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No. 3606: February 8, 1936

THE TEXAS MATHEMATICS TEACHERS' BULLETIN

Volume XX

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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

Mirabeau B. Lamar

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Volume XX

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NUMBERS OR MATHEMATICS*

By H. J. ETTLINGER
The University of Texas

As the University representative in mathematics of the State Curriculum Revision Committee, it is my purpose to survey in brief the mathematics part of the work of this Committee. On this program there will be several reports with respect to particular grades and subdivisions such as the junior and senior high school. The teachers themselves are sometimes expected to "look up" toward the work which is to follow and which may possibly culminate in work at the University. May I say at the outset that I agree most heartily with those who maintain that possibility of the student's future attendance at a university or college should not dominate, nay, even affect in any way the content of the course in hand in the lower schools. I would be willing to defend this statement not alone with respect to the internal construction of the mathematics curriculum throughout the grades but also with reference to the definite place that mathematics should have in the curriculum itself. I accept whole-heartedly the platform laid down some years ago by the National Committee on Mathematics Requirements that each grade or each year of mathematics should be based entirely on what the student is prepared to receive at that stage, without reference to what he will do later. In a program which is planned to develop to the fullest the social adaptability of each child at every stage, number work must have a prominent place. Number situations are so intertwined with everyday life and experience, with necessities and comforts, with intelligent voting, with personal requirements involving the elementary physical situations as well as the problem of living and working together happily, that there can be

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***Report addressed to the Mathematics Section of the Texas State Teachers Association, San Antonio, Texas, Nov. 29, 1935.**

only one answer to the question "Should number work or mathematics have a place in the curriculum?"

The objections to the study of mathematics, which have now become jokes almost of the horse-chestnut variety, are aimed at weaknesses which are present partly in content and partly in the teaching of the subject. I shall content myself with pointing out some simple examples. The old courses of formal manipulation and "dry as dust" mechanical technique belong to the middle ages of education. The learning of proofs by memory and the acquirement of any other kind of information, such as formulas, by rote as an end in itself, must go the route of the extinct dodo bird into oblivion.

Without any further preliminaries, we will state then that it is the purpose of mathematics to train students to think quantitatively and spatially. These are the lowest terms to which one can reduce the objectives of mathematics. There is no clash between the above statement and the assertion that studies should have social ends or should be concerned with real life situations, or that education should be functional, or that courses of study should be interesting. It is a trite saying to maintain that every person at every stage of existence is confronted by number situations. You may be a poet by temperament, you may be an artist by inclination, you may be a literary genius by heredity, but you cannot escape the everyday number situations which confront you as a human being, as a member of a community, as an individual with problems involving an understanding of numbers. The history of the individual as well as the history of the race could be told in numbers not necessarily limited to the seven ages delineated by Shakespeare. One can easily parallel the entire history of the development of the number system with the human history either of the whole race or of a single person. Similarly, but in a no less manifest manner, the sense of appreciation of form or spatial relationships is very important in the everyday life of the individual. These two notions of number relationships and spatial relationships must have their threads or strands completely shot through our courses of study. We must eliminate memory work, we must eliminate tedious manipulation, we must translate everything into such terminology as can be understood by the student at each particular stage, but the teacher who would successfully impart mathematical knowledge must at all times be conscious of the fact that mathematics is numbers and form.

The structure of our courses, as well as their content, should reveal the nature of the study of numbers itself. We are forced to expand our remarks particularly concerning the drill side of number work. The vast mass of criticism levelled against mathematics is based on the drill or manipulative side of mathematics. Particularly at the high school stage, algebra and geometry draw the fire of those who themselves are completely untutored in the fundamental concepts of number work. They claim that there are no relationships between real life situations and factoring in algebra and little or none in the demonstrations usually given in plane geometry. These skillful teachers will, however, definitely show the very close relationship between the number situations or spatial relationships involved and those of everyday life. For after all, in order to know mathematics, one must know how to do it. The only way to learn to do is by doing it. Consequently, drill work must be given in order that skill may be developed. One may give the following illustrations from the very beginning in the lowest grades. It is of significance that in the beginning we count objects such as one apple, two apples, three apples, etc.; or one sheep, two sheep, three sheep, etc.; or one window in a courthouse, two windows in a courthouse, three windows in a courthouse, etc. It is easy to see how tedious this sort of thing may become. In order to avoid losing the interest of the pupil we are forced to simplify a drill of the above kind to the familiar succession of one, two, three, etc., and thereby we gain in pupil interest. This, perhaps the simplest mathematical situation, is typical of a great many others. The acquirement of skill is impossible without drill. Drill should be simplified so as not to become too involved or tedious, but even here a great deal depends upon the atmosphere that has been created as to whether even the simplest kind of drill remains a mumbo-jumbo performance or becomes meaningful in terms of common experience. The kind of problems which may be used for drill can easily be made to include both the abstract kind and the concrete kind. A proper mixture will help to conserve the number background behind all abstract situations.

A word of caution is necessary from the point of view of simplification, applications, and concreteness. Any one of these may be overdone to the consequent detriment of any given group of students. The attempt to over-simplify sometimes results in involving the situation, because of length of explanation. At times the applications brought in may become mere repetition and, therefore, very boresome. Concreteness may wander so far away from the student's experience as to introduce new difficulties much weightier than those involved in the numbers themselves, for, after all, a number situation or a space relationship is very, very simple compared with the complications involved in the most elementary so-called life situation. Most life situations involve the conduct of human beings. Who is there who will say that the problem of two and two is not infinitely simpler than the problem of predicting human behavior, even though all the motives and influences be given?

Considerable attention should be given to ways and means for articulating the complete set of courses mapped out from the first to the eleventh or twelfth grade. This problem is especially important at the junior or senior high school stage where the old line courses in algebra and geometry mark off these two subjects as separate and distinct. Complete articulation may be obtained by the use of the so-called spiral method which consists of a return to the same topic at several stages but at a higher level each time. This conception is completely in line with the dynamic or developmental point of view with regard to numbers.

Finally it should be pointed out that a serious problem confronting the teachers in this day of many classes and large classes is to appeal to all the members of the class and not merely to the average member. The talented pupils in the class have every right to have their capabilities stimulated, and the work should be so designed as to afford the widest opportunities for developing the best students in the class. I close with sounding a warning that there is grave danger in building the course entirely around unit projects, for that may make it merely vocational with no depth at all and only a fair amount of surface.

AN INSIGHT INTO THE MATHEMATICAL SITUATION IN TEXAS

By BEE GRISSOM
Austin High School

I know of no better way in which to present an insight into the mathematical situation in Texas to the teachers of mathematics and to other persons interested in mathematics than by offering a resumé of the program presented at the meeting of the Mathematics Section of the Texas State Teachers Association on November 29, 1935. The program presented at that time was rather unusual in that it not only consisted of reports from the State Production Committee in the Field of Mathematics, but it also included the experiences, opinions, and ideas from the faculties of some 200 schools in Texas.

In November, 1934, the mathematics section realized its responsibility with regard to curriculum revision, its obligation to the teachers of mathematics to lend assistance and encouragement in the Texas Curriculum Revision Movement, and caused a committee to be appointed to study the mathematical situation in the public schools in Texas. The committee had for its purposes the securing of some idea as to the existing conditions in the mathematics curriculum, what is being done in the way of curriculum revision in mathematics in our high schools, and the securing of some expression from the teachers of mathematics as to what changes, if any, should be made. A questionnaire was prepared and mailed to approximately 200 high schools of all sizes and located in all sections of the state.

The questions were answered and returned by 188 schools. The returned questionnaires were first grouped according to the number of students graduating each year—that is, according to the enrollment. There were 106 schools graduating 49 students or fewer, 42 schools graduating from 50 to 99 students, 22 schools graduating from 100 to 199 students, and 18 schools graduating over 200 students each year. However, after the tabulation of answers was made by groups, the similarity between the various groups was such that it did not warrant their separation in the final report.

In a survey of this type one of the first questions usually considered is the credits required for graduation from high school. Although more than half of the schools require either 2 units of algebra and 1 unit of geometry (73 schools to be exact), or 1 unit of algebra and 1 unit of geometry (41 schools), there is considerable variation in the graduation requirements of the remainder of the schools. Several schools offer different types of diplomas, namely, a diploma issued to students who have prepared themselves to enter a college or university, another diploma for students who have prepared for a trade, and another for vocational or commercial work. It is interesting to note that this cross-section of our schools reveals 5 schools which do not require any mathematics at all for graduation and 9 schools which require only 1 unit for graduation.

"Are you contemplating a change in the graduation requirements in mathematics?" This question yielded interesting results. Only 27 per cent of the schools canvassed by the questionnaire planned to continue the requirement of 3 units in mathematics for graduation. The remaining schools varied considerably in their requirements for graduation, with 29 per cent requiring 2 units and 11 per cent requiring 2½ units. It is interesting to note that 7 schools plan to raise requirements (one school requires as many as 4 units) while 29 per cent of the schools plan to lower requirements, with the remaining schools having a variation from no units required to 3½ units required. The very large number of schools requiring less than 3 units is due to the fact that many of these schools lowered graduation requirements simultaneously with the lowering of entrance requirements by The University of Texas and other institutions of higher learning.

The most popular new courses are general mathematics and advanced or commercial arithmetic. Most of the general mathematics courses are to be added in the eighth grade, while there is considerable variation in the grade placement of the arithmetic. Each of these new courses in the majority of cases is to be considered as an optional course by the students.

In answer to the question as to how much of the text used in first-year algebra is actually taught, we find a surprising amount of variation. Half of the schools teach through linear systems while only 11 per cent of the schools teach through ratio and proportion. Attention is called to the fact that students taking only one year of algebra in these schools which teach only through linear systems do not have an opportunity to gain any facility in the use of exponents, radicals, or in the solution of quadratic equations—some of the most important concepts in algebra.

Whether plane geometry should be offered to students who have had only one year of algebra is a prevalent question in some sections of our state. Eighty-four per cent of the schools in Texas, according to this questionnaire, which have tried the plan of presenting plane geometry immediately following a one-year course in algebra recommend it and continue to use it. In practically all of these schools the algebra course is offered in the eighth grade and the plane geometry course in the ninth grade. Hence, in view of our present trend toward lowering graduation requirements in mathematics, this is a very interesting and important bit of information. We believe the plan should be very practical for use in the schools of Texas provided the one-year course in algebra is made sufficiently inclusive.

Eighty-seven per cent of the schools answering the questionnaire favored making our high school mathematics more practical. The most frequent suggestions as to how this might be accomplished, which I merely submit to you without comment, were:

1. To use problems met by the child in actual life situations.
2. To give more practical problems.
3. To offer more courses in arithmetic and general mathematics.
4. To use the laboratory method and field trips.
5. To obtain better textbooks.
6. To correlate mathematics with other subjects in the high school.

The majority of the "permanent drops" from school occur in the eighth and ninth grades; this, we believe, is a very strong argument for placing the more "practical courses" such as general mathematics and commercial arithmetic in the early high school grades.

The per cent of high school graduates who attend institutions of higher learning, of course, varies largely with the requirements of the school and its proximity to such an institution; also, large numbers of schools report that the per cent of graduates attending colleges and universities was considerably higher in more prosperous times. However, 55 per cent of the schools which answered this questionnaire now send less than 40 per cent of their students to institutions of higher learning. This, it seems, is another argument for making our mathematics courses fit the needs of those students who will never attend a college or university. Nevertheless, we must at the same time make our mathematics courses fit the needs of those students who will attend a college or university.

Attention is called to the fact that only 11.7 per cent of the schools have a detailed course of study and only 15 per cent are now writing or plan to write detailed courses of study. A large majority of the schools, then, teach by outline only and seemingly are content to continue this practice. We believe that the Texas Curriculum Revision Movement will tend to help the teachers of mathematics to write useful, detailed courses of study.

Finally, we asked for any suggestions or criticisms which would tend to improve the mathematical situation in Texas.

The responses were many and varied and space will not permit our presenting them all to you. However, they were mostly centered about the following themes:

1. More uniformity in the state course of study with agreement among the teachers as to what should be taught and how it should be taught.
2. Continue the teaching of arithmetic along with other courses in mathematics in the high school.
3. Make the course more practical, that is, more suited to the needs of the student.
4. Require teachers of mathematics to be majors in that field.
5. Make more of the course elective, especially plane geometry.
6. Revise the entire curriculum of mathematics from elementary grades through the high school.
7. Differentiate between the students preparing for college and those preparing for trades or vocations.
8. Work for the complete coöperation of the teachers of mathematics over the entire state.

It seems then, that the teachers of mathematics are anxiously awaiting a curriculum revision program. That program is here in the form of the Texas Curriculum Revision Movement which has been functioning for more than a year and a half. The influence of this movement upon our present program will be in direct proportion to the teacher participation in it. Many teachers are already at work in this coöperative movement and many others are being enrolled in this voluntary endeavor each day. It is to be hoped that a large majority of the teachers will contribute not only a generous portion of their time and energy but many accounts of successful teaching units gleaned from their vast store of experience. It is only through teacher participation in such a coöperative plan as outlined and promulgated by the State Department of Education and the Texas State Teachers Association through the Executive Committee that a real Texas program for Texas schools can be successfully realized.

The teachers of Texas are urged to keep before them the point of view upon which this dynamic program was conceived and is being developed. It is well for each teacher to review frequently the general basis upon which this program is being built and which is so capably expressed by Dr. Fred C. Ayer in his article on the "Major Purposes and Guiding Principles in the Curriculum Revision Movement" and which is so clearly adapted to the various fields of education by the State Production Committees.

The program of the Mathematics Section was of particular importance, and its value was greatly enhanced by the willingness of the speakers to make last minute adjustments and additions to their papers in order to bring to the teachers present a full report of the work of the State Production Committee, even to the very last meeting of that committee which was held in San Antonio just before the convening of the Association.

The viewpoint of the State Committee was carefully reviewed. Teachers of mathematics were especially urged to develop as many units of teaching as possible during the remainder of the present school year and to submit as many contributions toward the development of this revision movement as possible. Realizing that certain portions of the mathematics field could not be satisfactorily adapted to the "Account of Unit of Teaching" form as suggested by the State Department of Education, the following steps were suggested as an alternative outline for reporting a unit of teaching:

I. General Information.
   1. School.
   2. County.
   3. Post Office address.
   4. Grade.
   5. Teacher.
   6. Subject.

7. Title of unit.
8. Date begun.
9. Date completed.
10. Total number of days covered by unit.
11. What college training in mathematics has the teacher had?
12. How much teaching experience in mathematics has the teacher had?

II. The setting (type of school situation in which unit was developed, type of students, previous work, etc.).

III. Initial planning or bases for unit.

IV. How unit was started and methods of securing general group interest.

V. Description of development of unit (with reference to subject matter and procedure in considerable detail, narrative or outline).

VI. Statement of expected outcomes of unit.

VII. Evaluation of unit in terms of attainment and outcomes.

VIII. Articulation with other subjects:
1. Mathematics.
2. Language Arts.
3. Science.
4. Physical and Health.
5. Fine Arts.
6. Industrial education and homemaking.
7. Social studies.
8. Other subjects.
9. Vocabulary.

IX. Leads to other units.

X. Bibliographies and materials (pupils and teacher, separate).

XI. Recommendations for:
1. Other units that may have been used.
2. Other approaches to unit.
3. Changes for the future.

As a suggested aid to the articulation of mathematics with mathematics, and of mathematics with the other "strands" involved in our curriculum, a tentative summary of the strands in mathematics was offered to the teachers for their consideration. The following suggested strands in mathematics were presented and will appear in the

"Curriculum News Bulletin" in the form of a chart for all grades with the hope that the teachers of mathematics as well as the curriculum makers will carefully consider at all times both the "vertical and horizontal" educational growth of the child:

I. Numeration.
II. Notation.
III. Fundamental Operations (addition, subtraction, multiplication, and division).
  1. Integers.
  2. Fractions.
IV. Numeration.
V. Graphs.
VI. Use of Tables.
VII. Functional Concepts.
VIII. Language Arts.
IX. Home and Vocational Arts.
X. Social Relations (including the civic).
XI. Nature and Science.
XII. Creative and Recreative Arts (individual development).

"We should never think of mathematics as a mastery of facts and skills, isolated and independent of life situations, but rather as a meaningful growth in abstract modes of thought which can be used for more thorough understanding and appreciation of life itself at the present time as well as in later experiences. We possess this growth only when we are able to grasp and understand mathematical situations and to cope with them with confidence, sureness, and quantitative intelligence. . . . Every educational field involves mathematical ideas, notions, or concepts. May the present trend in education not overlook the value of both the quantitative and social aspect of the subject of mathematics. Neither aspect will follow the other, but both must be taught separately and together. That is, a conscientious effort must be made on the part of all the teachers to present mathematics as a method of quantitative thinking, mathematics as mathematics, and as a means of social participation with distinct correlation at every step of the educational progress."

This article was made possible by the very able assistance and coöperative spirit of K. L. Carter, Austin High School, Austin, Texas; J. W. O'Banion, Assistant State Supervisor and Director of Supervision; and the following members of the State Production Committee: H. J. Ettlinger, The University of Texas, Austin, Texas; Miss Minnie Behrens, Sam Houston State Teachers College, Huntsville, Texas; W. Wingo Hamilton, Greenville, Texas; Dr. Lorena Stretch, Baylor University, Waco, Texas; Mrs. Margaret G. Savage, Beaumont, Texas; Miss Edna McCormick, Southwest Texas State Teachers College, San Marcos, Texas; Mrs. W. W. Hair, Belton, Texas.

MEANS OR AVERAGES AND THEIR USES*

By E. L. DODD
The University of Texas

§ 1. *Introduction. Common average, median, mode*

What I have to say in the earlier part of this talk is fairly well known, I am sure, to many members of the Science Club. But a mathematician should be careful to keep his airplane in the lower altitudes, at least for a while. Otherwise, he may soon be lost in the clouds. So I shall try to begin at the beginning.

Most of you received early this year a 60-page booklet from the Office of the President, entitled: "The University of Texas." Herein are presented statistics selected as suitable for consideration by the Legislature. Pages 28 and 29 are devoted to salaries; and the three forms of means or averages most frequently employed by statisticians to describe what is called central tendency are here used; viz., the common average—technically known as the arithmetic mean—the median, and the mode. On page 29, the *average* full-time salary for professors at The University of Texas in 1934–35 is set down at $3550. On page 28 appear statistics taken from Pamphlet No. 58, United States Office of Education, 1934. Here the "most common salaries" are given for "deans and professors at twenty-eight state universities, 1934–35." Among these, The University of Texas ranked sixteenth—a little below the middle—with $3,325 as the "most common" salary for professors. This is what statisticians would call the *modal* salary. Here the mode, $3,325, is less than the average or arithmetic mean, $3,550. The last figure given on page 28 is $3,434, given as the *median* salary of professors in 71 universities and colleges. By the median salary we understand the *middle* salary

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***Presented to the Science Club of The University of Texas, November 4, 1935.**

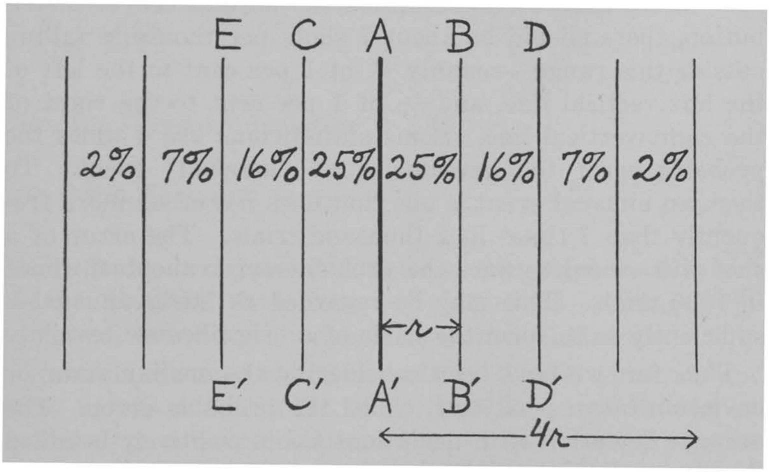
when all salaries are arranged in the order of their magnitude. Thus this median salary $3,434 as obtained from the records of 71 colleges and universities happens to be just about midway between the mode and the arithmetic mean of salaries for professors at The University of Texas in 1934–35.

Everyone would understand what an average salary is—the total budget available for salaries payable to a given group, divided by the number of the group. Perhaps not everyone is so familiar with the median or the mode. But statisticians use these forms of averages extensively—especially in unsymmetrical distributions, and distributions with considerable irregularity in the small sizes or the large sizes.

§ 2. *Measures of dispersion or scattering*

A good deal of information about a set of numbers or variates or measurements is given by specifying some central or average value, such as median, mode, or common average. But another question which naturally arises is this: Just how do the various values of the set cluster about this central value? Are they all close to the central value, or is there great scattering or dispersion? To illustrate the treatment of this problem, let us consider a simple diagram that is commonly used in connection with artillery fire. Suppose a hundred shots fired at a target large enough so that all the hundred shots are recorded as bullet holes. Draw a vertical line so that 50 shots lie to the left, and 50 shots to the right of this vertical line. This line is then the vertical center of fire. It may not pass through the bull's-eye aimed at, as there may be a constant error or deflection caused perhaps by the wind; but this line is what is called the center of fire. It turns out that the distribution of shots is usually so symmetrical about this line that this line represents at the same time average, median, and mode—although primarily it is a median. Now place to the right of this center of fire, AA', a parallel line BB' so that 25 shots or 25 per cent of the total number of shots fall between AA' and

BB'. And put a similar line CC' to the left of AA'. The distance from AA' to BB' will in practice be essentially that from AA' to CC'. This distance r is the so-called *probable error*. The name *probable error* is not very well chosen, but it has probably come to stay.



The idea is this: If the 100 bullet holes are numbered 1, 2, . . . 100, and if from 100 tickets numbered 1, 2, . . . 100, one is drawn at random, there is an equal chance that the corresponding hole will lie inside the band from CC' to BB' or outside. Or, if just one man fired the 100 shots, and he were to fire again, it would be fair to assume that he had equal chances of hitting inside or outside the band from CC' to BB'. If we call the horizontal distance from a bullet hole to the line AA', taken positively, its "error," then the so-called probable error is the *median* error. For half of the errors are less than the probable error and half are greater. It is interesting to note how shots outside the band from CC' to BB' are distributed. Let a system of parallel lines be drawn at intervals of r. Just outside a 25 per cent band there is a 16 per cent band; then a 7 per cent band; and finally a 2 per cent band. This is in accord with the so-called "normal" distribution—which artillery fire is found to follow closely. These numbers can be easily remembered; for $25 = 5 \times 5$; $16 = 4 \times 4$; and $9 = 3 \times 3$, with $7 + 2 = 9$.

The diagram indicates that practically all shots fall within a range extending $4r$ on each side of the center of fire. More accurately, assuming the normal law of distribution, there should be about 7 shots per thousand falling outside this range—roughly $\frac{1}{3}$ of 1 per cent to the left of the left vertical line, and $\frac{1}{3}$ of 1 per cent to the right of the right vertical line. Some statisticians use 4 times the probable error to characterize an "unusual" event. To them an unusual event is one that does not occur more frequently than 7 times in a thousand trials. The error of a shot will exceed 3 times the probable error about 43 times in 1000 trials. This may be regarded as fairly unusual—sufficiently so to form the basis of a "significance test."

Thus far, we have been considering the median error or deviation taken positively, called the probable error. The *average* deviation with deviations taken positively is called the *mean deviation.* This is generally somewhat larger than the probable error. Indeed, in the case of a normal distribution, as illustrated by artillery fire, it is fairly easy to see why the average deviation should exceed the probable error. If all the 25 per cent of shots to the right of BB' were to be consolidated uniformly in the band from BB' to DD', then as BB' would lie in the middle of the band extending from AA' to DD', we might well expect that the distance of BB' from AA' would be just about the average distance of all shots to the right of AA'. But with shots actually spreading out to the right of DD' we would expect an average deviation greater than r.

The probable error and the mean or average deviation—with deviations considered as positive—are both measures of dispersion or scattering. But there is another measure of dispersion, more important than either of these two, viz.: the *standard deviation.* To illustrate this, consider the following very simple numerical case.

Measurements X	Deviations from Aver. x	Absolute Deviation $\|x\|$	Squares x^2
21	-2	2	4
22	-1	1	1
26	3	3	9
3)69	3)0	3)6	3)14
Aver. = 23	0	2	$s^2 = 4.6667$
			$s = 2.16$

Here the standard deviation is 2.16.

The mean deviation is $\frac{|-2|+|-1|+|3|}{3} = 2.$

The probable error also is 2, since in the sequence of values of the absolute deviations, viz., 1, 2, 3, arranged in numerical order, the number 2 is in the middle.

Now the assumed measurements 21, 22, 26, above, do not form a "normal distribution." But if, in an analogous manner, we make calculations from a fairly large number of measurements of a physical quantity, we would find that the probable error is about ⅔ the standard deviation; and the mean deviation is about ⅘ the standard deviation. Likewise, these relations would be found valid for shots in artillery fire, and for many biological measurements—such as measurements of stature for a group of individuals of like race and sex.

As a matter of fact, it is customary to obtain first the standard deviation, and then find the probable error from:

Probable Error $= .6745 \times$ Standard Deviation. This assumes normality in the distribution. It may not give a good approximation when the distribution is known to be not normal.

The question often arises: Why not compute the average or mean deviation, and then take:

$$\text{Probable Error} = .84533 \times \text{Mean Deviation.}$$

It was pointed out by Helmert* that if a distribution is normal, there is a greater probability that a computed standard deviation will be close to the theoretic standard deviation that it represents than that the computed mean deviation should be close to its corresponding theoretic mean deviation.

Statisticians do rather tenaciously cling to the standard deviation in preference to the mean deviation. It may have been proved somewhere that the standard deviation is preferable even in cases where the distribution is obviously and grossly abnormal, but I have not seen a reference to such proof.

§ 3. *Power means*

If we let $\bar{x}$ be the arithmetic mean or common average of $X_1, X_2, \ldots, X_n$, the standard deviation s is given by $s = [(X_1 - \bar{x})^2 + (X_2 - \bar{x})^2 + \ldots + (X_n - \bar{x})^2]^{1/2}.n^{-1/2}$. It is the square-root of the average squared deviation from the arithmetic mean—briefly, it is a root-mean-square. Likewise, statisticians use the cube-root of the mean cube to measure the skewness or lack of symmetry of a distribution, and the fourth-root of the mean fourth power to measure the peakedness or bluntness of the distribution. These are all special cases of the power mean. Given a set of numbers $x_1, x_2, \ldots, x_n$, and a number p not zero; then, if p is a whole number, the power mean is

$$[(x_1^p + x_2^p + \ldots + x_n^p)]^{1/p}.$$

If p is not a whole number, the absolute or positive value of each x_i is taken before raising it to the p^{th} power.

§ 4. *Indirect averages. Geometrical interpretation*

A great many means, including these power means, are what may be called indirect averages. An averaging

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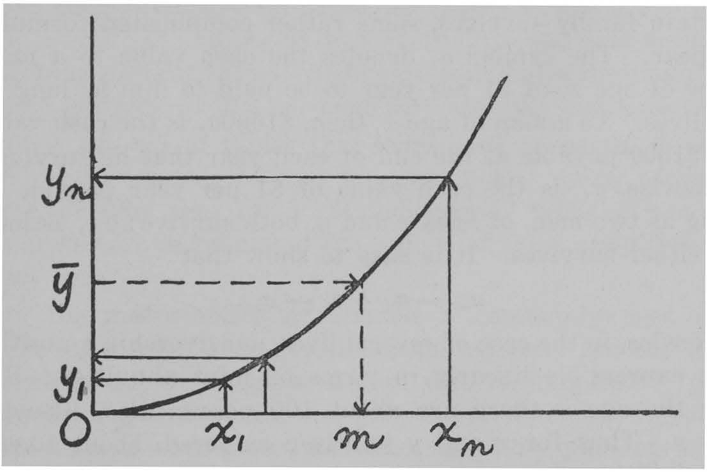
*Über die Wahrscheinlichkeit der Potenzsummen usw., Zeitschrift für Mathematik und Physik 21 (1876)—as cited in Czuber—Wahrscheinlichkeitsrechnung I (1914) p. 312.

process is performed, not on the data or given numbers, but on some specified function of the data. Let us consider the root-mean-square from the geometric standpoint. Construct the curve $y = x^2$. Here the ordinate is the square



of the abscissa. Let a number of points be laid off on the X-axis at distances of $x_1, x_2, \ldots, n_n$, to the right of the origin O. Erect ordinates to the curve $y = x^2$. From the points found thus on the curve extend horizonal lines to the Y-axis, marking the points thus found $y_1, y_2, \ldots, y_n$. Find the actual average $\bar{y}$ of these y points:

$$\bar{y} = (y_1 + y_2 + \ldots + y_n)/n.$$

From $\bar{y}$ carry a line horizontally to the curve $y = x^2$, and then vertically down to the X-axis. The point m thus obtained is the root-mean-square of the data, $x_1, x_2, \ldots, x_n$.

The curves to find the root-mean-cube and root-mean-fourth-power look much like that for root-mean-square, but rise more rapidly to the right of $x = 1$.

§ 5. *Averages in insurance and finance*

In the mathematics of finance and insurance there appear means of a different character, exponential means, involving an exponential function, instead of a power function.

The subject of life insurance, annuities, and pensions is not so very difficult as long as we are concerned with but a single life. But when we consider joint life insurance, payable upon the first death in a group—e.g., partners in a firm—or a life annuity payable as long as any one of a certain family survives, some rather complicated formulas appear. The symbol a_x denotes the cash value to a man, now of age x, of \$1 per year to be paid to him as long as he lives. To a man of age x, then, \$1000a_x is the cash value of \$1000 payable at the end of each year that he survives. Likewise, a_{xy} is the cash value of \$1 per year payable as long as two men, of ages x and y, both survive; $a_{\overline{xy}}$, as long as either survives. It is easy to show that

$$a_{\overline{xy}} = a_x + a_y - a_{xy}.$$

Likewise, in the case of several lives, survivorship annuities are expressible linearly in terms of joint annuities. But for the age x there are about 100 possibilities, likewise for y. Thus for x and y jointly considered, about 10,000 possibilities; for x, y, and z about a million possibilities; an so on,—for five lives, ten billion possibilities. The total number of values to be computed for joint life annuities would be reduced somewhat by relations like $a_{xy} = a_{yx}$; but still the number is prohibitive. It has been found possible, however, to avoid all this stupendous computation by the use of certain exponential means. Given ages x, y, and z, in general different, there is found a sort of average age w, so that

$$a_{www} = a_{xyz}.$$

With certain mortality tables, such an average age w would be hard to find. But with the American Experience Table, commonly used in the United States, with the English Healthy Males table, and with all tables that can be "Makehamized," the average w can be found from the equation:

$$c^w + c^w + c^w = c^x + c^y + c^z;$$

$$c^w = (c^x + c^y + c^z)/3;$$

$$w = [\log\ (c^x + c^y + c^z) - \log 3]/\log c.$$

This w is *not* the common average $(x+y+z)/3$. But the common average of c^x, c^y, and c^z is taken,—with log $c = .04$ approximately. As an *indirect* average, this w may be found geometrically, as just described. The curve used is the compound interest curve for money at about 9½ per cent. This has nothing to do with the interest rate used in obtaining present values. Insurance companies have usually used 3 per cent or 3½ per cent for valuation purposes.

With the aid of tables that have been prepared, it is a fairly easy matter to find an exponential mean of two, three, or four ages for a "Makehamized" table. Then on one page can be tabulated the values of joint annuities on equal ages.

In the mathematics of finance, a commonly used exponential mean is the so-called equated time for the discharge of debts. Suppose a man who is obligated to pay $100 at the end of the second, third, and fourth year, respectively, wishes to substitute the lump sum of $300 payable at the end of m years. Find m. Now the common average of 2, 3, and 4, is 3; and in practice, the equated time is taken as three years, although this is known to favor the borrower. With a stated interest rate—say 6 per cent—the exact value of m is found from

$$v^2 + v^3 + v^4 = v^m + v^m + v^m,$$
$$v^m = (v^2 + v^3 + v^4)/3,$$

with $v = 1/1.06$. This gives $m = 2.98$ years. Here, instead of averaging 2, 3, and 4, we average the second, third, and fourth power of v. The curve $y = (1.06)^{-x}$ gives the graphical representation.

One more illustration will be taken from the mathematics of finance. The so-called composite life of an industrial plant is an indirect average of the lives of its constituent parts, based upon the use of a sinking fund. The symbol $1/s_{x_1}$ signifies the annual deposit in a sinking fund which in x_1 years will yield $1. If a certain machine is to be replaced in 10 years at a cost of $5000, then $5000/s_{10}$ is the annual deposit which in 10 years would produce the required \$5000. Suppose a plant has n parts valued at $C_1, C_2, \ldots, C_n$ respectively, and estimated to last $x_1, x_2, \ldots, x_n$ years respectively. Then the composite life x of the plant is given by

$$C_1/s_{x1} + C_2/s_{x2} + \ldots + C_n/s_{xn} = C_1/s_x + C_2/s_x + \ldots + C_n/s_x = C/s_x, \text{ where } C = C_1 + C_2 + \ldots + C_n,$$

the value of the whole plant. Thus, to find $1/s_x$, we find a weighted average of the original sinking fund functions $1/s_{x_1}, 1/s_{x_2}, \ldots 1/s_{x_n}$;and then from the value of $1/s_x$, the composite life would be obtained. At 4 per cent, $1/s_{x_1}$ means

$$\frac{.04}{(1.04)^{x_1} - 1}.$$

As a whole, it is not an exponential function, but it involves the exponential function $(1.04)^{x_1}$.

§ 6. *Properties of means or averages*

From the illustrations that I have given, it should be evident that a good many different kinds of averages or means have been found useful. The question arises: Just in what way do these means differ? What different properties do they have?

Let us examine, in particular, the common average. Suppose that you have averaged a number of items, and then find that one of the items was listed incorrectly and should be increased. You would know that the increase of this item to its proper value would increase your first-formed average. We say that the common average or arithmetic mean is an increasing function of its elements. But not all means are increasing—e.g., the contraharmonic mean $(x_1^2 + x_2^2 + \ldots + x_n^2)/(x_1 + x_2 + \ldots + x_n)$, one of the oldest known means, is not increasing. The contraharmonic mean of 0 and 1 is 1; but that of ½ and 1 is only ⅚. When the first item here is increased by ½, the mean drops by ⅙.

Again, suppose that in a certain dairy the production of milk each day is listed in quarts, and that for a certain month, these daily items are averaged, and the average production found to be 1000 quarts. You would know that if the items were relisted in pints, and averaged, the result would be 2000 pints. That is, if each item in a set is multiplied by 2, the common average will be multiplied by 2. We say that the common average is a *homogeneous* function of the items. A great many averages in common use are homogeneous; but the special averages which are the most important in the mathematics of finance and insurance are not homogeneous.

Without attempting further* to describe various important properties of means, we may say that the arithmetic mean or common average is internal, unique, continuous, increasing, homogeneous, translative, symmetrical, associative, transitive. But it is *not potentive,* although the geometric mean *is potentive.* If all these ten properties were completely independent, there would be $2^{10} = 1024$ different kinds of means in accordance with a classification based upon these properties. Perhaps an analogy will help to make this clear. In a wind shield to our car, we should like to find three properties, viz.: that the wind shield be transparent, non-fragile, and inexpensive. In connection with these three properties, there are $2^3 = 8$ possibilities which may be diagrammed as follows:

T	F	E	T	F	E
		Not E	T	F	Not E
	Not F	E	T	Not F	E
		Not E	T	Not F	Not E
Not T	F	E	Not T	F	E
		Not E	Not T	F	Not E
	Not F	E	Not T	Not F	E
		Not E	Not T	Not F	Not E

*See Appendix for definitions. Also a paper by the author, "The complete independence of certain properties of means," *Annals of Mathematics,* Vol. 35 (1934), pp. 740–747.

What we usually get is a transparent, fragile, fairly inexpensive wind shield. Eventually, we shall have, perhaps, a transparent non-fragile inexpensive wind shield. Some progress has been made toward this goal.

Likewise, if a chemist has ten different requirements to satisfy, then if these are completely independent, a diagram like the foregoing will show that there are $2^{10} - 1 = 1023$ different ways of *not* completely satisfying these requirements.

§ 7. *The mean as an implicit function*

I shall not attempt to explain how the many various kinds of means or averages can be constructed to satisfy specified requirements. For I am presenting this to you mainly as introductory to the question: "In statistics just what is a mean or an average?" It turns out that not one of the ten properties that I have mentioned is common to all averages. To characterize a *statistical* average,* we must look elsewhere. Of course, it is partly a matter of judgment as to what is expedient; but it would be unfortunate to set up such a restrictive definition as to exclude functions which seem to serve the purpose of averages; just as it might be unfortunate in a definition of a blackbird to require that it be black, for if an albino should appear, we might hesitate to say that on account of its color, the bird should not be called a blackbird.

The examination of some simple cases may help us in seeing what after all is most important in the process of averaging. Suppose a man earns $4 one day and $6 the next, what are his average earnings for the two days? The figures may be placed thus:

		Solution
$ 4	$( )	$(5)
6	( )	(5)
10	10	10

*In general analysis, a mean is characterized by the single property of internality.

The problem before us is to place in each parenthesis the same number so that the new sum will be 10 just as the old sum is 10. Stated otherwise, it is to find the unknown x so that

$$4 + 6 = x + x$$

Thus $$x = (4 + 6)/2 = 5.$$

The main feature here is the replacement of each number of the data, 4, 6, by a single number 5 so that in a specified operation—here addition—the result will be unaltered. In this problem about earnings, addition is the most natural operation to consider. If a man earns $5 a day on two consecutiye days, he will be in the same financial position—so far as earnings go—as if he earned $4 the first day and $6 the second.

Suppose, now, that a man has a rectangular tract of land, 4 miles by 9 miles, and he wishes to exchange it for a square tract of similar fertility and advantages and with equal area. In this case, we have to average the data 4 and 9 by another method.

We write

$$4 \times 9 = x \times x;$$
$$x = \sqrt{4 \times 9} = 6.$$

Thus we get the geometric mean or mean proportional between 4 and 9. This is one of the earliest means to be used by mathematicians. It is especially important in certain work with populations where a constant percentage of increase is observed. In the United States from the earliest census in 1790 to the census of 1860 just before the Civil War the population increased almost 34 per cent per decade, which corresponds to 3 per cent per year. Thus, a 3 per cent compound interest table can be used very accurately to interpolate for the population between census dates. The population in 1855 must be taken as the geometric mean of the populations at 1850 and 1860—not the arithmetic mean or common average. For n variates $x_1, x_2, \ldots, x_n$, the geometric mean is

$$G = (x_1 \cdot x_2 \cdots x_n)^{1/n}.$$

Even this can be represented as an indirect average by taking logarithms.

$$\log G = (\log x_1 + \log x_2 + \ . \ . \ . \ + \log x_n)/n.$$

Thus the logarithms of the numbers are averaged, instead of the numbers themselves. This may be written in another form which is instructive:

$$\log G + \log G + \ldots + \log G = \log x_1 + \log x_2 + \ldots + \log x_n.$$

For all averages that I have considered tonight, except the median and mode, it is possible to regard the mean m of n numbers $x_1, x_2, \ldots, x_n$, as the solution of an equation

$$f(m) + f(m) + \ldots + f(m) = f(x_1) + f(x_2) + \ldots + f(x_n),$$

or $f(m) = [f(x_1) + f(x_2) + \ldots + f(x_n)]/n.$

where $f(x_1)$ signifies a given function of x_1. Such an equation gives *m implicitly.* More generally, we say that m is a mean of $x_1, x_2, \ldots, x_n$ if

$$F(m, m, \ldots, m) = F(x_1, x_2, \ldots, x_n).$$

This conception of a mean was presented by O. Chisini,* who regarded the possibility of the substitution of the single value m for a set of values $x_1, x_2, \ldots, x_n$ as the one characterizing feature of a mean. He gave an example of an *external* mean—that is, a mean not intermediate between the smallest and largest number of the set. The phrase "external mean" sounds about as peculiar at first hearing as the phrase a "white blackbird." But it indicates that from the *statistical* standpoint, internality or intermediacy for a mean is incidental. *Internality is not the main characteristic of a statistical mean.*

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*Sul concetto di media. Periodico di Matematico, Ser. 4, Vol. 9 (1929), pp. 106–116.

§ 8. *The mean as an explicit function*

Working from a different standpoint, O. Suto† had already used as *one* of three conditions to characterize the arithmetic mean, the requirement that the mean

$$m = f(x_1, x_2, \ldots, x_n)$$

should be such a function that in case all the elements should be alike—say each equal to x—we should have

$$f(x, x, \ldots, x) = x.$$

This is closely connected with the Chisini conception, and it turns out that some such condition as this is the *only condition that all statistical means satisfy.* In the case of multiple-valuedness, all that we can require is that *at least one value* of $f(x, x, \ldots, x)$ shall be x.

In the simple cases, we have only a finite number of variates to average; but the extension is immediate to the case of a countably infinite number of variates.

To get greater generality we can use the language of point-sets. Consider two sets E and H of real numbers, where E may be any set in a collection which includes individual numbers x. Then the function $f(E, H)$ is a mean, if the condition is imposed that for each x,

$$f(x, H) = x,$$

or at least one value of $f(x, H)$ is x. A somewhat more complicated function* is needed in some cases; but a discussion of this in detail would be out of place in this talk.

§ 9. *Conclusion*

In conclusion, I wish to repeat that many properties of statistical means commonly thought of as essential to means

†Law of the arithmetic mean. Tohoku Mathematical Journal, Vol. 6 (1914), pp. 79–81.

*Discussed in more detail in the author's paper, "A survey of statistical means or averages," Seventh American Scientific Congress, Mexico City, September 8–17, 1935.

or averages are after all only incidental—that is, there are means which do not have these properties. As regards the function $f(x_1, x_2, \ldots, x_n)$ as a mean of a finite number of elements, the only condition that is satisfied by every sort of average or mean is that at least one value of $f(x, x, \ldots, x)$ be x, itself.

Appendix

Definitions

A mean m of variates, $x_1, x_2, \ldots, x_n$, is:

Internal, if minimum of $x_1, \ldots, x_n \leqq m \leqq$ maximum of $x_1, x_2, \ldots, x_n$;

Increasing, if m increases when one variate (taken at pleasure) increases, the others remaining constant;

Homogeneous, if when each variate is multiplied by a constant k (taken at pleasure), the mean is multiplied by k;

Translative, if when each variate is increased by an arbitrary constant k, the mean is also increased by k;

Symmetrical, if the interchange of any two variates (at pleasure) does not alter the mean;

Associative, if the mean m remains unchanged when in any subset of its elements or variates (taken at pleasure) each variate is replaced by the mean for that subset;

Transitive, if the mean of means of sets of variates—each set containing the same number of variates—is the mean of all the variates considered individually;

Potentive, if when each element is raised to the same power t, the mean is raised to this power t.

WHY DO STUDENTS FAIL?

BY MARY E. DECHERD
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I tried an experiment the first semester of the current session to determine the reasons for the failures in some of my freshmen sections in mathematics.

In Mathematics 304, which, alas!, partakes largely of the nature of a "rehash" of high school algebra, and in both of my sections of Mathematics 307, Elementary Mathematics of Finance, it so happened that I was teaching fractions, both the fundamental operations on fractions and the solution of fractional equations. The texts emphasized clearly both (1) the *Fundamental Principle* of fractions: The value of a fraction is not changed if both numerator and denominator are multiplied or divided by the same number (not 0); and (2) the *Axiom* underlying "clearing" an equation of fractions: Equals multiplied by equals give equals.

In class we discussed all processes involved in fractions, and I had numerous exercises applying these processes handed in, for correction by my student assistant. These papers were returned to the students, who were invited to conferences if their grades were less than 9 (on the scale of 10). Wentworth was used as a reference text. The response of the class was, on the whole, very good.

After due warning, I gave a short examination on these two chapters on fractions, using two exercises from Wentworth:

1. Simplify $$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}$$

2. Solve $\dfrac{2x-3}{2x-4}-6=\dfrac{x+5}{3x-6}-\dfrac{11}{2}$

I had optimistically persuaded myself that the students realized that it was shorter, easier, and more logical to multiply both the numerator and the denominator in *1* by *abc* than to use the process of bringing both numerator and denominator to a common denominator and then inverting and multiplying. Indeed, not one among them could explain the reason for inversion, anyway. I had emphasized that reducing to a common denominator is a long, awkward procedure, to be used only as a last resort. Before the examination they had seemed to feel that the more satisfactory way to simplify a fraction is to apply the Fundamental Principle.

In the second question, I believed that the weakest among them knew that both "sides" of the equation could be "cleared" of denominators by multiplying by $6(x-2)$. I had warned them against reducing to a common denominator and "discarding" denominators, as some were prone to do.

The results did not entirely disappoint me, though far too many, after reducing the fraction in question *1* to the form

$$\frac{bc+ac+ab}{a^2c+b^2a+c^2b},$$

cancelled terms in a most distressing fashion, with a great variety of results, despite the fact that the meaning of *term* and *factor* had been carefully considered with both arithmetic and algebraic illustrations. In solving the equation, $2(2x-4)(3x-6)$ was used by some for the L.C.M. They found $x=13$ and 2, either not checking to test roots or else undaunted by the appearance of 0 in the denominators. The idea of combining 6 and 11/2 before clearing of fractions occurred to only two or three.

When I returned the corrected examination papers, I carefully worked both problems with the class, and requested everyone who had not done well on the quiz to rework each of the problems several times until he was sure that he had mastered every difficulty and clearly understood every principle involved.

I also suggested that everybody who had failed on the quiz should come to my office and be again examined on the same chapters. A number of students acted on my suggestion.

At least twice later in the semester when occasion presented itself I gave these same two problems in examinations, and each time again explained them briefly in class.

When the final examination came, I again gave these same two problems. As I am a believer in a wide choice of questions on final examinations, I could request that all students who expected an *A* or *B* on the course should not select these two much-used exercises, which I considered merely "life preservers" for the weak students. I wanted to know whether these poorer students were studying, and these problems on which they had had so much drill afforded an excellent test. The results were as follows: Of the nine students who failed in my section of Mathematics 304, five completed the first correctly, while only three were successful in finding $x = 13$ in number *2*, and two of the three failed to use the *least* common denominator. On the other hand, of those who made 60 points or more on the final only three failed on number *1*, and the same number, though not the same students, failed on number *2*. In one section of Mathematics 307, of the nine failures only one worked number *1* and three number *2*, while in the other section none of the four failures worked the first correctly and only one solved the equation.

What, then, am I to conclude from this experience? Can I not say that these students made very little effort on the course? If they lacked initiative in studying, could not any student follow the simple repeated directions given in regard to these two exercises?

Obviously the exercises were selected because they applied in rather simple fashion two processes fundamental in algebra, trigonometry, etc. This fact was also made clear to the classes. From this and many other experiences I am becoming more and more convinced that the only reason why any student fails in freshman mathematics is that he does not apply himself to his simple task.

THE BROWN UNIVERSITY PRIZE EXAMINATION

The Brown University Prize Examination for freshmen was given on Saturday, October 12, 1935. The questions were as follows:

1. Given that the volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius, find the inner and outer radii of a hollow spherical shell of thickness d and volume v.

2. Solve the equation:

$$\frac{3^{x^2}}{9^{4x-1}} = \frac{1}{243}$$

3. In a right triangle whose legs are a and b, prove that the segment of the bisector of the right angle inside the triangle has the length $\frac{ab\sqrt{2}}{a+b}$.

4. From a point A inside a circle a segment AB is drawn to the circle. Prove that the locus of the mid-point M of AB is a circle.

No prizes were awarded this year. One contestant worked question 2 correctly and one question 3; none of the others worked a single problem.