Parabolic Anderson Model on $\mathbb{R}^2$

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Contents

1 Abstract

For my thesis project we have been studying the analysis of the parabolic Anderson model in 2 spatial dimensions on the whole plane, performed by Hairer and Labbé in early 2015. This problem is a nice example as it requires renormalization to control the singularities and weighted spaces to control the divergence at infinity. After adding the necessary logarithmic counter term and posing the problem in the correct space we are then able to prove existence and uniqueness of the solution. Our main contribution is to offer a more explicit account than was previously available, and to correct some typos in the original work. This work is of importance because the parabolic Anderson model, which models a random walk driven by a random potential, can be used to study several topics such as spectral theory and some variational problems. Moreover, this analysis is of interest because it presents a particularly clean example, in that there is no need for any complicated (though more general) renormalization procedures. Rather, we use a trick from the analysis of smooth partial differential equations to identify the diverging terms and then add an appropriate counter term.

2 Background and Introduction

2.1 Stochastic PDE

In this work we study a class of problems known as stochastic partial differential equations (S)PDE. That is, partial differential equations (equations involving a function and its derivatives in multiple variables) where one of the functions in the equation is a stochastic process. Where a stochastic process is defined to be a collection of, not necessarily continuous, functions (or rather random variables) indexed by time. The difficulty is then that our solution depends on these typically highly irregular random inputs and so may not be differentiable. This creates a problem because our solution is defined using derivatives in a PDE.

To make this discussion more clear consider the class of semilinear (S)PDE, that is (S)PDE of the form:

$$\mathcal{L}u(x,t) = F(u(x,t), \xi(x,t)),$$

where $\mathcal{L}$ is a differential operator, ξ is the random input, F is a nonlinearity and where $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Assume further $\mathcal{L} = \partial_t - L$ where L is some spatial operator (involves no derivatives in t). In this case, a standard theorem in the theory of PDEs, Duhamel's formula, states that (for 0 initial condition):

$$u(x,t) = \int_0^t e^{(t-t')L} F(u,\xi)(x,t')dt' =: \mathcal{I}(F(u))$$

One natural thing to do now is insert a trial solution into the right hand side and see what happens. If we insert a trial solution $a(x,t)$ then we expect our true solution $u(x,t)$ to include terms (in a series) of the form $\mathcal{I}(F(a))$. But if the solution includes terms of the form $\mathcal{I}(F(a))$, it must contain terms of the form $\mathcal{I}(F(\mathcal{I}(F(a)))$ (and others). Thus, we must iterate this porcess over and over again infinitely many times, generating an infinite series trial solution. Herein lies the difficulty, because F depends on ξ which is very irregular, some of the terms in our expansion will be highly irregular. So irregular that they have divergences in the plane (one can think of these terms as like $\frac{1}{x^m}$ for some negative m). As such, they are not solutions to the equation at the divergent point.

In general there are two approaches to such problems. To understand large scale properties such as steady states and scaling properties one can consider a regularized equation where the random input is replaced by one which is smooth on small scales. Alternatively, one could consider small scale questions, such as the existence and uniqueness of solutions without regularizations. We will be taking the latter approach.

2.2 Introduction to the Problem

In this article, we present the 2015 work by Martin Hairer and Cyril Labbé [5] in which the authors renormalize and construct a solution to the parabolic Anderson model on $\mathbb{R}^2$:

$$\partial_t u = \Delta u + u \cdot \xi, \qquad u(0,x) = u_0(x) \tag{1}$$

where u is a function of $t \geq 0$, where $x \in \mathbb{R}^2$ and where ξ is a white noise on $\mathbb{R}^2$. There are 2 main difficulties associated with this problem: $u \cdot \xi$ is not classically well-defined (equivalent to the diverging terms described in the previous section) and, because we are working on $\mathbb{R}^2$, we need to properly weight our functions in order to prevent blow-up at infinity. That is, when solving a pde on an unbounded domain we require our solution to go to zero at infinity. As this may not be the case, we need to "weight" our functions *i.e* divide all of them by a function. Otherwise put, if we were to expand our solution into terms of a given homogeneity (where homogeneity refers to the rate at which a function goes to infinity: a homogeneity n implies the function goes to infinity like x^n), we need to remove the singularities from the negative homogeneity terms and we need to ensure that the sum of all the positive homogeneity terms does not diverge either. The first problem will be solved by renormalizing the equation. Fortunately for this presentation, we can reformulate this equation in terms of a stationary solution which will allow us to see clearly the diverging terms and so subtract them (thus, it is not necessary for this model to employ a more complex procedure such as the theory of regularity structures[3]). The second problem is dealt with by properly weighing our solution spaces.

Because of these difficulties, this work contributed in two significant ways to the general theory. Firstly, to our knowledge, it is the first incorporation of weighted Hölder spaces into the theory of (S)PDEs allowing one to characterize solutions on the whole space. This would then allow the same authors in [4] to extend this work to dimensions 1 and 3, where the authors need to renomalize more terms and so, make use of the methodology of regularity structures and moreover, incorporate weights into that same theory. The second contribution to the general theory is the stationary solution argument which is a nice way to rewrite this equation to isolate the singularity.

This work by Hairer and Labbé offers a nice example of how to solve an (S)PDE without the need to fully get into the theory of regularity structures. In this article we aim to expand this example in full detail, correct a few typos in the previous work, and present a complete account of the methodology needed for this equation: starting by characterizing white noise and presenting the necessary properties we will make use of. Then, we present the expanded Hölder spaces used to make sense of negative homogeneity terms like ξ and present a valuable convergence criterion. Then, we present several extremely useful properties of smooth functions with singularities. We next present the stationary solution argument which will allow us to renomalize the equation easily. Finally, we present the fixed point argument which brings this all together and solves the equation.

Remark 1
A key typo we appear to have found, which leads to a mismatch of some exponents in our exposition as compared to the original work is the following. In the proof of (what appears here as) proposition 3, the authors omit a factor of 2 in the exponent of 2 when applying lemma 3 (i.e Lemma 3 implies, $\mathbb{E}[L(\psi_x^n)^2] \lesssim 2^{-nd+2\kappa n}$ *and not* $2^{-nd+\kappa n}$ *). This will later manfest itself in the fixed point argument where* $\kappa \in (0, 1/4)$ *not* $(0, 1/2)$*. Otherwise the integral in (146) diverges.*

Remark 2
All of the theorems and proofs have been taken from several sources ([3][5]). A main objective in our work is to expand many of the arguments, fill in the details of the proofs and to give a very accessible account of the important work of Hairer and Labbé.

3 results

3.1 White Noise

Definition 1 (White Noise)
We denote by "white noise" a Gaussian stochastic process $\xi(t)(\omega)$ *with covariance* $\mathbb{E}\xi(s)\xi(t) = \delta(t-s)$*. That is, if we take* $\langle \cdot, \cdot \rangle$ *to be the scalar product in* $L^2(\mathbb{R})$*, this implies*

$$\mathbb{E}\langle g, \xi \rangle \langle h, \xi \rangle = \int\int g(s)h(t)\xi(s)\xi(t)dsdt = \int\int g(s)h(t)\delta(t-s)dsdt = \langle g, h \rangle \tag{2}$$

To give meaning to this definition we begin by proving the statement "white noise is the derivative of Brownian motion". Integrals of the form $\int g(s)\xi(s)ds$, for $g \in L^2(\mathbb{R})$ are well-defined random variables, although proving this requires a discussion of Gaussian measures beyond the scope of this review, for the interested reader we suggest [M. Hairer, lecture notes: Introduction to Stochastic PDEs]. Hence, if we take $g(s) = \mathbb{1}_{[0,t]}(s)$, then $B(t) = \int_0^t \xi ds$ is in $L^2(\mathbb{R})$. Moreover, we will show it is a Brownian motion on some space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2 (Brownian Motion)
A Brownian motion is a stochastic process $W = \{W(t)\}_{t\geq 0}$*, that is, a collection of random variables* $W(t)$ *defined on some probability space* $(\Omega, \mathcal{F}, P)$*, such that:*

1. *The function* $W(t)(\omega)$ *is almost surely continuous in* t *for every* $\omega \in \Omega$

2. *W has stationary and independent increments, that is, for any positive n and any* $0 = t_0 < t_1 < ... < t_n$*, the random variables* $W(t_i) - W(t_{i-1})$*,* $i \in \{1, ..., n\}$ *are mutually independent, and*

3. $W(s+t) - W(s)$ *has the same distribution as* $W(t)$ *for any* $s, t > 0$

The first property is clear for our process $B(t)$. To show the increments are independent, consider for all $A, C \in \mathcal{B}(\mathbb{R})$ (the Borel σ-algebra on $\mathbb{R}$) and all $0 \leq i < j \leq n$

$$\mathbb{P}(\{B(t_i) - B(t_{i-1}) \in A\} \cap \{B(t_j) - B(t_{j-1}) \in C\}) = \tag{3}$$

$$\mathbb{P}(\{\int_{t_{i-1}}^{t_i} \xi ds \in A\} \cap \{\int_{t_{j-1}}^{t_j} \xi ds \in C\}) \tag{4}$$

These 2 integrals are evaluating ξ over disjoint intervals and because the covariance is $\delta(t-s)$ these integrals are independent. Thus,

$$\mathbb{P}(\{\int_{t_{i-1}}^{t_i} \xi ds \in A\} \cap \{\int_{t_{j-1}}^{t_j} \xi ds \in C\}) = \mathbb{P}(\{\int_{t_{i-1}}^{t_i} \xi ds \in A\})\mathbb{P}(\{\int_{t_{j-1}}^{t_j} \xi ds \in C\}) \tag{5}$$

and so the random variables $B(t_i) - B(t_{i-1})$ are independent.

To show that our stochastic process has stationary increments we need to show

$$\mu_{B(t+s)-B(s)} = \mu_{B(t)} \tag{6}$$

$$\mathbb{P}[(B(t+s) - B(s))^{-1}(C)] = \mathbb{P}[B(t)^{-1}(C)] \tag{7}$$

$$\mathbb{P}[(\int_s^{s+t} \xi dt')^{-1}(C)] = \mathbb{P}[(\int_0^t \xi dt')^{-1}(C)] \tag{8}$$

for every $C \in \mathcal{B}(\mathbb{R})$. Because ξ is equally distributed in time the last probability shows the equality. So $B(t)$ is a Brownian motion, therefore, the statement "white noise is the derivative of Brownian motion" is justified.

At this stage we will prove a crucial proposition in the theory of (S)PDEs, the equivalence of Gaussian moments. To do so we will need the following theorem of Fernique, a proof of which can be found in [M. Hairer, lecture notes: Introduction to Stochastic PDEs].

Theorem 1 (Fernique 1970)
Let μ *be any probability measure on a separable Banach space* $\mathcal{B}$ *such that* $\mu \otimes \mu$ *is invarient under rotation by* $\pi/4$ *. That is for* $R_\psi : \mathcal{B}^2 \to \mathcal{B}^2$ *defined by*

$$R_\psi(x, y) = (x \sin\psi + y \cos\psi, x \cos\psi - y \sin\psi) \tag{9}$$

we require $R_\psi^*(\mu \otimes \mu) = \mu \otimes \mu$ *for* $\psi = \pi/4$*. Then, there exists an* $\alpha > 0$ *such that:*

$$\int_{\mathcal{B}} exp(\alpha||x||^2)\mu(dx) < \infty. \tag{10}$$

In particular, for $\mu(||x|| \leq \tau) \geq 3/4$*:*

$$\int exp\left(\frac{\alpha||x||^2}{\tau^2}\right)\mu(dx) \leq e^\alpha + 2\alpha \int_1^\infty te^{-\alpha t^2} dt \tag{11}$$

Proposition 1 (Equivalence of Gaussian Moments)
There exist universal constants α, $K > 0$ with the following properties. Let μ be a Gaussian measure on a separable Banach space $\mathcal{B}$ and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be any measurable function such that $f(x) \leq C_f exp(\alpha x^2)$ for every $x \geq 0$. Define furthermore the first moment of μ by $M = \int_{\mathcal{B}} ||x||\mu(dx)$. Then, one has the bound $\int_{\mathcal{B}} f(\frac{||x||}{M})\mu(dx) \leq KC_f$.

In particular, the higher moments of μ are bounded by $\int_{\mathcal{B}} ||x||^{2n}\mu(dx) \leq n!K\alpha^{-n}M^{2n}$

Proof. Fernique's theorem states that, for $\mu(||x|| \leq \tau) \geq 3/4$, for any $\beta > 0$:

$$\int \exp\left(\frac{\beta||x||^2}{\tau^2}\right)\mu(dx) \leq e^{\beta} + 2\beta\int_1^{\infty} te^{-\beta t^2}dt \tag{12}$$

Note that the right hand side is independent of τ.

Chebyshev's inequality states that:

$$\mu(|x - M| \geq k) \leq \frac{\sigma^2}{k^2} \tag{13}$$

For some $k > 0$ and where σ is the variance of x. The variance of x must be less than M because $x > 0$. Thus, we have:

$$\begin{aligned} \mu(||x|| \leq \tau) &= 1 - \mu(|x - M| \geq \tau - M) && (14)\\ &\geq 1 - \frac{\sigma^2}{(\tau - M)^2} && (15) \end{aligned}$$

Taking for instance $\tau = 4M$ gives

$$\mu(||x|| \leq \tau) \geq 1 - \frac{\sigma^2}{9M^2} \geq \frac{3}{4}. \tag{16}$$

Thus, in equation (10) we may take $\tau = 4M$. And so, taking $\beta = 16\alpha$ in equation (11) gives:

$$\begin{aligned} \int_{\mathcal{B}} f\left(\frac{||x||}{M}\right)\mu(dx) &\leq C_f\int_{\mathcal{B}} \exp\left(\frac{\alpha||x||^2}{M^2}\right)\mu(dx) && (17)\\ &\leq C_f\int_{\mathcal{B}} \exp\left(\frac{\beta||x||^2}{(4M)^2}\right)\mu(dx) && (18)\\ &\leq C_f\left(e^{\beta} + \frac{1}{2}\int_1^{\infty} te^{-\beta t^2}dt\right) && (19)\\ &\leq C_fK && (20) \end{aligned}$$

In order to prove the last inequality we use the fact:

$$e^{\alpha x^2} \geq \frac{\alpha^n x^{2n}}{n!}, \tag{21}$$

(evident from the Taylor expansion of $e^{\alpha x^2}$). Therefore,

$$\begin{aligned} \int_{\mathcal{B}} ||x||^{2n}M^{-2n}\mu(dx) &\leq \int_{\mathcal{B}} e^{\alpha x^2}n!\alpha^{-n}\mu(dx) && (22)\\ \int_{\mathcal{B}} ||x||^{2n}M^{2n}\mu(dx) &\leq n!K\alpha^{-n}M^{2n} && (23) \end{aligned}$$

□

3.2 Weighted Hölder Spaces

At this stage, we will introduce a collection of weighted Hölder spaces, in which negative Hölder regularity spaces will define distributions, separated by degrees of regularity. From there we will be able to formulate a fixed point arguement associated to the (PAM). Throughout this subsection we work in $\mathbb{R}^d$ for $d \in \mathbb{N}$, later on we will apply the results to the case $d = 2$.

Definition 3 (Weights)
A function $w : \mathbb{R}^d \to (0, \infty)$ is a weight if there exists a positive constant $C > 0$ such that:

$$C^{-1} \leq \sup_{|x-y| \leq 1} \frac{w(x)}{w(y)} \leq C \tag{24}$$

To construct the necessary Hölder spaces we begin with the usual definition:

Definition 4 ($\mathcal{C}^\alpha_w(\mathbb{R}^d)$ - $\alpha > 0$)
For $\alpha \in (0, 1)$, let $\mathcal{C}^\alpha_w(\mathbb{R}^d)$ be the space of functions $f : \mathbb{R}^d \to \mathbb{R}$ such that:

$$||f||_{\alpha,w} := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{w(x)} + \sup_{|x-y| \leq 1} \frac{|f(x) - f(y)|}{w(x)|x-y|^\alpha} < \infty \tag{25}$$

For $\alpha > 1$, define $\mathcal{C}^\alpha_w$ recursively as the space of differentiable functions f such that:

$$||f||_{\alpha,w} := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{w(x)} + \sum_i^d ||D_{x_i} f||_{\alpha-1,w} < \infty \tag{26}$$

To define the space for negative α we need a space of test functions. For $r \in \mathbb{N}$, let $\mathcal{B}^r_1$ be the space of smooth, compactly supported in a unit ball, functions on $\mathbb{R}^d$ whose $\mathcal{C}^r$ (usual unweighted Hölder) norm is less than 1. Moreover, let $\eta^\lambda_x : y \mapsto \lambda^{-d} \eta \left(\frac{y-x}{\lambda} \right)$.

Definition 5 ($\mathcal{C}^\alpha_w(\mathbb{R}^d)$ - $\alpha < 0$)
For every $\alpha < 0$, set $r := -\lfloor \alpha \rfloor$ and define $\mathcal{C}^\alpha_w(\mathbb{R}^d)$ as the space of distributions f on $\mathbb{R}^d$ such that:

$$||f||_{\alpha,w} := \sup_{x \in \mathbb{R}^d} \sup_{\eta \in \mathcal{B}^r_1} \sup_{\lambda \in (0,1]} \frac{|f(\eta^\lambda_x)|}{w(x)\lambda^\alpha} < \infty \tag{27}$$

Moreover we can extend the classical multiplication map of functions to the space $\mathcal{C}^\alpha_w$ as follows:

Theorem 2 (Multiplication of $\mathcal{C}^\alpha_w$ spaces)
Let $f \in \mathcal{C}^\alpha_{w_f}$ and $g \in \mathcal{C}^\alpha_{w_g}$ with $\alpha < 0$, $\beta > 0$ and $\alpha + \beta > 0$. Then there exists a continuous bilinear multiplication map $(f, g) \mapsto f \cdot g$ from $\mathcal{C}^\alpha_{w_f} \times \mathcal{C}^\beta_{w_g}$ into $\mathcal{C}^\alpha_{w_f w_g}$ that extends the classical multiplication map.

Proof. The proof of this lemma is not integral to this presentation; to see how the theorem is true for non-weighted space we direct the reader here: [[2] - Theorem 2.52]. To see how that theorem can be extended to the weighted spaces see [[5] - Theorem 2.6]. □

To state a useful convergence criteria we need a countable number of test functions to test against. Hence, for any $\psi \in \mathcal{C}^r$, set:

$$\psi_x^n(y) := 2^{\frac{nd}{2}}\psi((y_1 - x_1)2^n, ..., (y_d - x_d)2^n).\ x, y \in \mathbb{R}^d,\ n \geq 0. \tag{28}$$

Moreover define $\Lambda_n := \{(2^{-n}k_1, ..., 2^{-n}k_d) : k_i \in \mathbb{Z},\ \forall\, 1 \leq i \leq d\}$.

Proposition 2 (Convergence in $\mathcal{C}_w^\alpha(\mathbb{R}^d)$)
Let $\alpha < 0$ and $r > |\alpha|$. There exits a finite set Ψ of compactly supported functions $\psi, \varphi \in \mathcal{C}^r$ such that $\{\varphi_x^0, x \in \Lambda_0\} \cup \{\psi_x^n, n \geq 0, x \in \Lambda_n, \psi \in \Psi\}$ forms an orthonormal basis of $\mathbb{R}^d$, such that for every distribution ξ on $\mathbb{R}^d$, the following equivalence holds: $\xi \in \mathcal{C}_w^\alpha$ if and only if ξ belongs to the dual of $\mathcal{C}^r$ and

$$\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\frac{|\langle \xi, \psi_x^n)\rangle|}{w(x)2^{-\frac{nd}{2}-n\alpha}} + \sup_{x\in\Lambda_0}\frac{|\langle \xi, \varphi_x^0\rangle|}{w(x)} < \infty \tag{29}$$

Proof. The proof relies on several theorems from wavelet analysis by Ingrid Daubechies and Ives Meyer in 1988 and 1992 (citations from Prop 3.20 Hai14). A proof of the theorem for non-weighted spaces can be found here [[3] Prop 3.20]. Because all the arguements needed for the proof are localized on the support of the test functions the fact that $\frac{w(x)}{w(y)}$ is bounded uniformy for all $x, y \in \mathbb{R}$ such that $|x - y| \leq 1$ ensures that the proof applies.

□

Note that if ξ satisfies (29) and is a linear transformation on the linear span of φ_x^0 and ψ_x^n, then the bound (29) will hold for linear combinations of φ_x^0 and ψ_x^n. Therefore ξ belongs to $\mathcal{C}_w^\alpha$.

To characterize white noise, we define two families of weight functions indexed by $a, l \in \mathbb{R}$:

$$p_a(x) := (1 + |x|)^a \tag{30}$$

$$e_l(x) := \exp(l(1 + |x|)) \tag{31}$$

Take ξ to be a white noise on $\mathbb{R}^2$, let ϱ be a smooth, compactly supported, even function on $\mathbb{R}^2$ which integrates to 1, and let

$$\varrho_\epsilon(x) := \epsilon^{-2}\varrho(\frac{x}{\epsilon}) \text{ for all } x \in \mathbb{R}^2, \tag{32}$$

then we can define the mollified noise to be $\xi_\epsilon = \xi * \varrho_\epsilon$. At this stage we are ready to classify the regularity properties of white noise, ξ:

Lemma 1 (Regularity of ξ)
For any $a, \epsilon, \kappa > 0$, ξ_ϵ belongs almost surely to $\mathcal{C}_{p_a}^{-1-\kappa}(\mathbb{R}^2)$ and, as $\epsilon \to 0$, ξ_ϵ converges in probability to ξ in $\mathcal{C}_{p_a}^{-1-\kappa}$.

Proof. Working in dimension 2, by Proposition 2, to show the first statement, it suffices to show:

$$\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\frac{|\langle \xi, \psi_x^n)\rangle|}{p_a(x)2^{-\frac{nd}{2}-n\alpha}} \lesssim 1 \tag{33}$$

$$\sup_{x\in\Lambda_0}\frac{|\langle \xi, \varphi_x^0\rangle|}{p_a(x)} \lesssim 1 \tag{34}$$

To achieve the first bound take p to be an integer greater than 0 and write:

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\left(\frac{|\langle\xi,\psi_x^n\rangle|}{2^{-n(1+\alpha)}p_a(x)}\right)^{2p}\right]\lesssim\sum_{n\geq 0}\sum_{\psi\in\Psi}\sum_{x\in\Lambda_n}\frac{2^{2pn(1+\alpha)}\mathbb{E}[\langle\xi,\psi_x^n\rangle^{2p}]}{p_a(x)} \quad (35)$$

At this stage we'd like to use the equivalence of Gaussian moments to conclude that we can take the $2p$ out of the expectation, but as it is written in proposition 1 we do not have a bound uniform in p. Because we have convolved our Gaussian random variable with a compactly supported function, equation (21) can be made stronger: $e^{\alpha x^2}\geq Cx^{2n}$ where the constant depends on the domain of x and α only. Therefore, we can bound the higher moments of $\langle\xi,\psi_x^n\rangle$ uniformly in p: $\mathbb{E}[\langle\xi,\psi_x^n\rangle^{2p}]\leq K[\mathbb{E}\langle\xi,\psi_x^n\rangle]^{2p}$ where K is independent of p. Using this, the fact that the lattice Λ_n has less than 2^{2n} points in the unit ball (the number of points in the enclosing square) and the fact that there are finitly many $\psi\in\Psi$, we get:

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\left(\frac{|\langle\xi,\psi_x^n\rangle|}{2^{-n(1+\alpha)}p_a(x)}\right)^{2p}\right]\lesssim\sum_{x\in\mathbb{Z}}\sum_{n\geq 0}\frac{2^{2pn(1+\alpha)+2n}(\mathbb{E}[\langle\xi,\psi_x^n\rangle^2])^p}{p_a(x)^{2p}}. \quad (36)$$

And so, because the L^2 norm of ψ_x^n is 1, we can take p large enough for the sum to converge, and because our bound is uniform in p this shows we have convergence in L^1 and so the bound (33) holds. The same proof can be used when ξ is tested against φ_x^0. Thus, ξ belongs to $\mathcal{C}_{p_a}^{\alpha}$ for evvery $\alpha<-1$ (or rather $\langle\xi,\cdot\rangle$ does).

Turning now to $||\xi_\epsilon-\xi||_{\alpha,p_a}$ the proof is identical except for the L^2 moment of $\langle\xi,\psi_x^n\rangle$ is replaced with

$$\mathbb{E}\langle\xi-\xi_\epsilon,\psi_x^n\rangle^2=||\psi_0^n-\varrho_\epsilon*\psi_0^n||_{L^2}^2\lesssim 1\wedge(\epsilon^2 2^{2n}) \quad (37)$$

(where the $\wedge$ denote the "meet" or infimum of the elements). Hence, for p large enough we have:

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\left(\frac{|\langle\xi_\epsilon-\xi,\psi_x^n\rangle|}{2^{-n(1+\alpha)}p_a(x)}\right)^{2p}\right]\lesssim\sum_{x\in\mathbb{Z}}\sum_{n\geq 0}\frac{2^{2pn(1+\alpha)+2n}(1\wedge\epsilon^{2p}2^{2np})}{p_a(x)^{2p}}. \quad (38)$$

Treating each case of the supremum seperately and taking p as large as necessary gives:

$$\epsilon^{2p}\sum_{x\in\mathbb{Z}}\sum_{n\geq 0}\frac{2^{2pn(2+\alpha)+2n}}{p_a(x)^{2p}}\lesssim\epsilon^{2p}\sum_{n\geq 0}2^{2pn(2+\alpha)+2n} \quad (39)$$

$$\lesssim\frac{\epsilon^{2p}}{1-2^{2p(\alpha+2)+1}} \quad (40)$$

which, for an appropriate choice of p and ϵ, can be written

$$\frac{\epsilon^{2p}}{1-2^{2p(\alpha+2)+1}}\lesssim\epsilon^{2p}\,|\log_2\epsilon| \quad (41)$$

For the other case,

$$\sum_{x\in\mathbb{Z}}\sum_{n\geq 0}\frac{2^{2pn(1+\alpha)+2n}}{p_a(x)^{2p}} \lesssim \sum_{n\geq 0} 2^{2pn(1+\alpha)+2n} \tag{42}$$

$$\lesssim \frac{1}{1-2^{2p(\alpha+1)+1}} \tag{43}$$

$$\lesssim \epsilon^{-(1+2p(\alpha+1))}. \tag{44}$$

Therefore,

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{\psi\in\Psi}\sup_{x\in\Lambda_n}\left(\frac{|\langle\xi_\epsilon-\xi,\psi_x^n\rangle|}{2^{-n(1+\alpha)}p_a(x)}\right)^{2p}\right] \lesssim \epsilon^{-(1+2p(\alpha+1))} \vee \epsilon^{2p}\,|\log_2\epsilon| \tag{45}$$

As such, $\mathbb{E}||\xi_\epsilon-\xi||_{\alpha,p_a}\to 0$ as $\epsilon\to 0$. □

We close this subsection with a lemma we will make use of later on. Let $P_t(x) := (2\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$ be the heat kernel in dimension d. Let $P_t * f$ denote spacial convolution only. Then:

Lemma 2
For every $\beta\geq\alpha$ and every $f\in\mathcal{C}^\alpha_{e_l}$, we have

$$||P_t * f||_{\beta,e_l} \lesssim t^{-\frac{\beta-\alpha}{2}}||f||_{\alpha,e_l}, \tag{46}$$

uniformly over all l in a compact set of $\mathbb{R}$ and all t in a compact set of $[0,\infty)$.

Proof. To see this decompose $P_t(x) = P_+(x,t) + P_-(x,t)$ into 2 components: P_+, which is supported in the unit ball around 0 and P_-, which is smooth, i.e we isolate the singularity at 0 in P_+. Regarding P_-, we can take $t>0$:

$$\sup_{x,\eta,\lambda}\frac{|P_- * f(\eta_x^\lambda)|}{e_l(x)\lambda^\beta} \lesssim \sup_{x,\eta,\lambda}\frac{|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2\setminus B_1(0)} t^{-d/2}e^{-\frac{|z|^2}{4t}}f(y)\lambda^{-d}\eta(\frac{x-z-y}{\lambda})dydz|}{e_l(x)\lambda^\beta} \tag{47}$$

$$\lesssim \sup_{x,\eta,\lambda}\frac{|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}}f(y)\lambda^{-d}\eta(\frac{x-zt^{\frac{1}{2}}-y}{\lambda})dydz|}{e_l(x)\lambda^\beta} \tag{48}$$

$$\lesssim \sup_{x,\eta,\lambda} t^{\frac{d}{2}}\frac{|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}}f(yt^{\frac{1}{2}})\lambda^{-d}\eta(\frac{t^{\frac{1}{2}}(x-z-y)}{\lambda})dydz|}{e_l(xt^{\frac{1}{2}})\lambda^\beta} \tag{49}$$

$$\lesssim \sup_{x,\eta,\lambda}\frac{t^{-\frac{\beta}{2}}|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(yt^{\frac{1}{2}})\lambda^{-d}\eta(\frac{(x-z-y)}{\lambda})dydz|}{e_l(xt^{\frac{1}{2}})\lambda^\beta} \tag{50}$$

$$\lesssim \sup_{x,\eta,\lambda}\frac{t^{-\frac{\beta}{2}+\frac{\alpha}{2}}|\langle f,\eta_x^\lambda\rangle|}{e_l(xt^{\frac{1}{2}})\lambda^\beta} \tag{51}$$

$$\lesssim t^{-\frac{\beta-\alpha}{2}}||f||_{\alpha,e_l}, \tag{52}$$

Addressing P_+, set $P_+ = \sum_{n\in\mathbb{N}} P_n$, where P_n is a smooth function supported in the annulus: $\{(t,x) : 2^{-n-1} \leq |t|^{\frac{1}{2}} + |x| \leq 2^{-n+1}\}$. Therefore, $P_n(t,x) = 2^{dn}P_0(2^{2n}t, 2^n x)$. Then we have the following inequalities:

$$|\langle f,\eta_x^\lambda(\cdot - y)\rangle| \lesssim \lambda^\alpha e_l(x+y) \tag{53}$$

for $f \in \mathcal{C}^{\alpha}_{e_l}$, and

$$|\langle f, D^k_x P_n(t, \cdot - y)\rangle| \lesssim |\int f(z) D^k_x P_n(t, z-y) dz| \lesssim e_l(y) 2^{-n\alpha + n|k|} \tag{54}$$

uniformly for all $\eta \in \mathcal{B}^r_1$, $x, y \in \mathbb{R}^d$, $t > 0$, $n \geq 0$ and $k \in \mathbb{N}^2$. Notice that $P_n(t, \cdot)$ vanishes at $|t|^{\frac{1}{2}} \geq 2^{-n+1}$, or $n \geq 1 - \frac{1}{2}\log_2 t$. Hence:

$$|\langle P_+(t, \cdot) * f, \eta^\lambda_x\rangle| \lesssim e_l(x)(\lambda^\alpha \wedge t^{\alpha/2}) \tag{55}$$

$$|\langle f, D^k_x P_+(t, \cdot - x)\rangle| \lesssim e_l(x) t^{\frac{\alpha - |k|}{2}}. \tag{56}$$

If $\beta < 0$, (57) shows that the statement is true for P_+. For $\beta > 0$, (58) ensures that the sum in definition 4 converges appropriately.

□

3.3 Properties of Smooth Functions with Singularities at the Origin

Definition 6 (Order of Smooth Functions Away From the Origin)
Let $K : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ be a smooth function. K is of order ζ if, for all multi-indices k, there exists a constant $C > 0$ such that $|D^k K(x)| \leq C||x||^{\zeta - |k|}$ holds for all x such that $||x|| < 1$.

For all $m \geq 0$ we write,

$$|||K|||_{\zeta;m} := \sup_{|k|<m} \sup_{x \in \mathbb{R}^d} ||x||^{|k|-\zeta} |D^k K(x)|. \tag{57}$$

With that we can equivalantly say K is of order ζ if $|||K|||_{\zeta;m} < \infty$ for all $m \in \mathbb{N}$.

Note that if K is of order ζ, K is of order $\overline{\zeta}$ for all $\overline{\zeta} < \zeta$. Another very powerful set of tools in theory of (S)PDEs are the following properties of smooth functions away from the origin:

Theorem 3 (Properties of $K : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$)
For any $m \in \mathbb{N}$:

1. *If K_1 and K_2 are of order ζ_1 and ζ_2 respectively, then $K_1 K_2$ is of order $\zeta = \zeta_1 + \zeta_2$ and the bound:*

$$|||K_1 K_2|||_{\zeta;m} \leq C |||K_1|||_{\zeta_1;m} |||K_2|||_{\zeta_2;m} \tag{58}$$

   holds for some $C > 0$.

2. *If we assume $\zeta_1, \zeta_2 > -d$ and set $\zeta = \zeta_1 + \zeta_2 + d$. If $\zeta < 0$ then $K_1 * K_2$ is of order ζ and the bound:*

$$|||K_1 * K_2|||_{\zeta;m} \leq C |||K_1|||_{\zeta_1;m} |||K_2|||_{\zeta_2;m} \tag{59}$$

   holds. And for $\zeta > 0$, such that $\zeta \notin \mathbb{N}$, if K_1 and K_2 are compactly supported, then the function:

$$K(x) = (K_1 * K_2)(x) - \sum_{|k|<\zeta} \frac{x^k}{k!} D^k_x (K_1 * K_2)(0) \tag{60}$$

is of order ζ and a similar bound to (60) holds with the constant depending on the supports of K_1 and K_2.

3. *Lastly, for ϱ_ϵ as defined earlier, define $K_\epsilon = K * \varrho_\epsilon$. In that case, if K is of order $\zeta \in (-d, 0)$ then for all $m \in \mathbb{N}$, there exists a constant $C > 0$ such that $|||K_\epsilon|||_{\zeta,m} \leq C|||K|||_{\zeta,m}$ uniformly over all $\epsilon \in (0, 1]$. Moreover, for all $\overline{\zeta} \in [\zeta - 1, \zeta)$, there exists a constant $C > 0$ such that:*

$$|||K - K_\epsilon|||_{\overline{\zeta};m} \leq C\epsilon^{\zeta-\overline{\zeta}}|||K|||_{\zeta;m} \tag{61}$$

holds.

Proof. To prove 1. simply apply the general Leibnitz rule:

$$|||K_1K_2|||_{\zeta;m} := \sup_{|k|<m} \sup_{x\in\mathbb{R}^d} ||x||^{|k|-\zeta}|D^k(K_1K_2)(x)| \tag{62}$$

$$\leq C \left(\sup_{|k|<m} \sup_{x\in\mathbb{R}^d} ||x||^{|k|-\zeta_1}|D^k(K_1)(x)| \right) \left(\sup_{|j|<m} \sup_{y\in\mathbb{R}^d} ||y||^{|k|-\zeta_2}|D^j(K_2)(y)| \right) \tag{63}$$

$$\leq C|||K_1|||_{\zeta_1;m}|||K_2|||_{\zeta_2;m} \tag{64}$$

To prove statement 2. requires more work. First for every $x \neq 0$ and k such that $\overline{\zeta} < |k|$ we assume:

$$|D^k(K_1 * K_2)(x)| \lesssim ||x||^{\overline{\zeta}-|k|_s}|||K_1|||_{\zeta_1;|k|}|||K_2|||_{\zeta_2;|k|} \tag{65}$$

Inserting this into the definition of the $|||\cdot|||_{\zeta;m}$, because $\zeta < 0$, this proves the statement for $\overline{\zeta} < |k|$.

To prove the statement concerning $K(x)$ (as defined in (52)), note that $D^kK = D^k(K_1*K_2)$ for all $|k| > \overline{\zeta}$. Hence, by our assumption, these derivatives are appropriately bounded. It remains to show that there exists a set of reals, which we call $D^k(K_1 * K_2)(0)$ such that the same sort of bound applies for $\overline{\zeta} \geq |k|$.

To do this consider a set of multi-indices $\mathcal{A}_{\overline{\zeta}} = \{k : |k| < \overline{\zeta}\}$, we enumerate these indices in decreasing order: $\mathcal{A}_{\overline{\zeta}} = \{k_0, k_1, ..., k_M\}$ where, $|k_i| \geq |k_j|$ for $i < j$. Then, fix $K^{(0)}(x) := (K_1 * K_2)(x)$. For a given $n > 0$, assume we have the bound:

$$|D^{k_n+e_i}K^{(n)}(x)| \lesssim ||x||^{\overline{\zeta}-|k_n|-1}, \tag{66}$$

(which is true for $n = 0$ by assumption (67)). This implies that one can find constants C_n such that:

$$|D^{k_n}K^{(n)}(x) - C_n| \lesssim ||x||^{\overline{\zeta}-|k_n|}. \tag{67}$$

Now, set $K^{(n+1)}(x) = K^{(n)}(x) - C_n\frac{x^{k_n}}{k_n!}$, and set $K(x) = K^{(M)}(x)$. As such,

$$|||K|||_{\overline{\zeta};m} = \sup_{|k|\leq m} \sup_{x\in\mathbb{R}^d} ||x||^{|k|-\zeta}|D^kK^{(M)}(x)| \tag{68}$$

$$\lesssim \sup_{x\in\mathbb{R}^d} ||x||^{|k_M|-\zeta}|D^{k_M}K^{(0)}(x) - C_n\frac{x^{k_0}}{k_0!}| \tag{69}$$

$$\lesssim |||K_1|||_{\zeta_1;m}|||K_2|||_{\zeta_2;m}. \tag{70}$$

All that remains is to prove that assumption (67) applies. To do so, let $\varphi : \mathbb{R}^d \to [0,1]$ be a smooth function such that $\varphi(x) = 0$ for all $||x|| \geq 1$ and $\varphi(x) = 1$ for $||x|| \leq \frac{1}{2}$. For $r > 0$, set $\varphi_r(x) = \varphi(r^{-1}x_1, r^{-1}x_2)$. Assume without loss of generality, that for each $i = 1, 2$, $|||K_i|||_{\zeta_i;|k|} = 1$. Then we have:

$$
\begin{aligned}
(K_1 * K_2)(x) &= \int_{\mathbb{R}^d} \varphi_r(y) K_1(x-y) K_2(y) dy && (71)\\
&+ \int_{\mathbb{R}^d} \varphi_r(x-y) K_1(x-y) K_2(y) dy && (72)\\
&+ \int_{\mathbb{R}^d} (1 - \varphi_r(y) - \varphi_r(x-y)) K_1(x-y) K_2(y) dy. && (73)
\end{aligned}
$$

Changing variables in the second term gives,

$$
\begin{aligned}
(K_1 * K_2)(x) &= \int_{\mathbb{R}^d} \varphi_r(y) K_1(x-y) K_2(y) dy && (74)\\
&+ \int_{\mathbb{R}^d} \varphi_r(y) K_1(y) K_2(x-y) dy && (75)\\
&+ \int_{\mathbb{R}^d} (1 - \varphi_r(y) - \varphi_r(x-y)) K_1(x-y) K_2(y) dy. && (76)
\end{aligned}
$$

Applying derivatives in x to both sides gives:

$$
\begin{aligned}
& D^k(K_1 * K_2)(x) && (77)\\
= & \int_{\mathbb{R}^d} \varphi_r(y) D^k(K_1(x-y)) K_2(y) dy && (78)\\
+ & \int_{\mathbb{R}^d} \varphi_r(y) K_1(y) D^k(K_2(x-y)) dy && (79)\\
+ & \int_{\mathbb{R}^d} (1 - \varphi_r(y) - \varphi_r(x-y)) D^k(K_1(x-y)) K_2(y) dy && (80)\\
- & \sum_{l<k} \frac{k!}{l!(k-l)!} \int_{\mathbb{R}^d} D^l \varphi_r(x-y) D^{k-l}(K_1(x-y)) K_2(y) dy && (81)
\end{aligned}
$$

Of course, the above equality holds for any $r > 0$ so set $r \leq \frac{||x||}{2}$ and consider the terms seperately. Therefore, the integrand in the first term is supported in $\{y : ||y|| \leq \frac{||x||}{2}\}$. Hence,

$$
\begin{aligned}
|D^k K_1(x-y)| &\lesssim ||x||^{\zeta_1 - |k|} && (82)\\
|K_2(y)| &\leq ||y||^{\zeta_2} && (83)
\end{aligned}
$$

By assumption, $\zeta > -2 = -|s|$, so we have:

$$
\int_{||y|| \leq r} ||y||^{\zeta} dy \lesssim r^{2+\zeta} \qquad (84)
$$

Therefore, the first term is bounded by $||x||^{\zeta_1+\zeta_2+2-|k|} = ||x||^{\overline{\zeta}-|k|}$. The same holds true for the 2nd term.

For the third term, we know that $||y|| \geq \frac{||x||}{4}$ and $||x-y|| \geq \frac{||x||}{4}$. Through this and the triangle inequality, $||x-y|| \geq ||y|| - ||x||$ it is clear that:

$$||x-y|| \geq \epsilon||y|| + \left(\frac{1-\epsilon}{4} - \epsilon\right)||x||, \tag{85}$$

for any $\epsilon > 0$. Choosing an appropriate ϵ gives, $||x-y|| \geq ||y||$ for some constant $C > 0$. As such, using that K_i are compactly supported, we have the following bound:

$$\begin{aligned} & \int_{\mathbb{R}^d} (1 - \varphi_r(y) - \varphi_r(x-y)) D^k(K_1(x-y)) K_2(y) dy & (86) \\ \lesssim\ & \int_{C \geq ||y|| \geq \frac{||x||}{4}} ||y||^{\zeta_1 + \zeta_2 - |k|} dy & (87) \\ \lesssim\ & ||x||^{\bar{\zeta} - |k|}. & (88) \end{aligned}$$

The 4th bound can be dealt with in the same manor as the 3rd.

We turn now to the proof of statement 3. First observe that:

$$D^k K_\epsilon - D^k K(x) = \int_{\mathbb{R}^d} (D^k K(x-y) - D^k(x)) \varrho_\epsilon(y) dy \tag{89}$$

For $||x|| \geq 2\epsilon$. We know that the support of ϱ_ϵ is contained in a ball of radius ϵ. Moreover the integrand is 0 unless $||x-y|| \geq \frac{||x||}{2}$. Therefore, because ϱ_ϵ integrates to one we can bound $D^k K(x-y) \leq ||x||^{\zeta - |k|} |||K|||_{\zeta;|k|}$. Hence:

$$\begin{aligned} |D^k K_\epsilon(x) - D^k K(x)| &\lesssim ||x||^{\zeta - |k|} |||K|||_{\zeta;|k|} & (90) \\ &\lesssim |||K|||_{\zeta,|k|+1} \sum_{i=1}^{d} |y_i| ||x||^{\zeta - |k| - 1}. & (91) \end{aligned}$$

Integrating against ϱ_ϵ gives the bound:

$$\begin{aligned} & |D^k K(x-y) - D^k K_\epsilon(x)| & (92) \\ \lesssim\ & |||K|||_{\zeta,|k|+1} \sum_{i=1}^{d} \epsilon ||x||^{\zeta - |k| - 1} \lesssim \epsilon^{\zeta - \bar{\zeta}} |||K|||_{\zeta;|k|+1} ||x||^{\bar{\zeta} - |k|}. & (93) \end{aligned}$$

For $||x|| \leq 2\epsilon$, write

$$D^k K_\epsilon(x) = \int_{\mathbb{R}^d} K(y) D^k \varrho_\epsilon(x-y) dy. \tag{94}$$

Note that the integrand is supported in a ball of radius 3ϵ and that $|D^k \varrho_\epsilon|$ is bounded by $C\epsilon^{-2-|k|}$ for some constant C. Therefore, we can bound:

$$\begin{aligned} |D^k K_\epsilon(x)| &\lesssim \epsilon^{-2-|k|} |||K|||_{\zeta;0} \int_{||y|| \leq 3\epsilon} ||y||^\zeta dy & (95) \\ &\lesssim (||x|| + \epsilon)^{\zeta - |k|} |||K|||_{\zeta;|k|} & (96) \end{aligned}$$

As such, (similarly to the previous case) we have the bound:

$$|D^k K(x) - D^k K_\epsilon(x)| \lesssim |||K||_{\zeta;|k|}||x||^{\zeta-|k|} \tag{97}$$
$$\lesssim \epsilon^{\zeta-\bar{\zeta}}|||K|||_{\zeta;|k|}||x||^{\bar{\zeta}-|k|}, \tag{98}$$

as desired. □

3.4 Stationary Solution Argument

As explained in the introduction, there are two difficulties in dealing with the PAM in two dimensions on the continuum. The first is that on the continuum solutions need to be controlled at infinity, that is why we have incorporated weights into our Hölder spaces. These time-increasing weights will allow us to bound the terms in the fixed point argument by exchanging weights in the norms (see below). Essentially we would like to find a solution by integrating against the kernel countably many times and then summing. The weights ensure that the infinite sum of positive homogeneity terms does not make the sum diverge.

The second difficulty is the negative homogeneity terms we will create. Otherwise put, $u \cdot \xi$ is not well-defined because the sum of their Hölder regularities is below 0. We have already given meaning to this product through the expanded Hölder spaces but it remains to suitably renormalize the equation and remove the singularity.

Remark 3
A very good example in which to see what renormalization coresponds to for (S)PDE is the recent work in regularity structures by M. Hairer [cite hai14] . In his analysis of the 2D PAM on the torus Hairer deals with the same renormalization difficulty without the need for weights. But rather than renormalize using a stationary solution, as we do here, he instead uses the theory of regularity structures. This allows him to write his solution as an expansion analogous to a truncated Taylor series. Essentially by integrating the noise against the kernel and then multiplying by the noise (or not) and integrating again one is able to create a vector space graded by homogeneity. Expressing the fixed point argument in this space then allows one to write the solution as a sum of vectors that can be controlled. Because the authors work on the torus they need not be concerned with summing the positive homogeneity terms. For the PAM in 2d there are 2 negative homogeneity terms each of which needs to be expanded into its Weiner Chaos decomposition. Then allowing the author to pin-point the source of the blow up and subtract it.

Fortunately we need not delve into the theory of regularity structures thanks to a trick which is used in the non-stochastic case as well: we introduce the "stationary solution" Y and solve the PDE associated to $v = ue^Y$.

Remark 4
The analogue to the non-stochastic case is for $\partial_t u - \Delta u = bu$ for $b > 0$, we can solve this by inserting $v = ue^{bt}$, doing so shows v solves the heat equation which we can use to find u.

Take a to be some arbitrary small constant. Let PAM_ϵ denote:

$$\partial_t u_\epsilon = \Delta u_\epsilon + u_\epsilon(\xi_\epsilon - C_\epsilon), \qquad u_\epsilon(0,x) = u_0(x).$$

To construct the stationary solution let G be a compactly supported, even, smooth function on $\mathbb{R}^2\backslash\{0\}$ such that $G(x) = \frac{-\log|x|}{2\pi}$ whenever $|x| \leq \frac{1}{2}$, i.e G corresponds to the 2D heat kernel in a ball of radius $\frac{1}{2}$ and is even, compactly supported and smooth otherwise. Applying the Laplace operator to G, in a distributional sense, gives us a smooth compactly supported function F on the whole of $\mathbb{R}^2$ such that:

$$\Delta G(x) = \delta_0(x) + F(x). \tag{99}$$

With this we can introduce a process: $Y_\epsilon = G * \xi_\epsilon$. By definition Y_ϵ is smooth and stationary on $\mathbb{R}^2$ and is such that:

$$\Delta Y_\epsilon(x) = \xi_\epsilon(x) + F * \xi_\epsilon \tag{100}$$

With that we can now discuss the limiting nature of Y_ϵ in the following Corollary of Lemma 1:

Corollary 1 (Stationary Solution Limit)
For any $\kappa \in (0, \frac{1}{2})$, the sequence of processes Y_ϵ (and $D_{x_i}Y_\epsilon$ respectively), as $\epsilon \to 0$ converges in probability in $\mathcal{C}^{1-\kappa}_{p_a}(\mathbb{R}^2)$ ($\mathcal{C}^{-\kappa}(\mathbb{R}^2)$ respectively) towards the process Y (respectively $D_{x_i}Y$) defined by:

$$Y := G * \xi \tag{101}$$

$$D_{x_i}Y := D_{x_i}G * \xi \tag{102}$$

Proof. G is compactly supported and identical to the Green's function for the Laplacian around the origin. Therefore we can apply some classical Schauder estimates which can be found here [6], which state that for all $\alpha \in \mathbb{R}$ the bounds:

$$||G * f||_{\alpha+2} \lesssim ||f||_\alpha \tag{103}$$

$$||D_{x_i}G * f||_{\alpha+1} \lesssim ||f||_\alpha, \tag{104}$$

hold uniformly over all $f \in \mathcal{C}^\alpha$. Unfortunately we are not done here because we need these bounds to hold for $f \in \mathcal{C}^\alpha_w$.

Let χ be a compactly supported function on $\mathbb{R}^d$ such that $\sum_{k\in\mathbb{Z}^d} \chi(x-k) = 1$ for all $x \in \mathbb{R}^d$. For convenience let $\chi_k(x) = \chi(x - k)$. Given that, because G is compactly supported,

$$||G * (f\chi_k)||_{\alpha+2} \lesssim w(k)||f||_{\alpha,w} \tag{105}$$

$$||D_{x_i}G * (f\chi_k)||_{\alpha+1} \lesssim w(k)||f||_{\alpha,w} \tag{106}$$

uniformly over all $k \in \mathbb{Z}^d$ and $f \in \mathcal{C}^\alpha_w$. But for a fixed x, only a bounded number of $\{\chi_k(x), k \in \mathbb{Z}^d\}$ are non-zero, uniformly over all $x \in \mathbb{R}^d$ (because χ is compactly supported). Since $f = \sum_{x\in\mathbb{Z}^d} f\chi_k$:

$$||G * f||_{\alpha+2,w} \lesssim ||f||_{\alpha,w} \tag{107}$$

$$||D_{x_i}G * f||_{\alpha+1,w} \lesssim ||f||_{\alpha,w} \tag{108}$$

uniformly over all $f \in \mathcal{C}^\alpha_w$. Hence the statement follows from Lemma 1. □

Let $v_\epsilon(t,x) := u_\epsilon(t,x)e^{Y_\epsilon(x)}$ for $x \in \mathbb{R}^2$ and $t \geq 0$. As such, v_ϵ solves:

$$\partial_t v_\epsilon = \Delta v_\epsilon + v_\epsilon(Z_\epsilon - F * \xi_\epsilon) - 2 \bigtriangledown v_\epsilon \cdot \bigtriangledown Y_\epsilon, \tag{109}$$
$$v_\epsilon(0,x) = u_0(x)e^{Y_\epsilon(x)}, \tag{110}$$

where:

$$Z_\epsilon(x) := |\bigtriangledown Y_\epsilon(x)|^2 - C_\epsilon. \tag{111}$$

Because Y and F are smooth, equation (113) shows that the only source of blow up is the process Z_ϵ. With that in mind, we set:

$$C_\epsilon := \mathbb{E}[|\bigtriangledown Y_\epsilon|^2], \tag{112}$$

thereby removing the divergent 0th order term in the Wiener chaos decomposition. From there we simply apply the definitions:

$$\begin{aligned}
\mathbb{E}[|\bigtriangledown Y_\epsilon|^2] &= \mathbb{E}[\bigtriangledown G * \varrho_\epsilon * \xi|^2] && (113)\\
&= [|K_\epsilon * \xi|^2] && (114)\\
&= \int K_\epsilon(x-y)\xi(y)K_\epsilon(x-y')\xi(y')dydy' && (115)\\
&= \int K_\epsilon(x-y)\delta(y-y')K_\epsilon(x-y')dydy' && (116)\\
&= \int |K_\epsilon(y)|^2 dy && (117)\\
&= \int |\bigtriangledown G * \varrho_\epsilon(y)|^2 dy && (118)\\
&\sim \frac{1}{2\pi}\int \left(\frac{1}{|y|+\epsilon}\right)^2 \chi^2_{[0,1]}(|y|)dy && (119)\\
&\sim \frac{1}{2\pi}|\log\epsilon|. && (120)
\end{aligned}$$

Therefore:

$$C_\epsilon = -\frac{1}{2\pi}\log\epsilon + \mathcal{O}(1) \tag{121}$$

where $\mathcal{O}(1)$ converges to a constant as $\epsilon \to 0$. To close this subsection we would like to show that this renormalized sequence converges in the appropriate space.

Proposition 3 (Convergence of Z_ϵ)
For any $\kappa \in (0,1/4)$, the collection of processes Z_ϵ converges in probability as $\epsilon \to 0$, in the space $\mathcal{C}^{-2\kappa}_{p_a}(\mathbb{R}^2)$, towards the generalised process Z defined as follows: for every test function η, $\langle Z,\eta\rangle$ is a random variable in the second homogeneous Wiener chaos associated to ξ represented by the L^2 function:

$$(z,\tilde{z}) \mapsto \int \sum_{i=1,2} D_{x_i}G(z-x)D_{x_i}G(\tilde{z}-x)\eta(x)dx \tag{122}$$

To prove this proposition we will need the following lemma bounding Z and Z_ϵ. But first, note that because G is smooth away from the origin, compactly supported and is equal to the Green's function for the Laplacian in a neighborhood around the origin, therefore we can characterize its singularity as of order ζ for all $\zeta < 0$. Moreover, set $\varrho^{*2} = \varrho * \varrho$ and assume without lose of generality that ϱ and ϱ^{*2} are compactly supported in $\mathbb{R}^2$. In which case:

Lemma 3 (Bounds on Z and Z_ϵ)
For a given $\kappa \in (0, 1/2)$, we have the bounds:

$$\mathbb{E}\left[|Z(\eta_x^\lambda)|^2\right]^{\frac{1}{2}} \lesssim \lambda^{-\kappa} \tag{123}$$

$$\mathbb{E}\left[|Z_\epsilon(\eta_x^\lambda)|^2\right]^{\frac{1}{2}} \lesssim \lambda^{-\kappa} \tag{124}$$

$$\mathbb{E}\left[|(Z_\epsilon - Z)(\eta_x^\lambda)|^2\right]^{\frac{1}{2}} \lesssim \lambda^{-5\kappa}\epsilon^{\kappa}, \tag{125}$$

uniformly over all ϵ, $\lambda \in (0,1)$, all $x \in \mathbb{R}^2$ and all $\eta \in \mathcal{B}_1^r$.

Proof. By translation invariance it suffices to consider the case $x = 0$. Because C_ϵ corresponds to the 0 order Wiener chaos component in the decomposition of $|\bigtriangledown Y_\epsilon|$, the terms $Z(\eta^\lambda)$, $Z_\epsilon(\eta^\lambda)$ and $Z_\epsilon(\eta^\lambda) - Z(\eta^\lambda)$ must belong to the second homogeneous Wiener chaos associate to ξ. We begin by addressing the second bound:

$$\mathbb{E}_z[|Z_\epsilon(\eta^\lambda)|^2] \tag{126}$$

$$=\mathbb{E}\left[\left|\int_x (|\bigtriangledown Y_\epsilon|^2 - C_\epsilon)\eta^\lambda(x)dx\right|^2\right] \tag{127}$$

$$=\mathbb{E}\left[\left|\sum_{i=1}^{2}\int_x (D_{x_i}Y_\epsilon(z-x))^2\eta^\lambda(x)dx - \int_x\int_{\overline{z}}(D_{x_i}Y_\epsilon(\overline{z}-x))^2\eta^\lambda(x)d\overline{z}dx\right|^2\right] \tag{128}$$

$$=\sum_{i=1}^{2}\int_z\int_{\overline{x}}\int_x\int_{\overline{z}}\eta^\lambda(x)\eta^\lambda(\overline{x})(D_{x_i}Y_\epsilon(z-x))^2 - D_{x_i}Y_\epsilon(\overline{z}-x))^2) \tag{129}$$

$$(D_{x_i}Y_\epsilon(z-\overline{x}))^2 - D_{x_i}Y_\epsilon(\overline{z}-\overline{x}))^2)d\overline{z}dxd\overline{x}dz \tag{130}$$

$$=\sum_{i=1}^{2}\int_z\int_{\overline{x}}\int_x\int_{\overline{z}}2\eta^\lambda(x)\eta^\lambda(\overline{x})(D_{x_i}Y_\epsilon(z-x))^2 D_{x_i}Y_\epsilon(\overline{z}-\overline{x}))^2)d\overline{z}dxd\overline{x}dz \tag{131}$$

Because $Y_\epsilon = G_\epsilon * \xi$, and using definition 1 of white noise:

$$\mathbb{E}[|Z_\epsilon(\eta^\lambda)|^2] \tag{132}$$

$$= 2\sum_{i=1}^{2}\int_z\int_{\overline{z}}\left(\int_x \eta^\lambda(x)(D_{x_i}G_\epsilon(z-x))D_{x_i}G_\epsilon(\overline{z}-x))dx\right)^2 d\overline{z}dz \tag{133}$$

$$= 2\sum_{i=1}^{2}\int_{x,\overline{x}}\eta^\lambda(x)\eta^\lambda(\overline{x})\left(\int_z (D_{x_i}G_\epsilon(z))D_{x_i}G_\epsilon(z-x+\overline{x}))dz\right)^2 d\overline{x}dx \tag{134}$$

$$= 2\sum_{i=1}^{2}\int_{x,\overline{x}}\eta^\lambda(x)\eta^\lambda(\overline{x})\left(\int_z (D_{x_i}G_\epsilon * D_{x_i}G_\epsilon(x-\overline{x}))\right)^2 d\overline{x}dx \tag{135}$$

$$\tag{136}$$

By theorem 3 (properties of functions with singularities) and corollary 1 (stationary solution limit), this expectation grows like $\mathbb{E}[|Z_\epsilon(\eta^\lambda)] \lesssim ||D_{x_i}G_\epsilon * \eta^\lambda||_{-\kappa}^2 \lesssim \lambda^{-2\kappa}$. Therefore, the bound (126) holds. And for (125) we can simply replace G_ϵ by G in the above proof.

Addressing the 3rd bound, write

$$\mathbb{E}\left[|Z_\epsilon(\eta^\lambda)^2|\right] = \sum_{i=1}^{2} \int\int \eta^\lambda(x)\eta^\lambda(\overline{x})H_{\epsilon,i}(x-\overline{x})dxd\overline{x}, \tag{137}$$

where

$$\begin{aligned} H_{\epsilon,i}(y) &= ((D_{x_i}(G_\epsilon - G) * D_{x_i}G_\epsilon) \cdot (D_{x_i}(G_\epsilon + G) * D_{x_i}G_\epsilon)) \\ &\quad - ((D_{x_i}(G_\epsilon - G) * D_{x_i}G) \cdot (D_{x_i}(G_\epsilon + G) * D_{x_i}G)) . \end{aligned}$$

Hence, using the bounds from Theorem 3:

$$||H_{\epsilon,i}(x-\overline{x})||_{0,m} \tag{138}$$

$$\lesssim \epsilon^\kappa ||(D_{x_i}(G_\epsilon - G) * D_{x_i}G)(D_{x_i}(G_\epsilon + G) * D_{x_i}G)||_{-\kappa,m} \tag{139}$$

$$\lesssim \epsilon^\kappa ||(D_{x_i}(G_\epsilon - G) * D_{x_i}G)||^2_{-\kappa,m} \tag{140}$$

$$\lesssim \epsilon^{2\kappa} ||(D_{x_i}(G) * D_{x_i}G)||^2_{-2\kappa,m} \tag{141}$$

And so

$$\mathbb{E}\left[|Z_\epsilon(\eta^\lambda)^2|\right] \lesssim \epsilon^{2\kappa}\lambda^{-10\kappa} \tag{142}$$

□

We turn now to the proof of Proposition 3:

Proof. Take $L \in \{Z, Z_\epsilon, Z - Z_\epsilon\}$. Then:

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{x\in\Lambda_n}\left(\frac{L(\psi_x^n)}{p_a(x)2^{-n+d/2}}\right)^{2p}\right] \lesssim \sum_{k\in\mathbb{Z}^2}\frac{1}{p_a(k)^{2p}}\sum_{n\geq 0}\sum_{x\in\Lambda\cap B(k,1)}\frac{\mathbb{E}[L(\psi_x^n)^2]^p}{2^{-2np(\alpha+d/2)}}. \tag{143}$$

By lemma 3, using $\lambda = 2^{-n}$, $\mathbb{E}[L(\psi_x^n)^2] \lesssim 2^{-dn+2\kappa n}$. Moreover the number of points in $(\Lambda_n \cap B(k,1)) \lesssim 2^{dn}$, hence

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{x\in\Lambda_n}\left(\frac{L(\psi_x^n)}{p_a(x)2^{-n+d/2}}\right)^{2p}\right] \tag{144}$$

$$\lesssim \sum_{k\in\mathbb{Z}^d}\frac{1}{p_a(k)^{2p}}\sum_{n\geq 0}2^{2np(\alpha+\frac{d}{2})-(dn+2\kappa n)p+dn} \tag{145}$$

$$\lesssim \sum_{k\in\mathbb{Z}^d}\frac{1}{p_a(k)^{2p}}\sum_{n\geq 0}2^{np(2\alpha+2\kappa)+2n} \tag{146}$$

$$\tag{147}$$

Which is finite for $\alpha = -2\kappa$ and large enough p. As such Z and Z_ϵ belong to $\mathcal{C}^{-2\kappa}_{p_a}$.

Regarding $Z - Z_\epsilon$, Lemma 3.1 ensures $\mathbb{E}[(Z - Z_\epsilon)(\psi_x^n)^2] \lesssim \epsilon^{2\kappa}2^{n(10\kappa-2)}$ uniformly over x, n, and ϵ. Therefore:

$$\mathbb{E}\left[\sup_{n\geq 0}\sup_{x\in\Lambda_n}\left(\frac{(Z-Z_\epsilon)(\psi_x^n)}{p_a(x)2^{-n\alpha-n}}\right)^{2p}\right] \lesssim \sum_{k\in\mathbb{Z}^2}\frac{1}{p_a(k)^{2p}}\sum_{n\geq 0}\epsilon^{2\kappa p}2^{n(2\alpha p+10\kappa p+2)} \tag{148}$$

As such, choosing for example $\alpha = -6\kappa$ and taking p large we can conclude that $\mathbb{E}[||Z - Z_\epsilon||_{-6\kappa,p_a}] \lesssim \epsilon^\kappa$ uniformly for $\epsilon \in (0,1]$.

□

3.5 Fixed Point Argument

Before discussing the central fixed point argument crucial to solving this PDE we introduce the Banach space $\mathcal{E}^r_{l,T}$. For any $r > 0$, $l \in \mathbb{R}$, and $T > 0$, let $\mathcal{E}^r_{l,T}$ define the set of continuous functions v on $(0, T] \times \mathbb{R}^2$ such that:

$$|||v|||_{l,T} := \sup_{t\in(0,T]} \frac{||v_t||_{r,e_{l+t}}}{t^{-1+\kappa}} < \infty. \tag{149}$$

To set up the fixed point argument, fix $\kappa \in (0, \frac{1}{4})$ and let $a \in (0, \kappa/2)$. For a given $g, h^{(1)}, h^{(2)} \in \mathcal{C}^{-2\kappa}_{p_a}$ and $f \in \mathcal{C}^{-1+4\kappa}_{e_l}$, define the $\mathcal{M}_{T,f}(v)$ as follows:

$$\mathcal{M}_{T,f}(v)_t := \int_0^t P_{t-s} * \left(v_s \cdot g + D_{x_i} v_s \cdot h^{(i)}\right) ds + P_t * f, \tag{150}$$

where we omit the summation over $i = 1, 2$ for convenience in the rest of this article.

Proposition 4 (Fixed Point of $\mathcal{M}_{T,f}(v)$)
For any given $\kappa \in (0, \frac{1}{4})$, $g, h^{(1)}, h^{(2)} \in \mathcal{C}^{-2\kappa}_{p_a}$ and any $f \in \mathcal{C}^{-1+4\kappa}_{e_{l_0}}$, the map $\mathcal{M}_{T,f}$ admits a unique fixed point $v \in \mathcal{E}^{1+2\kappa}_{l_0,T}$. Furthermore, the solution map $(g, h^{(1)}, h^{(2)}, f) \mapsto v$ is continuous.

Proof. Using Lemma 2 we get that $||P_t * f||_{1+2\kappa,e_{l+t}} \lesssim t^{-1+\kappa}||f||_{-1+4\kappa,e_l}$ uniformly over all t in a compact subset of $\mathbb{R}_+$.

Furthermore, observe:

$$\sup_{x\in\mathbb{R}} \frac{p_a(x)e_{l+s}(x)}{e_{l+t}(x)} = \sup_{x\in\mathbb{R}} |x|^a e^{(s-t)|x|}, \tag{151}$$

taking a derivative of the left hand side and maximizing gives

$$\sup_{x\in\mathbb{R}} \frac{p_a(x)e_{l+s}(x)}{e_{l+t}(x)} \le e^{-a} \left(\frac{a}{t-s}\right)^a \tag{152}$$

Using the multiplication rule from theorem 3 we deduce that:

$$||v_s \cdot g + D_{x_i} v_s \cdot h^{(i)}||_{-2\kappa,e_{l+t}} \tag{153}$$

$$\le \frac{p_a(x)e_{l+s}(x)}{e_{l+t}(x)} ||v_s \cdot g||_{-2\kappa,p_a e_{l+s}} + ||v_s \cdot h^{(i)}||_{-2\kappa,p_a e_{l+s}} \tag{154}$$

$$\lesssim (t-s)^{-a}||v_s||_{1+2\kappa,e_{l+s}}(||g||_{-2\kappa,p_a} + ||h^{(i)}||_{-2\kappa,e_{l+t}}) \tag{155}$$

$$= (t-s)^{-a}s^{-1+\kappa}|||v_s|||_{l,T}(||g||_{-2\kappa,p_a} + ||h^{(i)}||_{-2\kappa,e_{l+t}}) \tag{156}$$

uniformly over all s, t in a compact set of $\mathbb{R}_+$ and l in a compact set of $\mathbb{R}$. Using Lemma 2 and $a < \kappa/2$:

$$\left|\left|\int_0^t P_{t-s} * \left(v_s \cdot g + D_{x_i} v_s \cdot h^{(i)}\right) ds\right|\right|_{1+2\kappa,e_{l+t}} \tag{157}$$

$$\lesssim \int_0^t (t-s)^{-\frac{1}{2}-2\kappa}s^{-1+\kappa}ds|||v|||_{l,T}(||g||_{-2\kappa,p_a} + ||h^{(i)}||_{-2\kappa,p_a}) \tag{158}$$

$$\lesssim t^{-1+\kappa}T^{\frac{1}{2}-2\kappa}|||v|||_{l,T}(||g||_{-2\kappa,p_a} + ||h^{(i)}||_{-2\kappa,p_a}) \tag{159}$$

uniformly over all $t \in (0,T]$, ensuring that $\mathcal{M}_{T,f}(v) \in \mathcal{E}^{1+2\kappa}_{l,T}$. Therefore,

$$|||\mathcal{M}_{T,f}(v) - \mathcal{M}_{T,f}(\overline{v})|||_{l,T} \lesssim T^{\frac{1}{2}-2\kappa} |||v-\overline{v}|||_{l,T}(||g||_{-2\kappa,p_a} + ||h^{(i)}||_{-2\kappa,p_a}) \quad (160)$$

uniformly over all l in a compact subset of $\mathbb{R}$, all T in a compact subset of $\mathbb{R}_+$, all $f \in \mathcal{C}^{-1+4\kappa}_{e_l}$ and all $v, \overline{v} \in \mathcal{E}_{l,T}$ (where here and for the remainer of the article we write $\mathcal{E}_{l,T} = \mathcal{E}^{-1+4\kappa}_{l,T}$).

This implies that there exists a sufficiently small $T^* > 0$ such that $\mathcal{M}_{T^*,f}$ is a contraction on $\mathcal{E}_{l,T^*}$ uniformly over all $l \in [l_0, l_0+T]$ and $f \in \mathcal{C}^{-1+4\kappa}_{e_l}$. We then proceed by iterating using linearity of $\mathcal{M}_{T,f}$ to cover $[0,T]$: Let the fixed point of $\mathcal{M}_{T^*,f}$ be $v^* \in \mathcal{E}_{l_0,T^*}$. If $T^* > T$ we are done. If not, set $f^* = v^*_{T^*/2} \in \mathcal{C}^{1+2\kappa}_{e_{l_0^*}}$ where $l_0^* = l_0 + T^*$. We know $l_0^* \le l_0 + T$ and $\mathcal{M}_{T^*,f^*}$ is a contraction on $\mathcal{E}_{l_0^*,T^*}$, hence it admits a fixed point, $v^{**} \in \mathcal{E}_{l_0^*,T^*}$. Let $v_s = v^*_s$ for $s \in (0, T^*/2]$ and $v_s = v^{**}_{s-T^*/2}$ for all $s \in (T^*/2, 3T^*/2]$. By linearity, v is a fixed point of $\mathcal{M}_{\frac{3T^*}{2},f}$ and $v \in \mathcal{E}_{l_0,3T^*/2}$. Suppose $\overline{v}$ was another fixed point, by uniqueness of the fixed point on $(0,T^*]$, $\overline{v} = v^*$ there, and $\overline{v}$ is necessarily a fixed point of $\mathcal{M}_{T^*,f^*}$, so it must coincide with v^{**}. Iterating further ensures existance and uniqueness of a fixed point on any subinterval of $[0,T]$.

Addressing the continuity of the solution map with respect to $f, g, h^{(i)}$. Let $\overline{\mathcal{M}}$ be the fixed point map associated to some $\overline{g}$ and $\overline{h}$. For initial conditions, $f, \overline{f} \in \mathcal{C}^{-1+4\kappa}$, $\mathcal{M}_{T,f}$ and $\overline{\mathcal{M}}_{T,\overline{f}}$ admit unique fixed points, v and $\overline{v} \in \mathcal{E}^{1+2\kappa}_{l_0,T}$. Moreover,

$$v_t - \overline{v}_t = (\mathcal{M}_{T,f}(v) - \mathcal{M}_{T,f}(\overline{v})) \quad (161)$$

$$+ \int_0^t P_{t-s} * \Big((\overline{v}_s(g-\overline{g})) + D_{x_i}\overline{v}_s(h^{(i)} - \overline{h}^{(i)}) \Big) ds + P_t * (f - \overline{f}). \quad (162)$$

Therefore,

$$\begin{aligned} & |||v-\overline{v}|||_{l,T} \\ \lesssim \;& T^{\frac{1}{2}-2\kappa}|||v|||_{l,T}(||\overline{g}-g||_{-2\kappa,p_a} + ||\overline{h}^{(i)} - h^{(i)}||_{-2\kappa,p_a} + ||f-\overline{f}||_{-1+4\kappa}) \\ + \;& T^{\frac{1}{2}-2\kappa}|||v-\overline{v}|||_{l,T}(||g||_{-2\kappa,p_a} + ||\overline{g}||_{-2\kappa,p_a} + ||h^{(i)}||_{-2\kappa,p_a} + ||\overline{h}^{(i)}||_{-2\kappa,p_a}) \end{aligned}$$

uniformly over all l in a compact subset of $\mathbb{R}$ and all T in a compact set of $\mathbb{R}_+$.

For a given $R > 0$, there exists a $T > 0$ such that:

$$|||v-\overline{v}|||_{l,T} \lesssim ||f-\overline{f}||_{-1+4\kappa,e_l} + T^{\frac{1}{2}-2\kappa}(||\overline{g}-g||_{-2\kappa,p_a} + ||\overline{h}^{(i)} - h^{(i)}||_{-2\kappa,p_a}). \quad (163)$$

uniformly over all l in a compact subset of $\mathbb{R}$ and all $g, \overline{g}, h, \overline{h}$ such that $|||v|||_{l,T}$, $||g||_{-2\kappa,p_a}$, $||\overline{g}||_{-2\kappa,p_a}$, $||h||_{-2\kappa,p_a}$, and $||\overline{h}||_{-2\kappa,p_a}$ are all less than R. This implies continuity of the solution map for (0,T]. Iterating the same argument gives continuity on any bounded interval.

□

At this stage we are in a position to present the main theorem in this work:

Theorem 4 (Convergence in Probability)
Consider an initial condition $u_0 \in \mathcal{C}^{-1+4\kappa}_{e_l}$ and a final time $T > 0$. For all $l' > l$,

the sequence of processes v_ϵ converges in probability as $\epsilon \to 0$ in the space $\mathcal{E}^{1+2\kappa}_{l',T}$ to a limit v which is the unique solution of:

$$\partial_t v = \Delta v + v(Z - F * \xi) - 2 \bigtriangledown v \cdot \bigtriangledown Y \ , \ v(0,x) = u_0(x) e^{Y(x)}. \tag{164}$$

As such, u_ϵ converges in probability in $\mathcal{E}^{1-\kappa}_{l',T}$ towards the process $u = ve^{-Y}$

Proof. Take $u \in \mathcal{C}^{-1+4\kappa}_{e_l}$ for a given $l \in \mathbb{R}$ and define $f_\epsilon := u_0 e^{Y_\epsilon}$. By Corollary 1 (the Stationary Solution Limit) and Theorem 2 (Multiplication of $\mathcal{C}^\alpha_w$ spaces), $f_\epsilon \in \mathcal{C}^{-1+4\kappa}_{e''_l}$ for all $l'' > l$.

Let v_ϵ be the unique fixed point of $\mathcal{M}_{T,f_\epsilon}$ associated to $g_\epsilon = Z_\epsilon - F * \xi_\epsilon$ and $h^{(i)} = D_{x_i} Y_\epsilon$. By Corollary 1 and Proposition 3 (Convergence of Z_ϵ), we know g_ϵ and $h^{(i)}$ converge in probability to $g = Z - F * \xi$, $h^{(i)} = -2D_{x_i} Y$ in $\mathcal{C}^{-2\kappa}_{p_a}$. Moreover by Lemma 1, $F * \xi_\epsilon$ converges towards $F * \xi$, since F is smooth and compactly supported. Therefore, by Proposition 4, v_ϵ converges in probability to $\mathcal{E}^{1+2\kappa}_{l'',T}$ to a unique fixed point v of $\mathcal{M}_{T,f}$ associated to g and $h^{(i)}$. Furthermore, Theorem 2 ensures that for every $l' > l''$, $u_\epsilon = v_\epsilon e^{Y_\epsilon}$ converges to $u = ve^{-Y}$ in $\mathcal{E}^{1-\kappa}_{l',T}$

□

4 Discussion

As discussed in the introduction, there are two approaches to solving (S)PDEs. The first is to homogenize (smooth) the noise and to then look for properties like stationary states. The second is to add counter terms to the equation so that solutions can be presented in a suitable space. Herein we have done the latter by rewriting the equation in a "stationary" form and controlling the diverging terms. Which allowed us to prove existence and uniqueness of solutions to the new renormalized equation. As such the importance of this result (independent of its use as an example) depends on the importance of the parabolic Anderson model.

One motivating example from which one can derive the discrete parabolic Anderson model is through random walks and particle dynamics. Consider a system of two types of particles, an A (catalyst) type and a B (reactant) type.

- Suppose the A particles evolve independently with the random input being related to the number of A-particles at a site (x,t) (i.e the number of A particles, a is such that $\xi = \gamma a - \delta$ for some $\gamma, \delta > 0$ and a white noise ξ).
- Suppose the B particles perform independent random walks at a rate $2d$ (where d is the dimension) and the number of particles at a site doubles at a rate of γ times the number of A particles.
- Suppose futher, that B particles die at a rate of δ
- Lastly, assume that the number of B particles at a site $(x,0)$ is $u_0(x)$

In this setting the solution to the PAM:

$$\partial_t u(x,t) = \Delta u(x,t) + \xi(x,t) u(x,t), \tag{165}$$

u, represents the number of B particles at a point (x,t) conditioned on the behavior of the A particles [1].

Therefore, one can see that the discrete parabolic Anderson model has relevence to (at least one) physical model and as such it is interesting to ask whether this model is realistic in the limit as the spacing goes to 0. Where by realistic we refer to the existence and uniqueness of solutions. *i.e* we have shown that this model is realistic in the sense that it has a unique solution.

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