nanotubes [15], and semiconductor heterostructures [18, 19], many of which have potential applications to quantum computation and quantum information processing.

For instance, in quantum dots, quantum diodes, and superlattices, the process of photon-assisted tunneling (PAT) is caused by the presence of AC electromagnetic radiation, namely by the oscillating electric field component of that radiation [7, 16]. Such an oscillating electric field is the same kind of external field leading to the Autler-Townes (or AC Stark) effect, which is the phenomenon in which the spectral lines of a confined quantum system (usually an atom or a molecule) split into more lines, representing the formation of new energy states. From the perspective of non-relativistic quantum mechanics, due to its significant variation over time, the oscillating electric field allows particles in propagating modes of certain energies to have high probabilities of transition into evanescent modes, which are the modes through which quantum tunneling occurs. From the relativistic perspective, however, the electromagnetic radiation assists those same transitions by stimulating the particles to emit photons, and as it is well known, photons are characterized by the quanta of energy equal to $\hbar\omega$, where ω is the angular frequency of the incoming electromagnetic wave. Both interpretations end up predicting the same phenomenon of photon-assisted tunneling.

To predict the effects of introducing a time-periodic field into a certain closed quantum system, physicists have relied on the feature that due to the time-periodicity of the introduced field, the potential[2] is time-periodic as well, thereby allowing the application of Floquet's theorem (to be formulated in Section 1.2) [10, 16]. Floquet's theorem has allowed physicists to develop a so-called "Floquet formalism," formally treating systems with time-periodic potentials as if they contain states of definite "energy" [22]. Nonetheless, the formalism makes the distinction that the conserved quantity in those states is not the energy by assigning to that quantity the name "quasienergy" [10] or "Floquet eigenenergy" [16].

An early example of a method similar to the Floquet technique has been provided by the electrical engineer Ping K. Tien and physicist James P. Gordon in 1963 [25]. The method was a heuristic approach to understand the effect of time-periodic driving on the current-voltage characteristics of superconductor-insulator-superconductor junctions. Beginning in

[1] Mesoscopic refers to systems that are greater than nano-scale systems, but are still small enough for quantum effects to be significant. Mesoscopic physics typically encompasses the range of scales between the typical size of a virus (100 nm) and that of a bacterium (1 μm).

[2] In quantum mechanics, the word "potential" is often used to refer to the potential energy function and is not to be confused with the electric potential.

the 1980s, the Floquet formalism has been commonly employed to find expressions for probabilities that incident waves of definite energies have to transmit through specific examples of localized systems with time-periodic potentials [3, 5, 10, 16, 17, 20, 24, 26]. At times, there were resonance dips and peaks in those transmission probabilities that were employed to find what are called the "energies" of quasibound states [16], which are states such that for a finite amount of time, a particle's probability is concentrated inside the region where the system is localized.

Our study demonstrates another example (described in Section 1.3) of a localized time-periodically driven system in which transmission probabilities can be computed and plotted as functions of an incoming wave's incident quasienergy and in which the bound state energies can be found. Our system has interesting complexities involved in the relationships between wavefunctions in different regions. Furthermore, plotting graphs of transmission probabilities versus incident quasienergy can be a challenge for this system, but I set up a convenient method of setting up the necessary equations that helped the plotting done by computational software take a small amount of time. This method can serve as a basis for creating a program predicting the transmission properties of the system considered in this paper.

1.2 Theoretical Background

Given a quantum mechanical system with a certain potential defined over all space and time, the nature of the time dependence of the potential has a fundamental impact on the dynamics of particle scattering and on the conservation properties of physical observables, such as energy. Suppose that a quantum system is **closed**, meaning that the corresponding potential does not change with time. Then, in any quantum state in which a particle has a definite energy at one time, the particle's energy will remain constant at all times. Such states in which energy conservation holds are called **stationary states**. By analogy, in classical Newtonian mechanics, energy conservation in a system is only guaranteed when all of the force fields present are conservative (that is, they have a corresponding potential energy function) and have no explicit time dependence.

A quantum system subject to a net time-dependent field no longer permits the existence of stationary states, but if an otherwise closed system is only driven by a time-periodic field, then a conservation property similar to that of energy still holds. To understand this property, let us consider the mathematical consequences of a system with a time-periodic potential.

In non-relativistic quantum mechanics, the space of complex-valued solutions to the **Schrödinger equation** determines the set of all states (or wavefunctions) accessible to any particle in the system. In fact, the set of wavefunctions is a subset of the space of all possible solutions and is determined by the conditions that the wavefunction needs to be:

1. **Square-integrable** over all space, meaning that the volume integral over all space of the absolute value squared of the wavefunction exists.
2. Nonzero in some nonempty subset of space, for otherwise it represents no particle.
3. (a) **Continuously differentiable** at every point where the potential is finite, meaning

that its partial derivatives with respect to spatial variables all exist and are continuous at every such point, and (b) only **continuous** at points where the potential is infinite.

I would like to note that from this point forward, any function that satisfies condition 2 above, we will refer to it as a **nonzero function**, which should **not** be interpreted as a function that is nowhere zero. In addition, for the system we will consider in this paper, condition 3 will imply that a physically acceptable wavefunction must be **continuously differentiable everywhere** (i.e. case (a) must hold in all of space), because our chosen potential will be finite at all positions in space.

In one-dimensional space, the **single-particle Schrödinger equation** is the following partial differential equation:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t), \tag{1}$$

where x is the position along one-dimensional space, t is the time, $\hbar$ is the reduced Planck's constant, μ is the mass of the considered particle, $V(x,t)$ is the potential energy function determining the system in question and the corresponding Schrödinger equation, and $\Psi(x,t)$ is the wavefunction, which is a complex-valued function of space and time and is the tool that contains all of the information that can be known about the particle's state. The wavefunction is the most physically informative mathematical construct in quantum mechanics, because it determines a particle's probability distribution not only over position, but over any physical observable, such as momentum and energy.

The right-hand side of equation (1) is often written as

$$\hat{H}(x,t)\Psi(x,t),$$

where

$$\hat{H}(x,t) = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + V(x,t).$$

$\hat{H}(x,t)$ is called the Hamiltonian operator, and it is a linear function acting on the vector space of wavefunctions and assigning to every wavefunction Ψ some function $\hat{H}\Psi$, which according to the Schrödinger equation (Eq. (1)), must equal to the imaginary number $i\hbar$ times the partial derivative $\frac{\partial\Psi(x,t)}{\partial t}$ of the wavefunction with respect to time.

We are interested in the situation when the potential $V(x,t)$ is time-periodic with some period $T = 2\pi/\omega$, meaning that the potential satisfies $V(x,t) = V(x,t+T)$ for any time t. Such a property of the potential implies that the Hamiltonian operator $\hat{H}(x,t)$ fulfills the condition $\hat{H}(x,t) = \hat{H}(x,t+T)$ for all t as well, so the Schrödinger equation is of the form

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat{H}(x,t)\Psi(x,t), \tag{2}$$

such that $\hat{H}(x,t)$ is a time-periodic operator.

Now, it is time to introduce the formulation of Floquet's theorem in finite dimensions[3]:

[3]The version of Floquet's theorem presented here is a slightly weaker statement than the original Floquet's theorem, because most members of the Dean's Scholars Honors program might not be acquainted with some linear algebra concepts, such as bases and Jordan canonical form, necessary to state the full theorem.

Theorem 1 (Floquet's theorem in finite dimensions [21]): *Let n be a finite positive integer. Suppose that an $n \times n$ matrix-valued function $A(t)$ is periodic in variable t with a period T (i.e. for all t, $A(t) = A(t+T)$). Then, the system of n homogeneous linear ordinary differential equations*

$$\frac{dx(t)}{dt} = A(t)x(t), \tag{3}$$

where $x(t)$ is a column vector of length n and is the unknown to be found, has a space of solutions that fully consists of superpositions (or, more technically, ***linear combinations****) of solutions of the form*

$$x_\Omega(t) = e^{\Omega t} p_\Omega(t), \tag{4}$$

where $p_\Omega(t)$ is a nonzero function for at most n values of Ω [4] *and, for all Ω and t, p_Ω satisfies $p_\Omega(t) = p_\Omega(t+T)$. In other words, for no more than n values of Ω, there exist nonzero functions p_Ω that are periodic with period T/m for some positive integer m, such that the corresponding x_Ω solves Eq. (3). For every such special value of Ω, however, p_Ω is not necessarily unique up to multiplication by a constant.*

In the case of Eq. (2), one needs the infinite-dimensional version of Floquet's theorem, because unlike the linear operator corresponding to multiplication of column vectors by the matrix $A(t)$, the time-periodic Hamiltonian operator $\hat{H}(x,t)$ does not act on a finite-dimensional vector space, but rather on an infinite-dimensional vector space for which the wavefunctions are its elements. Fortunately, Floquet's theorem has a straightforward generalization to some infinite-dimensional cases, including our case with the Schrödinger equation:

Theorem 2 (Floquet's theorem in the case of the Schrödinger equation [6, 9, 16, 23]): *If the Hamiltonian operator is time-periodic with a period T, then the Schrödinger equation (expressed in both equations (1) and (2)) has a space of solutions that fully consists of superpositions of solutions of the form*

$$\Psi_\Omega(x,t) = e^{-i\Omega t/\hbar} \phi_\Omega(x,t), \tag{5}$$

where for any real number Ω, $\phi_\Omega(x,t)$ can be a nonzero function that is time-periodic with the period T/m for some positive integer m and is not necessarily uniquely determined up to multiplication by a constant. In the Floquet formalism, these solutions are called ***Floquet states*** [10]*. Moreover, the Floquet states form a* ***complete orthonormal basis*** *in the space of solutions* [10]*, which is a stronger version of the statement that the superpositions of Floquet states fully comprise the space of solutions.*

The conserved quantity in the Floquet states from Eq. (5) is Ω, and it is called a **quasienergy** (or **Floquet eigenenergy**). The **conservation of quasienergy** is precisely that aforementioned property replacing the conservation of energy in a time-periodically driven system, such that Floquet states serve as replacements for stationary states.

[4] The values of Ω for which p_Ω is allowed to be nonzero in order for x_Ω to satisfy Eq. (3) will not be discussed here, but they are discussed in the actual formulation of Floquet's theorem. These Ω are the eigenvalues of $1/T$ times the logarithm of the so-called monodromy matrix. The monodromy matrix relates the solutions of Eq. (3) at any time t to the solutions of the same equation at time $t+T$.

One important aspect to note about Floquet states is that their quasienergy is not uniquely determined. Denoting the angular frequency of the time-periodic oscillation of the potential by ω (so that period $T = 2\pi/\omega$), all that we can say about the quasienergy is that it is uniquely determined up to the addition of an integer multiple of $\hbar\omega$. Indeed, by Euler's formula that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we have that $e^{-i\omega t}$ is a time-periodic function over the period $T = 2\pi/\omega$. Hence, if we consider a Floquet state with quasienergy $\Omega + \hbar\omega$

$$\Psi_{\Omega+\hbar\omega}(x,t) = e^{-i(\Omega+\hbar\omega)t/\hbar}\phi_{\Omega+\hbar\omega}(x,t),$$

then we can also write that

$$\Psi_{\Omega+\hbar\omega}(x,t) = e^{-i\Omega t/\hbar} \cdot (e^{-i\omega t}\phi_{\Omega+\hbar\omega}(x,t)), \tag{6}$$

where $e^{-i\omega t}\phi_{\Omega+\hbar\omega}(x,t)$ is time-periodic over the period T/p for some positive integer p, so this equation implies that any Floquet state $\Psi_{\Omega+\hbar\omega}(x,t)$ of quasienergy $\Omega + \hbar\omega$ also has quasienergy Ω. By almost identical reasoning, any Floquet state of quasienergy Ω has quasienergy $\Omega + \hbar\omega$ as well. By induction, we conclude that a Floquet state of quasienergy Ω also has any quasienergy $\Omega + n\hbar\omega$, where n is any integer.

From now on, when we will state that *the* quasienergy of a Floquet state is Ω, we will think of Ω as representing the entire set of values $\Omega + n\hbar\omega$ for which n is some integer. Furthermore, the set of all $\Omega + n\hbar\omega$, such that Ω belongs to the interval $[0, \hbar\omega)$ and n is an integer, contains every real number. This means that the quasienergies Ω within the interval $[0, \hbar\omega)$ represent all real numbers, thereby accounting for all of the possible quasienergies that the Floquet states can have.

1.3 The Problem Considered

Recall that the potential $V(x,t)$ determines the one-dimensional system in question and the Schrödinger equation corresponding to that system (see Eq. (1) and the description below it). In this thesis, we are interested in a system with one spatial dimension and with the following time-periodic potential:

$$V(x,t) = \begin{cases} -V_0 + U_0(1-|x|)\cos(\omega t) & \text{for } -1 \le x \le 1 \\ 0 & \text{for } -\infty < x < -1 \text{ and } -1 < x < \infty \end{cases}, \tag{7}$$

where V_0 is some positive constant, while U_0 and ω are any real constants. In Section 3, the units that we will use for the parameters U_0, V_0, and ω and other physical quantities involved will be Hartree atomic units, so $x = 1$ will mean $x = 1$ Bohr radius in the future.

To understand the shape of the potential, let us first consider the following potential:

$$V_{nopert}(x) = \begin{cases} -V_0 & \text{for } -1 \le x \le 1 \\ 0 & \text{for } -\infty < x < -1 \text{ and } -1 < x < \infty \end{cases}. \tag{8}$$

Then, $V_{nopert}(x)$ is a special case of $V(x,t)$ for which the time dependence is eliminated by setting $U_0 = 0$. Hence, $V_{nopert}(x)$ is the version of $V(x,t)$ that is not perturbed by the time-periodic oscillation due to the term $U_0(1-|x|)\cos(\omega t)$ within the spatial interval between $x = -1$ and $x = 1$. The shape of the unperturbed potential $V_{nopert}(x)$ is a one-dimensional finite square well centered at $x = 0$ with a width of 2 and a depth of V_0. For this reason, we will refer to the parameter V_0 as the "unperturbed well depth."

While the bottom of the potential well depicted by $V_{nopert}(x)$ remains flat (i.e. square) over time, the bottom of the potential well in the case of $V(x,t)$ constantly deforms as depicted in Figure 1. Due to the time-periodic perturbation $U_0(1-|x|)\cos(\omega t)$ within the region of the well, the bottom of the new well has an oscillating V-shape. The time-periodic oscillation is such that the potential energy is held fixed at the endpoints $x = \pm 1$ of the well, forms the spike of the letter "V" that oscillates sinusoidally with amplitude U_0 in the middle at $x = 0$, and has its linearity with respect to position preserved over time at other positions within the well. The average of the oscillation is the flat bottom of the unperturbed potential well $V_{nopert}(x)$.

Our first goal will be to solve the Schrödinger equation for this potential, which is

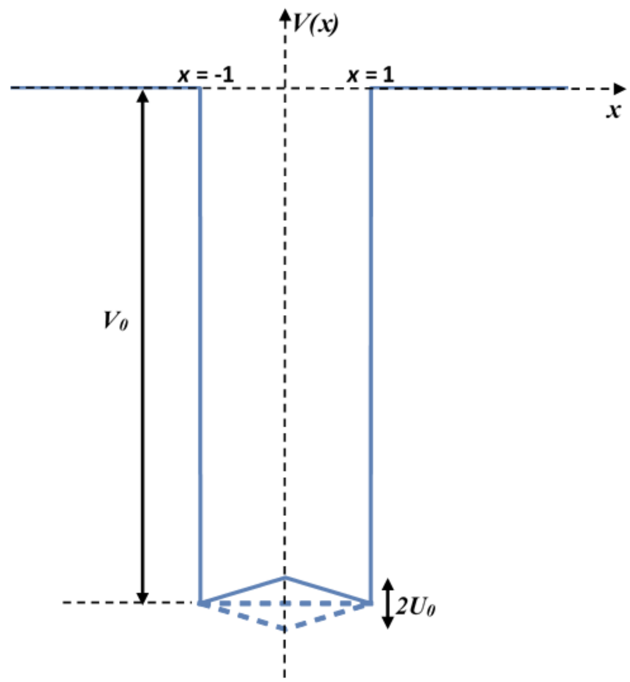

Figure 1: A schematic diagram of our chosen time-periodic potential $V(x,t)$ from Eq. (7). The bottom of the potential well is shown at three different times during one period of oscillation. V_0 is the "unperturbed" well depth and U_0 is the oscillation amplitude of the potential well's V-shaped bottom.

$$\begin{cases} i\hbar\dfrac{\partial\Psi(x,t)}{\partial t} = -\dfrac{\hbar^2}{2\mu}\dfrac{\partial^2\Psi(x,t)}{\partial x^2} + [-V_0 + U_0(1-|x|)\cos(\omega t)]\Psi(x,t) & \text{for } |x|\leq 1 \\ i\hbar\dfrac{\partial\Psi(x,t)}{\partial t} = -\dfrac{\hbar^2}{2\mu}\dfrac{\partial^2\Psi(x,t)}{\partial x^2} & \text{for } |x|> 1 \end{cases} \tag{9}$$

Next, we will exploit the solutions to this equation to analyze the scattering dynamics of this system.

2 Derivations and Results

2.1 Solving the Schrödinger Equation

By the generalization of Floquet's Theorem (Theorem 2 from Section 1.2), Eq. (9) has solutions of the form expressed in Eq. (5). If we substitute Eq. (5) into Eq. (2), then we have

$$\begin{aligned} i\hbar\frac{\partial\Psi_\Omega(x,t)}{\partial t} &= i\hbar(-i\Omega/\hbar)e^{-i\Omega t/\hbar}\phi_\Omega(x,t) + i\hbar e^{-i\Omega t/\hbar}\frac{\partial\phi_\Omega(x,t)}{\partial t} \\ &= e^{-i\Omega t/\hbar}\left(\Omega\phi_\Omega(x,t) + i\hbar\frac{\partial\phi_\Omega(x,t)}{\partial t}\right) \\ &= \hat{H}(x,t)\Psi_\Omega(x,t) = e^{-i\Omega t/\hbar}\hat{H}(x,t)[\phi_\Omega(x,t)]. \end{aligned}$$

Canceling out the exponential factors in the third and fifth expressions above, we obtain

$$\Omega\phi_\Omega(x,t) + i\hbar\frac{\partial\phi_\Omega(x,t)}{\partial t} = \hat{H}(x,t)\phi_\Omega(x,t). \tag{10}$$

The **Floquet Hamiltonian** is the following operator:

$$\hat{H}_F(x,t) = \hat{H}(x,t) - i\hbar\frac{\partial}{\partial t}.$$

Therefore, Eq. (10) can be written as

$$\hat{H}_F(x,t)\phi_\Omega(x,t) = \Omega\phi_\Omega(x,t). \tag{11}$$

This equation is a statement that for every Floquet state $\Psi_\Omega(x,t)$, the time-periodic function $\phi_\Omega(x,t)$ chosen in equation (5) is an eigenfunction of the Floquet Hamiltonian $\hat{H}_F$ with eigenvalue Ω. By the discussion following equation (6), the part of the solution that is time-periodic over period T/m for some positive integer m can be chosen differently such that it is an eigenfunction of $\hat{H}_F$ with an eigenvalue that differs from Ω by an integer multiple of $\hbar\omega$. Thus, via Eq. (11), we can define the quasienergy of a Floquet state $\Psi_\Omega(x,t)$ more rigorously as the set of eigenvalues of all possible time-periodic parts with period that is an integer divisor of T that $\Psi_\Omega(x,t)$ can have. For every $\Psi(x,t)$, such definition yields that the quasienergy is any value $\Omega + n\hbar\omega$ such that n is an integer. As mentioned in the last paragraph of Section 1.2, out of all of the values $\Omega + n\hbar\omega$, we can (and will) choose a representative value for the quasienergy that belongs to the interval $[0, \hbar\omega)$.

The aspect that any chosen time-periodic part (period T/m) of a Floquet state is an eigenfunction of $\hat{H}_F$ with the eigenvalue as the quasienergy explains quasienergy conservation in the same way that, for closed systems, the fact that the time-independent part of a stationary state is an eigenfunction of the Hamiltonian $\hat{H}$ with the eigenvalue as the energy explains energy conservation.

From this point forward, we will use the Hartree atomic units, in which we have the convenience that $\hbar = 1$. For our potential $V(x,t)$ given by Eq. (7), we have that Eq. (10) is

$$i\frac{\partial \phi_\Omega(x,t)}{\partial t} + \Omega\phi_\Omega(x,t) = \begin{cases} -\dfrac{1}{2\mu}\dfrac{\partial^2\phi_\Omega(x,t)}{\partial x^2} + [-V_0 + U_0(1-|x|)\cos(\omega t)]\phi_\Omega(x,t) & \text{for } |x|\leq 1 \\ -\dfrac{1}{2\mu}\dfrac{\partial^2\phi_\Omega(x,t)}{\partial x^2} & \text{for } |x|> 1 \end{cases}, \tag{12}$$

where we have also made the conversion to atomic units.

Our first step will be to find the solutions to Eq. (12) for each of the following four spatial regions: The intervals $(-1,0)$, $(0,1)$, $(-\infty,-1)$, and $(1,\infty)$. We are calling the interval $(-1,1)$ Region I, so we refer to the intervals $(-1,0)$ and $(0,1)$ as Region IL and Region IR, respectively. We assign to the remaining intervals $(-\infty,-1)$ and $(1,\infty)$ the names Region II and Region III, respectively.

• Region I ($|x|< 1$)

In Region I, let us define the function $f_\Omega(x,t) = e^{iU_0\sin(\omega t)/\omega}\phi_\Omega(x,t)$. Substituting $\phi_\Omega(x,t) = e^{-iU_0\sin(\omega t)/\omega}f_\Omega(x,t)$ into Eq. (12) for $|x|\leq 1$, we find after some cancellations that

$$i\frac{\partial f_\Omega(x,t)}{\partial t} + \Omega f_\Omega(x,t) = -\frac{1}{2\mu}\frac{\partial^2 f_\Omega(x,t)}{\partial x^2} - [V_0 + U_0|x|\cos(\omega t)]f_\Omega(x,t). \tag{13}$$

In Region IR, where $x \in (0,1)$, Eq. (13) takes the form:

$$i\frac{\partial f_\Omega(x,t)}{\partial t} + \Omega f_\Omega(x,t) = -\frac{1}{2\mu}\frac{\partial^2 f_\Omega(x,t)}{\partial x^2} - [V_0 + U_0 x\cos(\omega t)]f_\Omega(x,t). \tag{14}$$

Let us define another function that is written in terms of the solution:

$$g_\Omega(\xi_R,t) = e^{-iw(x,t)}f_\Omega(x,t), \tag{15}$$

where

$$w(x,t) = \frac{U_0 x\,\sin(\omega t)}{\omega} + \frac{U_0^2\sin(2\omega t)}{8\mu\omega^3} \tag{16}$$

and

$$\xi_R = x + \frac{U_0}{\mu\omega^2}\cos(\omega t). \tag{17}$$

If we also let

$$\chi_R(x,t) = e^{iw(x,t)}, \tag{18}$$

then the form of the Floquet state that we are searching for is

$$\Psi_\Omega^R(x,t) = e^{-i\Omega t/\hbar}\exp\left[-\frac{iU_0\sin(\omega t)}{\omega}\right]\chi_R(x,t)g_\Omega(\xi_R,t), \tag{19}$$

where the superscript R for Ψ_Ω indicates that we are trying to find the Floquet state in Region IR.

We want to substitute $f_\Omega(x,t) = \chi_R(x,t) g_\Omega(\xi_R, t)$ into Eq. (14). For convenience, let us first compute the necessary partial derivatives of $f_\Omega(x,t)$. The time partial derivative is

$$\frac{\partial f_\Omega}{\partial t} = \chi_R \frac{\partial g_\Omega}{\partial t} + \frac{\partial \chi_R}{\partial t} g_\Omega = \chi_R \frac{\partial g_\Omega}{\partial t} + i\frac{\partial w}{\partial t}\chi_R g_\Omega$$
$$= \chi_R \left(\frac{\partial g_\Omega}{\partial t} + i\left(U_0 x\cos(\omega t) + \frac{U_0^2 \cos(2\omega t)}{4\mu\omega^2} \right) g_\Omega \right)$$

The double-angle formula $\cos(2\omega t) = 1 - 2\sin^2(\omega t)$ gives

$$\frac{\partial f_\Omega}{\partial t} = \chi_R \left(\frac{\partial g_\Omega}{\partial t} + i\left(U_0 x\cos(\omega t) + \frac{U_0^2 (1-2\sin^2(\omega t))}{4\mu\omega^2} \right) g_\Omega \right) \tag{20}$$

The second-order spatial derivative is

$$\frac{\partial^2 f_\Omega}{\partial x^2} = \chi_R \frac{\partial^2 g_\Omega}{\partial x^2} + 2\frac{\partial \chi_R}{\partial x}\frac{\partial g_\Omega}{\partial x} + \frac{\partial^2 \chi_R}{\partial x^2} g_\Omega$$
$$= \chi_R \left(\frac{\partial^2 g_\Omega}{\partial x^2} + \frac{2iU_0 \sin(\omega t)}{\omega}\frac{\partial g_\Omega}{\partial x} - \frac{U_0^2 \sin^2(\omega t)}{\omega^2} g_\Omega \right) \tag{21}$$

Inserting $f_\Omega = \chi_R g_\Omega$ and the partial derivatives given by equations (20) and (21) into Eq. (14) and canceling out the χ_R yields

$$i\frac{\partial g_\Omega}{\partial t} - \left(U_0 x\cos(\omega t) + \frac{U_0^2 (1-2\sin^2(\omega t))}{4\mu\omega^2} \right) g_\Omega + \Omega g_\Omega$$
$$= -\frac{1}{2\mu}\frac{\partial^2 g_\Omega}{\partial x^2} - \frac{iU_0 \sin(\omega t)}{\mu\omega}\frac{\partial g_\Omega}{\partial x} + \frac{U_0^2 \sin^2(\omega t)}{2\mu\omega^2} g_\Omega - [V_0 + U_0 x \mathrm{cos}(\omega t)] g_\Omega.$$

Canceling the terms $-U_0 x\cos(\omega t) g_\Omega$ and $\frac{U_0^2 \sin^2(\omega t)}{2\mu\omega^2} g_\Omega$ existing on both sides of this equation lets us obtain

$$i\frac{\partial g_\Omega}{\partial t} + (\Omega - \frac{U_0^2}{4\mu\omega^2}) g_\Omega = -\frac{1}{2\mu}\frac{\partial^2 g_\Omega}{\partial x^2} - \frac{iU_0 \sin(\omega t)}{\mu\omega}\frac{\partial g_\Omega}{\partial x} - V_0 g_\Omega,$$

which we can also write as

$$i\left(\frac{\partial g_\Omega}{\partial t} + \frac{U_0 \sin(\omega t)}{\mu\omega}\frac{\partial g_\Omega}{\partial x} \right) = -\frac{1}{2\mu}\frac{\partial^2 g_\Omega}{\partial x^2} - (\Omega + V_0 - \frac{U_0^2}{4\mu\omega^2}) g_\Omega. \tag{22}$$

Next, we want to perform a coordinate transformation from (x,t) to (ξ_R, t), so we note that

$$\begin{cases} \dfrac{\partial g_\Omega}{\partial x} = \dfrac{\partial g_\Omega}{\partial \xi_R} \\ \dfrac{\partial g_\Omega}{\partial t} = \left(\dfrac{\partial g_\Omega}{\partial t} \right)_{\xi_R} - \dfrac{\partial g_\Omega}{\partial \xi_R}\dfrac{\partial \xi_R}{\partial t} = \left(\dfrac{\partial g_\Omega}{\partial t} \right)_{\xi_R} - \dfrac{U_0 \mathrm{sin}(\omega t)}{\mu\omega}\dfrac{\partial g_\Omega}{\partial \xi_R} \end{cases} \tag{23}$$

The coordinate conversion allows us to turn Eq. (22) to an equation of the same form as a free-particle Schrödinger equation:

$$i\left(\frac{\partial g_\Omega}{\partial t} \right)_{\xi_R} = -\frac{1}{2\mu}\frac{\partial^2 g_\Omega}{\partial \xi_R^2} - (\Omega + V_0 - \frac{U_0^2}{4\mu\omega^2}) g_\Omega. \tag{24}$$

This partial differential equation is separable into two ordinary differential equations (ODEs):

$$\begin{cases} -\dfrac{1}{2\mu}\dfrac{d^2\Xi_\Omega}{d\xi_R^2} = (\Omega + E_{pseudo} + V_0 - \dfrac{U_0^2}{4\mu\omega^2})\Xi_\Omega \\ i\left(\dfrac{\partial \tau_\Omega}{\partial t}\right)_{\xi_R} = E_{pseudo}\tau_\Omega \end{cases}, \tag{25}$$

where we have let $g_\Omega(\xi_R, t) = \Xi_\Omega(\xi_R)\tau_\Omega(t)$ be the form of the solution and E_{pseudo} be the separation constant. The general solutions to these ODEs for a given E_{pseudo} are clearly

$$\begin{cases} \Xi_\Omega(\xi_R) = \alpha^R\ e^{ik\xi_R} + \beta^R\ e^{-ik\xi_R} \\ \tau_\Omega(t) = Ce^{-iE_{pseudo}t} \end{cases},$$

where

$$k = \sqrt{2\mu\left(\Omega + E_{pseudo} + V_0 - \frac{U_0^2}{4\mu\omega^2}\right)}$$

and α^R, β^R, and C are arbitrary constants. Without loss of generality, we can set $C = 1$. Thus, we obtain for any fixed E_{pseudo} that

$$g_\Omega(\xi_R, t) = e^{-iE_{pseudo}t}\left(\alpha^R\ e^{ik\xi_R} + \beta^R\ e^{-ik\xi_R}\right). \tag{26}$$

The solution that we are searching for, however, is of Floquet type (as in Eq. (5)). Therefore, $\phi_\Omega(x, t)$ must satisfy $\phi_\Omega(x, t) = \phi_\Omega(x, t + T)$, forcing

$$g_\Omega(\xi_R, t) = \exp\left[i\left(\frac{U_0\sin(\omega t)}{\omega} - w(x, t)\right)\right]\phi_\Omega(x, t)$$

to be time-periodic with period T. Since ξ_R is time-periodic with period T (see Eq.(17) where we defined it), so is $\Xi_\Omega(\xi_R)$ regardless of E_{pseudo}. This signifies that what we have left to require is that $\tau_\Omega(t) = \tau_\Omega(t + T)$, a condition that holds only when $E_{pseudo} = \ell\omega$ for some integer ℓ. Hence, the general solution of Eq. (24) in the space of functions with period T/m for some positive integer m is

$$g_\Omega(\xi_R, t) = \sum_{\ell=-\infty}^{\infty} e^{-i\ell\omega t}\ \left(\alpha_\ell^R\ e^{ik_\ell\xi_R} + \beta_\ell^R\ e^{-ik_\ell\xi_R}\right), \tag{27}$$

where

$$k_\ell = \sqrt{2\mu\left(\Omega + \ell\omega + V_0 - \frac{U_0^2}{4\mu\omega^2}\right)}. \tag{28}$$

Finally, we can use Eq. (19) to construct the general form of a Floquet state in Region IR from what we have found:

$$\begin{aligned}\Psi_\Omega^R(x, t) = e^{-i\Omega t}\ &\exp\left[-i\frac{U_0}{\omega}(1 - x)\ \sin(\omega t)\right]\exp\left[\frac{iU_0^2\sin(2\omega t)}{8\mu\omega^3}\right] \\ \times \sum_{\ell=-\ell_{LB}}^{\infty} e^{-i\ell\omega t}\ &\left(\alpha_\ell^R\ e^{ik_\ell\left(x+\frac{U_0}{\mu\omega^2}\cos(\omega t)\right)} + \beta_\ell^R\ e^{-ik_\ell\left(x+\frac{U_0}{\mu\omega^2}\cos(\omega t)\right)}\right),\end{aligned} \tag{29}$$

where the lower bound $-\ell_{LB}$ of the index ℓ is chosen to keep k_ℓ real, so ℓ_{LB} is the largest integer satisfying

$$\ell_{LB} \leq \frac{1}{\omega}\left(\Omega + V_0 - \frac{U_0^2}{4\mu\omega^2}\right). \tag{30}$$

Solutions with imaginary k_ℓ are physically forbidden, because they turn $e^{\pm ik_\ell\left(x+\frac{U_0}{\mu\omega^2}\cos(\omega t)\right)}$ into real exponentials. To understand why the real exponentials make those solutions problematic to include, consider forming a basis set of solutions inside Region IR composed of every function

$$\Psi^R_{\Omega,\,\boldsymbol{\ell}}(x,t) = e^{-i\Omega t}\,\exp\left[-i\frac{U_0}{\omega}(1-x)\,\sin(\omega t)\right]\exp\left[\frac{iU_0^2\sin(2\omega t)}{8\mu\omega^3}\right]$$
$$\times e^{-i\ell\omega t}\,\left(\alpha^R_\ell\,e^{ik_\ell\left(x+\frac{U_0}{\mu\omega^2}\cos(\omega t)\right)} + \beta^R_\ell\,e^{-ik_\ell\left(x+\frac{U_0}{\mu\omega^2}\cos(\omega t)\right)}\right),$$

for which Ω is real and ℓ is an integer. The only superpositions of basis elements that are physically acceptable are those that, when matched to the solutions outside Region IR via the boundary conditions guaranteeing continuous differentiability, form a wavefunction that is square-integrable over the whole real line. If we consider a basis element with an imaginary k_ℓ, then due to its real exponentials, when it is matched to the external solutions, we obtain a wavefunction $\Psi(x,t)$ (defined over the whole real line) for which either $\lim_{x\to\infty}|\Psi(x,t)|^2 = \infty$ or $\lim_{x\to-\infty}|\Psi(x,t)|^2 = \infty$ holds[5]. As a result, if we take any superposition that includes this basis element and match it with the external solutions, then we will get a wavefunction that is not square integrable over all space.

In Region IL, where $x \in (-1,0)$, Eq. (13) takes the form:

$$i\frac{\partial f_\Omega(x,t)}{\partial t} + \Omega f_\Omega(x,t) = -\frac{1}{2\mu}\frac{\partial^2 f_\Omega(x,t)}{\partial x^2} - [V_0 - U_0 x\cos(\omega t)]f_\Omega(x,t). \tag{31}$$

The method of solving Eq. (31) is analogous to that for Eq. (14) and it yields that the general Floquet state in Region IL is

$$\Psi^L_\Omega(x,t) = e^{-i\Omega t}\,\exp\left[-i\frac{U_0}{\omega}(1+x)\,\sin(\omega t)\right]\exp\left[\frac{iU_0^2\sin(2\omega t)}{8\mu\omega^3}\right]$$
$$\times\sum_{\ell=-\ell_{LB}}^{\infty} e^{-i\ell\omega t}\,\left(\alpha^L_\ell\,e^{ik_\ell\left(x-\frac{U_0}{\mu\omega^2}\cos(\omega t)\right)} + \beta^L_\ell\,e^{-ik_\ell\left(x-\frac{U_0}{\mu\omega^2}\cos(\omega t)\right)}\right), \tag{32}$$

where the superscript L for Ψ_Ω is a reference to Region IL.

[5]That is because the Schrödinger equation imposes certain constraints on how the sign of the wavefunction's curvature depends on the value of the wavefunction.

- **Regions II and III ($|x|> 1$)**

Let us slightly rewrite Eq. (12) for $|x|> 1$ as follows:

$$i\frac{\partial\phi_\Omega(x,t)}{\partial t} = -\frac{1}{2\mu}\frac{\partial^2\phi_\Omega(x,t)}{\partial x^2} - \Omega\phi_\Omega(x,t). \tag{33}$$

Let E denote the separation constant for Eq. 33, then it is obvious that for a fixed E, the solution in Region II is

$$\phi_\Omega^{(II)}(x,t) = e^{-iEt}\left(\frac{A}{\sqrt{k^o}}\ e^{ik^o x} + \frac{B}{\sqrt{k^o}}\ e^{-ik^o x}\right),$$

and in Region III is

$$\phi_\Omega^{(III)}(x,t) = e^{-iEt}\left(\frac{C}{\sqrt{k^o}}\ e^{ik^o x} + \frac{D}{\sqrt{k^o}}\ e^{-ik^o x}\right),$$

where

$$k^o = +\sqrt{2\mu\,(\Omega + E)}.$$

Here, the function $+\sqrt{\cdot}$ is defined over the whole real line such that for all $x \geq 0$, $+\sqrt{x}$ is the positive square root of x and $+\sqrt{-x} = i\cdot(+\sqrt{x})$. For simplicity, we will denote $i\cdot(+\sqrt{x})$ by $+i\sqrt{x}$. We will also need the function $-\sqrt{\cdot}$, which just selects the square root that $+\sqrt{\cdot}$ does not choose.

Nevertheless, what we need are the Floquet states, so in Region II, we must have the time-periodicity condition $\phi_\Omega^{(II)}(x,t) = \phi_\Omega^{(II)}(x,t+T)$, an equation that holds only when $E = q\omega$ for some integer q. The general solution of Eq. (33) in Region II in the space of functions fulfilling the time-periodicity condition is thus

$$\phi_\Omega^{(II)}(x,t) = \sum_{q=-\infty}^{\infty} e^{-iq\omega t}\left(\frac{A_q}{\sqrt{k_q^o}}\ e^{ik_q^o x} + \frac{B_q}{\sqrt{k_q^o}}\ e^{-ik_q^o x}\right) \tag{34a}$$

$$= \sum_{q=0}^{\infty} e^{-iq\omega t}\left(\frac{A_q}{\sqrt{k_q^o}} e^{ik_q^o x} + \frac{B_q}{\sqrt{k_q^o}} e^{-ik_q^o x}\right) + \sum_{\widehat{q}=-\infty}^{-1} e^{-i\widehat{q}\omega t}\left(\frac{A_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{-K_{\widehat{q}}^o x} + \frac{B_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{K_{\widehat{q}}^o x}\right), \tag{34b}$$

where for any integer q,

$$k_q^o = iK_q^o = +\sqrt{2\mu(\Omega + q\omega)}, \tag{35}$$

i.e.

$$K_q^o = -ik_q^o = -\sqrt{-2\mu(\Omega + q\omega)}.$$

Note that in the last paragraph of Section 1.2, we said that for definiteness, we make the representative quasienergy values Ω lie in the interval $[0,\omega)$, which means that k_q^o is real for all nonnegative integers q and $K_{\widehat{q}}^o$ is real for all negative integers $\widehat{q}$.

Exactly the same should be said regarding the time-periodicity condition $\phi_\Omega^{(III)}(x,t) = \phi_\Omega^{(III)}(x,t+T)$ in Region III, so

$$\phi_\Omega^{(III)}(x,t) = \sum_{q=-\infty}^{\infty} e^{-iq\omega t}\left(\frac{C_q}{\sqrt{k_q^o}}\ e^{ik_q^o x} + \frac{D_q}{\sqrt{k_q^o}}\ e^{-ik_q^o x}\right) \tag{36a}$$

$$= \sum_{q=0}^{\infty} e^{-iq\omega t}\left(\frac{C_q}{\sqrt{k_q^o}} e^{ik_q^o x} + \frac{D_q}{\sqrt{k_q^o}} e^{-ik_q^o x}\right) + \sum_{\widehat{q}=-\infty}^{-1} e^{-i\widehat{q}\omega t}\left(\frac{C_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{-K_{\widehat{q}}^o x} + \frac{D_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{K_{\widehat{q}}^o x}\right). \tag{36b}$$

At this point, we simply have to attach the exponential factor $e^{-i\Omega t}$ to finally obtain the general Floquet states in Regions II and III. For Region II, we get

$$\Psi_\Omega^{(II)}(x,t) = e^{-i\Omega t}\sum_{q=-\infty}^{\infty} e^{-iq\omega t}\left(\frac{A_q}{\sqrt{k_q^o}}\ e^{ik_q^o x} + \frac{B_q}{\sqrt{k_q^o}}\ e^{-ik_q^o x}\right) \tag{37a}$$

$$= e^{-i\Omega t}\left[\sum_{q=0}^{\infty} e^{-iq\omega t}\left(\frac{A_q}{\sqrt{k_q^o}} e^{ik_q^o x} + \frac{B_q}{\sqrt{k_q^o}} e^{-ik_q^o x}\right) + \sum_{\widehat{q}=-\infty}^{-1} e^{-i\widehat{q}\omega t}\left(\frac{A_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{-K_{\widehat{q}}^o x} + \frac{B_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{K_{\widehat{q}}^o x}\right)\right], \tag{37b}$$

and for Region III, we get

$$\Psi_\Omega^{(III)}(x,t) = e^{-i\Omega t}\sum_{q=-\infty}^{\infty} e^{-iq\omega t}\left(\frac{C_q}{\sqrt{k_q^o}}\ e^{ik_q^o x} + \frac{D_q}{\sqrt{k_q^o}}\ e^{-ik_q^o x}\right) \tag{38a}$$

$$= e^{-i\Omega t}\left[\sum_{q=0}^{\infty} e^{-iq\omega t}\left(\frac{C_q}{\sqrt{k_q^o}} e^{ik_q^o x} + \frac{D_q}{\sqrt{k_q^o}} e^{-ik_q^o x}\right) + \sum_{\widehat{q}=-\infty}^{-1} e^{-i\widehat{q}\omega t}\left(\frac{C_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{-K_{\widehat{q}}^o x} + \frac{D_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{K_{\widehat{q}}^o x}\right)\right]. \tag{38b}$$

As in Region I, we must eliminate solutions that are impossible to superpose with other solutions such that the resulting state is square-integrable. Clearly, the solutions $e^{-K_{\widehat{q}}^o x}$ in Region II and $e^{K_{\widehat{q}}^o x}$ in Region III are forbidden for all negative integers $\widehat{q}$, because the former tends to infinity as $x \to -\infty$, while the latter does the same as $x \to \infty$. Therefore, we set $A_{\widehat{q}} = D_{\widehat{q}} = 0$ for all integers $\widehat{q} < 0$, which gives:

$$\Psi_\Omega^{(II)}(x,t) = e^{-i\Omega t}\left[\sum_{q=0}^{\infty} e^{-iq\omega t}\left(\frac{A_q}{\sqrt{k_q^o}}\ e^{ik_q^o x} + \frac{B_q}{\sqrt{k_q^o}}\ e^{-ik_q^o x}\right) + \sum_{\widehat{q}=-\infty}^{-1} e^{-i\widehat{q}\omega t}\left(\frac{B_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{K_{\widehat{q}}^o x}\right)\right], \tag{39}$$

and for Region III, we get

$$\Psi_\Omega^{(III)}(x,t) = e^{-i\Omega t}\left[\sum_{q=0}^{\infty} e^{-iq\omega t}\left(\frac{C_q}{\sqrt{k_q^o}} e^{ik_q^o x} + \frac{D_q}{\sqrt{k_q^o}} e^{-ik_q^o x}\right) + \sum_{\widehat{q}=-\infty}^{-1} e^{-i\widehat{q}\omega t}\left(\frac{C_{\widehat{q}}}{\sqrt{K_{\widehat{q}}^o}} e^{-K_{\widehat{q}}^o x}\right)\right]. \tag{40}$$

Equations (39) and (40) are general representations of Floquet states that are closer to depicting the space of physically acceptable wavefunctions outside the well than equations (37) and (38). In fact, every term in equations (39) and (40) has physical significance. Given that the potential is $V(x,t) = 0$ in Regions II and III, it is not surprising that the basis of solutions that we use to express the general Floquet states in those intervals consists of eigenfunctions of the kinetic energy operator, which are either propagating plane waves (or modes) if they have positive energy or evanescent modes if they have negative energy.

Notice that the time-independence of the potential in Regions II and III allows us to find solutions restricted to those regions that can be treated as states of definite energy (only outside the potential well, not for the whole system). In our problem, the Hamiltonian equals to the kinetic energy operator in the exterior of the well, so we get that the states of definite kinetic energy are the same as those with definite energy.

In each of the last two equations (Eqs. (39) and (40)), if we consider the first summation over index q ranging from 0 to ∞, then we can see that for $\Omega \in (0, \omega)$, it is fully composed of states with positive energies $\Omega + q\omega$, i.e. propagating plane waves, while for $\Omega = 0$, we still have that almost[6] all terms in the sum represent propagating modes (with energies $\Omega + q\omega = q\omega$). On the other hand, each of the remaining summations over index $\widehat{q}$ from $-\infty$ to -1 fully consists of states with negative energies $\Omega + \widehat{q}\omega$, i.e. evanescent modes.

Outside of Region I, there is a direct relationship between energy and quasienergy: Every Floquet state of quasienergy Ω, which is at the same time of quasienergy $\Omega + m\omega$ for all integers m, is a superposition of states with definite energies $\Omega + n\omega$ for all integers n. Consequently, for a particle in a Floquet state of a given quasienergy in the exterior of the well, the set of all possible energies that have nonzero probability of being measured is a subset of all quasienergies that can be assigned to the Floquet state. To establish the one-to-one correspondence between the general Floquet state's allowed energies and assignable quasienergies, Floquet formalism provides the concept called a channel [16]. A **channel** is a set of all states in either Region II or III that have definite energies and have their energies lie in the interval $[n\omega, (n+1)\omega)$ for some integer n. Given the integer n such that the energies for a channel lie in the interval $[n\omega, (n+1)\omega)$, the channel is referred to as the $\boldsymbol{n^{th}}$ **channel**. For example, for a quasienergy $\Omega \in [0, \omega)$, if a propagating or evanescent mode has energy $\Omega + n\omega$, then it is considered to be in the n^{th} channel. If we did not set $\hbar = 1$, all of the energies and assignable quasienergies that we consider in a Floquet state with quasienergy Ω would have been of the form $\Omega + n\hbar\omega$ instead of $\Omega + n\omega$.

Moreover, the terms in equations (39) and (40) with $e^{ik_q^o x}$ are rightward propagating waves and with $e^{-ik_q^o x}$ are leftward propagating waves. Hence, for $q \geq 0$, A_q is the probability amplitude of an incoming propagating mode (i.e. wave) traveling from the left in the q^{th} channel, B_q is the probability amplitude of an outgoing wave to the left in the q^{th} channel, C_q is the probability amplitude of an outgoing wave to the right in the q^{th} channel, and D_q

[6] There is only one exception when $\Omega = q = 0$, because that is when the energy $\Omega + q\omega$ is zero. The element from our chosen basis of solutions that corresponds to this case is a state in which a particle's kinetic energy is zero. Such a state is not relevant for our current purposes, because its zero kinetic energy causes its probability current to be zero, so it makes no contribution to the flux of probability into or out of the region of the well.

is the probability amplitude of an incoming wave from the right in the q^{th} channel. It is important to note that if an incoming plane wave is part of a Floquet state with quasienergy $\Omega \in [0, \omega)$ and is in the q^{th} channel, we state that it has an **incident quasienergy** Ω and an **incident energy** $\Omega + q\omega$. The remaining terms represent evanescent modes. Thus, for $q < 0$ [7], B_q is the probability amplitude of evanescent modes that tunnel into Region II in the q^{th} channel, while C_q is the same thing but for Region III.

One might be tempted to ask if there is a certain lower bound $-q_{LB} < 0$ to the summation index q such that for all $q < -q_{LB}$, $B_q = C_q = 0$. In other words, we might wonder if there are any evanescent modes with energies that are too low to be physically possible. Indeed, in reference [10] where a similar potential is considered, it is assumed that $q_{LB} = \ell_{LB}$, where ℓ_{LB} was introduced here on page 13 (in which we eliminated the Region I solutions with real exponentials). The reason for making such an assumption was based on the fact about systems with time-independent potentials that the energy of a stationary state (and thereby the energy expectation value of any state) must exceed the minimum potential energy [8, 14]. Nonetheless, there are three considerations in our time-periodically driven system that would make this rule **not** justified at this stage of our analysis:

1. The minimum in the potential energy changes over time.
2. Due to concerns about the divergence of the wavefunction under one of the limits $x \to \pm\infty$, it was justified to avoid solutions with real exponentials *inside* the region of the well. However, there are no such divergence concerns regarding any of the evanescent modes *outside* the well, because all of them have probabilities that exponentially decay as $x \to \pm\infty$.
3. The replacement of energy conservation by quasienergy conservation should make us doubt that it is impossible for an incoming wave of a given energy scattering off the potential well to be able to transition to *any* channel as long as it preserves its quasienergy. In other words, at this stage of our analysis, it is justified to hypothesize that in most cases, the time-periodic driving in Region I induces an incident wave with definite energy to have *at least some* nonzero probabilities[8] of transition to *every* energy that differs by an integer multiple of ω (if we did not set $\hbar = 1$, this would be $\hbar\omega$) from the initial energy. In fact, throwing out solutions corresponding to q less than some chosen integer might make it impossible for the Floquet state to be continuously differentiable at the boundaries.

Based on these considerations, the only way to find out whether or not q_{LB} exists is to examine what role the evanescent modes play in the boundary conditions guaranteeing the continuous differentiability of an entire Floquet state defined over all space. If it will turn out that for any chosen minimum energy threshold, it is impossible to always satisfy the boundary conditions when we exclude the evanescent modes of energies that are lower than the threshold, then q_{LB} does not exist. For these reasons, I think it is safest to keep the general Floquet states $\Psi_\Omega^{(II)}$ and $\Psi_\Omega^{(III)}$ the way they are expressed in Eqs. (39) and (40).

On the other hand, the choice of q_{LB} to be equal to ℓ_{LB} as was done in [10] can have

[7]Since $\widehat{q}$ is a "dummy" summation index, we can replace it by q to make the index q have any integer value, as in Eqs. (37a) and (38a). Henceforth, we will not use the index $\widehat{q}$ used in Eqs. (37b) and (38b).

[8]Many of those probabilities may be extremely close to zero, but still nonzero.

some practical use. It can serve as a guide to choose only a finite subset of all boundary conditions to account for such that the matrices needed to find the scattering matrix, or S-matrix, (defined in the next section) are reasonably truncated to matrices of finite size, making it possible to compute the elements of the S-matrix approximately. This choice is especially a good guide when the oscillation strength U_0 is small, because that is a situation for which the scattering dynamics should resemble the scattering in the case when $U_0 = 0$.

As a side-note, it is important to notice that as a consequence of Floquet's theorem, the energy transitions mentioned in consideration 3 are the *only* transitions that an incident plane wave is allowed to go through in a localized system with a time-periodic potential. If the driving field is an electromagnetic field, then our application of Floquet's theorem provides a non-relativistic prediction of the quantization of the electromagnetic field via photons (a consequence of which is photon emission and absorption), though it does not establish the true quantum electrodynamic explanation of what a photon field really is.

2.2 Derivation of the Floquet Scattering Matrix

In this section, our goal will be to derive the Floquet S-matrix, which for a given quasienergy Ω and for $q \geq 0$, relates the probability amplitudes A_q and D_q of incoming propagating modes to the amplitudes B_q and C_q of outgoing propagating modes as follows:

$$\begin{pmatrix} \overline{\mathbf{B}} \\ \overline{\mathbf{C}} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \overline{\mathbf{A}} \\ \overline{\mathbf{D}} \end{pmatrix}, \tag{41}$$

where

$$\begin{aligned} \overline{\mathbf{A}} &:= (A_0, A_1, A_2, \dots)^T, \ \overline{\mathbf{B}} := (B_0, B_1, B_2, \dots)^T, \\ \overline{\mathbf{C}} &:= (C_0, C_1, C_2, \dots)^T, \ \overline{\mathbf{D}} := (D_0, D_1, D_2, \dots)^T \end{aligned} \tag{42}$$

are column matrices of infinite size (i.e. column vectors with infinitely many components), which we will simply refer to as **infinite column vectors**, and

$$\boldsymbol{S} := \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{43}$$

is the Floquet S-matrix split into four matrix blocks S_{11}, S_{12}, S_{21} and S_{22}, all of which are of infinite size and represent *linear operators acting on infinite dimensional space*. We call such matrices of infinite size **infinite matrices**. Once we find expressions for these blocks, we will know how to compute the S-matrix.

In order to find the true expression for the S-matrix, we must account for not just amplitudes of propagating modes (for which $q \geq 0$), but also for amplitudes of evanescent modes (for which $q < 0$), because transitions from positive channels to negative channels are possible due to the time-dependence of the potential. To do this, we will first find a bigger matrix, which I call the **$\boldsymbol{\Sigma}$-matrix**. For a given quasienergy Ω, the $\boldsymbol{\Sigma}$-matrix relates the probability amplitudes A_q and D_q to amplitudes B_q and C_q *for all integers* q as follows:

$$\begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{C} \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}, \tag{44}$$

where

$$\begin{aligned} \boldsymbol{A} &:= (\ \dots\ , A_{-2}, A_{-1}, A_0, A_1, A_2, \dots\ )^T,\ \ \boldsymbol{B} := (\ \dots\ , B_{-2}, B_{-1}, B_0, B_1, B_2, \dots\ )^T, \\ \boldsymbol{C} &:= (\ \dots\ , C_{-2}, C_{-1}, C_0, C_1, C_2, \dots\ )^T,\ \ \boldsymbol{D} := (\ \dots\ , D_{-2}, D_{-1}, D_0, D_1, D_2, \dots\ )^T \end{aligned} \tag{45}$$

are infinite column vectors and

$$\boldsymbol{\Sigma} := \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \tag{46}$$

is the $\boldsymbol{\Sigma}$-matrix split into four blocks of infinite size. Since all of the amplitudes that the $\boldsymbol{\Sigma}$-matrix relates include the amplitudes that the S-matrix relates, the S-matrix is contained in the $\boldsymbol{\Sigma}$-matrix. In particular, every block $S_{\gamma\delta}$ is a submatrix of the block $\Sigma_{\gamma\delta}$ for all $\gamma, \delta = 1, 2$. Hence, the elements of the S-matrix are known once the $\boldsymbol{\Sigma}$-matrix is found.

The relationships between the amplitudes A_q, B_q, C_q, and D_q are determined by the **boundary conditions** guaranteeing the continuous differentiability of the Floquet solutions at all three boundaries between Regions II and IL, IL and IR, and IR and III. In other words, the conditions equate the **continuous extensions** of the Floquet solutions we already found and their spatial derivatives at boundaries located at $x = -1$, $x = 0$, and $x = 1$. These conditions are

$$\Psi_\Omega^{(II)}(-1, t) = \Psi_\Omega^L(-1, t), \tag{47}$$

$$\left.\frac{\partial \Psi_\Omega^{(II)}}{\partial x}\right|_{x=-1} = \left.\frac{\partial \Psi_\Omega^L}{\partial x}\right|_{x=-1}, \tag{48}$$

$$\Psi_\Omega^L(0, t) = \Psi_\Omega^R(0, t), \tag{49}$$

$$\left.\frac{\partial \Psi_\Omega^L}{\partial x}\right|_{x=0} = \left.\frac{\partial \Psi_\Omega^R}{\partial x}\right|_{x=0}, \tag{50}$$

$$\Psi_\Omega^{(III)}(1, t) = \Psi_\Omega^R(1, t), \tag{51}$$

$$\left.\frac{\partial \Psi_\Omega^{(III)}}{\partial x}\right|_{x=1} = \left.\frac{\partial \Psi_\Omega^R}{\partial x}\right|_{x=1}, \tag{52}$$

where we will use $\Psi_\Omega^{(II)}$ from Eq. (37a) and $\Psi_\Omega^{(III)}$ from Eq. (38a) for the purposes of computing the $\boldsymbol{\Sigma}$-matrix.

We mentioned before that Eqs. (39) and (40) are more representative of the space physically acceptable wavefunctions than Eqs. (37a) and (38a) are, because they account for the fact that for $q < 0$, A_q and D_q are amplitudes of **nonexistent** (i.e. physically unacceptable) modes that blow up at either $x = \pm\infty$. Nevertheless, we want to account for all boundary conditions as we derive the S-matrix by finding the $\boldsymbol{\Sigma}$-matrix first, so instead of setting $A_q = D_q = 0$ for all $q < 0$ right away, we treat the amplitudes of nonexistent modes as input variables. In fact, observe that the $\boldsymbol{\Sigma}$-matrix treats incoming propagating waves and nonexistent modes as inputs and outgoing propagating waves and evanescent modes as outputs, so this matrix will truly give us physical amplitudes if we input the amplitudes of nonexistent modes as equal to zero.

To facilitate the use of the boundary conditions, we want to note that Floquet solutions are of the form $\Psi_\Omega(x,t) = e^{-i\Omega t/\hbar}\phi_\Omega(x,t)$, such that $\phi_\Omega(x,t) = \phi_\Omega(x,t+T)$. Thus, we can perform Fourier series expansions of $\phi_\Omega(x,t)$ and $\frac{\partial}{\partial x}\phi_\Omega(x,t)$ in the time variable t in all four regions and then equate the continuous extensions of the resulting Fourier coefficients (which still depend on x) at the boundaries. In Regions II and III, there is no need to make any expansions, because $\phi_\Omega^{(II)}$ and $\phi_\Omega^{(III)}$ in equations (34) and (36) are already written in terms of their temporal Fourier series. In contrast, the time-periodic parts of the solutions in Region I are not expanded yet. To expand them, we employ the Jacobi-Anger identities [1]

$$\begin{cases} e^{is\sin(\omega t)} = \sum_{v=-\infty}^{\infty} J_v(s)\, e^{iv\omega t} \\ e^{-is\sin(\omega t)} = \sum_{v=-\infty}^{\infty} (-1)^v J_v(s)\, e^{iv\omega t} \\ e^{is\cos(\omega t)} = \sum_{v=-\infty}^{\infty} i^v J_v(s)\, e^{iv\omega t} \\ e^{-is\cos(\omega t)} = \sum_{v=-\infty}^{\infty} (-i)^v J_v(s)\, e^{iv\omega t} \end{cases} \tag{53}$$

all of which come from the following Laurent series (first used by the astronomer Peter Andreas Hansen) [1]:

$$\exp\left[\frac{z}{2}\left(a - \frac{1}{a}\right)\right] = \sum\nolimits_{\ell=-\infty}^{\infty} a^\ell J_\ell(z). \tag{54}$$

• The Boundary Conditions at $x = \pm 1$

From Eqs. (32) and (53), we get **for the left half of the well** that

$$\Psi_\Omega^L(x,t) = e^{-i\Omega t} \sum_{\ell=-\ell_{LB}}^{\infty} \left(\sum_{v=-\infty}^{\infty} (-1)^v J_v\left(\frac{U_0}{\omega}(1+x)\right) e^{iv\omega t}\right) \left(\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) e^{2iu\omega t}\right) \times$$
$$\times \quad \left(\alpha_\ell^L e^{ik_\ell x} \sum_{m=-\infty}^{\infty} (-i)^m J_m\left(\frac{k_\ell U_0}{\mu\omega^2}\right) e^{im\omega t} + \beta_\ell^L e^{-ik_\ell x} \sum_{m=-\infty}^{\infty} i^m J_m\left(\frac{k_\ell U_0}{\mu\omega^2}\right) e^{im\omega t}\right) e^{-i\ell\omega t}, \tag{55}$$

which via re-indexing (specifically, taking $n := m + 2u,\ p := n + v,$ and $q := \ell - p$, yielding $m = \ell - q - v - 2u$), can be written as

$$\begin{aligned} \Psi_\Omega^L(x,t) &= e^{-i\Omega t} \sum_{q=-\infty}^{\infty} \sum_{\ell=-\ell_{LB}}^{\infty} \sum_{v=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} (-1)^v J_v\left(\frac{U_0}{\omega}(1+x)\right) J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) \times \\ &\times \quad J_{\ell-q-v-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) \left((-i)^{\ell-q-v-2u}\alpha_\ell^L e^{ik_\ell x} + i^{\ell-q-v-2u}\beta_\ell^L e^{-ik_\ell x}\right) e^{-iq\omega t}. \end{aligned} \tag{56}$$

By evaluating this series at the boundary $x = -1$ and noting that $J_v(0) = \delta_{v0}$ (so that we can set index $v = 0$), we find that

$$\begin{aligned}\Psi_\Omega^L(-1,t) &= e^{-i\Omega t}\sum_{q=-\infty}^{\infty}\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) J_{\ell-q-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\times \\ &\times \ ((-i)^{\ell-q-2u}\alpha_\ell^L e^{-ik_\ell} + i^{\ell-q-2u}\beta_\ell^L e^{ik_\ell})e^{-iq\omega t}. \end{aligned} \tag{57}$$

Observe that my ability to use $J_v(0) = \delta_{v0}$ to eliminate infinitely many terms in the expression for Ψ_Ω^L at the boundary of the well is due to the simplifying property that the potential is fixed at that boundary. The same will happen when we will deal with the rightmost boundary where $x = 1$. By taking the spatial partial derivative of the expression in Eq. (56), we obtain

$$\begin{aligned}\frac{\partial \Psi_\Omega^L}{\partial x} &= e^{-i\Omega t}\sum_{q=-\infty}^{\infty}\sum_{\ell=-\ell_{LB}}^{\infty}\Bigg\{\alpha_\ell^L\Bigg[\sum_{v=-\infty}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) J_{\ell-q-v-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)(-i)^{\ell-q+v-2u}\times \\ &\times \left[\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1+x)\right) - J_{v+1}\left(\frac{U_0}{\omega}(1+x)\right)\right) + ik_\ell J_v\left(\frac{U_0}{\omega}(1+x)\right)\right]e^{ik_\ell x} + \\ &+ \beta_\ell^L\Bigg[\sum_{v=-\infty}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) J_{\ell-q-v-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) i^{\ell-q+v-2u}\times \\ &\times \left[\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1+x)\right) - J_{v+1}\left(\frac{U_0}{\omega}(1+x)\right)\right) - ik_\ell J_v\left(\frac{U_0}{\omega}(1+x)\right)\right]e^{-ik_\ell x}\Bigg\}e^{-iq\omega t}, \end{aligned} \tag{58}$$

where we have used

$$\frac{dJ_n(x)}{dx} = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)).$$

Evaluating the derivative in Eq. (58) at the boundary $x = -1$ and again using that $J_v(0) = \delta_{v0}$ gives

$$\begin{aligned}\left.\frac{\partial \Psi_\Omega^L}{\partial x}\right|_{x=-1} &= e^{-i\Omega t}\sum_{q=-\infty}^{\infty}\sum_{u=-\infty}^{\infty}\Bigg\{J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\times \\ &\times \sum_{\ell=-\ell_{LB}}^{\infty}\Bigg[\frac{U_0}{2\omega}\left(J_{\ell-q-2u-1}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) + J_{\ell-q-2u+1}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\right) - \\ &- k_\ell J_{\ell-q-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\Bigg]\left((-i)^{\ell-q-2u+1}\alpha_\ell^L e^{-ik_\ell} + i^{\ell-q-2u+1}\beta_\ell^L e^{ik_\ell}\right)\Bigg\}e^{-iq\omega t}. \end{aligned} \tag{59}$$

Similarly, one can derive from Eqs. (29) and (53) that **for the right half of the well,**

$$\begin{aligned}\Psi_\Omega^R(x,t) &= e^{-i\Omega t}\sum_{q=-\infty}^{\infty}\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{v=-\infty}^{\infty}\sum_{u=-\infty}^{\infty}(-1)^v J_v\left(\frac{U_0}{\omega}(1-x)\right) J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\times \\ &\times \ J_{\ell-q-v-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\left(i^{\ell-q-v-2u}\alpha_\ell^R e^{ik_\ell x} + (-i)^{\ell-q-v-2u}\beta_\ell^R e^{-ik_\ell x}\right)e^{-iq\omega t}, \end{aligned} \tag{60}$$

which is analogous to Eq. (56). Also, by analogy with Eq. (57),

$$\begin{aligned}\Psi_\Omega^R(1,t) &= e^{-i\Omega t}\sum_{q=-\infty}^{\infty}\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) J_{\ell-q-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\times \\ &\times\ (i^{\ell-q-2u}\alpha_\ell^R e^{ik_\ell}+(-i)^{\ell-q-2u}\beta_\ell^R e^{-ik_\ell})e^{-iq\omega t}. \qquad (61)\end{aligned}$$

By taking the spatial partial derivative of the expression in Eq. (60), we obtain

$$\begin{aligned}\frac{\partial\Psi_\Omega^R}{\partial x} = -e^{-i\Omega t}\sum_{q=-\infty}^{\infty}\sum_{\ell=-\ell_{LB}}^{\infty}\Bigg\{\alpha_\ell^R\Bigg[\sum_{v=-\infty}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) J_{\ell-q-v-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\ i^{\ell-q+v-2u}\times \\ \times\ \left[\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1-x)\right)-J_{v+1}\left(\frac{U_0}{\omega}(1-x)\right)\right)-ik_\ell J_v\left(\frac{U_0}{\omega}(1-x)\right)\right]e^{ik_\ell x}+ \\ +\ \beta_\ell^R\Bigg[\sum_{v=-\infty}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right) J_{\ell-q-v-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)(-i)^{\ell-q+v-2u}\times \\ \times\ \left[\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1-x)\right)-J_{v+1}\left(\frac{U_0}{\omega}(1-x)\right)\right)+ik_\ell J_v\left(\frac{U_0}{\omega}(1-x)\right)\right]e^{-ik_\ell x}\Bigg\}e^{-iq\omega t}. \quad (62)\end{aligned}$$

Therefore,

$$\begin{aligned}\left.\frac{\partial\Psi_\Omega^R}{\partial x}\right|_{x=1} = -e^{-i\Omega t}\sum_{q=-\infty}^{\infty}\sum_{u=-\infty}^{\infty}\Bigg\{J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\times \\ \times\ \sum_{\ell=-\ell_{LB}}^{\infty}\Bigg[\frac{U_0}{2\omega}\Big(J_{\ell-q-2u-1}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)+J_{\ell-q-2u+1}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\Big)- \\ -\ k_\ell J_{\ell-q-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\Bigg]\left(i^{\ell-q-2u+1}\alpha_\ell^R e^{ik_\ell}+(-i)^{\ell-q-2u+1}\beta_\ell^R e^{-ik_\ell}\right)\Bigg\}e^{-iq\omega t}. \qquad (63)\end{aligned}$$

From what we have found so far, it is easier to impose the boundary conditions (47) and (48) at $x=-1$ and (51) and (52) at $x=1$. For convenience, let us define the following constants:

$$\widehat{J}_{q\ell u} := J_{\ell-q-2u-1}\left(\frac{k_\ell U_0}{\mu\omega^2}\right); \qquad (64)$$

$$\widetilde{J}_{q\ell u} := \frac{U_0}{2\omega}\Big(J_{\ell-q-2u-1}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)+J_{\ell-q-2u+1}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\Big)-k_\ell J_{\ell-q-2u}\left(\frac{k_\ell U_0}{\mu\omega^2}\right). \qquad (65)$$

Now, we consider $x=-1$. By Eqs. (37a), (57), and (64), the continuity condition (47) implies that for all $q\in\mathbb{Z}$ that

$$\begin{aligned}\frac{A_q}{\sqrt{k_q^o}}e^{-ik_q^o}+\frac{B_q}{\sqrt{k_q^o}}e^{ik_q^o} &= \sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\widehat{J}_{q\ell u}\times \\ &\times\ ((-i)^{\ell-q-2u}\alpha_\ell^L e^{-ik_\ell}+i^{\ell-q-2u}\beta_\ell^L e^{ik_\ell}), \qquad (66)\end{aligned}$$

while by Eqs. (37a), (59), and (65), the differentiability condition (48) implies for all $q \in \mathbb{Z}$ that

$$\begin{aligned} i\sqrt{k_q^o}A_q e^{-ik_q^o} - i\sqrt{k_q^o}B_q e^{ik_q^o} &= \sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\widetilde{J}_{q\ell u} \times \\ &\times \left((-i)^{\ell-q-2u+1}\alpha_\ell^L e^{-ik_\ell} + i^{\ell-q-2u+1}\beta_\ell^L e^{ik_\ell}\right). \end{aligned} \tag{67}$$

Next, we consider $x = 1$. By Eqs. (38a), (61), and (64), the continuity condition (51) implies that for all $q \in \mathbb{Z}$ that

$$\begin{aligned} \frac{C_q}{\sqrt{k_q^o}}e^{ik_q^o} + \frac{D_q}{\sqrt{k_q^o}}e^{-ik_q^o} &= \sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\widehat{J}_{q\ell u} \times \\ &\times \left(i^{\ell-q-2u}\alpha_\ell^R e^{ik_\ell} + (-i)^{\ell-q-2u}\beta_\ell^R e^{-ik_\ell}\right), \end{aligned} \tag{68}$$

while by Eqs. (38a), (63), and (65), the differentiability condition (52) implies for all $q \in \mathbb{Z}$ that

$$\begin{aligned} -i\sqrt{k_q^o}C_q e^{ik_q^o} + i\sqrt{k_q^o}D_q e^{-ik_q^o} &= \sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\widetilde{J}_{q\ell u} \times \\ &\times \left(i^{\ell-q-2u+1}\alpha_\ell^R e^{ik_\ell} + (-i)^{\ell-q-2u+1}\beta_\ell^R e^{-ik_\ell}\right). \end{aligned} \tag{69}$$

Remark 3 : *In the process of deriving Eq.* (69)*, I multiplied both sides of the equation by* -1 *to make it look almost exactly like Eq.* (67)*.*

Eqs. (66) and (67) giving the boundary conditions at $x = -1$ can be written as a single matrix equation that holds all integers q:

$$\begin{aligned} &\begin{pmatrix} (e^{-ik_q^o})/\sqrt{k_q^o} & (e^{ik_q^o})/\sqrt{k_q^o} \\ i\sqrt{k_q^o}e^{-ik_q^o} & -i\sqrt{k_q^o}e^{ik_q^o} \end{pmatrix}\begin{pmatrix} A_q \\ B_q \end{pmatrix} = \sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty}\left\{J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\times\right. \\ &\times \left.\left[\begin{pmatrix} \widehat{J}_{q\ell u} \\ -i\widetilde{J}_{q\ell u} \end{pmatrix}(-i)^{\ell-q-2u}\alpha_\ell^L e^{-ik_\ell} + \begin{pmatrix} \widehat{J}_{q\ell u} \\ i\widetilde{J}_{q\ell u} \end{pmatrix} i^{\ell-q-2u}\beta_\ell^L e^{ik_\ell}\right]\right\}. \end{aligned} \tag{70}$$

Note that

$$\begin{vmatrix} (e^{-ik_q^o})/\sqrt{k_q^o} & (e^{ik_q^o})/\sqrt{k_q^o} \\ i\sqrt{k_q^o}e^{-ik_q^o} & -i\sqrt{k_q^o}e^{ik_q^o} \end{vmatrix} = -2i$$

is the determinant of the matrix on the left-hand side of Eq. (70). Thus, applying Cramer's Rule to (70), we obtain

$$\begin{aligned} A_q &= \frac{i}{2}\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\left\{\begin{vmatrix} \widehat{J}_{q\ell u} & (e^{ik_q^o})/\sqrt{k_q^o} \\ -i\widetilde{J}_{q\ell u} & -i\sqrt{k_q^o}e^{ik_q^o} \end{vmatrix}(-i)^{\ell-q-2u}\alpha_\ell^L e^{-ik_\ell} + \right. \\ &+ \left.\begin{vmatrix} \widehat{J}_{q\ell u} & (e^{ik_q^o})/\sqrt{k_q^o} \\ i\widetilde{J}_{q\ell u} & -i\sqrt{k_q^o}e^{ik_q^o} \end{vmatrix}(-i)^{\ell-q-2u}\beta_\ell^L e^{ik_\ell}\right\}; \end{aligned} \tag{71}$$

$$
\begin{aligned}
B_q &= \frac{i}{2}\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\left\{\begin{vmatrix}(e^{-ik_q^o})/\sqrt{k_q^o} & \widehat{J}_{q\ell u}\\ i\sqrt{k_q^o}e^{-ik_q^o} & -i\widetilde{J}_{q\ell u}\end{vmatrix}(-i)^{\ell-q-2u}\alpha_\ell^L e^{-ik_\ell}+\right.\\
&+ \left.\begin{vmatrix}(e^{-ik_q^o})/\sqrt{k_q^o} & \widehat{J}_{q\ell u}\\ i\sqrt{k_q^o}e^{-ik_q^o} & i\widetilde{J}_{q\ell u}\end{vmatrix}(-i)^{\ell-q-2u}\beta_\ell^L e^{ik_\ell}\right\}. \qquad (72)
\end{aligned}
$$

Eqs. (71) and (72) can be written in slightly better form:

$$
\begin{aligned}
A_q &= \frac{1}{2}\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\left\{\begin{vmatrix}\widehat{J}_{q\ell u} & (e^{ik_q^o})/\sqrt{k_q^o}\\ \widetilde{J}_{q\ell u} & \sqrt{k_q^o}e^{ik_q^o}\end{vmatrix}(-i)^{\ell-q-2u}\alpha_\ell^L e^{-ik_\ell}+\right.\\
&+ \left.\begin{vmatrix}\widehat{J}_{q\ell u} & (e^{ik_q^o})/\sqrt{k_q^o}\\ -\widetilde{J}_{q\ell u} & \sqrt{k_q^o}e^{ik_q^o}\end{vmatrix}(-i)^{\ell-q-2u}\beta_\ell^L e^{ik_\ell}\right\}; \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
B_q &= \frac{1}{2}\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\left\{\begin{vmatrix}(e^{-ik_q^o})/\sqrt{k_q^o} & \widehat{J}_{q\ell u}\\ -\sqrt{k_q^o}e^{-ik_q^o} & \widetilde{J}_{q\ell u}\end{vmatrix}(-i)^{\ell-q-2u}\alpha_\ell^L e^{-ik_\ell}+\right.\\
&+ \left.\begin{vmatrix}(e^{-ik_q^o})/\sqrt{k_q^o} & \widehat{J}_{q\ell u}\\ -\sqrt{k_q^o}e^{-ik_q^o} & -\widetilde{J}_{q\ell u}\end{vmatrix}(-i)^{\ell-q-2u}\beta_\ell^L e^{ik_\ell}\right\}. \qquad (74)
\end{aligned}
$$

Eqs. (73) and (74) can easily be combined into the following matrix equation:

$$
\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{B}\end{pmatrix} = \begin{pmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha^L}\\ \boldsymbol{\beta^L}\end{pmatrix}, \qquad (75)
$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined as in (45), $\boldsymbol{\alpha^L}$ and $\boldsymbol{\beta^L}$ are infinite column vectors defined as

$$
\begin{aligned}
\boldsymbol{\alpha^L} &:= (\alpha_{-\ell_{LB}}^L, \dots, \alpha_{-2}^L, \alpha_{-1}^L, \alpha_0^L, \alpha_1^L, \alpha_2^L, \dots)^T,\\
\boldsymbol{\beta^L} &:= (\beta_{-\ell_{LB}}^L, \dots, \beta_{-2}^L, \beta_{-1}^L, \beta_0^L, \beta_1^L, \beta_2^L, \dots)^T,
\end{aligned} \qquad (76)
$$

and X_{11}, X_{12}, X_{21}, and X_{22} are infinite matrices with (q,ℓ) entries that are

$$
(X_{11})_{q\ell} := \frac{1}{2}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix}\widehat{J}_{q\ell u} & \frac{e^{ik_q^0}}{\sqrt{k_q^0}}\\ \widetilde{J}_{q\ell u} & \sqrt{k_q^0}e^{ik_q^0}\end{vmatrix}(-i)^{\ell-q-2u}e^{-ik_\ell}, \qquad (77)
$$

$$
(X_{12})_{q\ell} := \frac{1}{2}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix}\widehat{J}_{q\ell u} & \frac{e^{ik_q^0}}{\sqrt{k_q^0}}\\ -\widetilde{J}_{q\ell u} & \sqrt{k_q^0}e^{ik_q^0}\end{vmatrix} i^{\ell-q-2u}e^{ik_\ell}, \qquad (78)
$$

$$
(X_{21})_{q\ell} := \frac{1}{2}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix}\frac{e^{-ik_q^0}}{\sqrt{k_q^0}} & \widehat{J}_{q\ell u}\\ -\sqrt{k_q^0}e^{-ik_q^0} & \widetilde{J}_{q\ell u}\end{vmatrix}(-i)^{\ell-q-2u}e^{-ik_\ell}, \qquad (79)
$$

$$
(X_{22})_{q\ell} := \frac{1}{2}\sum_{u=-\infty}^{\infty} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix}\frac{e^{-ik_q^0}}{\sqrt{k_q^0}} & \widehat{J}_{q\ell u}\\ -\sqrt{k_q^0}e^{-ik_q^0} & -\widetilde{J}_{q\ell u}\end{vmatrix} i^{\ell-q-2u}e^{ik_\ell}, \qquad (80)
$$

respectively.

Repeating the same process (delineated by Eqs. (70) $-$ (75)) for Eqs. (68) and (69) giving the boundary conditions at $x = 1$ yields the following matrix equation (analogous to Eq. (75)):

$$\begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} X_{22} & X_{21} \\ X_{12} & X_{11} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha^R} \\ \boldsymbol{\beta^R} \end{pmatrix} \tag{81}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are defined as in (45), $\boldsymbol{\alpha^R}$ and $\boldsymbol{\beta^R}$ are infinite column vectors defined as

$$\begin{aligned} \boldsymbol{\alpha^R} &:= (\alpha^R_{-\ell_{LB}}, \dots, \alpha^R_{-2}, \alpha^R_{-1}, \alpha^R_0, \alpha^R_1, \alpha^R_2, \dots)^T, \\ \boldsymbol{\beta^R} &:= (\beta^R_{-\ell_{LB}}, \dots, \beta^R_{-2}, \beta^R_{-1}, \beta^R_0, \beta^R_1, \beta^R_2, \dots)^T, \end{aligned} \tag{82}$$

and X_{11}, X_{12}, X_{21}, and X_{22} are still the matrix blocks defined in Eqs. (77)$-$(80).

It will soon be useful for us to know what is the inverse

$$\begin{pmatrix} X_{22} & X_{21} \\ X_{12} & X_{11} \end{pmatrix}^{-1},$$

so that we can rewrite equation (81) as

$$\begin{pmatrix} \boldsymbol{\alpha^R} \\ \boldsymbol{\beta^R} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{pmatrix}, \tag{83}$$

where we define

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} := \begin{pmatrix} X_{22} & X_{21} \\ X_{12} & X_{11} \end{pmatrix}^{-1} \tag{84}$$

Theorem 4 : *Suppose that we are given any square matrix $\boldsymbol{Q}$ of finite size split into four square blocks as follows:*

$$\boldsymbol{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

If we assume that all four blocks are invertible [9]*, then there is a convenient formula for the inverse of $\boldsymbol{Q}$:*

$$\boldsymbol{Q}^{-1} = \begin{pmatrix} (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})^{-1} & (Q_{21} - Q_{22}Q_{12}^{-1}Q_{11})^{-1} \\ (Q_{12} - Q_{11}Q_{21}^{-1}Q_{22})^{-1} & (Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})^{-1}. \end{pmatrix} \tag{85}$$

Hence, assuming that the linear operators represented by X_{11}, X_{12}, X_{21}, and X_{22} are invertible, we can apply the infinite-dimensional version of Theorem 4 to write the expressions for

[9] It is also possible to find an expression for the inverse of $\boldsymbol{Q}$ by only assuming that only two out of the four matrix blocks are square and only one of the square blocks and some composition of blocks is invertible. Though this expression would have a slightly larger scale of application, it is not as nice and symmetric as Eq. (85). Furthermore, it would expand the range of application insignificantly, because the set of invertible linear maps from a finite-dimensional vector space V to itself is open and dense in the set of all linear maps from V to itself.

matrices Z_{11}, Z_{12}, Z_{21}, and Z_{22} more explicitly as follows:

$$\begin{cases} Z_{11} &= (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1} \\ Z_{12} &= (X_{12} - X_{11}X_{21}^{-1}X_{22})^{-1} \\ Z_{21} &= (X_{21} - X_{22}X_{12}^{-1}X_{11})^{-1} \\ Z_{22} &= (X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1} \end{cases} . \tag{86}$$

• The Boundary Conditions at $x = 0$

Now, we need to consider the remaining boundary conditions (49) and (50) at $x = 0$. Recall that (29) and (32) have the exponential factor

$$\exp\left[\frac{iU_0^2\sin(2\omega t)}{8\mu\omega^3}\right]$$

in common. Thus, let us reconsider Eq. (55) written in less expanded form:

$$\Psi_\Omega^L(x,t) = e^{-i\Omega t}\exp\left[\frac{iU_0^2\sin(2\omega t)}{8\mu\omega^3}\right]\sum_{\ell=-\ell_{LB}}^{\infty}\left(\sum_{v=-\infty}^{\infty}(-1)^v J_v\left(\frac{U_0}{\omega}(1+x)\right)e^{iv\omega t}\right)\times$$
$$\times\left(\alpha_\ell^L e^{ik_\ell x}\sum_{m=-\infty}^{\infty}(-i)^m J_m\left(\frac{k_\ell U_0}{\mu\omega^2}\right)e^{im\omega t} + \beta_\ell^L e^{-ik_\ell x}\sum_{m=-\infty}^{\infty} i^m J_m\left(\frac{k_\ell U_0}{\mu\omega^2}\right)e^{im\omega t}\right)e^{-i\ell\omega t}, \tag{87}$$

Via re-indexing $p := m + v$ and $q := \ell - p$ (yielding $m = \ell - q - v$), we can turn Eq. (87) into

$$\begin{aligned} \Psi_\Omega^L(x,t) &= e^{-i\Omega t}e^{i\frac{U_0^2}{8\mu\omega^3}\sin(2\omega t)}\sum_{q=-\infty}^{\infty}\left[\left(\sum_{\ell=-\ell_{LB}}^{\infty}\sum_{v=-\infty}^{\infty}J_v\left(\frac{U_0}{\omega}(1+x)\right)J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\times\right.\right. \\ &\times \left.\left.\left((-i)^{\ell-q+v}\alpha_\ell^L e^{ik_\ell x} + i^{\ell-q+v}\beta_\ell^L e^{-ik_\ell x}\right)\right)e^{-iq\omega t}\right]. \end{aligned} \tag{88}$$

Similarly, one can get

$$\begin{aligned} \Psi_\Omega^R(x,t) &= e^{-i\Omega t}e^{i\frac{U_0^2}{8\mu\omega^3}\sin(2\omega t)}\sum_{q=-\infty}^{\infty}\left[\left(\sum_{r=-\ell_{LB}}^{\infty}\sum_{v=-\infty}^{\infty}J_v\left(\frac{U_0}{\omega}(1-x)\right)J_{r-q-v}\left(\frac{k_r U_0}{\mu\omega^2}\right)\times\right.\right. \\ &\times \left.\left.\left(i^{r-q+v}\alpha_r^R e^{ik_r x} + (-i)^{r-q+v}\beta_r^R e^{-ik_r x}\right)\right)e^{-iq\omega t}\right], \end{aligned} \tag{89}$$

where the index ℓ is replaced by index r so that the indices are distinguished upon the application of boundary conditions.

By evaluating the expressions in equations (88) and (89) at $x = 0$, it is clear that the continuity boundary condition (49) implies that for every $q \in \mathbb{Z}$,

$$
\begin{aligned}
& \sum_{\ell=-\ell_{LB}}^{\infty} \sum_{v=-\infty}^{\infty} J_v\left(\frac{U_0}{\omega}\right) J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)\left((-i)^{\ell-q+v}\alpha_\ell^L + i^{\ell-q+v}\beta_\ell^L\right) \\
= & \sum_{r=-\ell_{LB}}^{\infty} \sum_{v=-\infty}^{\infty} J_v\left(\frac{U_0}{\omega}\right) J_{r-q-v}\left(\frac{k_r U_0}{\mu\omega^2}\right)\left(i^{r-q+v}\alpha_r^R + (-i)^{r-q+v}\beta_r^R\right)
\end{aligned}
\tag{90}
$$

Taking the spatial partial derivative of the expression in Eq. (88) yields

$$
\begin{aligned}
& \frac{\partial \Psi_\Omega^L}{\partial x} = e^{-i\Omega t} e^{i\frac{U_0^2}{8\mu\omega^3}\sin(2\omega t)} \times \\
\times & \sum_{q=-\infty}^{\infty} \sum_{\ell=-\ell_{LB}}^{\infty} \sum_{v=-\infty}^{\infty} \left\{ \left[\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1+x)\right) - J_{v+1}\left(\frac{U_0}{\omega}(1+x)\right)\right) + \right.\right. \\
+ & \left. ik_\ell J_v\left(\frac{U_0}{\omega}(1+x)\right)\right] J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)(-i)^{\ell-q+v}\alpha_\ell^L e^{ik_\ell x} + \\
+ & \left[\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1+x)\right) - J_{v+1}\left(\frac{U_0}{\omega}(1+x)\right)\right) - ik_\ell J_v\left(\frac{U_0}{\omega}(1+x)\right)\right] \times \\
\times & \left. J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) i^{\ell-q+v}\beta_\ell^L e^{-ik_\ell x}\right\} e^{-iq\omega t}
\end{aligned}
\tag{91}
$$

Similarly,

$$
\begin{aligned}
& \frac{\partial \Psi_\Omega^R}{\partial x} = e^{-i\Omega t} e^{i\frac{U_0^2}{8\mu\omega^3}\sin(2\omega t)} \times \\
\times & \sum_{q=-\infty}^{\infty} \sum_{r=-\ell_{LB}}^{\infty} \sum_{v=-\infty}^{\infty} \left\{ \left[-\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1-x)\right) - J_{v+1}\left(\frac{U_0}{\omega}(1-x)\right)\right) + \right.\right. \\
+ & \left. ik_r J_v\left(\frac{U_0}{\omega}(1-x)\right)\right] J_{r-q-v}\left(\frac{k_r U_0}{\mu\omega^2}\right) i^{r-q+v}\alpha_r^R e^{ik_r x} + \\
+ & \left[-\frac{U_0}{2\omega}\left(J_{v-1}\left(\frac{U_0}{\omega}(1-x)\right) - J_{v+1}\left(\frac{U_0}{\omega}(1-x)\right)\right) - ik_r J_v\left(\frac{U_0}{\omega}(1-x)\right)\right] \times \\
\times & \left. J_{r-q-v}\left(\frac{k_r U_0}{\mu\omega^2}\right)(-i)^{r-q+v}\beta_r^R e^{-ik_r x}\right\} e^{-iq\omega t}
\end{aligned}
\tag{92}
$$

Hence, by evaluating the derivatives provided by equations (91) and (92) at $x = 0$, we see that the differentiability boundary condition (50) means that for every $q \in \mathbb{Z}$,

$$\begin{aligned}
&\sum_{\ell=-\ell_{LB}}^{\infty} \sum_{v=-\infty}^{\infty} \Bigg\{ \left[\frac{U_0}{2\omega} \left(J_{v-1}\left(\frac{U_0}{\omega}\right) - J_{v+1}\left(\frac{U_0}{\omega}\right) \right) + ik_\ell J_v\left(\frac{U_0}{\omega}\right) \right] J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) (-i)^{\ell-q+v} \alpha_\ell^L + \\
+ & \left[\frac{U_0}{2\omega} \left(J_{v-1}\left(\frac{U_0}{\omega}\right) - J_{v+1}\left(\frac{U_0}{\omega}\right) \right) - ik_\ell J_v\left(\frac{U_0}{\omega}\right) \right] J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) i^{\ell-q+v} \beta_\ell^L \Bigg\} \\
= & -\sum_{r=-\ell_{LB}}^{\infty} \sum_{v=-\infty}^{\infty} \Bigg\{ \left[\frac{U_0}{2\omega} \left(J_{v-1}\left(\frac{U_0}{\omega}\right) - J_{v+1}\left(\frac{U_0}{\omega}\right) \right) - ik_r J_v\left(\frac{U_0}{\omega}\right) \right] J_{r-q-v}\left(\frac{k_r U_0}{\mu\omega^2}\right) i^{r-q+v} \alpha_r^R + \\
+ & \left[\frac{U_0}{2\omega} \left(J_{v-1}\left(\frac{U_0}{\omega}\right) - J_{v+1}\left(\frac{U_0}{\omega}\right) \right) + ik_r J_v\left(\frac{U_0}{\omega}\right) \right] J_{r-q-v}\left(\frac{k_r U_0}{\mu\omega^2}\right) (-i)^{r-q+v} \beta_r^R \Bigg\}
\end{aligned} \tag{93}$$

For convenience, we define the following constant:

$$K_{\ell v} := \frac{U_0}{2\omega} \left(J_{v-1}\left(\frac{U_0}{\omega}\right) - J_{v+1}\left(\frac{U_0}{\omega}\right) \right) + ik_\ell J_v\left(\frac{U_0}{\omega}\right) \tag{94}$$

Since we require that k_ℓ is real, the complex conjugate of $K_{\ell v}$ must be

$$\overline{K}_{\ell v} = \frac{U_0}{2\omega} \left(J_{v-1}\left(\frac{U_0}{\omega}\right) - J_{v+1}\left(\frac{U_0}{\omega}\right) \right) - ik_\ell J_v\left(\frac{U_0}{\omega}\right) \tag{95}$$

Let us introduce the following notation as well:

$$(W_{11})_{q\ell} := \sum_{v=-\infty}^{\infty} J_v\left(\frac{U_0}{\omega}\right) J_{\ell-q-v}\left(\frac{k_l U_0}{\mu\omega^2}\right) (-i)^{\ell-q+v}; \tag{96}$$

$$(W_{12})_{q\ell} := \overline{(W_{11})_{q\ell}}; \tag{97}$$

$$(W_{21})_{q\ell} := \sum_{v=-\infty}^{\infty} K_{\ell v} J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) (-i)^{\ell-q+v}; \tag{98}$$

$$(W_{22})_{q\ell} := \overline{(W_{21})_{q\ell}} \tag{99}$$

With this new notation, it is evident that Eqs. (90) and (93) giving the boundary conditions at $x = 0$ give us that for all $q \in \mathbb{Z}$,

$$\sum_{\ell=-\ell_{LB}}^{\infty} (W_{11})_{ql} \alpha_\ell^L + (W_{12})_{ql} \beta_\ell^L = \sum_{r=-\ell_{LB}}^{\infty} (W_{12})_{qr} \alpha_r^R + (W_{11})_{qr} \beta_r^R \tag{100}$$

$$\sum_{\ell=-\ell_{LB}}^{\infty} (W_{21})_{q\ell} \alpha_\ell^L + (W_{22})_{q\ell} \beta_\ell^L = -\left(\sum_{r=-\ell_{LB}}^{\infty} (W_{22})_{qr} \alpha_r^R + (W_{21})_{qr} \beta_r^R \right) \tag{101}$$

Therefore, if let W_{11} be the infinite matrix with (q, ℓ) entries equal to $(W_{11})_{ql}$, W_{12} be the infinite matrix with (q, ℓ) entries equal to $(W_{12})_{ql}$, W_{21} be the infinite matrix with (q, ℓ)

entries equal to $(W_{21})_{ql}$, and W_{22} be the infinite matrix with (q, ℓ) entries equal to $(W_{22})_{ql}$, then Eqs. (100) and (101) can also be written as a single matrix equation:

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha^L} \\ \boldsymbol{\beta^L} \end{pmatrix} = \begin{pmatrix} W_{12} & W_{11} \\ -W_{22} & -W_{21} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha^R} \\ \boldsymbol{\beta^R} \end{pmatrix}, \tag{102}$$

where $\boldsymbol{\alpha^L}$ and $\boldsymbol{\beta^L}$ are defined as in (76) and $\boldsymbol{\alpha^R}$ and $\boldsymbol{\beta^R}$ are defined as in (82).

Again, we apply the infinite-dimensional version of Theorem 4 to obtain that

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}^{-1} = \begin{pmatrix} (W_{11} - W_{12}W_{22}^{-1}W_{21})^{-1} & (W_{21} - W_{22}W_{12}^{-1}W_{11})^{-1} \\ (W_{12} - W_{11}W_{21}^{-1}W_{22})^{-1} & (W_{22} - W_{21}W_{11}^{-1}W_{12})^{-1}. \end{pmatrix} \tag{103}$$

Let us introduce the following matrix:

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} := \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}^{-1} \begin{pmatrix} W_{12} & W_{11} \\ -W_{22} & -W_{21} \end{pmatrix}, \tag{104}$$

so that Eq. (102) becomes

$$\begin{pmatrix} \boldsymbol{\alpha^L} \\ \boldsymbol{\beta^L} \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha^R} \\ \boldsymbol{\beta^R} \end{pmatrix}, \tag{105}$$

By plugging in Eq. (103) into Eq. (104), one can derive that

$$\begin{cases} Y_{11} &= 2(W_{12}^{-1}W_{11} - W_{22}^{-1}W_{21})^{-1} \\ Y_{12} &= (I + W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})(I - W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})^{-1} \\ Y_{21} &= -(I + W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})(I - W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})^{-1} \\ Y_{22} &= -2(W_{21}^{-1}W_{22} - W_{11}^{-1}W_{12})^{-1} \end{cases} \tag{106}$$

• The Transfer Matrix, Σ-Matrix, and S-Matrix

The transfer matrix is a tool that could either be chosen to relate the modes in Region III to the modes in Region II *or vice versa*. We choose our transfer matrix

$$\boldsymbol{T} := \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

to relate the modes in Region III to the modes in Region II, so

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{pmatrix}. \tag{107}$$

From Eqs. (75), (83), and (105), we already know what is the transfer matrix. It is simply

$$\boldsymbol{T} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}, \tag{108}$$

Therefore, we have that for $i = 1, 2$,

$$T_{1i} = (X_{11}Y_{11} + X_{12}Y_{21})Z_{1i} + (X_{11}Y_{12} + X_{12}Y_{22})Z_{2i}. \tag{109}$$

Hence, from Eqs (86) and (106), it can be computed that

$$\begin{aligned}T_{11} &= \left[2X_{11}(W_{12}^{-1}W_{11} - W_{22}^{-1}W_{21})^{-1} - X_{12}(I + W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})(I - W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})^{-1}\right] \times \\ &\times \; (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1} + [X_{11}(I + W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})(I - W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})^{-1} - \\ &- \; 2X_{12}(W_{21}^{-1}W_{22} - W_{11}^{-1}W_{12})^{-1}](X_{21} - X_{22}X_{12}^{-1}X_{11})^{-1} \end{aligned} \tag{110}$$

and

$$\begin{aligned}T_{12} &= \left[2X_{11}(W_{12}^{-1}W_{11} - W_{22}^{-1}W_{21})^{-1} - X_{12}(I + W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})(I - W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})^{-1}\right] \times \\ &\times \; (X_{12} - X_{11}X_{21}^{-1}X_{22})^{-1} + [X_{11}(I + W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})(I - W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})^{-1} - \\ &- \; 2X_{12}(W_{21}^{-1}W_{22} - W_{11}^{-1}W_{12})^{-1}](X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1}. \end{aligned} \tag{111}$$

Now that we know how to relate $\boldsymbol{C}$ and $\boldsymbol{D}$ to $\boldsymbol{A}$ and $\boldsymbol{B}$, we can make use of this relationship to derive how $\boldsymbol{A}$ and $\boldsymbol{D}$ are related to $\boldsymbol{B}$ and $\boldsymbol{C}$, allowing us to obtain the Σ-matrix. By equation (107), $\boldsymbol{A} = T_{11}\boldsymbol{C} + T_{12}\boldsymbol{D}$, so

$$\boldsymbol{C} = T_{11}^{-1}\boldsymbol{A} - T_{11}^{-1}T_{12}\boldsymbol{D}. \tag{112}$$

From this equation, it is obvious that $\Sigma_{21} = T_{11}^{-1}$ and $\Sigma_{22} = -T_{11}^{-1}T_{12}$. Regarding the remaining two blocks Σ_{11} and Σ_{12} of the Σ-matrix, their relationship with the matrix blocks of the transfer matrix that we found is not as simple *generally speaking*, but it is just as simple with the matrix blocks of the other choice of transfer matrix, which relates $\boldsymbol{B}$ and $\boldsymbol{A}$ to $\boldsymbol{D}$ and $\boldsymbol{C}$ instead. However, given that our potential is an even function, our system has a reflection symmetry about $x = 0$, so there is no need to try to derive anything new. It is clear from the physical perspective that, by analogy with Eq. (112), we also have

$$\boldsymbol{B} = -T_{11}^{-1}T_{12}\boldsymbol{A} + T_{11}^{-1}\boldsymbol{D}, \tag{113}$$

so the four blocks of the Σ-matrix are:

$$\begin{cases} \Sigma_{11} = \Sigma_{22} = -T_{11}^{-1}T_{12} \\ \Sigma_{12} = \Sigma_{21} = T_{11}^{-1} \end{cases} \tag{114}$$

Hence, the two blocks T_{11} and T_{12} of the transfer matrix given by Eqs. (110) and (111) provide the full Σ-matrix of our system.

Suppose that we label the (λ, ν) entries of the four blocks of the Σ-matrix as follows:

$$\begin{cases} (\Sigma_{11})_{\lambda\nu} = (\Sigma_{22})_{\lambda\nu} = r_{\lambda\nu} \\ (\Sigma_{12})_{\lambda\nu} = (\Sigma_{21})_{\lambda\nu} = t_{\lambda\nu} \end{cases} \tag{115}$$

Then, the S-matrix is simply a submatrix of the Σ-matrix that only contains the entries of each block with nonnegative row and column indices λ and ν, so it is

$$\boldsymbol{S} = \begin{pmatrix} r_{00} & r_{01} & \dots & t_{00} & t_{01} & \dots \\ r_{10} & r_{11} & \dots & t_{10} & t_{11} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ t_{00} & t_{01} & \dots & r_{00} & r_{01} & \dots \\ t_{10} & t_{11} & \dots & r_{10} & r_{11} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{116}$$

2.3 Transmission Probabilities

• The Physical Interpretation and Graphs of the Elements of the S-Matrix

If we take a look at Eq. (41), we see that S_{21} relates the amplitudes of incoming propagating waves traveling from the left represented by $\overline{\mathbf{A}}$ to those of the outgoing propagating waves traveling to the right represented by $\overline{\mathbf{C}}$, while S_{12} relates the amplitudes of incoming waves ($\overline{\mathbf{D}}$) from the right to those of the outgoing waves ($\overline{\mathbf{B}}$) to the left. This signifies that for an incoming wave traveling from some direction (either from the left or from the right) in the ν^{th} channel, every (λ, ν) entry $t_{\lambda\nu}$ of the matrix blocks $S_{21} = S_{12}$ is the amplitude of both transmission through the region of the well (Region I) and transition into the λ^{th} channel. Hence, the absolute value squared $|t_{\lambda\nu}|^2$ of that amplitude represents the probability that an incoming wave traveling from some direction in the ν^{th} channel has to both transmit through Region I and to transition into the λ^{th} channel as an outgoing wave on the other side of the well. In Figures 2-5, we illustrate graphs of the combined transmission and transition probabilities $|t_{\lambda\nu}|^2$ as functions of the incident quasienergy Ω for some chosen nonnegative integers λ and ν and for the following set of parameters (expressed in Hartree atomic units): $\mu = 1$, $V_0 = 10$, $U_0 = 0.5$, and $\omega = 4$.

By similar analysis, the modulus squared $|r_{\lambda\nu}|^2$ of the (λ, ν) entry of the matrix blocks $S_{11} = S_{22}$ is the probability that an incoming wave traveling from some direction in the ν^{th} channel has to both reflect off the well and to transition to the λ^{th} channel as an outgoing wave in the same region outside of the well.

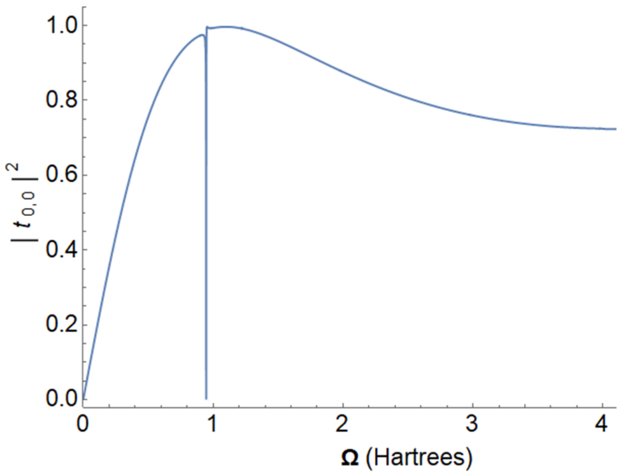

Figure 2: A plot of the probability $|t_{0,0}|^2$ that a particle from the zeroth channel on one side of the well will transmit into the zeroth channel on the other side. Here and in the rest of the figures of this thesis, the particle's mass is $\mu = 1$, the unperturbed well depth is $V_0 = 10$, the amplitude of the oscillation of the bottom of potential well is $U_0 = 0.5$, and the frequency of that same oscillation is $\omega = 4$ (all expressed in Hartree atomic units).

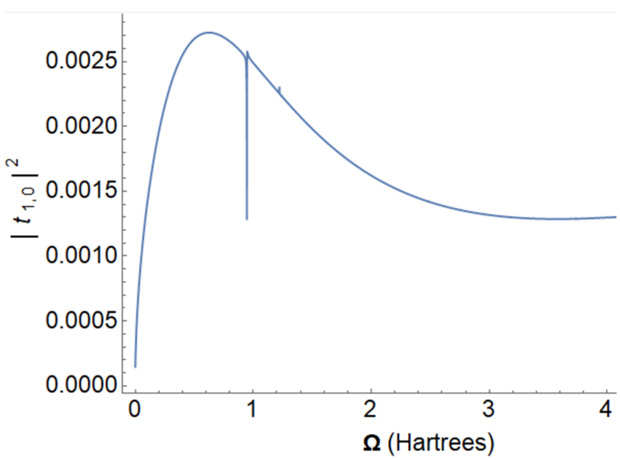

Figure 3: A plot of the probability $|t_{1,0}|^2$ that a particle from the zeroth channel on one side of the well will transmit into the first channel on the other side.

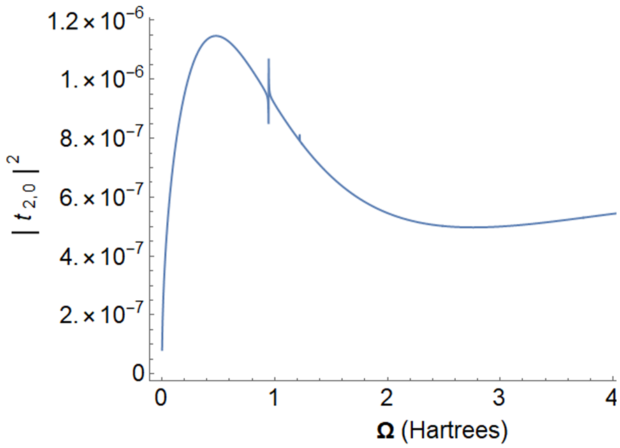

Figure 4: A plot of the probability $|t_{2,0}|^2$ that a particle from the zeroth channel on one side of the well will transmit into the second channel on the other side.

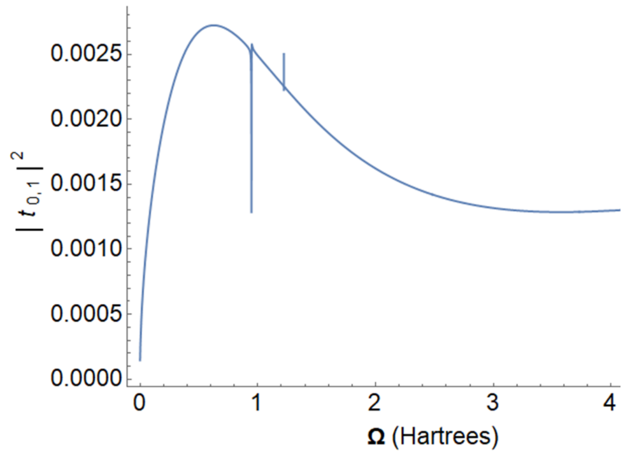

Figure 5: A plot of the probability $|t_{0,1}|^2$ that a particle from the first channel on one side of the well will transmit into the zeroth channel on the other side.

• Extending Our Interpretation Beyond the S-Matrix

The S-matrix is a submatrix of the **Σ**-matrix, so we should also address the physicality of the entries of the **Σ**-matrix that are excluded from the S-matrix. As we have restricted ourselves to consider the meaning of the S-matrix, our discussion has been limited to what happens in the nonnegative channels. Thus far, we only explained the physical meaning of $|t_{\lambda\nu}|^2$ and $|r_{\lambda\nu}|^2$ for nonnegative λ and ν.

Now, suppose that we took $|t_{\lambda\nu}|^2$ for which ν is negative, then it is the probability for a nonexistent mode (defined on page 19) to transmit through the well and transition into the λ^{th} channel. Of course, this makes no physical sense, because we are never given a nonexistent mode in the first place (as we have seen toward the bottom of page 15). It is evident that we would also get a nonsensical interpretation if we were to consider $|r_{\lambda\nu}|^2$ with negative ν. Thus, we cannot allow ν to be negative in the realms of physics.

On the other hand, there are also $|t_{\lambda\nu}|^2$ and $|r_{\lambda\nu}|^2$ for which λ is negative and ν is nonnegative. In this case, $|t_{\lambda\nu}|^2$ represents the probability for an incoming propagating mode in the ν^{th} channel to transmit through the well and to form an evanescent mode in the λ^{th} channel on the other side of the well, while $|r_{\lambda\nu}|^2$ with the same λ and ν is the probability for that same incoming wave to stay on the same side of the well and to create an evanescent mode in the λ^{th} channel.

Note that an evanescent mode, unlike a propagating mode, has a zero probability current, i.e. it exhibits no flux (or propagation) of probability in any direction. This implies that when the incident particle hits the region of the well and is induced by the driving field to access an evanescent mode, it does not propagate away from the well through that mode. Instead, the accessed evanescent mode blocks the probability from “leaking” out of the well, causing the particle’s positional probability distribution to accumulate in the region of the well.

We can also think about this as follows: as an evanescent mode is a part of the wavefunction that exponentially decays as $x \to \pm\infty$, the mode's contribution to the probability of the particle being measured at a position in the well's exterior tends to zero as the measurement becomes farther away from the well. Indeed, the contributions by evanescent modes to the particle's probability of being measured outside the well tend to be much smaller than the particle's probability to be measured inside the well and the probabilistic contributions by propagating modes in the well's exterior. Therefore, if there is a certain incident quasienergy for which some $|t_{\lambda\nu}|^2$ with a negative λ and a nonnegative ν has a local maximum, then we have evidence that the particle's probability accumulates inside the region of our well, because the inside of the well should have the greater share of the probability than the evanescent mode. For quasienergies close to this maximum, if some other transmission probabilities $|t_{\lambda'\nu}|^2$ with positive λ' and with the same ν have local minima followed by local maxima, we can claim that we have found an incident energy for which the particle's probability not only concentrates inside the well, but is also limited in how it leaks out of the well via propagating modes. In other words, the peak in the probability that an incident particle transitions to an evanescent mode induces the dip in the probability that this particle accesses an outgoing propagating mode, causing a significant portion of the particle's probability distribution to get "trapped" within the region of the well for a finite amount of time.

The phenomenon in which a peak in transmission into one mode induces a dip in transmission into another mode is called a **transmission resonance**. Furthermore, a Floquet state that is induced by a resonance between propagating and evanescent modes and exhibits the property that the particle's probability is localized for a finite amount of time is called a **quasibound state**. If the incident quasienergy that leads to the quasibound state is converted to the energy corresponding to the negative channel in which the *greatest* peak in the amplitude of the evanescent mode occurs, then what we will find is approximately equal to what is referred to as the **energy of the quasibound state**.

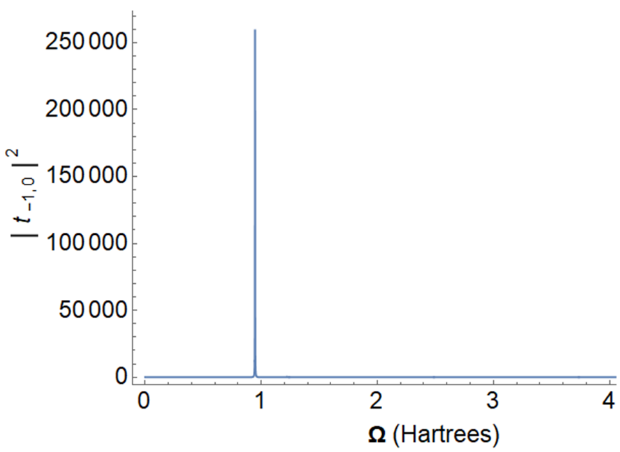

Figure 6: A plot of the probability $|t_{-1,0}|^2$ that a particle from the zeroth channel on one side of the well will transmit into the negative first channel on the other side.

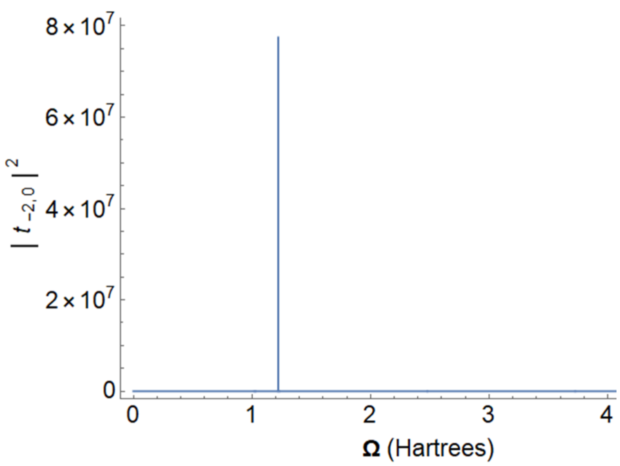

Figure 7: A plot of the probability $|t_{-2,0}|^2$ that a particle from the zeroth channel on one side of the well will transmit into the negative second channel on the other side.

Remark 5 *Keep in mind that a quasibound state is a Floquet state over all four regions, so it cannot actually have a definite energy and instead has a definite quasienergy. Thus, when we say "energy of the quasibound state," the word "energy" has a different meaning from what it formally means. Here, it actually refers to a specific choice of quasienergy.*

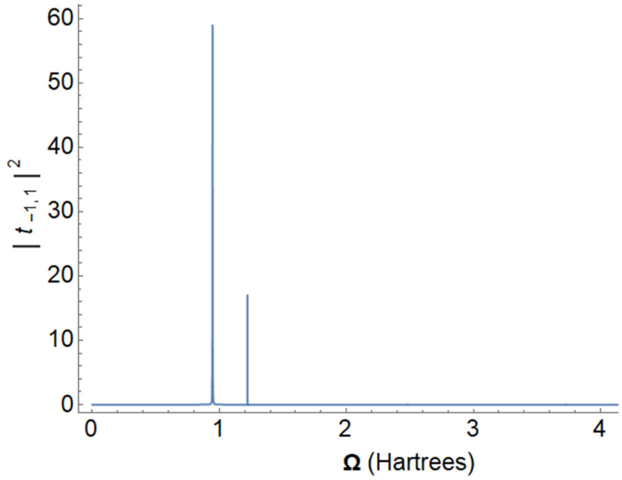

Figure 8: A plot of the probability $|t_{-1,1}|^2$ that a particle from the first channel on one side of the well will transmit into the negative first channel on the other side.

Suppose that we consider a case in which the driving field is weak (i.e. U_0 is small). If we gradually stop the time-periodic driving field to make the potential time-independent (that is, slowly diminish U_0 down to zero), the quasibound states will be replaced by **bound states**, which are stationary states of negative energy that are localized in the potential well forever. Bound states are completely trapped in the well because they are stationary states, so their energies are conserved. That makes it impossible for a particle to transition from a bound state to a propagating mode through which it can leak out, unless the potential gets somehow perturbed again.

In summary, every entry $t_{\lambda\nu}$ and entry $r_{\lambda\nu}$ of the Σ-matrix for which ν is nonnegative has physical significance, while other entries with negative index ν are unphysical. Figures 6-8 are graphs of $|t_{\lambda\nu}|^2$ as functions of the incident quasienergy Ω for some chosen negative integer λ, nonnegative integer ν, and again the following set of parameters: $\mu = 1$, $V_0 = 10$, $U_0 = 0.5$, and $\omega = 4$, all expressed in Hartree atomic units.

• The Method Employed to Create Figures 2-8

The computational software I used to generate the plots of probabilities in Figs. 2-8 is Wolfram Mathematica 11.0, Student Edition. Since all of the Figures 2-8 only represent entries from the off-diagonal blocks Σ_{12} and Σ_{21} of the Σ-matrix, we have that by (114), I only needed Eq. (110) for T_{11} to find the necessary blocks of the Σ-matrix. Thus, I combined Eq. (110) and the second equation in (114) into

$$\begin{aligned} \Sigma_{12} &= \Sigma_{21} = \\ &= \Big\{ \big[2X_{11}(W_{12}^{-1}W_{11} - W_{22}^{-1}W_{21})^{-1} - X_{12}(I + W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})(I - W_{22}^{-1}W_{21}W_{11}^{-1}W_{12})^{-1}\big] \times \\ &\times\ (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1} + [X_{11}(I + W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})(I - W_{11}^{-1}W_{12}W_{22}^{-1}W_{21})^{-1} - \\ &-\ 2X_{12}(W_{21}^{-1}W_{22} - W_{11}^{-1}W_{12})^{-1}](X_{21} - X_{22}X_{12}^{-1}X_{11})^{-1}\Big\}^{-1}. \end{aligned} \tag{117}$$

The first question to ask is, given the parameters $\mu = 1$, $V_0 = 10$, $U_0 = 0.5$, and $\omega = 4$, how can we manage to compute the elements of Σ_{12} if Eq. (117) involves products of matrices of infinite size? Since computational softwares are limited to dealing with matrices of finite size, our method of computation must involve reasonable truncations of the matrices X_{11}, X_{12}, X_{21}, X_{22}, W_{11}, W_{12}, W_{21}, and W_{22} to finite size. Eqs. (77)−(80) and (96)−(99) provide the (q, ℓ) entries of these eight matrices, and we need to choose certain finite ranges for indices q and ℓ in such a way that the matrices end up having the appropriate numbers of rows and columns for them to be multipliable and invertible in Eq. (117). If the we choose the ranges for q and for ℓ to be the same length for all eight matrices, then Eq. (117) will fully consists of square matrices, which would obviously be of appropriate size to be multipliable and invertible. In other words, truncating the eight matrices to square matrices of same size is always an option. Note that it is also possible to truncate some of the eight matrices to rectangular matrices such that Eq. (117) can still be used, but this thesis only considers truncations to square matrices.

To perform the truncation to square matrices reasonably, we need to make some physical considerations (or, if one would like to be more rigorous, mathematically derive the convergence properties as matrices are increased up back up to infinite size). Suppose that an incoming plane wave with high energy is introduced into our system. If its energy is high enough, then the unperturbed depth V_0 and oscillation amplitude U_0 of the well is negligibly small to the incident energy of the plane wave. Thus, the plane wave should keep propagating almost as if it is unaffected by the well. This justifies bounding q and ℓ from above for the purposes of truncation.

Recall that if we set $U_0 = 0$ so that the bottom of potential well does not oscillate anymore, then the evanescent modes with energies below the minimum potential energy cannot exist due to the impossibility for a particle in one energy state to transition to another energy state. Therefore, if U_0 is slightly greater than zero, the probability that a particle will access evanescent modes of very low energy should be extremely low, which justifies bounding q from below for truncation purposes (note that index ℓ is already bounded from below by $-\ell_{LB}$).

In the first paragraph of page 18, I mentioned that ℓ_{LB} can serve as a good guide for bounding q from below. If we take our parameters $\mu = 1$, $V_0 = 10$, $U_0 = 0.5$, and $\omega = 4$, then according to Eq. (30), if $\Omega \in [0, 2.0039)$, then $\ell_{LB} = 2$, while if $\Omega \in [2.0039, 4)$, then $\ell_{LB} = 3$. Based on this, I chose -2 to be the lower bound for both q and ℓ indices[10]. Regarding the upper bound to indices q and ℓ, I figured that given that the strength of the oscillation $U_0 = 0.5$ is weak, the channels above the positive second channel are accessed by an an incoming particle in the zeroth and first channels with very low probability, so I decided to bound both q and ℓ from above by 2.

In all, we have chosen $q, \ell \in \{-2, -2, 0, 1, 2\}$, thereby truncating the eight infinite matrices X_{11},...,X_{22}, W_{11},...,W_{22} down to 5×5 matrices. As a result, the resulting Σ_{12} matrix is 5×5, while its submatrix S_{12} is 3×3.

[10] I have selected a lower bound of -2 just for the purposes of providing some sort of demo. Clearly, it would have been better to bound from below by -3, and of course, that is something that the reader can try if he or she wishes to practice using my method. In this paper, we will find quasibound states resembling the first and second excited states of the unperturbed version of the well (with $U_0 = 0$), but if you bound from below by -3, one should also find a quasibound state resembling the ground state.

The next computational challenge to address is that the eight matrices as defined in Eqs. (77)−(80) and (96)−(99) have entries that are series (i.e. infinite sums), so in order to make computation possible, the series must be reduced to finite sums. Fortunately, all eight series converge extremely quickly. For example, this can be seen in the following figure:

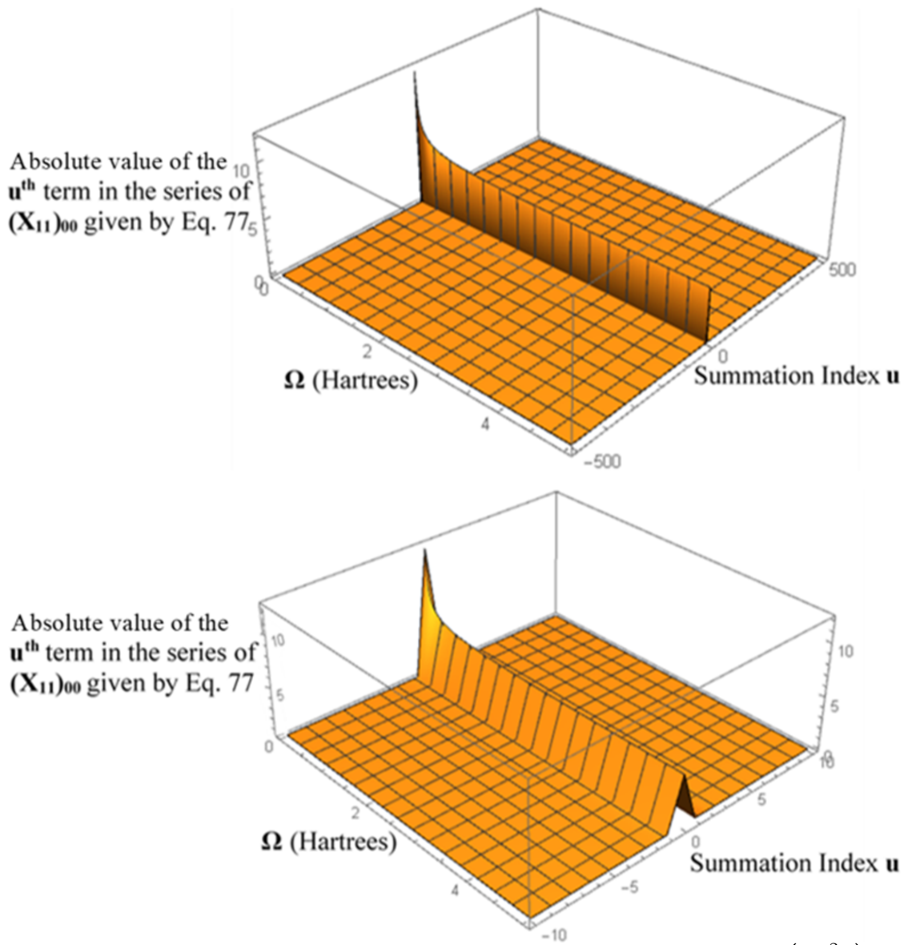

Figure 9: Both figures plot the absolute value of the summand $J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)(\widehat{J}_{q\ell u}\sqrt{k_q^0}e^{ik_q^0} - \widetilde{J}_{q\ell u}\frac{e^{ik_q^0}}{\sqrt{k_q^0}})(-i)^{\ell-q-2u}e^{-ik_\ell}$ of the series in Eq. (77) with $q = \ell = 0$. The plots show very quick convergence to zero as the summation index u gets farther away from zero. The lower plot is more zoomed in than the upper one.

Therefore, it was justified to truncate the series in Eqs. (77)−(80) and (96)−(99) as follows:

$$(X_{11})_{q\ell} := \frac{1}{2}\sum_{u=-10}^{10} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix} \widehat{J}_{q\ell u} & \frac{e^{ik_q^0}}{\sqrt{k_q^0}} \\ \widetilde{J}_{q\ell u} & \sqrt{k_q^0}e^{ik_q^0} \end{vmatrix}(-i)^{\ell-q-2u}e^{-ik_\ell}; \tag{118}$$

$$(X_{12})_{q\ell} := \frac{1}{2}\sum_{u=-10}^{10} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix} \widehat{J}_{q\ell u} & \frac{e^{ik_q^0}}{\sqrt{k_q^0}} \\ -\widetilde{J}_{q\ell u} & \sqrt{k_q^0}e^{ik_q^0} \end{vmatrix} i^{\ell-q-2u}e^{ik_\ell}; \tag{119}$$

$$(X_{21})_{q\ell} := \frac{1}{2}\sum_{u=-10}^{10} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix} \frac{e^{-ik_q^0}}{\sqrt{k_q^0}} & \widehat{J}_{q\ell u} \\ -\sqrt{k_q^0}e^{-ik_q^0} & \widetilde{J}_{q\ell u} \end{vmatrix}(-i)^{\ell-q-2u}e^{-ik_\ell}; \tag{120}$$

$$(X_{22})_{q\ell} := \frac{1}{2}\sum_{u=-10}^{10} J_u\left(\frac{U_0^2}{8\mu\omega^3}\right)\begin{vmatrix} \frac{e^{-ik_q^0}}{\sqrt{k_q^0}} & \widehat{J}_{q\ell u} \\ -\sqrt{k_q^0}e^{-ik_q^0} & -\widetilde{J}_{q\ell u} \end{vmatrix} i^{\ell-q-2u}e^{ik_\ell}; \tag{121}$$

$$(W_{11})_{q\ell} := \sum_{u=-10}^{10} J_v\left(\frac{U_0}{\omega}\right)J_{\ell-q-v}\left(\frac{k_l U_0}{\mu\omega^2}\right)(-i)^{\ell-q+v}; \tag{122}$$

$$(W_{12})_{q\ell} := \sum_{u=-10}^{10} J_v\left(\frac{U_0}{\omega}\right)J_{\ell-q-v}\left(\frac{k_l U_0}{\mu\omega^2}\right) i^{\ell-q+v}; \tag{123}$$

$$(W_{21})_{q\ell} := \sum_{u=-10}^{10} K_{\ell v}J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right)(-i)^{\ell-q+v}; \tag{124}$$

$$(W_{22})_{q\ell} := \sum_{u=-10}^{10} \overline{K_{\ell v}}J_{\ell-q-v}\left(\frac{k_\ell U_0}{\mu\omega^2}\right) i^{\ell-q+v}. \tag{125}$$

While experimenting with truncations via Wolfram Mathematica 11.0, I found that replacing $\sum_{u=-\infty}^{\infty}$ by $\sum_{u=-10}^{10}$ yielded results for $|t_{\lambda\nu}|^2$ that are exactly the same up to at least seven decimal places as when I replaced $\sum_{u=-\infty}^{\infty}$ by $\sum_{u=-1000}^{1000}$, which is a strong justification for the truncation I used. I suppose that even replacing $\sum_{u=-\infty}^{\infty}$ by $\sum_{u=-5}^{5}$ would lead to suitably accurate results.

I discovered that the truncations of the series played a significant role in how much time it took to create the plots in Figs. 2-8. If I truncated with $\sum_{u=-1000}^{1000}$, then it took two-and-a-half minutes for Mathematica to compute $t_{0,0}$ for a single value of incident quasienergy Ω, which meant that a plot with $|t_{0,0}|^2$ for 500 values of Ω would have to take about 21 hours to make. In contrast, if I truncated with $\sum_{u=-10}^{10}$, then it would only take less than 15 minutes to plot $|t_{0,0}|^2$ for 500 values of Ω with equally suitable accuracy. In Figures 2-8, I plotted each $|t_{\lambda\nu}|^2$ for about 1500 to 3000 values of Ω.

In conclusion, my plotting method would help one to accomplish suitable computational precision, accuracy, and efficiency when using Eq. (117) to make computational software generate graphs of the transmission probabilities as functions of incident quasienergy. An analogous method can be used to efficiently create graphs of reflection probabilities $|r_{\lambda\nu}|^2$.

3 Discussion

Overall, this analysis of our chosen localized one-dimensional quantum system that is subject to time-periodic driving revealed that the space of explicit solutions to the Schrödinger equation can be found, that an elegant analytic expression allowing us to compute the elements of the scattering matrix can be obtained, and that there is a convenient method to generate plots (Figs. 2-8) representing transmission probabilities.

Figures 2-8 all represent our system with the following parameters (all in Hartree atomic units): well width is 2, particle mass is $\mu = 1$, amplitude of the oscillation of the potential bottom is $U_0 = 0.5$, angular frequency of that same oscillation is $\omega = 4$, and unperturbed well depth is $V_0 = 10$. Notice how the amplitude U_0 is a lot smaller than the well depth V_0 and driving frequency ω. This is an indication we chose the version of our system with a weak driving field, so it should resemble the time-independent case when $U_0 = 0$ in some fundamental aspects. In other words, a system subject to a weak perturbing field like ours should still preserve some general properties of the unperturbed system.

Indeed, if we analyze Figures 2-5, we see that all of them have a resonance dip at either $\Omega \approx 1.22$ Hartrees or ≈ 0.945 Hartrees, or both. Both of those dips are always followed by a peak at a slightly higher incident quasienergy Ω. At the same time, Figs. 5-8 all show a resonance peak at quasienergies that are either 1.2212 or 0.9466 Hartrees, or both. Recall that in Section 2.3 (specifically on page 33), we discussed a phenomenon in which at around the same incident quasienergy, there is both a peak in probability that an incoming particle accesses an evanescent mode and a dip in the probability (followed by a peak) that the particle transitions to an outgoing propagating mode. As a reminder, this phenomenon is called a transmission resonance and the state induced by the incident quasienergy for which the resonance occurs is called a quasibound state. In our case, it is evident that transmission resonances and quasibound states occur at two quasienergies: 1.2212 and 0.9466 Hartrees. Comparing Figures 7 and 8, we can suspect (though not with complete confidence, for more graphs would have to be plotted) that at $\Omega = 1.2212$, the evanescent mode with the greatest peak in the probability amplitude is the one in the negative second channel. Hence, we can approximate that the energy of the quasibound state is

$$\Omega - 2\omega = 1.2212 - 8 = -6.7788 \text{ Hartrees.}$$

Comparing Figure 6 with Figs. 7 and 8, we can also suspect that at $\Omega = 0.9466$ the evanescent mode with the highest amplitude is the one in the negative first channel, so we can find another energy of a quasibound state:

$$\Omega - \omega = 0.9466 - 4 = -3.0534 \text{ Hartrees.}$$

When again the well width is 2 and the well depth $V_0 = 10$, but $U_0 = 0$ instead, there are three bound states in the well, which in order of ascending energy, are known as following: The ground state, first excited state, and second excited state. Here comes the important observation: The energy of the first excited state, which is -6.7791 Hartrees, is really close to the energy of the quasibound state corresponding to the transmission resonance at $\Omega =$

1.2212, while the energy of the second excited state, which is -3.0542 Hartrees, is very close to the energy of the other quasibound state that we found. From this resemblance between our driven system and the unperturbed system, we can infer that my method of finding transmission probabilities gives sensible results.

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