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Pinched manifolds becoming dull

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Pinched manifolds becoming dull

by

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Pinched manifolds becoming dull

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In this thesis, we prove short-time existence for Ricci flow, for a class of metrics with unbounded curvature.

Our primary motivation in investigating this class of metrics is that it includes many final-time limits of Ricci flow singularities. Well known examples include neckpinches and degenerate neckpinches. We provide an example of Ricci flow modifying a neighborhood of a manifold with the topological change $D^{1+p} \times S^q \to \mathrm{Cone}(S^p \times S^q) \to D^{1+q} \times S^p$, although we only rigorously deal with the second part of the transformation.

We also provide forward evolution from some manifolds with ends of infinite length and unbounded curvature, such as the submanifold given by $x_1^2 + x_2^2 + x_3^2 = (1 + x_4)^{-2}$ in $\mathbb{R}^4$. In this example, the two ends with unbounded curvature immediately become compact and with bounded curvature, so the topology of the forward evolution is S^3.

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Introduction

1.1 Background and Setting

A time-dependent family of Riemannian metrics $g(t)$ on a manifold M evolves by Ricci flow if

$$\partial_t g(t) = -2\,\mathrm{Rc}[g(t)]. \tag{1.1}$$

Here $\mathrm{Rc}[g(t)]$ is the Ricci curvature of the metric $g(t)$. If g_{init} is a complete metric with bounded curvature, then there is a solution $g(t)$ to Ricci flow on some time interval $[0, T_2)$ with $g(0) = g_{init}$. The results in this dissertation are about the possibility of starting Ricci flow from a certain class of metrics, to which the general theory does not apply.

For the reader less familiar with Ricci flow, we offer an intuitive picture. Ricci flow behaves as a reaction-diffusion equation. The *diffusion* part means that $g(t)$ gets smoother in short time, in ways that we can make precise. Also, the diffusion part tries to make the curvature more constant. It may not be immediately obvious from (1.1) that this behavior exists, but essentially it comes down to the fact that if we consider Rc as a second-order differential operator acting on g, it is kind of elliptic. The *reaction* part means that regions with large positive curvature get larger curvature. This is easier to guess from

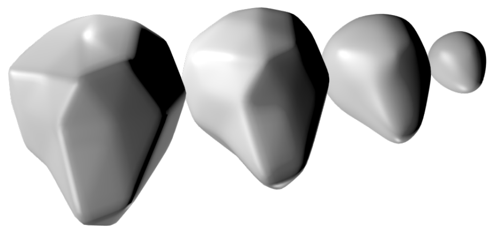

Figure 1.1: A bumpy sphere becoming round. The pictures of Ricci flow in this thesis are sketches, not accurately computed drawings.

(1.1): if the Ricci curvature is positive then (1.1) tells us that g should shrink, which makes the norm of the curvature larger.

Consider Figure 1.1. The initial metric is a bumpy sphere. The sequence of pictures is a series of snapshots of Ricci flow. In a short time, the metric loses many of its bumps and becomes smoother. The curvature becomes closer to being constant. Meanwhile, the sphere is shrinking overall, and the metric goes to zero everywhere at some finite time.

As another example of Ricci flow, consider Figure 1.2. The initial manifold is topologically S^3, and has a long, thin, necklike region. After a short time, the whole manifold changes. The most drastic change is that the necklike region shrinks because it is close to a part of $\mathbb{R} \times S^2$, which has positive Ricci curvature on the S^2 factor. In finite time the S^2 factor collapses on a lower-dimensional set; the metric is degenerate on a closed subset of S^3, and has unbounded curvature on its complement.

In general, for any manifold $(M, g(0))$ with bounded curvature tensor, the Ricci flow exists at least on some time interval $[0, T_2)$ where $T_2 \in (0, \infty]$.

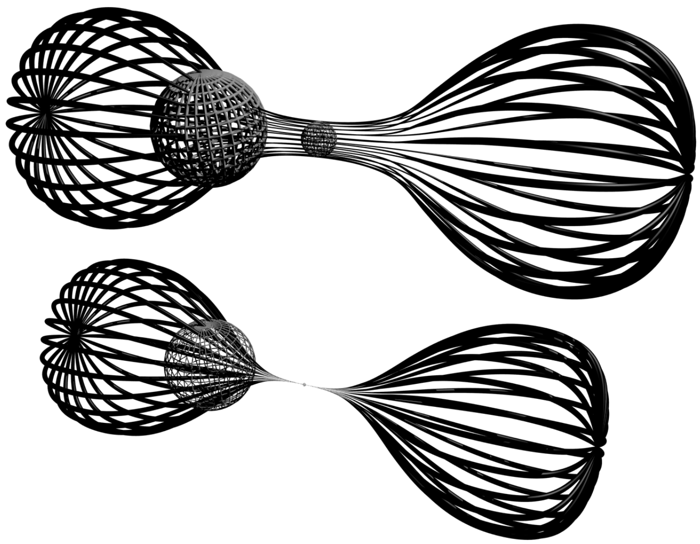

Figure 1.2: A three-dimensional rotationally symmetric neckpinch singularity

If $T_2 < \infty$ then [1]

$$\limsup_{t \to T_2} \sup_{p \in M} |\mathrm{Rm}| = \infty.$$

Refinements of this fact exist, for example one may replace Rm above with Rc [Š05].

The driving example of this thesis is the possibility of continuing the Ricci flow after the singularity in Figure 1.2. For simplicity, erase the "center" points in the last snapshot of Figure 1.2. Then we are left with a smooth, but incomplete with unbounded curvature, metric g_{sing} on two copies of $\mathbb{R}^3$. Just consider each connected component separately. There is a forward evolution depicted in Figure 1.3. Immediately after the singular time, each connected

[1] Here, $|\mathrm{Rm}| = |\mathrm{Rm}[g(t)]|_{g(t)}$ is the norm of the full curvature tensor of $g(t)$ measured with respect to $g(t)$. We hope it's always clear enough which metrics are in play.

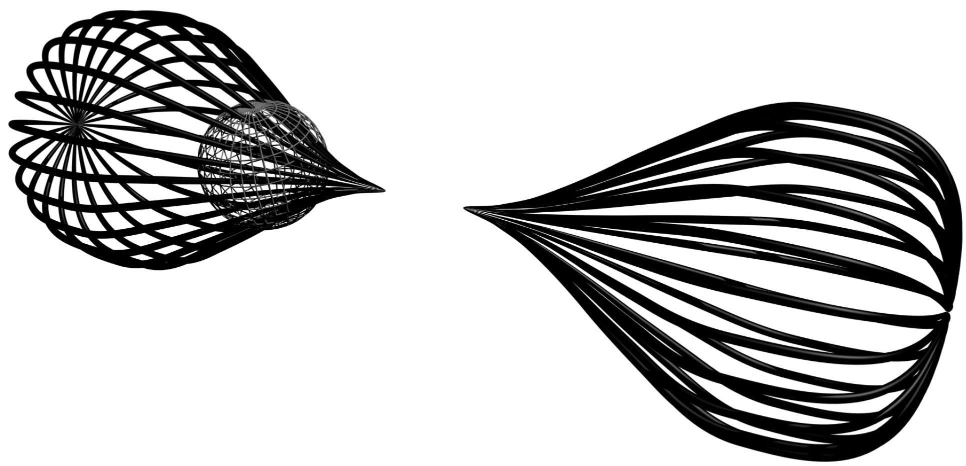

Figure 1.3: Forward flow from a neckpinch. Although they have high curvature, the tips are smooth.

component becomes topologically S^3. If we identify $\mathbb{R}^3$ with a punctured S^3, and set $M = \mathbb{R}^3$, $\bar{M} = S^3$, then the forward evolution can be simplistically described as a smooth, complete, Ricci flow $(\bar{M}, g(t))$ such that in $C^\infty_{loc}(M)$, $g(t) \to g_{sing}$. We also have Gromov-Hausdorff convergence of the metrics spaces $(\bar{M}, g(t))$ to (M, g_{sing}).

1.1.1 Ricci flow without singularities

A principle pursuit in geometric analysis is finding global implications of pointwise curvature assumptions about a Riemannian manifold. Ricci flow has contributed much to this topic. The general hope comes from the idea that Ricci flow should make the curvature of a metric more constant, and so it is possible to turn an inequality on the curvature into an equality, if the Ricci flow exists for a long time. This idea (specifically for Ricci flow) was introduced by

Hamilton in [Ham82], where he showed that if we start Ricci flow from a metric on a three-manifold with positive Ricci curvature then it approaches (after some scaling) a metric of constant positive Riemannian curvature. Therefore, a manifold which admits a metric of positive Ricci curvature also admits a metric of constant Riemannian curvature, and so it must be a quotient of a sphere. The same result was proven in [Ham86] in dimension four, but with "positive Ricci curvature" replaced with "positive curvature operator" (the standard metric on $S^2 \times S^2$ has positive Ricci curvature but is not a quotient of S^4). Note that this method immediately proves the result that the manifold is not just topologically, but also diffeomorphically, a quotient of a sphere.

Another celebrated result of this type is the resolution of the differentiable quarter-pinched curvature conjecture by Brendle and Schoen [BS09]: they show that any manifold with sectional curvature strictly between one and four flows, under Ricci flow, to a manifold with constant sectional curvature. (In fact, they only require that for some $f : M \to \mathbb{R}_+$ the sectional curvature of each plane in T_pM is between $f(p)$ and $4f(p)$.)

1.1.2 Ricci flow with surgery

The most widely known application of Ricci flow is the resolution of Thurston's geometrization conjecture by Perelman in [Per02] and [Per03]. This application has a complication not present in the aforementioned works. The Ricci flow exists at least up to a time when the curvature goes to infinity. In the situations visited by [Ham82], [Ham86], and [BS09], the curvature goes to infinity everywhere on the manifold, and if we correctly scale the metric in time, it smoothly approaches a metric of constant curvature. In contrast, in the generality needed for the geometrization conjecture, the metric forms local singularities; i.e. the curvature goes to infinity on a strict subset of the manifold.

Possible local singularities, like those in Figure 1.2, were correctly predicted in Section 3 of [Ham95]. These local singularities actually gave hope to the plan of using Ricci flow to resolve the geometrization conjecture: they are able to disconnect pieces of the manifold and perform the surgeries allowed by the conjecture. (Here one should imagine that the two bulbs in Figure 1.2 may be replaced with arbitrarily complicated three manifolds with relatively low curvature compared to the neck.)

For dealing with these singularities, one can use a Ricci flow with surgery. The idea is to classify what the metric looks like in regions of high enough curvature. While we cannot expect the curvature to be completely diffused as in [Ham82], in regions of high curvature it is locally diffused. This implies, also, a topological understanding of these regions. Therefore, before a singularity as pictured in Figure 1.2, one can perform a surgery to the manifold which allows the Ricci flow to continue.

We mention some successes of Ricci flow with surgery besides the resolution of the geometrization conjecture. The first was [Ham97], in which Hamilton classified compact four-manifolds with positive isotropic curvature and no essential incompressible space forms: every such manifold (M, g) is topologically

$$S^4, \quad \mathbb{RP}^4, \quad S^3 \times S^1, \quad S^3 \tilde{\times} S^1, \tag{1.2}$$

or a connect sum thereof. The idea is to prove that every high-curvature region is either a sphere, or one of two possible types of neck. We run Ricci flow, cutting out necks and throwing out any piece of one of the topologies in line (1.2), and eventually we are left with nothing. Going in reverse, we start from some finite number of pieces, and make connected sums, eventually returning to our original manifold.

The condition in [Ham97] that M has no essential incompressible space form is that there is no three-dimensional submanifold $N = S^3/\Gamma$ of M such that $\pi_1(N)$ injects into $\pi_1(M)$, besides the case $\Gamma = \{1\}$ and $\Gamma = \{\pm 1\}$. This is needed to rule out the possibility of necks with topology $(S^3/\Gamma) \times (-1, 1)$. We cannot cut such a neck and cap it off with a smooth manifold: the procedure results in a generalization called an orbifold. Chen, Tang, and Zhu [CTZ12] have carried out the complete classification of four-dimensional manifolds with positive isotropic curvature, using Ricci flow on orbifolds.

Most recently, Brendle [Bre18] defined a new curvature condition and used Ricci flow to get topological implications in any dimension. The result is analogous to [Ham97]: any manifold with no non-trivial incompressible space form which satisfies the curvature condition is topologically a connected sum of pieces of topology

$$S^n/\sim \quad \text{or} \quad S^{n-1} \times \mathbb{R}/\sim \,.$$

Here the quotients are by isometries of the standard metrics such that the result is compact.

Analogues of these results hold for mean curvature flow of hypersurfaces as well. For instance [HS09] shows that a two-convex hypersurface in $\mathbb{R}^{n+1}$ is topologically either S^n or the connected sum of copies of $S^{n-1} \times S^1$.

All of these results ([Ham97], [Bre18], [HS09]) rely on finding a condition on curvature which is preserved by the flow and rules out all but a few singularity models. For instance, the curvature condition in [Bre18] is that the curvature tensor at each point lies in a certain cone. The interior of this cone includes the curvature of the standard metric on $S^{n-1} \times \mathbb{R}$, which is why neck regions may appear. However, the standard metric on $S^{n-2} \times \mathbb{R}^2$ lies on the boundary of the cone, and since the flow immediately moves into the interior of the cone $S^{n-2} \times \mathbb{R}^2$ cannot occur as a singularity model. An interesting result would be to find a preserved curvature condition which also allows for $S^{n-2} \times \mathbb{R}^2$, but also rules out other problematic singularities. In this thesis we explore getting around a singularity modeled on $S^{n-2} \times \mathbb{R}^2$ with understandable topological change (see the example in section 1.3.3.) The central density introduced in [CHI04] for dimension four suggests that, at least in dimension four, we might be able to find conditions which only allow singularities modeled on $S^2 \times \mathbb{R}^2$, $S^3 \times \mathbb{R}$, and S^4.

1.1.3 Ricci flow through singularities

Construction of Ricci flow with surgery relies on some choices of parameters. Here's a rough idea of a procedure, of course the full procedure is more complicated. Hopefully, we can prove a structure theorem for regions where the norm of the curvature, $|\mathrm{Rm}|$, satisfies $|\mathrm{Rm}| > C_{\mathrm{Rm}}$ for some large C_{Rm} (which depends on the initial metric)- something like every neighborhood of such high curvature will have to look cylindrical or like a small sphere. (See e.g. Theorem 5.1 of [Ham97].) Then we wait until $|\mathrm{Rm}|$, reaches $4C_{\mathrm{Rm}}$ somewhere on the manifold. Cut out the region $\{p : |\mathrm{Rm}|_{g(t)}(p) > C_{\mathrm{Rm}}\}$ and, using our analytical and topological understanding of that region, replace it with a region satisfying $|\mathrm{Rm}| < 2C_{\mathrm{Rm}}$, in a way so that the manifold continues to satisfy certain estimates.

This procedure works if C_{Rm} is chosen large enough depending on the initial metric. If we choose C_{Rm} larger, then we cut out smaller regions. The natural question is whether we can get rid of the parameter C_{Rm} by sending it to infinity. Can we construct a Ricci flow through the singularity, which exists as a smooth manifold up to the singular time, is some sort of singular object at the singular time, and after the singular time is instantaneously a smooth, complete, manifold? This question has been answered affirmatively in various cases, and we give an overview here.

The first constructions of Ricci flow through singularities were done in the case of singly warped products of spheres over intervals. (Equivalently, a Riemannian $n-$manifold with a cohomogeneity-one $SO(n)$ symmetry.) In

[AK04] and [AK07] Angenent and Knopf gave a lot of information about the asymptotics of singly warped products undergoing standard neckpinches. In particular, they found an approximation for the final-time metric, i.e. the shape of the bottom picture in Figure 1.2. In [ACK12], Angenent, Caputo and Knopf constructed forward evolutions from these singly warped product metrics. The approach is to construct mollified metrics with bounded curvature, and prove uniform short-time existence for the Ricci flow from the mollified metrics. Then we can construct a flow as a limit of the mollified flows. Together with the evolution before the singularity, these provided the first example of Ricci flow through a singularity.[2]

Another example of a singularity on a singly warped product is the degenerate neckpinch. These unstable examples were constructed by Angenent, Isenberg, and Knopf in [AIK15]. In [Car16], the author constructed a forward-evolution from the singularity.

As mentioned, Perelman carried out the program of Ricci flow with surgeries in dimension three. In [KL14], Kleiner and Lott successfully took a limit of the surgery parameters and constructed an object called a singular Ricci flow. Roughly, space-time is packed into one manifold which is smooth everywhere. The approach taken is that the singular points (e.g. the center of a neckpinch, at the singular time) do not belong to the space-time manifold.

[2]Really, [FIK03] previously provided examples of Ricci flow through a singular metric, by constructing self-similar shrinking solutions that shrink to a singular metric, and self-similar expanding solutions which come out of the same singular metric. It would be more proper to say this was the first example of non-self-similar Ricci flow through a singularity.

Instead, the space-time manifold is smooth and satisfies Ricci flow everywhere. It is not complete, but the Cauchy sequences without limit have curvature going to infinity. The work [KL14] also proves many structural properties of singular Ricci flows.

This is satisfying as a Ricci flow that makes no arbitrary choices (of surgery parameter). Even more so because of work by Bamler and Kleiner [BK17] showing the uniqueness of the singular flows. This implies in particular that the limit of Ricci flow with surgery, as we take the parameters to their limit, is independent of subsequence. The work in proving uniqueness also has application to the stability of singular Ricci flows.

The introductions of both [KL14] and [BK17] are both in-depth and readable. We mention here the "boundary condition" taken in [BK17], since it is relevant to the study of Ricci flow through singularities as a whole. (Generally, as with PDE in euclidean space, we should not expect uniqueness without some extra condition.) An important breakthrough of Perelman was a structure theorem for regions of high curvature in a smooth three-dimensional Ricci flow, called the $\epsilon-$canonical neighborhood theorem (Theorem 12.1 of [Per02]). This says that every point with high enough curvature has a space-time neighborhood where it is close to a $\kappa-$solution. A $\kappa-$solution is a complete Ricci flow $(M, g(t))$ for $t \in (-\infty, 0]$ which has nonnegative curvature and is $\kappa-$noncollapsed on all scales[3]. Without any boundary condition, we should

[3] $\kappa-$noncollapsed means for any $r > 0$ and any $(p,t) \in M \times (-\infty, 0]$ with $|\mathrm{Rm}_{g(t)}|(p) < r^{-2}$, the volume of the ball of radius r around p is at least κr^d where $d = \dim(M)$.

expect wild solutions of Ricci flow coming out of singularities. In [BK17], Bamler and Kleiner assume that the solution satisfies the $\epsilon-$canonical neighborhood assumption; this gives a sort of asymptotic boundary condition near all singular points which is sufficient for uniqueness.

All of these constructions rely on a classification of the nature of singularities and high curvature regions of the flow. The $\epsilon-$canonical neighborhood theorem does not directly generalize to higher dimensions, and relies on pinching estimates which come from the possible algebraic properties of the curvature tensor. Theorem 5.2 of [Bre18] gives a generalization to higher dimensions in a case where the possible curvature tensors are restricted.

1.2 Results

1.2.1 Intuitive Overview

Let us give an overview of results, before stating conditions precisely. Our main theorem below will require some different conditions than what we state in this intuitive introduction.

We prove existence of Ricci flow starting from a class of singular initial metrics. Let $q \geq 2$, and let (S^q, g_{S^q}) be a metric on S^q with constant sectional curvature 1, and let $I = (0, \infty)$. As a first example, consider the metric on $I \times S^q$ given by

$$dx^2 + \phi(x)^2 g_{S^q} \tag{1.3}$$

where the function $\phi : I \to \mathbb{R}_+$ satisfies $\phi(x) = o(x)$ as $s \searrow 0$. Make the metric well behaved as $x \to \infty$, say ϕ is strictly positive and C^∞ for $x > 1$. This is a warped product metric, with an incomplete end at $x = 0$ where the metric has a cusp and the curvature goes to infinity.

Identify $(0, \infty) \times S^q$ with $\mathbb{R}^{1+q} \setminus \{0\}$ and consider these metrics as metrics on $\mathbb{R}^{1+q} \setminus \{0\}$. We prove the existence of a smooth, complete, Ricci flow on $\mathbb{R}^{1+q}$, for $t \in (0, T_*)$. As $t \searrow 0$, the Ricci flow limits to the metric (1.3) on $\mathbb{R}^{1+q} \setminus \{0\}$ (smoothly on compact sets).

As a first generalization, let $p \geq 1$ and consider a metric on $I \times S^q \times S^p$ of the form

$$dx^2 + \phi(x)^2 g_{S^q} + \psi(x)^2 g_{S^p}$$

Assume that $\psi/\phi \to \infty$ as $x \searrow 0$. Consequently, the g_{S^q} factor has the largest curvature near $x = 0$. Intuitively, these metrics have a p dimensional set of points on cusps. We construct a forward evolution with topology $\mathbb{R}^{1+q} \times S^p$. Even if ψ goes to zero at $s = 0$, for $t > 0$ the size of the S^p factor is strictly positive everywhere.

In this example we may replace (S^p, g_{S^p}) with any Einstein manifold, with any sign on the scalar curvature. The curvature of the S^p factor plays a small role, because the S^p factor is relatively large in size compared to the S^q factor. (Our full result actually allows the initial size of the S^p factor to be on the same order but slightly larger than the initial size of the S^q factor.)

Here is another interesting generalization we make. Change the interval I to be $I = (-\infty, \infty)$. Assume that for some $\beta > 0$, $\phi = o(|x|^{-\beta})$ as $x \searrow -\infty$.

Now the metric $dx^2 + \phi(x)^2 g_{S^q}$ has a complete, noncompact end at $x = -\infty$ where the curvature goes to infinity. Again, we construct a Ricci flow on $\mathbb{R}^{1+q}$, with bounded curvature for $t > 0$, which limits to the initial metric on $\mathbb{R}^{1+q} \setminus \{0\}$ as $t \searrow 0$. So, the infinite-length cusp at the left end compactifies. To our knowledge this is the first example of this type of behavior in Ricci flow in dimension larger than two. Topping [Top11] constructed similar (and more general) examples in two dimensions, but the situation is quite different analytically in two dimensions because g_{S^1} has zero curvature.

Finally, we have a short-time stability result for these warped-product forward evolutions, which allows us to remove the global symmetry assumptions. Consider any manifold (M, g), which has a neighborhood U outside of which the curvature is bounded, and a diffeomorphism $\Phi : U \to (0, L) \times S^q$. If Φ is close enough to being an isometry to a neighborhood of the left end of the warped product metrics described above, then we have a forward evolution from (M, g) which stays close to the forward evolution from the warped product metric.

1.2.2 Precise statements

To state our theorem precisely, we define the class of warped-product metrics from which we may flow. We will call them "model pinches". It is useful in the description to change coordinates. The metrics will be doubly-warped products of the form

$$g_{mp} = dx^2 + \phi(x)^2 g_{S^q} + \psi(x)^2 g_F$$

where ϕ is an increasing function of x. Therefore $u = \phi^2$ is invertible and we may write

$$g_{mp} = \frac{du^2}{uV_0(u)} + ug_{S^q} + W_0(u)g_F$$

where

$$V_0(u(x)) = \frac{|\nabla u|^2(x)}{u(x)}, \quad W_0(u(x)) = \psi(x)^2.$$

Here V_0 is normalized so it is invariant under scaling the metric. We have $u \in (0, u_{max})$ for some $u_{max} \in (0, \infty]$. For simplicity let's just say $u_{max} = \infty$, since we remove this assumption in our second theorem[4]. The distance between the sets $\{u = u_1\}$ and $\{u = u_2\}$ is given by $\int_{u_1}^{u_2} \frac{1}{\sqrt{uV_0(u)}} du$, so the compactness of the end where $u \searrow 0$ is hidden in the integrability of $\frac{1}{\sqrt{uV_0(u)}}$ near 0.

Generally, we use $v = u^{-1}|\nabla u|^2$ and $w = \psi^2$ to refer to the corresponding functions on some generic doubly warped product. We use capital V and W to refer to specific functions considered as functions of u.

In the definition below, $q \geq 2$, g_{S^q} is the metric of constant sectional curvature 1 on S^q, $\mu = 2(q-1)$ so that $2\,\mathrm{Rc}_{g_{S^q}} = \mu g_{S^q}$, and (F, g_F) is an Einstein manifold with $2\,\mathrm{Rc}_{g_F} = \mu_F g_F$. Finally, $I = (0, \infty)$.

Definition 1.2.1. *Let $V_0 : I \to \mathbb{R}_+$ and $W_0 : I \to \mathbb{R}_+$ be smooth functions. We call the metric g_{mp} on $I \times S^q \times F$ given by*

$$g_{mp} = \frac{du^2}{uV_0(u)} + ug_{S^q} + W_0(u)g_F$$

a model pinch *if the following conditions hold.*

[4]Most of the global assumptions, like ϕ increasing everywhere, can be removed.

(MP1) For any $u_1 > 0$, the curvature of g_{mp} is bounded on the set $\{u > u_1\}$, and w is strictly positive on that set.

(MP2) As $u \searrow 0$, $V_0(u) \searrow 0$.

(MP3) If $\mu_F > 0$ then for some $c > 0$, $W_0(u) \geq (1 + c)\frac{\mu_F}{\mu} u$.

(MP4) For some $C > 0$ and for $k = 1, 2, 3, 4, 5$:

$$\left| {V_0}^{[k]} \right| + \left| {W_0}^{[k]} \right| \leq C.$$

Here $F^{[k]} = \frac{1}{F} u^k \partial_u^k F$.

One implication that helps interpret some of these conditions is the following. At any point, we can write the curvature $\mathrm{Rm}_{g_{mp}}$ of g_{mp} as

$$\mathrm{Rm}_{g_{mp}} = u \, \mathrm{Rm}_{S^q} + w \, \mathrm{Rm}_F + \mathrm{Rm}_{warp} \tag{1.4}$$

where Rm_{S^q} is the curvature tensor of (S^q, g_{S^q}), Rm_F is the curvature tensor of (F, g_F), and the tensor Rm_{warp} is defined by (1.4). Condition (MP4) implies (after some calculation) Rm_{warp} satisfies $|\mathrm{Rm}_{warp}|_{g_{mp}} \leq C u^{-1} v$ (for some bigger C). Note that $|u \, \mathrm{Rm}_{S^q}|_{g_{S^q}} = C_q u^{-1} \gg C u^{-1} v$ by (MP2). Therefore the curvature at $u = u_\sharp$ is approximately the curvature of the product metric $ds^2 + u_\sharp g_{S^q} + W_0(u_\sharp) g_F$. Furthermore, this relationship holds for some derivatives as well: for $k = 1, 2, 3$,

$$u^{1+2/k} |\nabla^k \mathrm{Rm}_{g_{mp}}| \leq C v^{1+2/k}.$$

We've written this so that the left hand side is invariant under scaling g_{mp}.

Our main theorem constructs a forward evolution from model pinches. In the statement of the theorem we identify $M := I \times S^q \times F$ with $(\mathbb{R}^{1+q} \setminus \{0\}) \times F$, and we let $\bar{M} = \mathbb{R}^{1+q} \times F$. Lemmas 2.2.2 and 2.3.2 give us extra information about the forward evolution. For now, just know that this gives us a full description of the asymptotic shape of the Ricci flow; we give an overview and corollaries in Section 1.4.

Theorem 1.2.2. *Let g_{mp} be a model pinch. There is a Ricci flow $g_{wp}(t)$ on $\bar{M}$ and for some time interval $t \in (0, T_2)$, such that as $t \searrow 0$, $g_{wp}(t) \searrow g_{mp}$ in $C^\infty_{loc}(M)$. There are choices of the parameters of Definitions 2.2.1 and 2.3.1 such that g_{mp} is controlled in the productish region and in the tip region.*

The next theorem removes the global part of the model pinch assumption. For this, we need some additional assumptions on the factor F. Let $\Lambda_F = \sup_{p \in F} \max_{h \in Sym_2(T_pF), |h|=1} (\mathrm{Rm}_{g_F})_{abcd} h^{ac} h^{bd}$. (For example, if F has dimension p and constant sectional curvature k then $\Lambda_F = k(p-1)$). In particular, $2\Lambda_{S^q} = \mu$.

Definition 1.2.3. *We call a model pinch F-*reasonable *if*

(MR1) $\frac{W_0(u)}{u} \geq \frac{\Lambda_F}{\Lambda_{S^q}}$

(MR2) If $\mu_F = 0$ then $\frac{W_0(u)}{uV_0(u)} \to \infty$ as $u \searrow 0$.

Since $V_0(u)$ goes to zero as $u \searrow 0$, the second assumption (MR2) is automatic vacuous unless $\Lambda_F = \mu_F = 0$.

Theorem 1.2.4. *Let g_{mp} be an F-reasonable model pinch. There is an ϵ_0 depending on g_{mp} with the following property.*

Let (M^n, g) be a (possibly non-complete) Riemannian manifold. Let $U \subset M$ be open, and assume that $(M \setminus U, g)$ is a complete compact manifold with boundary, satisfying

$$\sup_{p \in M \setminus U} |\mathrm{Rm}| < C.$$

Suppose that $u_1 > 0$ and $\Phi : U \to (0, u_1) \times S^q \times F$ is a diffeomorphism such that

$$|g - \Phi^* g_{mp}| \leq \epsilon_0 \Phi^*(V_0) = \epsilon_0 V_0 \circ \Phi$$

and

$$\sum_{i=1}^{5} (\Phi^* u_0)^{i/2} \left| \left(\nabla^{\Phi^* g_{mp}}\right)^i g \right| \leq C.$$

Let $\bar{M} \supset M$ be the differential manifold obtained by replacing $U \sim (L, L') \times S^q \times F$ with $\bar{U} \sim D^{1+q} \times F$. For some $T_* > 0$, there is a Ricci flow $g(t), t \in [0, T_*)$ on $\bar{M}$ such that $g(t) \to g$ on M as $t \searrow 0$.

Of course both of these theorems also hold when the factor F is not there. (To be fancy, they hold for $\dim(F) = 0$.) We mention two further extensions to the above theorems which are immediate; the only extra difficulty is in writing down notation. First, we may consider extra Einstein manifolds $(F^{(i)}, g_{F^{(i)}})$ and put extra warped product factors on g_{wp} which satisfy the same assumptions. Second, in Theorem 1.2.4, we may consider a manifold with multiple disjoint neighborhoods $U^{(i)}$, each of which is close enough to being a model pinch.

1.3 Pinches that arise as final-time limits of Ricci flow

Our initial motivation for this project was to investigate the continuation of Ricci flow after certain finite-time singularities. Here we list some examples of smooth Ricci flows which have a model pinch has final-time limits.

1.3.1 The standard neckpinch

In [AK07], Angenent and Knopf considered neckpinches occuring on singly warped products over an interval. They proved that the warping function of the final-time limit of a neckpinch satisfies the asymptotics $\phi(x) = \sqrt{u(x)} \sim \frac{x}{|\log x|}$, or $V_0 \sim \frac{4}{\log u}$, near the singular end. In [ACK12], which was the main inspiration for our first theorem, Angenent, Caputo, and Knopf constructed Ricci flows emerging from any metric with that asymptotic profile.

1.3.2 Degenerate neckpinches

Another singularity that may arise in the category of warped products of spheres over an interval is the degenerate neckpinch. In this case, Angenent, Isenberg, and Knopf showed in [AIK15] that the final-time limit has the asymptotics $\phi(s) \sim s^{\beta_k}$ where $\beta_k = s^{\frac{2}{2k+1}}$, $k \in \mathbb{N} \setminus \{0\}$. In [Car16] the author constructed flows emerging from metrics with these asymptotic profiles.

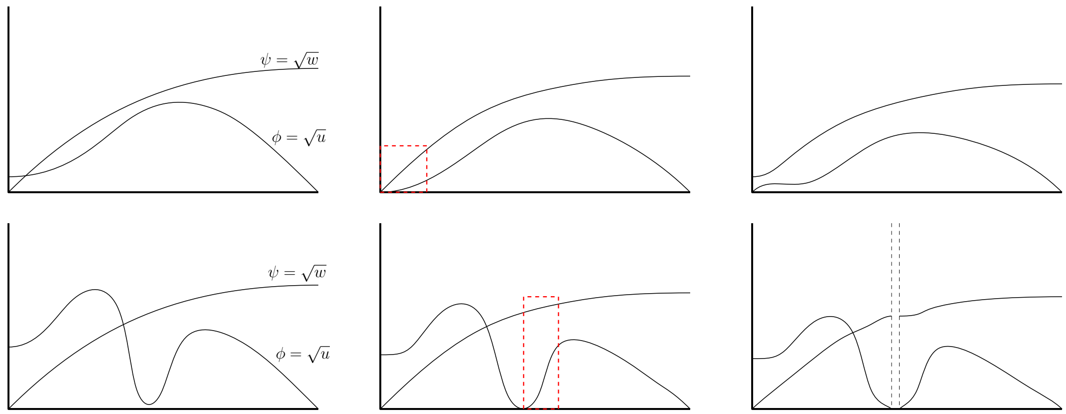

Figure 1.4: Top: the singularity described in Section 1.3.3. Bottom: the singularity described in Section 1.3.4. The pictures depict the manifold, from left to right, before, during, and after the singular time. On each row, the rectangle in the middle picture shows a neighborhood which is a part of a model pinch. (In the second row, there are actually two model pinches: to the left and to the right). In each picture the horizontal axis is the arclength from the left side. The dashed lines in the lower-right figure indicate that the manifold has two connected components.

1.3.3 Generalized cylinder singularities

For a third example of a singularity, consider the doubly-warped product depicted in the top row of Figure 1.4. A more stylized picture of a neighborhood of the singularity is Figure 1.5. The metric is a doubly warped product over an interval, with $(F, g_F) = (S^p, g_{S^p})$, and the singularity occurs at the left endpoint of the interval. Before the singular time, the metric satisfies the following boundary conditions at the left endpoint:

$$\phi > 0, \quad \partial_s \phi = 0, \quad \psi = 0, \quad \partial_s \psi = 0.$$

Here s is the distance from the left endpoint. A neighborhood of the left endpoint has topology $S^q \times D^{1+p}$ before the singular time. For the initial metric, the size of the S^q factor has a deep minimum at the center of the D^{1+p}.

As time goes on, the S^q factor shrinks drastically, and the metric encounters a singularity which can be rescaled to a generalized cylinder $S^q \times \mathbb{R}^{1+p}$. Without rescaling, at the singular time the metric takes on the topology of the cone over $S^q \times S^p$ (but is not asymptotically a metric cone). This singularity has not been rigorously constructed, but we provide a formal argument in Appendix C. We claim that the singular pinched metric should have asymptotics

$$\phi \sim \frac{s}{\log s}, \quad \psi \sim s. \tag{1.5}$$

This is an unsurprising result. The factor corresponding to the S^q behaves similarly to a standard neckpinch. The $1 + p$ dimensional part of the metric, $dx^2 + \psi^2 g_{S^p}$, is close to being a flat D^{1+p}, which corresponds to $\psi = x$ exactly. The flat metric is stable enough that the perturbation from the pinching factor does not affect it too much.

In the forward evolution of metrics with asymptotics (1.5), which we investigate here, the size of the S^p factor expands and the neighborhood takes on the topology $D^{1+q} \times S^p$.

1.3.4 Submanifolds of neckpinches

We can also consider a doubly-warped product over an interval where ϕ has a neck somewhere in the interior of the interval. Then we can force a singularity to occur in the interior of the interval modeled on $\mathbb{R}^{1+p} \times S^q$.

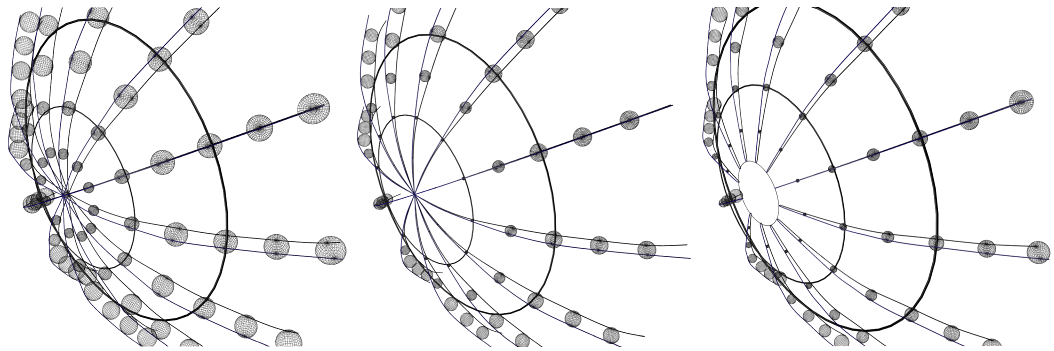

Figure 1.5: A Ricci flow through a model pinch with $q = 2$ and $(F, g_F) = (S^1, g_{S^1})$. The initial picture is a neighborhood with topology $S^2 \times D^{1+1}$, the middle picture has topology of the cone over $S^1 \times S^2$, and last picture has topology $D^{1+2} \times S^1$.

Here there is an S^p worth of singular points. While the previous example was also modeled on $\mathbb{R}^{1+p} \times S^q$, this one will have different asymptotics before the singularity because ϕ it is constant in p directions. Rather than seeing the operator $\Delta_{\mathbb{R}^{p+1}}$ acting on rotationally symmetric functions in Section C.2.1, we will see $\Delta_{\mathbb{R}}$ with p negligible directions.

This type of singularity should be stable in the class of doubly warped products; perturbations leave ϕ with a neck near $x = 0$. However, it should not be stable in the full class of riemannian metrics. It is not even stable in the class of singly warped products

$$g_B + \phi(b)^2 g_{S^q}$$

where $B = \mathbb{R} \times S^p$ and the original metric has $g_B = dx^2 + \psi(x)^2 g_{S^p}$. Indeed, if we allow ϕ to also depend on the S^p factor and perturb it so it has a strict local minimum at some point on that factor, we should approach a singularity at a single point on the S^p factor.

1.3.5 Scarred neckpinches

Here is an example which leads to a metric which is not quite a model pinch. Consider a standard singly warped neckpinch with spheres of dimension S^q: the initial metric is of the form $dx^2+u(x)g_{S^q}$ and the metric at the singular time is a model pinch. This has a forward evolution, which recovers with a smooth (but highly curved) disc of dimension $1+q$ at the tip. So, we have a Ricci flow of a singly warped product, at least on an open subset of $[-1,1]\times S^q$, for times $t\in[T_1,T_2]$, $T_1<0<T_2$.

Now, the Ricci flow of warped products with Einstein fibers does not care about the Riemannian curvature tensor of the fiber metric, it only cares about the Ricci curvature. In other words: suppose we have a Ricci flow on $B\times F$ of the form

$$g_B(t)+u(t)g_{F_1}$$

(where for each t, $u(t):B\to\mathbb{R}_+$) and $\mathrm{Rc}_{g_{F_1}}=\mu g_{F_1}$. Suppose (F_2,g_{F_2}) is another Einstein manifold with $\mathrm{Rc}_{g_{F_2}}=\mu g_{F_2}$. Then

$$g_B(t)+u(t)g_{F_2}$$

is also a Ricci flow.

Therefore, in the Ricci flow through a standard neckpinch, we can swap out g_{S^q} with any Einstein manifold (F_2^q,g_{F_2}) of our choosing, provided it has the same scalar curvature as g_{S^q}. The resulting object satisfies Ricci flow wherever $u > 0$, but is not a manifold for $t > 0$. Around the new points at the tip, the result has the topology of the cone over F_2. The forward evolution has a scar as a result of its surgery.

A special case of this situation is when g_{F_2} is the round metric on $F_2 = S^q/\Gamma$ for some group Γ. This case is important because it clearly cannot be ruled out by a pointwise curvature condition, and so it is relevant to the situations described in Section 1.1.2. The resulting object after the singularity is an orbifold. As we mentioned in Section 1.1.2, this was dealt with in four-dimensions in [CTZ12].

Of relevance to us is the case $q = 2k$ and $F_2 = S^k \times S^k$ (or, $q = k_1 + k_2$ with the correct scaling of two sphere factors). In this case, the metric at the singular time has the form [5]

$$g = dx^2 + u(x)g_{S^k} + u(x)g_{S^k}.$$

It satisfies all of the conditions of a model pinch except for (MP3), since $u = w$ and $\mu_F = \mu$. Since $S^k \times S^k$ is unstable under Ricci flow (we can perturb the size of one of the factors) we thought perhaps there could be two alternative forward evolutions where either of the factors becomes positive after tlahe singular time. We now believe that this is not possible, see Section 1.5.3.

[5] As you may have noticed, we are always lazy with writing the lifts of metrics and tensors etc. This is a place where it looks funny, because $u(x)g_{S^k} = u(x)g_{S^k}$. What we mean is: one of the g_{S^k} is the lift of the standard g_{S^k} under the map $\mathbb{R} \times S^k \times S^k$ to the second factor, and the other is the lift of the standard g_{S^k} under the map $\mathbb{R} \times S^k \times S^k$ to the third factor. Et cetera.

1.4 Shape of the forward evolution

In this section we describe various properties of the forward evolution $g(t)$ of a model pinch. As time goes on, the metric continues to be a doubly warped product:

$$g(t) = a(x,t)dx^2 + u(x,t)g_{S^q} + w(x,t)g_F.$$

Furthermore, we prove that u continues to be increasing in x. Therefore we may write w and $v = u^{-1}|\nabla u|^2$ as functions of u, and

$$g(t) = \frac{du^2}{uv} + ug_{S^q} + wg_F.$$

An advantage of this description is that it is diffeomorphism invariant. It is good to also keep in mind that the function v is invariant under scaling the metric.

For our initial metric, the derivatives of u and w are relatively small. Therefore (after investigating the curvature of warped products) we see that

$$-2\,\mathrm{Rc}(X,Y) \approx -2(q-1)u^{-1}\left(ug_{S^q}\right) - \mu_F w^{-1}\left(wg_F\right) = -\mu g_{S^q} - \mu_F g_F \tag{1.6}$$

Forward in time, this approximation continues to hold for a short time, while the derivatives of u and w continue to be small. We call the region where $v = u^{-1}|\nabla u|^2$ continues to be small the "productish" region. Let $\nu(t) = V_0(\mu t)$. The productish region is the set

$$\left\{(x,t) : \frac{u(x,t)}{t\nu(t)} \geq \sigma_* \text{ and } u < u_*\right\}$$

for some sufficiently large σ_*. In this region, we have $v \leq \epsilon$; by choosing σ_* large and u_* small we can force ϵ as small as we wish.

In the productish region, we get the approximations

$$v \approx V_{prish} := \frac{u + \mu t}{u} V_0 \left(u + \mu t\right) \tag{1.7}$$

$$w \approx W_{prish} := W_0 \left(u + \mu t\right) - \mu_F t. \tag{1.8}$$

Note that these approximations would be exact if the approximation (1.6) were exact and hence $u(x,t) = u(x,0) - \mu t$, $w(x,t) = w(x,0) - \mu_F t$, and

$$v(x,t) = \frac{|\nabla u(x,t)|^2}{u(x,t)} = \frac{|\nabla u(x,0)|^2}{u(x,0)} \frac{u(x,0)}{u(x,t)} = V_0(u + \mu t)\frac{u + \mu t}{u}.$$

In Section 2.2.4 we give some corollaries of our control in the productish region.

Now we come to a crucial juncture in the calculation of our approximate solution. We claim that the approximations (1.7) and (1.8) work for $u(x,t) \geq t\nu(t)$- in particular they work for $u \ll t$. To understand the approximations for small u, write

$$\nu(t) = V_0(\mu t), \quad \hat{V}_0(u/t,t) = \frac{V_0(\mu t(1 + \mu^{-1} u/t))}{\nu(t)},$$

$$\omega(t) = W_0(\mu t), \quad \hat{W}_0(u/t,t) = \frac{W_0(\mu t(1 + \mu^{-1} u/t))}{\omega(t)}.$$

Using our assumptions on V_0 and W_0, particularly (MP4), we can prove $\hat{V}_0(u/t,t) \approx 1 + \mu^{-1}(u/t)\nu^{[1]}(t)$ and $\hat{W}_0(u/t) \approx 1 + \mu^{-1}(u/t)\omega^{[1]}(t)$. Then our approximations say

$$v \approx \mu\sigma^{-1}(1+\mu^{-1}(1+\nu^{[1]}(t))\nu(t)\sigma) \tag{1.9}$$

$$w \approx \omega(t)(1+\mu^{-1}\omega^{[1]}(t)\nu(t)\sigma) - \mu_F t \tag{1.10}$$

where $\sigma = u/(t\nu(t))$.

If the left end of the manifold is to be smooth and compact, v cannot be small up to $u = 0$. In fact, $v = 4$ is a necessary condition at the left endpoint. At the left end, on the factor $I \times S^q$, we glue in a steady Bryant soliton of size $\approx t\nu(t) =: \alpha(t)$. This is a metric on $\mathbb{R}^{1+q}$ that moves only by diffeomorphisms under Ricci flow. We call the region where σ stays bounded, where we see the Bryant soliton, the "tip region". The asymptotics of the Bryant soliton as $u \to \infty$ match with the term $\mu\sigma^{-1}$ in (1.9). That we see a steady soliton is in accordance with the fact that we are scaling at a rate faster than t: as a general principle, if we scaled at rate t we would expect an expanding soliton, whereas if we scale at a faster rate we find a steady soliton.

For the factor F, the warping function is approximately constant. Therefore we expect to be able to attach a large F factor to our Bryant soliton. The approximate size of the unrescaled F factor is $\omega(t) - \mu_F t = W(\mu t) - \mu_F t$. Taking for simplicity the case $\mu_F \neq 0$, our assumptions imply that $\omega - \mu_F t \gtrsim t \gg t\nu(t)$. Therefore when we scale by $t\nu(t)$ the size of this factor goes to infinity, and around any point it approaches a Euclidean factor.

Thus, the zeroth order approximation of the metric near the tip (in other words, the expected limit of the rescaled metric as $t \searrow 0$) is (Bryant Soliton)$\times$

(Euclidean metric). We can get this approximation in a region of the form

$$\left\{(x,t) : \sigma < \nu^{-1/2}\right\}.$$

As $t \searrow 0$ (so $\nu \searrow 0$) this region covers the whole Bryant soliton.

We also need to find the first order approximation near the tip. The perturbation has size $\approx \nu$. The equation we get in space is

$$\text{(Linearization of Ricci Flow)}[g_1] = g_0,$$

where g_0 and g_1 represent the zeroth and first order approximations. This gives us an equation to solve for g_1. Of interest is that on the F factor, the solution coincides with the soliton potential, times g_F. Our first order approximation matches with all of the terms in (1.9), (1.10).

1.5 Sharpness and further questions

1.5.1 Regularity conditions (MP4)

The regularity condition is not too strong, in particular some of the assumptions should be implied if we assume that enough derivatives exist and are monotone. For example, if W_0 can be written as a power of u times something monotonic that grows or dies slower than a power, then ${W_0}^{[1]}$ is finite. Note that an implication of $|{W_0}^{[1]}|+|{W_0}^{[2]}|< C$ is that $\frac{W_0(ru)}{W_0(u)}$ can be bounded for small r, independently of u. In particular, $W_0(u) = e^{u^{-1}}$ and $W_0(u) = e^{-u^{-1}}$ both do not satisfy our assumptions. We cannot offer any guess as to whether our results hold for these functions.

As an example of a more wild profile for W_0, consider $W_0(u) = 2 + \sin(\log(u))$. Note that if the initial metric has bounded length near $u = 0$, then this may appear quite nasty. The curvature of the g_F factor of the metric is rapidly oscillating between positive and negative extremes, but the curvature of the g_{S^q} factor will be dominant. Around any point where $u = u_\sharp$, rescaling by $u_\sharp$ we will see a product metric on a long (size $\approx 1/V_0(u_\sharp)$) scale.

We can think of the conditions on V_0 in the same way, but it may be more reasonable to look at examples in terms of the arclength coordinate. So, consider the $I \times S^q$ part of the metric written as

$$dx^2 + u(x)^2 g_{S^q}, \quad x \in (L_0, \infty), \quad L_0 = 0 \text{ or } L_0 - \infty.$$

The condition that $\frac{u\partial_u V_0(u)}{V_0(u)} < C$ actually says, in a sense, that u must be small enough in terms of x. (Written in terms of x, this condition will involve the functional inverse of u.) The following functions satisfy the regularity conditions on V:

- $L_0 = 0$ and $u(x) = x^p|\log(x)|^q$, where $p > 2$ and $q \in \mathbb{R}$, or $p = 2$ and $q < 0$.
- $L_0 = -\infty$ and $u(x) = |x|^{-p}\log(|x|)^q$, where $p > 0$ and $q \in \mathbb{R}$.
- If we write $u(x) = \exp(-f)$ where $f \to \infty$ as $x \searrow L$, then the condition that $|V_0^{[1]}| < C$ is equivalent to $(1/f')' < C$. For example, $u(x) = \exp(-1/x), L_0 = 0$ or $u(x) = \exp(x), L_0 = -\infty$ are both valid model pinches.

It is possible to drop the requirement that k goes to 5 to k going to $2+\alpha$ in the Hölder sense. As is typical, the regularity theorems we apply (and Taylor's theorem) require a bit more than integer regularity. We are lazy and just drop an integer each application. The same applies in Theorem 1.2.4.

1.5.2 The profile $\phi(x) = \log(|x|)^{-1}$

Our results do not provide a forward evolution from the initial metric with $I = (-\infty, \infty)$, and $u(x) \sim \log(|x|)^{-2}$ at $x = -\infty$. Note in that case

$$v = u^{-1}|\nabla u|^2 = 4\log(|x|)^{-4}x^{-2}$$

so $V_0(u) = u^2 \exp(-2/u)$. Then $V_0^{[1]} = 2u^{-1} + 2$, which violates condition (MP4).

We conjecture that there is no forward evolution from this profile. Here is a possible reason. For any $r > 0$, the region which looks approximately like a skinny cylinder of radius r is quite long in comparison to r. More precisely, fixing ϵ there is a $C_\epsilon > 1$ such that for any r we have the following. The region where the radius ϕ is within a factor of $(1 \pm \epsilon)$ of r has length $(B_\epsilon)^{1/r^2}$, which is very large for small r. The Bryant soliton outside of this initial region does not have enough time to come save it from collapsing before time $t \approx r^2$. See Figure 1.6.

1.5.3 The conditions on the size of W_0 (MP3)

The condition (MP3) is vacuous if $\mu_F \leq 0$. For simplicity say $\mu_F = \mu$.

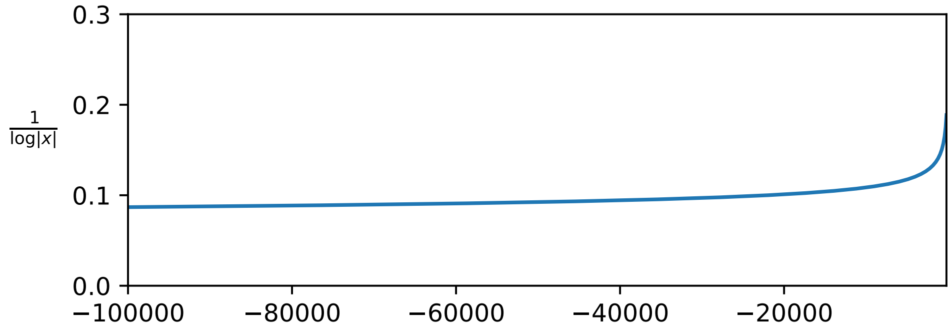

Figure 1.6: The graph of $1/|\log x|$. There is no salvation in sight for such a long cylindrical region.

We believe that it is possible to relax the condition (MP3) and still have a forward evolution with the same asymptotics. Let's rapidly go through a calculation. Suppose $W_0(u) = (1 + H_0(u))u$, where $H_0(u) \searrow 0$ (violating (MP3)). Calculating from (1.8), in the productish region where $u > Ct\nu(t)$,

$$\begin{aligned} w &\approx (1 + H_0(u + \mu t))(u + \mu t) - \mu t \\ &= u + H_0(u + \mu t)(u + \mu t). \end{aligned}$$

If we write $\eta(t) = H_0(\mu t)$ then for points where $Ct\nu(t) < u \ll t$ we have (recall $\sigma := \frac{u}{t\nu(t)}$):

$$\frac{w}{t\nu(t)} \approx \sigma + \mu \frac{\eta(t)}{\nu(t)}. \tag{1.11}$$

First consider the case $H_0(u) \gg V_0(u)$ (i.e. $\eta(t) \gg \nu(t)$). Then scaling w in the same way we scale u sends it to infinity, and w is approximately a constant. I expect this case to behave similarly to the case that is rigorously dealt with in this thesis. The major road block in dealing with it, for us, is reproving Lemma 2.3.5 which controls the derivative of w and therefore controls the level of interaction between the evolution of v and w. Unfortunately our method gives us no more wiggle room in this lemma.

To continue with our speculation, consider the case when $H_0(u) = c_0 V_0(u)$. Then in (1.11) we find

$$\frac{w}{t\nu(t)} = \sigma + c_0\mu.$$

We still would have the approximation (1.9) for v. This gives us the asymptotics for an *Ivey soliton* [Ive94], which is a complete soliton on $\mathbb{R}^{1+q} \times F$ of the form $dx^2 + u_{sol}(x)g_{S^q} + w_{sol}(x)g_F$. (The function $u(x)$ goes to zero at $x = 0$, and $w(x)$ stays positive.) So, in this case I expect to see the Ivey soliton in the rescaled limit at the tip. This case should be more difficult, because the system is more strongly coupled.

In the case when $H_0(u) \ll V_0(u)$, I do not expect a smooth forward evolution, but there may be a forward evolution with bounded Ricci curvature everywhere. In this forward evolution we glue in a Bryant of dimension $1 + (q + \dim(F))$, but with the sphere fibers $S^{q+\dim(F)}$ replaced with the Einstein manifold $S^q \times F$. Indeed, the case $H_0(u) = 0$ is the situation discussed in Section 1.3.5.

The reason I do not expect a smooth forward evolution is the following: consider $H_0(u) = \epsilon V_0(u)$. Then, we are in the case when we expect the Ivey soliton. The exact asymptotics of the Ivey soliton we get are determined by ϵ, and as $\epsilon \searrow 0$, this family of Ivey solitons approaches the Bryant soliton with $S^{q+\dim(F)}$ replaced with $S^q \times F$. Therefore, even trying to approximate the singular initial metric with smooth ones should lead us to the nonsmooth case.

1.5.4 The F-reasonable assumption

These conditions were an annoying thing to organize.

First, something like (MR1) is expected: if we allow perturbations which are not warped products, then we end up dealing with the full curvature of the metric, and we have to control it somehow.

Now, why do we have to make the annoying assumption (MR2) when $\mu_F = 0$? Consider the zeroth order approximation in the tip region, $w \approx W_0(\mu t) - \mu_F t$. Note if $\mu_F > 0$ then by our assumption (MP3), $W_0(\mu t) - \mu_F t \geq c\mu_F t$. If $\mu_F < 0$ then $W_0(\mu t) - \mu_F t \geq \mu_F t$ automatically. However, if $\mu_F = 0$ we don't have a lower bound on w/t.

This makes the case

$$\mu_F = 0, \quad \mathrm{Rm}_{g_F} \not\equiv 0, \quad W_0(u) = o(uV_0(u))$$

interestingly annoying. In this case, the dominant sectional curvatures are those of the F factor, but they have an extreme cancellation in the Ricci flow and they are not in charge of the evolution. The rescaled metrics $\frac{1}{\alpha(t)}g(t)$ actually do *not* converge at the tip to a Bryant soliton with a euclidean factor attached. Instead, the Ricci-flat factor collapses as $t \searrow 0$, even with the rescaling.

We need the assumption (MR2) because we use the convergence to a Bryant Soliton in the proof of the asymmetric case (Theorem 1.2.4).

1.5.5 The closeness required in the asymmetric case

Our condition for Theorem 1.2.4 is that the distance between the asymmetric metric and the model pinch goes to zero near the tip at least as fast as a specific rate ($\epsilon V_0(u)$). (To understand this result it is good to note that the quantity $|g - \Phi^* g_{mp}|$ is scale-invariant.) There is a sense in which this is probably not optimal. Our proof technique yields more than is stated in Theorem 1.2.4: it says that $g(t)$ actually stays close to the forward evolution from g_{mp}. We make no attempt to update the approximate model pinch.

Another theorem that we can compare Theorem 1.2.4 to is Theorem 1.1 of [GS16], and specifically equation (1.1). That theorem constructs forward evolution from metrics close to having conical singularities. There, g_c is a cone and the requirement (1.1) is that near the singularity the singular metric g satisfies $|g - \Phi^* g_c| = o(1)$ as we approach the singularity. This seems stronger than our theorem, because it makes no assumption on the rate at which it approaches the singularity. On the other hand, the case of a cone (which our theorem does not handle) is the case when V_0 is constant, so perhaps our condition is not dissimilar.

1.6 Reaction-diffusion equations

We compare our results to results on reaction diffusion equations:

$$\Box w = aw^p \tag{1.12}$$

Here $\Box$ is the heat operator $\Box = \partial_t - \Delta w$, and we are considering $w : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. The laplacian is the standard euclidean one.

The mathematical interest of such equations revolves around the fact that smooth solutions typically encounter singularities in finite time. By a singularity, we mean a time T for which $\sup_x w(x,t) \nearrow \infty$ as $t \nearrow T$ (if $a > 0$) or $\inf_x w(x,t) \searrow 0$ as $t \searrow T$ (if $a < 0$). For example, if the equation (1.12) is given constant initial data, it reduces to an ordinary differential equation. This can be explicitly solved and has a finite time singularity which occurs in all of space at once.

If the initial value of w is strictly positive, then we can apply standard parabolic theory to conclude that there is a smooth solution w, which will exist up to some time when w is no longer strictly positive. (The basic idea is: as long as w is positive, we can consider (1.12) as a linear equation with a coefficient that happens to depends on w, but is bounded.) However, we should expect that w may hit zero at some finite time- we call this a singularity. Indeed, if w is initially bounded a singularity must occur: the maximum principle tells us that the maximum of w decreases at least as fast as the corresponding ODE for constant solutions. The central goals are to understand what the singularities look like, and whether there is a solution which continues past the singularity.

Next we make the observation that the nature of the singularity is not obvious. Rewrite the equation (1.12) for clarity:

$$\partial_t w = \Delta w - |a| w^p$$

the term $|a|w^p$ is the "reaction" term and the term with the laplacian is the "diffusion" term. Consider the equation with only the reaction term:

$$\partial_t w(x,t) = |a| w^p(x,t).$$

If $p < 0$, then the smaller w is, the faster it decreases. Thus, the evolution makes local minima more pronounced. On the other hand, the diffusion term Δu causes the solution to become more constant. This means there is a fight between the reaction term and the diffusion term, and it's not immediately clear, for example, whether w will typically go to zero at one location or in an interval. In fact, the possible qualitative descriptions of singularities depend heavily on both p and the dimension.

1.6.1 Comparing to warped-product Ricci flow

Let us make an observation about (1.12). If we set $u = w^{1-p}$ then u satisfies, for some constant $a' > 0$ depending on a and p,

$$\Box u = a' \left(-1 + \frac{p}{1-p} u^{-1} |\nabla u|^2 \right) \tag{1.13}$$

This works for both $p < 1$ and $p > 1$, so specifying the exponent p in (1.12) is equivalent to specifying the coefficient $\frac{p}{p-1}$ on $u^{-1}|\nabla u|^2$ in 1.13. An advantage of 1.13 is that performing scaling is computationally easier: it is easy to check that we can scale space like α, and both u and t by α^2, and arrive at the same equation.

Thinking about this transformation in the opposite direction, one makes the following observation. If one encounters an equation like (1.13), or any equation like

$$\Box f = af^r + bf^{-1}|\nabla f|^2$$

the equation can be put into the form (1.12) with a power p depending on both r and b/a. Therefore one must be careful when guessing the qualitative properties of solutions, as they depend on the exact coefficient b.

Now, consider the Ricci flow of a metric of the form $g = g_B + ug_{S^q}$ where $u : B \times [0, T) \to \mathbb{R}_+$. Under Ricci flow, u satisfies ((B.3))

$$\Box_{g_B} u = -\mu + \tfrac{1}{4}(\mu - 2)u^{-1}|\nabla u|^2. \tag{1.14}$$

So u is the choice of function which puts the evolution in the form (1.13). ($Vol := u^{q/2}$ is the choice of function which puts the evolution in the form (1.12).) It is hard to come up with a good comparison to reaction-diffusion equations, because Δ_{g_B} is changing in time and coupled with u. Furthermore, if we chose to consider $\Box_g$ instead of $\Box_{g_B}$, we change the coefficient on $u^{-1}|\nabla u|^2$. We have found that *in our situation* the results match closest to the case (1.13) with $p/(1 - p) > 0$, i.e., $p \in (0, 1)$. To emphasize again that this metaphor is not perfect, note if $q = 2$ then $\mu = 2$ and the right hand side of (1.14) is -2.

1.6.2 Singularity recovery

In [GV97], Galaktionov and Vazquez constructed solutions to (1.12) which continue past singularities. The solution may take values ∞ or 0 on some set. Sometimes, the solution is even constantly infinite or constantly 0 right after the singular time, even if the singularity doesn't occur everywhere. However, they prove that in some cases, the continuation is non-trivial. The construction uses tools from semigroup theory, and relies on comparison for the heat operator is that is not directly applicable here.

Consider the case $n = 1$ and $p \in (-1, 1)$, Galaktionov and Vazquez show, in some cases, a nontrivial continuation. At the singular time, the function gets a zero set which starts expanding. At the edge of the zero-set, the solution looks like a scaled-down version of a traveling wave- a solution to (1.12) which moves by translation only. In the metaphor with warped-product Ricci flow, this traveling-wave plays the role of the Bryant soliton.

1.7 Related Work

1.7.1 Short-time existence results for Ricci flow

We have mentioned [ACK12], which constructed forward evolutions from some specific model pinches on singly warped products and was the original motivation for the current thesis. Recent work that is very close in spirit to ours is [Der16] and [GS16]. In [Der16], Deruelle showed that for any cone with positive curvature [6] , there is an expanding Ricci soliton which limits, backwards in time, to the cone. This can be considered as Ricci flow starting from the singular conical space. In [GS16], Gianniotis and Schulze allow us to start Ricci flow from any manifold which has singularities modeled on some of these cones, similarly to our Theorem 1.2.4.

Another work that deals with forward evolution from singular metrics is [ACF15]. First, Alexakis, Chen, and Fournodavlos prove that there is a singular steady Ricci soliton of the form

$$dx^2 + \phi(x)^2 g_{S^q}$$

where $\phi(x) \sim x^{1/\sqrt{q}}$ at 0 and $q \geq 2$. Furthermore, they show stability of the soliton, so that if $dx^2 + \tilde{\phi}(x)^2 g_{S^q}$ is a metric with $\tilde{\phi}(x)$ close enough to $\phi(x)$, it has a forward evolution which stays close to the soliton. The metric becomes a smooth manifold with boundary, with the boundary growing as time goes on.

Let's summarize the results for metrics of the form $dx^2 + \phi(x) g_{S^q}$, $q \geq 2$, ignoring requirements on derivatives of ϕ. The case $\phi(x) \sim x$ is the case when the metric is smooth. The case $\phi(x) \sim ax$ for $a < 1$ is covered by [GS16]. The case $\phi(x) \lesssim ax$ for all $a < 1$ is covered here. For $\phi \gtrsim x$, we just have [ACF15] which deals with $\phi \sim x^{1/\sqrt{q}}$.

The first result on existence for Ricci flow was Hamilton's original paper [Ham82]. There, short-time existence for complete compact manifolds was

[6] i.e. a metric $dx^2 + x^2 g_X$, where $\mathrm{Rm}[g_X] \geq 1$.

proven using the Nash-Moser inverse function theorem. A more complicated argument than is standard was needed because the equation, considered locally in coordinates, is not strictly parabolic. In [DeT83], DeTurck used a method now known as the DeTurck trick to turn the equation into a strictly parabolic one, yielding a shorter proof. DeTurck had previously done similar work on the local prescribed Ricci problem, i.e. the elliptic version of the problem [DeT81]. We deal more with the DeTurck trick in Section 3.1.1.

In [Shi89], Shi proved existence for (possibly noncompact) complete manifolds with bounded curvature. Shi solves the Dirichlect problem on compact sets (using DeTurck's trick) and takes a limit. Furthermore, he gave local derivative estimates- analogues of those generally available for parabolic equations- which are widely used. A nontrivial consequence of the result is that any metric with bounded curvature has a comparable metric with all covariant derivatives of the curvature bounded (where the bounds involved only depend on the dimension and the initial bound on the curvature). The new metric is found by running Ricci flow for a short time from the initial metric with bounded curvature.

As it is here, existence for Ricci flow from a class of metrics is usually tied to a uniform bound on existence time for some "approximating" class. Supposing we can show uniform short-time existence and estimates for a class of Riemannian manifolds, then any manifold which lies in the "boundary" of that class should have a Ricci flow obtained by taking the limit of Ricci flows in that class.

In [Sim02], Simon provides an evolution of C^0 metrics by Ricci curvature. The requirement is that the C^0 metric has bounded curvature (in that it is the limit of metrics with bounded curvature). This has topological implications, for example, such a space must topologically be a manifold.

There has been success for showing existence of Ricci flow assuming just a lower bound on (some flavor of) curvature. The lower bound may be non-negative curvature, almost non-negative curvature [7], or just the existence of some (possibly negative) lower bound. The metrics which we deal with in this thesis and are relevant to recovery from singularities satisfy

$$\mathrm{Rm} \geq -f(|\mathrm{Rm}|)|\mathrm{Rm}| \tag{1.15}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is some function satisfying $f(x) \to 0$ as $x \to \infty$. (Hence they have positive scalar curvature, although they do not have positive Ricci curvature.) We don't know of any work that deals with just the assumption (1.15). Furthermore, we can make model pinches (unrelated to recovery from singularites) which have no such bound (for example, by taking $\mu_F < 0$ and making $w \ll u$).

In [Sim09], Simon shows that one can flow a three-dimensional metric space with an upper bound on diameter, lower bound on the volume of balls, and nonnegative curvature. This has topological implications for 3-manifolds with nearly nonnegative curvature. [Sim12] starts the flow from

[7] E.g., the curvature is larger than $-\epsilon_0$ where ϵ_0 depends on the dimension and a lower bound on the volume of balls of radius 1. In some places in the literature almost non-negative is taken to mean (1.15), but that is not the case in the works cited here.

an arbitrary 3-manifold with only a (possibly negative) lower bound on Ricci curvature, assuming also a very mild condition on the curvature growth at infinity. Cabezas-Rivas and Wilking [CRW11] show short-time existence assuming non-negative complex sectional curvature. They use the fact that such manifolds have an exhaustion by compact convex sets, and they cosntruct a Ricci flow on the compact sets. Xu [Xu13] shows short-time existence assuming just a lower bound on Ricci curvature, and an integral estimate on the full curvature tensor in balls.

Bamler, Cabezas-Rivas, and Wilking [BCRW17] deal with Riemannian manifolds with a variety of almost-nonnegative curvature assumptions, including almost non-negative curvature operator and almost non-negative complex sectional curvature. They show that the almost non-negativity is preserved for a short time, with estimates only depending on the lower bound on curvature and the lower bound on the volume of balls. They also provide existence for non-compact manifolds with no upper bound on curvature (by taking limits of Ricci flows on an exhaustion by open sets). In particular [BCRW17] gives an alternative route to dealing with some of the conical singularities in [GS16].

In a different flavor, in [Top10] Topping created Ricci flows where the initial metric is an incomplete surface, and the flow instantanteously becomes complete with the same topology. As an example, a plane with a point removed instantaneously devolops a cusp of infinite length. By now, Topping and others have deveoloped a strong well-posedness theory for instantaneously complete Ricci flow in dimension two. Any surface, possibly incomplete and possibly with unbounded curvature, has a unique forward flow which is instantaneously complete [GT11], [Top15].

Topping also constructed, in [Top11], a two-dimensional Ricci flow on $\mathbb{R}^2$ for $t \in (0, T_2)$, called a contracting cusp. The metric is smooth and complete, but on $\mathbb{R}^2 \setminus \{0\}$, as $t \searrow 0$ the metric limits to a metric with an infinite-length cusp. (In fact, the construction in [Top11] is more general than stated here.) Therefore, from the infinite-length cusp, there are two reasonable forward evolutions. One is the unique complete forward evolution on the same topology, and the other is the topology-changing evolution which compactifies the cusp. The results in this thesis include similar contracting cusp flows in higher dimensions, e.g. a metric $\mathbb{R}^3$ which limits to a cusp on $\mathbb{R}^3 \setminus \{0\}$. I suspect that in dimensions higher than two, there is no complete forward evolution from a cusp on the same topology.

1.7.2 Warped Products and Ricci Flow

Singly and doubly warped products are important sources of examples in riemannian geometry and Ricci flow. The metrics are on the topology $M = B^m \times N^q$, for some manifold B which we call the base. The metrics have the form

$$g = g_B + \phi^2(b) g_N,$$

where g_B is a metric on B, g_N is a metric on N, and $\phi : B \to \mathbb{R}_+$. We use N for the fiber manifold here because it doesn't necessarily represent either the S^q

factor or the F factor in our model pinches. However, we need to assume that g_N is an Einstein manifold: $2\operatorname{Rc}[g_N] = \mu_N g_N$. If we don't assume this then the metric ceases to be a warped product under Ricci flow: even pretending that the Ricci tensor on each fiber is just the Ricci tensor of g_N, the fibers would perform Ricci flow at different rates and therefore cease to be scalings of each other.

The equation for Ricci flow on a warped product becomes, with $q = \dim(N)$,

$$\partial_t g = -2\operatorname{Rc}[g_B] + 2q\phi^{-1}\nabla\nabla\phi$$
$$\partial_t \phi = \Delta_B \phi - \tfrac{1}{2}\mu_N(1 - |\nabla\phi|^2)$$

In this thesis we are mostly concerned with doubly warped products over intervals, i.e. metrics of the form

$$a(x)dx^2 + \phi^2(x)g_{S^q} + \psi^2(x)g_F, \quad x \in I.$$

These are singly warped products in two ways: with base $I \times S^q$ and fiber F or with base $I \times F$ and fiber S^q. Both points of view have been useful for our intuition. A big simplification for a doubly warped product over an interval is that the hessian of a function of x is much simpler than that of a function of a general base.

Many Einstein manifolds and Ricci solitons are doubly-warped products. See for example [Böh98], [LPP04] in the Einstein case and [Bry], [Ive94] for Ricci solitons. A similar class of metrics which leads to many examples is cohomogeneity-one metrics [Cao96], [FIK03], [B9̈9], [DW09], [DHW11], [DW11], [BDGW15], [Sto15], [Win17], [App17]. There has also been a lot of work classifying Einstein manifolds or Ricci solitons which are singly-warped products over higher-dimensional bases, see [KK03], [CSW11], [PW10], [HPW12], [MZ15].

Furthermore, as we have mentioned warped products over intervals give many examples of singularity formation in Ricci flow, with many types of singularities. We have mentioned [Sim00], [AK04], [AK07], [AIK15]. Other examples on cohomogeneity-one metrics are [IKŠ16], [IKŠ17]. In [LŠ14], Lott and Šešum give a long-time result for Ricci flow with flat fibers over two-dimensional manifolds, a similar result is [Mar17].

Chapter 2

Forward flow from symmetric metrics

2.1 Overview of the proof and tools

Before anything, I want to make the reader aware of Appendix D, which starts on page 208, and densely lists a lot of the notation we use. I hope it is of use.

In this chapter we prove Theorem 1.2.2. We construct the forward evolution $g(t)$ as a limit of mollified flows. This is completed in Section 2.4.

As we mentioned in Section 1.4, the forward evolution has two regions: the productish region where the metric continues to look like a product (as the initial metric does) and the tip region where we glue in a steady soliton. Really, there is a third region, the "outer" region, which is the complement of these two. Since the initial metric is assumed to be smooth in the outer region, it is not a burden to control the metric there.

Appendix A proves generic estimates for the type of PDE encountered in the productish region. The results of Appendix A are used in Section 2.2 and also much later in Section 3.3.3. Results in Sections 2.2 and 2.3 will control our mollified flows in the productish and tip regions, respectively. For a pleasant reading, we recommend skipping the internals of Appendix A, and

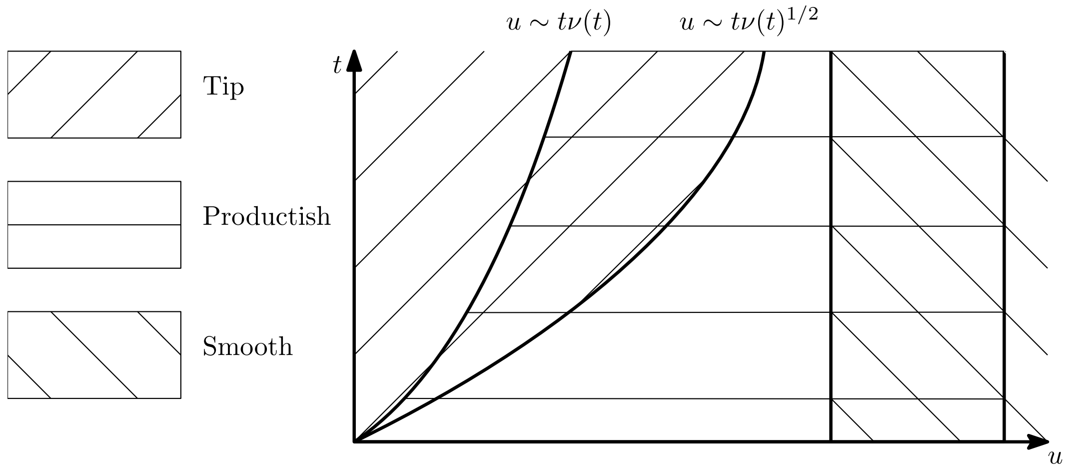

Figure 2.1: Map of the tip, productish, and outer regions

Sections 2.2 and 2.3 at first. Hopefully our overview here and summarizing remarks in Section 2.4 suffice.

In each region, we find an approximate solution. The main lemma of Section 2.2 is Lemma 2.2.2. This lemma assumes:

- A Ricci flow is close to our approximate solution in the productish region at its initial time.
- The Ricci flow satisfies a priori control around the left boundary of the productish region, which is strictly within tip region.
- The Ricci flow satisfies a priori control around the right boundary of the productish region, which is strictly within the outer region.

and implies control within the productish region. Similarly, the main Lemma of Section 2.3 is Lemma 2.3.2. This lemma assumes:

- A Ricci flow is close to our approximate solution in the tip region at its initial time.
- The Ricci flow satisfies a priori control around the right boundary of the tip region, which is strictly within the productish region.

and implies control within the tip region.

The control that we get in both cases is of the form $V^- < v < V^+$ and $W^- < w < W^+$, where $v = u^{-1}|\nabla u|^2$ and w are the functions associated to the evolving warped product metric $a(x,t)dx^2 + u(x,t)g_{S^q} + w(x,t)g_F$, and $V^\pm$ and $W^\pm$ are functions of u.

In Section 2.4.1, we remove some of the a priori assumptions of Lemmas 2.2.2 and 2.3.2. We show that, if the time is small enough, a metric satisfying the conclusion of Lemma 2.3.2 satisfies the a priori estimate at the left edge of the productish region which is required to apply Lemma 2.2.2, Also, a metric satisfying the conclusion of Lemma 2.2.2 satisfies the a priori estimate at the right edge of the tip region which is required to apply Lemma 2.3.2. It might seem like we are stuck with a circular argument, but we can get around it. We go through the motions here, since it's a type of argument we use often. Assuming we have the required bounds at an initial time T_1, consider the maximal interval $[T_1, T_2)$ such that the conclusions of Lemmas 2.2.2 and 2.3.2 hold. By continuity $T_2 > T_1$ (since we have a bit of room in the estimates at the beginning) and at time T_2 the conclusions hold with some strict inequalities replaced by non-strict ones. These non-strict ones are enough to show the a priori estimates required; so we would get a contradiction if T_2 was small enough for us to apply our lemmas.

In Section 2.4.2, for $m > 0$ we define mollified initial metrics $g_{init}^{(m)}$ which satisfy the initial assumptions of Lemmas 2.3.2 and 2.2.2. These mollified initial metrics agree with the metric g_{mp} on the set $\{u > m\}$. They have bounded curvature, so there is a Ricci flow $g^{(m)}(t)$ starting from $g_{init}^{(m)}$ for some time $T_{exist}^{(m)}$ which we would like to show is bounded from below independently of m.

In Section 2.4.3 we control the curvature of the mollified initial metrics, first in the outer region. This control allows us to prove the final a priori estimates required by Lemma 2.2.2 at the right-hand side of the productish region. Therefore the forward evolution of our mollified metrics satisfies the conclusions of Lemmas 2.2.2 and 2.3.2. These conclusions also imply that the forward evolution has bounded curvature in the productish and tip regions as well, so we get a lower bound on the existence time.

Lemma 2.4.11 proves Theorem 1.2.2 by constructing a (subsequential) limit of the Ricci flows from $g_{init}^{(m)}$.

2.1.1 Equations

Under Ricci flow, u evolves by

$$\Box_B u = -\mu + \tfrac{1}{4}(\mu - 2)v.$$

Equivalently,

$$\Box_M u = -\mu - v.$$

The function w which controls the size of g_F evolves by

$$\Box_M w = -\mu_F - y = -\mu_F - w^{-1}|\nabla w|^2.$$

We use this point of view to find the approximate solutions in the productish region. For an exposition of these equations for Ricci flow on warped products, see Section B.1.

For finer control, we need the evolution of v and w as functions of u. These are derived in Sections B.2.1 and B.2.2. We have

$$\begin{aligned}\partial_{t;u} v = {} & uv\partial_u^2 v - \tfrac{1}{2}u(\partial_u v)^2 \\ & + \mu\left(1 - \tfrac{1}{4}v\right)u^{-1}v + \mu\partial_u v \\ & - 2(\kappa^2)v,\end{aligned} \tag{2.1}$$

where $\kappa^2 = \frac{1}{4}(dim(F))w^{-2}u^2v^2(\partial_u w)^2$, and

$$\partial_{t;u} w - uv\partial_u^2 w = -\mu_F - y + \mu\partial_u w - \mu/2v\partial_u w. \tag{2.2}$$

2.1.2 The Maximum Principle

The maximum principle is a basic tool in the study of elliptic and parabolic PDE. It lets us control solutions to a PDE by functions, called *supersolutions* and *subsolutions*, which oversolve and undersolve the equation. Some of the points in this section might be interesting even if you knew that.

Here's the most basic statement in the parabolic case. We consider the domain $(x,t) \in [a,b] \times [0,T]$, for some $a,b \in \mathbb{R}$. We have a function $v : [a,b] \times [0,T] \to \mathbb{R}$. A simple example of a parabolic equation is $(\partial_t - \partial_x^2)v = F(v, \partial_x v)$, for some function $F : \mathbb{R} \to \mathbb{R}$. Suppose that v is a solution to this equation, and v^+ satisfies $(\partial_t - \partial_x^2)v^+ > F(v^+, \partial_x v^+)$ (so v^+ is a "strict supersolution" to this equation). Furthermore, suppose $v > v^+$ at time 0, and for all times $v \geq v^+$ on the boundary of (a,b). Then $v > v^+$ everywhere in $(0,T) \times (a,b)$.

To prove this, consider a first time t_0 when $v = v^+$ doesn't hold on the entire interval, and any location x_0 in (a,b) where $v = v^+$. A contradiction follows from

$$\partial_t v \geq \partial_t v^+, \quad v = v^+, \quad \partial_x v = \partial_x v^+, \quad \partial_x^2 v \leq \partial_x^2 v^+.$$

The generalization to the case $v : M \to \mathbb{R}$ for a smooth manifold M is immediate. In this case ∂_x^2 is replaced with some elliptic operator. For instance, we think in this way when dealing with our equations in the tip region, because there the left endpoint where $u = 0$ is a fake endpoint: it can be considered as in the interior of the completion of the warped product manifold. Other important generalizations deal with the case when we only know $(\partial_t - \partial_x^2)v^+ \geq F(v, \partial_x v)$ instead of a strict inequality, but we won't need that here.

What we would like to talk about here is dealing with $\boldsymbol{v} : M \to \mathbb{R}^2$ and other systems. In our situation this bold face $\boldsymbol{v}$ encapsulates the two evolving functions v and w. In such a general situation we have to be more careful with what we mean by our inequalities. In our case, we need nothing fancy and $\boldsymbol{v} < \boldsymbol{v}^+$ just means that it is true componentwise, so $v < v^+$ and $w < w^+$.

Now, though, dealing with gradient terms is not so easy. At a first time when either $v < v^+$ or $w < w^+$ is violated, we cannot say that $\nabla v = \nabla v^+$ and $\nabla w = \nabla w^+$! Only one of them will be true. If our equation is of the form $(\partial_t - \partial_x^2)\boldsymbol{v} = F(\boldsymbol{v}, \nabla \boldsymbol{v})$ then the simple version of the maximum principle doesn't carry through word for word. The problem is in terms where the gradients interact. For a simple example consider

$$(\partial_t - \Delta_M)v = 7v + |\nabla w|^2$$
$$(\partial_t - \Delta_M)w = 13w + v(\nabla w)^2.$$

(For a real example, consider (2.1) wherein κ^2 is a term containing some derivatives of w.) The last term in the first line causes difficulty, because $v = v^+$ does not give us any information on $|\nabla w|^2$. The last term in the second line is not as much of a problem. If we assume that in fact we have control from both sides, $v^- < v < v^+$ for some $v^\pm$ which are close together, then we can control the v part, and when w touches w^+ we learn the value of ∇w.

In our situation, we deal with all of these terms by just getting a bound on ∇w. These bounds come from regularity, and are presented in Lemma 2.2.5 (for the productish region) and Lemma 2.3.5 (for the tip region). This step was the largest stumbling block at each generalization of the requirements on the function W_0 for the model pinch. In fact, as mentioned in Section 1.5.3, I feel that it is still not optimal.

2.1.3 Regularity

Regularity theory is fundamental to the study of elliptic and parabolic PDE. In our case, we are interested in using it to get bounds which are somehow uniform. Of course it is quite technical, but it is extraordinarily beautiful once one takes a step back and realizes how non-obvious yet important the statements are. Generally, citations of regularity are dismal, and we don't help. As a user of regularity theory, it is usually safe to say that the result appears somewhere in [Lie96] or [OAL95], which are 452 and 648 pages[1] respectively and do not contain indices for their (differing) notation. The notes [Kry91] are easier to read but less general. The situation is especially bad for users of parabolic regularity; everything is just a standard generalization of the elliptic case. If the reader is lucky, the relevant case appears as an exercise.

Bamler wrote a refreshingly clean statement of the interior Schauder estimates he needed in [Bam14] (Section 2.5). We co-opt this statement, because it is exactly what we need except for standard generalizations. His statement does not allow for the time-dependence of the coefficients that we will have, but in fact the proof carries through exactly; the time dependence enters in the estimate on the $C^{2m-2,2\alpha;m-1,\alpha}$ norm of f_i in the middle of page 424. Furthermore, his statement does not allow the parabolic ball to hit the initial time, as we will need to. Accounting for this is also standard. In the proof of

[1] Single-spaced, not in double-spaced wide-equation thesis format

Lemma 2.6 of [Bam14], one may apply Exercise 9.2.5 of [Kry91] rather than Theorem 8.11.1 of [Kry91].

2.2 Control in the productish region

We define the productish region as a region on our forward evolutions of the form

$$\Omega_{prish} = \left\{ (u,t) : u + \mu t < u_* \text{ and } \sigma = \frac{u}{tV_0(\mu t)} > \sigma_* \right\}.$$

We will prove that in this region, the scale-invariant form of the gradient of u, namely $v = u^{-1}|\nabla u|^2$, is bounded by $C\max(\sigma^{-1}, V_0(u))$, which we can make as small as we like by choosing σ_* large and u_* small. We tested the name "productish" across many markets, and it was overwhelmingly met with confusion. To us, though, it is clear: the productish region is where the metric is nearly a product. We take the further liberty of using the abbreviation "prish" as a subscript.

All constants and definitions in this section implicitly depend on dimensions, g_F, and the chosen functions satisfying the model pinch conditions V_0 and W_0. In the productish region, we will have approximations of the form

$$v \approx V := \left(\frac{u + \mu t}{u} \right) V_0(u + \mu t)$$

and

$$w \approx W := W_0(u + \mu t) - \mu_F t.$$

These come directly from the calculations in Appendix A. They may be guessed by ignoring all terms in the evolution of v or w which depend on space derivatives of v or w.

More precisely, we will prove that v is between V^- and V^+, and w is between W^- and W^+, where

$$V^{\pm} = (1 \pm DV)V, \quad W^{\pm} = (1 \pm DV)W_0(u + \mu t) - \mu_F t. \tag{2.3}$$

We make some definitions to state the main result of this section. We will assume that $g(t) = a(x,t)dx^2 + u(x,t)g_{S^q} + w(x,t)g_F$ is a solution to Ricci flow on $[T_1, T_2]$. Our definitions depend on constants u_*, σ_*, and D, as well as c_{safe} and C_{reg}.

Definition 2.2.1. *We say that $g(t)$ is* barricaded *(by the productish barriers)* [2] *at a point if it satisfies*

$$V^- < v < V^+, \quad W^- < w < W^+$$

at that point.

We say that $g(t)$ is initially controlled in the productish region *if at $t = T_1$ and for all points satisfying $(1/2)\sigma_* T_1 \nu(T_1) < u < 2u_*$ it is barricaded and, for $k = 1, 2, 3$,*

$$u^k \partial_u^k v < u^k \partial_u^k V + c_{safe} C_{reg} D V^2, ,$$

$$u^k \partial_u^k w < u^k \partial_u^k W + c_{safe} C_{reg} D V W..$$

[2] In this section we only say "barricaded" but in Section 2.4 we will have to refer to either barricaded by the productish barriers, or barricaded by the tip barriers.

We say that $g(t)$ *is* barricaded at the left of the productish region *if it is barricaded for all points satisfying* $(1/2)\sigma_* t\nu(t) < u < \sigma_* t\nu(t)$ *and* $t \in [T_1, T_2]$.

We say that $g(t)$ *is* barricaded at the right of the productish region *if it is barricaded for all points satisfying* $u_* < u < 2u_*$ *and* $t \in [T_1, T_2]$.

We say that $g(t)$ *is* controlled in the productish region *if*

(P1) For all points in Ω_{prish}*, the solution is barricaded.*

(P2) For all points in Ω_{prish}*, and for* $k = 1, 2$,

$$u^k \partial_u^k v < u^k \partial_u^k V + C_{reg} D V^2,$$
$$u^k \partial_u^k w < u^k \partial_u^k W + C_{reg} D V W.$$

Lemma 2.2.2. *Let* $c_{safe} < \overline{c}_{safe} < 1$, $C_{reg} > \underline{C}_{reg}$, $D > \underline{D}$, $u_* < \overline{u}_*(D, C_{reg})$, *and* $\sigma_* > \underline{\sigma}_*(D, C_{reg})$. *Suppose* $0 < T_1 < T_2 < T_*$ *where* T_* *may depend on all other constants.*

Suppose $g(t)$ *is initially controlled, and barricaded at the left and the right, of the productish region. Then* $g(t)$ *is controlled in the productish region.*

In proving the conclusions of Lemma 2.2.2, we can assume that they hold on the interval $[T_1, T_2)$. This is because they are both true at the initial time by our assumption, and if they would fail at some time, by continuity of the functions involved there is a first time $T_{bad} > T_1$ such that at least one of them fails and the strict inequality becomes equality somewhere. Therefore the conclusions hold on the interval $[T_1, T_{bad})$. If this implies that they hold at T_{bad}

as well, then we have a contradiction. This extra assumption is usually useful for controlling terms when we don't care about the exact constant involved, because in any case we can choose our constants u_*, σ_*, and T_* so that it is as small as we want (see e.g. Lemma 2.2.5).

With this in mind, Lemma 2.2.2 will be proven by Lemmas 2.2.6 and 2.2.7 below, which show items (P1) and (P2) respectively. First, in Section 2.2.1, we inspect our approximations V and W more closely.

2.2.1 Examining our approximate solution

We are claiming that $v(p,t) \approx V(u(p,t),t)$ where V is the function

$$\mathrm{V}(u,t) = \frac{u+\mu t}{u}\mathrm{V}_0(u+\mu t) = \left(1+\mu\frac{t}{u}\right)\mathrm{V}_0(u+\mu t) \tag{2.4}$$

The effectiveness of the barriers defined in (2.3) is dependent on V staying small. In this section, we prove Lemma 2.2.4 which tells us that V does stays small exactly in the productish region Ω_{prish}, and also gives another description of V and W. The proof is elementary, but the reformulation of V is key to how the productish region hooks up with the tip region.

We aim to understand where V stays small. An apparent scary term in (2.4) is t/u. Defining $\rho = u/t$, we can write

$$\mathrm{V} = \left(1+\mu\rho^{-1}\right)\mathrm{V}_0(u+\mu t).$$

If we keep in mind that our main assumption on V_0 is that $\mathrm{V}_0(u) = o(1, u \to 0)$, then the following lemma, which says something about where V is small, is immediately apparent.

Lemma 2.2.3. *Let ϵ be given. For any ρ_* there is $u_*(\rho_*, \epsilon, V_0, \mu)$ so that*

$$\{(u,t) : u + \mu t < u_* \text{ and } \rho > \rho_*\} \subset \{(u,t) : V < \epsilon\}.$$

The discussion is not over: V does not get large if we fix ρ and send $u+\mu t \searrow 0$, as the factor $V_0(u+\mu t)$ helps us. To understand this factor better, let

$$\nu(t) = V_0(\mu t), \quad \hat{V}_0(\rho,t) = \frac{V_0\left(\mu t(1+\mu^{-1}\rho)\right)}{V_0(\mu t)}.$$

Then by definition,

$$V_0(u+\mu t) = \nu(t)\hat{V}_0(\rho,t).$$

By a straightforward calculation with Taylor's theorem, given in Lemma B.3.3,

$$\hat{V}_0(\rho,t) = 1 + \frac{1}{\mu}\rho\nu^{[1]}(t) + O(\rho^2; \rho \to 0)$$

where for t uniformly bounded (i.e. $0 \le t \le T_*$) the $O(\rho^2)$ term is uniform in t. This calculation uses the bound on ${V_0}^{[1]}$ and ${V_0}^{[2]}$. (The bound on ${V_0}^{[2]}$ is needed to control a remainder bound in Taylor's theorem.) Note that the assumption that ${V_0}^{[1]}(u) = \frac{u\partial_u V_0(u)}{V_0(u)}$ is bounded is equivalent to a bound on $\nu^{[1]}(t) = \frac{t\nu'(t)}{\nu(t)}$.

Let $\sigma = \rho/\nu(t) = u/(t\nu(t))$. Now we can write,

$$\begin{aligned} V &= \left(\mu\sigma^{-1} + \nu\right) V_1(\rho,t) \\ &= \mu\sigma^{-1}\left(1 + (1+\nu^{[1]})\mu^{-1}\nu\sigma + O((\nu\sigma)^2)\right). \end{aligned} \tag{2.5}$$

This makes it apparent that if we look at where $\sigma > \sigma_*$ for some large σ_*, V is still small. We present Lemma 2.2.4.

Lemma 2.2.4. *Let ϵ be given. There is $\sigma_*(\epsilon)$ and $u_*(\sigma_*, \epsilon)$ so that*

$$\{(u,t) : u + \mu t < u_* \text{ and } \frac{u}{t\nu(t)} > \sigma_*\} \subset \{(u,t) : V < \epsilon\}$$

Proof. (Lemma 2.2.4). First, choose $\underline{\sigma}_*$ small enough, and $\overline{u}_*$ at least small enough, so that $(\sigma^{-1} + \nu(t)) < \epsilon/100$ for all u, t satisfying $\sigma > \underline{\sigma}_*$ and $u + \mu t < \overline{u}_*$.

Next, by the expression (2.5), we can choose ρ_*, and decrease $\overline{u}_*$, so that for $\sigma > \sigma_*$ and $\rho = \nu\sigma < \rho_*$, we have $V < \epsilon/50$.

Finally, by Lemma 2.2.3 we can chose $\overline{u}_*$ so that $V < \epsilon$ for all u, t satisfying $\rho > \rho_*$ and $u + \mu t < \overline{u}_*$. □

We also examine the approximate solution for w, namely $W = W_0 \circ U_0 - \mu_F t$. Recall that we assume that $W_0(u)/u$ goes to infinity as $u \searrow 0$. This implies that W/t goes to infinity as $t \searrow 0$. Similarly to how we handled V, we may write

$$W = \omega(t)\left(\hat{W}(\rho, t) - \mu_F \frac{t}{\omega(t)}\right)$$

where

$$\omega(t) = W_0(\mu t), \quad \hat{W}(\rho, t) = \frac{W_0\left(\mu t\left(1 + \mu^{-1}\rho\right)\right)}{W_0\left(\mu t\right)}.$$

By Lemma B.3.3, we will have

$$\hat{W}(\rho, t) = 1 + \frac{\rho}{\mu}\omega^{[1]}(t) + O(\rho^2)$$

with the big-oh uniform for small t. Now we can write, with asymptotics as $\rho \searrow 0$,

$$\begin{aligned} W &= \omega(t)\left(1+\mu^{-1}\rho\omega^{[1]}(t)+O(\rho^2)-\mu_F\frac{t}{\omega(t)}\right) \\ &= \omega(t)\left(1+\mu^{-1}\nu\sigma\omega^{[1]}(t)+O((\nu\sigma)^2)-\mu_F\frac{t}{\omega(t)}\right) \end{aligned} \tag{2.6}$$

2.2.2 Trapping between barriers

We recall our equations. The metric $g(t)$ satisfies Ricci flow, the warping function u satisfies

$$\Box_M u = -\mu u + c_v v,$$

and if we set $\hat{w} = w + \mu_F t$ then $\hat{w}$ satisfies

$$\Box_M \hat{w} = -y = -\frac{|\nabla \hat{w}|^2}{\hat{w} - \mu_F t}. \tag{2.7}$$

We wish to apply Lemma A.1.9. To do this, we need to prove the required bound on the Hessian of u. This will be implied by an estimate on $y = w^{-1}|\nabla w|^2$, given below.

Lemma 2.2.5. *Suppose we are in the setting of Lemma 2.2.2. Assume additionally that items (P1) and (P2) hold on $[T_1, T_2)$. If $\sigma_* > \underline{\sigma}_*(D, C_{reg})$ and $u_* < \underline{u}_*(D, C_{reg})$ then*

$$\frac{uy}{vw} < C_{ybnd}$$

in Ω_{prish}, where C_{ybnd} only depends on the initial data.

Proof. Note that

$$\begin{aligned}\frac{uy}{vw} &= \frac{u^2|\nabla w|^2}{|\nabla u|^2 w^2} \\ &= \left(\frac{u\partial_u w}{w}\right)^2\end{aligned}$$

So by item (P2),

$$\frac{uy}{vw} \leq \left(\frac{u\partial_u W}{w} + C_{reg} DV \frac{W}{w}\right)^2. \tag{2.8}$$

By Lemma 2.2.4 we can decrease u_* and increase σ_* so that $C_{reg}DV < 1$ and $\frac{W}{W^-} < 2$. Then since w is between its barriers, we can bound w in (2.8) in terms of W.

$$\begin{aligned}\frac{uy}{vw} &\leq 4\left(\frac{u\partial_u W}{W} + 1\right)^2 = 4\left(\frac{u\partial_u W_0(u+\mu t)}{W_0(u+\mu t) - \mu_F t} + 1\right)^2 \\ &= 4\left(\frac{W_0(u+\mu t)}{W_0(u+\mu t) - \mu_F t} {W_0}^{[1]}(u+\mu t) + 1\right)^2.\end{aligned} \tag{2.9}$$

By the assumption (MP3) on W_0,

$$\begin{aligned}\frac{W_0(u+\mu t)}{W_0(u+\mu t) - \mu_F t} &= \frac{1}{1 - \frac{\mu_F t}{W_0(u+\mu t)}} \\ &\leq \frac{1}{1 - \frac{\mu_F t}{(1+c)\frac{\mu_F}{\mu} u + (1+c)\mu_F t}} \\ &\leq \frac{1}{1 - \frac{1}{1+c}}\end{aligned}$$

Therefore (2.9) is bounded by a constant depending only on the initial data, using also our assumption (MP4) that $W_0^{[1]} = \frac{u\partial_u W_0}{W_0}$ is bounded. □

Now we are in the position to prove that (P1) continues to hold.

Lemma 2.2.6. *Suppose we are in the setting of Lemma 2.2.2, and items (P1) and (P2) holds on* $[T_1, T_2)$.

If $D > \underline{D}$, $u_* < \overline{u}_*(D, C_{reg})$, $\sigma_* > \underline{\sigma}_*(D, C_{reg})$, $T_* < \overline{T}_*(D, u_*, \sigma_*)$ *then (P2) holds at* $t = T_2$.

Proof. Lemma A.1.9 proves the statement for v, assuming a bound on $|\nabla\nabla u|^2$. For a metric of the form given, the hessian of a function f which depends only on x satisfies

$$|\nabla\nabla f|^2 = \tfrac{1}{4}|\nabla f|^{-4}\langle\nabla|\nabla f|^2, \nabla f\rangle^2 + |\nabla f|^2|A|^2$$

where $|A|^2$ is the norm-squared of the second fundamental form of the level sets, namely

$$|A|^2 = \tfrac{1}{4}qu^{-1}v + \tfrac{1}{4}\dim(F)w^{-1}y.$$

Therefore we find,

$$\begin{aligned}
|\nabla\nabla u|^2 &\leq C\left(|\nabla u|^{-4}|\nabla|\nabla u|^2||\nabla u|^2 + u^{-1}v|\nabla u|^2 + w^{-1}y|\nabla u|^2\right) \\
&= C\left(u^{-1}v^{-1}|\nabla(uv)|^2 + v^2 + w^{-1}yuv\right) \\
&\leq C\left(uv^{-1}|\nabla v|^2 + \left(1 + \frac{u}{v}\frac{y}{w}\right)v^2\right)
\end{aligned}$$

Since we have $\frac{u}{v}\frac{y}{w} \leq C_{ybnd}$ from Lemma 2.2.5, the Hessian bound necessary in Lemma A.1.9 is taken care of.

Now, we can write (2.7) as,

$$\Box_M \hat{w} = -\frac{uy}{v\hat{w}}(u^{-1}v\hat{w}) = -\frac{\hat{w} - \mu_F t}{\hat{w}}\frac{uy}{vw}(u^{-1}v\hat{w})$$

We can bound the factor $\frac{uy}{vw}$ by C_{ybnd}. Furthermore we can bound $\frac{\hat{w}-\mu_F t}{\hat{w}}$ using (MP3) as in the proof of Lemma 2.2.5. So,

$$|\Box_M \hat{w}| \leq CC_{ybnd}(u^{-1}v\hat{w})$$

Lemma A.1.7 tells us that by choosing $\underline{D}$ large enough, we will have that $(1 \pm DV)W_0(u + \mu t)$ are sub- and supersolutions to this equation. $\Box$

2.2.3 Regularity

Lemma 2.2.7. *Suppose we are in the setting of Lemma 2.2.2. We can choose $\overline{c}_{safe}$, $\underline{C}_{reg}$, $\underline{u}_*$, and $\underline{T}_*$ such that if (P1) holds for $t \in [T_1, T_2)$ then (P2) holds for $t \in [T_1, T_2]$.*

Proof. We prove this theorem by applying parabolic regularity to the equations solved by v and w in terms of u. From (2.1) and (2.2), we have the equations

$$\begin{aligned}\partial_{t;u}v - \mu\partial_u v - \mu u^{-1}v = (uv)\,\partial_u^2 v - \tfrac{1}{2}u\,(\partial_u v)^2 \\ + a_1 v\partial_u v + a_2 u^{-1}v^2 + a_3\left(\frac{vu}{w}\right)^2(\partial_u w)^2\end{aligned}$$

and

$$\partial_{t;u}w - \mu\partial_u w = (uv)\,\partial_u^2 w - \mu_F + b_1 v\partial_u w + b_2\left(\frac{vu}{w}\right)(\partial_u w)^2,$$

where a_1, a_2, a_3 and b_1, b_2 are constants.

We let $z = u + \mu t$, $\hat{v} = z^{-1}uv$, and similarly $\hat{V} = z^{-1}uV = V_0(z)$. Also, let $\hat{w} = w + \mu_F t$, and similarly $\hat{W} = W + \mu_F t = W_0(z)$. Calculate,

$$\begin{aligned}
v &= u^{-1} z \hat{v}, \\
\partial_u v &= -\mu t u^{-2} \hat{v} + u^{-1} z \partial_z \hat{v} \\
\partial_u^2 v &= 2\mu t u^{-3} \hat{v} - 2\mu t u^{-2} \partial_z \hat{v} + u^{-1} z \partial_z^2 \hat{v} \\
\partial_u \hat{w} &= \partial_z \hat{w}, \quad \partial_u^2 \hat{w} = \partial_z^2 \hat{w}.
\end{aligned}$$

Also note that

$$\partial_{t;u} v - \mu \partial_u v - \mu u^{-1} v = u^{-1} z \partial_{t;z} \hat{v}$$

and

$$\partial_{t;u} w - \mu \partial_u w + \mu_F = \partial_{t;z} \hat{w}.$$

Therefore,

$$\begin{aligned}
(u^{-1} z) \partial_{t;z} \hat{v} = (z\hat{v}) & \left(u^{-1} z \partial_z^2 \hat{v} + 2\mu t u^{-3} \hat{v} - 2\mu t u^{-2} \partial_z \hat{v} \right) \\
& - \tfrac{1}{2} u \left(-\mu t u^{-2} \hat{v} + u^{-1} z \partial_z \hat{v} \right)^2 \\
& + a_1 (u^{-1} z \hat{v})(-\mu t u^{-2} \hat{v} + u^{-1} z \partial_z \hat{v}) \\
& + a_2 u^{-1} (u^{-1} z \hat{v})^2 + a_3 \left(\frac{vu}{w} \right)^2 (\partial_u \hat{w})^2
\end{aligned}$$

which simplifies to, for some constants c_1, c_2, c_3, c_4,

$$\begin{aligned}
(u^{-1} z) \partial_{t;z} \hat{v} = (z\hat{v}) & u^{-1} z \partial_z^2 \hat{v} \\
& + c_1 t u^{-3} (z\hat{v}) \hat{v} + c_2 t u^{-2} (z\hat{v}) \partial_z \hat{v} \\
& + c_3 u^{-1} z^2 (\partial_z \hat{v})^2 + c_4 \left(\frac{vu}{w} \right)^2 (\partial_u \hat{w})^2
\end{aligned}$$

and then, multiplying by uz^{-1},

$$\begin{aligned}\partial_{t;z}\hat{v} &= z\hat{v}\partial_z^2\hat{v} \\ &\quad + c_1 t u^{-2}\hat{v}^2 + c_2 t u^{-1}\hat{v}\partial_z\hat{v} \\ &\quad + c_3 z(\partial_z\hat{v})^2 + c_4 u z^{-1}\left(\frac{vu}{w}\right)^2(\partial_u\hat{w})^2\end{aligned}$$

We also derive the evolution for $\hat{w}$:

$$\partial_{t;w}\hat{w} = (z\hat{v})\partial_z^2\hat{w} + b_1(u^{-1}z\hat{v})\partial_z\hat{w} + b_2\left(\frac{z\hat{v}}{w}\right)(\partial_z\hat{w})^2.$$

Now, let u_1, t_1 be any point in the productish region, let $z_1 = u_1 + \mu t_1$, $\hat{v}_1 = \hat{v}(u_1, t_1)$, and $\hat{w}_1 = \hat{w}(u_1, t_1)$. Divide through in both equations by $z_1\hat{v}_1$. Also divide the equation for $\hat{v}$ by $\hat{V}_1 = \hat{V}(u_1, t_1) = V_0(z_1)$ and the equation for $\hat{w}$ by $\hat{W}_1 = \hat{W}(u_1, t_1) = W_0(z_1)$.

$$\begin{aligned}\frac{1}{z_1\hat{V}_1}\partial_{t;z}\left(\frac{\hat{v}}{\hat{V}_1}\right) &= \left[\frac{z\hat{v}}{z_1\hat{V}_1}\right]\partial_z^2\hat{v} \\ &\quad + c_1\left[\frac{t}{z_1}\frac{\hat{v}}{\hat{V}_1}\frac{u_1^2}{u^2}\right]u_1^{-2}\left(\frac{\hat{v}}{\hat{V}_1}\right) + c_2\left[\frac{t}{z_1}\frac{\hat{v}}{\hat{V}_1}\frac{u_1}{u}\right]u_1^{-1}\partial_z\left(\frac{\hat{v}}{\hat{V}_1}\right) \\ &\quad + c_3\left[\frac{z}{z_1}\right]\left(\partial_z\left(\frac{\hat{v}}{\hat{V}_1}\right)\right)^2 + c_4\left[\frac{v}{v_1}\frac{u^2}{zz_1}\frac{w_1^2}{w^2}\frac{v}{v_1}\right]\left(\partial_u\left(\frac{w}{w_1}\right)\right)^2\end{aligned}$$

$$\begin{aligned}\frac{1}{z_1\hat{V}_1}\partial_{t;z}\left(\frac{\hat{w}}{\hat{W}_1}\right) &= \left[\frac{z\hat{v}}{z_1\hat{V}_1}\right]\partial_z^2\left(\frac{\hat{w}}{\hat{W}_1}\right) \\ &\quad + b_1\left[\frac{z\hat{v}}{z_1\hat{V}_1}\right]u^{-1}\partial_z\left(\frac{\hat{w}}{\hat{W}_1}\right) + b_2\left[\frac{z}{z_1}\frac{\hat{v}}{\hat{V}_1}\frac{\hat{w}}{w}\frac{\hat{w}}{\hat{W}_1}\right]\left(\partial_z\left(\frac{\hat{w}}{\hat{W}_1}\right)\right)^2\end{aligned}$$

We will apply interior parabolic regularity to these equations, in the region

$$\Xi = \{(z,t) : (z,t) \in [z_1 - \delta u_1, z_1 + \delta u_1] \times [t_1 - \max(T_1, t_1 - \delta z_1^{-1}v_1^{-1}u_1^2), t_1],\}$$

which is a parabolic ball around (z_1, t_1) of radius δu_1, if we were to scale time to $\hat{t} = z_1 \hat{V}_1 t$. We choose $\delta < \frac{1}{2}$, so that this parabolic ball lies in the region Ω'_{prish} from Lemma 2.2.2. We have written the equation so that the factors in square brackets are smooth functions of u, t, $\frac{\hat{v}}{\hat{V}_1}$, and $\frac{\hat{w}}{\hat{W}_1}$ in this parabolic ball- this requires the knowledge that v and w are trapped between our barriers, so for example $\frac{w}{w_1}$ is not too far from 1 within Ξ. The important thing about this smoothness is that we have bounds on relevant quantities (e.g., the C^3 norm of the functions) are not dependent on u_1 or t_1.

In [Bam14], Bamler wrote a cleanly-stated regularity theorem for non-linear systems taking this form. See the discussion in Section 2.1.3.

All in all, we can apply regularity to bound the z derivatives of the functions $\frac{\hat{v}}{\hat{V}_1} - \frac{\hat{V}}{\hat{V}_1}$ and $\frac{\hat{w}}{\hat{W}_1} - \frac{\hat{W}}{\hat{W}_1}$. Our barriers tell us that the C^0 norm for both of these, in Ξ, is bounded by $CDV(u_1, t_1)$, where C depends on on the initial functions only. This implies, for some bigger constant C we have,

$$\begin{aligned} &\left| u_1 \partial_z \left(\frac{\hat{v} - \hat{V}}{\hat{V}_1} \right) \right| + \left| u_1^2 \partial_z^2 \left(\frac{\hat{v} - \hat{V}}{\hat{V}_1} \right) \right| \\ &+ \left| u_1 \partial_z \left(\frac{\hat{w} - \hat{W}}{\hat{W}_1} \right) \right| + \left| u_1^2 \partial_z^2 \left(\frac{\hat{w} - \hat{W}}{\hat{W}_1} \right) \right| \le CDV(u_1, t_1), \end{aligned}$$

for all points in Ξ- in particular for $u = u_1$, $t = t_1$. The point (u_1, t_1) was not special, so we have this inequality at all points in the productish region.

Now we convert this back to a statement in terms of u. Just using the definition of the quantities, calculate

$$
\begin{aligned}
\frac{u}{V}|\partial_u(v-V)| &= \frac{u}{V}|\partial_u\left(\frac{u+\mu t}{u}(\hat{v}-\hat{V})\right)| \\
&\leq \frac{u}{V}\mu t u^{-2}|\hat{v}-\hat{V}|+\frac{u}{V}|\partial_u(\hat{v}-\hat{V})| \\
&= \mu\frac{u^{-1}t}{V}|\hat{v}-\hat{V}|+\frac{u}{V}|\partial_u(\hat{v}-\hat{V})| \\
&= \mu\frac{z^{-1}t}{V}|v-V|+\frac{u}{z}\frac{u}{\hat{V}}|\partial_u(\hat{v}-\hat{V})|.
\end{aligned}
$$

Now using our barriers for the first term, and using the bound for regularity on the second term, as well as $\frac{t}{z} \leq 1$ and $\frac{u}{z} < 1$,

$$\frac{u}{V}|\partial_u(v-V)| \leq CDV$$

Performing similar calculations, we can make the bounds

$$\frac{u^2}{V}|\partial_u^2(v-V)|+\frac{u}{W}|\partial_u(w-W)|+\frac{u^2}{W}|\partial_u^2(w-W)| \leq CDV$$

□

2.2.4 Corollaries of control

The following corollaries state some precise results which hold for a metric satisfying the conclusions of Lemma 2.2.2. The corollaries above are just a matter of checking various derivatives and bounds. For Corollary 2.2.9 one can use the calculations of the curvatures for warped products in Appendix B.1.4.1.

First we rephrase our results in terms of how close the metric is to a cylinder.

Corollary 2.2.8. *Suppose that $g(t)$ is controlled in the productish region at time $t = t_{\#}$.*

For $u_{\#}$ such that $(u_{\#}, t_{\#})$ is in the productish region, let

$$g_{cyl} = dx^2 + g_{S^q} + \frac{W_{prish}(u_{\#}, t_{\#})}{u_{\#}} g_F.$$

Let L be given such that $\epsilon = L\sqrt{V_{prish}(u_{\#}, t_{\#})} < 1$. There is a map $\Phi : [-L, L] \times S^q \times F \to M$ which is the identity on the second two factors such that $u(\Phi(0, \cdot, \cdot), t_{\#}) = u_{\#}$ and

$$\left| g_{cyl} - \Phi^* \left(u_{\#}^{(-1)} g(t_{\#}) \right) \right|_{C^2([-L,L]\times S^q \times F)} \leq \epsilon C$$

We also state a result in terms of the curvature of the metrics.

Corollary 2.2.9. *Suppose $g(t)$ is controlled in the productish region. Then there is a constant C such that for all points in the productish region the curvature of $g(t)$ satisfies*

$$\begin{aligned} \mathrm{Rm} &= u^{-1} \left(u g_{S^q} \owedge u g_{S^q} \right) + w \, \mathrm{Rm}_{g_F} + \mathrm{Rm}_{warp} \\ &= u \, \mathrm{Rm}_{g_{S^q}} + w \, \mathrm{Rm}_{g_F} + \mathrm{Rm}_{warp} \end{aligned}$$

where $|\mathrm{Rm}_{warp}| \leq C u^{-1} v$.

One more basic statement about the curvature is the following.

Corollary 2.2.10. *Suppose $g(t)$ is controlled in the productish region. If $\mu_F = 0$, suppose F has constant curvature. Then for a larger C, for all points in the productish region,*

$$|\mathrm{Rm}| \leq \frac{C}{t\nu(t)}$$

Proof. In Corollary 2.2.9, since V is uniformly bounded in the productish region, we get $|\mathrm{Rm}_{warp}| \leq Cu^{-1}$. We also have $|u\,\mathrm{Rm}_{g_{S^q}}| \leq Cu^{-1}$. (In fact, it is exactly $C_q u^{-1}$ for some constant C_q depending on q. In the productish region, we have $u \geq t\nu(t)\sigma_*$, so $u^{-1} \leq \frac{1}{\sigma_*}\frac{1}{t\nu(t)}$.

We have $|w\,\mathrm{Rm}_{g_F}| = C_F w^{-1}$. If $\mu_F < 0$, then $W_0(u+\mu t) - \mu_F t \geq (-\mu_F)t$. If $\mu_F > 0$, then our assumption (MP3) tells us $W_0(u+\mu t) - \mu_F t \geq c\mu_F t$. If $\mu_F = 0$, then seince we assumed F has constant sectional curvature, $C_F = 0$. $\square$

2.3 Control in the tip region

We are still considering a Ricci flow of the form

$$g(t) = a(x,t)dx^2 + u(x,t)g_{S^q} + w(x,t)g_F, \quad t \in [T_1, T_2].$$

Section 2.2.1 shows that our approximate solutions in the productish region work up to where $\sigma = \frac{u}{t\nu(t)}$ stays very large. In order to examine the solution where σ is bounded, we will rescale the metric g by $\alpha = t\nu(t)$: set $\tilde{g} = \alpha^{-1}g$.

Since the approximate solution for w, coming into the tip region, is on the order of $\omega(t) \gg t\nu(t) = \alpha(t)$, rescaling w to $\tilde{w} = \alpha^{-1}w$ will cause $\tilde{w}$ to be unbounded. Instead, we will work with the function $\bar{w} = \omega^{-1}(w + \mu_F t)$, and since $\omega \gtrsim t \gg \alpha$ the effects of w on our equation for v will "scale away". (Some of this sentence is false for certain model pinches where $\mu_F \leq 0$, but everything works out in any case.)

For other functions related to g, we will decorate them with a tilde for their scaled version. For example, $L = u^{-1}(1-\frac{1}{4}v)$ is the sectional curvature of g for a plane tangent to the S^q factor, and $\tilde{L} = \sigma^{-1}(1-\frac{1}{4}v)$ is the corresponding sectional curvature for $\tilde{g}$. We also introduce a rescaled time derivative $\partial_\theta = \alpha\partial_t$.

In this section, we find the approximate solutions for v and $\bar{w}$ [3] :

$$V := V_{Bry}(\sigma) + \beta V_{Pert}(\sigma)$$
$$\bar{W} := 1 + (\log\omega)_\theta W_{Pert}(\sigma)$$

where $\beta = \alpha'$, $(\log\omega)_\theta = \partial_\theta(\log\omega)$, and V_{Bry}, V_{Pert}, and W_{Pert} are functions which are to be defined. In Lemmas 2.3.7 and 2.3.8 we define functions $V^\pm$ and $W^\pm$, which satisfy $V^- < V < V^+$ and $\bar{W}^- < \bar{W} < \bar{W}^+$, and will serve as barriers for v and $\bar{w}$. These functions depend on constants ϵ_v, ϵ_w, and δ.

The barriers V^- and V^+ are carefully defined so that if $V^- < v < V^+$ then $\tilde{L}$ is bounded near $\sigma = 0$. We write $L_{approx} = \sigma^{-1}(1 - \frac{1}{4}V)$. Finally, we introduce the notation $x^{a,b} = x^a(1+x)^{b-a}$; which is approximately x^a near $x = 0$ and x^b near $x = \infty$.

We make definitions similar to Definition 2.2.1. The tip region will be, for a constant ζ_* to be determined,

$$\Omega_{tip} = \left\{(u,t) : \frac{u}{t\nu(t)} < \frac{\zeta_*}{\nu^{1/2}},\ t \in [T_1, T_2]\right\}.$$

[3] We use the same notation V and $V^\pm$ here for different functions than the barriers in Section 2.2. In the following section, where we need to refer to both the functions defined here and the functions from Section 2.2, we will use e.g. V_{tip} for the function defined here and V_{prish} for the function defined there.

Definition 2.3.1. *We say that $g(t)$ is* barricaded (by the tip barriers) *at a point if it satisfies*

$$V^- < v < V^+, \quad W^- < w < W^+$$

at that point.

We say that $g(t)$ is initially controlled in the tip region *if at $t = T_1$ for all points satisfying $\sigma \le 2\zeta_* \nu^{-1/2}(T_1)$ it is barricaded, for $\sigma > 1$ and for $k = 1, 2, 3$,*

$$\begin{aligned} |\sigma^k \partial_\sigma^k v - \sigma^k \partial_\sigma^k V| &\le c_{safe} C_{reg} \left(\delta^{-1}\epsilon_v\right) \nu^{1/2} \sigma^{-1}, \\ |\sigma^k \partial_\sigma^k \bar{w} - \partial_\sigma^k \bar{w}| &\le c_{safe} C_{reg} \epsilon_w \nu^{1/2}. \end{aligned} \tag{2.10}$$

and for $\sigma \le 1$ and for $k = 1, 2, 3$,

$$\begin{aligned} |\sigma^{k/2} \partial_\sigma^k \tilde{L} - \sigma^{k/2} \partial_\sigma^k \tilde{L}| &\le c_{safe} C_{reg} \left(\delta^{-1}\epsilon_v\right) \nu^{1/2} \\ |\sigma^{k/2} \partial_\sigma^k \bar{w} - \sigma^{k/2} \partial_\sigma^k \bar{W}| &\le c_{safe} C_{reg} \epsilon_w \nu^{1/2} \end{aligned}$$

We say that $g(t)$ is barricaded at the right of the tip region *if it is barricaded for all points satisfying $\zeta_* \nu^{-1/2} < \sigma < 2\zeta_* \nu^{-1/2}$.*

We say that $g(t)$ is controlled in the tip region *if*

(T1) For all points in Ω_{tip}, the solution is barricaded.

(T2) For all points in Ω_{tip} with $\sigma \ge 1$ and for $k = 1, 2$,

$$\begin{aligned} |\partial_\sigma^k v - \partial_\sigma^k V| &\le C_{reg} \left(\delta^{-1}\epsilon_v\right) \nu^{1/2} \sigma^{-1} \\ |\partial_\sigma^k \bar{w} - \partial_\sigma^k \bar{W}| &\le C_{reg} \epsilon_w \nu^{1/2} \end{aligned}$$

(T3) For all points in Ω_{tip} with $\sigma \leq 1$ and for $k = 1, 2$,

$$|\sigma^{k/2}\partial_\sigma^k \tilde{L} - \sigma^{k/2}\partial_\sigma^k \tilde{L}| \leq C_{reg}\left(\delta^{-1}\epsilon_v\right)\nu^{1/2}$$
$$|\sigma^{k/2}\partial_\sigma^k \bar{w} - \sigma^{k/2}\partial_\sigma^k \bar{W}| \leq C_{reg}\epsilon_w \nu^{1/2}$$

Remark 1. V satisfies $(1/C)\sigma^{0,-1} < V < C\sigma^{0,-1}$ and W satisfies $(1/C) < W < C$. This is the reason for the factor σ^{-1} in (2.10). Furthermore, $(\delta^{-1}\epsilon_v)$ controls the separation between the barriers for v, whereas ϵ_w controls the separation between the barriers for w- this explains is the reason for the appearance of those constants.

The following is the main result of this section.

Lemma 2.3.2. *Let $0 < c_{safe} < \bar{c}_{safe} < 1$, $C_{reg} > \underline{C}_{reg}$, ϵ_v, $\epsilon_w < \bar{\epsilon}_w(\epsilon_v)$, ζ_*, and $\delta < \bar{\delta}(\zeta_*)$ be given. Suppose $0 < T_1 < T_2 < T_*$ where T_* may depend on all other constants.*

Suppose $g(t)$ is initially controlled in the tip region, and barricaded at the right of the tip region. Then $g(t)$ is controlled in the tip region.

2.3.1 A summary of functions

We will be introducing many functions of σ. Here, we provide the reader with a little cheat sheet to recall the asymptotics of the functions. This makes us feel better about possibly using the asymptotics without warning.

We use the notation $\sigma^{a,b} = \sigma^a(1+\sigma)^{b-a}$ and $|F|_2 = F + \sigma\partial_\sigma F + \sigma^2\partial_\sigma^2 F$. As usual, $c < C$ are constants depending only on the given model pinch.

We have

$$c\sigma^{0,-1} < |V_{Bry}|_2 < C\sigma^{0,-1}$$
$$c\sigma^{1,0} < |V_{Pert}|_2 < C\sigma^{1,0}$$
$$c\sigma^{0,1} < |W_{Pert}|_2 < C\sigma^{0,1}.$$

As $\sigma \to \infty$ we have

$$V_{Bry} = \mu\sigma^{-1} + O(\sigma^{-2}), \quad V_{Pert} = \tfrac{1}{2} + O(\sigma^{-1}), \quad W_{Pert} = \tfrac{1}{2}\mu\sigma + O(\log\sigma)$$

Our approximate solutions are $V = V_{Bry} + \beta V_{Pert}$ and $W = 1 + (\log\omega)_\theta W_{Pert}$. Here are crude bounds on our barriers: for $\nu^{1/2}\sigma < \zeta_*$

$$\tfrac{1}{2}V_{Bry} < V^- < V < V^+ < 2V_{Bry}$$
$$\tfrac{1}{2} < \bar{W}^- < \bar{W} < \bar{W}^+ < 2.$$

More precise bounds are given in Lemma 2.3.6: for $V_{diff} = V^+ - V$ or $V_{diff} = V - V^-$

$$c\left(\delta^{-1}\epsilon_v\right)\nu^{1/2}\sigma^{1,-1} \leq V_{diff} \leq C\left(\delta^{-1}\epsilon_v\right)\nu^{1/2}\sigma^{1,-1},$$

and for $W_{diff} = W^+ - W$ or $W_{diff} = W - W^-$

$$c\epsilon_w\nu^{1/2} \leq W_{diff} \leq C\epsilon_w\nu^{1/2}.$$

2.3.2 Type-II rescaling

Define $\alpha(t) = t\nu(t)$, $\partial_\theta = \alpha\partial_t$, and $\beta = \alpha'$. Recall $\tilde{g} = \frac{1}{\alpha}g$ and $\sigma = \frac{1}{\alpha}u$. Note that $v = u^{-1}|\nabla u|_g^2 = \sigma^{-1}|\nabla\sigma|_{\tilde{g}}^2$.

We continually use the notation

$$\partial_\sigma = |\nabla\sigma|_{\tilde{g}}^{-2}\tilde{\operatorname{grad}}\sigma,$$

$$\partial_{\theta;\sigma} = \partial_\theta - (\partial_\theta\sigma)\partial_\sigma.$$

Note that $\tilde{\operatorname{grad}}\sigma = \operatorname{grad} u$ and $|\nabla\sigma|_{\tilde{g}}^2 = \alpha|\nabla u|_g^2$ so $\partial_u = \alpha^{-1}\partial_\sigma$. We can also calculate,

$$\begin{aligned}
\partial_{t;u} &= \partial_t - (\partial_t u)\partial_u \\
&= \alpha^{-1}\partial_\theta - (\partial_t(\alpha\sigma))(\alpha^{-1}\partial_\sigma) \\
&= \alpha^{-1}\partial_\theta - \alpha^{-1}\beta\sigma\partial_\sigma - \alpha^{-1}(\partial_\theta\sigma)\partial_\sigma \\
&= \alpha^{-1}\left(\partial_{\theta;\sigma} - \beta\sigma\partial_\sigma\right) \qquad (2.11)
\end{aligned}$$

We define $\mathcal{Q}_\sigma$ and $\mathcal{L}_\sigma$ to be $\mathcal{Q}$ and $\mathcal{L}$, from (B.10), with ∂_u replaced with ∂_σ. Then using our equation for $\partial_{t;u}v$, (B.10), we find

$$\begin{aligned}
\partial_{\theta;\sigma}v &= \sigma^{-1}\mathcal{Q}_\sigma[v,v] + \sigma^{-1}\mathcal{L}_\sigma[v] + \beta\sigma\partial_\sigma v \qquad (2.12) \\
&\quad - 2\tilde{\kappa}^2 v
\end{aligned}$$

where $\tilde{\kappa} = \frac{1}{4}\dim(F)\tilde{w}^{-1}y$. Let

$$\mathcal{F}_\sigma[v,\tilde{\kappa}] = \left(\sigma^{-1}\mathcal{Q}_\sigma[v,v] + \sigma^{-1}\mathcal{L}_\sigma[v] - 2\tilde{\kappa}^2 v\right).$$

So (2.12) is $\partial_{\theta;\sigma}v - \mathcal{F}_\sigma[v,\tilde{\kappa}] - \beta\sigma\partial_\sigma v = 0$.

If v converges, as a function of σ, as $\theta \searrow -\infty$, and κ goes to 0, then this equation tells us that v converges to a solution v_S to $\mathcal{F}_\sigma[v_S, 0] = 0$. This is the equation for a singly-warped steady soliton.

2.3.3 The Bryant Soliton

The Bryant soliton (Bry, g_{Bry}, f_{Bry}) is a steady Ricci soliton on the topology $Bry = \mathbb{R}^{q+1}$. The metric is a singly warped product over the interval $[0, \infty)$

$$g_{Bry} = ds^2 + \sigma_{Bry}(s) g_{S^q}$$

which we may equivalently write as

$$g_{Bry} = \frac{du^2}{uv_{Bry}} + ug_{S^q}.$$

This soliton is the only steady Ricci soliton with this structure, besides the Euclidean metric on $\mathbb{R}^{q+1}$. [4]

For the Bryant soliton, u is strictly increasing as a function of s. As a steady soliton, under Ricci flow it moves only by diffeomorphisms, which fix the warped product structure. Since u increasing is a diffeomorphism-invariant property, u remains increasing under Ricci flow. The value of $v = u^{-1}|\nabla u|^2$ at a point where $u = u_*$ is also a diffeomorphism-invariant property, so $\partial_{t;u} v_{Bry} = 0$.

In other words, considering the function V_{Bry} so that $v_{Bry}(p) = V_{Bry}(u(p))$, we have

$$\mathcal{Q}[V_{Bry}, V_{Bry}] + \mathcal{L}[V_{Bry}] = 0.$$

The Bryant soliton has strictly positive sectional curvature, and its scalar curvature has a maximum at $u = 0$. The soliton is defined up to scaling

[4] TODO cite

and diffeomorphism, so let's say we have chosen the scaling with maximum scalar curvature μ. As $u \to \infty$, V_{Bry} has the asymptotics

$$V_{Bry}(u) = (1 + O(\sigma^{-1}))\mu u^{-1} \tag{2.13}$$

and as $u \to 0$, V_{Bry} has the asymptotics

$$V_{Bry}(u) = 4\left(1 - \frac{\mu}{q(q-1)}u + o(u)\right). \tag{2.14}$$

For any $k > 0$ we may scale the metric by k^{-1}, resulting in the Bryant soliton with maximum scalar curvature $k\mu$. The corresponding function V_{kBry} is related by

$$V_{kBry}(u) = V_{Bry}(ku).$$

2.3.4 Approximation for v

Suppose that v satisfies (2.12), and also converges sufficiently smoothly to a limit v_0 as $\theta \searrow -\infty$. Suppose also that $\tilde{\kappa}^2$ converges to zero as $\theta \searrow 0$. Then we learn,

$$\mathcal{Q}_\sigma[v_0, v_0] + \mathcal{L}_\sigma[v_0] = 0.$$

That is, v_0 describes a steady soliton.

If the limit metric has $\sigma \in [0, \infty)$, and has nonzero curvature, then we learn that as a function of u, $v_0 = V_{kBry}(u)$ for some scaling factor k. Comparing the asymptotics (2.13) with our approximate solution in the parabolic region (2.5), we choose $k = 1$.

Now we address the term $\beta\sigma v_\sigma$. This term suggests that our approximation $v \approx v_0$ for small θ is off by a term of order β. Write $\tilde{v}(\sigma,\theta) = v_0(\sigma) + \beta v_1(\sigma)$, and plug into $\partial_{\theta;\sigma} v - \mathcal{F}_\sigma[v,\kappa] - \beta\sigma\partial_\sigma v = 0$. This gives us,

$$\begin{aligned}\partial_{\theta;\sigma} v - \mathcal{F}_\sigma[v,\kappa] - \beta\sigma\partial_\sigma v = \beta_\theta v_1 \\ - \beta\left(2\sigma^{-1}\mathcal{Q}[v_0,v_1] + \sigma^{-1}\mathcal{L}[v_1] + \sigma\partial_\sigma v_0\right) \\ + (\dots)\end{aligned}$$

Here the term $(\dots)$ is bounded by

$$|\dots| \leq C\beta^2\left(\sigma^{-1}|v_1|_2^2 + \beta^{-2}\kappa^2|v_0| + \beta^{-1}\kappa^2|v_1|\right)$$

By elementary calculations using our assumed bound on $\nu^{[1]} + \nu^{[2]}$ we can also bound $\beta_\theta v_1$. (See Section B.3.3.) We have

$$\begin{aligned}\partial_{\theta;\sigma} v - \mathcal{F}_\sigma[v,\kappa] - \beta\sigma\partial_\sigma v = -\beta\left(2\sigma^{-1}\mathcal{Q}[v_0,v_1] + \sigma^{-1}\mathcal{L}[v_1] + \sigma\partial_\sigma v_0\right) \\ + \beta^2 E\end{aligned}$$

where

$$E \leq C\left(|v_1| + \sigma^{-1}|v_1|_2^2 + \beta^{-2}|\tilde{A}|^2|v_0| + \beta^{-1}\tilde{\kappa}^2|v_1|\right)$$

Concerning the equation approximately satisfied by v_1, we have the following lemma, which is Lemma 4 of [ACK12]. Here we use the notation $x^{a,b} = x^a(1+x)^{b-a}$.

Lemma 2.3.3. *There is a solution V_{Pert} to*

$$2\sigma^{-1}\mathcal{Q}_\sigma[V_{Bry}, V_{Pert}] + \sigma^{-1}\mathcal{L}[V_{Pert}] = -\sigma\partial_\sigma V_{Bry}.$$

on $[0,\infty)$, which extends to a smooth even function on $(-\infty,\infty)$.

The function $V_{kPert}(u) = k^{-1}V_{Pert}(ku)$ is a solution to

$$2\sigma^{-1}\mathcal{Q}_\sigma[V_{kBry}, V_{kPert}] + \sigma^{-1}\mathcal{L}[V_{kPert}] = -\sigma\partial_\sigma V_{kBry}.$$

As $\sigma \to \infty$, V_{kPert} has the asymptotics

$$V_{kPert} = (1 + O(\sigma^{-1}))k^{-1}. \tag{2.15}$$

There is a $C > 0$ depending on the dimension such that

$$|V_{Pert}|_2 < C\sigma^{1,0} \tag{2.16}$$

This invites the choice of approximate solution

$$V = V_{Bry} + \beta V_{Pert}.$$

2.3.5 Approximation for w

The expression for our approximation in the productish region (2.6) suggests that, in the tip region, $\bar{w} = \omega^{-1}(w + \mu_F t)$ is approximately constant in space.

We derive an equation for $\bar{w}$, to find the next order term. We can come from the evolution of w in terms of u, (B.15).

$$\partial_{t;u}(w - \mu_F t) = uv\partial_u^2(w - \mu_F t) - y + \mu\partial_u(w - \mu_F t) - \mu/2v\partial_u(w - \mu_F t).$$

Multiplying by $\omega^{-1}\alpha$, we have

$$\alpha\partial_{t;u}\bar{w} = \sigma^{-1}\mathcal{R}[\bar{w}, v] - (\alpha\omega^{-1})y - (\log\omega)_\theta\bar{w},$$

where

$$\mathcal{R}[W, v] = \sigma^2 v \partial_\sigma^2 W + (\mu - (c_v - \tfrac{1}{2}q)v)\sigma\partial_\sigma W.$$

Then using (2.11),

$$\begin{aligned}\partial_{\theta;\sigma}\bar{w} &= \sigma^{-1}\mathcal{R}[\bar{w}, v] - (\alpha\omega^{-1})y - (\log\omega)_\theta\bar{w} - \beta\sigma\partial_\sigma\bar{w} \\ &= \sigma^{-1}\mathcal{R}[\bar{w}, v] - \frac{1}{\bar{w} - \mu_F\frac{t}{\omega}}v\sigma(\partial_\sigma\bar{w})^2 - (\log\omega)_\theta\bar{w} - \beta\sigma\partial_\sigma\bar{w}. \quad (2.17)\end{aligned}$$

Concerning the operator $\mathcal{R}$, we have the following Lemma.

Lemma 2.3.4. *There is a solution $W_{Pert}(\sigma)$ to*

$$\sigma^{-1}\mathcal{R}[W_{Pert}, V_{Bry}] = 1$$

which extends to a smooth even function on $(-\infty, \infty)$.

The function $W_{kPert}(\sigma) = W_{Pert}(k\sigma)$ is a solution to

$$\sigma^{-1}\mathcal{R}[W_{kPert}, V_{kBry}] = 1$$

As $\sigma \to \infty$, W_{kPert} has the asymptotics

$$W_{kPert} = (1 + o(1))\tfrac{1}{2}\mu k\sigma \quad (2.18)$$

Proof. The main idea is that W_{Pert} is just a scaling of the gradient potential function f. This is because the gradient potential function (on any soliton) satisfies

$$\Delta_X f = 1$$

and the operator $\sigma^{-1}\mathcal{R}$ is a recasting of the laplacian in these coordinates. For the derivation see page 186. □

This suggests the approximate solution $\bar{W} = 1 + (\log\omega)_\theta W_{Pert}$. To find this, plug in $\bar{w}(\sigma, t) = 1 + \bar{w}_1(\sigma, t)$ as an initial approximation and assume that $\bar{w}_1$ goes to zero as $t \searrow 0$. Then taking the highest order terms in the limit $t \searrow 0$ we are left with the equation

$$-\left(\sigma^{-1}\mathcal{R}[\bar{w}_1, V_{Bry}] - (\log\omega)_\theta \cdot 1\right)$$

for $\bar{w}_1$.

Note $\log(\omega)_\theta = t\nu\frac{\partial_t\omega}{\omega} = \nu\omega^{[1]}$

2.3.6 y control

One tricky term which appears in the evolution of v, (2.12), is $\tilde{\kappa}^2 = \frac{1}{4}\dim(F)w^{-1}y$. This is difficult because it cannot be controlled with simple barrier arguments: at a point where w is trapped between barriers for w, and v touches barriers for v, we only know that the derivative v matches the derivative of the barrier for v, but we do not get a free bound on the derivative of w. For this reason we need to use regularity.

Lemma 2.3.5. *Suppose we are in the setting of Lemma 2.3.2. Suppose that (T1), (T2), and (T3) hold for $t \in [T_1, T_2)$. Then, if T_* is sufficiently small,*

$$\tilde{\kappa}^2 \leq CC_{reg}^2\epsilon_w^2\sigma^{1,0}\nu$$

Proof. We can use items (T3) and (T2) to control $\tilde{\kappa}^2$ along the flow. We rewrite $\tilde{\kappa}^2$ as,

$$\begin{aligned}\tilde{\kappa}^2 &= C\tilde{w}^{-1}y = C\frac{|\nabla \tilde{w}|^2_{\tilde{g}}}{\tilde{w}^2} \\ &= Cv\frac{|\nabla \tilde{w}|^2_{\tilde{g}}}{\tilde{w}^2}\frac{1}{|\nabla\tilde{\phi}|^2_{\tilde{g}}} = Cv\frac{1}{\tilde{w}^2}\left(\partial_{\tilde{\phi}}\tilde{w}\right)^2\end{aligned}$$

Here, we used that y and v are scale-invariant, and that $v = \frac{1}{4}|\nabla\tilde{\phi}|^2_{\tilde{g}}$. Now use $\tilde{w} = (\alpha\omega^{-1})(\bar{w} - \mu_F t/\omega)$.

$$\tilde{\kappa}^2 = Cv\left(\frac{1}{\bar{w} - \mu_F\frac{t}{\omega}}\right)^2\left(\partial_{\tilde{\phi}}\bar{w}\right)^2$$

Using the assumption (MP3) on W_0, $\omega(t) > (1+c)\mu t$, so we have

$$\tilde{\kappa}^2 \leq Cv\left(\frac{1}{\bar{w} - \frac{1}{1+c}}\right)^2\left(\partial_{\tilde{\phi}}\bar{w}\right)^2.$$

In the region under consideration, we can take T_* small enough so that $\bar{W}^- > 1 - \frac{1}{2}\frac{c}{1+c}$ and in particular since $\bar{w} > \bar{W}^-$, $\frac{1}{\bar{w}-\frac{1}{1+c}} < 2\frac{1+c}{c}$. Therefore, increasing C,

$$\tilde{\kappa}^2 \leq Cv\left(\partial_{\tilde{\phi}}\bar{w}\right)^2. \tag{2.19}$$

To control κ^2 in $\{\sigma < 1\}$, we need to use that $\partial_\phi\bar{w} = 0$ at $\phi = 0$. Copying (T3) for $k = 2$,

$$\partial^2_{\tilde{\phi}}\bar{w} < \partial^2_{\tilde{\phi}}\bar{W} + C_{reg}\epsilon_w\nu^{1/2}$$

which we can integrate from $\phi = 0$ to find,

$$\partial_{\tilde{\phi}}\bar{w} < \partial_{\tilde{\phi}}\bar{W} + C_{reg}\epsilon_w\nu^{1/2}\tilde{\phi}.$$

By the definition of $\bar{W}$, and using that $\phi^{-1}\partial_\phi W_{Pert}$ is bounded near $\phi = 0$,

$$\begin{aligned} \partial_{\tilde{\phi}}\bar{w} &< \nu\partial_\phi W_{Pert} + C_{reg}\epsilon_w \nu^{1/2}\tilde{\phi} \\ &< \left(C\nu + C_{reg}\epsilon_w\nu^{1/2}\right)\tilde{\phi}. \end{aligned}$$

Therefore, restricting T_* to be small enough so that the second term dominates, we find $(\partial_{\tilde{\phi}}\bar{w})^2 \leq CC_{reg}^2\epsilon_w^2\nu\tilde{\phi}^2$ for a larger C. Using (2.19) proves the claim in the region $\{\sigma < 1\}$.

To control κ^2 in $\{\sigma > 1\}$, we just copy (T2):

$$\partial_\sigma \bar{w} \leq \partial_\sigma \bar{W} + C_{reg}\epsilon_w\nu^{1/2}.$$

We have $\partial_\sigma \bar{W} \leq C\nu$ since $\partial_\sigma W_{pert}$ is bounded. So for small times,

$$\partial_\sigma \bar{w} \leq CC_{reg}\epsilon_w\nu^{1/2}.$$

This implies, using $v \leq V^+ \leq C\sigma^{0,-1}$,

$$\tilde{\kappa}^2 \leq CC_{reg}^2\epsilon_w^2\nu$$

$\square$

2.3.7 Barriers

In this section we define the barriers $V^\pm$ and $\bar{W}^\pm$ and prove that item (T1) continues to hold. The barriers are defined as follows. Let $k(t)^\pm = 1 \mp \delta^{-1}\epsilon_v\nu^{1/2}$ and then set

$$V^\pm = V_{k^\pm(t)Bry} + (\beta \mp \epsilon_v\nu)\, V_{k^\pm(t)Pert} \tag{2.20}$$

$$\bar{W}^\pm = 1 \pm \epsilon_w\nu^{1/2} + ((\log\omega)_\theta \mp \delta\epsilon_w\nu)\, W_{Pert}. \tag{2.21}$$

We will prove that these are sub- and supersolutions to the equations satisfied by $\bar{w}$ and v. The power $\nu^{1/2}$ is a bit mysterious here, but it is the best possible for barriers of this form. We discuss its derivation after Lemma 2.4.2. It is helpful to note that the assumptions on our model pinch imply that $\beta \sim \nu$, and that $(\log \omega)_\theta \lesssim \nu$. (Straightforward calculation, see Section B.3.3.)

The terms $-\epsilon_v \nu V_{k^\pm Pert}$ and $-\nu\delta\epsilon_w W_{Pert}$ are the terms which will give us that V^+ and W^+ are strict supersolutions to their equations. They are chosen by taking the approximate solution, which is found by starting from a limit at $t = 0$ and adding a perturbation which solves an elliptic equation, and then fiddling with the size of the perturbation.

Because the extra amount of the perturbation needed for a supersolution comes with a negative sign in both cases, we need to add something else to ensure that the supersolution lies above the intended approximate solution. This is the role of $k^\pm(t)$ and of $\pm\epsilon_w \nu^{1/2}$. (If it's not clear what's going on with k^+, recall that V_{Bry} is decreasing so $V_{k^+ Bry}(\sigma) = V_{Bry}(k^+\sigma) > V_{Bry}(\sigma)$.) The role of δ in both equations is to control the ratio of the extra positive term used to make the supersolution bigger than the approximate solution, to the extra negative term used to make the supersolution a supersolution to the equation.

Lemma 2.3.6 clarifies the role of δ. Recall the notation $\sigma^{a,b} = \sigma^a(1+\sigma)^{b-a}$. The significance of the factor $\sigma^{1,-1}$ in the inequalities for V in this lemma is the following. At infinity, $V \sim \sigma$ so this is a normalization. At 0, $V^+ - V^- \sim \sigma$ is necessary to ensure smoothness of a solution with V trapped between V^- and V^+. On the other hand $\bar{W} \sim 1$ everywhere so $\bar{W}$ requires no normalization.

Lemma 2.3.6. *There are constants $c < C$ depending only on the dimension such that the following holds.*

We have, for $V_{diff} = V^+ - V$ or $V_{diff} = V - V^-$,

$$c\delta^{-1}\epsilon_v \nu^{1/2}\sigma^{1,-1}\left(1 - C\delta\nu^{1/2}\sigma^{0,1}\right) \leq V_{diff} \leq C\delta^{-1}\epsilon_v\nu^{1/2}\sigma^{1,-1}.$$

Similarly, for $W_{diff} = \bar{W}^+ - \bar{W}$ or $W_{diff} = W - W^-$,

$$c\epsilon_w \nu^{1/2}\left(1 - C\delta\nu^{1/2}\sigma^{0,1}\right) \leq W_{diff} \leq C\epsilon_w \nu^{1/2}$$

In particular, if we choose $\delta < \frac{1}{2C}\zeta_*^{-1}$ then, renaming c, for all $\sigma < \zeta_*\nu^{-1/2}$ we have

$$c\delta^{-1}\epsilon_v\nu^{1/2}\sigma^{1,-1} \leq V_{diff} \leq C\delta^{-1}\epsilon_v\nu^{1/2}\sigma^{1,-1},$$
$$c\epsilon_w\nu^{1/2} \leq \bar{W}_{diff} \leq C\epsilon_w\nu^{1/2}.$$

Proof. The asymptotics of the V_{Bry} are given in (2.13) and (2.14). Also recall that $V_{kBry}(\sigma) = V_{Bry}(k\sigma)$. Using these asymptotics, for small enough σ,

$$c\delta^{-1}\epsilon_v\nu^{1/2}\sigma < V_{k^+Bry}(\sigma) - V_{Bry}(\sigma) < C\delta^{-1}\epsilon_v\nu^{1/2}\sigma,$$

and for large enough σ,

$$c\delta^{-1}\epsilon_v\nu^{1/2}\sigma^{-1} < V_{k^+Bry}(\sigma) - V_{Bry}(\sigma) < C\delta^{-1}\epsilon_v\nu^{1/2}\sigma^{-1}.$$

Furthermore, since V_{Bry} is strictly decreasing in any compact set away from the origin, for any $\sigma_1 < \sigma_2$ there are constants c_{σ_1,σ_2} and C_{σ_1,σ_2} such that

$$c_{\sigma_1,\sigma_2}\delta^{-1}\epsilon_v\nu^{1/2} < V_{k^+Bry}(\sigma) - V_{Bry}(\sigma) < C_{\sigma_1,\sigma_2}\delta^{-1}\epsilon_v\nu^{1/2}.$$

This is enough to prove that for some $c < C$,

$$c\delta^{-1}\epsilon_v\nu^{1/2}\sigma^{1,-1} < V_{k^+Bry}(\sigma) - V_{Bry}(\sigma) < C\delta^{-1}\epsilon_v\nu^{1/2}\sigma^{1,-1}.$$

Putting this together with the bound on V_{Pert}, (2.16), which says

$$-\epsilon_v\nu V_{pert}(\sigma) > -C\epsilon_v\nu\sigma^{1,0},$$

and using $\beta \leq C\nu$ (Section B.3.3), we have the claim for V.

The proof for $\bar{W}$ is similar but more straightforward. One needs to use the properties from Lemma 2.3.4. □

We now prove that $V^{\pm}$ and $\bar{W}^{\pm}$ are sub- and supersolutions to the equations satisfied by v and $\bar{w}$. In Lemma 2.3.9 we will summarize by saying that item (T1) continues to hold.

Lemma 2.3.7. *Let $\zeta_* > 0$, $\epsilon_v > 0$, and $\delta > 0$ be given. Let $V^{\pm}$ be the functions defined in* (2.20).

Suppose $\tilde{\kappa} = \frac{y}{\tilde{w}} = \alpha\frac{y}{w}$ satisfies $\tilde{\kappa}^2(\sigma, t) \leq c_{ytip}\epsilon_v\nu\sigma^{1,0}$ where c_{ytip} is a constant (chosen in the proof) depending only on dimensions.

Then there is a T_* depending on all parameters so that for $t < T_*$ and $\sigma < \zeta_*\nu^{-\frac{1}{2}}$ we have, for a constant c depending only on the dimensions,

$$\partial_{\theta;\sigma}V^+ - \mathcal{F}_\sigma[V^+, \tilde{\kappa}] - \beta\sigma\partial_\sigma V^+ \geq c\epsilon_v\nu\sigma^{1,-1} \tag{2.22}$$

and

$$\partial_{\theta;\sigma} V^- - \mathcal{F}_\sigma[V^-, \tilde{\kappa}] - \beta\sigma\partial_\sigma V^- \leq -c\epsilon_v \nu \sigma^{1,-1}.$$

Proof. Let us first demonstrate the main calculation, implicitly defining error terms E_1 and E_2. Calculate

$$\begin{aligned}
-\mathcal{F}_\sigma[V^+, 0] = &- \left(\sigma^{-1}\mathcal{Q}[V_{k^+Bry}, V_{k^+Bry}] + \sigma^{-1}\mathcal{L}[V_{k^+Bry}]\right) \\
&- (\beta - \epsilon_v \nu)\left(2\sigma^{-1}\mathcal{Q}[V_{k^+Bry}, V_{k^+Pert}] + \sigma^{-1}\mathcal{L}[V_{k^+Bry}]\right) \\
&- (\beta - \epsilon_v \nu)^2 \sigma^{-1}\mathcal{Q}[V_{k^+Pert}, V_{k^+Pert}].
\end{aligned}$$

The first line vanishes, and the second line can be computed from the equation solved by V_{k^+Pert}. The last line is error.

$$-\mathcal{F}_\sigma[V^+, 0] = +(\beta - \epsilon_v\nu)\sigma\partial_\sigma V_{k_+Bry} + E_1$$

Also calculate,

$$\begin{aligned}
-\beta\sigma\partial_\sigma V^+ &= -\beta\sigma\partial_\sigma V_{k^+Bry} - (\beta - \epsilon_v\nu)\beta\sigma\partial_\sigma V_{k^+Pert} \\
&= -\beta\sigma\partial_\sigma V_{k^+Bry} + E_2
\end{aligned}$$

Putting these together,

$$\begin{aligned}
-\mathcal{F}_\sigma[V^+, 0] - \beta\sigma\partial_\sigma V^+ &= -\epsilon_v \beta\sigma\partial_\sigma V_{k^+Bry} + C(\beta\sigma^{1,-1})\left(\beta\sigma^{0,1}\right) \\
&\geq c\epsilon_v\nu\sigma^{1,-1} + E_1 + E_2
\end{aligned}$$

where we used that $\sigma\partial_\sigma V_{k^+Bry} \leq -c\sigma^{1,-1}$ for some c. Therefore it remains to bound E_1 and E_2, as well as the other terms in (2.22), namely

$$\partial_{\theta;\sigma} V^+ \quad \text{and} \quad \mathcal{F}_\sigma[V^+, \tilde{\kappa}] - \mathcal{F}_\sigma[V^+, 0] = \tilde{\kappa}^2 V^+.$$

For the following, note We can assume that $k(t)$ is in $[1/2, 2]$. Using $\beta \sim \nu$, and recalling the notation $|f|_2 = |f| + \sigma|\partial_\sigma f| + \sigma^2|\partial_\sigma^2 f|$, we have the bound on E_1 and E_2,

$$\begin{aligned}
|E_1| + |E_2| &= \left|(\beta - \epsilon\nu)^2\sigma^{-1}\mathcal{Q}[V_{k^+Pert}, V_{k^+Pert}]\right| + |(\beta - \epsilon\nu)\beta\sigma\partial_\sigma V_{k^+Pert}| \\
&\leq C\nu^2\left(\sigma^{-1}|V_{Pert}|_2^2 + |V_{Pert}|_2\right) \\
&\leq C\nu^2\left(\sigma^{-1}\left(\sigma^{1,0}\right)^2 + \sigma^{1,0}\right) \leq C\nu^2\sigma^{1,0} = C\left(\nu\sigma^{1,-1}\right)\left(\nu\sigma^{0,1}\right)
\end{aligned}$$

Now we bound the time term, using $\partial_\theta\beta \lesssim \nu^2$ (straightforward calculation, Section B.3.3).

$$\begin{aligned}
\partial_{\theta;\sigma}V^+ &= \partial_{\theta;\sigma}\left(V_{Bry}(k(t)\sigma) + (1-\epsilon)\frac{\beta}{k(t)}V_{Pert}(k(t)\sigma)\right) \\
&\leq C\left(\sigma\partial_\sigma V_{Bry}\partial_\theta k + (\partial_\theta\beta + \beta\partial_\theta k)V_{Pert} + \beta\sigma\partial_\sigma V_{Pert}\partial_\theta k\right) \\
&\leq C\left(\sigma^{1,-1}\nu^{1+1/2} + (\nu^2 + \beta\nu^{1+1/2})\sigma^{1,0} + \beta\sigma^{1,0}\nu^{1+1/2}\right) \\
&\leq C\nu\sigma^{1,-1}\left(\nu^{1/2} + \nu\sigma^{0,1}\right)
\end{aligned}$$

Finally, we use our assumption on $\tilde{\kappa}$ to bound the term $\tilde{\kappa}^2V^+$ by

$$\begin{aligned}
\tilde{\kappa}^2V^+ &\leq C\left(c_{ytip}\epsilon_v\nu\sigma^{1,0}\right)\sigma^{0,-1} \\
&\leq C\nu\sigma^{1,-1}\left(c_{ytip}\epsilon_v\right)
\end{aligned}$$

All in all, we find

$$\partial_{\theta;\sigma} - \mathcal{F}[V^+, \tilde{\kappa}] - \beta\sigma\partial_\sigma V^+ \geq \nu\sigma^{1,-1}(c\epsilon_v - Cc_{ytip}\epsilon_v - o(1))$$

Here the term $o(1)$ goes to zero as $t \searrow 0$, in any region where $\sigma < \zeta_*\nu^{-1/2}$. The lemma follows by choosing the c in the statement to be one half of the

c above, choosing c_{ytip} to be sufficiently small, and choosing T_* to be small enough so that the $o(1)$ term is sufficiently small. $\square$

For arbitrary functions $\bar{w}$ and v we define

$$\mathcal{D}(\bar{w}, v) := \partial_{\theta;\sigma}\bar{w} - \left(\sigma^{-1}\mathcal{R}[\bar{w}, v] - \frac{1}{\bar{w} - \mu_F \frac{t}{\omega}} v\sigma(\partial_\sigma \bar{w})^2 - \beta\sigma\partial_\sigma\bar{w} - (\log\omega)_\theta \bar{w}\right).$$

The equation solved by w (2.17) is therefore $\mathcal{D}(\bar{w}, v) = 0$.

Lemma 2.3.8. *Let $\zeta_* > 0$, $\epsilon_v > 0$, $\epsilon_w > 0$, and $\delta > 0$ be given. Let $V^\pm$ and $W^\pm$ be the barriers defined in* (2.20) *and* (2.21)*.*

There is a T_* depending on all parameters such that for all $t < T_*$ and $\sigma < \zeta_* \nu^{-1/2}$ we have

$$\mathcal{D}(\overline{W}^+, v) > \tfrac{1}{2}\delta\epsilon_w\nu$$

and

$$\mathcal{D}(\overline{W}^-, v) < -\tfrac{1}{2}\delta\epsilon_w\nu$$

Proof. The main idea is that

$$\begin{aligned}(\log\omega)_\theta\overline{W}^+ - \sigma^{-1}\mathcal{R}[\overline{W}^+, v] = {} & (1 + \epsilon_w\nu^{1/2})(\log\omega)_\theta \\ & + (\log\omega)_\theta\left((\log\omega)_\theta - \delta\epsilon_w\nu\right)W_{Pert} \\ & - (\log\omega)_\theta\sigma^{-1}\mathcal{R}(W_{pert}, V_{Bry}) + \delta\epsilon_w\nu\sigma^{-1}\mathcal{R}(W_{Pert}, V_{Bry}) \\ & + \left((\log\omega)_\theta - \delta\epsilon_w\nu\right)\cdot\left(\sigma^{-1}\mathcal{R}(W_{pert}, V_{Bry}) - \sigma^{-1}\mathcal{R}(W_{Pert}, v)\right)\end{aligned}$$

We can simplify the first and third lines to find

$$
\begin{aligned}
(\log\omega)_\theta \overline{W}^+ - \sigma^{-1}\mathcal{R}[\overline{W}^+, v] \geq \epsilon_w(\log\omega)_\theta\nu^{1/2} + \delta\epsilon_w\nu \\
+ (\log\omega)_\theta\left((\log\omega)_\theta - \epsilon_w\nu\right) W_{Pert} \\
+ \left((\log\omega)_\theta - \epsilon_w\nu\right)\cdot\left(\sigma^{-1}\mathcal{R}(W_{pert}, V_{Bry}) - \sigma^{-1}\mathcal{R}(W_{Pert}, v)\right)
\end{aligned}
$$

The first line has the correct sign, we will use it to bound the other lines and the rest of the terms. First, let's bound the other lines above:

$$
\begin{aligned}
(\log\omega)_\theta \overline{W}^+ - \sigma^{-1}\mathcal{R}[\overline{W}^+, v] \geq \epsilon_w\nu \\
- C\nu^2\sigma^{0,1} \\
- C\nu(\delta^{-1}\epsilon_v)\nu^{1/2}\sigma^{1,0}
\end{aligned}
$$

Here we used the bound $|V_{Bry} - v| < c\delta^{-1}\epsilon_v\nu^p\sigma^{1,-1}$ together with $|\sigma\partial_\sigma W_{Pert}| + |\sigma^2\partial_\sigma^2 W_{pert}| \leq \sigma^{0,1}$. In the second inequality we also used $(\log\omega)_\theta = \nu\omega^{[1]} \leq C\nu$.

Next we find the term $\partial_{\theta;\sigma}\bar{W}^+$. The term $\partial_{\theta;\sigma}\epsilon_w\nu^{1/2}$ has the correct sign, so we ignore it. For the other time derivatives, we can use $|\partial_\theta^2(\log\omega)| + |\partial_\theta\nu| \leq C\nu^2$ (Section (B.3.1)):

$$
\begin{aligned}
\left|\partial_\theta\left((\log\omega)_\theta - \epsilon_w\nu\right) W_{Pert}\right| &\leq C\nu^2 W_{pert} \\
&\leq C\nu^2\sigma^{0,1}.
\end{aligned}
$$

To bound the remaining terms, note,

$$
|\sigma\partial_\sigma\bar{W}^+| \leq \nu\sigma^{1,1}
$$

and $v \leq C\sigma^{0,-1}$. Also, as in the proof of 2.3.5 we can bound $\frac{1}{\bar{W}^{\pm}-\mu_F t} \leq C$. So

$$\left| \frac{1}{\bar{W}^{+} - \mu_F \frac{t}{\omega}} v\sigma(\partial_\sigma \bar{W}^{+})^2 + \beta\sigma\partial_\sigma \bar{W}^{+} \right| \leq C \left(\sigma^{-1} v|\sigma\partial_\sigma \bar{W}^{+}|^2 + \nu|\sigma\partial_\sigma \bar{W}^{+}| \right)$$
$$\leq C \left(\nu^2 \sigma^{1,0} + \nu^2 \sigma^{1,1} \right) \leq C\nu \left(\nu\sigma^{1,1} \right)$$

Putting together all of the inequalities, we have

$$\mathcal{D}(\bar{W}^{+}, v) \geq \nu \left(\delta\epsilon_w - C\nu\sigma^{1,1} - C(\delta^{-1}\epsilon_v)\nu^{1/2}\sigma^{1,0} \right).$$

In the space-time region under consideration,

$$\mathcal{D}(\bar{W}^{+}, v) \geq \nu \left(\delta\epsilon_w - C\nu^{1/2}\zeta_* - C(\delta^{-1}\epsilon_v)\nu^{1/2} \right) = \nu \left(\delta\epsilon_w - C(\zeta_* + \delta^{-1}\epsilon_v)\nu^{1/2} \right).$$

For small enough T_*, the positive term dominates. □

Lemma 2.3.9. *Suppose we are in the setting of Lemma 2.3.2. Suppose* $\epsilon_w \leq \bar{\epsilon}_w(\epsilon_v, C_{reg})$. *There is a* T_* *depending on all parameters such that the following holds.*

If items (T1), (T2), and (T3) hold for $t \in [T_1, T_2)$*, then item (T1) holds for* $t \in [T_1, T_2]$.

Proof. Choose $\bar{\epsilon}_w$ small enough (i.e. $\lesssim \sqrt{\epsilon_v}$) so that Lemma 2.3.5 implies that we have the desired inequality $\tilde{\kappa}^2 \leq c_{ytip}\epsilon_v\beta\sigma^{1,0}$ needed to apply Lemma 2.3.7.

Now, suppose that v or w touches one of its barriers at time $t = T_2$. By Lemma 2.3.7 or 2.3.8, we get a contradiction to the maximum principle since these lemmas say that $V^{\pm}$ and $W^{\pm}$ are strict sub- and supersolutions to the corresponding equations. □

2.3.8 Regularity

Lemma 2.3.10. *Suppose we are in the setting of Lemma 2.3.2. Suppose $\delta < \overline{\delta}(\zeta_*)$ so that the conclusion of Lemma 2.3.6 holds. We can choose $\overline{c}_{safe}$ and $\underline{C}_{reg}$ depending only on the dimensions such that the following holds. Suppose item (T1) holds for $t \in [T_1, T_2)$. Then item (T2) holds for $t \in [T_1, T_2]$.*

Proof. Consider equation (2.12). Use

$$\begin{aligned}\tilde{\kappa}^2 &= \tfrac{1}{4}\dim(F)\tilde{w}^{-1}y \\ &= \tfrac{1}{4}\dim(F)(\bar{w} - \mu_F t/\omega)^{-2}|\nabla\bar{w}|^2_{\bar{g}} \\ &= \tfrac{1}{4}\dim(F)\sigma v(\bar{w} - \mu_F t/\omega)^{-2}\partial_\sigma^2\bar{w}\end{aligned}$$

to write

$$\begin{aligned}\partial_{\theta;\sigma} v = \sigma v\partial_\sigma^2 v + c_1\sigma^{-1}v + c_2\partial_\sigma v + c_3\sigma^{-1}v^2 + c_4\sigma(\partial_\sigma v)^2 \\ + \beta\sigma\partial_\sigma v - 2\sigma v\left(\frac{1}{\bar{w} - \mu_F t/\omega}\right)^2(\partial_\sigma\bar{w})^2 v.\end{aligned}$$

For σ_1 arbitrary, we multiply this by σ_1 to find,

$$\begin{aligned}\partial_{\theta;\sigma}(\sigma_1 v) = \left[\frac{\sigma}{\sigma_1}\right](\sigma_1 v)\,\partial_\sigma^2(\sigma_1 v) + c_1\left[\frac{\sigma_1}{\sigma}\right]\sigma_1^{-1}(\sigma_1 v) + c_2\partial_\sigma(\sigma_1 v) \\ + c_3\left[\frac{\sigma_1}{\sigma}\right]\sigma_1^{-2}(\sigma_1 v)^2 + c_4\left[\frac{\sigma}{\sigma_1}\right]\sigma_1^{-1}(\partial_\sigma(\sigma_1 v))^2 \\ + [\beta\sigma]\,\partial_\sigma(\sigma_1 v) + c_5\left[\frac{\sigma}{\sigma_1}(\sigma_1 v)^2\left(\frac{1}{\bar{w} - \mu_F t/\omega}\right)^2\right](\partial_\sigma\bar{w})^2.\end{aligned}$$

We also have the equation, from (2.17),

$$\begin{aligned}\partial_{\theta;\sigma}\bar{w} = [\sigma v]\,\partial_\sigma^2 w &+ \left[(\mu - (c_v - \tfrac{1}{2}q)v)\right]\partial_\sigma \bar{w}\\ &- \left[\frac{1}{\bar{w} - \mu_F \frac{t}{\omega}} v\right]\sigma(\partial_\sigma \bar{w})^2\\ &- (\log\omega)_\theta \bar{w} - [\beta\sigma]\,\partial_\sigma \bar{w}.\end{aligned}$$

For σ_1 and t_1 arbitrary but satisfying

$$1 < \sigma_1 < \zeta_* \nu^{-1/2}$$

we will apply parabolic regularity to $\sigma_1 v$ and w in the region

$$\Xi = (\sigma, \theta) \in [\sigma_1 - 1/2, \sigma_1 + 1/2] \times [\max(\theta(t_1) - 1/2, \theta(T_1)), \theta(t_1)]. \tag{2.23}$$

By Lemma 2.3.6, for $\frac{1}{2} < \sigma < \zeta_* \beta^{-1/2}$ we have

$$\sigma V^+ - \sigma V^- < C\delta^{-1}\epsilon_v \nu^{1/2},$$

$$\bar{W}^+ - \bar{W}^- < C\epsilon_w \nu^{1/2}.$$

Also, for functions between our barriers, the terms we have written in square brackets are smooth functions of σ, $\sigma_1 v$, and w.

Therefore, we may apply parabolic regularity to $\sigma_1 v - \sigma_1 V$ and $\bar{w} - \bar{W}$ to find that, for $k = 1, 2$,

$$\partial^k v \leq \partial^k V + \delta^{-1}\epsilon_v \nu^{1/2}\sigma_1^{-1}$$

and

$$\partial^k w \leq \partial^k W + \epsilon_w \nu^{1/2}$$

at $\sigma = \sigma_1$ and $t = t_1$. $\square$

Lemma 2.3.11. *Assume that we are in the setting of Lemma 2.3.2. We can choose $\overline{c}_{safe}$ and $\underline{C}_{reg}$ depending only on the dimensions such that the following holds. Suppose additionally that item (T1) and (T2) hold for $t \in [T_1, T_2)$. Then item (T3) holds for $t \in [T_1, T_2]$.*

Proof. We will control the scaled sectional curvature

$$\tilde{L} = \alpha L = \alpha u^{-1}(1 - \tfrac{1}{4}v) = \tilde{u}^{-1}(1 - \tfrac{1}{4}v).$$

and also controlled the scaled function $\bar{w}$. We will write the evolution equations in terms of $\tilde{\phi} = \alpha^{-1/2}\phi$.

We can derive the evolution for $\tilde{L}$ from (B.13).

$$\begin{aligned}
\partial_{\theta;\phi}\tilde{L} = &\left(1 - \tilde{\phi}^2\tilde{L}\right)\partial^2_{\tilde{\phi}}\tilde{L} + \tfrac{1}{2}\tilde{\phi}^2(\partial_{\tilde{\phi}}\tilde{L})^2 \\
&+ \tilde{\phi}^{-1}(\tfrac{1}{2}\mu + 5 - \tilde{\phi}^2\tilde{L})\partial_{\tilde{\phi}}\tilde{L} + (\mu+2)\tilde{L}^2 \\
&+ \tfrac{1}{8}\dim(F)\frac{\alpha}{\omega}v(\bar{w} - \mu_F t/\omega)^{-2}\partial^2_{\sigma}\bar{w} \\
&+ \beta\tilde{L} + \beta\sigma\partial_\sigma\tilde{L}.
\end{aligned}$$

$$\begin{aligned}
= &\left(1 - \tilde{\phi}^2\tilde{L}\right)\partial^2_{\tilde{\phi}}\tilde{L} + \tfrac{1}{2}\tilde{\phi}^2(\partial_{\tilde{\phi}}\tilde{L})^2 \\
&+ \tilde{\phi}^{-1}(\tfrac{1}{2}\mu + 5 - \tilde{\phi}^2\tilde{L})\partial_{\tilde{\phi}}\tilde{L} + (\mu+2)\tilde{L}^2 \\
&+ c\tilde{\phi}^{-2}\frac{\alpha}{\omega}v(\bar{w} - \mu_F t/\omega)^{-2}\partial^2_{\tilde{\phi}}\bar{w} \\
&+ \beta\tilde{L} + \tfrac{1}{2}\beta\phi\partial_\phi\tilde{L}.
\end{aligned}$$

We can also derive the equation for $\bar{w}$ in terms of $\tilde{\phi}$:

$$\partial_{\theta;\tilde{\phi}}\bar{w} = v\partial^2_{\tilde{\phi}}\bar{w} - y + (\tfrac{1}{2}\mu - (\tfrac{1}{4}\mu - 1)v)\tilde{\phi}^{-1}\partial_{\tilde{\phi}}\bar{w} \\ + (\log\omega)_\theta\bar{w} + \tfrac{1}{2}\beta\phi\partial_\phi\bar{w}$$

Let $\tilde{L}_{approx} = \sigma^{-1}(1 - \frac{1}{4}V) = \tilde{\phi}^{-2}(1 - \frac{1}{4}V)$ which is the approximation for $\tilde{L}$ given by the approximate solution for V. Our barriers tell us that, for $\sigma < 1$, we have

$$|\tilde{L} - \tilde{L}_{approx}| < c\delta^{-1}\epsilon_v\nu^{1/2}, \quad |\bar{w} - \bar{W}| < c\epsilon_w\nu^{1/2}.$$

Furthermore, the regularity up to time T_2 tells us that we may control the $C^{0,\alpha}$ norm of the terms $\tilde{\phi}^1\partial_{\tilde{\phi}}\bar{w}$ and $\tilde{\phi}^{-1}\partial_{\tilde{\phi}}\tilde{L}$. Since C_{reg} appears only as a coefficient of ν, they may be controlled independently of C_{reg} by taking t small enough.

So, applying regularity to $\tilde{L} - \tilde{L}_{approx}$ and $\bar{w} - \bar{W}$ proves the claim.

□

2.3.9 Corollary of control

The following corollary follows quickly from the control we have, by checking the curvatures of warped products.

Corollary 2.3.12. *Suppose $g_{wp}(t)$ is controlled in the tip region. If $\mu_F = 0$, suppose F has constant curvature. Then for some C, in the tip region,*

$$|\mathrm{Rm}| \leq \frac{C}{t\nu(t)}$$

We now give a specific result about the convergence in tip region as $t \searrow 0$. We assume that $g(t)$ is controlled in the tip region for $t \in (0, T_2)$. For each time, the scaled warping function $\sigma = \frac{u}{t\nu(t)}$ is a function $\sigma : I \to (0, \infty)$ which we extend by the identity to a map $\sigma : M = I \times S^q \times F \to (0, \infty) \times S^q \times F$. For each t, σ is a bijection if we restrict to some subset of I, i.e. we have an inverse

$$\sigma^{-1} : (0, \sigma_{max}(t)) \times S^q \times F \to I \times S^q \times F.$$

By our bounds on v, specifically since we keep it positive, $\sigma_{max}(t) \to \infty$ as $t \searrow 0$. We may define

$$G(t) = \frac{1}{\alpha(t)} \left(\sigma^{-1}\right)^* g(t).$$

As $t \searrow 0$ the domain of definition of G exhausts $(0, \infty) \times S^q \times F$. Essentially, we can use σ to find the diffeomorphisms such that neighborhoods of the tip converge to the Bryant soliton times a Euclidean factor.

Corollary 2.3.13. *Suppose that $g(t)$ is controlled in the tip region.*

The (for each t partially defined) metric $G(t)$, restricted to $(0, \infty) \times S^q$, converges in C^∞ as $t \searrow 0$ to the Bryant soliton metric

$$\frac{d\sigma_{Bry}^2}{\frac{1}{4}\sigma_{Bry} v_{Bry}} + \sigma_{Bry} g_{S^q}.$$

The pullback of the vector field $(\partial_\theta \sigma)\partial_\sigma$,

$$X(t) = \left(\sigma^{-1}\right)^* ((\partial_\theta \sigma)\partial_\sigma)$$

converges to the soliton vector field for the Bryant soliton.

Put $p = \dim(F)$*. Suppose additionally that* g_{mp} *is* F*-reasonable (Definition 1.2.3).*

For any point $P \in (0,\infty) \times S^q \times F$ *the pointed manifolds* $((0,\infty) \times S^q \times F, G(t), P)$ *converge, as* $t \searrow 0$*, to*

$$\left((0,\infty) \times S^q \times F_{lat}, \frac{d\sigma^2}{\sigma v_{Bry}} + \sigma^2 g_{S^q} + g_{F_{lat}}, \star \right).$$

The target point $\star$ *doesn't matter since the target manifold is homogeneous. The convergence is in the sense of pointed* C^∞ *riemannian manifolds, which allows a pullback by a time-dependent diffeomorphism.*

Proof. The convergence to the Bryant soliton in terms of σ happens up to some number of derivatives just because of the consequences of Lemma 2.3.2. To get C^∞ convergence, we need extra regularity, i.e. item (T2) and (T3) for larger k. To get this, we use interior parabolic regularity in the same way as Lemmas 2.3.10 and 2.3.11. In this situation, we no longer need estimates on the initial data. This is because the time variable θ goes to $-\infty$ as $t \searrow 0$, so the parabolic ball Ξ in (2.23) never touches $t = 0$, the initial time for $g(t)$.

Note that $\tilde{g}(t) = \alpha^{-1} g(t)$ satisfies

$$\partial_\theta \tilde{g} = -2\,\mathrm{Rc}[\tilde{g}] - \beta \tilde{g}.$$

So $G(t)$ satisfies

$$\partial_\theta G = -2\,\mathrm{Rc}[G] - \mathcal{L}_{(\partial_\theta \sigma)\partial_\sigma} G - \beta \tilde{g}.$$

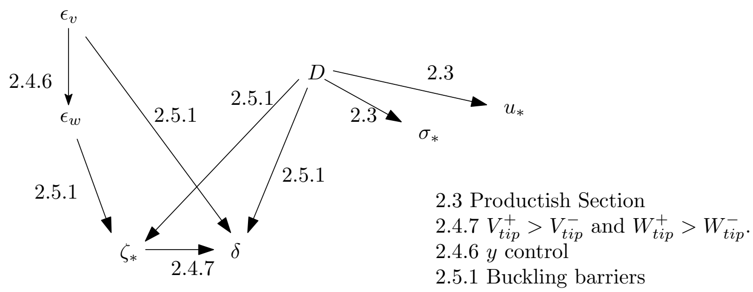

Figure 2.2: **Constant dependency graph.** All constants only depend on the constants which point to them. The arrows are marked with the Lemmas where the dependency arises. There are no cycles in the graph. T_* is allowed to depend on all constants.

As $t \searrow 0$, we have $\beta \searrow 0$, $G \to G_{Bry}$, and $\partial_\theta G \to 0$. This shows the convergence of $\partial_\theta \sigma$ to the soliton vector field.

To get the final convergence of the wg_F factor to $g_{\mathbb{R}^{\dim(F)}}$, note that we have

$$w \sim \omega - \mu_F t$$

so $\alpha^{-1} w \sim \alpha^{-1}\omega\,(1 - \mu_F t/\omega)$. In the case $\mu_F < 0$, this goes to ∞ at least as fast as $\frac{t}{\alpha} = \frac{1}{\nu}$ goes to infinity. In the case $\mu_F > 0$, this goes to infinity by the assumption (MP3). If $\mu_F = 0$, then this goes to infinity by the assumption that g_{mp} is F-reasonable. $\square$

2.4 Full flows of mollified metrics

In Sections 2.2 and 2.3, we studied the flow in two regions- the productish region and the tip region. We now want to start from one of our model pinches and create mollified initial metrics. The mollified metrics will exist for a uniform amount of time and satisfy the estimates from Lemmas 2.2.2 and 2.3.2. We will then take a limit of the mollified flows to construct a forward evolution from the model pinch.

In the previous two sections we constructed functions, which depend on u and time, and serve as barriers of the flow. Let V_{prish} and W_{prish} be the approximate solutions constructed in Section 2.2, and let V^{+}_{prish}, V^{-}_{prish}, W^{+}_{prish}, W^{-}_{prish} be the functions constructed in Lemma 2.2.6. Let V_{tip} and W_{tip} be the approximate solutions constructed in Section 2.3, and let V^{+}_{tip}, V^{-}_{tip}, W^{+}_{tip}, W^{-}_{tip} be the functions constructed in Section 2.3.7.

As a first step, the following lemma tells us how close the approximate solutions are to each other. Here, $|f|_3 = |f| + |\sigma\partial_\sigma f| + |\sigma^2\partial_\sigma^2 f| + |\sigma^3\partial_\sigma^3 f|$.

Lemma 2.4.1. *For $\sigma < \epsilon\rho_*\nu^{-1}$,*

$$\begin{aligned} \sigma|V_{prish} - V_{tip}|_3 &\leq C(\rho_*)\left(\nu^2\sigma^2 + \sigma^{-1} + \nu\right) \\ |\bar{W}_{prish} - \bar{W}_{tip}|_3 &\leq C(\rho_*)\left(\nu^2\sigma^2 + \nu\log\sigma\right) \end{aligned} \tag{2.24}$$

Proof. For V, the zeroth order statement follows from the approximation (2.5) for the parabolic approximation, the asymptotics (2.13) and (2.15) for V_{Bry} and V_{Pert}, and the fact that $\beta = (1 + \nu^{[1]})\nu$. For W, it follows from (2.6), and the asymptotics (2.18) for W_{Pert}

To get the higher order statement, we need to apply the higher derivative statement in Lemma B.3.3 to control the higher derivatives of the difference between V_{prish} and its approximation (2.5). We also control the higher derivatives of the difference between V_{tip} and its and the approximation coming from the asymptotics of V_{Bry} and V_{pert}, using analyticity of the relevant functions. $\square$

2.4.1 Buckling barriers

In this section, we prove Lemma 2.4.2. This shows that the barriers are ordered in a specific way: see Figure 2.3. The point is that this ordering means that boundary condition for the tip barriers is guaranteed by the productish barriers, and the left-hand boundary condition for the productish barriers is guaranteed by the tip barriers. We formalize this consequence in Lemma 2.4.3.

Lemma 2.4.2. *Let ϵ_v, ϵ_w, and σ_* be given. Assume $D > \underline{D}$, $\zeta > \underline{\zeta}_*(D, \epsilon_w)$, $\delta < \overline{\delta}(\zeta_*, \epsilon_v, D)$, and finally T_* is chosen depending on all other parameters.*

Then we have the following inequalities. For $\zeta_* \nu^{-1/2} \leq \sigma \leq 2\zeta_* \nu^{-1/2}$,

$$V^+_{tip} > V^+_{prish} \quad V^-_{tip} < V^-_{prish},$$
$$W^+_{tip} > W^+_{prish} \quad W^-_{tip} < W^-_{prish}.$$

For $\frac{1}{2}\sigma_* \leq \sigma \leq \sigma_*$,

$$V^+_{prish} > V^+_{tip} \quad V^-_{prish} < V^-_{tip},$$
$$W^+_{prish} > W^+_{tip} \quad W^-_{prish} < W^-_{tip}$$

Proof. We note the following inequalities:

$$cD\sigma^{-1} < \sigma V^+_{prish} - \sigma V_{prish} < CD\sigma^{-1},$$
$$cD\sigma^{-1} < \bar{W}^+_{prish} - \bar{W}_{prish} < CD\sigma^{-1}$$

This comes from the definition of the barriers $V^{\pm}_{prish} = (1 \pm DV)V$ and $\bar{W}^{\pm}_{prish} = (1 \pm DV)\bar{W}$, together with $V \sim \sigma^{-1}$ and $\bar{W} \sim 1$. Also, provided we take $\delta < c\zeta_*^{-1}$, by Lemma 2.3.6 we have

$$c\delta^{-1}\epsilon_v \nu^{1/2} \le \sigma V^+_{tip} - \sigma V_{tip} \le C\delta^{-1}\epsilon_v \nu^{1/2},$$
$$c\epsilon_w \nu^{1/2} \le \bar{W}^+_{tip} - \bar{W}_{tip} \le C\epsilon_w \nu^{1/2}.$$

We can put all these inequalities, together with (2.4.1), in terms of ζ:

$$\sigma|V_{prish} - V_{tip}| \le C(\rho_*)\left(\nu\zeta^2 + \zeta^{-1}\nu^{1/2} + \nu\right), \tag{2.25}$$
$$|\bar{W}_{prish} - \bar{W}_{tip}| \le C(\rho_*)\left(\nu\zeta^2 + \nu|\log\nu| + \nu|\log\zeta|\right),$$

$$cD\zeta^{-1}\nu^{1/2} < \sigma V^+_{prish} - \sigma V_{prish} < CD\zeta^{-1}\nu^{1/2}, \tag{2.26}$$
$$cD\zeta^{-1}\nu^{1/2} < W^+_{prish} - W_{prish} < CD\zeta^{-1}\nu^{1/2},$$

$$c\delta^{-1}\epsilon_v \nu^{1/2} \le \sigma V^+_{tip} - \sigma V_{tip} \le C\delta^{-1}\epsilon_v \nu^{1/2}, \tag{2.27}$$
$$c\epsilon_w \nu^{1/2} \le \bar{W}^+_{tip} - \bar{W}_{tip} \le C\epsilon_w \nu^{1/2}.$$

We now use the inequalities (2.25), (2.26), and (2.27) to prove the desired inequalities for the supersolutions. The desired inequalities for the subsolutions are similar.

First we deal with the inequality at $\sigma_*/2 < \sigma < \sigma_*$, where we wish to show that $V^+_{prish} > V^+_{tip}$. By applying (2.26), then (2.25), then (2.27) we find

$$\begin{aligned}
\sigma V^+_{prish} &\geq \sigma V_{prish} + cD\sigma^{-1} \\
&\geq \sigma V_{tip} + cD\sigma^{-1} \\
&- C\left(\nu^2\sigma^2 + \sigma^{-1} + \nu\right) \\
&\geq \sigma V^+_{tip} + cD\sigma^{-1} \\
&- C\left(\nu^2\sigma^2 + \sigma^{-1} + \nu\right) - C\delta^{-1}\epsilon_v\nu^{1/2}.
\end{aligned}$$

Choosing D such that $cD \geq 2C$ means that $\delta V^+_{prish} \geq \sigma V^+_{tip}$ at least for short time. Showing that $W^+_{prish} > W^+_{tip}$ is similar.

Now we deal with the inequalities for $\zeta_* \leq \zeta \leq 2\zeta_*$. First choose $\zeta_* \geq 10\frac{CD}{c\epsilon_w}$, and then chose $\delta \leq \frac{1}{10}(CD)^{-1}c\epsilon_v\zeta_*$. Then we have, using (2.27),

$$\begin{aligned}
\sigma V^+_{tip} &\geq \sigma V_{tip} + c\delta^{-1}\epsilon_v\nu^{1/2} \\
&\geq \sigma V_{tip} + 10CD\zeta^{-1}\nu^{1/2}.
\end{aligned}$$

Now using (2.25) and then (2.26), for $\zeta_* \leq \zeta \leq 2\zeta_*$,

$$\begin{aligned}
\sigma V^+_{tip} &\geq \sigma V_{prish} + 10CD\zeta^{-1}\nu^{1/2} \\
&- C\left(\nu\zeta^2 + \nu\right) - C\zeta^{-1}\nu^{1/2} \\
&\geq \sigma V^+_{prish} + 10CD\zeta^{-1}\nu^{1/2} \\
&- C\left(\nu\zeta^2 + \nu\right) - C\zeta^{-1}\nu^{1/2} - CD\zeta^{-1}\nu^{1/2} \\
&\geq \sigma V^+_{prish} + 8CD\zeta^{-1}\nu^{1/2}
\end{aligned}$$

with the last line valid for small enough times. Therefore, for small enough times, $V^+_{tip} \geq V^+_{prish}$ here. The calculation is similar for W; since $\zeta \geq 10\frac{CD}{c\epsilon_w}$ the upper bound $CD\zeta^{-1}\nu^{1/2}$ on $\bar{W}^+_{prish} - W_{prish}$ is dominated by the lower bound $c\epsilon_w\nu^{1/2}$ on $\bar{W}^+_{tip} - \bar{W}_{tip}$.

$\square$

We take a moment here to remark on the design of the tip barriers. To understand the term $\nu^{1/2}$ in the barriers' definitions, consider what would happen in Lemma 2.3.6 if we replaced $\nu^{1/2}$ with some function $f(\nu) \ll \nu^{1/2}$. We would still have

$$V_{diff} = V^+ - V \geq c\delta^{-1}\epsilon_v f(\nu)\sigma^{1,-1} - C\epsilon_v\nu\sigma^{1,0}$$

and upon pulling out the factor $\delta^{-1}\epsilon_v f(\nu)\sigma^{1,-1}$,

$$V_{diff} \geq c\delta^{-1}\epsilon_v f(\nu)\sigma^{1,-1}\left(1 - C\delta\frac{\nu}{f(\nu)}\sigma^{0,1}\right).$$

Since $f(\nu) \ll \nu^{1/2}$, $\frac{\nu}{f(\nu)} \gg f(\nu)$, so the region where $V_{diff} > 0$ is not contained in the region $f(\nu)\sigma \leq \zeta_*$ for any ζ_*.

However, in Lemma 2.4.2, it was important that the region where $V_{diff} > 0$ is contained in the region $f(\nu)\sigma \leq \zeta_*$. The reason is that, in approximating the first term of V^+_{tip}, we use the asymptotics of V_{Bry} to say

$$V_{k^+Bry} = (\mu + \delta^{-1}\epsilon_v f(\nu))\sigma^{-1} + O(\sigma^{-2}).$$

The term $\mu\sigma^{-1}$ matches with the leading order term of the approximation for V coming from the productish region (2.5). The $O(\sigma^{-2})$ term is essentially uncontrollable and falls into the error between V_{tip} and V_{prish} in (2.24). (We could find its sign by studying the Byrant soliton more closely, but that would only help us for either the sub- or supersolution.) Then we need the left over term $f(\nu)\sigma^{-1}$ to cover $O(\sigma^{-2})$- in other words, we need $f(\nu)\sigma \geq C$ for some C.

Therefore the $\nu^{1/2}$ is somehow optimal, at least for the technique that we are using.

The point of the inequalities in Lemma 2.4.2 is that they immediately imply Lemma 2.4.3 below. This says that we can remove the assumption in Lemma 2.2.2 which assumed that the solution stays within the productish barriers on the left edge of the productish region, and we can remove the assumption from 2.2.2 which assumed that the solution stays within the tip barriers on the right edge of the tip region.

Lemma 2.4.3. *Let $D > \underline{D}$, $C_{reg} > \underline{C}_{reg}$, $u_* < \overline{u}_*(D, C_{reg})$, $\sigma_* > \underline{\sigma}_*(D, C_{reg})$, ϵ_v, $\epsilon_w < \underline{\epsilon}_w(\epsilon_v)$, $\zeta_* \geq \underline{\zeta}_*(\epsilon_w, D)$, and $\delta < \overline{\delta}(\epsilon_v, D, \zeta_*)$ be given. There is a T_* depending on all parameters such that if $T_2 < T_*$ we have the following.*

Let $0 < T_1 < T_2 < T_*$. Assume that the initial metric is controlled at the initial time in the productish and tip regions, and also controlled at the right of the productish region. Then then conclusions of Lemmas 2.2.2 and 2.3.2 hold, i.e. we have (P1), (P2) and (T1), (T2), (T3).

Proof. Let $T_{bad} > T_1$ be the maximal time such that all the conclusions hold for $g(t)$. By Lemma 2.4.2, the assumption that the solution is barricaded on the right edge of the tip region is satisfied on $[T_1, T_{bad}]$, since the productish region barriers are tighter than the tip region barriers there. Similarly, the assumption

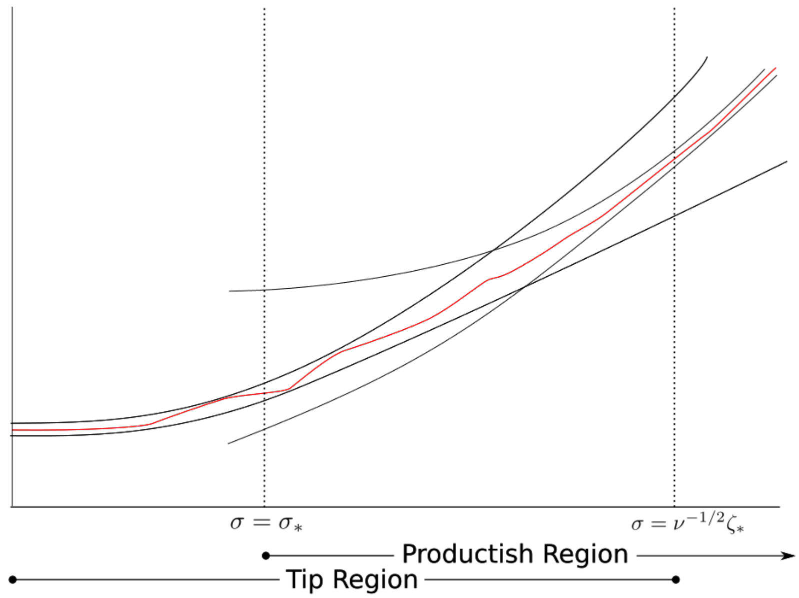

Figure 2.3: **Buckling barriers.** The red solution lies between the productish barriers in the productish region, and the tip barriers in the tip region. Because of the ordering of the barriers at $\sigma = \sigma_*$, the boundary conditions for the productish barriers are automatically satisfied. Similarly at $\sigma = \beta^{-p}\zeta_*$.

that the solution is barricaded on the left edge of the productish region holds on $[T_1, T_{bad}]$. By the assumptions of our lemma, all other assumptions needed to apply Lemmas 2.2.2 and 2.3.2 hold on $[T_1, T_{bad}]$. Therefore all the conclusions still hold at time $t = T_{bad}$. $\square$

The assumptions that g is well controlled in the productish and tip regions are all assumptions on the metric at time T_1. The only assumption left after Lemma 2.4.3 that is an a priori assumption on the forward evolution is that the metric is barricaded at the right of the productish region.

From now on we consider the constants D, C_{reg}, u_*, σ_*, ϵ_v, ϵ_w, ζ_*, and δ to be fixed and satisfying Lemma 2.4.3.

2.4.2 Mollifying metrics

In this section we will define mollified metrics, and prove some basic properties. We introduce a smooth cutoff function $\eta(x) : [0, \infty) \to [0, 1]$ which satisfies

$$\begin{cases} \eta(x) = 1 & x < 1 \\ \eta(x) \in [0,1] & 1 \leq x \leq 2 \\ \eta(x) = 0 & x > 2 \end{cases}$$

and define $\eta_r(x) = \eta(x/r)$.

Now, for arbitrary sufficiently small m, and $T_1^{(m)}$ to be determined, we define

$$V_{init}^{(m)} = \begin{cases} \eta_{2\zeta_*}(\zeta)V_{tip}(u, T_1^{(m)}) + (1 - \eta_{2\zeta_*}(\zeta))V_{prish}(u, T_1^{(m)}) & \zeta_* \nu^{-1/2} \leq \zeta \leq 4\zeta_* \nu^{-1/2} \\ V_{prish}(u, T_1^{(m)}) & 4\zeta_* t \nu^{1/2} \leq u \leq m \\ \eta_m(u) V_{prish}(u, T_1^{(m)}) + (1 - \eta_m(u)) V_0(u) & m \leq u \leq \infty \end{cases}$$

and define $W_{init}^{(m)}$ similarly. Therefore these functions agree with V_0 and W_0 for $u > 2m$, agree with the productish approximation (evaluated at time $T_1^{(m)}$) for $4\zeta_* t\nu^{1/2} < u \leq m$, and agree with the tip approximation (evaluated at time $T_1^{(m)}$) for $\zeta < 2\zeta_*$.

So far we have just been dealing with the diffeomorphism invariant considerations of v and w as functions of u and t. Now fix a model pinch metric g_{mp} on $M = I \times S^q \times F$, with the corresponding function $V_0(u)$ and

$W_0(u)$, and write it in coordinates so that:

$$g_{mp} = \frac{dx^2}{\frac{1}{4}xV_0(x)} + x^2 g_{S^q} + W_0(x) g_F.$$

In other words, the coordinate x is defined so that the value of $\phi = \sqrt{u}$ for g_{mp} is x. Now we define mollifications $g_{init}^{(m)}(t)$. We define them as

$$g_{init}^{(m)} = \frac{dx^2}{\frac{1}{4}xV_{init}^{(m)}(x)} + x^2 g_{S^q} + W_{init}^{(m)}(x,t) g_F.$$

Note that $g_{init}^{(m)}$ is equal to g_{mp} for $x > 2m$, and is smooth. It may seem that we have repeated ourselves, since we have already chosen $V_{init}^{(m)}$ and $W_{init}^{(m)}$. The point here is we are also fixing the coordinate of the interval factor.

The following Lemma says that $g_{init}^{(m)}$ satisfies all of the conditions on the initial metric required by Lemmas 2.2.2 and 2.3.2.

Lemma 2.4.4. *Let $m < \underline{m}$ and suppose $T_1^{(m)} < \underline{T}_1^{(m)}(m) < m$. Let $g_{init}^{(m)} = g^{(m)}(T_1^{(m)})$. Then for $T_1 = T_1^{(m)}$, the metric $g_{init}^{(m)}$ is initially controlled in the productish and tip regions.*

Proof. That $g_{init}^{(m)}$ is initially controlled in the tip region is immediate, because the functions v and w for $g_{init}^{(m)}$ exactly agree with with the functions V_{tip} and W_{tip} in the tip region.

Where v and w agree with V_{prish} and W_{tip}, the assumptions in the productish region are automatic. This is true for $4\zeta_* t\nu^{1/2} \leq u \leq m$. What's left is to check the assumptions for $\sigma_* t\nu \leq u \leq 2\zeta_* t\nu^{1/2}$ and $m \leq u \leq 2m$

Both conditions hold for $u \leq \zeta_* t\nu^{1/2}$ by Lemma 2.4.1, and the separation of the barriers. To check the conditions for $m \leq u \leq 2m$, note that they hold strictly in this compact set at time $t = 0$, so for sufficiently small $T_1^{(m)}$ they will continue to hold. □

2.4.3 Controlling curvature and convergence

Since $g_{init}^{(m)}$ is smooth, there is a solution to Ricci flow $g^{(m)}(t)$ on $[T_1^{(m)}, T_{final}^{(m)})$ with $g^{(m)}(T_1^{(m)}) = g_{init}$. We want to control $g^{(m)}(t)$. By Lemma 2.4.4 and 2.4.3, we have all of the conditions of Lemmas 2.2.2 and 2.3.2, except for the condition that the solution is between the barriers for $u_* < u < 2u_*$. Let $T_2^{(m)}$ be the maximal time such that this condition holds on $[T_1^{(m)}, T_2^{(m)})$. In Corollary 2.4.8 we will argue that we have a fixed lower bound on $T_2^{(m)}$.

As usual, in each lemma we may decrease T_*.

Lemma 2.4.5. *For any k, there is a constant C_k depending only on V_0, W_0, and u_* such that*

$$|\nabla^k \operatorname{Rm}_{g^{(m)}(t,x)}| < C_k$$

for any $m < \overline{m}$, any $x \in [\frac{1}{4}u_*, \infty)$, and any $t \in [T_1^{(m)}, \min(T_*, T_2^{(m)})]$.

Proof. The curvatures of the metrics $g_{init}^{(m)}$ have a uniform bound on their curvature and the volume of small enough balls in the subset $[\frac{1}{8}u_*, \infty) \times S^q \times F \subset M$. Therefore we can apply the pseudolocality theorem (Theorem 10.3 of [Per02]) at any point there, to get control on $|\mathrm{Rm}|$. Applying local derivative estimates (14.4.1 of [CCG+07b]) gives control on higher derivatives. □

Since our barrier control is in terms of u, we need to be able to transfer the set written in terms of u to being written in terms of x.

Lemma 2.4.6. *There is a $\overline{m}$ such that for any $m < \overline{m}$.*

$$\{p \in M : u^{(m)}(x,t) \in [u_*, 2u_*]\} \subset [\tfrac{1}{4}u_*, 4u_*] \times S^q \times F$$

for all $t \in [T_1^{(m)}, \min(T_*, T_2^{(m)})]$.

Proof. For $t = T_1^{(m)}$, we have $\frac{1}{4}u_* < u^{(m)}(x, t_*) < 4u_*(m)$ for $x \in [\frac{1}{4}u_*, 4u_*]$ (just from the definition). By Lemma 2.4.5, there is a uniform speed limit on u for all $x \in [\frac{1}{4}u_*, \infty]$. Therefore for $x \geq 4u_*$, u cannot decrease too fast and so we can get a time T_* so that u will not go below u_* before time T_*.

Also, we can decrease T_* so that u cannot go above u_* at $x = \frac{1}{4}u_*$. Since the conclusions of Lemmas 2.2.2 and 2.3.2 hold for $t \in [T_1^{(m)}, T_2^{(m)}]$, v is between its barriers for these times, and is in particular positive for $u \in [0, 2u_*]$. Therefore u is increasing up to $2u_*$. Therefore, u is smaller than u_* for $x < \frac{1}{4}u_*$. □

We do something sort of silly here. For a few lemmas, we assume that (F, g_F) has constant sectional curvature. This is so that we can have control on $|\mathrm{Rm}|$ via Corollaries 2.2.10 and 2.3.12. The control on $|\mathrm{Rm}|$ lets us use the powerful regularity theory set up by Shi [Shi89]. In the end, we can replace the constant sectional curvature fiber with anything we want, since the Ricci flow of warped products only cares about the Ricci curvature of the fiber.

Lemma 2.4.7. *Suppose (F, g_F) has constant sectional curvature. For any $t_0 > 0$, and $k \in \mathbb{N} \cup \{0\}$, there is a constant $C(t_0, k)$ such that*

$$|\nabla^k \operatorname{Rm}_{g^{(m)}}| \leq \frac{C(t_0, k)}{t\nu(t)}.$$

for all $t \in \left[\max(t_0, T_1^{(m)}), \min(T_*, T_2^{(m)})\right]$.

Proof. For $k = 0$, this is exactly Lemma 2.2.10, Lemma 2.3.12, and Lemma 2.4.5. For $k > 0$, we can apply Shi's derivative estimates (Theorem 1.1 of [Shi89]), using the result for $k - 1$ at time $t_0/2$. □

Corollary 2.4.8. $T_{final}^{(m)} > T_2^{(m)} > T_*$.

Proof. The Ricci curvature is bounded at time $T_2^{(m)}$, by Lemma 2.4.7. Therefore, $T_{final}^{(m)} > \min(T_*^{(m)}, T_2^{(m)})$.

By Lemmas 2.4.5 and 2.4.6 the curvature and its derivatives are bounded for $u^{(m)}(x,t) \in [u_*, 2u_*]$. This implies a speed limit on the functions $v^{(m)}$ and $w^{(m)}$ there. Since the functions are uniformly separated from the barriers are time $t = T_1^{(m)}$, they cannot pass the barriers for some fixed time. □

We now have all of the conclusions of Lemmas 2.2.2 and 2.3.2, for each $g^{(m)}(t)$, on $[T_1^{(m)}, T_*]$.

Lemma 2.4.9. *Possibly decreasing T_*, for any $x_0 > 0$, and $k \in \mathbb{N} \cup \{0\}$, there is a constant $C(x_0, k)$ such that*

$$|\nabla^k \operatorname{Rm}_{g^{(m)}}| \leq C(x_0, k)$$

for $x \in [x_0, \infty)$ and $t \in [T_1^{(m)}, T_*]$.

Proof. Once we prove the Lemma for $k = 0$, the result follows for $k > 0$ using local derivative estimates and the result for $k' = k - 1$ and $x_0' = 2x_0$.

Since $T_1^{(m)} < m$, at the beginning time $T_1^{(m)}$ the point x_0 lies in the productish region, which is defined as the points where $u \geq t\nu(t)\sigma_*$. (By restricting T_*, we can assume $\nu(t) < \frac{1}{\sigma_*}$). We begin by showing that we can control how long x_0 stays in the productish region.

The function u satisfies the evolution equation

$$\partial_t u = \Delta_M u - 2u^{-1}|\nabla u|^2 - \mu$$

and as long as u is in the productish region, we have the estimate

$$|\Delta_M u| + u^{-1}|\nabla u|^2 = |\Delta_M u| + v \leq Cv \leq c$$

where C and c are some constants depending only on the model pinch. (The bound on the laplacian comes from the regularity in conclusion (P2).) Furthermore, $u(x_0, T_1^{(m)}) = x_0$. Therefore,

$$u(x_0, t) \geq x_0 - (\mu + c)(t - T_1^{(m)}) \geq x_0 - (\mu + c)t$$

Now, x_0 continues to be a point in the productish region as long as $u \geq \sigma_* t\nu(t)$, so at least as long as

$$x_0 - (\mu + c)t \geq t\nu(t)\sigma_*$$

or for at least

$$t \leq \frac{x_0}{(\mu + c) + \nu(t)\sigma_*},$$

which, since we assume $\nu(t) < 1/\sigma_*$, will be implied if $t \leq t_0(x_0) := \frac{x_0}{\mu+c+1}$.

Now, for $t < t_0(x_0)$ we have

$$u_0 \geq x_0 - (\mu + c)t_0 \geq x_0 \left(1 - \frac{\mu + c}{\mu + c + 1}\right) = c' x_0.$$

So, since x_0 is in the productish region for $t < t_0(x_0)$ we have, by Lemma 2.2.9,

$$|\mathrm{Rm}| \leq \frac{C}{c'} \frac{1}{x_0}.$$

On the other hand, for $t > t_0(x_0)$ we have $|\mathrm{Rm}| \leq C \frac{1}{t_0(x_0)\nu(t_0(x_0))}$ by Lemma 2.4.7. □

Lemma 2.4.10. *Possibly decreasing T_*, for any x_0 and $k \in \mathbb{N} \cup \{0\}$ there is a constant $C(x_0, k)$ such that*

$$|(\nabla^{g_{mp}})^k g^{(m)}(x,t)|_{g_{mp}} \leq C(x_0, k).$$

for all $x \in [x_0, \infty)$ and $t \in [T_1^{(m)}, T_*]$.

Proof. In this proof we take all norms and covariant derivatives with respect to g_{mp} unless otherwise specified. At time $t = T_1^{(m)}$ we have some bound on the left hand side. Now, for $k = 0$, we integrate

$$\partial_t |(g^{(m)}(x) - g_{mp})| = |-2\,\mathrm{Rc}[g^{(m)}]| \leq 2C(x_0, 0)$$

where $C(x_0, k)$ is the constant from Lemma 2.4.9. This gives us the bound for $k = 0$.

For $k = 1$ we can differentiate the Ricci flow equation with $\nabla^{g_{mp}}$ to find

$$\partial_t \nabla g^{(m)} = -2\nabla \operatorname{Rc}_{g^{(m)}} .$$

Using the formula for writing one connection in terms of another, we find

$$|\nabla \operatorname{Rc}_{g^{(m)}}| \leq |\nabla^{g^{(m)}} \operatorname{Rc}_{g^{(m)}}| + C|\nabla g^{(m)}||\operatorname{Rc}_{g^{(m)}}|,$$

so

$$\begin{aligned}\partial_t |\nabla g^{(m)}| &\leq |\nabla^{(m)} \operatorname{Rc}_{g^{(m)}}| + C|\nabla^{g_{mp}} g^{(m)}||\operatorname{Rc}_{g^{(m)}}| \\ &\leq C(x_0, 1) + CC(x_0, 0)|\nabla^{g_{mp}} g^{(m)}|\end{aligned}$$

where $C(x_0, 1)$ and $C(x_0, 0)$ are the constants from 2.4.9. By Gronwall's inequality, the result follows. $\square$

Now we prove Theorem 1.2.2. Recall that we identify $M = I \times S^q \times F$ with $(\mathbb{R}^{1+q} \setminus \{0^{1+q}\}) \times F \subset \bar{M} := \mathbb{R}^{1+q} \times F$, where 0^{1+q} is the origin in $\mathbb{R}^{1+q}$. We will construct the Ricci flow $g(t)$ provided by Theorem 1.2.2 as a limit of our flows of mollified metrics $g^{(m)}(t)$. As a little notational annoyance, we set $g^{(m)}_{shift}(t) = g^{(m)}(t - T_1^{(m)})$, which is a Ricci flow for times at least $[0, T_*/2]$. ($g^{(m)}$ has the nice property that $g^{(m)}$ evaluated at time t is approximately our approximate solution at time t, whereas $g^{(m)}_{shift}$ is nice because it always starts at time 0.)

Lemma 2.4.11. *There is a sequence $m_k \searrow 0$, and a family of metrics $g(t)$, $t \in [0, T_*/2]$ such that*

$$g^{(m)}_{shift}(t) \to g(t)$$

in $C^\infty_{loc}(M \times [0, T_*/2])$*.*

We have $\partial_t g(t) = -2\,\mathrm{Rc}_{g(t)}$ *and* $g(0) = g_{mp}$*. Furthermore,* $g(t)$ *satisfies the conclusions of Lemmas 2.2.2 and 2.3.2, with non-strict inequalities. For* $t > 0$ *the metric* $g(t)$ *can be extended smoothly to* $\bar{M}$*.*

Proof. For any $x_0 > 0$ we have C^∞ control on the derivatives of $g^{(m)}_{shift}$ for $(x, t) \in [x_0, \infty) \times [0, T_*/2]$ (Lemma 2.4.10). By the Arzelà-Ascoli Theorem, we get convergence of a subsequence in any such region. By taking a diagonal subsequence, we get convergence to a metric on $g(t)$ as desired. Since the convergence happens in C^∞, the Ricci flow equation and all the estimates pass to the limit.

For any $t > 0$, the doubly-warped product metric $g(t)$ satisfies the inequalities the conclusion of Lemma 2.2.2 and 2.3.2. (Perhaps with non-strict inequalities, but we can make the constants worse to make the inequalities strict.) These imply that the metric has a extension to $\bar{M}$. $\square$

Chapter 3

Forward flow from asymmetric metrics

3.1 Overview of the proof and tools

In this chapter, we set up and prove Theorem 1.2.4. Before anything, I want to remind the reader of Appendix D, which starts on page 208, and densely lists a lot of the notation we use. I hope it is of use.

In Section 3.3, we prove Theorem 1.2.4. As in the proof of Theorem 1.2.2, we construct the forward flow as a limit of mollified metrics. Here, the mollified metrics will satisfy Ricci-DeTurck flow (see Section 3.1.1) around a background metric constructed from the forward evolution from the model pinch g_{mp}. The shape of the model pinch plays a role in being able to control the flow, and similarly to our proof of Theorem 1.2.2 we get control in the productish and tip regions separately.

In the productish region, we again use the inequalities built up in Section A to control the evolution. In the tip region, we use a contradiction-compactness argument to move the flow to Ricci flow near the Bryant soliton. Then we use Lemma 3.2.4, which is proved in Section 3.2. This constructs supersolutions to Ricci-DeTurck flow around the Bryant soliton. This was the hardest step to find, and is most dependent on the specific geometry.

3.1.1 Ricci-DeTurck flow

The Ricci flow is not a stictly parabolic system. In [DeT83], DeTurck introduced a modification of Ricci flow which is strictly parabolic. Furthermore, the modification is only the pullback by a family of diffeomorphisms, and so the Ricci flow can be reconstructed by undoing the pullback.

To implement Ricci-DeTurck flow, one chooses a background metric $\tilde{g}$. This background metric may itself evolve in time. Define, for any other metric g, the vector field

$$(V[g,\tilde{g}])^i = g^{ab}\left((\Gamma_g)^i_{ab} - (\Gamma_{\tilde{g}})^i_{ab}\right).$$

Here Γ_g and $\Gamma_{\tilde{g}}$ are the Christoffel symbols of g and $\tilde{g}$. A fancier definition of V is that it is the map laplacian of the identity map from (M,g) to $(M,\tilde{g})$. Now, we solve the initial value problem

$$\begin{aligned} g(0) &\quad \text{given}, \\ \partial_t g(t) &= -2\,\mathrm{Rc}[g] + \mathcal{L}_{V[g,\tilde{g}]}g. \end{aligned} \tag{3.1}$$

Here $\mathcal{L}$ is the Lie derivative. As it goes, (3.1) is a strictly parabolic quasilinear system, and may be solved by standard theory. "As it goes" is a poor description, see Chapter 4 of [AH11] for a good explanation why this is true. Once we have a solution to (3.1), we can recover a solution to Ricci flow. Define $\Phi : M \to M$ by integrating the vector field $-V[g,\tilde{g}]$:

$$\partial_t \Phi(p,t) = -V[g,\tilde{g}](p,t)$$

and then set $g_{RF} = \Phi^* g$. This "undoes" the lie derivative in (3.1), and g_{RF} will satisfy $\partial_t g_{RF} = -2\operatorname{Rc}[g_{RF}]$.

In Lemma 2.1 of [Shi89], Shi derived the following coordinate evolution equation for $g(t)$:

$$\partial_t g_{ij} = \Delta_{\tilde{g},g} g_{ij} - \left[g^{ab} g_{ip} \operatorname{Rm}^{p}_{ajb}\right]_{i\leftrightarrow j} + \operatorname{Cov}(g, \nabla g)_{ij}. \tag{3.2}$$

We explain the notation. Here, the convention is that all covariant derivatives and norms are by default with respect to the background metric $\tilde{g}$. We define

$$\Delta_{\tilde{g},g} = g^{ab}\nabla_a \nabla_b,$$

where we are taking covariant derivatives with respect to the background metric $\tilde{g}$, and tracing with g. The notation $[\cdot]_{i\leftrightarrow j}$ means the symmetrization of the tensor with respect to i and j, that is $[A_{ij}]_{i\leftrightarrow j} = A_{ij} + A_{ji}$. Finally, $\operatorname{Cov}(g, \nabla g)$ is a tensor contraction of two copies of g^{-1} and two copies of ∇g. Generally when dealing with questions of stabiltiy, one does not need its exact form and will only need that if $|g - \tilde{g}|_{\tilde{g}} < \frac{1}{2}$ then for some c_0 depending on the dimension,

$$|\operatorname{Cov}(g, \nabla g)| \leq c_0 |\nabla g|^2.$$

(The restriction $|g - \tilde{g}|_{\tilde{g}} < \frac{1}{2} < 1$ is needed to estimate the size of the inverse of g.)

It is useful to write (3.2) in terms of thinking of g as a perturbation of $\tilde{g}$. Furthermore, we will want to consider not just Ricci flow, but also Ricci flow modified by a time-dependent vector field. For a time-dependent vector field X, we write

$$\mathrm{Rf}_X[g] = \partial_t g - \left(-2\,\mathrm{Rc}[g] - \mathcal{L}_X g\right).$$

Now we write an equation for $h = \tilde{g} - g$. We have seen the evolution in Lemma 3.1.1 used in many places in many forms, but never written explicitly. We give the derivation in Appendix B.5. Please be patient with the dense notation, and maybe skip to the explanation after the statement of the following lemma.

Lemma 3.1.1. *Let $\tilde{g}$ be a time-dependent family of metrics, and let X be a time-dependent vector field. Let g be a metric satisfying*

$$\mathrm{Rf}_X[g] = \mathcal{L}_{V[g,\tilde{g}]}g. \tag{3.3}$$

Let $g = \tilde{g} + h$, $g^{-1} = \tilde{g}^{-1} - \bar{h}$, and $\hat{h}^{ij} = \tilde{g}^{ai}\tilde{g}^{bj}h_{ab} - \hat{h}^{ab}$.

Then

$$\begin{aligned}\Box_{X,\tilde{g},g} h = {} & 2\,\mathrm{Rm}[h] + \mathrm{UT}[h] + Q[h] + \mathrm{Cov}[g, \nabla h] \\ & - \mathrm{Rf}_X[\tilde{g}] - (\mathrm{Rf}_X[\tilde{g}] \cdot h)\end{aligned}$$

where all covariant derivatives and curvatures are with respect to $\tilde{g}$, and

$$\begin{aligned}(\Delta_{\tilde{g},g}h)_{ij} = g^{ab}\nabla_a\nabla_b h_{ij}, \quad \Delta_{X,\tilde{g},g} = \Delta_{\tilde{g},g} - \nabla_X, \quad \Box_{X,\tilde{g},g} = \partial_t - \Delta_{X,\tilde{g},g}, \\ \mathrm{Rm}[h] = \tilde{g}^{ac}\tilde{g}^{bd}\,\mathrm{Rm}_{ajbi}\,h_{cd}, \quad Q[h] = \left[\mathrm{Rm}_{ajb}^{p}\,h^{ab}h_{ip} - \mathrm{Rm}_{ajbi}\,\hat{h}^{ab}\right]_{i\leftrightarrow j}, \\ (A\cdot B)_{ij} = \tfrac{1}{2}\tfrac{1}{2}\left[\tilde{g}^{ab}A_{ai}B_{bj}\right]_{i\leftrightarrow j}, \quad \mathrm{UT}[B] = ((\partial_t\tilde{g})\cdot B)\end{aligned}$$

Remark 2. We can also write the statement of Lemma 3.1.1 as,

$$\begin{aligned}\mathrm{Rf}_{X+V[\tilde{g},g]}[g] = {} & \mathrm{Rf}_X[\tilde{g}] + (\mathrm{Rf}_X[\tilde{g}]\cdot h) \\ & + \Box_{X,\tilde{g},g}h - 2\,\mathrm{Rm}[h] - \mathrm{UT}[h] - Q(h) - Cov(g, \nabla h)\end{aligned}$$

The linear term $2\,\mathrm{Rm}[h]$ is the term of principle interest. The term $\mathrm{UT}[h]$ can be handled by the Uhlenbeck trick. This term comes from the change in background metric and the Uhlenbeck trick says that it may be removed by considering a changing orthonormal frame. We just use the following, which is a straightforward calculation and may be considered a cheap version of the Uhlenbeck trick.

Lemma 3.1.2. *If B is a time dependent two-tensor and $\tilde{g}$ is a time-dependent metric, then*

$$\partial_t |B|^2_{\tilde{g}} = 2\langle B, \partial_t B - \mathrm{UT}[B]\rangle_{\tilde{g}}$$

From this, we see that the term $\mathrm{UT}[h]$ cancels in the evolution of $|h|_{\tilde{g}}$. In our application, we will just bound the rest of the terms. As long as $|h| < \frac{1}{2}$ (we require $|h| < \frac{1}{2} < 1$ to control the norm of the inverse of g) we have

$$Q[h] \leq c_0 |\mathrm{Rm}|_{\tilde{g}} |h|^2_{\tilde{g}}, \quad Cov[g, \nabla h] \leq c_0 |\nabla h|^2_{\tilde{g}},$$

for some c_0 depending on the dimension.

For short-time existence and regularity, e.g. the work in [Shi89], it is enough to fix a background metric which doesn't change in time, and is close enough to the initial metric, provided that the background metric has bounded curvature. For long-time existence and stability, or short-time existence near metrics with unbounded curvature, one needs the background metric to satisfy Ricci flow at least to a high degree.

3.1.1.1 Evolution of the norm

The regularity theory of Ricci flow, which we use heavily, relies on the strictly parabolic nature of the evolution of h given in Lemma 3.1.1. However, the only consequence we directly use from the evolution is the following. Suppose $\mathrm{Rf}_X[\tilde{g}] = 0$ and let $y = |h|^2_{\tilde{g}}$. Let $\Lambda_{\mathrm{Rm}} : M \to \mathbb{R}$ be

$$\Lambda_{\mathrm{Rm}}(p) = \max_{h \in Sym_2(T_pM) : |h|=1} \langle \mathrm{Rm}[h], h \rangle.$$

Then (in this section c_0 is a floating constant depending on the dimension)

$$\begin{aligned} \Box_{X,\tilde{g},g} y \leq 4\Lambda_{\mathrm{Rm}} y - 2g^{ab}\tilde{g}^{cd}\tilde{g}^{ef}\nabla_a h_{ce}\nabla_b h_{df} \\ + c_0\left(|\mathrm{Rm}|y^{3/2} + |\nabla h|^2 y^{1/2}\right). \end{aligned}$$

The term $-2g^{ab}\tilde{g}^{cd}\tilde{g}^{ef}\nabla_a h_{ce}\nabla_b h_{df}$ is strictly negative. There are two (well known) ways to use it to our advantage, and one of them is sharp so we cannot use both at once. The simpler way is to write (assuming $|h| < \frac{1}{2}$, and letting c_0 be a floating constant depending on the dimension)

$$\begin{aligned} g^{ab}\tilde{g}^{cd}\tilde{g}^{ef}\nabla_a h_{ce}\nabla_b h_{df} &\geq (1 - c_0|h|)\tilde{g}^{ab}\tilde{g}^{cd}\tilde{g}^{ef}\nabla_a h_{ce}\nabla_b h_{df} \\ &= (1 - c_0 y^{1/2})|\nabla h|^2 \end{aligned}$$

and therefore we can write

$$\Box_{X,\tilde{g},g} y \leq 4\Lambda_{\mathrm{Rm}} y - 2(1 - c_0 y^{1/2})|\nabla h|^2 + c_0|\mathrm{Rm}|y^{3/2}. \tag{3.4}$$

This has the advantage that the derivative terms on the right hand side are strictly negative, provided y is sufficiently small.

The second way to use this good term is by deriving the equation for $z = y^{1/2} = |h|$:

$$\begin{aligned}\Box_{X,\tilde{g},g} z \leq{} & 2\Lambda_{\mathrm{Rm}} z \\ & - |h|^{-1} g^{ab} \tilde{g}^{cd} \tilde{g}^{ef} \nabla_a h_{ce} \nabla_b h_{df} + \tfrac{1}{4}|h|^{-3} g^{ab} \nabla_a |h|^2 \nabla_b |h|^2 \\ & + c_0 \left(|\mathrm{Rm}| z^2 + |\nabla h|^2 \right).\end{aligned}$$

This introduces the undesirable positive term $+\frac{1}{4}|h|^{-3} g^{ab} \nabla_a |h|^2 \nabla_b |h|^2$ from differentiating the square root twice. However, we can use the inequality

$$g^{ab} \nabla_a |h|^2 \nabla_b |h|^2 \leq 4|h|^2 g^{ab} \tilde{g}^{cd} \tilde{g}^{ef} \nabla_a h_{ce} \nabla_b h_{df},$$

to absorb that term. We end up with

$$\Box_{X,\tilde{g},g} z \leq 2\Lambda_{\mathrm{Rm}} z + c_0 \left(|\mathrm{Rm}| z^2 + |\nabla h|^2 \right). \tag{3.5}$$

Note now that the first term $2\Lambda_{\mathrm{Rm}} z$ has one power of h, whereas the other terms have two powers of h- so as long as the derivative is controlled the first term should dominate. In order to use this, we will need some estimate on $|\nabla h|^2$, probably from regularity theory. In the end, the regularity theory uses the first trick on the evolution of $|\nabla h|^2$.

3.2 Stability of the Bryant soliton

In this section, we will prove a sort-of stability result for Ricci DeTurck flow around the Bryant soliton, Lemma 3.2.4. This will be used to prove the short-time stability of flows from model pinches. It might be possible to bring this statement to a full C^2 stability statement for the Bryant soliton. For a complete stability result for the Bryant soliton, see [Der14]. Note that that result does not suffice here because being in the weighted L^2 space required there requires exponential decay at infinity.

We begin by proving a version of the Anderson-Chow estimate. In [AC05], Anderson and Chow proved an estimate in three dimensions for solutions to the linearization of Ricci-DeTurck flow, in terms of the scalar curvature. The inequality is

$$|\mathrm{Rc}|^2 - R\Lambda_{\mathrm{Rm}} \geq 0 \tag{3.6}$$

valid on any three-dimensional manifold. Recall the definition

$$\Lambda_{\mathrm{Rm}} = \max_{h \in Sym_2(M):|h|=1} \langle \mathrm{Rm}[h], h \rangle$$

from Section 3.1.1.1. This estimate is useful in classifying solitons [Bre13], and was also vital in [BK17]. In [WC16], Wu and Chen prove a higher-dimensional version of the Anderson-Chow estimate, assuming that the Weyl tensor vanishes identically along the flow (Claim 2.1 in [WC16]). For a singly-warped product, the Weyl tensor does vanish identically (since it is conformal to a cylinder) and therefore [WC16] applies. We also give a proof in the restricted setting we need, because it is more elementary.

For a singly warped product, $ds^2 + u(s)g_{S^q}$, we let L be the sectional curvature of a plane tangent to S^q, and K the sectional curvature of a plane spanned by ∂_s and a vector from S^q.

Lemma 3.2.1. *Let $g = ds^2 + ug_{S^q}$ be a warped product metric with nonnegative sectional curvature. Then the Anderson-Chow inequality* (3.6) *holds for* g. *The equality is achieved only at points where the sectional curvature is constant or the curvature $K = -u^{-1/2}\partial_s^2 u^{1/2}$ is* 0.

Proof. Note the calculation below is just done within the vector space $T_P M$ for an arbitrary $P \in M$. The scalar curvature of g is

$$R = 2qK + q(q-1)L.$$

The Ricci curvature of g is

$$\mathrm{Rc} = qK ds^2 + (K + (q-1)L)(ug_{S^q})$$

so

$$|\mathrm{Rc}|^2 = q^2K^2 + q(K + (q-1)L)^2.$$

Writing $\alpha = \frac{K}{(q-1)L}$ we can rewrite these as

$$\frac{R}{(q-1)L} = q(2\alpha + 1), \quad \frac{|\mathrm{Rc}|^2}{((q-1)L)^2} = q\left(q\alpha^2 + (\alpha+1)^2\right)$$

Now let's find h with $|h| = 1$ which maximizes $\mathrm{Rm}[h,h]$. Take an orthonormal basis $V_0 = \partial_s, V_1, \ldots, V_q$ for T_pM, such that h is diagonal with respect to $V_1 \ldots, V_q$, that is for i, j nonzero and distinct, $h_{ii} = \lambda_i$ and $h_{ij} = 0$.

Then we calculate,

$$\begin{aligned}
\mathrm{Rm}[h,h] &= \mathrm{Rm}_{aibj}\, h^{ij} h^{ab} \\
&= \sum_{a=1}^{n} h^{00} h^{aa}\, \mathrm{Rm}_{a0a0} + \sum_{i=1}^{n} h^{ii} h^{00}\, \mathrm{Rm}_{0i0i} + \sum_{i=1}^{n}\sum_{a=1}^{n} h^{ii} h^{aa}\, \mathrm{Rm}_{aiai} \\
&+ \sum_{j=1}^{n} h^{0j} h^{j0}\, \mathrm{Rm}_{j00j} + \sum_{i=1}^{n} h^{i0} h^{0i}\, \mathrm{Rm}_{0ii0} + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} h^{ij} h^{ji}\, \mathrm{Rm}_{ijji}\,.
\end{aligned}$$

The first line is the case when $i = j$: the first term is when $i = 0$, the second term is when $a = 0$, and the third term is when neither is 0. The second line is when $i \neq j$: the first term is when $i = 0$, the second term is when $i \neq 0$ but $j = 0$, and the third term is when neither is 0. Note that actually this last term vanishes since $h^{ij} = 0$, and since $\mathrm{Rm}_{0ii0} = \mathrm{Rm}_{j00j} = -L$, the second line is negative. Therefore to optimize h we will take $h^{0i} = 0$. Let $b = h_{00}$. Simplifying, we have

$$\mathrm{Rm}[h,h] = 2b(\sum \lambda_i)K + \left(\left(\sum \lambda_i\right)^2 - \sum \lambda_i^2\right) L$$

We can assume $b > 0$, since negating h does not change $\mathrm{Rm}[h,h]$. Then, to maximize either

$$\sum \lambda_i \quad \text{or} \quad \left(\left(\sum \lambda_i\right)^2 - \sum \lambda_i^2\right)$$

we would take the λ_i all equal. Since this maximizes either term, and since K and L are positive, it maximizes all of $\mathrm{Rm}[h,h]$. Define $\lambda = \sqrt{q}\lambda_i$, with the motivation that λ is the norm of the restriction of h to TS^q, so $b^2 + \lambda^2 = 1$. So, recalling the definition $\alpha = \frac{K}{(q-1)L}$ we arrive at

$$\frac{\mathrm{Rm}[h,h]}{(q-1)L} = 2\sqrt{q}\alpha(b\lambda) + (\lambda^2).$$

The positive eigenvalue of the matrix $\begin{pmatrix} 0 & \sqrt{q}\alpha \\ \sqrt{q}\alpha & 1 \end{pmatrix}$ is $\frac{1}{2}(1 + \sqrt{4q\alpha^2 + 1})$. Therefore, since b and λ optimize $2\sqrt{q}\alpha b\lambda + \lambda^2$ with $b^2 + \lambda^2 = 1$, we have,

$$\frac{\Lambda_{\mathrm{Rm}}}{(q-1)L} = \tfrac{1}{2}(1 + \sqrt{4q\alpha^2 + 1})$$

Therefore,

$$A = \frac{|\mathrm{Rc}|^2 - R\,\mathrm{Rm}[h,h]}{q(q-1)^2L^2} = q\alpha^2 + (\alpha+1)^2 - \tfrac{1}{2}(2\alpha+1)(1 + \sqrt{4q\alpha^2 + 1})$$

Now, for $q = 2$ and for each α, we have $A \geq 0$ by the three dimensional Anderson-Chow estimate. (We could also check by hand.) We claim A doesn't decrease as we increase q. Calculate,

$$\frac{dA}{dq} = \alpha^2 - (2\alpha+1)(4q\alpha^2+1)^{-1/2}\alpha^2$$

So for $q \geq 2$

$$\frac{dA}{dq} \geq \alpha^2\left(1 - (2\alpha+1)(8\alpha^2+1)^{-1/2}\right) \geq 0.$$

□

Now let $g_{\mathbb{R}^p}$ be the euclidean metric, and let g_{sol} be the product metric $g_{sol} = g_{Bry} + g_{\mathbb{R}^p}$. Here g_{Bry} is the Bryant soliton, and we let R_0 be its maximum scalar curvature..

Corollary 3.2.2. *The Anderson-Chow inequality* (3.6) *holds for* g_{sol}.

Proof. The extra flat factor does not affect any of the terms in (3.6). The Bryant soliton has nonnegative curvature, so the previous lemma applies. □

Using this estimate, we construct a supersolution to linearized Ricci-DeTurck flow around g_{sol}. We write $g_{Bry} = ds^2 + u(s) g_{S^q}$.

Lemma 3.2.3. *Let $(Bry \times \mathbb{R}^p, g_{sol}, X)$ be the Bryant soliton crossed with a euclidean factor, with maximum scalar curvature R_0.*

There is a function $F : Bry \times \mathbb{R}^p \to \mathbb{R}_{>0}$, which is just a function of u, with the following properties.

1. *For some $c > 0$, $\Delta_X F + 2\Lambda_{\mathrm{Rm}} F \leq -c u^{0,-2} \log(2+u) F$.*
2. *For some $c_1, c_2 > 0$, as $u \to \infty$, $F = c_1 u^{-1} \left(1 + c_2 \frac{\log u}{u}\right)(1 + o(1))$.*

Proof. First recall that if f is the soliton potential and $\bar{f}(p) = -\frac{f(p) - f(0)}{R_0}$ then $\bar{f}$ satisfies

$$\Delta_X \bar{f} = 1, \quad \bar{f}(0) = 0, \quad \nabla \bar{f}(0) = 0,$$

and has the asymptotics at ∞,

$$\bar{f} = \mu^{-1} u \left(1 - c_{\bar{f}} \frac{\log u}{u}\right)(1 + o(1; u \to \infty))$$

for some constant $c_{\bar{f}}$ (see Section B.4 and especially (B.20)). Also, $\bar{f}$ attains its minimum of 0 at $u = 0$.

Now let $F_1 = \left(\bar{f} + a\right)^{-1}$ for some $a > 0$ to be determined. Calculate,

$$\begin{aligned}\Delta_X F_1 &= -(F_1)^2 \Delta \bar{f} + 2F_1^3 |\nabla \bar{f}|^2 \\ &= -\left(F_1 - 2F_1^2 |\nabla \bar{f}|^2\right) F_1\end{aligned}$$

so,

$$-\left(\Delta_X + 2\Lambda_{\mathrm{Rm}}\right) F_1 = \left(F_1 - 2\Lambda_{\mathrm{Rm}} - 2F_1^2 |\nabla \bar{f}|^2\right) F_1 \tag{3.7}$$

We claim that for large enough B, the function $F = F_1 + BR$ satisfies the properties in the lemma. The asymptotics at infinity (i.e. item (2) of the conclusion) are immediate from the asymptotics for F_1 and $R = c_1 u^{-1} + O(u^{-2})$. Now calculate,

$$\begin{aligned}-(\Delta_X + 2\Lambda_{\mathrm{Rm}})F &= \left(\frac{-(\Delta_X + 2\Lambda_{\mathrm{Rm}})F_1}{F_1 + BR} + B\frac{-(\Delta_X + 2\Lambda_{\mathrm{Rm}})R}{F_1 + BR} \right) F \\ &=: (T_1 + T_2)\, F\end{aligned}$$

Note the term T_2 is positive everywhere by the singly-warped Anderson-Chow estimate:

$$\Delta_X R + 2\Lambda_{\mathrm{Rm}} R = -2|\mathrm{Rc}|^2 + 2\Lambda_{\mathrm{Rm}} R \leq 0.$$

Claim: Let K be a compact subset of Bry not containing the origin. If B is sufficiently large, then there is a c so that $T_1 + T_2 > c$ on $K \times \mathbb{R}^p$.

Proof of Claim: On $K \times \mathbb{R}^p$, the singly-warped Anderson-Chow estimate is not sharp, so $|\mathrm{Rc}|^2 - R\Lambda_{\mathrm{Rm}} > c_K$ in $K \times \mathbb{R}^p$. Therefore,

$$-\left(\Delta_X + 2\Lambda_{\mathrm{Rm}}\right) \geq c_K R \quad \mathrm{in} K \times \mathbb{R}^p$$

By compactness of K, everything is bounded from below on K. Therefore, examining the dependence of T_1 and T_2 on B, we can chose B large enough so that $T_1 \geq -c_K/4$ and $T_2 \geq c_K/2$ on K. ■

Claim: For sufficiently small a in the definition of F_1 (independent of B), and sufficiently small u_1 (independent of B), there is a c (which may depend on B) such that $T_1 > c$ in $\{u < u_1\}$.

Proof of Claim: Choose $a < \frac{1}{2\Lambda_{\mathrm{Rm}}(0)}$. Then $F_1 - 2\Lambda_{\mathrm{Rm}} > 0$ in a neighborhood of 0. Also, $|\nabla \bar{f}|^2(0) = 0$. The claim follows from (3.7) by choosing u_1 and c small enough. ■

Claim: For sufficiently large u_2 (depending on a, but independent of B) and sufficiently small c (depending on B and a), T_1 satisfies $T_1 \geq cu^{0,-2}\log(2+u)$ on the set $\{u > u_2\}$.

Proof of Claim: The Bryant soliton satisfies, as $u \to \infty$,

$$\mathrm{Rm} = u^{-1}\left(ug_{S^q} \odot ug_{S^q}\right) + O(u^{-2}|\nabla u|^2) = u^{-1}\left(ug_{S^q} \odot ug_{S^q}\right) + O(u^{-2})$$

Note that the largest eigenvalue of $u^{-1}\left(ug_{S^q} \odot ug_{S^q}\right)$ is $(q-1) = \frac{1}{2}\mu$. We can calculate the asymptotics of F_1 from the asymptotics of $\bar{f}$ from (B.20) in Section B.4.1:

$$F_1 = \mu u^{-1}\left(1 + c_{\bar{f}}\frac{\log u}{u}\right)(1 + o(1; u \to \infty)).$$

Also, $|\nabla \bar{f}|^2 = O(1; u \to \infty)$. From this we find,

$$(F_1 - 2\Lambda_{\mathrm{Rm}} - 2F_1^2|\nabla \bar{f}|^2) = \mu c_{\bar{f}} u^{-2}\log u + O(u^{-2}; u \to \infty)$$

The claim follows by choosing u_2 large enough and c small enough. ■

To prove the lemma, choose u_1 and u_2 in accordance with the second and third claims above, and then choose B large enough so the conclusion of the first claim holds on the complement of $\{u_1 < u < u_2\}$. Then the conclusion of the lemma holds (taking the minimum over the values of c). □

Lemma 3.2.4. *There is a constant $\overline{\epsilon} > 0$ depending only on the dimension with the following property. Suppose $\epsilon < \overline{\epsilon}$ and let $\bar{F} = \epsilon F$, where F is defined in Lemma 3.2.3.*

Suppose that $g(t) = g_{sol} + h(t)$ is a Ricci DeTurck flow around g_{sol} modified by X, on a time interval I (so (3.3) holds). Suppose that for all $P \in Bry \times \mathbb{R}^p$ and $t \in I$,

$$|h(P,t)| \leq \bar{F}(P).$$

Suppose that either $I = (-\infty, T]$ or $I = [0, T]$ with the condition at time $t = 0$ that

$$u^{0,1/2}|\nabla h| + u^{0,1}|\nabla^2 h| < \bar{F}. \tag{3.8}$$

Then the strict *inequality $|h(P,t)| < \bar{F}$ holds for all $P \in Bry$ and $t \in I$.*

We note that we could change (3.8) to having right hand side $C_h \bar{F}$ if we allow ϵ to depend on C_h.

Proof. In this proof the ever-increasing constant C is chosen independently of ϵ. First, we write the inequality solved by $\bar{F}$ in terms of the laplacian $\Delta_{X,g_{Bry},g}$.

By Lemma 3.2.3, we have

$$-\left(\Delta_X \bar{F} + 2\Lambda_{\mathrm{Rm}} \bar{F}\right) \geq c u^{0,-2} \log(2+u) \bar{F}. \tag{3.9}$$

Since $|\nabla\nabla F| \leq C u^{0,-3} \leq C u^{0,-2} F$, and $|h| \leq \epsilon F$, we have

$$|\Delta_{X,g_{Bry},g} F - \Delta_X F| \leq C u^{0,-2} \epsilon F^2 = C\epsilon u^{0,-3} F$$

and multiplying through by ϵ, $|\Delta_{X,g,\bar{g}} \bar{F} - \Delta_X \bar{F}| \leq C\epsilon u^{0,-3} \bar{F}$. Therefore, decreasing c and demanding that ϵ is sufficiently small, we can replace (3.9) with

$$-\left(\Delta_{X,g_{Bry},g} \bar{F} + 2\Lambda_{\mathrm{Rm}} \bar{F}\right) \geq c u^{0,-2} \log(2+u) \bar{F}. \tag{3.10}$$

Next we note the regularity available. We claim that for some C (independent of ϵ, P_*, and t_*), we have $|\nabla h|(P_*, t_*) < C u^{0,-1/2} \bar{F}(P_*)$. To see this, let $a = u^{0,-1}(P_*)$ and scale the parabolic system by a:

$$\tilde{g}_{Bry} = ag, \quad \tilde{h} = ah, \quad \tilde{t} = at, \quad \tilde{X} = a^{-1}X, \quad \tilde{u} = au.$$

We want to apply regularity in a parabolic neighborhood of some sufficiently small size $r > 0$. The Bryant soliton has a bound $|\nabla u|^2 \leq C$ for some C. So, for any r, for all $P \in B_{\tilde{g}}(P_*, r) = B_g(P_*, r/\sqrt{a})$ we have

$$\begin{aligned} |\tilde{u}(P) - \tilde{u}(P_*)| &= a|u(P) - u(P_*)| \\ &\leq Ca\frac{r}{\sqrt{a}} = Cr\sqrt{a} = Cru^{0,-1/2} \leq Cr. \end{aligned} \tag{3.11}$$

Therefore for sufficiently small r, the ball of radius r around P_*, with respect to $\tilde{g}$, is close to a euclidean ball, uniformly in P_*.

To continue the regularity argument, in the case when $I = [0, T]$, the parabolic neighborhood of size r around P_* may see the initial condition. We need to check what the bounds on the initial condition (3.8) says about $\tilde{h}$. At the initial time,

$$|\nabla \tilde{h}|_{\tilde{g}}(P, 0) = a^{-1/2} |\nabla h|_{g}(P, 0) \leq c_h \frac{u^{0,-1/2}(P)}{u^{0,-1/2}(P_*)} \bar{F} = c_h \frac{u^{0,1/2}(P_*)}{u^{0,1/2}(P)} \bar{F} \leq C\bar{F}$$

where we used (3.11) and forced r sufficiently small. Similarly scaling shows $|\nabla\nabla\tilde{h}|_{\tilde{g}}(P, 0) \leq C\bar{F}$.

Therefore, in the parabolic neighborhood of size r we may apply parabolic regularity to find that $|\nabla\tilde{h}|_{\tilde{g}_{Bry}}$ is bounded by $C\bar{F}$. Scaling back, we find $|\nabla h_{g_{Bry}}|(P_*, t_*) = a^{1/2}|\nabla h_{g_{Bry}}|(P_*, t_*) \leq Cu^{0,-1/2}(P_*, t_*)\bar{F}(P_*, t_*)$.

Now, by the bound on the evolution of $|h|$ (3.5), we have that $Z = |h|$ satisfies

$$\square_{X,g_{Bry},g} Z - 2\Lambda_{\mathrm{Rm}} Z \leq C|\mathrm{Rm}_{g_{Bry}}|Z^2 + C|\nabla h|^2.$$

Or, since we have assumed $Z \leq \bar{F}$, and also $|\nabla h| < Cu^{0,-\frac{1}{2}}\bar{F}$ by the discussion on regularity,

$$\square_{X,g_{Bry},g} Z - 2\Lambda_{\mathrm{Rm}} Z \leq Cu^{0,-1} Z^2 + Cu^{0,-1}\bar{F}^2.$$

Then since $Z \leq \bar{F} \leq \epsilon C u^{0,-1}$,

$$\square_{X,g_{Bry},g} Z - 2\Lambda_{\mathrm{Rm}} Z \leq \epsilon C u^{0,-2}\bar{F}.$$

In particular, we can choose ϵ sufficiently small so that

$$\square_{X,g_{Bry},g} Z - 2\Lambda_{\mathrm{Rm}} Z \leq (c/2) u^{0,-2}\bar{F}.$$

where c is the constant from (3.10).

Therefore

$$\Box_{X,g_{Bry},g}(\bar{F}-Z)-2\Lambda_{\mathrm{Rm}}(\bar{F}-Z)\geq(c/2)u^{0,-2}\bar{F}>0$$

and the lemma follows by the maximum principle. $\Box$

3.3 Asymmetric metrics

In this section, we carry out the proof of Theorem 1.2.4, which shows that we can flow from metrics with neighborhoods close to model pinches.

3.3.1 Setup of the background metric

Let (M, g_{init}) be a manifold satisfying the assumptions of the theorem. To summarize, there is an open set $U \subset M$, and a diffeomorphism $\Phi : U \to (L, L') \times S^q \times F$, and a model pinch g_{mp} such that

$$|g_{init}-\Phi^* g_{mp}|\leq \epsilon_0 V_0.$$

We also have some regularity. We will forget about the diffeomorphism Φ and just use the coordinates for $(L, L')\times S^q\times F$ on U. The metric g_{mp} has a forward Ricci flow $g_{wp}(t)$ (where "wp" stands for warped product) from Theorem 1.2.2. We may restrict this flow to (L, L') and then view it as a flow on U, and now (remember we are forgetting about Φ) we have functions $u : U \times [0, T_*) \to \mathbb{R}_+$ and $w : U \times [0, T_*) \to \mathbb{R}_+$ defined on U coming from the warped product $g_{wp}(t)$. We write u_0 for the initial value of u.

For any $u_\sharp > 0$, let $\Omega_{<u_\sharp} = \{p : u_0(p) < u_\sharp\}$ and $\Omega_{>x} = \{p \in U : u_0(p) > u_\sharp\} \cup (M \setminus U)$. Note that while $\{p : u < u_\sharp\}$ is a subset of space-time which is different for each time-slice, $\Omega_{<u_\sharp}$ is a fixed subset of M.

Now, we wish to set up a background metric to use for Ricci-DeTurck flow. Below, we will chose constants $0 < u_{**} < u_* < u_\dagger$. In the region $\Omega_{<u_*}$ we will want the background metric to be $g_{wp}(t)$ and we will control the solution using barriers. In the region $\Omega_{>u_{**}}$, we will be able to control the solution more crudely, since the initial metric has bounded curvature there. Because our control is cruder, it will not be so important exactly what the background metric is.

Let $\eta : [0, \infty) \to \mathbb{R}$ be a smooth cutoff function satisfying $\eta(x) \in [0, 1]$ and

$$\eta(x) = 1 \text{ for } x < 1 \quad \eta(x) = 0 \text{ for } x > 2,$$

and define $\eta_r(x) = \eta(x/r)$. Then define

$$g_{bg}(t) = \eta_{u_\dagger}(u_0)\, g_{wp}(t) + \big(1 - \eta_{u_\dagger}(u_0)\big)\, g_{init}.$$

Here, we abuse notation and define $\eta_{u_\dagger}(u(p, 0)) = 0$ where u is undefined, i.e. outside of the set U. So, $g_{bg}(t)$ is a time-dependent metric which agrees with $g_{wp}(t)$ for points $p \in \Omega_{<u_\dagger}$, and agrees with g_{init} for points $p \in \Omega_{>2u_\dagger}$.

Note that we can always choose T_* small enough (depending on u_* and $u_\dagger$) so that for $t < T_*$ we have $u_0(p) < u_\dagger$ wherever $u(p, t) < u_*$. Therefore $g(p, t) = g_{wp}(t)$ on the set $\{(p, t) : u(p, t) < u_*\}$.

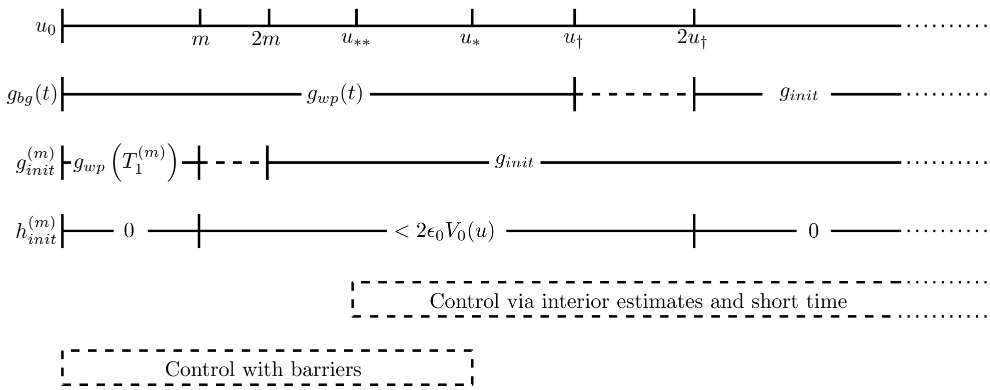

Figure 3.1: A map of our background metric and mollified metrics. The background metric g_{bg} is defined in Section 3.3.1 and the mollified metric $g_{init}^{(m)}$ is defined in Section 3.3.2. The dashed lines between m and $2m$ and between $u_\dagger$ and $u_{2\dagger}$ indicate that the metric is being interpolated between the value on the left and the value on the right. The control is performed in Lemma 3.3.5.

3.3.2 Construction of mollified initial metrics

As in the proof of Theorem 1.2.2, we will construct the forward evolution from g_{init} as a limit of mollified flows. A parameter $m \in [0,1]$ determines the space scale of the mollification. Define $T_1^{(m)} = \epsilon_{time} m V_0(m)$, where we will choose ϵ_{time} later. We define the mollified initial metric $g_{init}^{(m)}$ by

$$g_{init}^{(m)} = \eta_m(u_0)\, g_{wp}(T_1^{(m)}) + (1 - \eta_m(u_0))\, g_{init}.$$

Then $g_{init}^{(m)}$ can be extended to be a smooth complete metric with bounded curvature on $\bar{M}$ (since $g_{wp}(t)$ can). (Recall the notation $\bar{M}$ from the statement of Theorem 1.2.4.) Therefore, there is a solution to Ricci-DeTurck flow with background metric g_{bg} on a time interval $[T_1^{(m)}, T_2^{(m)}]$, with initial value $g_{init}^{(m)}$:

$$g^{(m)}(T_1^{(m)}) = g_{init}^{(m)}$$
$$\partial_t g^{(m)}(t) = -2\operatorname{Rc}_{g^{(m)}(t)} + \mathcal{L}_{V[g^{(m)}(t),\, g_{bg}(t)]} g_m(t).$$

We let $h^{(m)}(t)$ be the difference

$$h^{(m)}(t) = g^{(m)}(t) - g_{bg}(t),$$

and $h_{init}^{(m)} = h^{(m)}(T_1^{(m)}) = g_{init}^{(m)} - g_{bg}(T_1^{(m)})$. For the remainder of this section, we derive bounds on $h_{init}^{(m)}$ and its derivatives.

In $\Omega_{<m}$, both the mollified metric $g_{init}^{(m)}$ and the background metric $g_{bg}(T_1^{(m)})$ are equal to $g_{wp}(T_1^{(m)})$, so $h_{init}^{(m)} = 0$ in $\Omega_{<m}$. Similarly, in $\Omega_{>2u_\dagger}$, both $g_{init}^{(m)}$ and $g_{bg}(T_1^{(m)})$ are equal to g_{init}, so $h_{init}^{(m)} = 0$ in $\Omega_{>2u_\dagger}$.

Note that $|\mathrm{Rm}[g_{wp}(t)]| < Cu(p,t)^{-1}$ in $\Omega_{>m} \cap \Omega_{<u_\dagger}$. Therefore for $t < T_1^{(m)} = \epsilon_{time} C V_0(m)$

$$|g_{wp}(t) - g_{wp}(0)| < \epsilon_{time} C V_0(m) < \epsilon_{time} C V_0(u)$$

for all points in $\Omega_{>m} \cap \Omega_{<u_\dagger}$. Subsequently

$$\begin{aligned}|g_{wp}(T_1^{(m)}) - g_{init}(T_1^{(m)})| &< |g_{wp}(T_1^{(m)}) - g_{wp}(0)| + |g_{wp}(0) - g_{init}| \\ &< (\epsilon_{time} C + \epsilon_0) V_0(u).\end{aligned}$$

Here we used the assumption that $|g_{wp}(0) - g_{init}| < \epsilon_0 V_0(u)$. By choosing $\epsilon_{time} = (\epsilon_0 / C)$ we get

$$|g_{wp}(T_1^{(m)}) - g_{init}(T_1^{(m)})| < 2\epsilon_0 V_0(u) \tag{3.12}$$

for all points in $\Omega_{>m} \cup \Omega_{<u_\dagger}$.

Therefore, coming back to the definition of $g^{(m)}$ and the definition of g_{bg}, we find

$$|h^{(m)}_{init}| = |g^{(m)}_{init} - g_{bg}(T_1^{(m)})| < 2\epsilon_0 V_0(u) \tag{3.13}$$

everywhere.

We can also control the higher derivatives of $h^{(m)}$ with respect to g_{bg}. Here the covariant derivative ∇ is defined to be, by default, with respect to g_{bg}. Consider,

$$\nabla h^{(m)}_{init} = \eta_m(u_0)\nabla g_{wp}(T_1^{(m)}) + (1-\eta_m(u_0))\nabla g_{init} + \frac{1}{m}\eta'(u_0/m)(\nabla u_0)\left(g_{wp}(T_1^{(m)}) - g_{init}\right)$$

Note that wherever $\eta_m \neq 0$, $g_{bg}(T_1^{(m)}) = g_{wp}(T_1^{(m)})$, so the first term vanishes. Also, the region $\eta' \neq 0$ is the region $u_0 \in [m, 2m]$, so up to a constant we can replace the m in the denominator of the third term with u_0. Furthermore, η' is uniformly bounded. Therefore,

$$\begin{aligned} |\nabla h^{(m)}| &\leq |\nabla g_{init}| + C\frac{|\nabla u_0|}{u_0}|g_{wp} - g_{init}|\chi_{u_0 \in [m,2m]} \\ &= |\nabla g_{init}| + C\frac{1}{\sqrt{u_0}}\sqrt{v_0}|g_{wp} - g_{init}|\chi_{u_0 \in [m,2m]} \\ &\leq |\nabla g_{init}| + C\frac{1}{\sqrt{u_0}}v_0^{3/2}. \end{aligned}$$

In the last line we use (3.12). Now, considering only the region $u_0 < u_\dagger$, where $g_{bg} = g_{wp}(t)$, we can use our assumption on the higher derivatives in Theorem

1.2.4 to get

$$
\begin{aligned}
|\nabla h^{(m)}| &\leq |\nabla^{g_{wp}} g_{init}| + C\frac{1}{\sqrt{u_0}} v_0^{3/2} \\
&\leq C\frac{1}{\sqrt{u_0}} + C\frac{1}{\sqrt{u_0}} v_0^{3/2} \leq C\frac{1}{\sqrt{u_0}}.
\end{aligned}
$$

Finally, we can bound $|\nabla g_{init}|$ in the region $u_0 > u_\dagger$ by some constant, so this estimate extends to that region as well.

Doing similar computations for higher derivatives, we find

$$
\sum_{i=1}^{5} u_0^{i/2} |\nabla^i h_{init}^{(m)}| \leq C. \tag{3.14}
$$

This implies that the curvature of $g_{init}^{(m)}$ satisfies, in the region $u_0 > u_{**}$,

$$
|\mathrm{Rm}_{g_{init}^{(m)}}| + |\nabla \mathrm{Rm}_{g_{init}^{(m)}}| + |\nabla\nabla \mathrm{Rm}_{g_{init}^{(m)}}| \leq C.
$$

and in the region $u_0 \in [m, u_*]$,

$$
|\mathrm{Rm}_{g_{init}^{(m)}}| + u_0^{1/2} |\nabla \mathrm{Rm}_{g_{init}^{(m)}}| + u_0 |\nabla\nabla \mathrm{Rm}_{g_{init}^{(m)}}| \leq \frac{C}{u_0}
$$

3.3.3 Control in the productish region

In this section we control the Ricci DeTurck flow in the productish region of the warped-product solution $g_{wp}(t)$. This uses the general sub- and supersolutions from Appendix A. Note that since $u_* < u_\dagger$, in $\{u < u_*\}$ the background metric g_{bg} agrees with the warped product solution to Ricci flow g_{wp} (for small enough times), so $g^{(m)}(t)$ solves Ricci DeTurck flow around g_{wp}.

Our warped product metric $g_{wp}(t)$ satisfies the following in the productish region. (See the definition of "controlled in the productish region" (Definition 2.2.1) and Corollary 2.2.9.)

- We have

  $(1-DV)V \leq v \leq (1+DV)V$ and $(1-DV)\hat{W}t \leq w+\mu_F t \leq (1+DV)\hat{W}$

  where $V = Q \cdot V_0 \circ U_0 = \frac{u+\mu t}{u} V_0(u+\mu t)$, and $\hat{W} = W_0(u+\mu t)$.

- $|\nabla\nabla u| < Cv$.

- The curvature Rm can be written as

$$\begin{aligned} \mathrm{Rm} &= u\,\mathrm{Rm}_{g_{S^q}} + w\,\mathrm{Rm}_F + \mathrm{Rm}_{warp} \\ &= u^{-1}\left((ug_{S^q}) \owedge (ug_{S^q})\right) + w\,\mathrm{Rm}_F + \mathrm{Rm}_{warp}, \end{aligned} \tag{3.15}$$

  where Rm_{warp} satisfies $|\mathrm{Rm}_{warp}| \leq Cu^{-1}v$.

Recall the definition $\Lambda_{\mathrm{Rm}} = \max_{h \in Sym_2(M):|h|=1} \langle \mathrm{Rm}[h], h\rangle_{g_{mp}}$ from Section 3.1.1.1. Here, and in this section, by default we are taking all inner products and curvatures with respect to $g_{wp}(t)$. From the second point, we will get the following Lemma.

Lemma 3.3.1. *There is a constant C depending on the model pinch g_{mp} such that the forward evolution $g_{wp}(t)$ from g_{mp} satisfies*

$$\Lambda_{\mathrm{Rm}} \leq (q-1)u^{-1} + Cu^{-1}v = \tfrac{1}{2}\mu u^{-1} + Cu^{-1}v.$$

Proof. Note the use of the metric in $\mathrm{Rm}[h]$ to contract tensors. Let us write

$$\left(\mathrm{Rm}_{g_1}[h;g_2]\right)_{ef} = (g_2)^{ab}(g_2)^{cd}\left(\mathrm{Rm}_{g_1}\right)_{acef} h_{bd}$$

so that $\mathrm{Rm}[h] = \mathrm{Rm}_{g_{wp}}[h;g_{wp}]$. Now we can compute some scaling for the components of $\mathrm{Rm}_{g_{wp}}$ in (3.15).

$$\begin{aligned}\left(u\,\mathrm{Rm}_{g_{S^q}}\right)[h;g_{wp}] &= \left(u\,\mathrm{Rm}_{g_{S^q}}\right)[h;ug_{S^q}]\\ &= u^{-1}\left(\mathrm{Rm}_{g_{S^q}}\right)[h;g_{S^q}].\end{aligned}$$

Therefore

$$\begin{aligned}\max_{|h|_{g_{wp}}=1}\langle(u\,\mathrm{Rm}_{g_{S^q}})[h;g_{wp}],h\rangle_{g_{wp}} &= u^{-1}\max_{|h|_{g_{S^q}}=1}\langle\mathrm{Rm}_{g_{S^q}}[h;g_{S^q}],h\rangle_{g_{S^q}}\\ &:= u^{-1}\Lambda_{S^q} = u^{-1}(q-1).\end{aligned} \tag{3.16}$$

Similarly,

$$\max_{|h|_{g_{wp}}=1}\langle(w\,\mathrm{Rm}_{g_F})[h;g_{wp}],h\rangle_{g_{wp}} = w^{-1}\Lambda_F. \tag{3.17}$$

Now, we use our assumed bounds on W_0,

$$\begin{aligned}\frac{w}{u} &\geq \frac{W_0(u+\mu t)-\mu_F t}{u} + \frac{DVW_0(u+\mu t)}{u}\\ &\geq \frac{k\cdot(u+\mu t)-\mu_F t}{u} + \frac{kDV\cdot(u+\mu t)}{u},\end{aligned}$$

where $k = \max\left(\frac{\Lambda_F}{\Lambda_{S^q}},(1+c)\frac{\mu_F}{\mu}\right)$ by the assumption (MR1) of Theorem 1.2.4 and the assumption (MP3) of model pinches. First using $k \geq (1+c)\frac{\mu_F}{\mu}$ and then $k \geq \frac{\Lambda_F}{\Lambda_{S^q}}$,

$$\frac{w}{u} \geq k + \frac{kDV\cdot(u+\mu t)}{u} \geq \frac{\Lambda_F}{\Lambda_{S^q}} + Cu^{-1}V$$

for some C depending only on g_{wp}. Therefore, coming back to (3.17), and increasing C,

$$\max_{|h|_{g_{wp}}=1} \langle (w\,\mathrm{Rm}_{g_F})[h; g_{wp}], h\rangle_{g_{wp}} \leq u^{-1}\Lambda_{S^q} + Cu^{-1}V. \tag{3.18}$$

Now we put together (3.16) and (3.18). Since Rm_{g_F} and $\mathrm{Rm}_{g_{S^q}}$ act only on the orthogonal components $Sym_2(TF) \subset Sym_2(TM)$ and $Sym_2(TS^q) \subset Sym_2(TM)$ respectively, we can take the maximum of the two pieces to find

$$\max_{|h|_{g_{wp}}=1} \langle (u\,\mathrm{Rm}_{g_S^q} + w\,\mathrm{Rm}_{g_F})[h; g_{wp}], h\rangle_{g_{wp}} \leq u^{-1}\Lambda_{S^q} + Cu^{-1}V.$$

Finally, adding in Rm_{warp} can only change this result by something proportional to its norm. So, increasing C,

$$\begin{aligned}\max_{|h|_{g_{wp}}=1} \langle \mathrm{Rm}_{g_{wp}}[h; g_{wp}], h\rangle_{g_{wp}} &= \max_{|h|_{g_{wp}}=1} \langle (u\,\mathrm{Rm}_{g_S^q} + w\,\mathrm{Rm}_{g_F} + \mathrm{Rm}_{warp})[h; g_{wp}], h\rangle_{g_{wp}} \\ &\leq u^{-1}\Lambda_{S^q} + Cu^{-1}V.\end{aligned}$$

□

Let $y = |h^{(m)}(t)|^2$. By the equation for the evolution of the norm of the perturbation (3.4), in the productish region y satisfies the inequality

$$\Box_{g_{wp},g}\, y \leq u^{-1}\left(2\mu + Cv + Cy^{1/2}\right) y, \tag{3.19}$$

or, just rewriting,

$$(\Box_{g_{wp},g} - 2\mu u^{-1}) y \leq Cu^{-1}\left(v + y^{1/2}\right) y. \tag{3.20}$$

We now use the supersolutions found in Appendix A to control y in the productish region.

Lemma 3.3.2. *Suppose $D' > \overline{D}'(g_{wp})$, $0 < \epsilon < 1$ is given, $u_* < \overline{u}_*(g_{wp}, D')$, $\sigma'_* < \underline{\sigma}'_*(g_{wp}, D')$, and $T_* < \overline{T}_*(g_{wp}, D')$. Set*

$$\Omega'_{prish} = \left\{(p,t) : u < u_*, \sigma = \frac{u}{t\nu(t)} > \sigma'_*, t < T_*\right\}$$

Let $Y^+ = (1 + D'V)Q^2 (V_0 \circ U_0)^2 = (1 + D'V)V^2$ and $\bar{Y}^+ = \epsilon^2 Y^+$. If $y < \bar{Y}^+$ on the parabolic boundary of Ω, then $y < \bar{Y}^+$ in Ω.

The factor of ϵ^2 may seem superfluous, but it is needed for the argument in the tip region.

Proof. In the end, we will choose $\overline{u}_*$, $\underline{\sigma}_*$, and $\overline{T}_*$ to ensure that $\bar{Y}^+$ is certainly smaller than $\frac{1}{2}$ in the region Ω. (We may do this since we can make V arbitrarily small by Lemma 2.2.4.) Therefore equation (3.4) is valid.

By Lemma A.1.5 we have that, for some $c > 0$,

$$\Box_{g_{wp}} \bar{Y}^+ - 2\mu u^{-1} \bar{Y}^+ \geq (cD')u^{-1} v \bar{Y}^+.$$

Since $(\bar{Y}^+)^{1/2} \leq CV$, we find (possibly decreasing c),

$$\Box_{g_{wp}} \bar{Y}^+ - 2\mu u^{-1} \bar{Y}^+ \geq (cD')u^{-1} \left(v + (\bar{Y}^+)^{1/2}\right) \bar{Y}^+$$

We can change the $\Box_{g_{wp}}$ to $\Box_{g_{wp}, g^{(m)}}$. (Recall the definition of $\Box_{g_{wp}, g^{(m)}}$ in Lemma 3.1.1.) As long as $y < Y^+$ we have

$$|\Box_{g_{wp}} Y^+ - \Box_{g_{wp}, g^{(m)}} Y^+| \leq C(\bar{Y}^+)^{1/2} |\nabla\nabla Y^+| \leq Cu^{-1} v (\bar{Y}^+)^{3/2}.$$

In the second inequality we use Lemma A.1.6, and our bound $|\nabla\nabla u| < Cv$. Again decreasing c, we have

$$\Box_{g_{wp},g^{(m)}} Y^+ - 2\mu u^{-1} Y^+ \geq (cD')u^{-1}\left(v + (\bar{Y}^+)^{1/2}\right)\bar{Y}^+ \tag{3.21}$$

The lemma follows by comparing (3.21) to the evolution for y (3.20), choosing D' large enough, and applying the maximum principle. $\square$

3.3.4 Control in the tip region

Lemma 3.3.3. *Let σ_*, ζ_*, and $\epsilon < \bar{\epsilon}(g_{wp})$ be given. Let F be the function from Lemma 3.2.3 and let $\bar{F} = \epsilon F$. There is a $T_*(\sigma_*, g_{wp})$ such that we have the following.*

Suppose $g(t) = g_{bg}(t) + h(t)$ is a solution to Ricci-DeTurck flow around a background metric $g_{bg}(t)$, on a time interval $[T_1, T_2]$, and $T_2 < T_*$. Suppose that $g_{bg}(t) = g_{wp}(t)$ for $u < u_\dagger$, for some $u_\dagger > 0$ and for some metric $g_{wp}(t)$ satisfying the conclusions of Theorem 1.2.2. Suppose that

$$h = 0 \quad \text{for} \quad t = T_1 \text{ and } \sigma < \nu^{-1/2}(T_1)\zeta_*,$$

$$|h|_{g_{wp}} \leq \bar{F} \quad \text{for} \quad t \in [T_1, T_2] \text{ and } \sigma \in [\sigma_*, \nu^{-1/2}\zeta_*],$$

and

$$\max_M |\mathrm{Rm}_g| \leq C_{\mathrm{Rm}} \frac{1}{t\nu(t)}.$$

Then $|h|_{g_{wp}} \leq \bar{F}$ for $\sigma < \sigma_*$ and $t \in [T_1, T_*]$.

Proof. We will choose $\bar{\epsilon}$ sufficiently small in the end. We use a contradiction-compactness argument to move the situation to Ricci-DeTurck flow around the Bryant soliton crossed with a euclidean factor, $g_{sol} = g_{Bry} + g_{\mathbb{R}^{dim(F)}}$.

For contradiction, assume that there is no such T_*. This means that there is a sequence of counterexamples: there are solutions $g^{(i)} = g_{wp} + h^{(i)}$ to the Ricci DeTurk flow around g_{wp}, defined on intervals $[T_1^{(i)}, T_2^{(i)}]$, satisfying the conditions of the Lemma, but $|h^{(i)}(p^{(i)}, T_2^{(i)})| = \bar{F}(\sigma(p^{(i)}, T_2^{(i)}))$ for some sequence $p^{(i)}$ with $\sigma(p^{(i)}, T_2^{(i)}) \leq \sigma_*$ and $T_2^{(i)} \searrow 0$. Let $\sigma^{(i)} = \sigma(p^{(i)}, T_2^{(i)})$. We may pass to subsequence so that the $\sigma^{(i)}$ converge to some $\sigma^{(\infty)} \leq \sigma_*$.

Let $\alpha^{(i)} = \alpha(T_1^{(i)})$. We claim that there is a T_{**} depending on g_{wp} such that $T_2^{(i)} - T_1^{(i)} \geq \alpha^{(i)} T_{**}$. Indeed, we can let $g^{(i)}_{scaled}(0) = \frac{1}{\alpha^{(i)}} g^{(i)}(0)$. This will have uniformly bounded curvature, and so for some fixed time its Ricci flow will have uniformly bounded curvature, and therefore can only move so far. Also, $g^{(i)}_{wpscaled} = \frac{1}{\alpha^{(i)}} g_{wp}$ has the same property. By regularity the covariant derivatives of h with respect to g_{wp} are uniformly bounded for bounded time, and hence the DeTurk diffeomorphisms can only move so much. All in all, for some fixed time, the scaled version of h can only move so far on the compact set $\{\sigma < \sigma_*\}$, so scaling back we get the result.

Let G_{wp} be the family of metrics $G_{wp} = \alpha^{-1}(\sigma^{-1})^* g_{wp}$ which is g_{wp} modified by scaling by α^{-1} and pulling back for σ. Also let $G^{(i)} = \alpha^{-1}(\sigma^{-1})^* g^{(i)}$ and $H^{(i)} = G^{(i)} - G_{wp} = \alpha^{-1}(\sigma^{-1})^* h^{(i)}$.

Now $G^{(i)}$ satisfies

$$\partial_\theta G^{(i)} = -2\,\mathrm{Rc}[G^{(i)}] - \mathcal{L}_{X+V[G^{(i)},G_{wp}]} G^{(i)} - \beta G^{(i)}, \tag{3.22}$$

for $\theta \in [\theta(T_1^{(i)}), \theta(T_2^{(i)})]$. Here X is the vector field $\partial_\theta \sigma$. Note that

$$\theta(T_2^{(i)}) - \theta(T_1^{(i)}) = \int_{T_1^{(i)}}^{T_2^{(i)}} \alpha^{-1} dt \geq \int_{T_1^{(i)}-\alpha^{(i)}T_{**}}^{T_1^{(i)}} \alpha^{-1}(t)dt \geq T_{**} \frac{\alpha\left(T_1^{(i)}\right)}{\alpha\left(T_1^{(i)} + \alpha(T^{(1)}(i))T_{**}\right)}$$

By Lemma B.3.2, this right hand side is $(1 + o(1; T_1 \searrow 0))T_{**}$. So, passing to a subsequence, the sequence $\theta(T_2^{(i)}) - \theta(T_1^{(i)})$ either converges to ∞ or converges to some $\Theta_1 > 0$.

Translate the θ intervals so that the times $\theta(T_2^{(i)})$ all land at time 0. By Corollary 2.3.13, the background metrics G_{wp} converge to the Bryant soliton crossed with $\mathbb{R}^{dim(F)}$, and the vector field X converges to the soliton vector field. Passing to a subsequence, the $H^{(i)}$ converge to a solution H of Ricci-DeTurck flow around the Bryant soliton, modified by the Bryant soliton vector field X. (Note that the term βG in (3.22) converges to zero.) The time interval is either $\theta \in (-\infty, 0]$, or $\theta \in [\Theta_1, 0]$. In the second case, $H(\Theta_1) = 0$. Furthermore, since $h^{(i)}$ satisfy the hypotheses of the lemma, the bounds in Lemma 3.2.4 are satisfied, provided $\bar{\epsilon}$ is small enough.

However, at time 0 and at some point $p \in Bry \times \mathbb{R}^{dim(F)}$ with $\sigma_{Bry}(p) = \sigma^{(\infty)}$, we will have $|H| = |\bar{F}|$. This contradicts the strict inequality in the conclusion of Lemma 3.2.4. $\square$

3.3.5 Buckling Barriers

In this lemma, we show that the function

$$Y^+ = (1 + D'V)Q^2(V_0 \circ U_0)^2,$$

which we use as a barrier for $|h|^2$ in the productish region, crosses the function F^2, which we use as a barrier in the tip region. This shows that they ensure each others' boundary conditions.

The following Lemma deals with the unscaled functions Y^+ and F^2. Of course, the inequalities (3.23) and (3.24) also hold for $\bar{Y}^+ = \epsilon^2 Y$ and $\bar{F}^2 = \epsilon^2 F^2$.

Lemma 3.3.4. *Let the constant D', in the definition of Y^+ be given. There are $\sigma_* > 0$, $\sigma_2 > 0$, $\zeta_* > 0$, and $b \in \mathbb{R}_+$ such that we have the following inequalities.*

For $t < T_*$, at $\sigma = \sigma_*$, we have

$$bF^2 < Y^+. \tag{3.23}$$

For $t < T_*$, and $\sigma \in [\sigma_2, \zeta_* \nu^{-1/2}]$, we have

$$Y^+ < bF^2. \tag{3.24}$$

Proof. Below c_i are positive constants, and all asymptotics are as $\sigma \to \infty$ and $t \searrow 0$.

Recall the asymptotics of F:

$$F = c_1 \sigma^{-1} - c_2 \sigma^{-2} \log \sigma + o(\sigma^{-2} \log \sigma),$$

so,

$$F^2 = \sigma^{-2}\left(c_3 - c_4\sigma^{-1}\log\sigma + o(\sigma^{-1}\log\sigma)\right).$$

Recall the asymptotics of V:

$$V = c_5\sigma^{-1}\left(1 + O(\nu+\nu^2\sigma)\right),$$

so,

$$\begin{aligned} Y^+ &= (1+D'V)V^2 \\ &= \sigma^{-2}\left(c_6 + c_7 D'\sigma^{-1} + O(\nu+\nu^2\sigma) + O(D'\sigma^{-2})\right). \end{aligned}$$

Letting $d = bc_3 - c_6$, we find,

$$\sigma^2(bF^2 - Y^+) = d - \sigma^{-1}\left(c_4\log\sigma + c_7D'\sigma^{-1} + o(\log\sigma) + O(D'\sigma^{-2})\right) + O(\beta+\beta^2\sigma)$$

Now choose σ_* large enough so that for $\sigma > \sigma_*$ the asymptotic terms $o(\log\sigma)$ and $O(D'\sigma^{-2})$ above apply well. Furthermore, since $\sigma\nu^2 < \zeta_*\nu^{3/2}$ for $\sigma \leq \zeta_*\nu^{-1/2}$, we can choose T_* small enough so that the $O(\nu+\nu^2\sigma)$ term is smaller, in norm, than $d/2$. Specifically, for $\sigma \in [\sigma_*, \zeta_*\nu^{-p}]$ and $t < T_*$ we have

$$\begin{aligned} &\tfrac{1}{2}d - \tfrac{3}{2}\sigma^{-1}\left(c_4\log\sigma + c_7D'\sigma^{-1}\right) \\ &\leq \sigma^2(bF^2 - Y^+) \\ &\leq \tfrac{3}{2}d - \tfrac{1}{2}\sigma^{-1}\left(c_4\log\sigma + c_7D'\sigma^{-1}\right). \end{aligned}$$

Now choose $b = b(\sigma_*, D')$ so that $d = bc_3 - c_6$ is positive but small enough that

$$\tfrac{3}{2}d - \tfrac{1}{2}\sigma_*^{-1}\left(c_4\log\sigma_* + c_7D'\sigma_*^{-1}\right) < 0$$

so the desired inequality holds at $\sigma = \sigma_*$. Then choose σ_1 large enough so that

$$\tfrac{1}{2}d - \tfrac{3}{2}\sigma^{-1}\left(c_4 \log \sigma + c_7 D' \sigma^{-1}\right) > 0$$

for $\sigma > \sigma_1$. Then the desired inequality for $\sigma \in [\sigma_1, \zeta_* \nu^{-1/2}]$ also holds, for small enough times. $\square$

3.3.6 Global control

We come back to our mollified initial metrics $g_{init}^{(m)}$ defined in section 3.3.2. Recall that for each m, we have defined $g^{(m)}(t)$ to be the Ricci-DeTurck flow around $g_{bg}(t)$, on some interval $[T_1^{(m)}, T_2^{(m)}]$, starting from $g_{init}^{(m)}$.

By now, we are set up with the functions

$$\bar{Y}^+ = \epsilon^2 Y^+ = \epsilon^2 (1 + D'V) Q^2 (V_0 \circ U_0)^2$$

which serves as an upper barrier for $|h|^2$ in the productish region, and

$$\bar{F}^2 = b\epsilon^2 F^2$$

which serves as an upper barrier in the tip region. By Lemma 3.3.4, we only need to ensure the boundary conditions for the barriers at the initial time and at the $u = u_*$ boundary of the productish region. The following Lemma summarizes this.

Recall that ϵ_0 controls how close we assume the singular initial metric g is to the model pinch g_{mp}. On the other hand ϵ controls how tight our barriers are, and needs to be small for Lemma 3.3.3. ϵ_0 will be smaller than ϵ so that the initial metric has a little room below the barriers.

Lemma 3.3.5. *Let the constants D', σ_*, ζ_*, ϵ, and b, used to define the barriers $\bar{Y}^+$ and $\bar{F}^2$, be chosen in accordance with Lemmas 3.3.2, 3.3.3, and 3.3.4. Let $\epsilon_0 = (1/C_\epsilon)\epsilon$, where ϵ_0 is in the assumption of Theorem 1.2.4 and C_ϵ depends only on the model pinch. Let $u_* < \overline{u}_*(\epsilon_0, D')$, in accordance with Lemma 3.3.2, and let $\delta = \sqrt{\frac{1}{2}\bar{Y}(u_*, 0)}$. Then, decreasing T_* depending on everything else, the following holds.*

Let $g_{bg}(t)$ be defined as in Section 3.3.1 with $u_\dagger = 2u_*$, and let $g^{(m)}$ and $h^{(m)}$ be defined as in Section 3.3.2. At the initial time $T_1^{(m)}$, we have

$$|h^{(m)}_{init}| < \tfrac{1}{2}\delta \quad \text{in} \quad \Omega_{u>u_*/4}.$$

Let $T^{(m)}_{bad}$ be the first time such that

$$|h^{(m)}| < \delta \quad \text{in} \quad \Omega_{u>u_*/2} \tag{3.25}$$

fails to hold. Then on $[T_1^{(m)}, \min(T^{(m)}_{bad}, T_*)]$,

1. *For $\sigma < \zeta_* \beta^{-1/2}$, we have $|h^{(m)}|^2 < b\bar{F}^2$.*
2. *For $\sigma > \sigma_*$ and $u < u_*$, we have $|h^{(m)}|^2 < \bar{Y}^+$.*

Proof. Choose $C_\epsilon > 2$ at least large enough so that with, $\epsilon_0 = \frac{1}{C_\epsilon}\epsilon$ the definition of $g^{(m)}_{init}$ (and in particular (3.13)) implies items 1 and 2 at time $t = T_1^{(m)}$. We even have $h^{(m)}(T_1^{(m)}) = 0$ for $\sigma < \beta^{-1/2}\sigma_*$. Then, by Lemmas 3.3.2, 3.3.3, and 3.3.4, items 1 and 2 hold as long as

$$|h^{(m)}|^2 < \bar{Y}^+ \tag{3.26}$$

continues to hold at $u = u_*$.

Choose $u_\dagger = 2u_*$, and possibly decrease u_* and increase C_ϵ so that

$$4\epsilon_0 V_0(2u_\dagger) < \delta := \sqrt{\tfrac{1}{2}\bar{Y}^+(u_*, 0)}.$$

To do this, we first possibly decrease u_* so that $V_0(2u_\dagger) = V_0(4u_*) < \frac{1}{4}\delta\epsilon_0^{-1}$. Then note

$$\begin{aligned}
\sqrt{\tfrac{1}{2}\bar{Y}^+(u_*, 0)} &= \frac{\epsilon_1}{\sqrt{2}} V_0(u_*) \sqrt{1 + D' V_0(u_*)} \\
&\geq \frac{\epsilon_1}{\sqrt{2}} V_0(u_*) = \frac{C_\epsilon \epsilon_0}{\sqrt{2}} V_0(u_*) \\
&= \frac{C_\epsilon}{4\sqrt{2}} \frac{V_0(\frac{1}{2}u_\dagger)}{V_0(2u_\dagger)} \left(4\epsilon_0 V_0(2u_\dagger)\right).
\end{aligned}$$

So, choosing C_ϵ large enough, we get the desired inequality. Note that it can be chosen independently of $u_\dagger = 2u_*$ by Lemma B.3.2, which is just some calculus with the regularity assumption on V_0.

By (3.13) we have

$$|h_{init}^{(m)}| \leq 2\epsilon_0 V_0(u) \leq 2\epsilon_0 V_0(2u_\dagger) < \tfrac{1}{2}\delta \quad \text{in} \quad \Omega_{>u_*/4}.$$

which proves the claim for the initial time $T_1^{(m)}$. We have $\delta = \sqrt{\frac{1}{2}\bar{Y}^+(u_*, 0)}$, so possibly restricting T_*, we can get that $\delta < \sqrt{\bar{Y}^+(u_*, t)}$ for all $t < T_*$. Therefore, as long as

$$|h^{(m)}| < \delta \quad \text{in} \quad \Omega_{\geq u_*/2},$$

we would have (3.26) and therefore prove items 1 and 2. $\square$

The final desired inequality (3.25) can be shown to hold for a short time just by regularity of the Ricci DeTurck flow.

Lemma 3.3.6. *With the setup of Lemma 3.3.5, possibly decreasing T_* and u_*, $T_{bad}^{(m)} > T_*$.*

Proof. The set $\Omega_{u>u_*/4}$ is a compact manifold with boundary. On it, g_{bg} has bounded curvature for $t \in [0, T_*]$. On $[T_1^{(m)}, T_{bad}^{(m)}]$, $|h^{(m)}|$ is bounded by $C\delta$ for some C depending only on the model pinch. (This uses conclusion (2) of Lemma 3.3.5.) Also on this set, up to five derivatives (with respect to g_{bg}) of $h^{(m)}$ are controlled at the initial time (from (3.14)).

We decrease u_* so that $\delta = \sqrt{\frac{1}{2}\bar{Y}(u_*, 0)}$ is small enough so that we can apply interior Schauder estimates to the Ricci DeTurck flow within $\Omega_{u>u_*/4}$. This gives us bounds on up to four derivatives of $h^{(m)}$ in $\Omega_{u>(3/8)u_*}$, which are independent of m. Therefore we have a bound on the time derivative of $|h^{(m)}|$, so restricting T_* we get the claim. □

Lemma 3.3.7. *There is a constant C such that*

$$|\mathrm{Rm}_{g_{wp}}|^{-i/2} |\nabla^i h| \leq C$$

in $u < u_\dagger$ and $t < T_*$, for $i = 1, 2, 3, 4$. Furthermore, the final time $T_2^{(m)}$ of the Ricci-DeTurck flow of $g^{(m)}$ satisfies $T_2^{(m)} > T_*$.

Proof. As long as $|h^{(m)}(t)|$ stays bounded everywhere, we can apply interior regularity to control the derivatives of $h^{(m)}$, using our derivative bounds for the initial metric (3.14).

Once we have derivative bounds on $h^{(m)}$, we can also control the curvature of $g^{(m)}$. This gives us a full bound on the curvature, up to time T_*, so $T_2^{(m)} > T_*$. $\square$

Lemma 3.3.8. *As $m \searrow 0$, a subsequence of the time-dependent metrics $g^{(m)}(t)$ converge to a solution $g(t)$ of Ricci-DeTurck flow around g_{bg}, with $g(0) = g_{init}$. The convergence happens in $C^3_{loc}\left(\bar{M} \times [0, T_*] \setminus P \times \{0\}\right)$, where $P = \bar{M} \setminus M$.*

Furthermore, the DeTurck vector fields $V[g^{(m)}, g_{bg}]$ converge, in C^2, to $V[g(t), g_{bg}]$.

Proof. Let K_i, $i \in \mathbb{N}$ be an increasing sequence of compact sets whose union is $\bar{M} \times [0, T_*] \setminus P \times \{0\}$. On each K_i, up to four derivatives of $h^{(m)}$ are controlled by a constant which is independent of m, so we can apply Arzela-Ascoli to find a convergent subsequence on K_i in C^3. The diagonalization argument gives convergence on $C^3_{loc}\left(\bar{M} \times [0, T_*] \setminus P \times \{0\}\right)$.

Since the convergence happens in C^3, the equation passes to the limit. Since $V[g^{(m)}, g_{bg}]$ depends on one derivative of $g^{(m)}$ with respect to g_{bg}, we get the convergence of the vector fields. $\square$

Appendices

Appendix A

Nearly constant regions

A.1 Nearly constant regions of reaction-diffusion equations

Let $\mu > 0$ and $c_v \in \mathbb{R}$. We study solutions to

$$\begin{aligned}\Box u &= -\mu + c_v u^{-1}|\nabla u|^2 \\ &= -\mu + c_v v \end{aligned} \tag{A.1}$$

where we have defined $v = u^{-1}|\nabla u|^2$. We are interested investigating regions where v is small, and controlling other functions in terms of u.

In this section we consider $u : M \times [0, T) \to \mathbb{R}$ which satisfies (A.1), on an evolving Riemannian manifold $(M, g(t))$ which satisfies Ricci flow. (If $(M, g(t))$ does not satisfy Ricci flow, there is another term in (A.2) below.) The value of c_v does not come into play very much here. (Recall from the paragraphs after (1.13) in Section 1.6 that in some situations it is important.)

Lemma A.1.1. *Suppose u satisfies* (A.1)*, and suppose g satisfies Ricci flow. Then v satisfies*

$$\Box v = u^{-1}\left(\mu + E_{error}\right) v$$

where E_{error} is a function of space-time satisfying

$$-C\left(v+\frac{|\nabla\nabla u|^2}{v}\right)\leq E_{error}\leq Cv \tag{A.2}$$

and C is a constant depending on c_v.

A more precise derivation of the evolution of v is in Lemma B.2.2, but this is all we need in this section.

Proof. This is a consequence of the parabolic Bochner formula ((1.6) of [HN15]), valid whenever we have an evolving metric and evolving function:

$$\Box|\nabla w|^2 = 2\langle\nabla w,\nabla\Box w\rangle - 2|\nabla\nabla w|^2 - \mathrm{Rf}(\nabla w,\nabla w).$$

Here $\mathrm{Rf}(\nabla w,\nabla w)=\partial_t g(\nabla w,\nabla w)--2\,\mathrm{Rc}_g(\nabla w,\nabla w)$. The upper bound on the error is stronger because we can throw away the norm of the hessian which comes in the Bochner formula. $\square$

A.1.1 Two dimensional first order PDE

We will use functions dependent on u and t to control v. If F depends on u and t alone then

$$\Box F=\left((\Box u)F^{[1]}+F^{[t]}-vF^{[2]}\right)\left(u^{-1}F\right).$$

Here, we use the notation $F^{[k]}=u^k\frac{1}{F}\partial_u^k F$ and $F^{[t]}=u\frac{1}{F}\partial_t F$. These are both invariant under scaling the system or F.

Given the equation for $\Box u$ we can calculate further,

$$\Box F=\left(F^{[t]}-\mu F^{[1]}+v\left(c_vF^{[1]}-F^{[2]}\right)\right)\left(u^{-1}F\right). \tag{A.3}$$

This formula tells us that when v is small and $F^{[1]}, F^{[2]}$ are controlled, $\Box F$ is approximately the first order linear operator $L[F] := (\partial_{t;u} - \mu\partial_u) F = \left(F^{[t]} - \mu F^{[1]}\right)(u^{-1}F)$.

A relevant function that we will use is $U_0(u,t) := u + \mu t$. The inspiration for the name is that if $u(p,0) =: u_0(p)$ were constant in space then $U_0(p,t) = u_0(p)$. We will also use $Q(u,t) := u^{-1}U_0$. These are related to the linear operator $L[F]$. U_0 gives the characteristic curves of the equation, and Q is a solution to $L[F] = \mu u^{-1}F$ with constant initial data 1. The following lemma collects these facts.

Lemma A.1.2. *The function U_0 satisfies $U_0^{[1]} = Q^{-1}$, $U_0^{[2]} = 0$, $U_0^{[t]} = \mu Q^{-1}$, and in particular*

$$U_0^{[t]} - \mu U_0^{[1]} = 0.$$

The function Q satisfies $Q^{[1]} = -(1 - Q^{-1})$, $Q^{[2]} = 2(1 - Q^{-1})$, $Q^{[t]} = \mu Q^{-1}$, and in particular

$$Q^{[t]} - \mu Q^{[1]} = \mu, \quad Q(u,0) = 1, \quad Q(u,t) \geq 1.$$

Proof. The calculation for U_0 is small. To ease the calculation for Q, let $F(x) = x^{-1}$ which satisfies

$$\mathrm{F}^{[1]} = -1, \quad \mathrm{F}^{[2]} = 2, \quad \mathrm{F}^{[t]} = 0.$$

Then we can use the (standard calculus) formulae in Lemma B.3.1. $\square$

We can use Q and U_0 to solve more linear equations involving $L[F] = (F^{[t]} - \mu F^{[1]})F$. The following calculation is immediate.

Lemma A.1.3. *If $Z_0 : \mathbb{R}^+ \to \mathbb{R}_+$ is a differentiable function and $Z = Q^p(Z_0 \circ U_0)$ then Z satisfies*

$$\partial_t Z - \mu \partial_u Z = p\mu u^{-1} Z,$$

$$Z(x, 0) = Z_0(x).$$

If Z_0 satisfies $|Z_0^{[1]}| + |Z_0^{[2]}| < K$, then Z satisfies $|Z^{[1]}| + |Z^{[2]}| < K'$ for some K' depending on p and K.

Now that we have given the solutions to the first-order equation which approximates $\square$ for a (u, t)-dependent function, we quantify the approximation. The following lemma describes the degree to which

$$Z(p, t) = Q^p \cdot (Z_0 \circ U_0) = \left(\frac{u(p, t) + \mu t}{u(p, t)}\right)^p Z_0(u(p, t) + \mu t)$$

is approximately a solution to the linear parabolic PDE $(\square - p\mu u^{-1})w = 0$ on the evolving manifold.

Lemma A.1.4. *Let Z_0 be a given differentiable function $Z_0 : \mathbb{R}_+ \to \mathbb{R}_+$. Then $Z = Q^p \cdot (Z_0 \circ U_0)$ satisfies*

$$\square Z - p\mu u^{-1} Z = E u^{-1} v Z \tag{A.4}$$

$$Z(p, 0) = Z_0(u(p, 0))$$

where $|E| \leq C(|{Z_0}^{[1]}| + |{Z_0}^{[2]}|)$, and C is a constant depending only on c_v and p.

Proof. Apply equation (A.3), using Lemma A.1.3 to find $Z^{[t]} - \mu Z^{[1]}$. $\square$

A.1.2 Sub- and super-solutions

We are still assuming that u satisfies (A.1) and g satisfies Ricci flow. Suppose the error term in Lemma A.1.1 is small. Then we may expect v itself to be approximately given by a solution to $\square v - \mu u^{-1} v = 0$. By Lemma A.1.4 we find that v should be approximately given by $V := Q \cdot (V_0 \circ U_0)$. This, in turn, will give us control on the error term in Lemma A.1.4, which told us that $Z = Q^p \cdot (Z_0 \circ U_0)$ is approximately a solution to $\square z = p\mu u^{-1} z$.

In this lemma we create sub- and supersolutions to $\square z = p\mu u^{-1} Z$, which beat the error in this approximate solution. The supersolution is defined as $Z^+ = (1 + DV)Z$, and the subsolution as $Z^- = (1 - DV)Z$, for some $D > 0$.

Lemma A.1.5. *(Supersolutions to linear parabolic equations) Let Z_0 and V_0 be given differentiable functions $Z_0, V_0 : \mathbb{R}_+ \to \mathbb{R}_+$. Define $V = Q \cdot (V_0 \circ U_0)$ and $Z = Q^p \cdot (Z_0 \circ U_0)$.*

There is $\underline{D} > 0$ and $c > 0$ depending on

$$\sup|V_0^{[1]}|,\ \sup|V_0^{[2]}|,\ \sup|Z_0^{[1]}|,\ \sup|Z_0^{[2]}|,\ c_v,\ p,\ \mu. \tag{A.5}$$

such that for any $D > \underline{D}$, there is $\epsilon > 0$ depending on (A.5) *and D with the following property.*

Suppose u satisfies (A.1)*, and let Ω be a subset of space-time where $\frac{1}{2}V \leq v(p,t) \leq 2V$ and $V < \epsilon$. Then Z^+ is a supersolution to $(\square - p\mu u^{-1})$ on*

Ω. *Even better,*

$$\Box Z^+ - p\mu u^{-1} Z^+ \geq (cD) u^{-1} v Z^+, \quad \text{on } \Omega, \tag{A.6}$$

and,

$$\Box Z^- - p\mu u^{-1} Z^- \leq -(cD) u^{-1} v Z^-, \quad \text{on } \Omega.$$

Proof. Write $Z^+ = Z + Z_2$ with $Z_2 = DVZ = DQ^{p+1}((V_0 \cdot Z_0) \circ U_0)$. Then we can use Lemma A.1.4 and in particular (A.4) to calculate the heat operator applied to Z_2:

$$\Box Z_2 - (p+1)\mu u^{-1} Z_2 = E_2 u^{-1} v Z_2$$

where E_2 is some error which is absolutely bounded depending on $V_0^{[1]}, V_0^{[2]}, Z_0^{[1]}$, and $Z_0^{[2]}$. In terms of the linear equation we are interested in, this means

$$\begin{aligned}\Box Z_2 - p\mu u^{-1} Z_2 &= u^{-1}\left(1 + E_2 v\right) Z_2 \\ &= D u^{-1}\left(1 + E_2 v\right) V Z.\end{aligned}$$

By choosing ϵ small enough we can force $1 + E_2 v \geq \frac{1}{2}$. Now using again equation (A.4) from Lemma A.1.4, but now applied to Z, we find

$$\begin{aligned}(\Box - p\mu u^{-1})(Z^+) &= (\Box - p\mu u^{-1})(Z + Z_2) \\ &= u^{-1}\left(Ev + D\left(1 + E_2 v\right) V\right) Z \\ &\geq u^{-1}\left(Ev + D(1-\delta) V\right) Z \\ &= D u^{-1}\left(\frac{E}{D} + \frac{1}{2}\frac{V}{v}\right) v \frac{Z^+}{1 + DV} \\ &\geq D u^{-1}\left(\frac{E}{D} + \frac{1}{4}\right) v \frac{Z^+}{1 + DV}\end{aligned}$$

Here, E (another error term with unknown sign, bounded in terms of (A.5)) is independent of D. (This uses Lemma A.1.6 below.) In the last line we used the assumption that $v \leq \frac{1}{2}V$. We can choose $\underline{D}$ large to force $\frac{E}{D}$ to be at least $-\frac{1}{8}$, and then choose ϵ small enough so that $\frac{1}{1+DV}$ is at least $\frac{1}{2}$. Then we take $c = \frac{1}{16}$. $\square$

Lemma A.1.6. *There is a constant C depending on the items in line* (A.5)*, and in particular independent of D, such that*

$$|(Z^+)^{[1]}|+|(Z^+)^{[2]}|\leq C,$$

and similarly for the subsolution Z^-.

If in addition we assume that $|\nabla\nabla u|\leq C_{hess}v$, then $|\nabla\nabla Z^+|\leq C$ for a constant depending on line (A.5) *and C_{hess}.*

Proof. First, derive bound for $V = QV_0 \circ U_0$ and $Z = Q^p Z_0 \circ U_0$.

$$\begin{aligned} V^{[1]} &= Q^{[1]} + \left((V_0)^{[1]} \circ U_0\right) (U_0)^{[1]} \\ &= -(1 - Q^{-1}) + \left((V_0)^{[1]} \circ U_0\right) Q^{-1}, \end{aligned}$$

so $|V^{[1]}|\leq 1 + \sup|V_0^{[1]}|$. Similarly, we can bound $Z^{[1]}$ by $p + \sup|Z_0^{[1]}|$.

Now calculate,

$$\begin{aligned} (1 + DV)^{[1]} &= \frac{u\partial_u(1 + DV)}{1 + DV} \\ &= \frac{DV}{1 + DV}\frac{u\partial_u V}{V} \leq V^{[1]}. \end{aligned}$$

Once we have this, the full bound on $(Z^+)^{[1]}$ follows from

$$(Z^+)^{[1]} = ((1+DV)Z)^{[1]} = (1+DV)^{[1]} + Z^{[1]}.$$

The bound on $(Z^+)^{[2]}$ is similar.

To get the second claim, use (B.16):

$$\frac{u}{v}\frac{|\nabla\nabla Z^+|}{Z^+} \leq \left(Z^{+[2]} + Z^{+[1]}\frac{|\nabla\nabla u|}{v}\right) \leq \left(Z^{+[2]} + CZ^{+[1]}\right).$$

□

Corollary A.1.7. *With the setup of Lemma A.1.5, by decreasing c we actually have*

$$\Box Z^+ - p\mu u^{-1}Z^+ \geq (cD)u^{-1}vZ^+\left(1 + u\frac{|\partial_u Z^+|}{Z^+} + \frac{u}{v}\frac{|\nabla Z^+|^2}{(Z^+)^2}\right).$$

Proof. Note that

$$u\frac{\partial_u Z^+}{Z^+} = Z^{+[1]}$$

and unraveling definitions,

$$\frac{u}{v}\frac{|\nabla Z^+|^2}{(Z^+)^2} = \frac{u}{u^{-1}|\nabla u|^2}\frac{(\partial_u Z^+)^2|\nabla u|^2}{(Z^+)^2} = \left(Z^{+[1]}\right)^2.$$

Therefore the inequality follows from the bounds in Lemma A.1.6. □

A.1.3 Bounding v

In this section we estimate v, using barrier arguments. An upper bound on v is easier, because the upper inequality in Lemma A.1.1 is independent of second derivatives of u. Lemma A.1.8 shows how one may get such an upper bound. However, in order to glue our approximation in the region where v is small to something where v is not small, we need a lower bound on v in terms of u as well. This follows from an assumption on the Hessian of u, which will be understood later.

Lemma A.1.8. *If $D > \underline{D}(V_0, c_v)$ and $\epsilon < \overline{\epsilon}(V_0, c_v, D)$ the following holds. Let Ω be a smooth subset of space-time where $V < \epsilon$.*

Set $V^+ = (1 + DV)V$. If $v < V^+$ on the parabolic boundary of Ω, then $v < V^+$ in Ω

Proof. By our assumption and the upper bound in Lemma A.1.1, v satisfies

$$\Box v - \mu u^{-1} v \leq C u^{-1} v^2$$

for some constant C. We consider this as $A(v) \leq 0$ where A is the linear operator

$$A(f) = \left(\Box - \mu u^{-1} - C u^{-1} v\right) f.$$

We can apply Lemma A.1.5, with $p = 1$ and $Z_0 = V_0$. By (A.6) we have

$$\Box V^+ - \mu u^{-1} V^+ \geq D u^{-1} v V^+.$$

so we find $A(V^+) \geq 0$ provided $D \geq C$.

The claim follows from the maximum principle. $\Box$

In order to bound v from below, we also need an upper bound on the hessian of u.

Lemma A.1.9. *Let $C_{hessbnd} > 0$ be given, $D > \underline{D}(V_0, c_v, C_{hessbnd})$ and $\epsilon < \overline{\epsilon}(V_0, c_v, D, C_{hessbnd})$ the following holds. Suppose $|\nabla\nabla u|^2 \leq (C + C_{hessbnd})(v^2 + uv^{-1}|\nabla v|^2)$.*

Set $V^- = (1 - DV)V$. If $V^- < v < V^+$ on the parabolic boundary of Ω, then $V^- < v < V^+$ in Ω.

Proof. Applying Lemma A.1.1, we learn that

$$\Box v - \mu u^{-1} v \geq -(C + C_{hessbnd}) u^{-1} \left(v^2 + uv^{-1}|\nabla v|^2\right)$$

In other words, $A(v) \geq 0$ where

$$A(f) = \Box f - \mu u^{-1} f + (C + C_{hessbnd}) u^{-1} v f \left(1 + \frac{u}{v} \frac{|\nabla f|^2}{f^2}\right)$$

By the first part of Lemma A.1.7 we can choose D large enough so that V^- satisfies $A(V^-) \leq 0$. The maximum principle is strong enough to deal with the extra term $|\nabla v|^2$, because at a first point where $v = V^-$ we know $\nabla v = \nabla V^-$. $\square$

Appendix B

Equations and derivations

B.1 Equations for warped products and Ricci flow

In this section we review some of the properties of Ricci flow on warped products. The metrics are on the topology $M = B^m \times N^q$, for some manifold B which we call the base. The metrics have the form

$$g = g_B + \phi^2(b) g_N,$$

where g_B is a metric on B, g_N is a metric on N, and $\phi : B \to \mathbb{R}_+$. We assume that g_N is an Einstein manifold: $2\,\mathrm{Rc}[g_N] = \mu_N g_N$.

In this thesis we are mostly concerned with doubly warped products over intervals, i.e. metrics of the form

$$a(x)dx^2 + \phi^2(x) g_{S^q} + \phi^2(x) g_F, \quad x \in I.$$

These are singly warped products in two ways: with base $I \times S^q$ and fiber F or with base $I \times F$ and fiber S^q. Both points of view have been useful for our intuition. A big simplification for a doubly warped product over an interval is that the hessian of a function of x is much simpler than that of a function of a general base.

We use the convention that X and Y are lifts of vector fields on B to the product $B \times F$, while U and V are lifts of vector fields on F. Furthermore, we will forevermore not say "X is a lift of a vector field on B to the product" and rather just say "X is a vector field on B" with the understanding that it is lifted to the product whenever we use it as such.

Everything here can be found or derived from Section 7 of [O'N83].

B.1.1 The connection on vector fields

We first describe how the Levi-Civita connection of g acts on vector fields. If X and Y are vector fields on B, then

$$\nabla_X Y = \nabla^B_X Y$$

where ∇^B is the Levi-Civita connection of g_B.

Now, *if* we were dealing with a product (e.g. $\phi = \text{const}$), then for a vector field U on F, we would have that U is parallel with respect to vector fields from B, so that $\nabla_X U = 0$. For a warped product, we have

$$\nabla_X U = \phi^{-1} d\phi(X) U.$$

However, there is a way to create a parallel (for B) vector field from U: if we normalize U with respect to g. That is, let

$$\hat{U} = \phi^{-1} U.$$

Then we can immediately check the properties

$$g(\hat{U}, \hat{U}) = g_F(U, U), \quad \nabla_X \hat{U} = 0$$

which say that $\hat{U}$ has constant norm and is parallel as we move on B.

As a consequence, we find that submanifolds of the form $B \times \{p\}$ for $p \in F$ are totally geodesic submanifolds of (M, g).

Now consider taking covariant derivatives with respect to vector fields on F. We have

$$\nabla_U X = \phi^{-1} d\phi(X) U$$
$$\nabla_U V = \nabla^F_U V - g(U, V)\phi^{-1} \operatorname{grad} \phi$$

The way we remember the sign above is by drawing $\mathbb{R}^2$ with polar coordinates, in which the metric can be written

$$dr^2 + r^2 g_{S^q}$$

B.1.2 Curvatures

The curvature of a warped product can be described as follows. If U and V are perpendicular unit vectors on the fiber, then

$$R(U, V, U, V) = \frac{R_N(U, V, U, V) - |\nabla\phi|^2}{\phi^2}.$$

In particular, if (g_N, N) is the metric of constant sectional curvature Sec, then

$$R(U, V, U, V) = \frac{Sec - |\nabla\phi|^2}{\phi^2}.$$

For vectors U on the fiber and X, Y on the base, we have

$$R(U, X, U, Y) = -\frac{\nabla_X \nabla_Y \phi}{\phi}, \tag{B.1}$$

and if both W, X, Y, and Z are all vectors on the base, then

$$R(X,Y,Z,W) = R_B(X,Y,Z,W).$$

From these formulae, we can calculate the Ricci curvature directly from definition.Using $2\,\mathrm{Rc}[g_N] = \mu_N g_F$,

$$\begin{aligned}
\mathrm{Rc}(U,V) &= -\phi\Delta_B\phi + \tfrac{1}{2}\mu\left(1 - \frac{2(q-1)}{\mu}|\nabla\phi|^2\right), \\
\mathrm{Rc}(U,X) &= 0, \\
\mathrm{Rc}(X,Y) &= \mathrm{Rc}_B(X,Y) - q\phi^{-1}\nabla_X\nabla_Y\phi.
\end{aligned}$$

B.1.3 Ricci flow for warped products

If g evolves by Ricci flow, then

$$\begin{aligned}
\partial_t g_B &= -2\,\mathrm{Rc}[g_B] + 2q\phi^{-1}\nabla\nabla\phi \\
\partial_t\phi &= \Delta_B\phi - \tfrac{1}{2}\mu\phi^{-1}\left(1 - \frac{2(q-1)}{\mu}|\nabla\phi|^2\right) \\
&= \Delta_M\phi - \phi^{-1}|\nabla\phi|^2 - \tfrac{1}{2}\mu\phi^{-1}
\end{aligned}$$

In a different notation,

$$\begin{aligned}
\mathrm{Rf}[g_B] &= 2q\phi^{-1}\nabla\nabla\phi \\
\Box_B\phi &= -\tfrac{1}{2}\mu\phi^{-1}\left(1 - \frac{2(q-1)}{\mu}|\nabla\phi|^2\right) \\
\Box_M\phi &= -\phi^{-1}|\nabla\phi|^2 - \tfrac{1}{2}\mu\phi^{-1}
\end{aligned}$$

and $u = \phi^2$ satisfies,

$$\begin{aligned} \Box_M u &= 2\phi\Box_M\phi - 2|\nabla\phi|^2 \\ &= -\mu - 4|\nabla\phi|^2 \qquad \text{(B.2)} \\ &= \left(-u^{-1}\mu\right) u - 2\langle\nabla u, \nabla\log\phi\rangle \end{aligned}$$

or

$$\begin{aligned} \Box_B u &= 2\phi\Box_B\phi - 2|\nabla\phi|^2 \\ &= -\mu + \tfrac{1}{4}(2(q-1)-2)v \qquad \text{(B.3)} \end{aligned}$$

where $v = u^{-1}|\nabla u|^2$.

Recall in our case, for $g_N = g_{S^q}$ we have $\mu = 2(q-1)$.

B.1.4 Doubly warped products over an interval

Now consider a metric of the form

$$g = a(x)dx^2 + \phi^2 g_{F_1} + \psi^2 g_{F_2}, \quad x \in I.$$

We define an arclength coordinate s (up to a constant) by $ds^2 = adx^2$. We can view g as a warped product with fiber g_{F_1} over base $I \times F_2$, as well as a warped product with fiber g_{F_2} over the base $I \times F_2$. Consider for simplicity the case when g_{F_1} has constant sectional curvature Sec_1 and g_{F_2} has constant sectional curvature Sec_2. Then there are five special sectional curvatures:

$$\begin{aligned} L_1 &= \frac{Sec_1 - |\nabla\phi|^2}{\phi^2} = \frac{Sec_1 - \phi_s^2}{\phi^2}, \quad L_2 = \frac{Sec_2 - \psi_s^2}{\psi^2}, \\ K_1 &= -\frac{\phi_{ss}}{\phi}, \quad K_2 = -\frac{\psi_{ss}}{\psi}, \quad K_{mix} = -\frac{\phi_s\psi_s}{\phi\psi}. \end{aligned}$$

The curvatures L_1 and L_2 are those that we get from planes spanned by two perpendicular vectors tangent to the same fiber. K_1 and K_2 come from planes spanned by ∂_s and a vector on one of the fibers. K_{mix} comes from a plane spanned by a vector on F_1 and a vector on F_2; this comes from the extra terms (compared to a product) in computing the hessian in (B.1).

B.1.4.1 Curvatures in terms of u, v and w

We put the curvatures of a doubly warped product in terms of v and w, and their u derivatives. Recall the definitions

$$u = \phi^2, \quad w = \psi^2, \quad v = u^{-1}|\nabla u|^2 = 4|\nabla\phi|^2$$

First, we have

$$L_1 = u^{-1}Sec_1 - \tfrac{1}{4}u^{-1}v.$$

Now calculate,

$$\partial_s u \partial_u v = (\partial_s u)4(\partial_s\phi)(\partial_u\partial_s\phi) = 4(\partial_s\phi)(\partial_s^2\phi)$$

so

$$2\phi\partial_u v = 4(\partial_s^2\phi)$$

and

$$K_1 = -\tfrac{1}{2}\partial_u v = -\frac{\partial_s^2\phi}{\phi}.$$

Now we calculate the curvatures involving ψ.

$$\psi_s = \tfrac{1}{2}w^{-1/2}w_s = \tfrac{1}{2}w^{-1/2}w_u u_s = \tfrac{1}{2}w^{-1/2}u^{1/2}v^{1/2}w_u$$

so

$$\begin{aligned} L_2 &= \frac{Sec_2}{w} - \tfrac{1}{4}u^{-1}v\left(u^2w^{-2}w_u^2\right) \\ K_{mix} &= (\tfrac{1}{2}u^{-1/2}v^{1/2})(\tfrac{1}{2}w^{-1}u^{1/2}v^{1/2}w_u) = \tfrac{1}{4}u^{-1}v\left(uw^{-1}w_u\right). \end{aligned}$$

Finally, we calculate

$$\begin{aligned} \psi_{ss} &= \tfrac{1}{4}\left(w^{-3/2}u^{1/2}v^{1/2}w_uw_s + w^{-1/2}u^{-1/2}v^{1/2}w_uu_s + w^{1/2}u^{1/2}v^{-1/2}w_uv_s + w^{-1/2}u^{1/2}v^{-1/2}w_{us}\right) \\ &= \tfrac{1}{4}w^{-1/2}uv(w^{-1}w_u^2 + u^{-1}w_u + v^{-1}w_uv_u + w_{uu}). \end{aligned}$$

Therefore,

$$K_2 = -\tfrac{1}{4}u^{-1}v(u^2w^{-2}w_u^2 + uw^{-1}w_u + u^2v^{-1}w^{-1}w_uv_u + u^2w^{-1}w_{uu})$$

B.2 Deriving equations

B.2.1 Deriving equations for v.

In this Lemma, $\mathrm{Rf}[g_B] = \partial_t g_B - (-2\,\mathrm{Rc}_{g_B})$.

Lemma B.2.1. *Suppose (B, g_B) is an evolving Riemannian manifold and $\phi :$ $B \times [T_1, T_2] \to \mathbb{R}_+$ is an evolving function on B. Suppose g_B and ϕ satisfy*

$$\begin{aligned} \mathrm{Rf}[g_B] &= T + 2c_1\phi^{-1}\nabla\nabla\phi \\ \Box_B\phi &= \tfrac{1}{2}\phi^{-1}\cdot(-\mu + c_z z) \end{aligned}$$

where $z = |\nabla\phi|^2$.

Let $\kappa(p,t)$ *be the norm of the second fundamental form of the level set of* u *passing through* p *at time* t.

Then z *satisfies*

$$\begin{aligned}\Box z &= \phi^{-2}(\mu - c_z z)z \\ &\quad + (c_z - c_1)\langle\nabla z, \nabla\log\phi\rangle - z^{-1}|\nabla z|^2 + \tfrac{1}{2}\phi^2 z^{-2}\left(\langle\nabla z, \nabla\log\phi\rangle\right)^2 \\ &\quad - 2z\kappa^2 - \phi^2 T(\nabla\log\phi, \nabla\log\phi)\end{aligned}$$

Proof. We can apply the parabolic Bochner formula ((1.6) of [HN15]) to these equations to find

$$\begin{aligned}\Box|\nabla\phi|^2 &= 2\langle\nabla\Box\phi, \nabla\phi\rangle - 2|\nabla\nabla\phi|_B^2 - \mathrm{Rf}(\nabla\phi, \nabla\phi) \\ &= 2\langle\nabla\Box\phi, \nabla\phi\rangle - 2|\nabla\nabla\phi|_B^2 - 2c_1\phi^{-1}\nabla_{\nabla\phi}\nabla_{\nabla\phi}\phi - T(\nabla\phi, \nabla\phi)\end{aligned}$$

We calculate the first term:

$$\begin{aligned}2\langle\nabla\Box\phi, \nabla\phi\rangle &= \phi^{-2}(\mu - c_z z)|\nabla\phi|^2 + c_z\phi^{-1}\langle\nabla z, \nabla\phi\rangle \\ &= \phi^{-2}(\mu - c_z z)z + c_z\langle\nabla z, \nabla\log\phi\rangle\end{aligned}$$

For the second term, we can change the hessian to

$$\begin{aligned}-2|\nabla\nabla\phi|^2 &= -2z\kappa^2 - z^{-1}|\nabla z|^2 + \tfrac{1}{2}z^{-2}\langle\nabla z, \nabla\phi\rangle^2 \\ &= -2z\kappa^2 - z^{-1}|\nabla z|^2 + \tfrac{1}{2}z^{-2}\phi^2\langle\nabla z, \nabla\log\phi\rangle^2\end{aligned}$$

And for the third term, we can change the hessian using

$$-2c_1\phi^{-1}\nabla_{\nabla\phi}\nabla_{\nabla\phi}\phi = -c_1\phi^{-1}\langle\nabla z, \nabla\phi\rangle = -c_1\langle\nabla z, \nabla\log\phi\rangle$$

Putting everything together, we find the desired equation. $\square$

Lemma B.2.2. *Suppose* (B, g_B) *is an evolving Riemannian manifold and* u *is a function on* B*. Suppose* g_B *and* u *satisfy*

$$\mathrm{Rf}[g_B] = T + 2c_1 u^{-1/2}\nabla\nabla u^{1/2}$$
$$\Box_B u = -\mu + c_v v \tag{B.4}$$

where $v = u^{-1}|\nabla u|^2$*.*

Let $\kappa^2(p,t)$ *be the norm of the second fundamental form of the level set of* u *passing through* p *at time* t*. Define the constants* $c_z = (4c_v + 2)$*,* $c_v' = \frac{1}{4}c_z$*, and* $c_3 = \frac{1}{2}(c_z - c_1)$*.*

Then v *satisfies*

$$\begin{aligned}\Box v = {} & u^{-1}(\mu - c_v' v)v - 2v\kappa^2 - T(\nabla u, \nabla u)\\ & + c_3\langle \nabla v, \nabla \log u\rangle - v^{-1}|\nabla v|^2 + \tfrac{1}{2}uv^{-2}\left(\langle \nabla v, \nabla \log u\rangle\right)^2\end{aligned}$$

Proof. Let $v = u^{-1}|\nabla u|^2 = 4|\nabla\phi|^2$. Define $z = |\nabla\phi|^2$. Calculating the evolution for ϕ we find,

$$\begin{aligned}\Box_B \phi &= -\tfrac{1}{2}\mu\phi^{-1} + (\tfrac{1}{2})(c_z)\phi^{-1}z\\ &= -\tfrac{1}{2}\phi^{-1}\left(\mu - (c_z)z\right)\end{aligned}$$

where $c_z = 4c_v - 2$. To check this, write

$$\begin{aligned}
\Box_B u^{1/2} &= \tfrac{1}{2}u^{-1/2}\Box_B u + \tfrac{1}{4}u^{-3/2}|\nabla u|^2 \\
&= -\tfrac{1}{2}\phi^{-1}\mu + \tfrac{1}{2}c_v\phi^{-1}v + \tfrac{1}{4}\phi^{-1}v \\
&= -\tfrac{1}{2}\phi^{-1}\mu + 2c_v\phi^{-1}z + \phi^{-1}z \\
&= -\tfrac{1}{2}\phi^{-1}\mu + (2c_v+1)\phi^{-1}z \\
&= \tfrac{1}{2}\phi^{-1}\left(-\mu + (4c_v+2)\phi^{-1}z\right)
\end{aligned}$$

Then, apply Lemma B.2.1, and use $v = 4z$ and $\log u = 2\log\phi$. $\Box$

B.2.1.1 Equidistant Level Sets

Now, suppose that the level sets of u are equidistant. Then v is dependent on u and t alone so we find $\nabla v = |\nabla u|^{-1}\langle\nabla v, \nabla u\rangle$. Then we find, from Lemma B.2.2,

$$\begin{aligned}
\Box_B v = u^{-1}(\mu - c_v' v)v - 2\kappa^2 v - u^{-1}\bar{T}v \\
+ c_3 v(\partial_u v) - \tfrac{1}{2}u(\partial_u v)^2
\end{aligned} \tag{B.5}$$

On the other hand, since the level sets of u are equidistant, we can use that v is a function of u and t to calculate $\Box_B v$ in terms of derivatives with respect to u, using (B.4). (This is applying the general formula (B.17).)

$$\Box_B v = (-\mu + c_v v)\partial_u v + \partial_{t;u} v - uv\partial_u^2 v. \tag{B.6}$$

From (B.5) and (B.6) it follows that,

$$\begin{aligned}
\partial_{t;u} v = uv\partial_u^2 v - \tfrac{1}{2}u(\partial_u v)^2 + u^{-1}(\mu - c_v' v)v + (c_4 v + \mu)\partial_u v \\
- 2\kappa^2 v - u^{-1}\bar{T}v
\end{aligned} \tag{B.7}$$

where $c_4 = c_3 - c_v$.

B.2.1.2 The case of warped product Ricci flow

In the case of Ricci flow of a metric $g = g_B + u g_{S^q}$, where the Ricci curvature of g_{S^q} is $\mu g_{S^q} = 2(q-1)g_{S^q}$, we have

$$\begin{aligned} \mathrm{Rf}[g_B] &= 2qu^{-1/2}\nabla\nabla u^{1/2} \\ \Box_B u &= -\mu + \tfrac{1}{4}(\mu - 2)v \end{aligned} \tag{B.8}$$

Therefore in Lemma B.2.2 we have $c_v = \frac{1}{4}(\mu - 2)$ and $c_1 = q = \frac{1}{2}\mu + 1$. Then we find

$$\begin{aligned} c_z &= 4c_v + 2 = \mu \\ c_v' &= \tfrac{1}{4}c_z = \tfrac{1}{4}\mu \\ c_3 &= \tfrac{1}{2}(c_z - c_1) = \tfrac{1}{2}(\mu - (\tfrac{1}{2}\mu + 1)) = \tfrac{1}{4}\mu - \tfrac{1}{2} \end{aligned}$$

Finally, $c_4 = c_3 - c_v = \frac{1}{4}\mu - \frac{1}{2} - (\frac{1}{4}\mu - \frac{1}{2}) = 0$.

So, from (B.7),

$$\begin{aligned} \partial_{t;u} v = {} & uv\partial_u^2 v - \tfrac{1}{2}u(\partial_u v)^2 \\ & + \mu\left(1 - \tfrac{1}{4}v\right)u^{-1}v + \mu\partial_u v \\ & - 2(\kappa^2)v \end{aligned} \tag{B.9}$$

One convenient way to write this is as,

$$\partial_{t;u} v = u^{-1}\mathcal{Q}[v,v] + u^{-1}\mathcal{L}[v] - 2(\kappa^2)v \tag{B.10}$$

where $\mathcal{L}$ and $\mathcal{Q}$ are the operators

$$\mathcal{L}[w] = L(w, \partial_u w)$$
$$L(A, B) = \mu A + \mu B$$

and

$$\mathcal{Q}[w, w] = Q(w, u\partial_u w, u^2 \partial_u w_{uu})$$
$$Q(A, B, C) = AC - \tfrac{1}{2}B^2 - \tfrac{1}{4}\mu C^2.$$

For w_1 and w_2 different functions, we define $\mathcal{Q}[w_1, w_2]$ to be the extension of $\mathcal{Q}$ to a symmetric bilinear operator.

The properties relevant to the analysis are

1. L is linear in its arguments, and Q is quadratic in its arguments.

2. The coefficient on $u^2 V_{uu}$ is V.

3. The strictly first-order terms are $u^{-1}\mu(1 - \frac{1}{4}V)V$.

B.2.1.3 Writing the evolution in terms of L and ϕ

It is also convenient to consider the evolution of $L = \frac{1-\frac{1}{4}v}{u}$. L is a sectional curvature, so it is a geometrically natural quantity to consider. If the metric is smooth near $u = 0$ then L will be bounded there, which gives us more information that v being bounded.

Coming from (B.9), replace $v = 4(1 - uL)$ and divide through by $-4u$ to find

$$\partial_{t;u} L = 4u\,(1 - uL)\,\partial_u^2 L + 2u^2(\partial_u L)^2 \\ + (\mu + 8 - 4uL)\partial_u L + (\mu + 2)L^2 + \tfrac{1}{2}u^{-1}\kappa^2 v.$$

An important point here is that the terms $u^{-1}L$ cancel. This is somewhat expected, since for example the sphere has constant non-zero curvature L despite u going to zero. Let us also put this in terms of derivatives with respect to $\phi = \sqrt{u}$. Note that

$$\partial_u = \tfrac{1}{2}\phi^{-1}\partial_\phi \tag{B.11}$$

and

$$u\partial_u^2 = \tfrac{1}{4}\left(\partial_\phi^2 - \phi^{-1}\partial_\phi\right). \tag{B.12}$$

Since ϕ is a function of u, $\partial_{t;u} = \partial_{t;\phi}$. So, we have

$$\begin{aligned}\partial_{t;\phi} L &= \left(1 - \phi^2 L\right)(\partial_\phi^2 L - \phi^{-1}\partial_\phi L) + \tfrac{1}{2}\phi^2(\partial_\phi L)^2 \\ &\quad + \phi^{-1}(\tfrac{1}{2}\mu + 4 - 2\phi^2 L)(\partial_\phi L) + (\mu + 2)L^2 + \tfrac{1}{2}\kappa^2 v \\ &= \left(1 - \phi^2 L\right)\partial_\phi^2 L + \tfrac{1}{2}\phi^2(\partial_\phi L)^2 \\ &\quad + \phi^{-1}(\tfrac{1}{2}\mu + 5 - \phi^2 L)\partial_\phi L + (\mu + 2)L^2 + \tfrac{1}{2}\kappa^2 v.\end{aligned} \tag{B.13}$$

The advantage of this is the clear regularity around $\phi = 0$ provided L is bounded.

B.2.2 Additional warped product factors

We continue considering the Ricci flow of a metric of the form $g = g_B + ug_{S^q}$:

$$\mathrm{Rf}[g_B] = 2c_1 u^{-1/2}\nabla\nabla u^{1/2},$$

$$\square_B u = -\mu + c_v v.\text{(B.8)}$$

Here $c_v = \frac{1}{4}(\mu - 2)$. Suppose that g_B itself has a warped product factor: $B = B_2 \times F^p$ and $g_B = g_{B_2} + wg_F$. Take $y = w^{-1}|\nabla w|^2$ and suppose that $2\,\mathrm{Rc}[g_F] = \mu_F g_F$. We make no assumptions on the sign on μ_F.

To quickly derive an equation for h in terms of $\square_B$, go from (B.2) which says

$$\square_{B_2\times F\times S^q} w = -\mu_F - y$$

where $y = w^{-1}|\nabla w|^2$. Since

$$\begin{aligned}
\square_{B_2\times F\times S^q} w &= \partial_t w - (\Delta_{B_2\times F\times S^q} w) \\
&= \partial_t w - \left(\Delta_{B_2\times F} w + \tfrac{1}{2}qu^{-1}\langle\nabla u, \nabla w\rangle\right) \\
&= \partial_t w - \left(\Delta_B w + \tfrac{1}{2}qu^{-1}\langle\nabla u, \nabla w\rangle\right) \\
&= \square_B w - \tfrac{1}{2}qu^{-1}\langle\nabla u, \nabla w\rangle
\end{aligned}$$

So we find,

$$\begin{aligned}
\square_B w &= -\mu_F - y - \tfrac{1}{2}qu^{-1}\langle\nabla u, \nabla w\rangle \\
&= -\mu_F - y - \tfrac{1}{2}qv\partial_u w. \qquad \text{(B.14)}
\end{aligned}$$

Now, using and the fact that w is a function of u and t,

$$\Box_B w = (-\mu + c_v v)\partial_u w + \partial_{t;u} w - uv\partial_u^2 w$$

so by (B.14) we find

$$\begin{aligned}\partial_{t;u} w - uv\partial_u^2 w &= -\mu_F - y + \mu\partial_u w - c_v v\partial_u w - \tfrac{1}{2}qv\partial_u w \\ &= -\mu_F - y + \mu\partial_u w - \mu/2v\partial_u w. \qquad \text{(B.15)}\end{aligned}$$

Note we may also write $y = w^{-1}|\nabla w|^2 = w^{-1}uv(\partial_u w)^2$.

B.2.2.1 Writing the evolution in terms of ϕ

We also write (B.15) in terms of ϕ. Using (B.11) and (B.12) we have

$$\begin{aligned}\partial_{t;u} w = v &\left(\partial_\phi^2 w - \phi^{-1}\partial_\phi w\right) \\ &- \mu_F - y + (\mu - \mu/2v)\tfrac{1}{2}\phi^{-1}\partial_\phi w\end{aligned}$$

Simplifying,

$$\partial_{t;\phi} w = v\partial_\phi^2 w - \mu_F - y + (\tfrac{1}{2}\mu - (\tfrac{1}{4}\mu - 1)v)\phi^{-1}\partial_\phi w$$

B.2.3 Second fundamental form for doubly warped products

Consider the case of a doubly warped product over an interval,

$$ds^2 + ug_{S^q} + wg_F$$

the second fundamental form κ in Section B.2.1 is the second fundamental form of a surface $(s, p) \times F$, which is

$$\tfrac{1}{4}dim(F)w^{-1}y = \tfrac{1}{4}dim(F)w^{-2}u^2v(\partial_u w)^2.$$

Therefore the term $-2(\kappa^2)v$ is $-\frac{1}{2}dim(F)w^{-2}u^2v^2(\partial_u w)^2 = -\frac{1}{8}dim(F)w^{-2}v^2\phi^2(\partial_\phi w)^2$.

Using this we can change (B.13) to

$$\begin{aligned}
\partial_t L &= \left(1-\phi^2 L\right)\partial_\phi^2 L + \tfrac{1}{2}\phi^2(\partial_\phi L)^2 \\
&\quad + \phi^{-1}(\tfrac{1}{2}\mu + 5 - \phi^2 L)\partial_\phi L + (\mu+2)L^2 - \tfrac{1}{8}w^{-2}v^2\phi^2(\partial_\phi w)^2 \\
&= \left(1-\phi^2 L\right)\partial_\phi^2 L + \tfrac{1}{2}\phi^2(\partial_\phi L)^2 \\
&\quad + \phi^{-1}(\tfrac{1}{2}\mu + 5 - \phi^2 L)\partial_\phi L + (\mu+2)L^2 + w^{-2}(1-uL)^2(\partial_\phi w)^2
\end{aligned}$$

B.3 Calculus

B.3.1 Calculus with functions of functions on manifolds

Suppose $(M, g(t))$ is a manifold with an evolving metric, and $u : M \times [0,T] \to \mathbb{R}$ is a function on the manifold. For a function $F : M \times [0,T]$ we use $\partial_u F = |\nabla u|^{-2}\nabla_{\mathrm{grad}\, u} F$. Note that $\partial_u F$ is defined where $|\nabla u| \neq 0$ (and maybe elsewhere). We also define,

$$\partial_{t;u} = \partial_t - (\partial_t u)\partial_u.$$

The derivative ∂_u is the derivative along a curve which moves perpendicular to the level sets of u at unit speed. The derivative $\partial_{t;u}$ is the derivative along a curve in $M \times [0,T]$ which moves so that the time-derivative of the t component is 1, and also moves in M to always stay on the level set of u (always choosing to move perpendicularly to the level set).

Say that a function $F : M \times [0,T] \to \mathbb{R}$ is (u,t)-dependent if $F(p,t) = f(u(p,t),t)$ for some $f : \mathbb{R} \times [0,T] \to \mathbb{R}$. If F is (u,t)-dependent then

$$\partial_u F(p,t) = |\nabla u|^{-2} \nabla_{\operatorname{grad} u} F = |\nabla u|^{-2} f_1(u(p,t),t) \nabla_{\operatorname{grad} u} \operatorname{grad} u = f_1(u(p,t),t),$$

where we use subscript 1 to denote derivative with respect to the first component. Also,

$$\partial_{t;u} F(p,t) = f_2(u(p,t),t).$$

In particular, $\partial_u u = 1$ and $\partial_{t;u} u = 0$.

Now suppose $u : M \times [0,T] \to \mathbb{R}_+$ is positive and understood. Then we define the following. For $F : M \times [0,T] \to \mathbb{R}_+$ set

$$F^{[1]} = \frac{u \partial_u F}{F}, \quad F^{[2]} = \frac{u^2 \partial_u^2 F}{F}, \quad F^{[t]} = \frac{u \partial_t F}{F}.$$

In the case that F is (u,t)-dependent then $\partial_u F$ gives information about the full derivative of F:

$$\nabla F(p,t) = (\partial_u F) \nabla u = (F^{[1]}) \left(u^{-1} F\right) \nabla u.$$

Also,

$$\begin{aligned} \nabla\nabla F(p,t) &= (\partial_u^2 F) \nabla u \otimes \nabla u + \partial_u F \nabla\nabla u \\ &= u^{-1} F \cdot \left((F^{[2]}) u^{-1} \nabla u \otimes \nabla u + F^{[1]} \nabla\nabla u \right) \end{aligned}$$

so

$$|\nabla\nabla F| \leq (u^{-1} v F) \left(F^{[2]} + F^{[1]} \frac{|\nabla\nabla u|}{v} \right) \tag{B.16}$$

We can also calculate $\Box F$ as

$$\Box F = \partial_{t;u} F + (\Box u)\, \partial_u F - |\nabla u|^2 \partial_u^2 F.$$

If $v = u^{-1}|\nabla v|^2$ we have

$$\Box F = \left((\Box u)\, F^{[1]} + F^{[t]} - vF^{[2]}\right) u^{-1} F. \tag{B.17}$$

The following lemma states standard calculus formulae.

Lemma B.3.1. *Suppose F and G are (u,t)-dependent functions. Then*

$$\begin{aligned}
(FG)^{[1]} &= F^{[1]} + G^{[1]} \\
(FG)^{[2]} &= F^{[2]} + G^{[2]} + F^{[1]} G^{[1]} \\
FG^{[t]} &= F^{[t]} + G^{[t]}
\end{aligned}$$

Suppose G is a (u,t)-dependent function and $F : \mathbb{R} \to \mathbb{R}$. Then the (u,t)-dependent function $H(p,t) = F(G(p,t),t)$ satisfies

$$\begin{aligned}
H^{[1]} &= \left(F^{[1]} \circ G\right) G^{[1]} \\
H^{[2]} &= \left(F^{[2]} \circ G\right) \left(G^{[1]}\right)^2 + \left(F^{[1]} \circ G\right) G^{[2]} \\
H^{[t]} &= \left(F^{[1]} \circ G\right) G^{[t]}
\end{aligned}$$

B.3.2 One dimensional calculus

In this section, we put down facts for functions $f : [0, x_{\max}] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. We assume for $x > 0$, f is smooth and $f(x) \in \mathbb{R}_+$. We really only care about what is happening in any open neighborhood of 0, where f may go to ∞ or 0.

We use the notation

$$f^{[k]}(x) = \frac{x^k f^{(k)}(x)}{f(x)}$$

where $f^{(k)}$ is the k^{th} derivative. For example, if $f(x) = x^p \log(x)^q$, for $p, q \in \mathbb{R}$, then $f^{[1]}(0) = p$. Note that $f^{[k]}$ is invariant under scaling either f or the interval $[0, x_{\max}]$.

Let

$$\hat{f}(t, r) = \frac{f(t(1+r))}{f(t)}$$

Lemma B.3.2. *Suppose $f^{[1]}$ is bounded. Then for any $r' > 0$, $\hat{f}$ is bounded (independently of t) for $r \leq r'$.*

Proof. Calculate,

$$\begin{aligned}
\partial_r \hat{f}(t, r) &= \frac{t}{f(t)} f'(t(1+r)) \\
&= \frac{1}{1+r} \hat{f}(t, r) f^{[1]}(t(1+r)).
\end{aligned}$$

Therefore by Gronwall's inequality,

$$\begin{aligned}
\hat{f}(t, r) &\leq \hat{f}(t, 0) \exp\left(\int_0^r \frac{1}{1+r} f^{[1]}(t(1+r)) dr \right) \\
&= \exp\left(\int_0^r \frac{1}{1+r} f^{[1]}(t(1+r)) dr \right)
\end{aligned}$$

□

Lemma B.3.3. *Suppose that both $f^{[1]}$ and $f^{[2]}$ are bounded. Then as $r \searrow 0$,*

$$\hat{f}(r,t) = 1 + rf^{[1]}(t) + O(r^2).$$

If $k > 1$ and both $f^{[k]}$ and $f^{[k+1]}$ are bounded, then as $r \searrow 0$,

$$\partial_r^k \hat{f}(r,t) = f^{[k]}(t) + O(r).$$

The big-oh terms in these statements are independent of t.

Proof. By Taylor's theorem, there is an $r_* \in [0, r]$ such that

$$\begin{aligned}
\hat{f}(r,t) &= 1 + rt\frac{f'(t)}{f(t)} + (rt)^2 \tfrac{1}{2} \frac{f''((1+r_*)t)}{f(t)} \\
&= 1 + rf^{[1]}(t) + r^2 \frac{f((1+r_*)t)}{f(t)} \tfrac{1}{2} f^{[2]}((1+r_*)t) \\
&= 1 + rf^{[1]}(t) + r^2 \hat{f}(r_*, t) \tfrac{1}{2} f^{[2]}((1+r_*)t)
\end{aligned}$$

By Lemma B.3.2, $\hat{f}(r_*, t)$ can be bounded independently of t. The statement follows. □

$$\begin{aligned}
\partial_r \hat{f}(r,t) &= \frac{t}{f(t)} f'(t(1+r)) \\
&= \frac{t}{f(t)} f'(t) + rt\frac{t}{f(t)} f''(t) + \tfrac{1}{2}(rt)^2 \frac{t}{f(t)} f^{(3)}(t(1+r_*)) \\
&= f^{[1]}(t) + rf^{[2]}(t) + \tfrac{1}{2} r^2 f^{[3]}(t(1+r_*))
\end{aligned}$$

$$
\begin{aligned}
r^k \partial_r^k \hat{f}(r,t) &= \frac{t^k}{f(t)} f^{(k)}(t(1+r)) \\
&= \frac{r^k t^k}{f(t)} \left(f^{(k)}(t) + (rt) f^{(k+1)}(t(1+r_*)) \right) \\
&= r^k f^{[k]}(t) + r^{k+1} f^{[k]}(t(1+r_*))
\end{aligned}
$$

B.3.3 Time derivatives

We examine the size of time derivatives appearing in type two rescalings. We have $\alpha = t\nu(t)$, $\beta = \alpha'$, and $\partial_\theta = \alpha \partial_t$.

First calculate,

$$
\begin{aligned}
\beta &= \nu + t\nu' \\
&= \nu \left(1 + \nu^{[1]}\right).
\end{aligned}
$$

Therefore $\beta \sim \nu$.

Next,

$$
\begin{aligned}
\alpha'' = \beta' &= 2\nu' + t\nu'' \\
&= t^{-1} \nu \left(2\nu^{[1]} + \nu^{[2]}\right).
\end{aligned}
$$

Therefore, $\partial_\theta \beta = \alpha \partial_t \beta = \nu^2 (2\nu^{[1]} + \nu^{[2]}) \lesssim \nu^2$. We also calculate the second derivative of $\log \omega$.

$$
\begin{aligned}
\partial_\theta(\log \omega) &= \frac{\partial_\theta \omega}{\omega} \\
&= \nu \omega^{[1]}.
\end{aligned}
$$

$$
\begin{aligned}
\partial_\theta^2(\log\omega) &= \frac{\partial_\theta^2\omega}{\omega} - \frac{(\partial_\theta\omega)^2}{\omega^2} \\
&= \frac{t\nu\partial_t(t\nu\partial_t\omega)}{\omega} - \nu^2(\omega^{[1]})^2 \\
&= \nu^2\omega^{[1]} + t\nu\nu_t\omega^{[1]} + \nu^2\omega^{[2]} - \beta^2(\omega^{[1]})^2 \\
&= \nu^2\left(\omega^{[1]} + \nu^{[1]}\omega^{[1]} + \omega^{[2]} - (\omega^{[1]})^2\right)
\end{aligned}
$$

In particular, $\partial_\theta^2(\log\omega) \lesssim \nu^2$.

B.4 Facts about the Bryant soliton

Let (Bry, g_{Bry}, X) be the Bryant steady soliton with minimum scalar curvature R_0. Bryant's original work is [Bry], see also Section 1.4 of [CCG+07a] for an exposition of the construction. The extra analysis carried out here is generally justified by the analyticity of the solution. Let

$$g_{Bry} = ds^2 + u_{Bry}g_{S^q} = ds^2 + \phi_{Bry}^2 g_{S^q}$$

and

$$X = \operatorname{grad} f.$$

On any steady soliton we have $R + |\nabla f|^2 = R_0$ (Corollary 1.16 in [CCG+07a]). Since the Bryant soliton is a singly warped product, we have more precisely $df = -\sqrt{R_0 - R}\,ds$. Taking the trace of the soliton equation we have $R + \Delta f = 0$, so we find

$$\Delta_f(-f) = R_0.$$

We know that $\phi_{Bry} = O(\sqrt{s})$ as $s \to \infty$. To find the exact coefficient use

$$0 = \phi_{ss} - f_s \phi_s - (q-1)\frac{1 - |\nabla\phi|^2}{\phi}$$

so $\phi \sim R_0^{-1/4}\sqrt{\mu s}$ and $u \sim R_0^{-1/2}\mu s$ at ∞.

B.4.1 Next order approximation

So far we have found as $s \to \infty$

$$f = -(1 + o(1))R_0^{-1/2} s$$
$$u = (1 + o(1))\mu R_0^{-1/2} s.$$

Now we seek the next term in the asymptotic expansion.

The function u satisfies

$$0 = u_{ss} - f_s u_s + c_v u^{-1} u_s^2 - \mu$$

where $c_v = \frac{1}{2}(\frac{1}{2}\mu - 1)$. We also have $\Delta_f(-f) = R_0$ or

$$0 = (-f)_{ss} - f_s(-f)_s + q\phi^{-1}\phi_s(-f_s) = R_0.$$

Strictly in terms of u and $\bar{f} = -f/R_0$ we have

$$u_{ss} + R_0 \bar{f}_s u_s + c_v u^{-1} u_s^2 = \mu$$
$$\bar{f}_{ss} + R_0 \bar{f}_s^2 + \tfrac{1}{2} q u^{-1} u_s \bar{f}_s = 1$$

Write $G = \bar{f}_s$.

$$u_{ss} + R_0 G u_s + c_v u^{-1} u_s^2 = \mu \tag{B.18}$$
$$G_s + R_0 G^2 + \tfrac{1}{2} q u^{-1} u_s G = 1 \tag{B.19}$$

Now write $u = \mu R_0^{-1/2} s + u_1$ and $G = R_0^{-1/2} + G_1$. Partially writing out (B.18) and (B.19),

$$u_{1,ss} + R_0 \left(R_0^{-1} \mu + \mu R_0^{-1/2} G_1 + R_0^{-1/2} u_{1,s} + u_{1,s} G_1 \right) + c_v u^{-1} u_s^2 = \mu$$
$$G_{1,s} + R_0 \left(R_0^{-1} + 2R_0^{-1/2} G_1 + G_1^2 \right) + \tfrac{1}{2} q u^{-1} u_s G = 1$$

Simplifying,

$$u_{1,ss} + \mu R_0^{1/2} G_1 + R_0^{1/2} u_{1,s} + R_0 u_{1,s} G_1 + c_v u^{-1} u_s^2 = 0$$
$$G_{1,s} + 2R_0^{1/2} G_1 + R_0 G_1^2 + \tfrac{1}{2} q u^{-1} u_s G = 0$$

We have $u^{-1} = \mu^{-1} R_0^{1/2} s^{-1} \left(1 - u_1 + o(u_1)\right)$. Now, the highest order terms in the equation for G_1 are

$$2R_0^{1/2} G_1 + \tfrac{1}{2} q R_0^{-1/2} s^{-1},$$

therefore

$$G_1 = (1 + o(1)) \left(-\tfrac{1}{4} R_0^{-1} s^{-1} \right).$$

Then the highest order terms in the equation for u_1 are

$$\mu R_0^{1/2} G_1 + R_0^{1/2} u_{1,s} - c_v \mu R_0^{-1/2} s^{-1}$$

which gives

$$u_1 = (1 + o(1)) R_0^{-1} \left(\tfrac{1}{4} q \mu + c_v \mu \right) \log s$$

Unravelling definitions, we have found

$$\bar{f} = R_0^{-1/2} s + \tfrac{1}{4} q R_0^{-1} \log s + o(\log s)$$
$$u = \mu R_0^{-1/2} s + R_0^{-1} \left(\tfrac{1}{4} q + c_v \right) \mu \log s + o(\log s)$$

Therefore,

$$\bar{f} = \mu^{-1}u - \tfrac{1}{4}qR_0^{-1}\log u + o(\log u) \tag{B.20}$$

B.4.2 Continuation of the proof of Lemma 2.3.4

For the Bryant soliton, We have that

$$u_{ss} - u_s f_s - \mu + c_v v = 0$$

Equivalently,

$$\Delta_X u - \mu + (c_v - \tfrac{1}{2}q)v_{Bry} = 0. \tag{B.21}$$

(The $-\frac{1}{2}qv_{Bry}$ comes because $\Delta u = u_{ss} + \frac{1}{2}qu^{-1}u_s^2$.)

We also have that

$$\Delta_X \bar{f} = \Delta_X - f/R_0 = 1.$$

Thinking in terms of u this says,

$$f_u\Delta_X u + f_{uu}u_s^2 = -R_0$$

$$f_u\Delta_X u + f_{uu}uv_{Bry} = -R_0$$

Then, using (B.21)

$$f_u(\mu - (c_v - \tfrac{1}{2}q)v_{Bry}) + f_{uu}u_s^2 = -R_0$$

$$f_u\mu - (c_v - \tfrac{1}{2}q)f_u v_{Bry} + f_{uu}uv_{Bry} = -R_0$$

The asymptotics claimed in the Lemma are given in (B.20).

B.5 Deriving the evolution of Ricci-DeTurck flow

Here we prove Lemma 3.1.1, which gives the evolution of a perturbation of the background metric under Ricci-DeTurck flow.

Proof. (Lemma 3.1.1) The convention in this proof is that all curvatures and covariant derivatives are taken with respect to $\tilde{g}$.

By Lemma 2.1 of [Shi89] we have

$$\partial_t g_{ij} = g^{ab}\nabla_a\nabla_b g_{ij} - \left[g^{ab}g_{ip}\operatorname{Rm}^p_{ajb}\right]_{i\leftrightarrow j} - (\mathcal{L}_X g)_{ij} + \operatorname{Cov}(g, \nabla g)$$

Since $g = \tilde{g} + h$ and $\tilde{g}$ is parallel with respect to ∇, we find

$$\begin{aligned} \partial_t h_{ij} &= g^{ab}\nabla_a\nabla_b h_{ij} && \text{(B.22)} \\ &- \partial_t\tilde{g} - \left[g^{ab}g_{ip}\operatorname{Rm}^p_{ajb}\right]_{i\leftrightarrow j} - (\mathcal{L}_X g)_{ij} && \text{(B.23)} \\ &+ \operatorname{Cov}(g, \nabla h) \end{aligned}$$

Rewriting the curvature term. Let $g^{ij} = \tilde{g}^{ij} - \bar{h}^{ij}$. Expand $g^{ab}g_{ip} = (\tilde{g}^{ab} - \bar{h}^{ab})(\tilde{g}_{ip} + h_{ip})$ in the curvature term.

$$\begin{aligned} -g^{ab}g_{ip}\operatorname{Rm}^p_{ajb} &= \left(-\tilde{g}^{ab}\tilde{g}_{ip} + \bar{h}^{ab}\tilde{g}_{ip} - \tilde{g}^{ab}h_{ip} + \bar{h}^{ab}h_{ip}\right)\operatorname{Rm}^p_{ajb} \\ &= -\operatorname{Rc}_{ij} + \bar{h}^{ab}\operatorname{Rm}_{ajbi} - \operatorname{Rc}^p_j h_{ip} + \bar{h}^{ab}h_{ip}\operatorname{Rm}^p_{ajb} \quad \text{(B.24)} \end{aligned}$$

Now let

$$\hat{h}^{ij} = \tilde{g}^{ci}\tilde{g}^{dj}h_{cd} - \bar{h}^{ab}$$

so that

$$\bar{h}^{ab}\operatorname{Rm}_{ajbi} = \tilde{g}^{ca}\tilde{g}^{db}\operatorname{Rm}_{cjdi} h_{ab} - \operatorname{Rm}_{ajbi}\hat{h}^{ab}.$$

Putting this together with (B.24) we have

$$\begin{aligned}
-g^{ab}g_{ip}\operatorname{Rm}_{ajb}^{p} &= -\operatorname{Rc}_{ij} + \tilde{g}^{ac}\tilde{g}^{bd}\operatorname{Rm}_{cjdi} h_{ab} - \operatorname{Rc}_{j}^{p} h_{ip} \\
&\quad + h^{ab}h_{ip}\operatorname{Rm}_{ajb}^{p} - \operatorname{Rm}_{ajbi}\hat{h}^{ab}.
\end{aligned}$$

Finally taking the symmetrization we find

$$\left[-g^{ab}g_{ip}\operatorname{Rm}_{ajb}^{p}\right]_{i\leftrightarrow j} = -2\operatorname{Rc}_{ij} + 2\operatorname{Rm}[h]_{ij} - (\operatorname{Rc}\cdot h)_{ij} + Q(h)_{ij} \quad \text{(B.25)}$$

Rewriting the Lie term We have

$$-\mathcal{L}_X g = -\mathcal{L}_X\tilde{g} - \mathcal{L}_X h$$

We use the formula relating the lie derivative with the covariant derivative and the lie derivative of the metric,

$$\begin{aligned}
(-\mathcal{L}_X h)_{ij} &= (-\nabla_X h)_{ij} - \tfrac{1}{2}\left[h_{pi}\tilde{g}^{pq}(\mathcal{L}_X\tilde{g})_{qj}\right]_{i\leftrightarrow j} \\
&= (-\nabla_X h)_{ij} - \tfrac{1}{2}\left((\mathcal{L}_X\tilde{g})\cdot h\right)_{ij}
\end{aligned}$$

(The first line is true in general, the second line uses that h_{ij} is symmetric.) Thus

$$-\mathcal{L}_X g = -\mathcal{L}_X\tilde{g} - \nabla_X h - \tfrac{1}{2}\left(\mathcal{L}_X\tilde{g}\right)\cdot h \quad \text{(B.26)}$$

Coming back to the evolution. Using (B.25) and (B.26), the evolution (B.22)-(B.23) becomes

$$\begin{aligned}
\partial_t h = \hat{\Delta} h - \nabla_X h \\
&- \partial_t \tilde{g} - 2\,\mathrm{Rc}[\tilde{g}] - \mathcal{L}_X \tilde{g} \\
&- \tfrac{1}{2}\left((2\,\mathrm{Rc} + \mathcal{L}_X \tilde{g}) \cdot h\right) \\
&+ 2\,\mathrm{Rm}[h] + Q(h) + \mathrm{Cov}(g, h).
\end{aligned}$$

So unraveling definitions,

$$\begin{aligned}
\hat{\Box}_X h = - \mathrm{Rf}_X[\tilde{g}] \\
&+ \tfrac{1}{2}\left((\partial_t g) \cdot h\right) - \tfrac{1}{2}\left((\partial_t g + 2\,\mathrm{Rc} + \mathcal{L}_X \tilde{g}) \cdot h\right) \\
&+ 2\,\mathrm{Rm}[h] + Q(h) + \mathrm{Cov}(g, h) \\
\\
&= - \mathrm{Rf}_X[\tilde{g}] \\
&+ \tfrac{1}{2}\,\mathrm{UT}[h] - \tfrac{1}{2}\left(\mathrm{Rf}_X[\tilde{g}] \cdot h\right) \\
&+ 2\,\mathrm{Rm}[h] + Q(h) + \mathrm{Cov}(g, h)
\end{aligned}$$

as desired. $\square$

Appendix C

Formal asymptotics before a singularity

In this section we formally derive the asymptotics of a flow into a singular metric of the form assumed in Theorem 1.2.2. This was described in Section 1.3.3. We work in the s coordinate, which is the arclength from the tip. In other words, we write our warped product metrics as

$$(ds(x,t))^2 + \phi(s,t)^2 g_{S^q} + \psi(s,t)^2 g_{S^p}$$

Under Ricci flow,

$$\begin{aligned}
\partial_t|_x \psi &= \psi_{ss} + \left(p\frac{\psi_s}{\psi} + q\frac{\phi_s}{\phi} \right) \psi_s - \psi^{-1}\psi_s^2 - (p-1)\psi^{-1}, \\
\partial_t|_x \phi &= \phi_{ss} + \left(p\frac{\psi_s}{\psi} + q\frac{\phi_s}{\phi} \right) \phi_s - \phi^{-1}\phi_s^2 - (q-1)\phi^{-1}, \\
\partial_t|_x \log s' &= p\frac{\psi_{ss}}{\psi} + q\frac{\phi_{ss}}{\phi}.
\end{aligned}$$

To convert the time derivatives to the s coordinate we may use that for any evolving function f,

$$\begin{aligned}
\partial_t|_s f &= \partial_t|_x f - [\partial_t|_x s]\, \partial_s f \\
&= \partial_t|_x f - \left[\int_0^s \frac{\partial_t|_x s'}{s'} ds \right] \partial_s f \\
&= \partial_t|_x f - I[\psi, \phi] \partial_s f
\end{aligned}$$

Where in the last line we have named $I[\psi,\phi] = \int_0^s p\frac{\psi_{ss}}{\psi} + q\frac{\phi_{ss}}{\phi} ds$. Using this we find,

$$\partial_t|_s\psi = \psi_{ss} + \left(p\frac{\psi_s}{\psi} + q\frac{\phi_s}{\phi} - I[\psi,\phi]\right)\psi_s - \psi^{-1}\psi_s^2 - (p-1)\psi^{-1}$$
$$\partial_t|_s\phi = \phi_{ss} + \left(p\frac{\psi_s}{\psi} + q\frac{\phi_s}{\phi} - I[\psi,\phi]\right)\phi_s - \phi^{-1}\phi_s^2 - (q-1)\phi^{-1}$$

Before the singular time, these functions will have the boundary conditions at $s = 0$:

$$\phi > 0, \quad \psi = 0,$$
$$\partial_s\phi = 0, \quad \partial_s\psi = 1.$$

Given this, we rewrite $\psi = s(1+\tilde{\psi})$ where now ψ will have $\tilde{\psi}_s = 0$ at $s = 0$, if the metric is smooth. We may integrate $I[\phi, s(1+\tilde{\psi})]$ by parts to find

$$\left[\left(p\frac{\psi_s}{\psi} + q\frac{\phi_s}{\phi}\right) - I[\psi,\phi]\right] = \left[\frac{p}{s} - \int_0^s \left(q\frac{\phi_s^2}{\phi^2} + p\left(\frac{\tilde{\psi}_s^2}{(1+\tilde{\psi})^2} + \frac{2\tilde{\psi}_s}{s(1+\tilde{\psi})}\right)\right) ds\right].$$

One may then compute,

$$\begin{aligned}
\partial_t|_s\tilde{\psi} &= \tilde{\psi}_{ss} + p\frac{\tilde{\psi}_s}{s} - 2p\frac{1}{s}\int_0^s \frac{\tilde{\psi}_s}{s} ds + 2(p-1)\frac{\tilde{\psi}}{s^2} \\
&- 2p\frac{\tilde{\psi}}{s}\int_0^s \frac{\tilde{\psi}_s}{s(1+\tilde{\psi})} ds - 2p\frac{1}{s}\int_0^s \frac{\tilde{\psi}_s}{s}\left(\frac{1}{(1+\tilde{\psi})} - 1\right) ds \\
&+ \frac{(p-1)}{s^2}\left(\left(1+\tilde{\psi}\right) - \frac{1}{1+\tilde{\psi}} - 2\tilde{\psi}\right) \\
&- \frac{1+\tilde{\psi}}{s}\int_0^s \left(q\frac{\phi_s^2}{\phi^2} + p\frac{\tilde{\psi}_s^2}{(1+\tilde{\psi})^2}\right) ds \\
&- \tilde{\psi}_s\int_0^s \left(q\frac{\phi_s^2}{\phi^2} + p\left(\frac{\tilde{\psi}_s^2}{(1+\tilde{\psi})^2} + \frac{2\tilde{\psi}_s}{s(1+\tilde{\psi})}\right)\right) ds - \frac{\tilde{\psi}_s^2}{s},
\end{aligned}$$

$$
\begin{aligned}
\partial_t|_s \phi = \phi_{ss} &+ \frac{p}{s}\phi_s - (q-1)\phi^{-1} \\
&- \int_0^s \left(q\frac{\phi_s^2}{\phi^2} + p\left(\frac{\tilde{\psi}_s^2}{(1+\tilde{\psi})^2} + \frac{2\tilde{\psi}_s}{s(1+\tilde{\psi})} \right) \right) \phi_s - \frac{\phi_s^2}{\phi}.
\end{aligned}
$$

Here we have organized the equations so that the first lines are linear.

C.1 Overview of formal asymptotics

Our inspection of the shape of the Ricci flow starts with what we are most confident in. Our primary assumption is that the flow develops a Type-I singularity modeled on $\mathbb{R}^{p+1} \times S^q$. This means that under a rescaled flow, the metric approaches the standard metric on $\mathbb{R}^{p+1} \times S^q$, which is a fixed point for the flow.

In order to study the flow more closely, we expand the solution around the fixed point. We assume that the solution approaches the fixed point at the same rate as in previously studied cases (the case $p = 1$), which gives us an asymptotic expansion. Our goal is to be able to use this asymptotic expansion to learn something about the "naked-eye" final time profile by looking at this asymptotic expansion. (Note however that the "neck" region where this asymptotic expansion is valid becomes a single point at the singular time.)

The asymptotic expansion around the fixed point is actually not enough to tell us about the naked-eye profile. It is, however, enough to tell us about a region farther from the neck, which still disappears at the singular time. Then, information from this region is enough to tell us about the naked-eye profile.

C.2 The neck region

Our primary assumption is as follows:

Assumption C.2.1. *On the submanifold $\{s = 0\}$, and at time T, the metric has a type-I singularity modeled on $\mathbb{R}^{p+1} \times S^q$. Precisely,*

- *At $s = 0$, $|\mathrm{Rm}| = O\left(\frac{1}{T-t}\right)$*
- *The metrics $G(t) = \frac{1}{T-t}(X(t))^* g(t)$ converge to the soliton metric G_{sol} on $\mathbb{R}^{p+1} \times S^q$, in compact neighborhoods of the submanifold $\{s = 0\}$. Here $X(t)$ is a family of diffeomorphisms which integrate the soliton vector field.*

In particular, Assumption C.2.1 implies the following on the level of the functions s, ϕ, ψ. Set:

$$\sigma = (T-t)^{-1/2} s \quad \Phi = (T-t)^{-1/2}\phi \quad \Psi = (T-t)^{-1/2}\psi \quad \tilde{\Psi} = \psi$$

Then $\Phi \to \sigma$ and $\Psi \to \sqrt{2(q-1)}$ in regions $\{\sigma < A\}$. We will call such a region $\{\sigma < A\}$ a neck region.

Notice that because s is a geometric coordinate for the metrics g (and σ is a geometric coordinate for G) we do not have to worry about the diffeomorphisms $X(t)$. The soliton metric is given by $\tilde{\Psi} = 0$, $\Phi = \sqrt{\mu} := \sqrt{2(q-1)}$, and is a fixed point of the rescaled system. Write $\Phi = \sqrt{\mu}(1 + \tilde{\Phi})$ so that now

$\tilde{\Psi} = 0, \tilde{\Phi} = 0$ is a fixed point. In full, the evolution of $\tilde{\Phi}, \tilde{\Psi}$ is

$$
\begin{aligned}
\partial_\tau|_\sigma \tilde{\Psi} = \tilde{\Psi}_{\sigma\sigma} + p\frac{1}{\sigma}\tilde{\Psi}_\sigma - \tfrac{1}{2}\sigma\tilde{\Psi}_\sigma + \frac{2(p-1)}{\sigma^2}\tilde{\Psi} - \frac{2p}{\sigma}\left[\int_0^\sigma \frac{\tilde{\Psi}_\sigma}{\sigma}\right] \\
- 2p\frac{\tilde{\Psi}}{s}\int_0^s \frac{\tilde{\Psi}_s}{s(1+\tilde{\Psi})}ds \\
- 2p\frac{1}{s}\int_0^s \frac{\tilde{\Psi}_s}{s}\left(\frac{1}{(1+\tilde{\Psi})} - 1\right)ds \\
+ \frac{(p-1)}{s^2}\left(\left(1+\tilde{\Psi}\right) - \frac{1}{1+\tilde{\Psi}} - 2\tilde{\Psi}\right) \\
- \frac{1+\tilde{\Psi}}{s}\int_0^s \left(q\frac{\tilde{\Phi}_s^2}{(1+\tilde{\Phi})^2} + p\frac{\tilde{\Psi}_s^2}{(1+\tilde{\Psi})^2}\right)ds \\
- \tilde{\Psi}_s \int_0^s \left(q\frac{\tilde{\Phi}_s^2}{\tilde{\Phi}^2} + p\left(\frac{\tilde{\Psi}_s^2}{(1+\tilde{\Psi})^2} + \frac{2\tilde{\Psi}_s}{s(1+\tilde{\Psi})}\right)\right)ds - \frac{\tilde{\Psi}_s^2}{s}
\end{aligned}
$$

$$
\begin{aligned}
\partial_\tau|_\sigma \tilde{\Phi} = \tilde{\Phi}_{\sigma\sigma} + p\frac{1}{\sigma}\tilde{\Phi}_\sigma - \tfrac{1}{2}\sigma\tilde{\Phi}_\sigma + \tilde{\Phi} \\
+ \tfrac{1}{2}\left(1 + \tilde{\Phi} - \frac{1}{1+\tilde{\Phi}} - 2\tilde{\Phi}\right) \\
+ \left[\int_0^\sigma q\frac{\tilde{\Phi}_\sigma^2}{(1+\tilde{\Phi})^2} + \frac{p\tilde{\Psi}_\sigma^2}{(1+\tilde{\Psi})^2} + \frac{2p\tilde{\Psi}_\sigma}{\sigma(1+\tilde{\Psi})}\right]\tilde{\Phi}_\sigma - \left[\frac{\Phi_\sigma}{1+\tilde{\Phi}}\right]\tilde{\Phi}_\sigma
\end{aligned}
$$

Here, the first lines in both evolutions are linear and the others are at least quadratic in $\tilde{\Phi}$ and $\tilde{\Psi}$.

We proceed to study these linearizations. Most familiar is the linearization of the evolution for $\tilde{\Phi}$.

C.2.1 The linearization for $\tilde{\Phi}$.

We study the operator

$$\partial_\sigma^2 + \frac{p}{\sigma}\partial_\sigma - \tfrac{1}{2}\sigma\partial_\sigma + 1. \tag{C.1}$$

For smoothness of the metric, $\tilde{\Phi}$ as a function of σ must extend to an even function around zero. On functions with this property the operator (C.1) can be recognized as the operator

$$L_\Phi = \Delta_{\mathbb{R}^{p+1}} - \tfrac{1}{2}\vec{x}\cdot\nabla + 1$$

acting on a rotationally symmetric function in $\mathbb{R}^{p+1}$. This operator is self-adjoint on

$$L^2(\mathbb{R}_+, \sigma^p e^{\sigma^2/4}) = \text{ rotationally symmetric functions of } L^2(\mathbb{R}^{p+1}, e^{|-\vec{x}|^2/4}).$$

The eigenvalues of $\Delta_{\mathbb{R}^{p+1}} - \frac{1}{2}\vec{x}\cdot\nabla$ in $L^2(\mathbb{R}^{p+1}, e^{-|\vec{x}|^2/4})$ are the $(p+1)$-dimensional Hermite polynomials, which are all given by products of one-dimensional hermite polynomials in each coordinate. The eigenvalues of $\partial_\sigma^2 + \frac{p}{\sigma}\partial_\sigma - \frac{1}{2}\sigma\partial_\sigma$ come from those hermite polynomials which happen to be rotationally symmetric. The eigenspaces of $\partial_\sigma^2 + \frac{p}{\sigma}\partial_\sigma - \frac{1}{2}\sigma\partial_\sigma + 1$ (including the $+1$ term which shifts eigenvalues) are as follows:

- Constants are eigenfunctions with eigenvalue 1.
- There is a one-dimensional nullspace. $f(x_1) = x_1^2 - 2$ is a one-dimensional hermite polynomial, and if $x_1, x_2, \ldots, x_{p+1}$ are the coordinates of $\mathbb{R}^{p+1}$

then

$$(x_1^2-2)+(x_2^2-2)+\cdots+(x_{p+1}^2-2)=(x_1^2+x_2^2+\cdots+x_{p+1}^2)-2(p+1)$$

is in the nullspace of $\Delta_{\mathbb{R}^{p+1}}-\frac{1}{2}x\cdot\nabla+1$, and

$$\sigma^2-2(p+1)$$

is in the nullspace of

$$\partial_\sigma^2+\frac{p}{\sigma}\partial_\sigma-\tfrac{1}{2}\sigma\partial_s+1$$

- All further eigenspaces are negative.

Remember that we are considering a flow in which Φ approaches $\sqrt{\mu}$, i.e. $\tilde{\Phi}$ approaches zero. Therefore the constant component of $\tilde{\Phi}$ should get smaller, and in the $\tau\to\infty$ limit should disappear. Therefore we expect the nullspace to play the biggest role; this was the case in [AK07] as well. Lemma C.2.2 studies the nullspace more explicitly.

Lemma C.2.2. *In any neighborhood of zero, the only solutions to*

$$\partial_\sigma^2 f+\left(\frac{p}{\sigma}-\frac{\sigma}{2}\right)\partial_\sigma f+1=0 \tag{C.2}$$

which are C^2 at zero are multiples of

$$f(\sigma)=(\sigma^2-2(p+1))$$

where k is arbitrary.

Proof. (C.2) is a degree two linear ODE with an isolated regular singular point at $\sigma = 0$. The method of Frobenius yields two independent solutions to (C.2) around zero: one is $\sigma^2 - 2(p+1)$ and the other blows up at $\sigma = 0$ with order $O(\sigma^{-(p+1)})$. $\square$

C.2.2 The linearization for $\tilde{\Psi}$

The linearization of the evolution for $\tilde{\Psi}$ is

$$L_\Psi[f] = \partial_\sigma^2 f + \frac{p}{\sigma}\partial_\sigma f - \frac{\sigma}{2}\partial_\sigma f + \frac{2(p-1)}{\sigma^2} f - \frac{2p}{\sigma}\left[\int_0^\sigma \frac{f_\sigma}{\sigma} d\sigma\right].$$

This is more complicated because there is a nonlocal term in the linearization. We can begin by computing it on monomials. Since Ψ must be even and vanish at zero, we just compute $L_\Psi[f]$ for $f = \sigma^{2k}$, $k \geq 1$.

$$\begin{aligned} L_\Psi[\sigma^{2k}] &= 2k(2k-1)\sigma^{2k-2} + p2k\sigma^{2k-2} \\ &\quad - \frac{1}{2}2k\sigma^{2k} + 2(p-1)\sigma^{2k-2} - \frac{2p}{\sigma}\int_0^\sigma 2k\sigma^{2k-2} \\ &= \left(2k(2k-1) + 2pk + 2(p-1) - \frac{4pk}{2k-1}\right)\sigma^{2k-2} - k\sigma^{2k} \end{aligned}$$

The pleasant part of the situation is that the operator acts on the monomials as an upper-triangular matrix. Therefore one can read off the eigenvalues in the span of the monomials. They are the coefficients of σ^{2k} above, that is $-k$ for $k \in \mathbb{N}, k \geq 1$. This method also gives a formula for computing the eigenfunctions.

One can also study the functions satisfying $L_\Psi[f] = \lambda f$ by multiplying by σ and differentiating with respect to σ, arriving at

$$\sigma\partial_\sigma^3 f + \left((p+2) - \frac{\sigma^2}{2}\right)\partial_\sigma^2 f + \left(\frac{(4p-2)}{\sigma} - (1+\lambda)\sigma\right)\partial_\sigma f - \left(\frac{2(p-1)}{\sigma^2} - \lambda\right) f = 0$$

In any case we find the following.

Lemma C.2.3. *The operator L_Ψ has a strictly negative spectrum in $L^2(\mathbb{R}_+, \sigma^p e^{-\sigma^2/4})$.*

C.2.3 The ansantz for $\tilde{\Phi}, \tilde{\Psi}$

Our assumption that the rescaled metric approaches the soliton $\mathbb{R}^{p+1} \times S^q$ says that $\tilde{\Phi}$ and $\tilde{\Psi}$ both approach zero as $\tau \to \infty$. We make two further assumptions about the rate of this convergence.

Assumption C.2.4. *The limits*

$$\tilde{\Psi}_1(\sigma) := \lim_{\tau\to\infty} \tau\tilde{\Psi}(\sigma,\tau), \quad \tilde{\Phi}_1(\sigma) := \lim_{\tau\to\infty} \tau\tilde{\Phi}(\sigma,\tau)$$

both exist. The convergence happens in C^2 on any region $\{\sigma < \sigma_\#\}$.

Assumption C.2.5. *As functions of σ, $\tilde{\Psi}_1$ and $\tilde{\Phi}_1$ are in $L^2(\mathbb{R}_+, \sigma^p e^{-\sigma^2/4})$.*

By Assumption C.2.4 we can write

$$\tilde{\Psi} = \tau^{-1}(\tilde{\Psi}_1(\sigma) + \tilde{\Psi}_1^{(err)}(\sigma,\tau)), \quad \tilde{\Phi} = \tau^{-1}(\tilde{\Phi}_1(\sigma) + \tilde{\Phi}_1^{(err)}(\sigma,\tau))$$

where $\Psi_1^{(err)}$ and $\Phi_1^{(err)}$ converge C^2_{loc} to zero as $\tau \to \infty$. Plugging this into the evolution equations and bounding nonlinear terms shows $L_\Phi[\tilde{\Phi}_1] = L_\Psi[\tilde{\Psi}_1] = 0$. Then Lemmas C.2.2 and C.2.3 with Assumption C.2.5 show that for some k_0

$$\tilde{\Psi}_1 = 0, \quad \tilde{\Phi}_1 = k_0(\sigma^2 - 2(p+1)).$$

Remark 3. In [AK07], the authors rigorously found the value of k_0 for the case $p = 0$. Formally one can find the value by analyzing the evolution of the inner product

$$k_0(\tau) = \int_{\mathbb{R}} \left(\tilde{\Phi}(\sigma, \tau)\right) \left(\sigma^2 - 2(p+1)\right) \left(\sigma^p e^{-\sigma^2/4}\right) d\sigma,$$

taking into account quadratic terms in the evolution of $\tilde{\Phi}$.

Story C.2.6. *I'll be honest, I think I could have written some of this better. If you want to talk about it, let me know. If you're reading this and need a break, here's a story.*

Telling of Story: A student really enjoyed Professor Gordon's algebraic topology class. As a thank-you, he brought a box of donut holes from Ken's Donuts to his office one day.

"What... are these?", Professor Gordon exclaimed. "They taste good, but... something is just off!"

At first, the student was confused. Ken's Donuts is well known for their high quality control and safety standards! But, the next morning, the student was enlightened.

The student brought the professor Gordon a box of Ken's finest, honest donuts. "I'm, I'm sorry for, for, for my faux paux yesterday, p-p-professor," the student stammered. The hardened Professor took the box suspiciously. He looked inside. His eyes glistened. He took a bite, and said, "tapology accepted." ■

C.2.4 Validity

We come back to

$$\Psi(\sigma,\tau) = \sigma\left(1 + \tau^{-1}\tilde{\Psi}_1^{(err)}(\sigma,\tau)\right), \tag{C.3}$$

$$\Phi(\sigma,\tau) = \sqrt{\mu}\left(1 + \tau^{-1}k(\sigma^2 - 2(p+1)) + \tau^{-1}\tilde{\Phi}_1^{(err)}(\sigma,\tau)\right), \tag{C.4}$$

and study the regions where it is valid for $\Psi_1^{(err)}$ and $\Phi_1^{(err)}$ to be small. Using (C.3), (C.4) we can derive evolution equations for $\Psi_1^{(err)}$ and $\Phi_1^{(err)}$. These will be parabolic equations with a source term which is $O((\tau^{-1}\sigma^2)^2, (\tau^{-1}\sigma^2) \to 0)$.

Therefore it is consistent to assume that

$$\tilde{\Psi}_1^{(err)}(\sigma,\tau),\ \tilde{\Phi}_1^{(err)}(\sigma,\tau) = o(1; \tau \to \infty) \text{ if } \sigma = o(\sqrt{\tau}, \tau \to \infty).$$

We study regions where $\sigma = O(\sqrt{\tau})$ in the next section.

C.3 The intermediate region

When $\sigma \sim \sqrt{\tau}$, the assumption that the error term for the neck approximation $\hat{\Phi}$ is small is no longer feasible. Let us introduce scaled functions

$$\xi = \sigma/\sqrt{\tau} \qquad Y = \Psi/\sqrt{\tau} \qquad \tilde{Y} = \tilde{\Psi} \qquad (\text{so } Y = \xi(1+\tilde{Y})\ )$$

Then

$$\begin{aligned}
\partial_\tau|_\sigma &= \partial_\tau|_\xi + (\partial_\tau\xi)\partial_\xi \\
&= \partial_\tau|_\xi - \tfrac{1}{2}\tau^{-1}\xi\partial_\xi \\
\partial_\sigma &= (\partial_\sigma\xi)\partial_\xi \\
&= \tau^{-1/2}\partial_\xi
\end{aligned}$$

Calculate the evolutions. Every term in the right hand side of $\partial_\tau|_\sigma \Psi$ scales to have a τ^{-1} coefficient except for the $-\frac{1}{2}\sigma\Psi_\sigma$ which just scales to $-\frac{1}{2}\xi\Psi_\xi$. The evolution of Φ also has reaction terms with no τ^{-1} coefficient.

We assume that Φ and $\tilde{Y}$ have limits as $\tau \to \infty$:

Assumption C.3.1. *The limits*

$$\Phi_{int0}(\xi) = \lim_{\tau\to\infty} \Phi(\tau,\xi) \quad \text{and} \quad \tilde{Y}_{int0}(\xi) = \lim_{\tau\to\infty} \tilde{Y}(\tau,\xi)$$

exist. The limit occurs in C^2 on regions $\{\xi < \xi_\#\}$.

Under Assumption C.3.1, Φ_{int0} and $\tilde{Y}_{int0}$ will solve

$$\begin{aligned} 0 &= -\tfrac{1}{2}\xi\tilde{Y}_{int0,\xi} \\ 0 &= -\tfrac{1}{2}\xi\Phi_{int0,\xi} - (q-1)\Phi_{int0}^{-1} + \tfrac{1}{2}\Phi_{int0} \end{aligned}$$

so they are

$$\begin{aligned} \tilde{Y}_{int0} &= k_0 \\ \tilde{\Phi}_{int0} &= \sqrt{k_1\xi^2 + \mu} \end{aligned}$$

(Recall $\mu = 2(q-1)$).

C.3.1 Matching

We want the intermediate approximations to be valid at the boundary of the neck region, and match the neck approximations. Therefore, let us say we hope the intermediate solutions to be valid on regions of the form

$$\{\sigma \geq \sigma_0 \text{ and } \xi < \xi_0\} = \{\sqrt{\tau}\xi \geq \sigma_0 \text{ and } \xi < \xi_0\} = \left\{\frac{\sigma_0}{\sqrt{\tau}} \leq \xi \leq \xi_0\right\}$$

Set

$$\tilde{Y}_{int} = k_0$$
$$\Phi_{int} = \sqrt{k_1 \xi^2 + \mu}$$

with k_0 and k_1 to be determined.

First, unravel the definition of $\tilde{Y}$.

$$Y_{int} = \xi(1 + \tilde{Y}_{int}) = \xi(1 + k_0)$$
$$\Psi_{int} = \sqrt{\tau} Y_{int} = \sigma(1 + k_0)$$

Matching Ψ_{int} with $\Psi_{neck} = \sigma$ gives $k_0 = 0$.

Putting Φ_{neck} in terms of ξ gives

$$\Phi_{neck} = \sqrt{\mu}(1 + k(\xi^2 - 2(p+1)\tau^{-1}))$$

so when ξ is small and τ is large

$$\Phi_{neck} \approx \sqrt{\mu}, \quad \Phi_{neck,\xi} \approx 0, \quad \Phi_{neck,\xi\xi} \approx 2k\sqrt{\mu},$$
$$\Phi_{int} \approx \sqrt{\mu}, \quad \Phi_{int,\xi} \approx 0, \quad \Phi_{int,\xi\xi} \approx \frac{k_1}{\sqrt{\mu}}.$$

So we choose $k_1 = 2k\mu$ and have

$$\Phi_{int} = \sqrt{\mu}\sqrt{2k\xi^2 + 1}$$

C.4 Outer region

Both the neck and intermediate regions shrink to the singular submanifold at $t = T$. Now we attempt to use the intermediate approximations to get information about the solution at time T, outside of the singular submanifold.

From our considerations in the intermediate region, with Assumption C.3.1 and the matching we can write

$$\psi = \left((T-t)^{1/2}|\log(T-t)|^{1/2}\right)\xi\left(1+\tilde{Y}_{int0}^{(err)}\right) = s\left(1+\tilde{Y}_{int0}^{(err)}\right), \quad \text{(C.5)}$$

$$\begin{aligned}\phi &= \sqrt{(T-t)}\sqrt{\mu}\left(\sqrt{2k\xi^2+1}+\Phi_{int0}^{(err)}\right)\\ &= e^{-\tau/2}\sqrt{\mu}\left(\sqrt{2k\xi^2+1}+\Phi_{int0}^{(err)}\right),\end{aligned} \quad \text{(C.6)}$$

where the error terms $Y_{int0}^{(err)}$ and $\Psi_{int0}^{(err)}$ are $o(1;\tau\to\infty)$ on sets $\{\xi<\xi_\#\}$. Recall that τ,ξ,s,t are related by

$$(T-t)\tau = e^{-\tau}\tau = \frac{s^2}{\xi^2}.$$

We will take $\xi_\#$ to infinity and s to 0, but still have the error terms go to zero. To do this, let $\tau_\#(\xi_\#)$ be chosen large enough depending on $\xi_\#$ so that as $\xi_\#\to\infty$

$$Y_{int0}^{(err)}(\xi_\#,\tau=\tau_\#(\xi_\#))\to 0, \quad \Phi_{int0}^{(err)}(\xi_\#,\tau=\tau_\#(\xi_\#))\to 0.$$

Because the error terms are continuous, it is possible to make $\tau_\#(\xi_\#)$ continuous and increasing. As a further requirement on $\tau_\#(\xi_\#)$ we ask that

$$\frac{\log(\xi_\#)}{\log(s_\#)} = \frac{\log\xi_\#}{\log\left(\sqrt{e^{-\tau_\#}\tau_\#}\xi_\#\right)}\to 0, \qquad \text{equivalently } \frac{\log\xi_\#}{\tau_\#}\to 0. \quad \text{(C.7)}$$

To recap, we are taking $\xi_\#$ to infinity and $\tau_\#$ to infinity, monotonically. From these we can find $t_\#$ and $s_\#$ by $(T-t_\#)=e^{-\tau_\#}$ and

$$e^{-\tau_\#}\tau_\# = \frac{s_\#^2}{\xi_\#^2}. \quad \text{(C.8)}$$

Find an expression for $(T - t_{\#}) = e^{-\tau_{\#}}$ by taking the logarithm of both sides of (C.8) and then dividing $e^{-\tau_{\#}}\tau_{\#}$ by $\tau_{\#}$:

$$-\tau_{\#} + \log \tau_{\#} = -2\log s_{\#} - 2\log \xi_{\#}$$

so,

$$\tau_{\#}(1 + o(1; t_{\#} \to T)) = -2\log s_{\#}\left(1 - \frac{\log \xi_{\#}}{\log s_{\#}}\right),$$

$$\tau_{\#} = (2\log s_{\#})\left(1 + o(1; t_{\#} \to T)\right)$$

$$e^{-\tau_{\#}} = \frac{s_{\#}^2}{\xi_{\#}^2}\frac{1}{-2\log s_{\#}}\left(1 + o(1; t_{\#} \to T)\right).$$

We used (C.7) in the second implication.

Now, evaluate (C.5), (C.6) for $s \le s_{\#}, t = t_{\#}$.

$$\begin{aligned}\psi(s, t_{\#}) &= s\left(1 + \tilde{Y}_{int0}^{(err)}\left(\frac{s}{e^{\tau_{\#}}\tau_{\#}}\xi_{\#}, t_{\#}\right)\right) \\ &= s\left(1 + o(1; t_{\#} \to T)\right) \qquad \text{(C.9)}\end{aligned}$$

In evaluating (C.6), we use (**??**) to evaluate $\sqrt{T - t_\#} = e^{-\tau_\#/2}$.

$$\begin{aligned}
\phi(s, t_\#) &= \frac{s_\#}{\xi_\#} \frac{1}{\sqrt{-2 \log s_\#}} (1 + o(1; t_\# \to \infty)) \sqrt{\mu} \\
&\quad \cdot \left(\sqrt{2k \frac{s^2}{e^{-\tau_\#} \tau_\#} + 1} + \Phi^{(err)}_{int0} \left(\xi_\#, \tau_\# \right) \right) \\
&= \frac{s_\#}{\xi_\#} \frac{1}{\sqrt{-2 \log s_\#}} \sqrt{\mu} \left(\sqrt{2k \frac{s^2}{e^{-\tau_\#} \tau_\#} + 1} \right) \\
&= \frac{s}{s_\#} \frac{s_\#}{\xi_\#} \frac{1}{\sqrt{-2 \log s_\#}} \sqrt{\mu} \left(\sqrt{2k \frac{s_\#^2}{e^{-\tau_\#} \tau_\#} + 1} \right) (1 + o(1; t_\# \to \infty)) \\
&= \sqrt{\mu k} \frac{s}{\log s} \frac{\log s}{\log s_\#} \sqrt{1 + \frac{1}{2k \xi_\#}} (1 + o(1; t_\# \to \infty)) \\
&= \sqrt{\mu k} \frac{s}{\log s} \frac{\log s}{\log s_\#} (1 + o(1; t_\# \to \infty)) \qquad \text{(C.10)}
\end{aligned}$$

We assume that the value of $\phi(s_\#, t)$ at $t = t_\#$ is a good approximation for its value at $t = T$.

Assumption C.4.1.

$$|\psi(s_\#, t_\#) - \psi(s_\#, T)| = o(1; s_\# \to 0) \psi(s_\#, t_\#),$$

$$|\phi(S_\#, t_\#) - \phi(s_\#, T)| = o(1; s_\# \to 0) \phi(s_\#, t_\#).$$

In particular, Assumption C.4.1 implies the asymptotic profile at the singular time $t = T$:

$$\psi(s, T) = s(1 + o(1; s \to 0)), \quad \phi(s, T) = \sqrt{\mu} \frac{s}{\sqrt{|\log s|}} (1 + o(1; s \to 0)).$$

The following lemma provides justification for Assumption C.4.1, by showing that it is at least true for the linearization of the system in time.

Lemma C.4.2. *With all assumptions before Assumption C.4.1, the time derivatives of ϕ and ψ at $(s,t) = (s_\#, t_\#)$ satisfy*

$$(T - t_\#) \cdot \partial_t|_s \psi(s_\#, t_\#) = o(1)\psi(s_\#, t_\#), \quad (T - t_\#) \cdot \partial_t|_s \phi(s_\#, t_\#) = o(1)\phi(s_\#, t_\#)$$

Proof. The evolution for ψ and ϕ (after performing an integration by parts) is

$$\partial_t|_s \psi = \psi_{ss} + \left(p\frac{\psi_s}{\psi} - p\int_0^s \frac{\psi_{ss}}{\psi} ds - q\int_0^s \frac{\phi_s^2}{\phi^2} ds \right) \psi_s - \psi^{-1}\psi_s^2 - (p-1)\psi^{-1},$$
$$\partial_t|_s \phi = \phi_{ss} + \left(p\frac{\psi_s}{\psi} - p\int_0^s \frac{\psi_{ss}}{\psi} ds - q\int_0^s \frac{\phi_s^2}{\phi^2} ds \right) \phi_s - \phi^{-1}\phi_s^2 - (q-1)\phi^{-1}.$$

To evaluate the terms involving just derivatives of ϕ and ψ we can use (C.9) and (C.10). For the nonlocal terms, we need more. To evaluate the nonlocal term involving ψ, note that $\psi \approx s$ is valid in the parabolic region as well. To evaluate $\int_0^s \frac{\phi_s^2}{\phi^2} ds$, we can apply Cauchy-Schwarz and integrate:

$$\begin{aligned}
\left| \int_0^s \frac{\phi_s^2}{\phi^2} \right| (s, t_\#) &\le \left(\max_{[0,s]} |\phi_s(s, t_\#)| \right) \left| \int_0^s \frac{\phi_s}{\phi^2} \right| \\
&= \left(\max_{[0,s]} |\phi_s(s, t_\#)| \right) \left| \frac{1}{\phi(0)} - \frac{1}{\phi(s)} \right| \\
&\le 1 \cdot 2\sqrt{\mu} \frac{1}{\sqrt{\mu}\sqrt{T - t_\#}} (1 + o(1; t_\# \to \infty))
\end{aligned}$$

□

C.5 Conclusion

Our conjecture is thus as follows: consider any Ricci flow in the space of metrics we are considering, which has a type-I singularity at $s = 0, t = T$ modeled on the standard soliton on $R^{p+1} \times S^q$ and which is either compact or has reasonable growth at infinity. Then the limit of the metrics as $t \to T$ will have the form

$$\psi = s(1 + o(1; s \to 0)),$$
$$\phi = \sqrt{\mu}\sqrt{k}\frac{s}{\sqrt{|\log s|}}\left(1 + o(1; s \to 0)\right).$$

This is an unsurprising conclusion if one considers the stability of $\mathbb{R}^{p+1}$ under Ricci flow, and compares with previous results in the $p = 0$ case. In fact the only effect that the value of p has, on the level of our asymptotics, is in on term in the neck region.

Appendix D

Notation

We adopt the shorthand that when stating hypotheses, the statement $x < \overline{x}(y, z)$ means "there exists an $\overline{x}$, depending on y and z, such that if $x < \overline{x}$, the following holds." This allows us to quickly state "if $x < \overline{x}(y, z)$ and $w < \overline{w}(x, y)$ then ...." For some reason, we might need to choose x strictly smaller than $\overline{x}(y, z)$ when apply the theorem, so the statement can not be reduced to implementing some constants depending on y and z alone.

The curvature tensors are Rm for the full riemannian $(0, 4)$ tensor, Rc for the Ricci curvature, and R for the scalar curvature. The indices of Rm are such that Rm_{ijij} is a sectional curvature in an orthonormal frame.

The vector field $V[g, \tilde{g}]$, the operator $\Delta_{g,\tilde{g}}$, and $\mathrm{Rf}[g]$ are defined in Section 3.1.1. There we also define $\mathrm{Rm}[h]$ for a symmetric two-tensor h, and $\Lambda_{\mathrm{Rm}} : M \to \mathbb{R}$.

Partial derivatives are denoted with $\partial_\cdot$. See Section B.3.1 for the notation $\partial_t f$, $\partial_{t;u} f$, and $f^{[1]}$, $f^{[t]}$, etc.

Everywhere g_{S^q} is the metric of sectional curvature 1 on the q dimensional sphere S^q. We define $\mu = 2(q-1)$ so that $2\,\mathrm{Rc}_{g_{S^q}} = \mu g_{S^q}$. We also have a general Einstein manifold (F, g_F) in play, its Ricci curvature satisfies

$2\,\mathrm{Rc}_F = \mu_F g_F$ for some $\mu_F \in \mathbb{R}$.

Usually we have a metric of the form

$$adx^2 + ug_{S^q} + wg_F$$

for x in some interval I. Here a, u, and w are functions of I. The functions a, u, and w may also depend on time. On these manifolds we have the derived functions $v = u^{-1}|\nabla u|^2$ and $y = w^{-1}|\nabla w|^2$. Rarely we also use $\phi = \sqrt{u}$ and $\psi = \sqrt{w}$.

The heat operator is $\Box u = \partial_t u - \Delta u$. This depends on a (usually time-dependent) riemannian metric. We may decorate $\Box$ or Δ with subscripts to specify which metric. If X is a vector field then $\Delta_X u = \Delta u - X(u)$ and $\Box_X = \partial_t - \Delta_X$.

We have a lot of scaling. Briefly:

$$\begin{aligned} \nu(t) = V_0(\mu t), \quad \omega(t) = W_0(\mu t), \quad \alpha(t) = t\nu(t), \quad \beta(t) = \alpha'(t) \\ \rho = t^{-1}u, \quad \sigma = (t\nu(t))^{-1}u, \quad \zeta = t\nu(t)^{-1/2}u = \nu(t)\sigma, \\ \bar{w} = \omega(t)^{-1}(w + \mu_F t). \end{aligned}$$

We have some functions which are written in terms of u. Generally capital letters denote known functions which are written in terms of u, whereas lowercase letters denote unknown functions. The functions V_0 and W_0 are the initial values for v and w in a model pinch. V_{prish} and W_{prish} are our approximations for v and w in the productish region, and $V^{\pm}_{prish}$ and $W^{\pm}_{prish}$ are upper and lower barriers for v and w based on these approximations. Similarly these names with the subscript *tip* are approximations and barriers in the tip region. In Section 2.2, we only refer to the functions for the produtish region, and therefore we drop the subscripts for cleanliness. Similarly in Section 2.3 we only refer to the tip functions, so we drop the subscript there as well.

Other functions of u and t are Q and U_0 (introduced in Lemma A.1.2) and $V_{Bry}, V_{pert}, W_{pert}$ (introduced in Section 2.3, and with an overview in Section 2.3.1).

We define $x^{a,b} = x^a(1+x)^{b-a}$. The point is that it's a smooth function on $(0, \infty)$ which behaves like x^a at 0 and x^b at ∞.

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Vita

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