Chance-Constrained Efficiency Analysis

By: **Kenneth C. Land, C.A. Knox Lovell, and Sten Thore**

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Keywords: data envelopment analysis (DEA), chance-constrained programming

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We show that the chance-constrained enveloping process leads to the definition of a conventional (certainty-equivalent) efficiency ratio (a ratio between weighted outputs and weighted inputs). Furthermore, extending the concept of Pareto and Koopmans efficiency to the case of chance-constrained dominance (to be defined), we establish the identity of the following two chance-constrained efficiency concepts: (i) the chance constrained DEA efficiency measure of a particular output-input point is unity, and all chance-constraints are binding; (ii) the point is efficient in the sense Pareto and Koopmans.

Finally we discuss the implications of our approach for econometric frontier analysis.

1. Introduction.

As the literature of data envelopment analysis (DEA) has grown (for surveys see Charnes and Cooper [10] and Banker, Charnes, Cooper, Swarts and Thomas ([3]), many researchers have felt a need to incorporate stochastic considerations in the model to accommodate the presence of measurement and specification errors.

By specification errors we mean here, as in statistical analysis generally, the presence of unspecified ("hidden", "intervening") causal factors so that the causal hypothesis at hand is not complete. The omission of these factors appears in the model as stochastic variation.

Production relationships are often stochastic in nature. In agriculture, weather is unpredictable. In manufacturing there may be considerable variability in the quality of output obtained, as attested by the need for statistical quality control. In product development there is uncertainty whether new designs will be technically viable and uncertainty about the prospective market.

Another kind of uncertainty relates to the nature of the economic system inside which production units are operating. A comparison of the efficiency of private firms and government-owned firms operating in the same industry might hinge upon the evaluation of such factors as the willingness of management to take risks, and incentives to cost control (or the possible failure to exercise such control). Both the nature of risk and the attitude toward risk are often different in private and public enterprise.

Uncertainty in the mind of the management regarding the availability of inputs or demand for outputs leads to a need to hold inventory - - stocks of raw materials, of semifinished goods, and of consumer goods ready for shipment. A manufacturing firm holds inventory of steel profiles or imported electric components. It stores output in a warehouse before delivery. In a market economy final demand may be quite volatile and management needs to put contingencies in place to deal with such variation. Under state socialism, managers hoard labor and resources to protect themselves against stochastic variation in supplies caused by abrupt changes ordered by state bureaucrats (Kornai [20] and [21]).

The concept of "efficiency" must somehow be related to how managers deal with uncertainty. Inefficient use of inputs is not just inappropriate proportions, as when some inputs are lacking entirely and the presence of others is excessive. Efficiency is an <u>ex ante</u> concept and should refer to the degree of preparedness that management has established to handle stochastic variation in production relationships.

In order to model efficiency in the face of uncertainty, we shall use the technique of chance-constrained programming (for early developments see Charnes, Cooper and Symonds [13] and Charnes and Cooper [6], [7], [8]). This programming method is attractive when the purpose of the programming is to avoid excessive one-sided stochastic variation. In deterministic DEA, and in conventional activity analysis, all observations are required to fall on one side of the efficiency frontier. Here we shall permit stochastic variation around the frontier, but the bulk of observations will still be required to fall behind it.

Our ideas have evolved, and been shared with the academic community, in the following manner. The starting point was the observation that the term "efficiency" was being used in the sociological literature somewhat loosely in comparisons between capitalist firms and state socialist firms (e.g. in Burawoy and Lukacs [5]). In order to bring out what seemed to be intended, Land, Stark, and Thore [23] suggested a chance-constrained format. At a National Science Foundation conference on parametric and nonparametric approaches to frontier analysis, University of North Carolina at Chapel Hill, September 1988, Land, Lovell and Thore [22] presented the basic chance-constrained DEA model for the first time. The equivalence of chance-constrained DEA efficiency and chance-constrained Pareto-Koopmans efficiency was stated and proven by Thore at a symposium on recent developments in evaluation of organisational productivity, Odense University, May 1989 [31]. None of the reports mentioned now have been printed previously and this material is now brought together here for the first time.

Additional aspects of chance-constrained DEA have been probed by Petersen and Olesen [24]. Alternative stochastic approaches to DEA have been reported by Banker [2] , Desai and Schinnar [14], Satish [25], Sengupta ([27], [28], [29]), and Sengupta and Sfeir [29]. Also, for a nonparametric study of optimizing behavior with measurement error see Varian [33].

Turning to the parametric (econometric) approach to frontier analysis, Aigner and Chu [1], p. 838 made cautions but correct reference to chance-constrained programming by noting that a deterministic production frontier relationship might be replaced with a probability statement involving a specified minimum probability with which the relationship is to hold. They did not implement their own suggestion, however. Three years later Timmer [31] did, in a way that was most unfortunate, suggesting that efficient observations be discarded from analysis until the desired minimum probability is met. For criticism, see Forsund, Lovell and Schmidt [16], p. 10. Discarding observations, ostensibly efficient or otherwise, is not sound econometric practice.

Our goal in this paper is to reintroduce chance-constrained programming into the frontier literature, to suggest that it offers a useful way of improving deterministic frontier models, and to use it to forge a link between the deterministic and the stochastic approaches to frontier analysis.

Sector 2 presents the basic chance-constrained DEA format. In the common fashion, the input and output observations of a particular DMU (decision-making unit) to be examined are compared with so-called "best practice". Adopting a standard input-oriented formulation, best practice is defined as the greatest possible radial contraction of the inputs, while still obtaining the considered vector of outputs. These conditions are now all written as chance-constraints, i.e. they are required to hold "most of the time". The radial contraction factor is the desired chance-constrained DEA measure of efficiency. It can be written as an efficiency ratio (a ratio between weighted outputs and weighted inputs).

A well known result in deterministic DEA states that a DEA-efficient point is also efficient in the sense of Pareto and Koopmans (Charnes, Cooper, Golany, Seiford, and Stutz [11]). Section 3 demonstrates a similar result for chance-constrained DEA. The conventional way of defining a Pareto and Koopmans efficient point is to state that it is an "undominated point". Extending the concept to a chance-constrained setting, we define chance-constrained dominance and identify chance-constrained Pareto-Koopmans efficiency with a chance-constrained undominated point. It is shown that chance-constrained DEA efficiency implies chance-constrained Pareto-Koopmans efficiency and conversely.

Section 4 ties Pareto-Koopmans efficiency to chance-constrained activity analysis (Thore [30]). Each DMU is reinterpreted as a separate "activity". An efficient activity can be represented as the optimal solution to a chance-constrained programming problem representing the conversion of resources (inputs) into consumer goods (outputs). For any activity, whether efficient or not, one can calculate the chance-constrained DEA efficiency measure. In the case that the efficiency measure equals unity, there is again Pareto-Koopmans efficiency. The efficiency concepts are the same.

Section 5 comments on mixed nonparametric- parametric approaches. There is a mathematical Appendix.

2. A Chance-Constrained Radial Contraction Formulation.

Use the notation

$i=1,\dots,I$	the collection of DMU's
$m=1,\dots,M$	inputs
$n=1,\dots,N$	outputs
$X = [x_{mi}]$	sample input matrix
$Y = [y_{ni}]$	sample output matrix
$X_0= [x_{m0}]$	column vector of inputs of the particular DMU investigated
$Y_0= [y_{n0}]$	column vector of outputs of the particular DMU investigated

It is assumed that all x_{mi} and all y_{ni} are stochastic with a known joint probability distribution. It should also be pointed out that what matters here is not the variability of these inputs and outputs in the past, but their expected variability. To simplify matters we shall assume throughout that this expected probability distribution is jointly normal. (This assumption is made for convenience only, and any known joint probability distribution will do.)

In the common manner, use the symbols X^m and Y^n to denote the rows of the matrices X and Y, and the symbols X_j and Y_j to denote the columns of the matrices.

Petersen and Olesen [24] also assume that X_0 and Y_0 are stochastic. In what follows, we shall prefer to regard X_0 and Y_0 as deterministic. That is, the question posed here is this: how is the particular observed outcome (X_0, Y_0) of the DMU under investigation to be rated in relation to the chance-constrained envelope?

Next, introduce the following unknowns to be determined:

θ	radial contraction factor
$\lambda = [\lambda_j]$	DMU loadings, determining "best practice"

and consider

(1) $\min \theta$

$$\text{subject to Prob } (Y^n\lambda \geq y_{n0}) \geq 0.95 \text{ , } n=1,\ldots,N$$

$$\text{Prob } (\theta x_{m0} \geq X^m\lambda) \geq 0.95 \text{ , } m=1,\ldots,M$$

$$\theta \text{ unrestricted in sign, } \lambda \geq 0$$

The program instructs us to minimize the contraction or shrinking factor θ, subject to two sets of chance constraints: "best practice" outputs should not fall short of the output considered, and the shrunken input cannot fall short of best practice inputs.

(Here and throughout the paper the additional restriction $\Sigma\lambda_i = 1$ can be adjoined to the model. See Grosskopf [17] for an exposition of the link between restrictions on the intensity vector and scale economies in production.)

Converting to certainty equivalents

$$
\begin{aligned}
(2)\quad & \min \theta \\
& \text{subject to } E(Y^n\lambda) - 1.645\ \text{s.d.}(Y^n\lambda) \geq y_{n0}\ , \quad n=1,\dots,N \\
& \qquad \theta x_{m0} - E(X^m\lambda) - 1.645\ \text{s.d.}(X^m\lambda) \geq 0\ , \quad m=1,\dots,M \\
& \qquad \theta \text{ unrestricted in sign, } \lambda \geq 0 \qquad ,
\end{aligned}
$$

The number 1.645 is $F^{-1}(0.95)$ where F is the distribution function of the normal distribution. (5 % of the observations of a normal distribution with mathematical expectation =0 and standard deviation = 1 exceed this number.) The operator E is the mathematical expectation. The operator s.d. is the standard deviation (the square root of the variance).

Define Lagrange multipliers

$v = [v_n]$ row vector of "virtual multipliers" of outputs

$u = [u_m]$ row vector of "virtual multipliers" of inputs

There is a Kuhn-Tucker condition that states (since θ is unrestricted in sign)

$$(3)\quad u^* X_0 = 1$$

Next, using the Theorem in the Appendix, one has (in the common fashion, the asterisk denotes an optimal value)

(4) $\theta^* = v^*Y_0$

so that, combining (3) and (4)

(5) $\theta^* = v^*Y_0/u^*X_0$

In words, the optimal contraction factor θ^* can be written as a deterministic Charnes-Cooper efficiency ratio.

In the deterministic case one has that $\theta^* \leq 1$. But no such result is available here. Instead, there are three possibilities:

$\theta^* < 1$, or $\theta^* = 1$ but some constraints remain slack at the point of optimum	the DMU being rated will be said to be "chance-constrained DEA inefficient" (or subefficient);
$\theta^* = 1$ and all constraints are tight at the point of optimum	the DMU being rated will be said to be "chance-constrained DEA efficient";
$\theta^* > 1$	the DMU being rated will be said to be "chance-constrained DEA hyperefficient".

The locus of efficient input-output combinations may be referred to as the chance-constrained efficiency frontier. Chgance-constrained DEA inefficient points are located "below" the frontier. Hyperefficient points lie "above" it. Hyperefficiency can occur precisely because some small threshold fraction of all observed points are permitted to cross the efficiency frontier.

The point $(Y_0, \theta^* X_0)$ is a point on the efficiency frontier (the chance-constrained envelope).

The formulations that we have provided up to this point have avoided possible difficulties arising if one or several virtual multipliers become zero. In order to deal with this matter (see Charnes and Cooper [9]), program (2) should be amended to read

$$
\begin{aligned}
(6)\quad & \min\ \theta - \epsilon(es^+ + es^-) \\
& \text{subject to } E(Y^n\lambda) \qquad\qquad - 1.645\ \text{s.d.}(Y^n\lambda) - s^+_n = y_{n0}\ ,\ n=1,\dots,N \\
& \qquad\qquad \theta x_{m0} - E(X^m\lambda) - 1.645\ \text{s.d.}(X^m\lambda) - s^-_m = 0\ ,\ m=1,\dots,M \\
& \qquad\qquad \theta \text{ unrestricted in sign, } \lambda,\ s^+,\ s^- \geq 0
\end{aligned}
$$

where $s^+ = [s^+_n]$ and $s^- = [s^-_m]$ are column vectors of slack variables and e is a row vector of suitable dimensions with unity in all positions, and $\epsilon > 0$ is a non-Archimedean infinitesimal. If $\theta^* = 1$ and $s^{+*} = s^{-*} = 0$, the DMU being tested is rated efficient.

Figures 1 and 2 illustrate the chance-constrained frontier that can be solved under varying circumstances from the problem formulation (2) (or from (6), as the case may be).[1] Figure 1 shows how the location of the frontier depends upon the expected stochastic nature of inputs and outputs. The bottom curve in the diagram illustrates a case where there is little random variation in the data so that the joint probability distribution is well concentrated around its mean. As explained, the chance-constrained frontier is defined so that at least 95 % of all data points are expected to fall below it. With increasing variability of data (but keeping the mathematical expectations unchanged) the frontier is shifted successively upward.

/ Figure 1 About Here/

Figure 2 demonstrates how the frontier depends on the preset probability threshold level employed in the chance-constrained formulation. The top curve in the diagram is the same as the top curve in Figure 1. It is drawn with that threshold being equal to 0.95. What happens if the chance-constraint becomes less stringent so that, say, only 90 % or more of the data points are required to fall below the frontier(but now keeping the probability distribution itself unchanged)? The answer is that the frontier is then parametrically shifted downward. The diagram illustrates two alternatives, with the threshold level equal to 90 % and 80 %, respectively.

/ Figure 2 About Here/

3. Pareto-Koopmans Efficiency.

Already in the pioneering paper of DEA (Charnes, Cooper and Rhodes [12]), it was pointed out that the conditions for efficiency in the sense of the efficiency ratio measure are also the conditions for Pareto-Koopmans efficiency (*ibid.*, p.433). The formal developments were supplied a couple of years later, in Charnes, Cooper, Golany, Seiford, and Stutz [11].

In a similar manner we now proceed to show that the conditions for efficiency developed in Section 2 can identically be interpreted as chance-constrained Pareto-Koopmans efficiency.

To develop these matters *ab initio*, it is convenient first to define the concept of chance-constrained dominance. We shall say that the stochastic point (x,y) dominates a given test point (x_0,y_0) in the sense of chance-constrained programming (on the tolerance level of 95 %) if

$$(7) \qquad \begin{aligned} &\text{Prob } (y_n \geq y_{n0}) \geq 0.95 \ , \ n=1,\ldots,N \\ &\text{Prob } (x_m \leq x_{m0}) \leq 0.95 \ , \ n=1,\ldots,N \end{aligned}$$

with at least one of the certainty equivalence constraints being strict.

In order to clarify these concepts, turn to the diagrams Figure 3a and Figure 3b. Figure 3a shows x-space (the space of inputs), and Figure 3b shows y-space (the space of outputs). The deterministic test point (x_0,y_0) is marked in the diagrams. A stochastic point (x,y) will dominate (x_0,y_0) in the chance-constrained sense if the vector of inputs x lies "below" the vector x_0 most of the time so that the use of inputs is "smaller", and if the vector of outputs y lies "above" the vector y_0 most of the time so that the resulting outputs obtained are "greater".

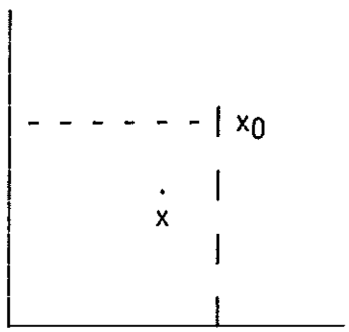


Figure 3a. Input space.

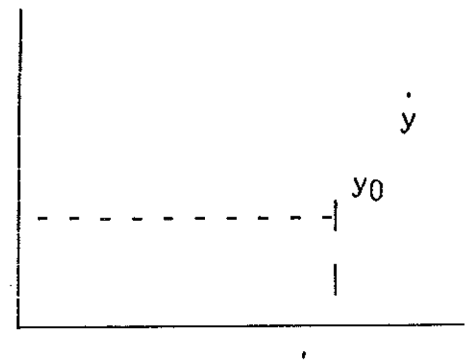


Figure 3b. Output space.

Thus prepared, we shall say that a given DMU under investigation (having outputs y_0 and inputs x_0), is Pareto-Koopmans efficient in the sense of chance-constrained programming (on the tolerance level 95 %) if and only if it is possible to determine a nonnegative vector λ of best practice so that the chance constraints

$$(8) \quad \begin{aligned} &\text{Prob } (Y^n\lambda \geq y_{n0}) \geq 0.95 \text{ , } n=1,\ldots,N \\ &\text{Prob } (X^m\lambda \leq x_{m0}) \geq 0.95 \text{ , } m=1,\ldots,M \end{aligned}$$

can be brought into tightness.

Converting into certainty equivalents, this is to require that the deterministic constraints

$$(9) \quad \begin{aligned} &E(Y^n\lambda) - 1.645 \text{ s.d.}(Y^n\lambda) \geq y_{n0} \text{ , } n=1,\ldots,N \\ &E(X^m\lambda) + 1.645 \text{ s.d.}(X^m\lambda) \leq x_{m0} \text{ , } m=1,\ldots,M \end{aligned}$$

can be brought into tightness.

Writing down the goal program ($\epsilon > 0$ is the same non-Archimedean used in program (6))

$$
(10)\quad \min -\epsilon(es^{+} + es^{-})
$$

$$
\begin{aligned}
\text{subject to } & E(Y^{n}\lambda) - 1.645 \text{ s.d.}(Y^{n}\lambda) - s^{+}_{n} = y_{n0}\ ,\ n=1,\ldots,N \\
& E(X^{m}\lambda) + 1.645 \text{ s.d.}(X^{m}\lambda) + s^{-}_{m} = x_{m0}\ ,\ m=1,\ldots,M \\
& \lambda, s^{+}, s^{-} \geq 0
\end{aligned}
$$

the DMU under investigation will be obviously be Pareto-Koopmans efficient if and and only if the goal program (10) has the optimal solution $s^{+*} = s^{-*} = 0$.

The main result is now that a DMU will be Pareto-Koopmans efficient in the sense of chance-constrained programming if and only if the DMU is chance-constrained DEA efficient. Compare first (10) with (6). Assume that the DMU under investigation is rated "efficient" in the sense that the optimal efficiency value θ to be solved from program (6) equals unity and that all enveloping constraints are tight. In other words, assume that program (6) has an optimal solution θ^{*}, λ^{*}, s^{+*}, s^{-*} with $\theta^{*} = 1$ and $s^{+*} = s^{-*} = 0$ and one sees that λ^{*}, $s^{+*} = s^{-*} = 0$ solves program (10). Hence, the DMU is Pareto-Koopmans efficient in the sense oif chance-constrained programming.

Conversely, assume that the DMU under investigation is Pareto-Koopmans efficient in the sense that it is possible to determine a nonnegative vector λ of best practice so that the constraints (8) are brought into tightness. Then program (10) has an optimal solution $\lambda^{\#}$, $s^{+\#}$, $s^{-\#}$ with $s^{+\#} = s^{-\#} = 0$ and one sees that $\theta = 1$, $\lambda^{\#}$, $s^{+\#} = s^{-\#} = 0$ solves program (6). Hence, the DMU is rated chance-constrained DEA efficient. Q.E.D.

4. Chance-Constrained Activity Analysis.

The concept of chance-constrained Pareto-Koopmans efficiency in its turn is related to chance-constrained activity analysis.

In order to bring out this relation, identify each DMU $i=1,\ldots,I$ with the "manager" of a separate activity. (Koopmans called the coordinator of the overall mathematical program the "helmsman", the administrator of each commodity a "custodian", and the administrator of each activity a "manager", see [18], p. 93 ff.) The sample input matrix X is interpreted as the matrix of unit input requirements of managers, i.e. x_{mi} is the amount of input m required to operated activity i at unit level. Similarly, the sample output matrix Y is interpreted as the matrix of unit outputs, with y_{ni} being the amount of output n obtained when activity is is operated at unit level. Further, the DMU loading factor λ_i of each activity i is interpreted as the "level" (or intensity) of operation of activity i. In the common manner of activity analysis, the vector product $X\lambda$ is then the vector of input requirements, and the vector $Y\lambda$ is the vector of outputs obtained.

The inputs $n=1,\ldots,N$ are now also referred to as "resources", and the outputs as "consumer goods" $m=1,\ldots,M$.

Also introduce the following additional notation

$p = [p_n]$	row vector of prices of consumer goods $n=1,\ldots,N$
$d = [d_n]$	column vector of demand for consumer goods $n=1,\ldots,N$
$q = [q_n]$	row vector of prices of resources $m=1,\ldots,M$
$w = [w_m]$	column vector of supplies of resources $m=1,\ldots,M$

The chance-constrained activity analysis model is (see Thore [30]):

(11) $\max \; pd - qw$

$$\text{subject to Prob } (d_n - Y^n\lambda \leq 0) \geq 0.95, \; n=1,\ldots,N$$

$$\text{Prob } (X^m\lambda - w_m \leq 0) \geq 0.95, \; m=1,\ldots,M$$

$$d,w,\lambda \geq 0$$

where the constraints have the certainty equivalents

(12) $d_n - EY^n\lambda + 1.645 \text{ s.d.}(Y^n\lambda) \leq 0, \; n=1,\ldots,N$

$$EX^m\lambda + 1.645 \text{ s.d.}(X^m\lambda) - w_m \leq 0, \; m=1,\ldots,M$$

The first set of constraints (12) states that if production is to be sufficient to cover actual consumer demand most of the time, as required by the chance-constraint, then expected output must be large enough to cover actual demand all of the time, plus a contingency term whose magnitude depends on technological output variability and managerial risk. The second set of constraints in (12) states that if the supply of a resource is to be sufficient to cover the input requirement most of the time, actual supply must be large enough to cover expected input all of the time, plus a contingency term depending upon technological input variability and managerial risk. Uncertainty about output coefficients leads management to hold inventories of finished goods. Uncertainty about input coefficients leads management to hoard resources and maintain excess capacity. Chance-constrained programming provides a way of formalizing these notions, and provides a solution technique for determining optimal activity levels and optimal stocks. See further Thore [30].

An efficient point of chance-constrained activity analysis is a solution point to program (11) with $p,q > 0$. More specifically, (d^*, w^*) is said to be efficient if and only if there exists $p,q > 0$ so that (d^*, w^*, λ^*) solves program (11).

It is immediately clear that this is the same as Pareto-Koopmans chance-constrained efficiency. For (d^*, w^*) is now the test point, and comparing the constraints of program (11) with (7) we see that $(X\lambda^*, Y\lambda^*)$ dominates (d^*, w^*) in the sense of chance-constrained programming. Conversely, given any test point (x_0, y_0) that happens to be Pareto-Koopmans chance-constrained efficient, the deterministic constraints (9) can then be brought to tightness. Hence, there exists some λ^* that together with $d^* = y_0$ and $w^* = x_0$ brings (12) into tightness. In other words, the point (d^*, w^*) solves the chance-constrained activity analysis model (11).

5. Chance-Constrained Frontiers and Parametric Stochastic Frontiers.

The chance-constrained efficiency measure is calculated through conversion of the original chance-constrained program to its certainty equivalent. In this sense it is still deterministic, just like best practice efficiency measurement is. No parameters are estimated in the process, and efficiency scores do not come with standard errors in parentheses. Nonetheless chance-constrained efficiency analysis takes on the appearance of, and much of the content of, a stochastic cost frontier model.

Consider first a conventional deterministic cost minimization problem

$$\text{(13)} \quad \min \; wX\lambda \quad \text{subject to} \quad Y\lambda \geq y_0, \quad \lambda \geq 0$$

where w is a row vector of input prices. The chance-constrained version of (13) reads on certainty-equivalent form (see Thore [30], eq. (2))

$$\text{(14)} \quad \min \; w(EX)\lambda \quad \text{subject to} \quad E(Y^n\lambda) - 1.645 \text{ s.d.}(Y^n\lambda) \geq y_0, \; n=1,\ldots,N, \quad \lambda \geq 0$$

The solution point lies on the chance-constrained Pareto-Koopmans efficiency frontier.

A generic parametric stochastic cost frontier model can be written as

$$\text{(15)} \quad wx = C(y,w) + T + u_c$$
$$x_m = D_m(y,w) + T_m + u_m \quad, m=1,\ldots,M$$

Here wx is total cost, $C(\ )$ and $D_m(\ )$ are specified functional expressions depending on one or several parameters, $T \geq 0$ is the cost of inefficiency, T_m is the amount of inefficiency, and (u_c, u_n) is a random disturbance vector. (The inefficiency term T may be broken up into a technical inefficiency component and an allocative inefficiency component.) If system (15) is expressed in natural logarithms of variables, input demands are replaced with input cost shares. Also, output equations can bew added to the system, although they rarely are. As Bauer [4] has noted, this system can be estimated in a variety of ways.

Note first the structural similarity between the two systems., Chance-constrained technology is described by an enveloping, or supporting, or extremal, best practice technology. Deviations from the frontier in both directions are permitted, but the frontier is determined so that the observations stay beneath it most of the time. Stochastic cost frontier technology, on the other hand, is described by a deterministic kernel plus a one-sided deviation above it plus a two-sided random deviation. The input demand appears as a deterministic kernel plus a two-sided deviation, plus a two-sided random deviation.

Consider also the information requirements of the two approaches. They both start out from given observations (X,Y,w). Chance-constrained efficiency measurement requires evaluator-supplied information on the joint probability distribution of all stochastic inputs and outputs. The parametric approach requires evaluator-supplied information of the functional forms C and D_m, and frequently for T, T_m, u_c, u_m as well, plus a linkage relationship between T and the T_m. Whether the requirements of the one approach are more or less onerous than of the other is an empirical matter worthy of examination.

Appendix.

This appendix states and proves a simple theorem for chance-constrained programming. Consider the chance-constrained program

$$\text{(A1)}\quad \min cx \quad \text{subject to } \mathrm{Prob}(A^i x \geq b_i) \geq 0.95,\ i=1,\ldots,m \quad x \geq 0$$

where x is a nx1 column vector of unknowns to be determined, A is a mxn matrix of random variables with a known and given joint probability distribution (for simplicity but without loss of generality assumed to be joint normal), and b = b(mx1) and c = c(1xn) are deterministic.

The certainty equivalent to (A1) is

$$\text{(A2)}\quad \min cx \quad \text{subject to } E(A^i x) - 1.645\ \mathrm{s.d.}(A^i x) \geq b_i,\ i=1,\ldots,m \quad x \geq 0$$

where

$$\text{(A3)}\quad (\mathrm{s.d.}(A^i x))^2 = \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} x_j x_k \,\mathrm{Cov}(a_{ij}, a_{ik}).$$

Define the vector u = u(1xm) of Lagrange multipliers of the constraints in (A2). Then

$$\text{(A4)}\quad cx^* = u^* b$$

(In the common manner, the asterisk denotes the value at an optimal point).

The desired result follows from application of the Kuhn-Tucker conditions to (A2). There are two sets of conditions that read

$$\text{(A5)}\quad \sum_{i=1}^{i=m} u_i^* \left(Ea_{ij} - 1.645 \frac{\sum_{k=1}^{k=n} x_k^* \operatorname{Cov}(a_{ij},a_{ik})}{\text{s.d.}(A^i x^*)} \right) \le c_j$$

$$x_j^* \sum_{i=1}^{i=m} u_i^* \left(Ea_{ij} - 1.645 \frac{\sum_{k=1}^{k=n} x_k^* \operatorname{Cov}(a_{ij},a_{ik})}{\text{s.d.}(A^i x^*)} \right) - c_j x_j^* = 0,$$

for all $j=1,\ldots,n$.

Using the last of these two sets of conditions, summing over $j=1,\ldots,n$ to form cx^*, and simplifying, the result (A4) follows.

Figure 1. Chance-Constrained Frontiers for Different Degrees of Expected Variability of Inputs and Outputs.

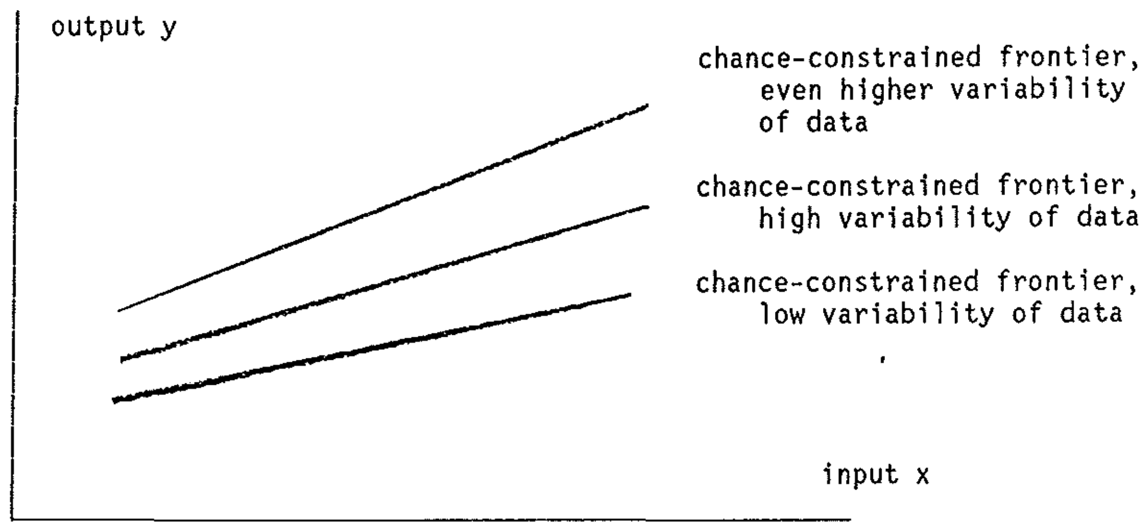
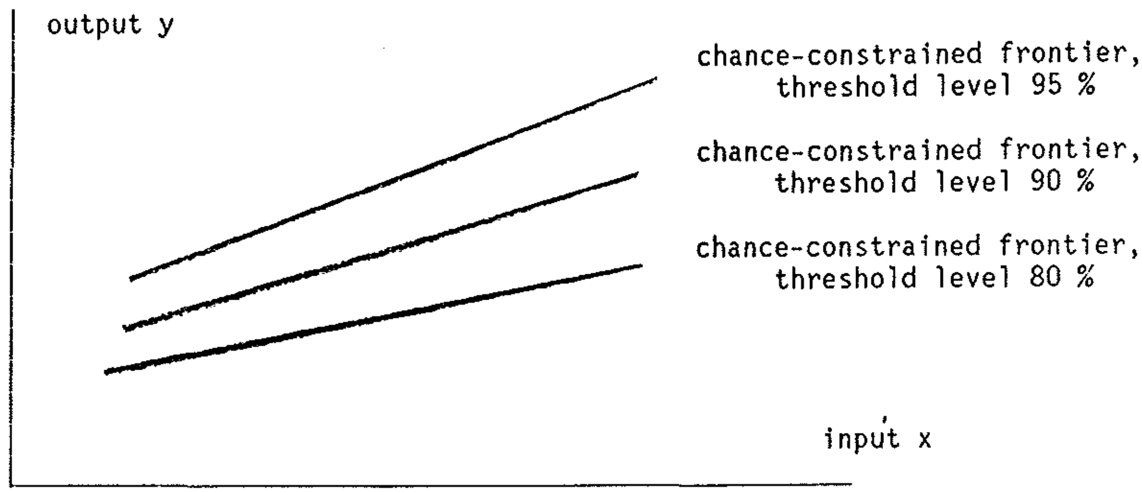


Figure 2. Chance-Constrained Frontiers at Varying Probabilistic Threshold Levels.



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