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Tropical Theta Functions and Log Calabi-Yau Surfaces

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Tropical Theta Functions and Log Calabi-Yau Surfaces

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DISSERTATION

Presented to the Faculty of the Graduate School of
The University of Texas at Austin
in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN
May 2014

Dedicated to my wife Katy and our daughter Nora.

Acknowledgments

I would like to thank my advisor, Sean Keel, for introducing me to this topic and providing suggestions, encouragement, and insights that guided me through graduate school. I also want to thank Mark Gross, David Helm, Andy Neitzke, and James Pascaleff for helpful conversations and instruction. I am also grateful towards my fellow graduate students, especially Yuan Yao and Yuecheng Zhu, from whom I learned a great deal. Finally, I want to thank my family, particularly my wife, Katy, for their love and support throughout our time in Austin.

Tropical Theta Functions and Log Calabi-Yau Surfaces

Publication No. __________________

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The University of Texas at Austin, 2014

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We describe combinatorial techniques for studying log Calabi-Yau surfaces. These can be viewed as generalizing the techniques for studying toric varieties in terms of their character and cocharacter lattices. These lattices are replaced by certain integral linear manifolds described in [GHK11], and monomials on toric varieties are replaced with the canonical theta functions defined in [GHK11] using ideas from mirror symmetry. We classify deformation classes of log Calabi-Yau surfaces in terms of the geometry of these integral linear manifolds. We then describe the tropicalizations of theta functions and use them to generalize the dual pairing between the character and cocharacter lattices. We use this to describe generalizations of dual cones, Newton and polar polytopes, Minkowski sums, and finite Fourier series expansions. We hope that these techniques will generalize to higher rank cluster varieties.

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Chapter 1

Introduction

The running theme of this thesis is that log Calabi-Yau surfaces (or in another language, fibers of rank 2 cluster $\mathcal{X}$-varieties) are a reasonably mild generalization of toric surfaces, so one can hope to better undertand them by applying techniques from toric geometry. Toric varieties are of course understood by studying their character and cocharacter lattices, denoted M and N, respectively. [GHK11] generalizes the cocharacter lattice by defining the tropicalization U^{trop} of a log Calabi-Yau surface U. They then use toric degenerations, modified by scattering diagrams, to construct a mirror family $\mathcal{V}$ of log Calabi-Yau surfaces, with the integer points of U^{trop} serving as a generalization of the character lattice for $\mathcal{V}$. That is, the global sections of the family $\mathcal{V}$ admit a canonical module-basis of "theta functions," parametrized by the integer points $U^{\text{trop}}(\mathbb{Z}) \subset U^{\text{trop}}$, which generalize monomials on toric varieties. In this thesis, we carefully examine the structure of U^{trop} and its relationship to U and $\mathcal{V}$ in order to better understand the log Calabi-Yau surface.

1.0.1 Some Main Results

As mentioned, a point $q \in U^{\text{trop}}(\mathbb{Z})$ corresponds to a boundary divisor D_q for some compactification of U, and also to a canonical theta function ϑ_q on the mirror $\mathcal{V}$. Let V be a generic fiber of the mirror ([GHK] shows that V is deformation equivalent to U).[1] We similarly have that $v \in V^{\mathrm{trop}}(\mathbb{Z})$ corresponds to a boundary divisor D_v of certain compactifications of V, and also to a theta function ϑ_v on the mirror $\mathcal{U}$ to V. We may view U as a fiber of $\mathcal{U}$. For f a regular funciton on U and $q \in U^{\mathrm{trop}}(\mathbb{Z})$, we define $f^{\mathrm{trop}}(q) := \mathrm{val}_{D_q}(f)$. Similarly with V. Define $\langle q, v\rangle := \vartheta_q^{\mathrm{trop}}(v)$, and similarly, $\langle q, v\rangle^{\vee} := \vartheta_v^{\mathrm{trop}}(q) := \mathrm{val}_{D_q}(\vartheta_v)$. These pairings generalize the dual pairing between M and N in the toric situation.

Theorem 1.0.1 (3.2.14)**.** $\langle \cdot, \cdot\rangle$ *and* $\langle \cdot, \cdot\rangle^{\vee}$ *are equivalent.*

Generalizations of this have been conjectured in [FG09] (their Conjecture 4.3, part 3) and [GHKK].

We define *tropical* functions on U^{trop} and V^{trop} to be the integral, piecewise-linear functions which are "convex along broken lines" (we show this is equivalent to [FG09]'s notion of "convex with respect to every seed" in the language of cluster varieties). The tropical functions form a min-plus algebra, and we call a tropical function φ *indecomposable* if it cannot be written as a minimum of two other tropical functions, neither of which is φ. The tropical functions generalize convex integral piecewise-linear functions on $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, and the indecomposable functions generalize the linear functions. [GHKK] conjectures that tropicalizations of regular functions are tropical for *any* log Calabi-Yau variety, and [FG09] conjectures that the theta functions (not their tropicalizations) satisfy a related indecomposability condition (now known to be false in general). For the log Calabi-Yau surface cases, we show:

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[1] We assume throughout this subsection that U is "positive," as defined in §1.0.2, although Remark 3.2.25 explains how to extend Theorem 1.0.2 to the non-positive cases.

Theorem 1.0.2 (3.2.24)**.** *The tropical functions are exactly the tropicalizations of regular functions, and the indecomposable tropical functions are exactly the tropicalizations of theta functions.*

We also generalize the notion of *Newton polytopes* by defining $\mathrm{Newt}(\sum_{q\in S}\vartheta_q)$ to be the "strong" convex hull of S in U^{trop}. We generalize notions of dual polytopes and show that their properties and relationships to the log Calabi-Yau surfaces are similar to in the toric case. We also define the Minkowski sum $\mathrm{Newt}(f)+\mathrm{Newt}(g)$ as $\mathrm{Newt}(fg)$. U^{trop} contains a singular point that prevents addition from being defined as easily as in the toric case. However, U^{trop} is covered by convex cones, and addition does of course make sense when restricting to these cones.

Theorem 1.0.3 (3.3.27)**.** *If the Minkowski sum of a collection $Q_1,\ldots,Q_s$ of polytopes contains the origin, then it can be computed by taking the convex hull of the union over all convex cones σ of all sums of s-tuples $q_i\in Q_i\cap\sigma$, $i=1,\ldots,s$.*

In fact, only finitely many convex cones and s-tuples are needed, so Minkowski sums really are computable. We view this as a tropicalized version of [GHK11]'s formula for multiplying theta functions.

1.0.2 Setup

Throughout this paper, Y will denote a smooth, projective, rational surface over an algebraically closed field $\Bbbk$ of characteristic 0. The *boundary* D will denote a choice of nodal anti-canonical divisor in Y, and U will denote $Y\setminus D$. Here, $D=D_1+\ldots D_n$ is a either a cycle of smooth irreducible rational curves D_i with normal crossings, or if $n = 1$, D is an irreducible curve with one node. By a *compactification* of U, we mean such a pair (Y, D) ([GHK] describes these as compactifications with "maximal boundary"). We call (Y, D) a *Looijenga pair*, as in [GHK11], and we call U a *log Calabi-Yau surface* or a *Looijenga interior*.

For a Looijenga pair (Y, D), we define a *toric blowup* to be a Looijenga pair $(\widetilde{Y}, \widetilde{D})$ together with a birational map $\widetilde{Y} \to Y$ which is a blowup at a nodal point of the boundary D, such that $\widetilde{D}$ is the preimage of D. Note that taking a toric blowup does not change the interior $U = Y \setminus D = \widetilde{Y} \setminus \widetilde{D}$. We also use the term toric blowup to refer to finite sequences of such blowups.

By a *non-toric blowup* $(\widetilde{Y}, \widetilde{D}) \to (Y, D)$, we will always mean a blowup $\widetilde{Y} \to Y$ at a non-nodal point of the boundary D such that $\widetilde{D}$ is the proper transform of D. Let $(\overline{Y}, \overline{D})$ be a Looijenga pair where $\overline{Y}$ is a toric variety and $\overline{D}$ is the toric boundary. We say that a birational map $Y \to \overline{Y}$ is a *toric model* of (Y, D) (or of U) if it is a finite sequence of non-toric blowups. Every Looijenga pair has a toric blowup which admits a toric model ([GHK11], Prop. 1.19).

According to [GHK], all deformations of U come from sliding the non-toric blowup points along the divisors $\overline{D}_i \subset D$ without ever moving them to the nodes of D. We call U *positive* if some deformation of U is affine. This is equivalent to saying that D supports an effective D-ample divisor, meaning a divisor whose intersection with each component of D is positive. We will always take the term D-ample to imply effective, unless otherwise stated. See §2.3.3 for equivalent characterizations of U being positive. We will assume that U is positive throughout Chapter 3.

1.0.3 Outline of the Paper

Cluster Varieties: §2.1 summarizes the relationship that [GHK13a] describes between Looijenga pairs and [FG09]'s cluster varieties. Briefly, [GHK13a] explains how to view cluster varieties as certain blowups of toric varieties. As already mentioned, log Calabi-Yau surfaces can also be constructed by blowing up toric varieties. As shown in §5 of [GHK13a], every log Calabi-Yau surface appears (up to codimension 2) as a symplectic leaf in what [FG09] calls a cluster $\mathcal{X}$-variety.

The Tropicalization of U: In §2.2, we review [GHK11]'s construction of the tropicalization of U, an integral linear manifold denoted U^{trop}. The integer points $U^{\text{trop}}(\mathbb{Z}) \subset U^{\text{trop}}$ generalize the cocharacter lattice N for toric varieties. If $q \in U^{\text{trop}}(\mathbb{Z})$ is primitive (i.e., nonzero and not a positive integral multiple of some other element of $U^{\text{trop}}(\mathbb{Z})$), then it corresponds to an irreducible divisor D_q in the boundary of some compactification of U. If q is a multiple $|q| \in \mathbb{Z}_{\geq 0}$ times a primitive element, then the corresponding divisor is $|q|D_q$. We call $|q|$ the *index* of q.

U^{trop} is homeomorphic to $\mathbb{R}^2$, and the integral linear structure that captures the intersection data of the boundary divisors. This structure is singular at a point $0 \in U^{\text{trop}}$, and we examine the monodromy around this point. We then analyze the integral piecewise-linear functions on U^{trop} using the intersection theory on compactifications of U: an integral piecewise-linear functions φ on U^{trop} corresponds to a Weil divisor $W_\varphi := \sum \varphi(v_i) D_{v_i}$ on a compactification $(Y, D = \sum D_{v_i})$ of U, and the "bending parameter" of φ across ρ_{v_i} is the intersection number $W_\varphi \cdot D_{v_i}$. Let $\beta_{v_1,\ldots,v_s}$ denote a function which has bending parameter $|v_i|$ along the ray ρ_{v_i} generated by v_i, for each i, and otherwise has no other bends. As a consequence of the symmetry of the intersection product, we find:

Proposition 1.0.4 (2.2.10)**.** *If the intersection matrix* $H = (D_i \cdot D_j)_{ij}$ *for some compactification of* U *is invertible, then* β_v *is uniquely defined for each* v*, and* $\beta_v(w) = \beta_w(v)$ *for all* $v, w \in U^{\mathrm{trop}}(\mathbb{Q})$*.*

The symmetry in Theorem 1.0.1 may be viewed as a consequence of this when both sides are negative (see Remark 3.2.15), but the proof we give actually follows a different approach. We also give a local coordinate description of the functions $\beta_{v_1,\ldots,v_s}$ which is very useful when proving Theorem 1.0.3. At the end of §2.2, we give the definitions of lines and polygons in U^{trop}, as introduced in [GHK].

Classification: §2.3 offers several equivalent classifications of log Calabi-Yau surfaces up to deformation, with characterizations based on the intersection data of D, the regular functions on U, the geometry of U^{trop} (including the monodromy and properties of lines), the intersection form on the lattice $D^{\perp} \subset A_1(Y, \mathbb{Z})$ of curve classes which do not intersect any component of D, and the properties of a seed S for the cluster variety containing U as a fiber.

For example, endpoint that U corresponds to an acyclic cluster variety (i.e., the quiver corresponding to some seed has no oriented cycles) if and only if some straight lines in U^{trop} do not wrap all the way around the origin. The cases where no lines wrap are fibers of "finite-type" cluster varieties, meaning that the underlying graphs of the corresponding quivers are simply-laced Dynkin diagrams. We show that the (inverse) monodromy of U^{trop} in these finite-type cases are Kodaira's monodromy matrices I_n,

II, III, and IV, from his classification of singular fibers in elliptic surfaces. Similarly, the non-acyclic positive cases correspond to Kodaira's matrices I_n^*, II^*, III^*, and IV^*.

Constructing the Mirror and the Theta Functions: In §3.1 we review [GHK11]'s construction of the mirror family $\mathcal{V}$ of U. The theta functions ϑ_q, $q \in U^{\mathrm{trop}}(\mathbb{Z})$, are defined in terms of broken lines, which are certain piecewise-straight lines in U^{trop} with attached monomials. At the end of §3.1, we review [GHK]'s construction of compactifications of $\mathcal{V}$.

Theta Functions and their Tropicalizations: In §3.2, we explicitely describe the tropicalizations of theta functions, as defined above in §1.0.1, and we investigate some of their properties. We begin by describing a way to identify U^{trop} with V^{trop} for computational purposes (analogous to using the standard inner product to identify $N_{\mathbb{R}}$ with $M_{\mathbb{R}}$ in the toric situation). We find an explicit description of $\langle \cdot, \cdot \rangle$ in §3.2.3 and §3.2.4. For example, as investigated in §3.2.6.1, tropical theta functions which are negative everywhere bend along at most a single ray. On the other hand, each seed from the cluster structure induces a different integral linear structure on U^{trop}, and the tropical theta functions which are positive somewhere are linear with respect to some seed.

In §3.2.5, we use these explicit descriptions to conclude Theorem 1.0.1. §3.2.7 introduces the tropical functions mentioned above in §1.0.1. Convexity along a broken line locally means convexity with respect to a linear structure in which the broken line is straight. Tropical functions are defined to be convex along all broken lines, and we show that this is equivalent to being convex with respect to the linear structure induced by each seed. We then prove Theorem 1.0.2 and make several conjectures about how this might generalize to higher dimensional cluster varieties and fibers of cluster varieties.

Toric Constructions for Log Calabi-Yau's:

In §3.3 we use the pairing $\langle \cdot, \cdot \rangle$ to generalize several constructions from toric geometry. §3.3.1 focuses on constructions involving polytopes. For example, we define the *strong convex hull* of a set $Q \subset U^{\mathrm{trop}}$ as

$$\mathbf{Conv}(Q) = \left\{ x \in U^{\mathrm{trop}} | \langle x, v \rangle \geq \inf_{q \in Q} \langle q, v \rangle \text{ for all } v \in V^{\mathrm{trop}} \right\}.$$

We call a polytope strongly convex if it equals its own strong convex hull. Such polytopes and their Minkowski sums also appear in the literature on cluster varieties (cf. [FG11] and [She12]). We show:

Theorem 1.0.5 (3.3.8)**.** *A rational polytope Q is strongly convex if and only if any broken line segment with endpoints in Q is entirely contained in Q.*

Consider a regular function $f := \sum_{q \in Q} a_q \vartheta_q$, $Q \subset U^{\mathrm{trop}}(\mathbb{Z})$, $a_q \neq 0$. The *Newton polytope* of f is defined to be $\mathbf{Conv}(Q)$. On the other hand, a Weil divisor W on a compactification of V corresponds to a piecewise-linear function φ_W on V^{trop}, hence to a polytope $\Delta_W := \{\varphi_W \leq 1\}$ in V^{trop}. $\Delta_W^{\vee} \subset U^{\mathrm{trop}}$ is then defined to be the Newton polytope of a generic section of $\mathcal{O}(W)$, and this agrees with the polar polytope

$$\Delta_W^{\circ} := \{ q \in U^{\mathrm{trop}} | \langle q, v \rangle \geq -1 \text{ for all } v \in \Delta_W \}$$

if W is effective. Note that the theta functions corresponding to integer points in $\Delta_W^{\vee}$ form a canonical basis of global sections for $\mathcal{O}(W)$. This relationship was previously examined in [GHK] for W strictly effective (i.e., for $\varphi_W \geq 0$, or for $\Delta_W^\vee$ containing the origin in its interior).

Other properties of polytopes from the toric situation now easily generalize. For example, we find exactly as in the toric situation that the number of lattice points on edges of $\Delta_W^\vee$ is related to certain intersection numbers of W with boundary divisors (see Proposition 3.3.16).

In §3.3.2 we note that the notion of *dual cones* also generalizes from toric varieties in a very straightforeward way: the dual to a cone $\sigma \subset V^{\mathrm{trop}}$ is the cone $\sigma^\vee := \{q \in U^{\mathrm{trop}} | \langle q, v\rangle \geq 0 \text{ for all } v \in V^{\mathrm{trop}}(\mathbb{Z})\}$. If $\sigma^\vee$ is two-dimensional, then Spec of the ring generated by the ϑ_q's with $q \in \sigma^\vee$ is an affine open subset of a compactification of V (see Corollary 3.3.21). This is analogous to the usual construction of toric varieties from fans, as seen in [Ful93].

In §3.3.3 we introduce the Minkowski sums mentioned above in §1.0.1. We prove Theorem 1.0.3, along with a closely related tropical multiplication formula along the way:

Theorem 1.0.6 (3.3.25)**.** *Assume we are not in one of the I_k ($k \neq 0$) cases of §2.3.4.1. Let $q_1, \ldots, q_s \in U^{\mathrm{trop}}(\mathbb{Z})$ be cyclically ordered, and let $+_i$ denote addition on the complement of the cone σ_i bounded by q_{i-1} and q_i. Suppose $\left(\prod_{i=1}^k \vartheta_{q_i}\right)^{\mathrm{trop}}(u) < 0$ for all $u \in \sigma_i$. Then*

$$\left(\prod_{i=1}^k \vartheta_{q_i}\right)^{\mathrm{trop}}\Bigg|_{\sigma_i} = \vartheta^{\mathrm{trop}}_{q_1 +_i \cdots +_i q_n}\big|_{\sigma_i}.$$

Consequently, if $\left(\prod_{i=1}^k \vartheta_{q_i}\right)^{\mathrm{trop}} \leq 0$ *everywhere and is* 0 *along at most a single ray, then*

$$\left(\prod_{i=1}^k \vartheta_{q_i}\right)^{\mathrm{trop}} = \left(\sum_{i=1}^n \vartheta_{q_1+_i\cdots+_iq_n}\right)^{\mathrm{trop}}.$$

The strategy of the proof is to apply §2.2.4.2's local coordinate description of the piecewise-linear functions $\beta_{q_1,\ldots,q_s}$ to the descriptions of tropical theta functions in terms of bending parameters given in §3.2.6.1.

Integral Formulas: In §3.4, we consider integrals of the form

$$Tr_q(f) := \int_\gamma f\vartheta_q^{-1}\Omega,$$

where γ is a certain canonical homology class in U defined in [GHK] (the class of a conjectural SYZ fiber), and Ω is a holomorphic volume form on U with simple poles along D, normalized so that $\int_\gamma \Omega = 1$. The Tr_0 case was examined in [GHK], where they showed that $Tr_0(\vartheta_r) = \delta_{0,r}$. It was suggested by V.V. Fock, based on examples he had computed, that one more generally has $Tr_q(\vartheta_r) : \delta_{q,r}$. S. Keel explained this for some cases, but found that it fails in general. In §3.4 I give the following general collection of conditions in which this relationship does hold:

Theorem 1.0.7. *Let* $f = \sum_q c_q\vartheta_q$ *be a function on* V*. Suppose that at least one of the following holds:*

- r *is not in the convex hull of any point* $q \in \mathrm{Newt}(f) \cap U^{\mathrm{trop}}(\mathbb{Z})$*, except possibly* $q = r$*. In particular, this includes cases where* r *is a vertex of* $\mathrm{Newt}(f)$*, as well as cases where* r *is in the complement of* $\mathrm{Newt}(f)$*.*

- *$r \in U^{\mathrm{trop}}(\mathbb{Z})$ is in the cluster complex (i.e., $r = 0$ or $\langle r, v\rangle > 0$ for some v).*

Then $c_r = Tr_r(f)$. In particular, if every point of $\mathrm{Newt}(f) \cap U^{\mathrm{trop}}(\mathbb{Z})$ *which is not a vertex is in the cluster complex, then*

$$f = \sum_{r \in U^{\mathrm{trop}}(\mathbb{Z})} Tr_r(f)\vartheta_r. \tag{1.1}$$

The proof for the first condition is based on the residue theorem and the relationship between convex hulls and the zeroes and poles of theta functions. The proof for the second condition follows from reducing to the toric case.

We think of Equation 3.7 as a generalization of the formula for Fourier series expansions. Indeed, in the case that V is a toric variety, applying this theorem to monomials and restricting to the orbits of the torus action recovers the usual formula for (finite) Fourier expansions.

Chapter 2

Classification of Rank 2 Cluster Varieties

2.1 Cluster Varieties as Blowups of Toric Varieties

In [FG09], Fock and Goncharov construct spaces called cluster varieties by gluing together algebraic tori via certain birational transformations called mutations. [GHK13a] interprets these mutations from the viewpoint of birational geometry, and thereby relates the log Calabi-Yau surfaces of [GHK11] to cluster varieties. This section will summarize some of the main ideas from [GHK13a].

2.1.1 Defining Cluster Varieties

The following construction is due to Fock and Goncharov [FG09].

Definition 2.1.1. A *seed* is a collection of data

$$S = (N, I, E := \{e_i\}_{i\in I}, F, \langle\cdot,\cdot\rangle, \{d_i\}_{i\in I}),$$

where N is a finitely generated free Abelian group, I is a finite set, E is a basis of N indexed by I, F is a subset of I, $\langle\cdot,\cdot\rangle$ is a skew-symmetric $\mathbb{Q}$-valued bilinear form, and the d_i's are positive rational numbers called *multipliers*. We call e_i a *frozen* vector if $i \in F$. The *rank* of a seed or of a cluster variety will mean the rank of $\langle\cdot,\cdot\rangle$.

We define another bilinear form on N by

$$(e_i, e_j) := \epsilon_{ij} := d_j\langle e_i, e_j\rangle,$$

and we require that $\epsilon_{ij} \in \mathbb{Z}$ for all[1] $i, j \in I$. Let $M = N^*$. Define

$$p^* : N \to M, \quad v \mapsto (v, \cdot).$$

Let $K = \ker(p^*)$, and $\overline{N} = \operatorname{im}(p^*) \subseteq M$. Note that $K = \ker[v \mapsto \langle v, \cdot \rangle]$. For each $i \in I$, define d_i' (the *modified multipliers*) by saying that $p^*(e_i)$ is d_i' times a primitive vector in M.

Remark 2.1.2. Given only the matrix (e_i, e_j) and the set F, we can recover the rest of the data, up to a rescaling of $\langle \cdot, \cdot \rangle$ and a corresponding rescaling of the d_i's. This rescaling does not affect the constructions below, and it is common take the scaling out of the picture by assuming that the d_i's are relatively prime integers (although we do not make this assumption). Also, notice that $\langle \cdot, \cdot \rangle$ and $\{d_i'\}$ together determine $\{d_i\}$, so when describing a seed we may at times give $\{d_i'\}$ instead of $\{d_i\}$.

Given a seed S as above and a choice of non-frozen vector $e_j \in E$, we can use a *mutation* to define a new seed $\mu_j(S) := (N, I, E' = \{e_i'\}_{i \in I}, F, \langle \cdot, \cdot \rangle, \{d_i\})$, where the (e_i')'s are defined by

$$e_i' = \mu_j(e_i) := \begin{cases} e_i + \epsilon_{ij} e_j & \text{if } \epsilon_{ij} > 0 \\ -e_i & \text{if } i = j \\ e_i & \text{otherwise.} \end{cases} \tag{2.1}$$

Mutation with respect to frozen vectors is not allowed.

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[1] The construction of cluster varieties does not depend on the values of $\langle e_i, e_j \rangle$ or ϵ_{ij} for $i, j \in F$, and so it is common to not include these coefficients in the data. When they are included in the data, as in [FG09] and [GHK13a], they are not typically requried to be integers. However, as [GHK13a] points out, if these are not integers, then the image of p^* is not contained in M. [GHK13a] takes a slightly different fix to this (in which the ϵ_{ij} with $i, j \in F$ are again irrelevant), but it is essentially equivalent to our fix if we dropped the requirement that $\langle e_i, e_j \rangle = -\langle e_j, e_i \rangle$ when $i, j \in F$.

Given a lattice L and some $v \in L^*$, we will denote by z^v the corresponding monomial on $T_L := L \otimes \Bbbk^* = \operatorname{Spec} \Bbbk[L^*]$ (or more precisely, the max-Spec of $\Bbbk[L^*]$). Corresponding to a seed S, we can define a so-called seed $\mathcal{X}$-torus $X_S := T_M = \operatorname{Spec} \Bbbk[N]$, and a seed $\mathcal{A}$-torus $A_S := T_N = \operatorname{Spec} \Bbbk[M]$. We define *cluster monomials* $X_i := z^{e_i} \in \Bbbk[N]$ and $A_i := z^{e_i^*} \in \Bbbk[M]$, where $\{e_i^*\}_{i \in I}$ is the dual basis to E.

Remark 2.1.3. We are departing somewhat from a common convention. In place of M, other authors often use the superlattice $(M)^\circ \subset M \otimes \mathbb{Q}$ spanned over $\mathbb{Z}$ by vectors $f_i := d_i^{-1} e_i^*$. They then take $A_i := (z^{f_i}) \in \Bbbk[M^\circ]$. It seems to this author that this significantly complicates the exposition and the formulas that follow, with little benefit, and so we do not follow this convention.

For any $j \in I$, we have a birational morphism $\mu_j^{\mathcal{X}} : \mathcal{X}_S \rightarrow \mathcal{X}_{\mu_j(S)}$ (called a cluster $\mathcal{X}$-mutation) defined by

$$(\mu_j^{\mathcal{X}})^* X_i' = X_i \left(1 + X_j^{\operatorname{sign}(-\epsilon_{ij})}\right)^{-\epsilon_{ij}} \quad \text{for } i \neq j; \qquad (\mu_j^{\mathcal{X}})^* X_j' = X_j^{-1}.$$

Similarly, we can define a cluster $\mathcal{A}$-mutation $\mu_j^{\mathcal{A}} : \mathcal{A}_S \rightarrow \mathcal{A}_{\mu_j(S)}$,

$$A_j (\mu_j^{\mathcal{A}})^* A_j' = \prod_{i : \epsilon_{ji} > 0} A_i^{\epsilon_{ji}} + \prod_{i : \epsilon_{ji} < 0} A_i^{-\epsilon_{ji}}; \qquad (\mu_j^{\mathcal{A}})^* A_i' = A_i \quad \text{for} \quad i \neq j.$$

Now, the cluster $\mathcal{X}$-variety $\mathcal{X}$ is defined by using compositions of $\mathcal{X}$-mutations to glue $\mathcal{X}_{S'}$ to $\mathcal{X}_S$ for every seed S' which is related to S by some sequence of mutations. Similarly for the cluster $\mathcal{A}$-variety $\mathcal{A}$, with $\mathcal{A}$-tori and $\mathcal{A}$-mutations. The *cluster algebra* is the subalgebra of $\Bbbk[M]$ generated by the the cluster variables A_i of every seed that we can get to by some sequence of mutations. In this context, the well-known Laurent phenomenon simply says[2] that all the cluster variables are regular functions on $\mathcal{A}$. The ring of all global regular functions on $\mathcal{A}$ is called the *upper cluster algebra*.

On the other hand, the X_i's do not always extend to global functions on $\mathcal{X}$. When a monomial on a seed torus (i.e., a monomial in the X_i's for a fixed seed) does extend to a global function on $\mathcal{X}$, we call it a *global monomial*, as in [GHK13a].

2.1.1.1 Quivers and Seeds

For future reference, we mention a standard way to represent the data of a seed with the data of a (decorated) quiver. Each seed vector e_i corresponds to a vertex v_i of the quiver. The number of arrows from v_i to v_j is equal to $\langle e_i, e_j \rangle$, with a negative sign meaning that the arrows actually go from v_j to v_i. Each vertex v_i is decorated with the number d_i. Furthermore, the vertices corresponding to frozen vectors are boxed. Observe that all the data of the seed can be recovered from the quiver.

Now, a seed is called *acyclic* if the corresponding quiver contains no directed paths that do not pass through any frozen (boxed) vertices. A cluster variety is called acyclic if any of the corresponding seeds are acyclic. It is easy to see that a seed S is acyclic if and only if there is some closed half-plane in $\overline{N}$ which contains $p^*(e_i)$ for every $i \in I \setminus F$.

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[2] [GHK13a] uses this observation to give a geometric proof of the Laurent phenomenon

2.1.2 The Geometric Interpretation

As in [GHK13a], for a lattice L with dual L^* and with $u \in L$, $\psi \in L^*$, define

$$\begin{aligned} m_{u,\psi,L} &: T_L \dashrightarrow T_L \\ m^*_{u,\psi,L}(z^{\varphi}) &= z^{\varphi}(1+z^{\psi})^{\varphi(u)} \quad \text{for } \varphi \in L^*. \end{aligned}$$

One can check that the mutations above satisfy

$$(\mu_j^{\mathcal{X}})^* = m^*_{-(\cdot,e_j),e_j,M} : \quad z^v \mapsto z^v(1+z^{e_j})^{-(v,e_j)} \tag{2.2}$$

$$(\mu_j^{\mathcal{A}})^* = m^*_{-e_j,(e_j,\cdot),N} : \quad z^{\gamma} \mapsto z^{\gamma}(1+z^{(e_j,\cdot)})^{-\gamma(e_j)}.$$

The following Lemma from [GHK13a] is what leads to the nice geometric interpretations of mutations and cluster varieties.

Lemma 2.1.4 ([GHK])**.** *Suppose that u is primitive in a lattice L. Let Σ be a fan in L with rays corresponding to u and $-u$. Recall that the toric variety $TV(\Sigma)$ admits a $\mathbb{P}^1$ fibration π with D_u and D_{-u} as sections, corresponding to the projection $L \to L/\mathbb{Z}\langle u\rangle$.*

The mutation $\mu_{u,\psi,L}$ is the birational map on $T_L \subset TV(\Sigma)$ coming from blowing up the "hypertorus"

$$H^- := \{1 + z^{\psi} = 0\} \cap D_{-u}$$

and then contracting the proper transforms of the fibers F of π which intersect this hypertorus. Furthermore, $\mu_j^{\mathcal{X}}$ (and under certain conditions, $\mu_j^{\mathcal{A}}$) preserve the centers of the blowups corresponding to $\mu_i^{\mathcal{X}}$ (and, respectively, $\mu_i^{\mathcal{A}}$) for each $i \neq j$.

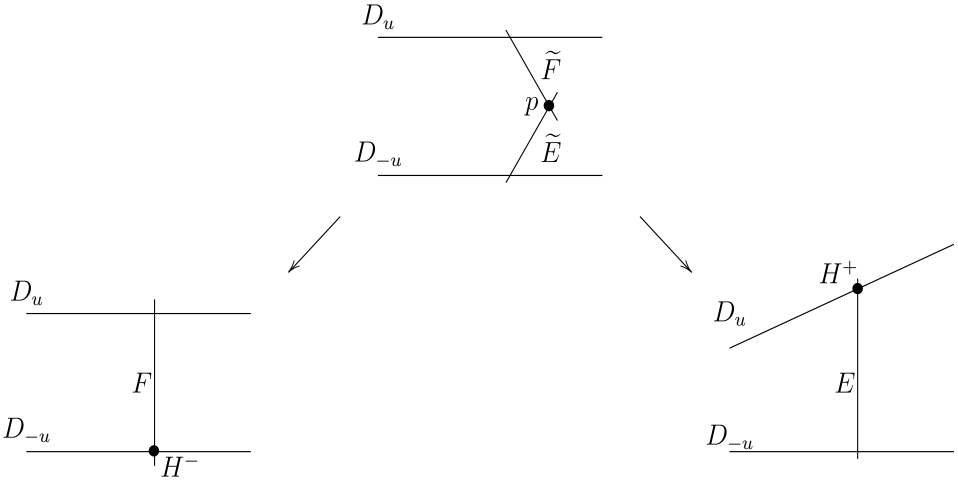

Figure 2.1: A mutation involves blowing up a hyportorus H^- in D_{-u} (left arrow) and then contracting the proper transform $\widetilde{F}$ of the fibers F which hit H_- (right arrow), down to a hypertorus H^+ in D_u. $\widetilde{E}$ denotes the exceptional divisor, with E being its image after the contraction of $\widetilde{F}$. The locus $p = \widetilde{E} \cap \widetilde{F}$ has codimension 2 and does not appear in the cluster variety.

Thus, a cluster $\mathcal{X}$-mutation $(\mu_j^{\mathcal{X}})^*$ corresponds to blowing up $\{\mathcal{X}_j = -1\} \cap D_{(\cdot, e_j)}$, followed by blowing down some fibers of a certain $\mathbb{P}^1$ fibration, and repeating d_j' times (since $(e_j, \cdot)$ is d_j' times a primitive vector). The new seed torus is only different from the old one in that it is missing the blown-down fiber of the initial $\mathbb{P}^1$ fibration, but has gained the exceptional divisor from the final blowup (except for the lower-dimensional set of points where this exceptional divisor intersects a blown-down fiber, represented by p in Figure 2.1).

Since the centers of the blowups corresponding to the other mutations have not changed, this shows that the cluster $\mathcal{X}$-variety can be constructed (up to codimension 2) as follows: For any seed S, take a fan in M with rays generated by $\pm(\cdot, e_i)$ for each i, and consider the corresponding toric variety. For each $i \in I \setminus F$, blow up the hypertorus $\{X_i = -1\} \cap D_{(\cdot,e_i)}$ d'_i times, and then remove the first $(d'_i - 1)$ exceptional divisors. The cluster $\mathcal{X}$ variety is then the complement of the proper transform of the toric boundary.

Remark 2.1.5. In this construction of $\mathcal{X}$, the centers for the hypertori we blow up may intersect if $(\cdot, e_i) = (\cdot, e_j)$ for some $i \neq j$, so some care must be taken regarding the ordering of the blowups. Fortunately, this issue only matters in codimension at least 2. See [GHK13a] for more details.

2.1.3 The Cluster Exact Sequence

Observe that for each seed there is an exact sequence

$$0 \to K \to N \xrightarrow{p^*} M \to M/p^*(N) \to 0.$$

This induces an exact sequence

$$1 \to \mathcal{H}'_{\mathcal{A}} \to \mathcal{A}_S \xrightarrow{p} \mathcal{X}_S \to \mathcal{H}_{\mathcal{X}} \to 1,$$

where $\mathcal{H}'_{\mathcal{A}} := \operatorname{Hom}(M/p^*(N), \Bbbk^*)$, and $\mathcal{H}_{\mathcal{X}} := \operatorname{Hom}(K, \Bbbk^*) = T_{K^*}$.

As observed in [FG09], the above exact sequence commutes with mutations.[3] We thus obtain the exact sequence

$$1 \to \mathcal{H}'_{\mathcal{A}} \to \mathcal{A} \xrightarrow{p} \mathcal{X} \xrightarrow{\lambda} \mathcal{H}_{\mathcal{X}} \to 1. \tag{2.3}$$

Let $\mathcal{U} := p(\mathcal{A}) \subset \mathcal{X}$. The sequence $1 \to \mathcal{H}'_{\mathcal{A}} \to \mathcal{A} \to \mathcal{U} \to 1$ should be viewed as a generalization of the construction of toric varieties as quotients, with $\mathcal{U}$ being the

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[3] This is one thing that does become easier to see with the conventions mentioned in Remark 2.1.3.

generalization of the toric variety.[4] In fact, Section 4 of [GHK13a] shows that the ring of global sections of $\mathcal{A}$ is (under certain conditions) the Cox ring of $\mathcal{U}$.

§5 of [GHK13a] shows that Looijenga interiors (i.e., log Calabi-Yau surfaces), as defined in §1.0.2, are exactly the surfaces (up to codimension 2 and contractible complete subvarieties) which arise as fibers of λ for rank 2 cluster varieties. We will explain this relationship now.

2.1.4 Looijenga Interiors

Let U be a Looijenga interior. Recall that U admits a compactification Y with boundary $D := Y \setminus U$, and we can choose this compactification to be one which admits a toric model $\pi : (Y, D) \rightarrow (\overline{Y}, \overline{D})$. Let $\overline{N}$ be the cocharacter lattice of $\overline{Y}$ (this will actually correspond to the saturation in M of what we called $\overline{N}$ before). Choose an orientation on $\overline{N}$ and let $(\cdot \wedge \cdot) : \overline{N}^2 \rightarrow \mathbb{Z}$ denote the corresponding standard wedge form. Take a set $\overline{E} := \{\overline{e_1}, \ldots, \overline{e_m}\} \subset \overline{N}$ of vectors generating $\overline{N}$ as a $\mathbb{Z}$-module, and a set $F \subset I := \{1, \ldots, m\}$, such that if $\{\overline{e_{i_k}}\}_{k=1,\ldots,s}$ are the vectors on a ray ρ corresponding to some boundary divisor $D_\rho \subset \overline{D}$, then $\sum_{i_k \notin F} |\overline{e_{i_k}}|$ is the number of non-toric blowups taken on D_ρ by π (recall that the *index* $|v|$ of a vector v is the positive integer such that v is $|v|$ times a primitive vector).

Now, let S be the seed with N freely generated by a set $E = \{e_1, \ldots, e_m\}$, $\langle e_i, e_j \rangle := \overline{e_i} \wedge \overline{e_j}$, I and F as above, and $d_i' := |\overline{e_i}|$. Recall from Remark 2.1.2 that

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[4] This sequence actually generalizes the construction for toric varieties without boundary (i.e., just algebraic tori). However, one may allow for boundary components by allowing compactifications of $\mathcal{A}$ and $\mathcal{U}$.

this data determines $\{d_i\}$. Using S to construct $\mathcal{A}$ and $\mathcal{X}$, the interpretation of $\mathcal{X}$-mutations from §2.1.2 reveals that the fibers of λ are, up to codimension 2 and a finite collection of interior (-2)-curves, deformation equivalent to U—the deformations just correspond to different choices of which points are blown up on each D_i. We note that these changes in codimension 2 and the removal of the complete subvarieties are unimportant to us, since these things do not affect global sections.

Example 2.1.6. Consider the case where Y is a cubic surface, obtained by blowing up 2 points on each boundary divisor of $(\overline{Y} \cong \mathbb{P}^2, \overline{D} = D_1 + D_2 + D_3)$. We can take

$$\overline{E} = \{(1,0),(1,0),(0,1),(0,1),(-1,-1),(-1,-1)\},$$

with each $d_i = d_i' = 1$ and F empty. Then the fibers of $\mathcal{X}$ correspond to the different possible choices of blowup points on the D_i's, up to automorphism. The fiber $\mathcal{U}$ is very special, having four (-2)-curves. If we instead take $\overline{E} = \{(1,0),(0,1),(-1,-1)\}$ with $\langle \cdot,\cdot \rangle$ given by $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, and each $d_i = d_i' = 2$, then the fibers of the resulting $\mathcal{X}$ include only the surfaces constructed by blowing up the same point twice on each D_i and then removing the three resulting (-2)-curves, up to automorphism. $\mathcal{U}$ is the fiber where the blowup points are colinear and so there is one remaining (-2)-curve.

The deformation type of the fibers of $\mathcal{X}$ has only changed by the removal of certain (-2)-curves. Thus, the deformation type of the ring of global sections of the fibers has not changed!

The above example demonstrates that we can often change the number of vectors in a seed without changing the deformation type of the affinizations.[5] More precisely, for a seed $\{N = \mathbb{Z}\langle E\rangle, I, E = \{e_1, \ldots, e_m\}, F, \langle\cdot,\cdot\rangle, \{d_i\}\}$, and a collection of partitions $d'_i = d'_{i,1} + \ldots + d'_{i,b_i}$, $d'_{i,j} \in \mathbb{Z}_{\geq 0}$, we can define a new seed S' as follows: Let $E' : \{e_{i,j}\}$, $i = 1, \ldots, m$, $j = 1, \ldots, b_i$, and $N' := \mathbb{Z}\langle E'\rangle$. Define $\langle e_{i_1,j_1}, e_{i_2,j_2}\rangle' := \langle e_{i_1}, e_{i_2}\rangle$. We say the pair $(i,j) \in F'$ if $i \in F$. Finally, $d'_{i,j}$ is as in the partitions, and this determines $\{d_{i,j}\}$. The corresponding space $\mathcal{X}'$ then contains $\mathcal{X}$ as a subfamily (up to codimension 2 and the removal of some contractible complete subvarieties), and the affinizations of the fibers of the two spaces are in the same deformation class.

Remark 2.1.7. We actually have more freedom with the frozen basis vectors, because we can change their multipliers without affecting the cluster varieties at all. Furthermore, we can actually remove frozen basis vectors without affecting the deformation type of the affinizations of the fibers, so long as this removal does not change $p^*(\mathbb{Z}\langle E\rangle)$.

Definition 2.1.8. For a seed S, if $i \neq j$ implies $(e_i, \cdot) \neq (e_j, \cdot)$, we call S *minimal* (this means that each d'_i, $i \notin F$, is the total number of non-toric blowups taken on the divisor corresponding to e_i). On the other hand, if each $d'_i = 1$, we will call S *maximal.* S_1 and S_2 will be called *equivalent* if the affinizations of the fibers of the corresponding $\mathcal{X}$-varieties $\mathcal{X}_1$ and $\mathcal{X}_2$ are of the same deformation type.

Note that every seed S is canonically equivalent to a minimal seed and to a maximal seed (up to changing the skew form and multipliers for frozen vectors).

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[5] The affinization of a scheme is defined to be Spec of its ring of global sections.

Example 2.1.9. The first seed for the cubic surface in Example 2.1.6 is maximal, while the second seed is minimal.

2.1.4.1 The Canonical Intersection Form

For S a rank 2 seed with each $d_i' = 1$, [GHK13a] describes a canonical way to identify K with $D^{\perp} := \{C \in A_1(Y, \mathbb{Z}) | C \cdot D_i = 0 \ \forall \ i\}$, thus inducing a canonical symmetric bilinear form on K. This identification of K with $D^{\perp}$ is as follows: an element v of K corresponds to a relation of the form $\sum a_i \overline{e_i} = 0$. Standard toric geometry says that this determines a unique curve class C_v in $\pi^*[A_1(\bar{Y})]$ such that $C_v \cdot D_i = \sum a_j$ for each i, where the sum is over all j such that $D_{(e_j, \cdot)} = D_i$. So we can canonically define an isomorphism $\iota : K \cong D^{\perp}$ by

$$v \mapsto C_v - \sum_i a_i E_i.$$

Finally, for $v_1, v_2 \in K$, define $Q(v_1, v_2) = \iota(v_1) \cdot \iota(v_2)$. We will see in §2.3 that $D^{\perp}$ together with this intersection pairing tells us quite a bit about the deformation type of U. In particular, [GHK13a] tells us that U is positive if and only if Q is negative definite.

Recall that varying the fiber of $\mathcal{X}$ corresponds to changing the choices of non-toric blowup points on D. For some choices of blowup points, certain classes C in $D^{\perp}$ may be represented by effective curves (e.g., this happens when we blowup the points where a representative of C intersects the boundary, with the number of blowups being at least the intersection multiplicity). Let $D^{\perp}_{\mathrm{Eff}} \subseteq D^{\perp}$ be the sublattice generated by the curve classes which are represented by an effective curve on some fiber.

Example 2.1.10. For the seed from Example 2.1.6, K is generated by $\{e_2 - e_1, e_4 - e_3, e_6 - e_5, e_1 + e_3 + e_5\}$. The corresponding curves in $D^\perp$ are $\{E_1 - E_2, E_3 - E_4, E_5 - E_6, L - E_1 - E_3 - E_5\}$, where E_i is the exceptional divisor of the blowup corresponding to e_i, and L is a generic line in $\overline{Y} \cong \mathbb{P}^2$. Using $E_i \cdot E_j = -\delta_{ij}$, $L \cdot L = 1$, and $L \cdot E_i = 0$ for each i, one easily checks that this lattice has type D_4. On the special fiber $\mathcal{U}$, these four curve classes are effective, so $D^\perp_{\mathrm{Eff}} = D^\perp$.

2.1.5 Tropicalizations of Cluster Varieties

[FG09] describes *tropicalizations* $\mathcal{A}^{\mathrm{trop}}$ and $\mathcal{X}^{\mathrm{trop}}$ of the spaces $\mathcal{A}$ and $\mathcal{X}$, respectively. Given a seed S, $\mathcal{A}^{\mathrm{trop}}$ can be canonically identified as an integral piecewise-linear manifold with $N_{\mathbb{R}}$, and the integer points $\mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$ of the tropicalization are identified with N. For a different seed $\mu_j(S)$, the identification is related by the integral piecewise-linear function $\overline{\mu_j} : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$, where we use the overline to indicate that e_j is mapped by the same piecewise-linear function as the other vectors, rather than getting a special treatment. Similarly for $\mathcal{X}^{\mathrm{trop}}$ and $\mathcal{X}^{\mathrm{trop}}(\mathbb{Z})$ using $M_{\mathbb{R}}$, M, and the dual seed mutations.

Our interest in this paper is primarily with the fibers U of λ in the cases where U is two-dimensional, so we will spend the next section analyzing U^{trop}. This may be canonically identified with $p^*(\mathcal{A}^{\mathrm{trop}}) \subset \mathcal{X}^{\mathrm{trop}}$. However, we will study U^{trop} primarily from the perspective of [GHK11], where it is seen to have a canonical integral linear structure which is closely related to the geometry of the compactifications (Y, D). We will briefly relate this to the cluster variety perspective in §3.1.5.

2.1.6 Dual Canonical Bases

The Fock-Goncharov dual basis conjectures from [FG09] predict that the points of $\mathcal{A}^{\mathrm{trop}}(\mathbb{Z})$ parameterize a basis of global functions on $\mathcal{X}$, and similarly, the points of $\mathcal{X}^{\mathrm{trop}}$ should parametrize global functions on $\mathcal{A}$.[6] [GHK13a] uses the above geometric interpretations of $\mathcal{X}$ and $\mathcal{A}$ to show that the conjecture as stated in [FG09] cannot hold in general because $\mathcal{X}$ may have too few global functions, and $\Gamma(\mathcal{A}, \mathcal{O})_{\mathcal{A}})$ may fail to be finitely generated. Still, [GHKK] proves a formal version of the Fock-Goncharov conjecture and examines the extent to which the formal version can be used to obtain the original prediction from [FG09]. This involves understanding functions which are "tropical" in the sense of our §3.2.7.

The construction in [GHK11] proves an analogue of the Fock-Goncharov conjecture relating the points of $U^{\mathrm{trop}}(\mathbb{Z})$ to canonical theta functions on a family $\mathcal{V}$ mirror to (Y, D). In general, this mirror is only formally defined, but if U is positive, it can be extended to an affine variety. Furthermore, this affine variety has (the affinizations of) deformations of U as fibers, so one may view this as saying that points in $p(\mathcal{A}^{\mathrm{trop}}(\mathbb{Z})) = U^{\mathrm{trop}}(\mathbb{Z})$ parametrize functions on fibers of $\mathcal{X}$, or in the other direction, points of $U^{\mathrm{trop}}(\mathbb{Z}) \subset \mathcal{X}^{\mathrm{trop}}(\mathbb{Z})$ parametrize functions on the quotient $p(\mathcal{A})$. Thus, this construction is a simplified version of the situation from the full Fock-Goncharov conjecture.

Conjectures 4.1, 4.2, and 4.3 of [FG09] predict not only the existence of the dual bases, but also several properties which they should satisfy. Much of this paper will

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[6] If $(\cdot,\cdot) \neq \langle\cdot,\cdot\rangle$, then we should actually use the Langland's dual spaces $\mathcal{X}^\vee$ and $\mathcal{A}^\vee$, respectively.

deal with proving analogues of these conjectures for two-dimensional U^{trop} and U. We hope that future work will generalize the methods here to understand the full, higher dimensional conjectures.

2.2 The Tropicalization of U

This section examines U^{trop} with its integral linear structure defined in [GHK11]. U^{trop} is a natural generalization of the cocharacter space $N_{\mathbb{R}}$ corresponding to a toric variety, and the relationship between U^{trop} and the mirror is a natural generalization of the character space $M_{\mathbb{R}}$.

2.2.1 Some Generalities on Integral Linear Stuctures

A manifold B is said to be *(oriented) integral linear* if it admits charts to $\mathbb{R}^n$ which have transition maps in $\mathrm{SL}_n(\mathbb{Z})$. We allow B to have a set O of singular points of codimension at least 2, meaning that these integral linear charts only cover $B' := B \setminus O$. Our space of interest, $B = U^{\text{trop}}$, will be homeomorphic to $\mathbb{R}^2$ and will typically have a singular point at 0.

B' admits a flat affine connection, defined using the charts to pull back the standard flat connection on $\mathbb{R}^n$. Furthermore, pulling back along these charts give a local system Λ of integral tangent vectors on B', along with a dual local system Λ^* in the cotangent bundle. Note that the monodromy μ of Λ is contained in $SL_n(\mathbb{Z})$, so the wedge form on any exterior product $\Lambda^{\bullet} TB$ commutes with parallel transport.

Call $\sigma \subset B'$ affine if it is connected and contained in the domain of some chart for the integral linear structure (e.g., σ might be identified with a cone in $\mathbb{R}^n$).

Note that a chart with σ in its domain induces an embedding of σ into T_pB' for any $p \in \sigma$, commuting with parallel transport in σ. When we talk about addition, scalar multiplication, or wedge products of points on σ, we will mean the operations induced by this identification with the tangent space, if well-defined. Because of the monodromy, these operations do depend on the choice of an affine σ, but not on the specific choice of map. B' also has designated *integral points* which come from using the charts to pull back $\mathbb{Z}^n \subset \mathbb{R}^n$, or alternatively, from lifting Λ to B' via the above identifications. These points are defined globally, independent of choices of charts.

By an *integral linear map* of integral linear manifolds, we mean a map which is linear in each chart and which maps integral points to integral points.

2.2.1.1 Integral Linear Functions

Let P^{gp} be a finite-rank lattice and $P^{gp}_{\mathbb{R}} := P^{gp} \otimes_{\mathbb{Z}} \mathbb{R}$. We say an function from $\mathbb{R}^n$ to $P^{gp}_{\mathbb{R}}$ is integral linear if it is linear as a map of $\mathbb{R}$-vector spaces and has integral slope, meaning it takes integral points to P^{gp}. On an integral linear manifold B, we can define a sheaf $\mathcal{L}_{P^{gp}}$ of integral linear functions on B' by saying that a function $f : V \to \mathbb{R}$ is integral linear if and only if, for each coordinate chart $\psi_U : U \to \mathbb{R}^n$, $f|_{V\cap U} = f_U \circ \psi_U|_{V\cap U}$ for some function f_U which is integral linear on $\mathbb{R}^n$. We similarly define a sheaf $\mathcal{PL}_{P^{gp}}$ of integral piecewise-linear functions. These definitions extend to all of B by requiring that the functions be continuous at the singular points.

We note that to specify an integral linear structure on an integral piecewise-linear manifold (i.e., a manifold where transition functions are piecewise-linear), it suffices to identify which piecewise-linear functions are actually linear. These functions can then be used to construct charts. It therefore also suffices to specify which piecewise-straight lines are straight, since (piecewise-)straight lines form the fibers of (piecewise-)linear functions. We will use this to define other linear structures on U^{trop} in §3.

2.2.2 Constructing U^{trop}

Notation 2.2.1. Given a toric model $(Y, D) \to (\overline{Y}, \overline{D})$, let N be the cocharacter lattice corresponding to $(\overline{Y}, \overline{D})$, and let $\Sigma \subset N_{\mathbb{R}}$ be the corresponding fan. Σ has cyclically ordered rays ρ_i, $i = 1, \ldots, n$, with primitive generators v_i, corresponding to boundary divisors $\overline{D_i} \subset \overline{D}$ and $D_i \subset D$. We choose an orientation[7] of $N_{\mathbb{R}}$ so that ρ_{i+1} is counterclockwise of ρ_i. Let $\sigma_{u,v}$ denote the closed cone bounded by two vectors u, v, with u being the clockwise-most boundary ray. In particular, if u and v lie on the same ray, we define $\sigma_{u,v}$ to be just that ray. Denote $\sigma_{i,i+1} := \sigma_{v_i, v_{i+1}}$. We may use variations of this notation, such as v_ρ for a primitive generator of some arbitrary ray ρ with rational slope, but these variations should be clear from context.

We now use (Y, D) to define an integral linear manifold U^{trop}. As a topoogical manifold, U^{trop} is the same as $N_{\mathbb{R}}$, and as smooth manifolds, $U_0^{\mathrm{trop}} := U^{\mathrm{trop}} \setminus \{0\}$ is the same as $N_{\mathbb{R}} \setminus \{0\}$. Note that an integral Σ-piecewise-linear (i.e., bending only on rays of Σ) function φ on U^{trop} can be identified with a Weil divisor of Y via $W_\varphi := a_1 D_1 + \ldots + a_n D_n$, where $a_i = \varphi(v_i) \in \mathbb{Z}$. We define the integer linear structure of U^{trop}

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[7] Choosing a cyclic ordering for the components of D (assuming D has at least three components) is equivalent to choosing an orientation for $N_{\mathbb{R}}$ or U^{trop}. It is also equivalent to fixing the sign for the holomorphic volume form Ω on U, which we will use in §3.4. We assume throughout the paper that such a choice has been fixed.

by saying that a function φ on the interior of $\sigma_{i-1,i} \cup \sigma_{i,i+1}$[8] is linear if it is Σ-piecewise linear and $W_{\varphi} \cdot D_i = 0$. This last condition is equivalent to

$$a_{i-1} + D_i^2 a_i + a_{i+1} = 0. \tag{2.4}$$

The set $U^{\mathrm{trop}}(\mathbb{Z})$ is equal to the set N as a subset of U^{trop}.

Remark 2.2.2. This construction of U^{trop} naturally generalizes to higher dimensions, but the two-dimensional case is special in that the linear structure on U^{trop} is canonically determined by (Y, D) (it does not depend on the choice of toric model). This is evident from the following atlas for U^{trop} (from [GHK11]): the chart on $\sigma_{i-1,i} \cup \sigma_{i,i+1}$ takes v_{i-1} to $(1,0)$, v_i to $(0,1)$, and v_{i+1} to $(-1, -D_i^2)$, and is linear in between.

Furthermore, toric blowups and blowdowns do not affect the integral linear structure, so as the notation suggests, U^{trop} and $U^{\mathrm{trop}}(\mathbb{Z})$ depend only on the interior U.

Example 2.2.3. If (Y, D) is toric, then U^{trop} is just $N_{\mathbb{R}}$ with its usual integral linear structure. This follows from the standard fact from toric geometry that $\sum_i (C \cdot D_i) v_i = 0$ for any curve class C. Taking non-toric blowups changes the intersection numbers, resulting in a non-trivial monodromy about the origin.

Remark 2.2.4. Recall from standard toric geometry that any primitive vector $v \in N$ corresponds to a prime divisor D_v supported on the boundary of some toric blowup of

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[8] We assume here that there are more than 3 rays in Σ, so that $\sigma_{i-1,i} \cup \sigma_{i,i+1}$ is not all of $N_{\mathbb{R}}$. This assumption can always be achieved by taking toric blowups of (Y, D). Alternatively, it is easy to avoid this assumption, but the notation and exposition becomes more complicated. We will therefore continue to implicitely assume that there are enough rays for whatever we are trying to do, without further comment.

$(\overline{Y}, \overline{D})$, and a general vector kv with $k \in \mathbb{Z}_{\geq 0}$ and v primitive corresponds to the divisor kD_v. Two divisors on different toric blowups are identified if they determine the same discrete valuation on the funciton field of $\overline{Y}$ (equivalently, if there is some common toric blowup on which their proper transforms are the same). Since taking proper transforms under the toric model gives a bijection between boundary components of (Y, D) and boundary components of $(\overline{Y}, \overline{D})$ (and similarly for the boundary components of toric blowups), we see that points of $U_0^{\text{trop}}(\mathbb{Z})$ correspond to the divisorial discrete valuations of (Y, D) along which Ω has a pole. 0 of course corresponds to the trivial valuation. Here, Ω is the canonical (up to scaling) holomorphic volume form on U with a simple pole along D, and *divisorial* means the valuation corresponds to a divisor on some toric blowup of (Y, D).

2.2.3 The Developing Map

We now describe a tool from [GHK11] that we use for doing explicit computations on U^{trop}. Consider the universal cover $\xi : \widetilde{U}_0^{\text{trop}} \to U_0^{\text{trop}}$. Note that $\widetilde{U}_0^{\text{trop}}$ also has a canonical integer linear structure pulled back from U_0^{trop}. The integer points are $\widetilde{U}_0^{\text{trop}}(\mathbb{Z}) := \xi^{-1}[U_0^{\text{trop}}(\mathbb{Z})]$. Furthermore, a ray $\rho \in U_0^{\text{trop}}$ pulls back to a family of rays ρ^j, $j \in \mathbb{Z}$, projecting to ρ (we arbitrarily choose a ray in $\widetilde{U}_0^{\text{trop}}$ to be ρ_0 and then assign the other indices so that they increase as we go counterclockwise). Note that wedge products on $\widetilde{U}_0^{\text{trop}}$ are well-defined, and sums of points are well-defined whenever the points share a convex cone.

Suppose that $v \in \rho_0$ and $v' \in \rho_0'$ are primitive vectors in $\widetilde{U}_0^{\text{trop}}$ spanning the integer points of $\sigma_{v,v'}$. Then there is a unique linear map $\delta_{\rho,\rho'} : \widetilde{U}_0^{\text{trop}} \to \mathbb{R}^2 \setminus \{0\}$ such that $\delta_{\rho,\rho'}(v) = (1,0)$ and $\delta_{\rho,\rho'}(v') = (0,1)$. We call this the *developing map* with respect to ρ and ρ'. We will often leave off the subscripts if they are not relavent, or we will write δ_ρ if only the image ρ of the first ray is relavent. δ is an integral linear immersion, and $\delta(\widetilde{U}_0^{\mathrm{trop}}(\mathbb{Z})) \subseteq \mathbb{Z}^2 \setminus \{(0,0)\}$.

Note that for any ray $\rho \subset U_0^{\mathrm{trop}}$, we have that $\xi^{-1}(U_0^{\mathrm{trop}} \setminus \rho)$ is a collection of (not necessarily convex) open cones $\sigma^\circ_{\rho^j,\rho^{j+1}}$ bounded by ρ^j and ρ^{j+1}, and $\xi|_{\sigma^\circ_{\rho^j,\rho^{j+1}}}$ is an isomorphism onto $U^{\mathrm{trop}} \setminus \rho$. We will use δ_ρ^i to denote δ_ρ restricted to $\sigma^\circ_{\rho^i,\rho^{i+1}}$. Thus, for each $i \in \mathbb{Z}$, $\delta^i_\rho \circ \xi^{-1}|_{U^{\mathrm{trop}}\setminus\rho}$ is an integral linear chart. In particular, δ^i_ρ induces via pullback a definition for addition and wedge products on $U^{\mathrm{trop}} \setminus \rho$, and the $SL_2(\mathbb{Z})$-invariance of these operations means that they do not depend on i.

Thus, wedge products are well-defined on the complement of any ray, and similarly on any subcone. Addition is well-defined on any convex cone.[9] Positive scalar multiplication is of course well-defined globally. If we write $u \wedge v$ without specifying on which affine open set $\wedge$ is defined, then we mean the form defined on $\sigma_{u,v}$.

Example 2.2.5. Consider the cubic surface (as in Example 2.1.6) constructed by taking two non-toric blowups on each of the three boundary divisors D_1, D_2, and D_3 of $\mathbb{P}^2$. The intersection matrix $H := (D_i \cdot D_j)$ is

$$H = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

---

[9] We will sometimes add vectors in a cone which is not convex. This is fine if we view the sum as living in the tangent spaces of points in the cone. In §3.3.3, we talk about a set of sums of points in cones which may not be convex, in which case we mean the set of sums which are well-defined in U^{trop}.

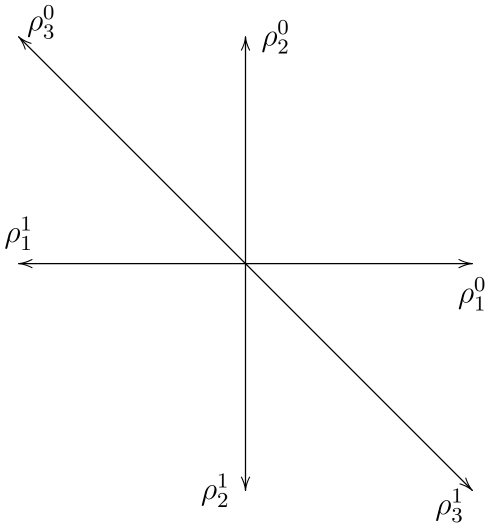

Figure 2.2: Cubic surface developing map. We let ρ_i^j denote $\delta^j_{\rho_{D_1},\rho_{D_2}}(\rho_{D_i})$.

and Equation 2.4 (or the construction from charts) implies that $\delta^0_{\rho_{D_1},\rho_{D_2}}(v_3) = (-1, 1)$, and $\delta^j(v) = (-1)^j\delta^0(v)$. See Figure 2.2.

Example 2.2.6. Consider $(\overline{\mathcal{M}_{0,5}}, D = D_1 + \ldots + D_5)$ constructed from the toric surface $(\mathbb{P}^2, \overline{D} = \overline{D_1} + \overline{D_2} + \overline{D_4})$ by making toric blowups at $D_1 \cap D_4$ and $D_2 \cap D_4$, as well as one non-toric blowup on each of $\overline{D_1}$ and $\overline{D_2}$. We then have five boundary components, each with self-intersection -1. A developing map takes the rays of the fan to $(1, 0), (0, 1), (-1, 1), (-1, 0)$, and $(0, -1)$, respectively, and then restarts with $(1, -1)$ and $(1, 0)$. See Figure 2.3.

2.2.3.1 Monodromy About the Origin

We now consider what happens when we parallel transport a tangent vector v in T_pU^{trop} counterclockwise around the origin. We use the embedding of a cone in the

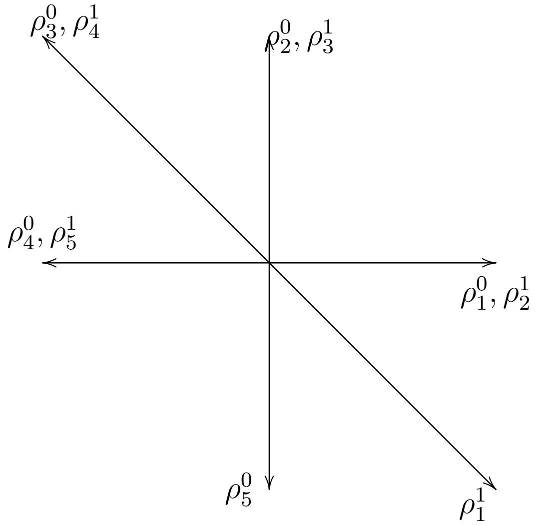

Figure 2.3: $\overline{\mathcal{M}}_{0,5}$ developing map, with ρ_i^j labelled for $j = 0, 1$.

tangent spaces of its points (which are all identified via parallel transport in the cone), and we use the notation $\delta^i := \delta^i_{\rho_{D_1}, \rho_{D_2}}$.

Example 2.2.7. Suppose $Y \to \overline{Y}$ consists of a single non-toric blowup on, say, D_1. Then $\delta^0(v_1) = \delta^1(v_1) = (1, 0)$. However, $\delta^0(v_2) = (0, 1)$ while $\delta^1(v_2) = (1, 1)$. We can view parallel transporting counterclockwise around the origin as parallel transporting up one sheet on the developing map, and then the monodromy tells us how to write the transported vector in terms of $\delta^1(v_1)$ and $\delta^1(v_2)$. Thus, the monodromy is

$$\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Similarly, the monodromy is in general given by $\mu = \left(\delta^1(v_1) \;\; \delta^1(v_2)\right)^{-1}$ with respect to the basis and developing map $\{\delta^0(v_1) = (1, 0), \delta^0(v_2) = (0, 1)\}$. We therefore use μ^{-k} to denote the map $\widetilde{U}_0^{\mathrm{trop}} \to \widetilde{U}_0^{\mathrm{trop}}$ which lifts vectors up k sheets. Note that the monodromy determines U^{trop} as an integral linear manifold: U^{trop} is the quotient of $\widetilde{U}_0^{\mathrm{trop}}$ by this $\mathbb{Z}$-action.

μ and μ^{-1} can always be factored into a product of unipotent matrices as follows: choose a toric model in which k_i non-toric blowups are taken on the divisor D_{v_i}, for $v_1, \ldots, v_s \in N$ cyclically ordered. Then we have the factorization

$$\mu^{-1} = \mu_{v_s}^{-k_s} \cdots \mu_{v_1}^{-k_1}, \tag{2.5}$$

where $\mu_{v_i}^{-k_i}$ is given in an oriented unimodular basis (v_i, v_i') by the matrix $\begin{pmatrix} 1 & k_i \\ 0 & 1 \end{pmatrix}$. More generally, in a basis where $v_i = (a, b)$, the corresponding contribution to μ^{-1} is

$$\mu_{(a,b)}^{-k_i} := \begin{pmatrix} 1 - k_i ab & k_i a^2 \\ -k_i b^2 & 1 + k_i ab \end{pmatrix}. \tag{2.6}$$

Now μ can of course be expressed as $\mu_{v_1}^{k_1} \cdots \mu_{v_s}^{k_s}$. Alternatively (following from the fact that $A\mu_v A^{-1} = \mu_{Av}$), the monodromy matrix is given by the product $\mu = (\mu'_{v_s})^{k_s} \cdots (\mu'_{v_1})^{k_1}$ of matrices of the form

$$(\mu'_{v_i})^{k_i} := \mu_{(a_i,b_i)}^{k_i} = \begin{pmatrix} 1 + k_i a_i b_i & -k_i a_i^2 \\ k_i b_i^2 & 1 - k_i a_i b_i \end{pmatrix}, \tag{2.7}$$

where $(a_1, b_1) := v_1$, and for $i > 1$, $(a_i, b_i) := (\mu'_{v_{i-1}})^{k_{i-1}} \cdots (\mu'_{v_1})^{k_1} v_i$. This can be interpreted by saying that before we can apply the monodromy contribution corresponding to v_i, we have to let the modifications we have made so far act on v_i.

Example 2.2.8. In Example 2.2.5, we have $\delta^1(v_1) = (-1, 0)$ and $\delta^1(v_2) = (0, -1)$, so we thus see that the monodromy for the cubic surface is $-\mathrm{Id}$.

Example 2.2.9. Similarly, for Example 2.2.6 we have $\delta^1(v_1) = (1,-1)$ and $\delta^1(v_2) = (1,0)$, so the monodromy is

$$\mu = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

with respect to the basis $\{\delta^0(v_1) = (1,0), \delta^0(v_2) = (0,1)\}$.

Note that U^{trop} is uniquely determined (as an integral linear manifold, up to isomorphism) by its monodromy, and that a factorization of the monodromy into unipotent elements with cyclically ordered *eigenrays* as above corresponds to a toric model for a Looijenga pair (and hence to a seed as in §2.1.4). By eigenray, we mean an eigenline with a chosen direction.

2.2.3.2 Mutations and Monodromy

We now describe the monodromy of U^{trop} directly in terms of seed data. Use $\mu_{i,S}$ to indicate that we are mutating a seed S with respect to a vector e_i. We consider the induced map on $\overline{N}$ (as in §2.1.4), which we denote by $\overline{\mu}_{i,S}$. This is not hard to describe—it is given by Equation 2.1, with the e_i's replaced by $\overline{e_i}$'s, and $(\cdot,\cdot)$ replaced by the induced non-degenerate bilinear form on $\overline{N}$. Assume that the $\overline{e_i}$'s are positively ordered with respect to the orientation induced by $\langle\cdot,\cdot\rangle$.

Now we observe that, in the notation of Equation 2.6, $\overline{\mu}_{i,S}^2 = \mu_{\overline{e_i}}^{-d'_i}$. Thus, the inverse monodromy μ^{-1} of U^{trop} is $\mu^{-1} = \prod \overline{\mu}_{i,S}^2$, where the product is taken over all i, with the $\overline{e_i}$'s being ordered counterclockwise as we move from right to left in the product. Note that the e_i's in this formula are not affected by the previous mutations!

Alternatively, by Equation 2.7, we have $\mu = \overline{\mu}_{n,S^n}^{-2} \circ \overline{\mu}_{n-1,S^{n-1}}^{-2} \circ \cdots \circ \overline{\mu}_{1,S^1}^{-2}$, where $S^1 := S$, and $S^k := \mu_{k-1,S^{n-1}}^{-2}(S^{k-1})$. That is, we apply the inverse mutation twice with respect to one vector, then twice with respect to the next vector in the new seed, and so on.

Now, using the above composition of mutations to compute the monodromy of U^{trop}, we can apply §2.3 to determine whether or not a cluster variety is positive: If $\mathrm{Tr}(\mu) > 2$, then we are in a negative definite case. If μ is $SL_2(\mathbb{Z})$-conjugate to a matrix of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a > 0$, then we are in a strictly negative semi-definite case. Otherwise (if $\mathrm{Tr}(\mu) < 2$ or if μ is $SL_2(\mathbb{Z})$-conjugate to $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \leq 0$), we are in a positive case.

2.2.4 Convex Integral Piecewise-Linear Functions on U^{trop}

If we choose a monoid P in our lattice P^{gp}, we can define what it means for a $\mathcal{PL}_{P^{gp}}$ function f to be convex along some ray ρ. Let σ^+ and σ^- denote disjoint open convex cones in U^{trop} with ρ contained in each of their boundaries. Let n_ρ be the unique primitive element of Λ^* which vanishes along the tangent space to ρ and is positive on vectors pointing from ρ into σ^+. We note that n_ρ may be viewed as $\pm v_\rho \wedge \cdot$, with the sign being positive if σ_+ is chosen to be counterclockwise of ρ.

Observe than any integral linear function f can be given on a cone σ by some $f_\sigma \in \Lambda^*$, using the local embedding of σ in its tangent spaces. Since the cotangent spaces on either side of ρ can be identified via parallel transport, we can compute

$$f_{\sigma^+} - f_{\sigma^-} = p_{\rho,f} n_\rho.$$

Here, $p_{\rho,f} \in P^{gp}$ is called the *bending parameter* of f along ρ. Note that this is independent of which side of ρ we call σ^+ and which we call σ^-. We say that f is *convex* (resp. strictly convex) along ρ if $p_{\rho,f} \in P$ (resp. $P \setminus P^\times$, where $P^\times$ denotes the invertible elements of P). We note that these notions naturally generalize to all integral linear manifolds.

For the rest of this section we will assume $P^{gp} = \mathbb{Z}$ and $P = \mathbb{Z}_{\leq 0}$.

2.2.4.1 Piecewise Linear Functions in terms of Weil Divisors

Let φ be a rational piecewise linear function on U^{trop} (that is, we are allowing rational values at integral points). We will always assume that we have taken enough toric blowups of (Y, D) so that $D_v \subset D$ for every ρ_v along which φ bends. As in §2.2.2, we define a rational Weil divisor

$$W_\varphi := \sum_i \varphi(v_i) D_i.$$

Then it follows from Equation 2.4 that $p_i := W_\varphi \cdot D_i$ is the bending parameter of φ along ρ_i. We see immediately from the definitions that φ is linear along ρ_i if and only if $p_i = 0$.

Conversely, for any nonsingular compactification (Y, D) of U and any rational Weil divisor $W = \sum_i w_i D_i$ supported on D, there is a unique rational piecewise linear function φ_W taking values w_i on v_i and bending only on the ρ_i's. φ_W is integral if and only if W is integral. The bending parameter at ρ_i is given by $W \cdot D_i$. That is, if we view W as a vector $W = (w_1, \ldots, w_n)$ in $\langle D \rangle$ (the lattice freely generated by the D_i's), then the bending parameters of φ_W are given by the vector

$$P = (p_1, \ldots, p_n) = HW,$$

where $H = (D_i \cdot D_j)$ is the intersection matrix. So given a collection of bending parameters p_i, there is a unique rational piecewise-linear function on U^{trop} with these bending parameters if and only if H is invertible, and it is given by the $\mathbb{Q}$-Weil divisor $W = H^{-1}P$. We will see in §2.2.4.2 that H being invertible is equivalent to $\mathrm{Id} - \mu^{-1}$ being invertible.

Assume for now that H is invertible over $\mathbb{Q}$. Let $v \in U_0^{\mathrm{trop}}(\mathbb{Z})$. We have $v = p_v v'$ for some non-negative integer p_v and some primitive vector v' on the ray ρ_v. Let β_v denote the unique rational piecewise linear function on U^{trop} which bends only on ρ_v with bending parameter $-p_v$. Note that the sums of functions of this form are exactly the convex rational piecewise linear functions on U^{trop} with integral bending parameters.

Let ψ_{ρ_v} denote the unique convex *integral* piecewise linear function which bends only on ρ_v with the smallest (in absolute value) possible nonzero bending parameter b_v (b_v may have to be less than -1 to ensure that ψ_{ρ_v} can be integral). The following proposition illustrates the utility of this Weil divisor perspective for understanding functions on U^{trop}, and we will later relate this proposition to a certain symmetry between U and its mirror (cf. Remark 3.2.15).

Proposition 2.2.10. *Assume H is invertible over $\mathbb{Q}$. For $v, w \in U^{\mathrm{trop}}(\mathbb{Z})$, we have $\beta_v(w) = \beta_w(v)$, and $\psi_{\rho_v}(b_w w') = \psi_{\rho_w}(b_v v')$*

Proof. Fix a compactification $(Y, D = D_1 + \ldots + D_n)$, and view $D_v = p_v D_{v'}$ and $D_w = p_w D_{w'}$ as vectors in $\langle D_1, \ldots, D_n \rangle$. Then $W_{\beta_v} = H^{-1} D_v$, and we have

$$\beta_v(w) = D_w^T H^{-1} D_v.$$

So the first part of the proposition follows from the fact that the intersection form is symmetric. The second part then follows because $\psi_{\rho_v} = \beta_{b_v v'}$. $\square$

2.2.4.2 Piecewise Linear Functions in Local Coordinates

We now use developing maps to describe rational piecewise linear functions in terms local coordinates of U^{trop}. We use the notation $v_i = p_i v_i'$, for $v_i' \in U^{\text{trop}}(\mathbb{Z})$ primitive and $-p_i \in \mathbb{Z}_{\leq 0}$ a bending parameter. $\beta_{v_1,\ldots,v_k}$ (cyclically ordered) will denote the space of piecewise linear functions with bending parameters $-p_i$ along the ray generated by $v_i' \in U^{\text{trop}}$ for each i (so $-p_i \leq 0$ for all i implies convexity). In fact, we could easily extend what follows to include rational or even real p_i's (viewing v_i with $p_i < 0$ as the formal data of the pair v_i', p_i, rather than as an element of U^{trop}).

Choose some $\rho \in \sigma_{v_k,v_1}$, generated by v_ρ, and identify the complement of ρ with its image under the developing map δ_ρ^0. Suppose $\varphi \in \beta_{v_1,\ldots,v_k}$. On σ_{v_ρ,v_1}, φ is given by $\varphi(w) = u \wedge w$ for some $u \in \mathbb{R}^2$. Then on σ_{v_1,v_2}, we see immediately from the definition of a bending parameter that φ is given by $\varphi(w) = (u - v_1) \wedge w$. By induction, on $\sigma_{v_i,v_{i+1}}$ we have $\varphi(w) = (u - v_1 - \ldots - v_i) \wedge w$. This description will be crucial for our proofs of the Minkowski sum formulas in §3.3.3.

Let $\widetilde{\varphi}$ be the lift of φ to $\widetilde{U}_0^{\text{trop}}$. For φ to be globally well-defined on U^{trop}, we must have $\widetilde{\varphi}(w) = \widetilde{\varphi}(\mu^{-1}(w))$ for all w (recalling that μ^{-1} just lifts w up a sheet). So for $w \in \sigma_{v_\rho, v_1}$, we must have

$$\begin{aligned} u \wedge w &= (u - v_1 - \ldots - v_k) \wedge \mu^{-1}(w) \\ &= \mu(u - v_1 - \ldots - v_k) \wedge w, \end{aligned}$$

and this suffices for all w. Since the wedge is non-degenerate, we can rearrange to find that

$$\left(\operatorname{Id} - \mu^{-1}\right) u = v_1 + \ldots + v_k. \tag{2.8}$$

So if $\operatorname{Id} - \mu^{-1}$ is invertible over $\mathbb{Q}$ (respectively, $\mathbb{Z}$), we see that any collection of integral bending parameters determines a unique rational (respectively, integral) piecewise linear function. One easily checks that

$$(\operatorname{Id} - \mu^{-1})^{-1} = \frac{1}{2 - \operatorname{Tr}(\mu)} (\operatorname{Id} - \mu),$$

unless $\operatorname{Tr}(\mu) = 2$, in which case both sides are undefined.

The nullity of $\operatorname{Id} - \mu^{-1}$ is equal to the dimension of $\beta_{v_1, \ldots, v_k}$, which is nonempty exactly when $-(v_1 + \ldots + v_k) \in (\operatorname{Id} - \mu^{-1})\mathbb{R}^2$ (or $(\operatorname{Id} - \mu^{-1})\mathbb{Z}^2$ if we restrict to integral functions). Comparing with our previous description of rational piecewise linear functions, we see that $\operatorname{Id} - \mu^{-1}$ must have the same nullity as the intersection matrix H (assuming D has at least 2 components). In particular, H is invertible if and only if $\operatorname{Tr}(\mu) \neq 2$. We will see in §2.3 that $\operatorname{Tr}(\mu)$ only equals 2 in what we call the I_k cases (which are the simplest cases) and in the cases where H is negative semi-definite (but not definite).

Examples 2.2.11.

- Considering the space of functions β_0 shows that the nullity of $\operatorname{Id} - \mu^{-1}$ is equal to the dimension of the space of global linear functions on U^{trop}.
- In the toric case, $\mu = \operatorname{Id}$, so we have a 2-dimensional space of linear functions. $\beta_{v_1,\ldots,v_k}$ is then nonempty if and only if $v_1 + \ldots + v_k = 0$.
- If the null space of $\mu - \operatorname{Id}$ is non-trivial (equivaently, if H is degenerate), then μ has some invariant direction (i.e., an eigenspace with eigenvalue 1). Such a μ must, up to cojugation, have the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.
- Consider the cubic surface described in 2.2.5, with $\delta^0(v_1) = (1,0)$, $\delta^0(v_2) = (0,1)$, and $\delta^0(v_3) = (-1,1)$. Let φ be the piecewise linear function with bending parameter -2 along ρ_3. That is, $\varphi = \beta_{(-2,2)}$. Recall $\mu = -\operatorname{Id}$. So

  $$v_0 = (\operatorname{Id} - \mu^{-1})^{-1} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{2.9}$$

  meaning that $\varphi|_{\sigma_{(1,0),(-1,1)}}(u) = (-1,1) \wedge u$, and $\varphi|_{\sigma_{(-1,1),(-1,0)}}(u) = [(-1,1) - (-2,2)] \wedge u = (1,-1) \wedge u$. We see that $W_\varphi = D_1 + D_2$, which does indeed have the correct intersection numbers. We also note that $\varphi = \psi_{\rho_3}$, since $\beta_{(-1,1)}$ is not integral.

2.2.5 Lines and Polygons in U^{trop}

Understanding lines and polygons in U^{trop} is important when studying compactifications of the mirror. This will be essential when we investigate the tropicalizations of the theta functions in §3.2.

2.2.5.1 Lines in U^{trop}

By a "line" in U^{trop}, we will mean a geodesic with respect to the canonical flat connection on U_0^{trop}. That is:

Definitions 2.2.12. A *parametrized line* in U^{trop} is a continuous map $L : \mathbb{R} \to U_0^{\mathrm{trop}}$ such that $L'(t_1)$ and $L'(t_2)$ are related by parallel transport along the image of L for all $t_1, t_2 \in \mathbb{R}$. A *line* is the data of the image $L(\mathbb{R})$ and the vectors $L'(t) \in TU_0^{\mathrm{trop}}$, $t \in \mathbb{R}$, for some parametrized line L (equivalently, a line is a parametrized line up to a choice of shift $t \mapsto t + c$ of the domain). We may abuse notation by letting L denote the unparametrized line or its image.

The *(signed) lattice distance* of a (parametrized) line from the origin is defined to be

$$\mathrm{dist}(L, 0) := L(t) \wedge L'(t)$$

where t is any point in $\mathbb{R}$, and the point $L(t)$ is identified with a vector in its tangent space. Note that $d(L, 0) > 0$ means L is going counterclockwise about the origin.

Now, for $q \in U_0^{\mathrm{trop}}$ and $d \in \mathbb{R}$, we define L_q^d to be the line which goes to infinity parallel to q and has lattice distance d from the origin. By *going to infinity parallel to* q we mean that for any open cone $\sigma \ni q$, there is some $t_\sigma \in \mathbb{R}$ such that $t > t_\sigma$ implies $L(t) \in \sigma$ and $L'(t) = q$ under parallel transport in σ.

We may similarly define *coming from infinity parallel to* q by replacing $t > t_\sigma$ with $t < t_\sigma$ and replacing $L'(t) = q$ with $-L'(t) = q$. We denote the directions in which a line L goes to and comes from infinity by $L(\infty)$ and $L(-\infty)$, respectively.

Note that the above definitions all make sense on $\widetilde{U}_0^{\mathrm{trop}}$. We will at times refer to lines in $\widetilde{U}_0^{\mathrm{trop}}$ using the same notation as for U^{trop}.

Remark 2.2.13. In general, a line need not go to or come from infinity at all. In fact, one characterization of U being positive is that every line in U^{trop} both goes to and comes from infinity, cf. §2.3.

Definition 2.2.14. We define L_q^0 to be the limit of L_q^d as d approaches 0 from below. In other words, it consists of the ray coming in from the direction $L_q^{d<0}(-\infty)$ hitting 0, as well as the ray leaving the origin in the direction $q = L_q^{d<0}(\infty)$. When we use the term "line," we will be excluding the $d = 0$ cases unless L_q^0 is invariant under the monodromy.

We say that a line L_q *wraps* if it intersects every ray, except possibly ρ_q, at least once. It wraps k times if it hits each ray at at least k times (except possibly for ρ_q, which it might only hit $k-1$ times).

We call the connected component of $U^{\mathrm{trop}} \setminus L$ containing the origin the 0-side of L, denoted $Z(L)$. We say a line L_q^d has 0 on the left if $d > 0$, and on the right if $d < 0$. We will write $L_q^{d>0}$ or $L_q^{d<0}$ when we want to clarify that 0 is on the left or right side, respectively, without having to specify d. Let $L_q^{d,0} \subseteq L_q^d$ denote the boundary of the 0-side. Note that $L_q^{d,0} = L_q^d$ exactly when the line does not self-intersect.

Examples 2.2.15.

- If (Y, D) is toric, then $U^{\mathrm{trop}} \cong \mathbb{R}^2$, and lines are just the usual notion of lines with a chosen constant velocity.

- If (Y, D) is the cubic surface introduced in Example 2.2.5, then for any ray $\rho \subset U^{\mathrm{trop}}$, $U^{\mathrm{trop}} \setminus \rho$ is isomorphic (as an integral linear manifold) to an open half-plane. Any line will go to and come from infinity in the same direction—we call such lines *self-parallel.* If we now make a non-toric blowup on some D_{ρ_q}, then in the new integral linear manifold, $L^d_{q'}$ $(d \neq 0)$ will self-intersect if $q' \neq q$, but will still be self-parallel if $q' = q$. We will see in §3.2 that $L^{d<0}_{q'}$ self-intersecting corresponds to the theta function $\vartheta_{q'}$ having poles along every boundary divisor.
- See Figure 3.1 for illustrations of some possible lines.

2.2.5.2 Polygons in U^{trop}

Definitions 2.2.16.

- A (convex) *polytope* $\Delta \subset U^{\mathrm{trop}}$ is the closure of a set homeomorphic to an open k-ball for some $k \leq 2$ such that the boundary is a finite union of line segments and rays. We also consider a point to be a polytope. By *polygon*, we will mean a 2-dimensional polytope.
- A polytope Δ is *convex* if any line segment in U^{trop} (including those which wrap around the origin) with endpoints Δ is entirely contained in Δ.
- A polytope is *integral* (resp. *rational*) if all of its vertices are integral (resp. rational) points.
- A polygon is *nonsingular* if at each vertex of the form $v = F_1 \cap F_2$ (F_i edges), we have that primitive generators of F_1 and F_2 generate the lattice Λ_p of integral tangent vectors at p.

We will be especially interested in polygons with 0 in their interiors.

Lemma/Definition 2.2.17. *Suppose that lines in* U^{trop} *all go to and come from infinity (i.e., U is "positive," see §2.3). Also, let* $P^{gp} = \mathbb{Z}$*, and* $P = \mathbb{Z}_{\leq 0}$*. We then have:*

- *A star-shaped (i.e., closed under multiplication by elements of* $[0,1]$*) polygon is a set* $\Delta_{\varphi} \subset U^{\mathrm{trop}}$ *of the form* $\varphi \geq -1$ *for some piecewise-linear function* φ *on* U^{trop}*.*
- Δ_{φ} *is convex if and only if* φ *is convex. Equivalently, the polygon is convex if it is the closure of the intersection of a finite number of* 0*-sides of lines in* U^{trop}*, or equivalently, if it is convex on some cone-neighborhood of each vertex in the usual sense.*
- Δ_{φ} *is bounded if and only if* $\varphi < 0$ *everywhere on* U_0^{trop}*.*

2.2.6 The Tropicalization Determines the Charge

One natural question to ask is to what extent U^{trop} determines U. We will see in the next section that in many cases, U is uniquely determined up to deformation by U^{trop}. This is not always the case though: for example, there are two degree 8 Del Pezzo's with an irreducible choice of anti-canonical divisor which have the same U^{trop} but are not deformation equivalent. This subsection shows that U^{trop} does at least determines the number of non-toric blowups, and this at least determines U up to homeomorphism.

Definition 2.2.18. The *charge*[10] of a Looijenga pair (Y, D) is the number of non-toric

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[10] More generally, the charge of a log Calabi-Yau variety $(Y, D = D_1 + \ldots + D_n)$ is given by $c(Y, D) := \dim(Y) + \mathrm{rank}(\mathrm{Pic}(Y)) - n$.

blowups in a toric model for some toric blowup of (Y, D).

Lemma 2.2.19. *A Looijenga pair $(Y, D = D_1 + \ldots + D_n)$ with $n > 1$ and intersection matrix H has charge*

$$c(Y, D) = 12 - 3n - \mathrm{Tr}(H) \tag{2.10}$$

Proof. First note that (for $n > 1$) toric blowups increase n by 1, decrease $\mathrm{Tr}(H)$ by 3, and keep the charge constant, so Equation 2.10 is unaffected by toric blowups and blowdowns. Similarly, non-toric blowups decrease $\mathrm{Tr}(H)$ by 1 and increase the charge by 1, so the validity of the equation is also unaffected by non-toric blowups. Since every Looijenga pair is related to a copy of the toric pair $(\mathbb{P}^2, D)$ by some sequence of toric blowups, toric blowdowns, and non-toric blowups, it now suffices to just check this case. We have $c = 0$, $n = 3$ and $\mathrm{Tr}(H) = 3$, so the equation holds. □

An similar formula appears in [GHK]: $c(Y, D) = 12 - (n + K^2)$.

Proposition 2.2.20. *Suppose that (Y, D) and (Y', D') are two Looijenga pairs with the same tropicalization U^{trop}. Then $c(Y, D) = c(Y', D')$.*

Proof. Let Σ_Y and $\Sigma_{Y'}$ be the corresponding fans in U^{trop}. There exists some non-singular common refinement Σ which is the fan for a toric blowup of both (Y, D) and (Y', D'). The intersection matrices for these two toric blowups are the same, since each can be determined from Σ, so the claim follows from Lemma 2.2.19. □

2.3 Classification

Here we give several equivalent classifications for the possible deformation classes of Looijenga pairs. These classifications are based on the intersection matrix H of D, the intersection form Q on $D_{\mathrm{Eff}}^{\perp} \subset D^{\perp} \cong K$ (see §2.1.4.1), the monodromy μ of U^{trop}, the properties of lines in U^{trop}, the global functions on U, the properties of the quiver for a corresponding cluster structure, and various other properties. This may be viewed as a classification of rank-2 cluster varieties up to the notion of equivalence given in Definition 2.1.8. The classification is not totally new—for example, the cases that we refer to as "no lines wrap" or "some lines wrap" are simply the finite-type or, respectively, acyclic cases in the cluster language. However, we do offer several new characterizations of these cases.

Throughout this section, D will be called *minimal* if it has no (-1)-components.

2.3.1 The Negative Definite Case

The following are equivalent, and have all appeared (along with some other equivalent statements) in in some form in [GHK11], [GHK], or [GHK13a].

- The **intersection matrix** $H = (D_i \cdot D_j)$ is negative definite.
- Any **developing map** δ as in §2.2.3 embeds the universal cover $\widetilde{U}_0^{\mathrm{trop}}$ of U_0^{trop} into a strictly convex cone of $\mathbb{R}^2$
- The **monodromy** satisfies $\mathrm{Tr}(\mu) > 2$.
- All **lines** in U^{trop} wrap infinitely many times around the origin, meaning that they hit each ray infinitely many times.

- The **quadratic form** Q is not negative semi-definite.
- U admits no non-constant global **functions**.
- D can be blown down to get a surface $\overline{Y}$ with a cusp singularity. If D is minimal, $D_i^2 \leq -2$ for all i, and $D_i^2 \leq -3$ for some i.

See Example 1.9 of [GHK11] for the relationship between μ^{-1} and the cusp singularity on $\overline{Y}$. In fact, much of [GHK11] is devoted to deformations of cusp singularities.

2.3.2 The Strictly Negative Semi-Definite Case

Once again, the following statements are all equivalent and can be found in [GHK11] and [GHK] (or follow easily).

- The **intersection matrix** H is negative semi-definite but not negative definite.
- Any **developing map** δ for U_0^{trop} identifies the universal cover of U_0^{trop} with a half-plane in $\mathbb{R}^2$.
- The monodromy μ is $SL_2(\mathbb{Z})$-conjugate to a matrix of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where $a > 0$.
- **Lines** in U^{trop} can be circles, or they can wrap infinitely many times around the origin.
- If D is minimal, then $D \in D^{\perp}$, meaning that either $D_i^2 = -2$ for all i, or D is irreducible with $D^2 = 0$.
- The **quadratic form** Q is negative semi-definite but not negative definite (since $Q(D) = 0$).

- (Y, D) is deformation equivalent to a Looijenga pair (Y', D') which admits an elliptic fibration having D' as a fiber.
- $\dim \operatorname{Spec}(\Gamma(Y, \mathcal{O}_Y)) = 1$.

As stated above, if D is minimal then it is either irreducible or consists of $n > 1$ (-2)-curves. The largest possible n here is 9. This follows from Lemma 2.2.19, which says that the charge is $c(Y, D) = 12 - 3n - \operatorname{Tr}(H) = 12 - n$. The charge is by definition non-negative, giving us $n \leq 12$. Furthermore, the classifications below then imply that some lines do not wrap if $c(Y, D) \leq 2$, so then $n \leq 9$. A case with $n = 9$ can be explicitely constructed.

2.3.3 The Positive Cases

As a converse to the above cases, we have that the following are equivalent:

- The **intersection matrix** H is not negative semi-definite.
- The **developing map** for U_0^{trop} is not injective.
- **Lines** in U^{trop} wrap at most finitely many times. Each line both goes to and comes from infinity.
- The **quadratic form** Q is negative definite.
- U is a minimal resolution of an affine surface with at worst Du Val singularities. U is deformation equivalent to an affine surface, and $\dim \operatorname{Spec}(\Gamma(Y, \mathcal{O}_Y)) = 2$.
- D supports a D-ample divisor.

If any of these conditions hold, we say that U is *positive*. We have the following sub-cases:

Proposition 2.3.1. All Lines Wrap (Finitely Many Times): *The following are equivalent:*

1. **Lines** *in* U^{trop} *all wrap, but only finitely many times.*
2. *Every sheet of the* **developing map** *is convex, but the developing map is not injective.*
3. *Non-zero global* **functions** *on* U *are not generically* 0 *along any boundary divisor of any compactification* (Y, D) *of* U *(i.e, the corresponding valuations are non-positive).*
4. *The inverse* **monodromy matrix** μ^{-1} *is conjugate to a Kodaira matrix*[11] *of type* I_k^**,* II^**,* III^**, or* IV^**.*
5. *If* D *is minimal, then either* $D = D_1 + D_2$ *with* $D_1^2 = 0$ *and* $-1 \neq D_2^2 \leq 0$ *(up to re-labelling), or* D *is irreducible with* $1 \leq D^2 \leq 4$*.*
6. U *can be constructed from* $(\mathbb{P}^2, D)$*, with* $D = D_1 + D_2 + D_3$ *a triagle of lines, by blowing up* d_i *times on* D_i *for each* i*, with* (d_1, d_2, d_3) *as in the final column of Table 2.1. Equivalently,* U *corresponds to a seed with* $E = (e_1, e_2, e_3)$*,* $F = \emptyset$*,* $\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$*, and multipliers* (d_1, d_2, d_3) *as in the final column of Table 2.1.*

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[11] In [Kod63], Kodaira listed the matrices which can appear as monodromies about singular fibers of elliptic fibrations of surfaces. See Tables 2.1 and 2.2 for a list of these matrices.

7. $D^{\perp}_{\mathrm{Eff}} = D^{\perp}$*, and the* **quadratic form** Q *is of type* D_n *(*$n \geq 4$*) or* E_n *(*$n = 6$*,* 7*, or* 8*).*

Proof. (1)⇔(2) is clear from the definitions. (1)⇔(3) will be clear from the description of tropical theta functions in terms of lines, given in §3.2.

For (1)⇒(5), using the construction of U^{trop} from charts in Remark 2.2.2, we can easily see that having any $D_i^2 > 0$ with D not irreducible would allow a line to not wrap. On the other hand, having every $D_i^2 \leq -2$ would mean we are in a negative semi-definite case. So if D is minimal and not irreducible, then D_i^2 must be 0 for some i. D having more than one additional component would allow a non-convex sheet of the developing map, so the claim follows, except for when D is irreducible. In these cases, if $D^2 > 4$, then the proper transform after taking a toric blowup would have positive self-intersection, which we have already ruled out, and $D^2 < 1$ would mean we are in a negative semi-definite case.

For (5)⇒(2), observe that in the $D_1^2 = D_2^2 = 0$ case, every sheet of any developing map is convex (but not strictly convex). The other cases come from non-toric blowups and toric blow-downs of this, so the sheets of their developing maps will of course still be convex (non-toric blowups make these sheets "more convex").

(5)⇔(4) is a straightforward check. Note that we now have the equivalence of (1) through (5).

(6)⇒(7) is also straightforward. For U generic, $D^{\perp}$ is generated by classes of the form $E_{i,j_1} - E_{i,j_2}$ (where $E_{i,j}$ denotes the exceptional divisor from a non-toric blowup on D_i), together with a class of the form $L - E_{1,j_1} - E_{2,j_2} - E_{3,j_3}$, where L is the class of a generic line in $\mathbb{P}^2$. If we choose all the blowup points on each D_i to be infinitely near, and choose the blowup points on different D_i's to be colinear, then $D^{\perp}$ is generated by effective divisors with the correct intersections.

(7)⇒(1) because Q of type D_n or E_n implies that Q is negative definite, so by the above characterizations, we are not in an H negative semi-definite case. We also cannot be in a some lines wrap case because, as we see below, $Q|_{D^{\perp}_{\mathrm{Eff}}}$ in these cases is a direct sum of A_{n_i}'s.

It now suffices to show that (5)⇒(6) (since (4)⇔(5), this means we are showing that U^{trop} really does determine the deformation type of U in these cases). For the I_0^* case, we have $\mu^{-1} = -\mathrm{Id}$. We will see in Example 3.3.18 that since such a U^{trop} contains a reflexive polytope with 3 integer points on the boundary, any surface with this U^{trop} as its tropicalization must be a degree 3 del Pezzo surface, i.e., a cubic surface.

Now for the I_k^* cases, we can choose a compactification (Y, D) of U with $D_1^2 = D_2^2 = -1$ and $D_3^2 = -1 - k$. The divisor $C := D_1 + D_2$ has $C \cdot D_1 = C \cdot D_2 = C^2 = 0$, and $C \cdot D_3 = 2$. By Riemann-Roch, $\dim |C| \geq 1$. If C is the only singular element of some $\mathbb{P}^1 \subset |C|$, then (for U generic in its deformation class) $Y \setminus C$ is a $\mathbb{P}^1$-bundle over $\mathbb{A}^1$, hence has Euler characteristic 2. So then Y has Euler characteristic 5. However, we know from §2.2.6 that U^{trop} determines the charge c of (Y, D), which in this situation is $6 + k$. One checks that the Euler characteristic of a Looijenga pair with n boundary components and charge c is $n + c$, which in this case is $9 + k > 5$. So $|C|$ must contain other singular curves. These must contain irreducible rational components E_1, E_2 with $E_i \cdot D_3 = 1$ and $E_i^2 = -1$. Blowing down either of these is a non-toric blowdown and reduces us to the I_{k-1}^* case, so the claim follows by induction.

For the IV^* case, we have a compactification of U with $D = D_1 + D_2 + D_3$, $D_1^2 = -1$, $D_2^2 = D_3^2 = -2$. Note that $D \cdot D_1 = 1$, while $D \cdot D_2 = D \cdot D_3 = 0$, so $\dim |D| \geq 1$. Thus, there is some point on D_1 which we can blow up to get a new pair $(\widetilde{Y}, \widetilde{D})$, with exceptional divisor E, such $\widetilde{Y}$ admits an elliptic fibration with $\widetilde{D}$ being a fiber and E being a section. Such a surface can be obtained by blowing up 9 base-points for a pencil of cubics in $\mathbb{P}^2$, with E being the exceptional divisor of the final blowup (cf. [HL02]). $\widetilde{D}$ then is the proper transform of one of the cubics $\overline{D}$ in the pencil, so there must have been 3 base-points on each component $\overline{D}_i$ of $\overline{D}$. Thus, after blowing E down, we see that Y must contian disjoint (-1)-curves hitting each component of D. Blowing down a (-1)-curve hitting, say, D_2, reduces to the I_1^* case we have already dealt with.

A similar argument works for the III^* case using a compactification of U with $D = D_1 + D_2$, $D_1^2 = -1$, $D_2^2 = -2$, and blowing up a point in D_1 to get a surface with an elliptic fibration. The II^* case is also similar, using D irreducible with self-intersection 1 and blowing up some point in D to get a surface with an elliptic fibration.

□

Table 2.1 summarizes the different cases from the above theorem.

2.3.4 Not All Lines Wrap

Proposition 2.3.2. *The following are equivalent:*

1. *U^{trop} contains a* **line** *which does not wrap.*

Kodaira Matrix	Cartan Form Q	**M**onodromy μ	(d_1, d_2, d_3)
I_k^* $(k \geq 0)$	D_{n+4}	$\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$	(2,2,2+n)
IV^*	E_6	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	(2,3,3)
III^*	E_7	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	(2,3,4)
II^*	E_8	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	(2,3,5)

Table 2.1: Cases where all lines wrap.

2. *Some compactification of U admits a toric model for which all the non-toric blowups are on divisors corresponding to rays in one half of $\overline{N}$ (the cocharacter lattice of the image). I.e, there is some seed for which all of the non-frozen vectors' images in $p^*(N)$ lie in one half of the plane.*

3. *Any cluster structure corresponding to U is acyclic.*

4. *The* **quadratic form** *Q on $D^{\perp}$ is negative definite, and $Q|_{D^{\perp}_{\mathrm{Eff}}}$ is a direct sum of A_{n_i}'s. In fact, it is $A_{d'_1-1} \oplus \cdots \oplus A_{d'_m-1}$, where the (d'_i)'s are the modified multipliers for a minimal acyclic seed corresponding to U (equivalently, d'_i is the number of non-toric blowups on D_i in a toric model for a compactification U).*

5. *There exists a global monomial on U (by which we mean on an $\mathcal{X}$-space containing U as a fiber).*

Proof. (1)⇔(2) follows immediately from Lemma 3.1.7. (2)⇔(3) was observed in §2.1.1.1. (1)⇔(5) follows from Theorem 3.1.10. (4)⇒(1) because if some line does wrap (possibly infinitely many times), then we have seen that either Q is not negative-definite or $Q|_{D^{\perp}_{\mathrm{Eff}}}$ is of type D_n or E_n.

For (2)⇒(4), first note that Q is negative definite on $D^{\perp}$ by positivity of U. Now, let $(Y, D) \to (\overline{Y}, \overline{D})$ be the toric model corresponding to a seed with the images of all rays in one half of the plane $\overline{N}_{\mathbb{R}}$ corresponding to $\overline{Y}$. For any curve $\overline{C}$ in $\overline{Y}$, $\sum \overline{C} \cdot \overline{D}_i v_i = 0$. If $\overline{C}$ is the image of a curve $C \in D^{\perp}$, then it can only intersect blowup points, so the only possibility is that C is supported on the exceptional divisors. Thus, $D^{\perp}_{\mathrm{Eff}}$ is generated by classes obtained by taking the d'_i blowups to be infinitely near, and then taking the $d'_i - 1$ exceptional divisors which do not intersect D. □

2.3.4.1 No Lines Wrap

Proposition 2.3.3. *The following are equivalent:*

1. *No* **Lines** *in* U^{trop} *wrap.*
2. *No sheet of the* **developing map** *is convex.*
3. *Every global* **function** *on* U *is generically* 0 *along some boundary divisor of some compactification (i.e, the corresponding valuations are positive). The Laurent phenomenon holds for the* $\mathcal{X}$*-space, meaning that each* X_i *is a global monomial. Furthermore, the global monomials form an additive basis for the global function on* U *(we will see that global monomials are theta functions, and in these cases, they are all the theta functions).*
4. *The inverse* **monodromy** *matrix* μ^{-1} *is a Kodaira matrix of type* I_k*, II, III, or IV.*

5. *U (or rather, the corresponding cluster variety) is of finite-type, meaning that it has only a finite number of seeds.*

6. *For some seed, the corresponding maximal quiver (after removing frozen vectors) is of type A_1^k ($k \in \mathbb{Z}_{\geq 0}$), A_2, A_3, or D_4.*

Proof. (1)⇔(2) is obvious. (1)⇔(3) follows from Theorem 3.1.10.

Now define $q_{\pm} = L_q^{d<0}(\pm\infty)$. To see that (1) implies (5), we need Lemma 3.1.7, which shows that there are only finitely many (-1)-curves hitting boundary divisors corresponding to rays in σ_{q_-,q_+}. Since no lines wrap, we can cover U^{trop} by finitely many cones of the form $\sigma_{q_-.q_+}$, and so there are only finitely many (-1)-curves in Y hitting the boundary. Since seeds correspond to certain finite subsets of this collection of (-1)-curves, the claim follows.

(5)⇔(6) follows from a well-known result of [FZ03], which says that a cluster algebra is of finite type if and only if the underlying graph of a quiver (minus the boxed vertices) corresponding to some seed is a simply laced (i.e., type ADE) Dynkin diagram—one easily checks that the type ADE quivers producing rank 2 cluster varieties are exactly those listed in the Proposition. One can easily check (6)⇒(4) by explicit computation: the A_1^k, A_2, A_3, and D_4 quivers correspond to the I_k, II, III, and IV matrices, respectively. (4)⇒(1) is now automatic.

□

Table 2.2 lists the cases where no lines wrap, along with their basic properties. We once again use the notation (d_1, d_2, d_3) to indicate that such a Looijenga pair can be

Quiver	Kodaira Matrix	Cartan Form Q	Monodromy μ	(d_1, d_2, d_3)
A_1^k $(k \geq 0)$	I_k	A_{k-1}	$\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$	(k,0,0)
A_2	II	A_0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	(1,1,0)
A_3	III	A_1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	(2,1,0)
D_4	IV	A_2	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	(3,1,0)

Table 2.2: Cases where no lines wrap.

obtained by starting with the toric variety $(\mathbb{P}^2, D = D_1 + D_2 + D_3)$, and then blowing up d_1, d_2, and d_3 points on D_1, D_2, and D_3, respectively.

2.3.4.2 Some Lines Wrap and Some Do Not

Proposition 2.3.4. *The following are equivalent:*

1. *Some* **Lines** *in* U^{trop} *wrap, while others do not.*
2. *Some (but not all) sheets of the* **developing map** *are convex.*
3. *The* **monodromy** *satisfies* $\operatorname{Tr}(\mu) \leq -2$*, and if there is equality, then* μ *is conjugate to* $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$ *for some* $a < 0$.

Proof. (1)⇔(2) is easy. (3) follows because all the other possibilities have been eliminated by the previous propositions. □

Chapter 3

Theta Functions and their Tropicalizations

3.1 Construction of Theta Functions and the Mirror

This section summarizes [GHK11]'s construction of the mirror family. We assume throughout this chapter that (Y, D) is positive, unless otherwise stated. This assumption simplifies the details of the construction, the notation, and the statements of the theorems from [GHK11], but the basic ideas of the construction are unchanged. In §3.1.7, we describe how to obtain compactifications of the mirror as in [GHK]. These compactifications really do require positivity.

3.1.1 Setup

Choose some lattice P^{gp} and some finitely generated submonoid $P \subseteq P^{gp}$. Recall that $\widetilde{U}_0^{\mathrm{trop}}$ denotes the universal cover of U_0^{trop}. Let us define $\widetilde{U}^{\mathrm{trop}}$ by adding a single point 0 to $\widetilde{U}_0^{\mathrm{trop}}$ which is the limit $\lim_{a \to 0} aq$ for every $q \in \widetilde{U}_0^{\mathrm{trop}}$. We say 0 is in $\widetilde{U}^{\mathrm{trop}}(\mathbb{Z})$, and we extend ξ to $\widetilde{U}^{\mathrm{trop}}$ by saying that $\xi(0) = 0$. Let $r : \mathcal{P} \to \widetilde{U}^{\mathrm{trop}}$ denote the trivial bundle $\widetilde{U}^{\mathrm{trop}} \times P^{gp}_{\mathbb{R}}$, and let $\mathcal{P}(\mathbb{Z})$ denote the subset $\widetilde{U}^{\mathrm{trop}}(\mathbb{Z}) \times P^{gp}$. We note that $\mathcal{P}$ is itself an integral linear manifold on the complement of the 0-fiber.

When constructing the mirror over $\operatorname{Spec} \Bbbk[P]$, we will need a choice of convex integral Σ-piecewise-linear function[1] $\varphi : \widetilde{U}^{\text{trop}} \to P^{gp}$ such that, if we think of the monodromy μ as shifting points down one sheet, $\varphi \circ \mu = \varphi + \varphi_0$ for some globally linear function φ_0 (we defined $\mu(0) = 0$ and say that piecewise-linear functions are 0 at 0). Equivalently, we require that for each ray $\rho \subset U^{\text{trop}}$, φ has the same bending parameter along every component of $\xi^{-1}(\rho)$. Note that we can view φ as a section of $\mathcal{P}$.

Note that, up to a choice of globally linear function, φ is determined by specifying the bending parameters for each ray ρ in the fan Σ in U^{trop}. φ will in fact only matter to us up to this choice of globally linear function, so specifying the bending parameters is enough.

For example, we may take $P^{gp} := A_1(Y, \mathbb{Z}) \cong \text{Pic}(Y)^*$ and P to be the Mori cone $\text{NE}(Y)$. We will want P to be finitely generated. For $P = \text{NE}(Y)$, this follows from the Cone Theorem and our assumption that (Y, D) is positive.[2] We can then take φ to have bending parameters $[D_i]$ along the preimages of ρ_i for each i, and we denote such a φ by $\varphi_{\text{NE}(Y)}$, despite it being defined only up to a linear function.

These choices for P^{gp}, P, and φ will lead to a construction of a mirror family which is in a sense universal (see [GHK13b]). Taking another choice of P^{gp} and P together with a monoid homomorphism $\eta : \text{NE}(Y) \to P$ will define another family with a map to this universal one. We will always assume we have such an η.

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[1] [GHK11] instead uses a “multi-valued” function on U^{trop}, but this difference is not significant for our purposes.

[2] When working without the positivity assumption, [GHK11] chooses a strictly convex rational polyhedral cone σ_P containing $\text{NE}(Y)_{\mathbb{R}_{\geq 0}}$ and lets $P = \sigma_P \cap A_1(Y, \mathbb{Z})$.

Example 3.1.1. Let $E_1, \ldots, E_s$ denote the exceptional divisors of some toric model π for (Y, D). Let $\mathrm{NE}_\pi(Y)$ denote the subcone of $A_1(Y, \mathbb{Z})$ spanned by $\mathrm{NE}(Y)$ and $-[E_i]$, $i = 1, \ldots, s$. We can then take η to be the inclusion $\mathrm{NE}(Y) \hookrightarrow \mathrm{NE}_\pi(Y)$. The base of the resulting family is what [GHK11] refers to as the Gross-Siebert locus.

3.1.1.1 The Cone Bounded by φ

Define $\tau_{\mathbb{R}} := \varphi(U^{\mathrm{trop}}) + P_{\mathbb{R}} := \{(x, \varphi(x)+y) \in \mathcal{P} | x \in \widetilde{U}^{\mathrm{trop}},\ y \in P_{\mathbb{R}}\}$, and let $\tau := \tau_{\mathbb{R}} \cap \mathcal{P}(\mathbb{Z})$. Consider consecutive rays $\rho_i,\ \rho_{i+1}$ in $\xi^{-1}(\Sigma) \subset \widetilde{U}^{\mathrm{trop}}$, with $\sigma_{i,i+1}$ denoting the closed cone they bound. We have a cones $\tau_{i,i+1,\mathbb{R}} := \tau_{\sigma_{i,i+1}} := \tau_{\mathbb{R}} \cap r^{-1}(\sigma_{i,i+1})$ with integer points $\tau_{i,i+1} := \tau \cap r^{-1}(\sigma_{i,i+1})$.

Let $\Lambda\mathcal{P}$ denote the bundle of integral tangent vectors in $T\mathcal{P}$. For any point $x \neq 0$ in a cone $\sigma \subset \widetilde{U}^{\mathrm{trop}}$ (not necessarily convex, but at least not surjecting to $\mathbb{R}^2 \setminus \{0\}$ under a developing map), we consider the canonical embedding of $r^{-1}(\sigma)$ into $T_x\mathcal{P}$. Note that this identifies points in $r^{-1}(\sigma) \cap \mathcal{P}(\mathbb{Z})$ with points in $\Lambda_x\mathcal{P}$. Furthermore, this identification commutes with parallel transport along any path contained in σ. We may therefore write $T_\sigma\mathcal{P}$ to mean $T_x\mathcal{P}$ for any $x \in \sigma$, and similarly with $\Lambda_\sigma\mathcal{P}$.

For example, we have embeddings of $\tau_{i,i+1,\mathbb{R}}$ and $\tau_{i,i+1}$ in $T_{\sigma_{i,i+1}}\mathcal{P}$ and $\Lambda_{\sigma_{i,i+1}}\mathcal{P}$, respectively. Similarly, we may view $\varphi|_\sigma$ as a map from σ to $T_\sigma\mathcal{P}$. To be clear, when viewing φ as locally embedding $\widetilde{U}^{\mathrm{trop}}$ into $T\mathcal{P}$, we will write $\widetilde{\varphi}$, wheras φ will denote the $P_{\mathbb{R}}^{gp}$-valued function. We also have an induced additive action of $P_{\mathbb{R}}^{gp}$ on the tangent spaces, and thus an identification of $P_{\mathbb{R}}^{gp}$ with $P_{\mathbb{R}}^{gp} + 0 \subset T_\sigma\mathcal{P}$. Note that we may view $\widetilde{\varphi}(u)$ as $u + \varphi(u)$ in $T_u\mathcal{P}$. We will abuse notation and use these identifications freely.

3.1.1.2 The Toric Case

If (Y, D) is a toric variety with its toric boundary, then we can choose $\varphi_{\mathrm{NE}(Y)}$ to satisfy $\varphi_{\mathrm{NE}(Y)} = \overline{\varphi_{\mathrm{NE}(Y)}} \circ \xi$ for some convex integral Σ-piecewise-linear function $\overline{\varphi_{\mathrm{NE}(Y)}} : U^{\mathrm{trop}} \to \mathrm{NE}(Y)$ (cf. [GHK11], Lemma 1.14). Similarly with any $\varphi = \eta \circ \varphi_{\mathrm{NE}(Y)}$ as above. We can therefore work with U^{trop} instead of $\widetilde{U}^{\mathrm{trop}}$ (but let us otherwise use the same notation as before). $U^{\mathrm{trop}} \times P_{\mathbb{R}}^{gp}$ is a vector space in the toric situation, and this induces a monoid structure on τ (usually, the monodromy about the 0-fiber prevents τ from admitting such a structure). In this case, the mirror family $\mathcal{V}$ is simply $\mathrm{Spec}(\Bbbk[\tau]) \to \mathrm{Spec}(\Bbbk[P])$, where the morphism comes from the inclusion of P into $r^{-1}(0)$. This is the well-known Mumford degeneration. The central fiber is $\mathbb{V}_n := \mathbb{A}^2_{x_1,x_2} \cup \mathbb{A}^2_{x_2,x_3} \cup \ldots \cup \mathbb{A}^2_{x_n,x_1} \subset \mathbb{A}^n_{x_1,\ldots,x_n}$ $(n \geq 3)$, and the general fiber is $(\Bbbk^*)^2$ (cf. [GHK11], §1.2).

Also in the toric case, given a convex integral polygon Δ in U^{trop}, we can define a convex integral polygon $\Delta^{\overline{\varphi}} := \overline{\varphi}(\Delta) + P_{\mathbb{R}} \subset \mathbb{P}$. The corresponding toric variety $\mathcal{V}_{\Delta^{\overline{\varphi}}}$ is then a (partial) compactification of $\mathcal{V}$.

In a non-toric case we do not have a natural global way to add points of τ. However, the identification with a cone in the tangent space does give us a natural monoid structure on $r^{-1}(\sigma)$ for any convex cone σ in $\widetilde{U}^{\mathrm{trop}}$. Consider $\tau_{\rho_i} := \tau_{i-1,i} + \tau_{i,i+1} \subset \mathcal{P}_{\rho_i}$. Now for any $\rho \subseteq \sigma \subset \widetilde{U}^{\mathrm{trop}}$ (ρ and σ cones of dimension 1 or 2), define

$$\tau_{\rho,\sigma} := \tau_\rho - \widetilde{\varphi}(\sigma \cap \widetilde{U}^{\mathrm{trop}}(\mathbb{Z})) = \{x - y \in T_\rho \mathcal{P} | x \in \tau_\rho, y \in \widetilde{\varphi}(\sigma \cap \widetilde{U}^{\mathrm{trop}}(\mathbb{Z}))\}. \tag{3.1}$$

That is, we allow negation of integer points on the image of $\widetilde{\varphi}|_\sigma$. Define $R_{\rho,\sigma} := \Bbbk[\tau_{\rho,\sigma}]$, and $\mathcal{V}_{\rho,\sigma} := \mathrm{Spec}(R_{\rho,\sigma})$. Note that $R_{\rho,\sigma}$ is the localization of $R_{\rho,\rho}$ by functions of the form $z^{\widetilde{\varphi}(x)}$ for $x \in \sigma \cap \widetilde{U}^{\mathrm{trop}}(\mathbb{Z})$.

The plan for constructing the mirror family is then to glue $\mathcal{V}_{\rho_i,\rho_i}$ to $\mathcal{V}_{\rho_{i+1},\rho_{i+1}}$ for each i, via an isomorphism $R_{\rho_i,\sigma_{i,i+1}} \tilde{\rightarrow} R_{\rho_{i+1},\sigma_{i,i+1}}$. Also, if $\xi(\rho) = \xi(\rho')$, then φ near ρ differs from φ near ρ' by a linear function, and this linear function induces an isomorphism between $R_{\rho,\rho}$ and $R_{\rho',\rho'}$. We use this to identify $\mathcal{V}_{\rho,\rho}$ with $\mathcal{V}_{\rho',\rho'}$ for each such pair of rays.

We do naturally have $R_{\rho_i,\sigma_{i,i+1}}$ identified with $R_{\rho_{i+1},\sigma_{i,i+1}}$ by parallel transport in $\sigma_{i-1,i} \cup \sigma_{i,i+1}$, but this naive identification is not the correct gluing: it gives a flat deformation of $\mathbb{V}_n^0 := \mathbb{V}_n \setminus \{0\}$, but this does not extend to a deformation of $\mathbb{V}_n$ (except in the toric case). The problem is essentially that locally defined functions generally do not commute with transportation around the origin. We therefore need a modified version of this gluing.

The correct modifications are defined in terms of a certain canonical *scattering diagram* in U^{trop}. We will also need an automorphism of R_{ρ_i,ρ_i} for each i, and we will think of these as isomorphisms between $R^+_{\rho_i,\rho_i} := R_{\rho_i,\rho_i}$ (thought of as corresponding to the cone $\sigma_{i,i+1}$) and $R^-_{\rho_i,\rho_i} := R_{\rho_i,\rho_i}$ (associated with the cone $\sigma_{i-1,i}$). Plus signs and minus signs as superscripts will always have these meanings for us.

3.1.2 The Consistent Scattering Diagram

A scattering diagram $\mathfrak{d}$ for us includes the data of a set of rays in $\widetilde{U}^{\mathrm{trop}}$ with associated functions which satisfy certain conditions. These functions are used to define certain ring automorphisms, and for the "consistent" scattering diagram which we will define, these automorphisms make it possible to construct the scheme we were after in the previous subsection.

For a ray $\rho \subset \widetilde{U}^{\mathrm{trop}}$ with rational slope, let $D_\rho := D_{\xi(\rho)}$ be the corresponding boundary divisor in $(\tilde{Y}, \tilde{D})$ (some toric blowup π of (Y, D)). Let $\beta \in H_2(\tilde{Y}, \mathbb{Z})$ with $k_\beta := \beta \cdot D_\rho \in \mathbb{Z}$, and $\beta \cdot D_{\rho'} = 0$ for $\xi(\rho) \neq \xi(\rho')$. Let $F_\rho := \overline{D \setminus D_\rho}$, $\tilde{Y}^\circ_\rho := \tilde{Y} \setminus F_\rho$, and $D^0_\rho := D \setminus F_\rho$.

Now, define $\overline{\mathcal{M}}(\tilde{Y}^\circ_\rho / D^\circ_\rho, \beta)$ to be the moduli space of stable relative maps[3] of genus 0 curves to $\tilde{Y}^\circ_\rho$, representing the class β and intersecting D°_ρ at one unspecified point with multiplicity k_β. This moduli space has a virtual fundamental class with virtual dimension 0. Furthermore, $\overline{\mathcal{M}}(\tilde{Y}^\circ_\rho / D^\circ_\rho, \beta)$ is proper[4] over $\operatorname{Spec} \Bbbk$. Thus, we can define the relative Gromov-Witten invariant N_β as

$$N_\beta := \int_{[\overline{\mathcal{M}}(\tilde{Y}_\rho / D_\rho, \beta)]^{vir}} 1.$$

This is a virtual count of the number of curves in $\tilde{Y}$ of class β which intersect D at precisely one point on D°_ρ. If $N_\beta \neq 0$, we call β an $\mathbb{A}^1$ class.

Recall that η denotes a homomorphism from $\mathrm{NE}(Y)$ to P. We now define

$$f_\rho := \exp\left[\sum_\beta k_\beta N_\beta z^{\eta(\pi_*(\beta)) - \widetilde{\varphi}(k_\beta v_\rho)}\right] \in R_{\rho,\rho}.$$

Here, the sum is over all $\beta \in \mathrm{NE}(\widetilde{Y})$ which have 0 intersection with all boundary divisors except for D_ρ.

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[3] For details on relative Gromov-Witten invariants, see [Li02], or see [GPS09] for a treatment of this particular situation.

[4] See Theorem 4.2 of [GPS09], or Lemma 3.2 of [GHK11].

Example 3.1.2. Consider $\tilde{Y} = \overline{\mathcal{M}_{0,5}}$ as in Example 2.2.6. Let $\beta = E_1$, the (-1)-curve which only hits D_1. Then $N_\beta = 1$. Due to the stacky nature of $\overline{\mathcal{M}}(\tilde{Y}_\rho/D_\rho, \beta)$, N_β might not always be a positive integer. For example, with $\tilde{Y}$ and β as above, we have $N_{k\beta} = \frac{(-1)^{k-1}}{k^2}$ (see [GPS09], Proposition 6.1).

These multiple covers of E_1 are the only $\mathbb{A}^1$ classes for D_1, so we can compute f_{ρ_1} (for $\xi(\rho_1)$ corresponding to D_1). Suppose $P^{gp} := A_1(Y)$ and $\eta := \mathrm{Id}$. We have

$$\begin{aligned} f_{\rho_1} &= \exp\left[\sum_{k \in \mathbb{Z}_{>0}} k \left(\frac{(-1)^{k-1}}{k^2}\right) z^{k[E_1] - \varphi(kv_{\rho_1}) - kv_{\rho_1}}\right] \\ &= 1 + z^{[E_1] - \varphi(v_{\rho_1}) - v_{\rho_1}}. \end{aligned}$$

Suppose we instead take $P := \mathbb{Z}_{\leq 0}$, $\eta(C) := -W \cdot C$ for the ample divisor $W = \sum D_i$. Let t denote the generator for P. We can take $\varphi(v_i) = -1 = t$ for each i, and the exponent becomes $(W \cdot [E_1])t - t - v_{\rho_1} = -v_{\rho_1}$. Then we have $f_{\rho_1} = 1 + z^{-v_{\rho_1}}$.

More generally, if the only $\mathbb{A}^1$-classes hitting $D_{\xi(\rho)}$ are a set $\{E_1, \ldots, E_k\}$ of (-1)-curves, along with their multiple covers, then

$$f_\rho = \prod_{i=1}^{k} \left(1 + z^{\eta(E_i) - \widetilde{\varphi}(v_\rho)}\right)$$

3.1.3 Constructing the Mirror Family

The family $\mathcal{V}$ we wish to construct will be a flat affine deformation of $\mathbb{V}_n$, but we will first construct a flat formal deformation $\hat{\mathcal{V}}$ of $\mathbb{V}_n$. This of course comes from an inverse system of infinitesimal deformations $\mathcal{V}_k$ of $\mathbb{V}_n$.

Note that $P \setminus 0$ corresponds to a maximal ideal $\mathfrak{m} \subset \Bbbk[P]$. Thus, for any $\Bbbk[P]$-algebra R and any $k \in \mathbb{Z}_{\geq 0}$, we have an ideal $\mathfrak{m}^k R$.

As explained in §3.1.1.2, we want to use the scattering diagram to glue $\mathcal{V}^{+}_{\rho_i,\rho_i}$ to $\mathcal{V}^{-}_{\rho_{i+1},\rho_{i+1}}$ by identifying $\mathcal{V}_{\rho_i,\sigma_{i,i+1}}$ with $\mathcal{V}_{\rho_{i+1},\sigma_{i,i+1}}$. Since the scattering diagram generally has infinitely many rays, we cannot usually do this directly.

Instead, we note that there are only finitely many rays ρ in the interior of $\sigma_{i,i+1}$ for which the function $f_\rho \not\equiv 1$ modulo $\mathfrak{m}^k R_{\rho,\rho} \subset \mathfrak{m}^k R_{\rho_i,\sigma_{i,i+1}} = \mathfrak{m}^k R_{\rho_{i+1},\sigma_{i,i+1}}$. This is because there are only finitely many points in $P \setminus k\mathfrak{m}_P$, and $\mathbb{A}^1$-classes with non-vanishing contributions live in $\mathrm{NE}(Y) \setminus k\mathfrak{m}_{\mathrm{NE}(Y)}$. We therefore replace each ring R of the construction with $R_k := R/\mathfrak{m}^k R$.

Now, given a curve $\gamma : [0,1] \to U_0^{\mathrm{trop}}$, we will define a corresponding homomorphism $\Pi_\gamma^{(\pm,\pm)} : \Bbbk[\Lambda_{\gamma(0)}\mathcal{P}_k^{\pm}] \to \Bbbk[\Lambda_{\gamma(1)}\mathcal{P}_k^{\pm}]$. The signs in the superscripts are explained below, and the subscript k's indicate that we are modding out by $\mathfrak{m}^k$. This homomorphism comes from using parallel transport of $\Lambda\mathcal{P}$ along γ, except whenever γ crosses a scattering ray ρ with $f_\rho \not\equiv 1$ modulo $\mathfrak{m}^k R_{\rho,\rho}$, we apply the $\Bbbk[\Lambda_\rho \mathcal{P}_k]$-automorphism

$$z^u \mapsto z^u f_\rho^{\langle n_\rho, r_*(u)\rangle}, \tag{3.2}$$

where n_ρ is a primitive generator of Λ_ρ^* which is 0 along ρ and positive on vectors pointing into the cone from which γ came, and $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Of course, if $\gamma(0)$ and/or $\gamma(1)$ are contained in scattering rays, we need to specify whether or not we apply the automorphisms corresponding to these rays. If the first sign of the superscript of $\Pi_\gamma^{(\pm,\pm)}$ is $+$ (resp. $-$), the decision of whether or not to begin with the scattering automorphism corresponding to $\gamma(0)$ is determined by viewing $\gamma(0)$ as lying infinitesimally counterclockwise (resp. clockwise) of the ray it sits on, and similarly for $\gamma(1)$ with the second sign.

Now, we can identify $U^+_{\rho_i,\sigma_{i,i+1},k} := \operatorname{Spec}(R^+_{\rho_i,\sigma_{i,i+1},k})$ with $U^-_{\rho_{i+1},\sigma_{i,i+1},k}$ using the $\Bbbk[\Lambda_{\sigma_{i,i+1}}\mathcal{P}]$-automorphism given by $\Pi^{+,-}_\gamma$, where $\gamma(0) \in \rho_i$, $\gamma(1) \in \rho_{i+1}$, and $\gamma \subset \sigma_{i,i+1}$. We thus glue $\mathcal{V}^+_{\rho_i,\rho_i,k}$ to $\mathcal{V}^-_{\rho_{i+1},\rho_{i+1},k}$ for all i. Similarly, for each i, we can glue $U^-_{\rho_i,\rho_i,k}$ to $U^+_{\rho_i,\rho_i,k}$ via the automorphism $\Pi^{-,+}_\gamma$, where $\gamma(t) = v_i \in \rho_i$ for all $t \in [0,1]$. Also, recall that we can canonically identify $U^\pm_{\rho,\rho,k}$ with $U^\pm_{\rho',\rho',k}$ whenever $\xi(\rho) = \xi(\rho')$.

Preforming all these gluings yields schemes $\mathcal{V}_k$ which are flat infinitesimal families over $\operatorname{Spec}(\Bbbk[P]/\mathfrak{m}^k)$. Taking the inverse limit with respect to k yields a flat formal deformation $\hat{\mathcal{V}}$ of $\mathbb{V}_n$. Finally, we take the affinization $\mathcal{V} := \operatorname{Spec}\Gamma(\hat{\mathcal{V}}, \mathcal{O}_{\hat{\mathcal{V}}})$.

3.1.4 Broken Lines and the Canonical Theta Functions

In this section we describe a canonical $\Bbbk[P]$-module basis for the global sections of $\mathcal{O}_{\mathcal{V}}$. These sections are called *theta functions*.

Definitions 3.1.3. Let $q \in \widetilde{U}^{\mathrm{trop}}(\mathbb{Z})$, and $Q \in \widetilde{U}^{\mathrm{trop}}$. A *broken line* γ with limits (q, Q) is the data of a continuous map $\gamma : (-\infty, 0] \to \widetilde{U}^{\mathrm{trop}}$, values $-\infty < t_0 < t_1 < \ldots < t_s = 0$, and for each $t \neq t_i$, $i = 0, \ldots, s$, an associated monomial $c_t z^{m_t} \in R_{\gamma(t)} := \Bbbk[\Lambda_{\gamma(t)}\mathcal{P}]$ with $c_t \in \Bbbk$ and $r_*(m_t) = -\gamma'(t)$, such that:

- $\gamma(0) = Q$
- $\gamma_0 := \gamma|_{(-\infty,t_0]}$ and $\gamma_i := \gamma|_{[t_{i-1},t_i]}$ are geodesics (i.e., straight lines with constant velocities).
- For all $t \ll t_0$, $\gamma(t)$ is in some fixed convex cone σ_q containing q, and $m_t = \widetilde{\varphi}(q)$ under parallel transport in σ_q.
- For all $a \in (t_{i-1}, t_i)$ (or $(-\infty, t_0)$ for $i = 0$) and $b \in (t_i, t_{i+1})$, and all relevant $R_{\gamma(t)}$'s identified using parallel transport along γ, we have that $\gamma(t_i)$ is contained

in a scattering ray ρ, and

$$c_b z^{m_b} = (c_a z^{m_a})(c_\rho z^{m_\rho})$$

where $c_\rho z^{m_\rho}$ is any term in the formal power series expansion of $f_\rho^{\langle n_\rho, r_*(m_a)\rangle}$ (so $c_b z^{m_b}$ is a monomial term from the expansion of Equation 3.2).

Remark 3.1.4. We call the choice of monomial $c_\rho z^{m_\rho}$ a *bend.* Note that broken lines in this setup can only bend away from the origin. If we say that a bend is maximal, we will mean that the broken line is bending away from the origin as much as possible (that is, the degree of z^{-v_ρ} in the chosen monomial was as large as possible, so in particular f_ρ must have been a polynomial). We may also call this the maximal bend away from the origin. In §3.1.5 we will see a related scattering diagram in U^{trop} equipped with a different linear structure. In this situation, some broken lines may bend towards the origin, and we will be interested in the broken lines with the maximal allowed bends towards the origin (which in our current setup are always straight lines).

We say that two broken γ and γ' with $\mathrm{Limits}(\gamma) = (q, Q)$ and $\mathrm{Limits}(\gamma') = (q, Q')$ are *equivalent* if they have the same bends (so there is a natural correspondence between the smooth segments of the broken lines, with corresponding segments being parallel). Let $[q, \gamma]$ denote the equivalence class of a broken line γ with limits (q, Q) (the inclusion of q in the notation here is meant to simplify notation in the formulas below).

We say that an equivalence class $[q, \gamma]$ is *infinitely near* a ray ρ ($[q, \gamma]\,\mathrm{IN}\,\rho$ for short) if given any open cone σ containing ρ, there exits a broken line $\gamma' \in [q, \gamma]$

with limits (q, Q') such that $Q' \in \sigma$. We say $[q,\gamma]$ is *positively infinitely near* ρ ($[q,\gamma]\,\mathrm{PIN}\,\rho$) if the same is true for any half-open cone σ^+ containing ρ as a clockwise-most boundary ray. Similarly for *negatively infinitely near* ($[q,\gamma]\,\mathrm{NIN}\,\rho$) with σ^- having ρ a counterclockwise-most boundary ray.

Given a class $[q,\gamma]$, let $c_\gamma z^{m_\gamma}$ denote the monomial attached to the last straight segment of each $\gamma' \in [q,\gamma]$. Now for any ray $\rho \subset \widetilde{U}^{\mathrm{trop}}$, we define

$$T_q^+(\rho) := \sum_{[q,\gamma]\,\mathrm{PIN}\,\rho} c_\gamma z^{m_\gamma}, \qquad \text{and} \qquad T_q^-(\rho) := \sum_{[q,\gamma]\,\mathrm{NIN}\,\rho} c_\gamma z^{m_\gamma}.$$

Now at last we define the *theta functions*. Define $\vartheta_0 = 1$. For $q \in U_0^{\mathrm{trop}}(\mathbb{Z})$ and $\rho \subset \widetilde{U}^{\mathrm{trop}}$, we define

$$\vartheta_q|_{U^{\pm}_{\rho,\rho}} = \sum_{\widetilde{q}|\xi(\widetilde{q})=q} T^{\pm}_{\widetilde{q}}(\rho)$$

Since the $\mathcal{V}^{\pm}_{\rho,\rho}$'s form an open cover of $\mathcal{V}$, this suffices to define the theta functions.

Remark 3.1.5. The scattering diagram we use is called "consistent" because [GHK11] shows that for any $q \in U^{\mathrm{trop}}(\mathbb{Z})$ and any curve γ in U_0^{trop} with $\gamma(0) \in \rho_0$ and $\gamma(1) \in \rho_1$, we have (modulo any positive integer power of $\mathfrak{m}$)

$$\Pi_\gamma^{(\pm_0,\pm_1)}[T_q^{\pm_0}(\rho_0)] = T_q^{\pm_1}(\rho_1). \tag{3.3}$$

That is, the sums of monomials determining the theta functions are "parallel" with respect to this modified parallel transport Π. Furthermore, for any ρ, ρ' with $\xi(\rho) = \xi(\rho')$, $\vartheta_q|_{U^{\pm}_{\rho,\rho}}$ agrees with $\vartheta_q|_{U^{\pm}_{\rho',\rho'}}$ under the canonical identification of $R^{\pm}_{\rho,\rho}$ with $R^{\pm}_{\rho',\rho'}$ (when using multi-valued functions on U^{trop} as in [GHK11] instead of our single-valued

φ on $\widetilde{U}^{\mathrm{trop}}$, Equation 3.3 holding for all γ implies this condition). These conditions are exactly what we need for the theta functions to be well-defined globally.

Theorem 3.1.6 ([GHK11]). *The theta functions form a canonical $\Bbbk[P]$-module basis for the space of global sections of $\mathcal{V}$. That is,*

$$\mathcal{V} = \operatorname{Spec}\left(\bigoplus_{q \in U^{\mathrm{trop}}(\mathbb{Z})} \Bbbk[P]\vartheta_q\right).$$

Furthermore, the multiplication rule can be described as follows: Given $q_1, q_2, q \in U^{\mathrm{trop}}(\mathbb{Q})$, the ϑ_q-coefficient of $\vartheta_{q_1} \cdot \vartheta_{q_2}$ is given by

$$\sum_{\substack{([q_1,\gamma_1],[q_2,\gamma_2]) \\ [q_i,\gamma_i]\,\mathrm{IN}\,\rho_q \\ m_{\gamma_1}+m_{\gamma_2}=q}} c_{Q_1}c_{Q_2}.$$

The part about the multiplication rule is easy to see after noting that ϑ_q is the only theta function with a z^q term along ρ_q.

3.1.5 Another Construction of U^{trop}

We discuss here another point of view on the construction of U^{trop} that will be helpful to us later on. Recall that each seed S induces a linear structure on U^{trop}. U^{trop} with this linear structure may be identified with $N_{\mathbb{R}} = N \otimes \mathbb{R}$, where $(Y, D) \to (\overline{Y}, \overline{D})$ is the toric model corresponding to S and N is the cocharacter lattice of $\overline{Y}$. Suppose that this toric model includes b_i non-toric blowups on D_{m_i}, with corresponding exceptional divisors E_{ij}, $j = 1, \ldots, b_i$.

Now, let $\mathfrak{d}_0$ be the scattering diagram in $N_{\mathbb{R}}$ with rays

$$\left\{ \mathbb{R}m_i, \prod_{j=1}^{b_i}\left(1 + z^{\widetilde{\varphi}(m_i) - \eta(E_{ij})}\right) \middle| \, i = 1, \ldots, n \right\},$$

where η is as in Example 3.1.1. One may use $\mathfrak{d}_0$ to construct a consistent scattering diagram $S(\mathfrak{d}_0)$ as in [KS06] and [GPS09]. All of the rays added to $\mathfrak{d}$ are outgoing, meaning that any broken line crossing these scattering rays can only bend away from the origin. Thus, it is only broken lines crossing $\mathbb{R}_{\geq 0}m_i$ that can bend towards the origin.

U^{trop} with its usual integral linear structure now comes from modifying $N_{\mathbb{R}}$ so that lines which take the maximal allowed bend towards the origin are actually straight (cf. §2.2.1.1). Furthermore, if we break our initial scattering rays up into two outgoing rays by negating the exponents of the $\mathbb{R}_{\geq 0}m_i$ parts of the initial rays, then $S(\mathfrak{d}_0)$ becomes our consistent scattering diagram $\mathfrak{d}$ in U^{trop} from before. This construction is carried out in detail in §3 of [GHK11].

3.1.6 The Cluster Complex

We will now show that lines which do not wrap (cf. §2.2.5.1) bound especially nice parts of the scattering diagram and correspond to particularly simple theta functions. Recall that $\sigma_{u,v} \subset U^{\mathrm{trop}}$ denotes the cone with u on the clockwise-most boundary ray and v on the counterclockwise-most boundary ray. Also recall our notation regarding lines in §2.2.5.1.

Lemma 3.1.7. *Let $q \in U^{\mathrm{trop}}(\mathbb{Z})$ and suppose $L_q^{d<0}$ does not wrap. Let $q_{\pm} := L_q^{d<0}(\pm\infty) \in U^{\mathrm{trop}}(\mathbb{Z})$ (so $q_+ = q$). There is some compactification (Y, D) of U which admits a toric model where all the non-toric blowdowns are on divisors D_u with $u \in \sigma_{q_-,q_+}$ (cf. Figure 3.1(a), where we write V_- and V_+ instead of q_- and q_+).*

Proof. Let v be any vector in σ_{q_+,q_-} forming nonsingular cones with $L(\infty)$ and $L(-\infty)$. Let (Y, D) have the form $D = D_{q_+} + D_v + D_{q_-} + \sum D_i$, where the D_i's correspond to vectors in σ_{q_-,q_+}. Note that $D_v^2 = 0$. Thus, $|D_v|$ gives a fibration of Y over $\mathbb{P}^1$ with rational fibers and with D_{q_+} and D_{q_-} as sections. Let F be the fiber containing $\sum D_i$. We can assume (by taking enough toric blowups) that (Y, D) was chosen so that the $\mathbb{P}^1$'s in F not contained in $\sum D_i$ do not hit nodal points of $\sum D_i$. These $\mathbb{P}^1$'s are then (-1)-curves (for U generic in its deformation class) and can be blown down. On the complement of D_v and F, each fiber is a chain of $\mathbb{P}^1$'s. We can contract all but one of these $\mathbb{P}^1$'s from each chain, and then what remains on the complement of D is just a $\Bbbk^*$ fibration over $\Bbbk^*$; i.e., $(\Bbbk^*)^2$. Thus, we have constructed a toric model of the desired type. □

Note that this toric model is unique except for the choices of exceptional divisors intersecting D_{q_-} and D_{q_+}.

Corollary 3.1.8. *If $L_q^{d<0}$ does not wrap, then for U generic, the only $\mathbb{A}^1$-classes corresponding to rays in σ_{q_-,q_+} are exceptional divisors in one of these toric models.*

Proof. Suppose $C \subset (Y, D)$ is an $\mathbb{A}^1$ class for some $v \in \sigma_{q_-,q_+}$ such that C is not contracted under one of these toric models. Then in this toric model, $\overline{C} \subset (\overline{Y}, \overline{D})$ intersects only divisors corresponding to rays in one half of the plane $N_{\mathbb{R}}$. Since $\sum(\overline{C} \cdot \overline{D}_v)v = 0$ for toric varieties, this is impossible unless $\overline{C}$ only intersects $\overline{D}_{q_-}$ and $\overline{D}_{q_+}$. In this case, C is a component of a fiber other than D_v and F in the above proof, and such fibers are chains of $(\mathbb{P}^1)$'s. Since U is generic, we can assume the fiber contains only two $\mathbb{P}^1$'s, and either one can be contracted in a toric model for the proof of the previous lemma. $\square$

Definition 3.1.9. The *cluster complex* is the union of the cones of the form σ_{q_-,q_+} as in Lemma 3.1.7.

Given this understanding of the scattering diagram in the cluster complex, we can describe many of the theta functions very explicitely. Let $L_q^{d<0}$ and σ_{q_-,q_+} be as above. Note that for any $x \in \sigma_{q_-,q_+}$, the only broken lines with initial direction q and endpoint x must be going clockwise about the origin, and so they will only hit the scattering rays in σ_{q_-,q_+}. Let σ_q be a top-dimensional non-singular cone with q as the clockwise-most endpoint and containing no scattering rays in its interior[5]. Then on $\mathcal{V}_{\sigma_q,\sigma_q} \subset \mathcal{V}$, ϑ_q is given by $z^{\widetilde{\varphi}(q)}$.

Suppose we cross clockwise past a scattering ray in the interior of σ_{q_-,q_+} to a cone corresponding to another patch of $\mathcal{V}$. Let e be a primitive generator of the scattering ray ρ_e, and suppose that a toric model as in Lemma 3.1.7 consists of b_v blowups along D_v. From Example 3.1.2, we know that the scattering automorphism for crossing ρ_v clockwise given by

$$z^v \mapsto z^v \left(\prod_{i=1}^{b_e} \left(1 + z^{\eta(E_i)-\varphi(e)-e}\right) \right)^{e \wedge r_*(v)}$$

If we choose a generic fiber of the mirror family, then that fiber can be identified (up to codimension 2) with a fiber of the $\mathcal{X}$-space.[6] The above scattering automorphism is

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[5] This is possible becuse of Lemma 3.1.7. Such cones make it possible to see patches of $\mathcal{V}$ without going through the whole inverse limit construction.

[6] In fact, [GHK13a] shows that the $\mathcal{X}$-space can be realized as a quotient of the universal mirror $\mathcal{V}$ by a certain torus action.

then exactly an $\mathcal{X}$-mutation formula restricted to this fiber!

In particular, we have:

Proposition 3.1.10. *For any q in the cluster complex, the theta function ϑ_q (restricted to a fiber V of the mirror) is the restriction of a global monomial on the $\mathcal{X}$-space.*

Proof. For q in the cluster complex, the line $L_q^{d<0}$ does not wrap, so the above observations apply. The intersection of $\mathcal{V}_{\sigma_q,\sigma_q}$ with V is a seed torus on which ϑ_q is a monomial. Since it extends to a global function, it is by definition a global monomial. □

3.1.7 Compactifications

Let Δ be a convex rational nonsingular polytope in U^{trop} such that each vertex of Δ is contained in a ray of Σ. Note that Σ induces a polyhedral decomposition $\Sigma\Delta$ on Δ. As in [GHK], we construct from $\Sigma\Delta$ a partial (full if Δ is bounded) compactification $\mathcal{V}_{\Sigma\Delta}$ of $\mathcal{V}$.

First we recall that in the toric situation, the compactified family is the toric variety corresponding to the polytope $Q_\Delta := \varphi(\Delta) + P_{\mathbb{R}}$ (with φ a function on $N_{\mathbb{R}} = U^{\mathrm{trop}}$ rather than on $\widetilde{U}^{\mathrm{trop}}$). The general fiber is the toric variety corresponding to $\Delta \subset N_{\mathbb{R}}$, while the central fiber is $\mathbb{V}_n(\Delta)$, a compactification of $\mathbb{V}_n$ where the irreducible components are the toric varieties corresponding to the cells of $\Sigma\Delta$ (cf. [GS11]).

As in the construction of $\mathcal{V}$, the idea behind the general construction is to do the toric construction locally on $\widetilde{U}^{\mathrm{trop}}$ and to use the scattering diagram for gluing. Let $\widetilde{\Sigma\Delta}$ be the lift $\widetilde{\Delta}$ of $\Delta \setminus \{0\}$ by ξ with the polyhedral decomposition coming from the lift $\widetilde{\Sigma}$ of $\Sigma\setminus\{0\}$. Given a maximal dimensional cell $\sigma \in \widetilde{\Sigma\Delta}$, let Q_σ denote the polytope

$\varphi(\sigma) + P_{\mathbb{R}}$ embedded in $T_\sigma \mathcal{P}$. For any cell ρ in $\widetilde{\Sigma\Delta}$, define $Q_\rho = \bigcup_{\sigma \supset \rho} Q_\sigma \subset T_\rho \mathcal{P}$, where the union is over the maximal dimensional cells containing ρ. Now we define a cone $\kappa_{\rho,\mathbb{R}} \subseteq T_\rho \mathcal{P}$ generated by

$$\{x - y \in T_\rho \mathcal{P} : x \in Q_\rho, y \in \varphi(\rho)\}.$$

Let κ_ρ denote the integer points of $\kappa_{\rho,\mathbb{R}}$. Note that if $\rho \in \widetilde{\Sigma}$, then κ_ρ is just $\tau_{\rho,\rho}$ from §3.1.1.1.

Thus, the new cones for this construction come from taking ρ to be in a boundary component of Δ. If $F_{i,i+1}$ denotes the edge $\sigma_{i,i+1} \cap \varphi(\partial\widetilde{\Delta})$, and $p_i = F_{i-1,i} \cap F_{i,i+1} = \rho_i \cap \varphi(\partial\widetilde{\Delta})$, then $\Bbbk[\kappa_{p_i}]$ is a toric subring of $\Bbbk[\tau_{\rho_i,\rho_i}]$. $\operatorname{Spec}(\Bbbk[\kappa_{p_i}]) \setminus \operatorname{Spec}(\Bbbk[\tau_{\rho_i}])$ contains two toric boundary divisors, corresponding to the faces sitting over $F_{i-1,i}$ and $F_{i,i+1}$.

Now, the construction of the compactified family $\mathcal{V}_\Delta$ proceeds as for $\mathcal{V}$, forming inverse systems of quotients of the $\Bbbk[\kappa_{p_i}]$'s and using the scattering automorphisms to glue. $\mathcal{V}_\Delta \setminus \mathcal{V}$ is a set of divisors $\{\mathcal{D}_i\}$ corresponding to the F_i's, with two divisors being identified whenever the corresponding faces of $\widetilde{\Delta}$ are related by some integer power of μ.

To show that this construction is well-defined and that each face really gives a single, well-defined boundary divisor, we have to check that $\mathcal{D}_{F_{i,i+1}} := \operatorname{Spec} \Bbbk[\kappa_{F_{i,i+1}}] \setminus \operatorname{Spec} \Bbbk[\kappa_{\sigma_{i,i+1}}]$ is preserved when crossing a scattering ray in $\sigma_{i,i+1}$. Let ρ_u be such a scattering ray, generated by primitive $u \in \sigma_{i,i+1}$. Let v be a primitive vectors tangent to $F_{i,i+1}$. Then $\Bbbk[\kappa_{F_{i,i+1}}] = \sqrt{\Bbbk[z^{\pm v}, z^{-u}]}$ (i.e., the radical of the subring of $\Bbbk[\sigma_{i,i+1}]$ generated by $z^{\pm v}$ and z^{-u}). $\mathcal{D}_{F_{i,i+1}}$ is the zero set of z^{-u}. This zero set is not changed by crossing ρ_u because z^{-u} is invariant under the corresponding scattering automorphism.

Let $L_{v_F}^{d_F>0}$ be the line containing some edge F of Δ. Let ρ be a ray intersecting F. The valuation (i.e., the order of vanishing) of some $z^{(q,p)} \in R_{\rho_i,\rho_i}$ $(q = r((q,p)))$ along the divisor $\mathcal{D}_F$ is

$$\mathrm{val}_{\mathcal{D}_F}\left(z^{(q,p)}\right) = v \wedge q. \tag{3.4}$$

We will use this to explicitely describe valuations of theta functions in the next section.

3.2 Tropical Theta Functions

3.2.1 Tropicalization of the Mirror

We know from [GHK] that generic fibers of the mirror $\mathcal{V}$ are deformation equivalent to our the original space U. Thus, the tropicalization V^{trop} of a generic fiber V is non-canonically isomorphic to U^{trop}, and any construction done using U and U^{trop} can similarly be done using V and V^{trop}. We describe here some ways to identifiy V^{trop} with U^{trop}.

Notation 3.2.1. We will always use gothic $\mathfrak{D}$'s to denote divisors on the boundary of a generic fiber V of the mirror. Script $\mathcal{D}$'s denote boundary divisors for the whole mirror family. We will use $(Z, \mathfrak{D})$ to denote a compactification of V.

Remark 3.2.2. Theta functions were defined using broken lines in $\widetilde{U}^{\mathrm{trop}}$, and compactifications were defined using polygons in $\widetilde{U}^{\mathrm{trop}}$ which are invariant under the monodromy. By the consistency of the scattering diagram and the monodromy invariance of the polygons, we can study the images of these things in U^{trop} rather than working in $\widetilde{U}^{\mathrm{trop}}$. Understanding the monomials attached to the theta functions is somewhat delicate (interpreting the exponents requires introducing a certain bundle over U^{trop} described in [GHK11]), but for the rest of this paper we only need to know the images of the exponents under r_*, which can easily be viewed as living in the tangent space to U^{trop}. We thus use U^{trop} instead of $\widetilde{U}^{\mathrm{trop}}$ throughout the rest of the paper.

As we just saw in §3.1.7, lines with rational slope in U^{trop} determine boundary divisors of $\mathcal{V}$. In the construction above, the divisor does not depend on the vector attached to the line or on the distance of the line from the origin. Given a primitive vector $v \in U^{\mathrm{trop}}$, we can associate the divisor $\mathfrak{D}_{L_v^{d>0}}$ corresponding to $L_v^{d>0}$. Similarly, for $v = |v|v'$ with v' primitive and $|v|$ a non-negative rational number, we associate the divisor $|v|\mathfrak{D}_{L_v^{d>0}}$. This gives an identification of $U^{\mathrm{trop}}(\mathbb{Q})$ with $V^{\mathrm{trop}}(\mathbb{Q})$ which restricts to an identification of $U^{\mathrm{trop}}(\mathbb{Z})$ with $V^{\mathrm{trop}}(\mathbb{Z})$. We will see that this extends to an integral linear identification $w_U : U^{\mathrm{trop}} \rightarrow V^{\mathrm{trop}}$. This is the identification we will primarily use.

Convention 3.2.3. We give V^{trop} the opposite orientation of that induced by w_U.

Alternatively, given $v = |v|v'$ as above, we can associate $|v|\mathfrak{D}_{L_v^{d<0}}$. This is equivalent to doing the above identification with the orientation of U^{trop} reversed (i.e., using the orientation of V^{trop}). We will not use this identification $U^{\mathrm{trop}} \rightarrow V^{\mathrm{trop}}$, but it is closely related to what we will call $w_V : V^{\mathrm{trop}} \rightarrow U^{\mathrm{trop}}$ in §3.2.5.

As another alternative, suppose that H is invertible over $\mathbb{Q}$, as in Lemma 2.2.10. From Example 2.2.11, we know that for U positive, this only fails in the I_k cases of §2.3.4.1 (which are the simplest cases anyways). Recall the notation ψ_v and b_v from Lemma 2.2.10. Given a primitive vector $v \in U^{\mathrm{trop}}(\mathbb{Z})$, we can associate an edge L_{ψ_v} defined by $\psi_{\rho_v} = d < 0$. We then define $w_\psi(v) \in V^{\mathrm{trop}}(\mathbb{Q})$ to be the point corresponding to $\frac{1}{b_v}\mathfrak{D}_{L_{\psi_v}}$. Scaling by $\mathbb{Q}$, this is easily extended to a bijection $w_\psi : U^{\text{trop}}(\mathbb{Q}) \to V^{\text{trop}}(\mathbb{Q})$, and one can show that this extends to rational linear isomorphism $w_\psi : U^{\text{trop}} \to V^{\text{trop}}$.

3.2.2 Tropicalizing Functions

For any rational function f on V, we define an integral piecewise-linear function $f^{\text{trop}} : V^{\text{trop}} \to \mathbb{R}$ as follows: for $v \in V^{\text{trop}}(\mathbb{Z})$, $f^{\text{trop}}(v) := \operatorname{val}_{\mathfrak{D}_v}(f)$. Then extend f^{trop} linearly to the real points of V^{trop}.

For this section, we once again call $\mathbb{R}$-valued functions convex if their bending parameters are non-positive (i.e., we take $P := \mathbb{Z}_{\leq 0}$).

Lemma 3.2.4. *If f is regular on V, then f^{trop} is convex.*

Proof. Let $(Z, \mathfrak{D})$ be a nonsingular compactification of V such that any ray on which f^{trop} is nonlinear corresponds to some component of $\mathfrak{D}$. The principal divisor corresponding to f is $(f) = \mathfrak{D}_f^0 - \mathfrak{D}_f^\infty + V(f)$, where $\mathfrak{D}_f^0$ denotes the divisor of zeroes of f on the boundary, $\mathfrak{D}_f^\infty$ denotes the divisor of poles of f on the boundary, and $V(f)$ denotes the interior zeroes of f. So f^{trop} is the integral piecewise-linear function on V^{trop} corresponding to the Weil divisor $\mathfrak{D}_f^0 - \mathfrak{D}_f^\infty$, and the bending parameter along some ρ_v is given by $\mathfrak{D}_v \cdot (\mathfrak{D}_f^0 - \mathfrak{D}_f^\infty) = -\mathfrak{D}_v \cdot V(f) \leq 0$. □

The properties of valuations give us the following relations for all rational functions on V:

$$(fg)^{\text{trop}} = f^{\text{trop}} + g^{\text{trop}}$$

$$(f+g)^{\mathrm{trop}} \geq \min(f^{\mathrm{trop}}, g^{\mathrm{trop}}) \tag{3.5}$$

Furthermore, the second relation is an equality at points where $f^{\mathrm{trop}} \neq g^{\mathrm{trop}}$. Suppose that there exists a $v \in U^{\mathrm{trop}}$ such that $(f+g)^{\mathrm{trop}}(v) > \min[f^{\mathrm{trop}}(v), g^{\mathrm{trop}}(v)]$. Then, by continuity, there must be some open cone σ in U^{trop} containing v where $f^{\mathrm{trop}} = g^{\mathrm{trop}}$. We will see that if f and g are theta functions, then having $f|_\sigma = g|_\sigma$ for open σ implies $f = g$. So the inequality in Equation 3.5 is an equality for theta functions, and similarly for any finite sum theta functions with positive coefficients.

Remark 3.2.5. We will need that the monomials attached to the broken lines contributing to a theta function do not cancel with each other when added together. This is proved in [GHKK].

3.2.3 The Valuation Functions

Given a vector $v \in U^{\mathrm{trop}}$, we define an integral piecewise-linear function $\mathrm{val}_v : U^{\mathrm{trop}} \to \mathbb{R}$ as follows. For $d \leq 0$, the fiber $\{\mathrm{val}_v = d\}$ is the set $L_v^{-d,0}$. If L_v^{-d} wraps, then this completely defines val_v.

If L_v^{-d} does not wrap, then these fibers with $d < 0$ miss some cone $\sigma \subset U^{\mathrm{trop}}$. In this case, for $d > 0$, the fiber $\{\mathrm{val}_v = d\}$ is the broken line with initial direction v and signed lattice distance $-d$ from the origin which takes the maximal allowed bend across every scattering ray that it crosses. By §3.1.6, there are only finitely many such scattering rays. We call this broken line $\mathfrak{L}_v^{-d}$.

By taking a toric model corresponding to scattering rays in σ as in Lemma 3.1.7, we can see that there is some seed S with respect to which each $L_v^{-d>0}$ and each $\mathfrak{L}_v^{-d<0}$ is straight and goes to ∞ parallel to v. Thus, val_v is indeed a well-defined integral convex piecewise-linear function. In fact, with respect to the linear structure corresponding to this seed, val_v is given by $v \wedge \cdot$.

Note that differentiating gives us a function $D\,\mathrm{val}_v : TU^{\mathrm{trop}}_{\mathrm{val}_v} \to \mathbb{R}$, where $U^{\mathrm{trop}}_{\mathrm{val}_V}$ denotes the complement in U_0^{trop} of the singular locus of $D\mathrm{val}_v|_{U_0^{\mathrm{trop}}}$. Note that if we identify q with a vector $\widetilde{q}$ in its tangent space, then $D\,\mathrm{val}_v(\widetilde{q}) = \mathrm{val}_v(q)$.

Lemma 3.2.6. *Let γ be a broken line with $m_t = -\gamma'(t)$ being (r_* of) the attached monomial at some time t. If $t_2 > t_1$, then $D\,\mathrm{val}_v(m_{t_2}) \geq D\,\mathrm{val}_v(m_{t_1})$ (assuming the t_i's are generic enough for each side to be defined).*

As in [GHKK], we say that functions satisfying this condition for all broken lines are *decreasing along broken lines* (since they decrease on the tangent directions of the broken lines).

Proof. First note that val_v being convex means that the bends of val_v while moving along γ will only increase $D\,\mathrm{val}_v(m_t)$, as desired. Now let ρ_u (the ray generated by some primitive u) be the only scattering ray where γ bends between times t_1 and $t_2 = t_1 + \epsilon$. Then $m_{t_2} = m_{t_1} - ku$ for some $k \in \mathbb{Z}_{\geq 0}$.

Suppose that $\mathrm{val}_v \leq 0$ everywhere. In particular, $\mathrm{val}_v(u) \leq 0$. Then

$$\begin{aligned} D\,\mathrm{val}_v(m_{t_2}) &\geq D\,\mathrm{val}_v(m_{t_1}) - kD\,\mathrm{val}_v(u) \\ &\qquad = D\,\mathrm{val}_v(m_{t_1}) - k\,\mathrm{val}_v(u) \geq D\,\mathrm{val}_v(m_t). \end{aligned}$$

On the other hand, suppose val_v is positive somewhere. Let σ be the cone on which it is non-negative, and S a corresponding seed as in Lemma 3.1.7. Let γ

bend along some ray ρ_u between times t_1 and $t_2 = t_1 + \epsilon$ as before. If $u \notin \sigma$, then $\mathrm{val}_v(u) \leq 0$, and we again see $D\,\mathrm{val}_v(m_{t_2}) \geq D\,\mathrm{val}_v(m_{t_1})$. Otherwise, we work with the linear structure and scattering diagram on U^{trop} corresponding to the seed S (cf. §3.1.5). With respect to this structure, broken lines in σ bend towards the origin, so $m_{t_2} = m_t + ku$, $k \in \mathbb{Z}_{\geq 0}$, and so we still have $D\,\mathrm{val}_v(m_{t_2}) \geq D\,\mathrm{val}_v(m_{t_1})$, as desired. $\square$

Define $\mathbf{val}_v(\vartheta_q) := \min_{[q,\gamma]} \left[\min_{t\in(-\infty,0]} D\,\mathrm{val}_v(-\gamma'(t))\right]$, where the first min is over all equivalence classes of broken lines with initial direction q. More generally, for a function $f = \sum_{i\in I} a_i\vartheta_{q_i}$ with $a_i \neq 0$ for each $i \in I$, define $\mathbf{val}_v(f) = \min_{i\in I} \mathbf{val}_v(\vartheta_{q_i})$. The above lemma implies:

Corollary 3.2.7. $\mathbf{val}_v(\vartheta_q) = \mathrm{val}_v(q)$.

Lemma 3.2.8. $\mathbf{val}_v(\vartheta_{q_1}\vartheta_{q_2}) = \mathbf{val}_v(\vartheta_{q_1}) + \mathbf{val}_v(\vartheta_{q_2})$.

Proof. Suppose that val_v is non-positive everywhere. Then it only bends along a single ray ρ. If we take a branch cut along ρ, U^{trop} can be identified with a convex cone on which val_v is linear. On the other hand, if val_v is positive somewhere then we have seen that there is some seed with respect to which val_v is linear.

In either case, Theorem 3.1.6 and Remark 3.2.5 imply that $\vartheta_{q_1}\vartheta_{q_2}$ has a $\vartheta_{q_1+q_2}$ term (addition performed with respect to the above-mentioned linear structure or branch cut on U^{trop} that makes val_v linear). The linearity of val_v then gives us $\mathbf{val}_v(\vartheta_{q_1+q_2}) = \mathbf{val}_v(\vartheta_{q_1}) + \mathbf{val}_v(\vartheta_{q_2})$. Similarly, Theorem 3.1.6 and Lemma 3.2.6 imply we cannot get any larger values, so the equality holds.

$\square$

Theorem 3.2.9. *Under the identification* w_U, $\mathrm{val}_v(q) = \mathrm{val}_{\mathfrak{D}_v}(\vartheta_q)$. *Thus,* $\mathbf{val}_v(f) = \mathrm{val}_{\mathfrak{D}_v}(f)$.

Proof. Suppose that $q = L_v^d(t_q) \in L_v^{d,0}$ for some $d > 0$. We see from Equation 3.4 and the definition of theta functions that

$$\mathrm{val}_{\mathfrak{D}_v}(\vartheta_q) = \min_{[q,\gamma] | \gamma(0) = q} v \wedge m_\gamma, \tag{3.6}$$

where v may be interpreted as $\gamma'(t_q)$. By the definition of $\mathbf{val}_v$ and the fact that $v \wedge m_\gamma = D\,\mathrm{val}_v(m_\gamma)$, the right-hand side is $\geq \mathbf{val}_v(\vartheta_q)$, which by Corollary 3.2.7 equals $\mathrm{val}_v(q)$. The straight broken line contained in ρ_q gives us equality.

Now suppose $\mathrm{val}_v(q) = d \geq 0$. Let $p \in L_v^c$ for some $c > 0$. Then, as in Equation 3.6, we have

$$\mathrm{val}_{\mathfrak{D}_v}(\vartheta_q) = \min_{[q,\gamma] | \gamma(0) = p} v \wedge m_\gamma,$$

and this is still $\geq \mathbf{val}_v(\vartheta_q) = \mathrm{val}_v(q) = d \geq 0$.

Now, pick any q' with $\mathrm{val}_v(q') < -d$. We can write $\vartheta_q \vartheta_{q'} = \sum_{r \in I} a_r \vartheta_r$, $a_r \neq 0$, for some $I \subset U^{\mathrm{trop}}(\mathbb{Z})$. Lemma 3.2.8 tells us that $\mathbf{val}_v(\vartheta_q \vartheta_{q'}) = d - d' < 0$. In particular, there is some $r \in I$ with $\mathrm{val}_{\mathfrak{D}_v}(\vartheta_r) = \mathbf{val}_v(\vartheta_r) = d - d' < 0$, so we do not need to worry about the $r \in I$ for which $\mathrm{val}_{\mathfrak{D}_v}(\vartheta_r) \geq 0$. The previous paragraph shows that these are the r for which $\mathrm{val}_v(r) \geq 0$.

Thus, we have

$$\mathrm{val}_{\mathfrak{D}_v}(\vartheta_q \vartheta_{q'}) = \min_{r \in I} \mathrm{val}_{\mathfrak{D}_v} \vartheta_r = \mathbf{val}_v(\vartheta_q \vartheta_{q'}) = d - d'.$$

Since valuations are additive, this implies that $\mathrm{val}_{\mathfrak{D}_v}(\vartheta_q) = d = \mathrm{val}_v(q)$, as desired. □

3.2.4 Tropical Theta Functions

The previous subsection tells us that $\vartheta_q^{\mathrm{trop}}(v) = \mathbf{val}_v(\vartheta_q) = \mathrm{val}_v(q)$. In this subsection we will explicitely describe the fibers of $\vartheta_q^{\mathrm{trop}}$ in V^{trop}.

Notation 3.2.10. We will use the notation $\wedge_{q^+}$ to indicate we are using the wedge product on defined on U^{trop} by cutting along ρ_q and then identifying ρ_q with the clockwise-most boundary ray of $U^{\mathrm{trop}} \setminus \rho_q$ (so $q \wedge v \geq 0$ for nearby v in $U^{\mathrm{trop}} \setminus \rho_q$). Similarly, for $\wedge_{q^-}$ we identify ρ_q with the counterclockwise-most boundary ray.

Lemma 3.2.11. *If* $\mathrm{val}_v(q) \leq 0$, *then*

$$\begin{aligned}
\mathrm{val}_v(q) &= \min_{t \in \mathbb{R} | L_v^{d>0}(t) \in \rho_q} \left\{ (L_v^{d>0})'(t) \wedge q \right\} \cup \{0\} \\
&= \min_{i=0,\ldots,k} \left\{ \mu^{-i} v \wedge_{v^+} q \right\} \cup \{0\} \\
&= \min_{i=0,\ldots,k} \left\{ v \wedge_{q_-} \mu^i q \right\} \cup \{0\} \\
&= \min_{t \in \mathbb{R} | L_q^{d<0}(t) \in \rho_v} \left\{ v \wedge (L_q^{d<0})'(t) \right\} \cup \{0\}
\end{aligned}$$

where k is the smallest non-negative integer such that $v \wedge_{q_-} \mu^{k+1} q \geq 0$.

Proof. Let $t_1, \ldots, t_k$ be the times at which $L_v^{d>0}(t)$ intersects ρ_q. For the first equality, note that if for some d_i, $L_v^{d_i>0}(t_i) = q$, then $(L_v^{d_i>0})'(t) \wedge q$ is negative the lattice distance of the line from the origin at that time[7] (i.e., $-d_i$). Since $L_v^{d>0,0}$ contains the point of $\rho_q \cap L_v^{d>0}$ closest to the origin, say, $L_v^{d>0}(t_m)$, we have that d_m is the largest d_i's. Hence, the min in the first equality is obtained at $L_v^{d>0}(t_m) \in L_v^{d>0,0}$. Since $(L_v^{d>0})'(t_m) \wedge q = \mathrm{val}_v(q)$, this proves the first equality.

---

[7] When we multiply d by a positive scalar c, we map $L_v^d(t)$ to $cL_v^d(t)$. That way the times $t_1, \ldots, t_k$ are unchanged.

The second equality follows by noting that each time we follow $L_v^{d>0}$ around the origin (moving backwards along the line), the tangent vector (initially v) is multiplied by μ^{-1}. Note that k as in the statement of the theorem is the number of times that the $L_v^{d>0}$ intersects ρ_q.

The third equality follows from the fact that $\mu \in SL_2(\mathbb{Z})$, and so $a \wedge b = \mu(a) \wedge \mu(b)$. The fourth equality follows symmetrically to the second equality. $\square$

Corollary 3.2.12. *Under the identification w_U of U^{trop} with V^{trop}, for $d < 0$, $L_q^{d,0}$ is the fiber $\{v \in U^{\mathrm{trop}} | \vartheta_q^{\mathrm{trop}}(v) = d\}$.*

Proposition 3.2.13. *Under the identification w_U of U^{trop} with V^{trop}, for $d > 0$, $\mathfrak{L}_q^{d,0}$ is the fiber $\{v \in U^{\mathrm{trop}} | \vartheta_q^{\mathrm{trop}}(v) = d\}$.*

Proof. The first statement is clear from what we have already said. For the second statement, let γ_q and γ_v be broken lines with initial tangent vectors q and v, respectively, which are supported on $\mathfrak{L}_q^{d,0}$ and $\mathfrak{L}_v^{-d,0}$, respectively. Let q_1, v_1 be negative of the tangent vectors to γ_q and γ_v, respectively, on the counterclockwise-side of a scattering ray ρ_v generated by primitive vector v, and similarly for q_2 and v_2 on the clockwise-side of ρ_u.

It suffices to show that $v_1 \wedge q_1 = v_2 \wedge q_2$. Let b_u be the degree of the scattering function attached to ρ_u (so for U generic, it is the number of (-1)-curves hitting D_u). Then when crossing in the counterclockwise direction, q_2 changes to $q_1 = q_2 + b_u(u \wedge q_2)u$, while v_2 changes to $v_1 = v_2 + b_u(u \wedge v_2)u$. So indeed,

$$v_1 \wedge q_1 = v_2 \wedge q_2 + b_u(u \wedge v_2)(u \wedge q_2) + b_u(v_2 \wedge u)(u \wedge q_2) = v_2 \wedge q_2.$$

$\square$

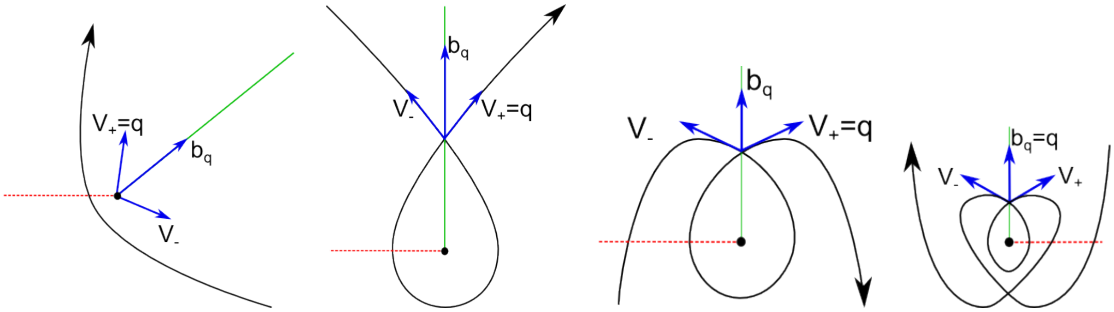
3.2.5 Symmetry of the Dual Pairing

Note that we have a canonical pairing $\langle \cdot, \cdot \rangle_{\mathbb{Z}} : U^{\mathrm{trop}}(\mathbb{Z}) \times V^{\mathrm{trop}}(\mathbb{Z}) \to \mathbb{Z}$ defined by $\langle q, v \rangle := \vartheta_q^{\mathrm{trop}}(v) = \mathrm{val}_{\mathfrak{D}_v}(\vartheta_q)$. This can be extended to a pairing $\langle \cdot, \cdot \rangle : U^{\mathrm{trop}} \times V^{\mathrm{trop}} \to \mathbb{R}$ as follows: extending to rational points is easy because the pairing is linear with respect to multiplication by non-negative rational (and real) numbers in either variable. Fixing one variable gives a piecewise-linear (in particular, continuous) function in the other, and so we can extend continuously to the real points for both variables.

On the other hand, since V is itself a log Calabi-Yau surface (deformation equivalent to U), we could apply the mirror constructions of §3.1 to V to construct a mirror family $\mathcal{U}$ to V, with points $v \in V^{\mathrm{trop}}(\mathbb{Z})$ corresponding to canonical theta functions ϑ_v on $\mathcal{U}$. U (or at least some deformation of U) may be identified with a fiber of $\mathcal{U}$, and so we obtain a map $w_V : V^{\mathrm{trop}} \to U^{\mathrm{trop}}$ analogously to how we defined w_U (here, it is important to remember that we take the orientation of V^{trop} to be opposite that induced by w_U). Corollary 3.2.12 and Proposition 3.2.13 hold as before with the roles of U^{trop} and V^{trop} interchanged. We see:

Theorem 3.2.14. *For $q \in U^{\mathrm{trop}}$ and $v \in V^{\mathrm{trop}}$, $\vartheta_q^{\mathrm{trop}}(v) = \vartheta_v^{\mathrm{trop}}(q)$. Thus, the pairing $\langle \cdot, \cdot \rangle$ does not depend on which side we view as the mirror.*

Proof. Note that the support of $w_U(L_q^{d,0})$ is the same as that of $L_{w_U(q)}^{-d,0}$, and similarly with $w_U(\mathfrak{L}_q^{d,0})$ and $\mathfrak{L}_{w_U(q)}^{-d,0}$. The negation of the distance comes from the difference in orientation between U^{trop} and V^{trop}. We want to show that $\vartheta_q^{\mathrm{trop}}(v) = \mathrm{val}_q(v)$. This follows immediately from comparing the definition of val_q in §3.2.3 to the descriptions of $\vartheta_q^{\mathrm{trop}}$ in Corollary 3.2.12 and Proposition 3.2.13. $\square$



(a) (b) (c) (d)

Figure 3.1: Some lines L_q which (a) do not wrap; (b) wrap once; (c) wrap twice (as in the E_7 case); and (d) wrap three times (as in the E_8 case). The dashed red rays indicate our chosen branch cuts. The blue vectors denote the boundary vectors and the bend b_q. The green rays are the rays along which the functions bend. The curved appearance of the lines occurs because the projection of U^{trop} onto the page is not an isometry.

Remark 3.2.15. If we use the identification w_ψ instead of w_U, then for $q, v \in U^{\mathrm{trop}}$ with $\langle q, w_\psi(v)\rangle \leq 0$, one could show that $\langle q, w_\psi(v)\rangle = \beta_v(q)$ (notation as in Lemma 2.2.10). Then the symmetry of Theorem 3.2.14 exactly means that $\beta_v(q) = \beta_q(v)$, which is precisely what Lemma 2.2.10 says. So from this perspective, the symmetry of the pairing is essentially a consequence of the symmetry of the intersection form.

3.2.6 Bending Parameters of Tropical Theta Functions

3.2.6.1 Bends of the Negative Fibers

Let $d < 0$. Recall that $L_q^{d,0}$ is the fiber $\vartheta_q^{\mathrm{trop}}(v) = d$ by Corollary 3.2.12. Either $L_q^{d,0}$ is unbounded as in Figure 3.1(a), or, if L_q^d self-intersects, then $L_q^{d,0}$ is bounded as in Figure 3.1(b,c,d). It is clear from these figures that there is some $b_q \in V^{\mathrm{trop}}(\mathbb{Z}) = w_U(U^{\mathrm{trop}}(\mathbb{Z}))$ such that $\vartheta_q^{\mathrm{trop}} = \beta_{b_q}$ whenever both are negative (β_{b_q} is as defined in §2.2.4.1). Furthermore, the ray ρ_{b_q} should intersect the vertex of $L_q^{d,0}$ (if there is one).

To find b_q, we first define the *boundary vectors* of $L_q^{d,0}$ (see Figure 3.1). If $L_q^{d,0}$ is unbounded, then we say the boundary vectors of $L_q^{d,0}$ are $V_+ := L_q^d(\infty) = q$ and $V_- := L_q^d(-\infty)$. Otherwise, let $t_1 < t_2 \in \mathbb{R}$ denote the initial and final times times for which $L_q^d(t) \in L_q^{d,0}$. Then the boundary vectors are $V_+ := (L_q^d)'(t_2)$ and $V_- := -(L_q^d)'(t_1)$ (so V_+ is the outward flow, and V_- is the inward flow, which we negate). Note that $V_- = -\mu(V_+)$. We can add these tangent vectors and identify the sum with a point in U^{trop}. We claim that $b_q := V_- + V_+$. In fact, this is easy to see: just observe that when we cross the ray ρ_{b_q} in the counterclockwise direction, $\vartheta_q^{\mathrm{trop}}$ changes from $\cdot \wedge (-V_-)$ to $\cdot \wedge V_+ = \cdot \wedge (-V_- + b_q)$, which indeed means that $\vartheta_q^{\mathrm{trop}}$ equals β_{b_q}.

It follows immediately from the above argument that if $L_q^{d>0}$ wraps at most once (Figure 3.1(a,b)), then $b_q = q - \mu q$ (where we choose a cut which hits L_q^d exactly once). In terms of our classification in §2.3, the $Q = E_7$ and E_8 cases are the only ones where lines wrap more than once (Figure 3.1(c,d)). We take cuts as in the figures. In the E_7 case, we still have that $b_q = q - \mu(q)$. In the E_8 case, we find $b_q = \mu(q) - \mu^2(q) = q$. In particular, we note:

Lemma 3.2.16. *The map* $b : U^{\mathrm{trop}}(\mathbb{Z}) \to U^{\mathrm{trop}}(\mathbb{Z})$, $q \mapsto b_q$, *extends to an integral linear endomorphism of* U^{trop}.

3.2.6.2 Bends of the Positive Fibers

We continue to use the identification w_U. Let $\rho \subset U^{\mathrm{trop}}$ be a ray along which $\vartheta_q^{\mathrm{trop}}$ is non-negative. Let b_ρ be the degree of the scattering function f_ρ (so for U generic, it is the number of (-1)-curves intersecting D_ρ).

Proposition 3.2.17. *Let* $v \in U_0^{\mathrm{trop}}(\mathbb{Z})$ *(identified with* $V_0^{\mathrm{trop}}(\mathbb{Z})$ *by* w_U*) be primitive, generating a ray* ρ_v*. Suppose* $\vartheta_q^{\mathrm{trop}}(v) \geq 0$*, and assume that* $\vartheta_q^{\mathrm{trop}}$ *is positive somewhere. Then the bending parameter of* $\vartheta_q^{\mathrm{trop}}$ *along* ρ *is* $-b_\rho \vartheta_q^{\mathrm{trop}}(v)$*.*

Proof. This follows immediately from Proposition 3.2.13, the definition of broken lines, and the description of the scattering diagram in §3.1.6. In fact, if U is generic and $E_1, \ldots, E_{b_i}$ are the (-1)-curves intersecting $\mathfrak{D}_\rho$, then it follows from the desription of ϑ_q in Proposition 3.1.10 that $\mathrm{val}_{E_i} \vartheta_q = \vartheta_q^{\mathrm{trop}}(v)$ for each i, and all the zeroes of ϑ_q are along (-1)-curves like this. The description of the bending parameters then follows from the relationship between bending parameters and intersection numbers in Lemma 3.2.4. $\square$

Remark 3.2.18. We note that the local coordinate description of piecewise-linear functions from §2.2.4.2 easily implies that the sum of the bends of $\vartheta_q^{\mathrm{trop}}$ must equal b_q of §3.2.6.1, and similarly for val_v.

We can now prove:

Lemma 3.2.19. *Suppose that two tropical theta functions* $\vartheta_{q_1}^{\mathrm{trop}}$ *and* $\vartheta_{q_2}^{\mathrm{trop}}$ *are equal on some open cone* $\sigma \subseteq U^{\mathrm{trop}}$*. Then* $q_1 = q_2$*.*

Proof. Suppose that there is some subcone of σ on which the functions are negative. Then the fiber $\vartheta_{q_1}^{\mathrm{trop}} = \vartheta_{q_2}^{\mathrm{trop}} = -1$ is a line segment L in U^{trop}, and extending this segment to ∞ (with 0 on the right) recovers $q_1 = q_2$.

Now suppose that $\vartheta_{q_1}^{\mathrm{trop}} = \vartheta_{q_2}^{\mathrm{trop}} \geq 0$ everywhere on σ. Recall that this means $\vartheta_{q_i}^{\mathrm{trop}}$ will bend along each $\rho_v \subset \sigma$ with bending parameter $-b_{\rho_{v_i}} \vartheta_{q_i}^{\mathrm{trop}}(v_i)$, where v_i is primitive on ρ_i and $b_{\rho_{v_i}}$ is as above. Thus, we know how to extend the fibers to infinity to determine the q_i's. $\square$

3.2.7 Convexity Properties

We saw in Lemma 3.2.4 that tropicalizations of regular functions are convex. [GHKK] defines a stronger version of convexity, namely, *convexity along broken lines.* Recall from §2.2.1.1 that to define a linear structure on a piecewise-linear manifold, it suffices to specify which piecewise-straight lines are straight.

Definition 3.2.20. Let $\gamma : (-\infty, 0] \to U^{\mathrm{trop}}$ be a broken line, and let φ be a rational piecewise linear function on U^{trop}. In a neighborhood of a point $\gamma(t_p) = p$ contained in a ray ρ, we can modify the linear structure of U^{trop} so that $\gamma'(t)$ is constant in a neighborhood of t_p (with adjacent tangent spaces identified using parallel transport along γ). Then φ is said to be convex along γ at the point p if it is convex across ρ with respect to this affine structure. We say that φ is *convex along broken lines* if it is convex along every broken line.

Note that the usual notion of convexity is just convexity along straight lines. Our definition is somewhat different from that used in [GHKK]. They say a function is convex along broken lines if it is decreasing along broken lines, in the sense of §3.2.3. These definitions are in fact equivalent:

Lemma 3.2.21. *Convex along broken lines is equivalent to decreasing along broken lines.*

Proof. This follows from recalling that the usual notion of convexity can be defined as decreasing along straight lines. $\square$

Lemma 3.2.6 thus implies that valuation functions, and hence tropical theta functions, are convex along broken lines. We will see this in another way below.

Definition 3.2.22. We call a function $\varphi : U^{\mathrm{trop}} \to \mathbb{R}$ *tropical* if it is integral piecewise-linear and convex along broken lines. Note that tropical functions are closed under addition and min. We say φ is an indecomposable tropical function if it cannot be written as a minimum of some finite collection S of tropical functions with $\varphi \notin S$.

[FG09] defines another notion of convexity:

Definition 3.2.23. Recall that every seed induces a vector space structure on U^{trop} (viewed as a subspace of $\mathcal{X}^{\mathrm{trop}}$). One says that a piecewise-linear function $\varphi : U^{\mathrm{trop}} \to \mathbb{R}$ is *convex with respect to every seed* if it is convex with respect to each of these vector space structures.

Recall that we can apply the mirror construction to V and V^{trop}, so the notion of convexity along broken lines makes sense in V^{trop}. Furthermore, w_U identifies broken lines in U^{trop} with broken lines in V^{trop} and thus preserves convexity along broken lines.

Theorem 3.2.24. *If $\varphi : U^{\mathrm{trop}} \to \mathbb{R}$ is piecewise-linear, then φ is convex along broken lines if and only if it is convex with respect to every seed. The tropical functions on V^{trop} are exactly the tropicalizations of regular functions on V, and the indecomposable tropical functions are exactly the tropicalizations of theta functions.*

Proof. In U^{trop} (hence V^{trop}) with its canonical integral linear structure, broken lines can only bend away from the origin. Let $\mathfrak{L}^d$ denote a fiber $\varphi = d$ for some piecewise-linear function φ. φ being convex means that when $d < 0$, $\mathfrak{L}^d$ only bends towards the origin, and when $d > 0$, $\mathfrak{L}^d$ only bends away from the origin. Locally changing to an affine structure in which some broken line is straight will only cause lines to bend more *towards* the origin. Thus, on a cone where φ is non-positive, convexity of φ along broken lines is equivalent to convexity along straight lines.

Now, suppose that φ is convex along straight lines and non-negative on some (necessarily convex) cone σ. We saw in §3.1.6 that σ must live in the cluster complex. Convexity of φ along broken lines is now equivalent to convexity along the broken lines which take the maximal allowed bend across each ray in σ. Any such broken line lives in some $\mathfrak{L}_q^{d>0}$, and it follows from Proposition 3.1.10 that there is a seed for which $\mathfrak{L}_q^{d>0}$ is straight in the corresponding linear structure.

In summary, convexity of φ along broken lines is equivalent to convexity along straight lines in V^{trop} and along maximally broken lines in the cluster complex. Any maximally broken line in the cluster complex is straight with respect to some seed, and the same is locally true for straight lines in V^{trop}. Thus, convexity with respect to every seed implies convexity along broken lines. On the other hand, every line which is straight with respect to some seed is a broken line, so convexity along broken lines implies convexity with respect to every seed.

Now, given any regular function f on V, we know that the restriction of f to any seed torus is regular, and so f^{trop} is convex with respect to any seed. This gives an alternative proof of the fact that tropicalizations of regular functions are convex along broken lines.

Now suppose that φ is not an indecomposable tropical function. Then $\varphi = \min(f_1, f_2)$ for two tropical functions f_1 and f_2, neither of which is globally equal to φ. So we can find cones σ_1, σ_2 sharing a boundary ray ρ such that $\varphi|_{\sigma_i} = f_i$ and $f_1(x) \neq f_2(x)$ for $x \in \sigma_2$. Suppose that φ is linear across ρ along some broken line γ crossing ρ. Since f_1 and f_2 are both convex along this broken line and φ is their minimum, they must both be equal to φ in a neighborhood of ρ. This contradicts our assumption that $f_1(x) \neq f_2(x)$ for $x \in \sigma_2$, so φ must bend across ρ along γ.

When crossing from σ_1 to σ_2 above, φ changed from f_1 to f_2. If we continue going around the origin in the same direction, φ must eventually change back to f_1 after crossing some ray (or else it would be identically equal to f_2), and so we find that there are in fact at least two rays ρ_1 and ρ_2 in V^{trop} such that φ bends nontrivially across ρ_i along any broken line crossing ρ_i, for each i.

If $\vartheta_q^{\mathrm{trop}}$ is non-positive everywhere, then there is only one ray across which $\vartheta_q^{\mathrm{trop}}$ bends nontrivially along straight lines. On the other hand, if $\vartheta_q^{\mathrm{trop}}$ is positive somewhere, then $\vartheta_q^{\mathrm{trop}}$ bends across straight lines only in the interior of σ_{q_-,q_+}, but it does not bend along $\mathfrak{L}_q^{d>0}$, which crosses any ray in the interior of σ_{q_-,q_+}. Thus, the tropicalization of any theta function is indecomposable.

On the other hand, let φ be an arbitrary tropical function on V^{trop}. Suppose that $\varphi \leq 0$ everywhere, but is not identically 0 (hence has a nontrivial bend along some ray). Then there is some compactification of V in which $-\varphi$ corresponds to an effective boundary divisor $W_{-\varphi}$ which has non-negative intersection with every boundary component, and positive intersection with some component. The linear system $|W_{-\varphi}|$ contains a pencil, and for V generic, this pencil gives a regular function on V. φ is the tropicalization of this function.

Now suppose that φ is a tropical function such that $\varphi|_\sigma = \vartheta_q^{\mathrm{trop}}|_\sigma$ for some $q \in U^{\mathrm{trop}}(\mathbb{Z})$ and some $\sigma \subset V^{\mathrm{trop}}$, and assume that $\vartheta_q^{\mathrm{trop}} > 0$ somewhere in V^{trop}. Then $\varphi \leq \vartheta_q^{\mathrm{trop}}$ everywhere in V^{trop}, because there is some seed with respect to which $\vartheta_q^{\mathrm{trop}}$ is linear and φ is convex, hence equal to the minimum of its linear parts.

Let φ be a tropial function on V^{trop} which is positive on some cone σ_+. Let $\sigma \subset \sigma_+$ be a subcone on which φ is linear. We can choose a covariantly constant integral section section q_σ of $T\sigma$ such that $\varphi(p) = p \wedge [q_\sigma(p)]$, where p is being identified with a vector in T_pV^{trop}. If we view the fiber $F_d := \{\varphi|_\sigma = d > 0\}$ as part of a broken line with q_σ giving the negative tangent direction, then we can extend F_d indefinitely each direction, taking the maximal allowed bend at each wall it crosses, to get a broken line $\mathfrak{L}_q^{d>0}$. $\varphi|_\sigma$ then must equal $\vartheta_q^{\mathrm{trop}}|_\sigma$, and so $\varphi \leq \vartheta_q^{\mathrm{trop}}$ everywhere on V^{trop}.

Now let σ be a cone outside of σ_+ on which φ is linear. We have a fiber $F_d := \{\varphi|_\sigma = d < 0\}$ as before, and extending indefinitely in either direction (without bends this time) gets us a line $L_{q'}^{d<0}$ containing F_d. If $L_{q'}^{d<0}$ does not wrap, then $\varphi|_\sigma = \vartheta_{q'}^{\mathrm{trop}}|_\sigma$, and this tropical theta function is positive somewhere on V^{trop}. So then $\varphi \leq \vartheta_{q'}^{\mathrm{trop}}$ every on V^{trop}. If $L_{q'}^{d<0}$ does wrap, then F_d extended in at least one direction will enter σ_+. Let $\widetilde{\sigma}$ be a cone containing F_d extended in this one direction to the point $p \in \sigma_+$. $\widetilde{\sigma}$ is convex since the extension of F_d hits both boundary rays. Let $\widetilde{\varphi_\sigma}$ be the function on $\widetilde{\sigma}$ obtained by extending φ_σ linearly, so $\widetilde{\varphi_\sigma}$ has the extension of F_d as a fiber. Hence, $\widetilde{\varphi_\sigma}$ is negative at $p \in \sigma_+$. Since $\varphi|_\sigma = \widetilde{\varphi_\sigma}$, $\varphi|_{\widetilde{\sigma}}$ is convex, and $\widetilde{\varphi_\sigma}$

is linear, we then have that φ is negative at $p \in \sigma_+$, a contradiction.

Thus, on every domain of linearity, φ is equal to the restriction of some tropical theta function which is somewhere positive. φ must therefore be equal to the min of these tropical theta functions.

Since every regular function can be written as a sum of theta functions, this implies that any tropical function is the tropicalization of some regular function, and also that the indecomposable tropical functions are exactly the tropical theta functions.

□

Remark 3.2.25. In the above theorem, we assumed U was positive. However, we can easily extend to the negative cases. In the negative definite cases, convex (along straight lines) functions on U^{trop} must be positive everywhere on U_0^{trop}. But for a positive function to be convex along broken lines, it must take the maximal possible bend along every broken line it passes. Since U^{trop} in any non finite-type case contains infinitely many scattering rays, this is impossible. So there are no non-trivial tropical functions on U^{trop} in these cases. This is what we expect since there are no non-constant regular funcitons on U in these cases.

In the strictly semi-definite cases, there are straight lines in U^{trop} which are circles, and these give fibers for a ray's worth of tropical functions. In general, U for these cases might not admit non-constant regular functions, so the theorem as stated does not quite hold here. However, it is possible to deform such a U to a surface admitting an elliptic fibration over $\mathbb{A}^1$, and powers of the fibration map give the desired regular functions. So the theorem does hold up to deformation of U.

Corollary 3.2.26. *The identification* $w_U : U^{\mathrm{trop}}(\mathbb{Q}) \to V^{\mathrm{trop}}(\mathbb{Q})$ *really does extend to an integral linear isomoprhism* $w_U : U^{\mathrm{trop}} \to V^{\mathrm{trop}}$*.*

Proof. w_U extends to an integral linear function because it pulls back tropical functions (restricted to the rational points) to tropical functions (restricted to the rational points). It is an isomorphism because w_V gives the inverse map. □

The notions of convexity along broken lines and convexity with respect to every seed make sense in more general situations related to cluster varieties (cf. [GHKK] and [FG09], respectively).

Conjecture 3.2.27. *Convexity along broken lines is always equivalent to convexity with respect to every seed.*

The key to proving this conjecture in dimension 2 was Lemma 3.1.7, which says that the following conjecture holds in dimension 2:

Conjecture 3.2.28. *If* φ *is a tropical function on the tropicalization of a cluster variety (or a fiber of a cluster variety)* $\mathcal{Y}$*, and if* φ *is positive at some point on a scattering wall* $\mathfrak{w}$*, then the wall-crossing formula for* $\mathfrak{w}$ *is the formula for some mutation in some cluster structure on* $\mathcal{Y}$ *(i.e.,* $\mathfrak{w}$ *lives in the cluster complex for some cluster structure on* $\mathcal{Y}$*).*

The following conjecture is from [GHKK]:

Conjecture 3.2.29 ([GHKK])**.** *The tropicalization of any regular function on any log Calabi-Yau variety is convex along broken lines.*

Proving Conjecture 3.2.27 would immediately imply this, because globally regular functions are of course regular on each seed torus, and they therefore give convex functions with respect to every seed. Of course, we also conjecture that the other parts of Theorem 3.2.24 generalize to other cluster situations (and more generally, to other log Calabi-Yau situations).

3.3 Toric Constructions for Log Calabi-Yau Surfaces

Throughout this section, it is always possible to switch the roles of U and V using the symmetry of the pairing $\langle \cdot, \cdot \rangle$. We will therefore only define and prove things for one side.

3.3.1 Polytopes

Definition 3.3.1. Let Q be any subset of V^{trop}. The *polar polytope* Q° is the set $\{q \in U^{\mathrm{trop}} | \langle q, v\rangle \geq -1 \text{ for all } v \in Q\}$.

The *strong convex hull*[8] of a set $Q \subset U^{\mathrm{trop}}$ is the set

$$\mathbf{Conv}(Q) = \left\{ x \in U^{\mathrm{trop}} | \langle x, v \rangle \geq \inf_{q \in Q} \langle q, v \rangle \text{ for all } v \in V^{\mathrm{trop}} \right\}.$$

Let $f = \sum_{q \in Q} a_q \vartheta_q \in \Gamma(V, \mathcal{O}_V)$, $a_q \neq 0$, for some finite set $Q \subset U^{\mathrm{trop}}(\mathbb{Z})$. The *Newton Polytope* of f is the set $\mathrm{Newt}(f) := \mathbf{Conv}(Q)$. Equivalently, $\mathrm{Newt}(f) = \{x \in U^{\mathrm{trop}} | \langle x, v \rangle \geq f^{\mathrm{trop}}(v) \text{ for all } v \in V^{\mathrm{trop}}\}$.

A set Q is called *strongly convex* if $Q = \mathbf{Conv}(Q)$.

---

[8] It follows from Theorem 3.2.14 that this is equivalent to the version of convex hull used in [FG11] and [She12]. Similarly for the Minkowski sums of §3.3.3.

The following lemma follows directly from the definitions.

Lemma 3.3.2. *For any set $Q \subseteq V^{\mathrm{trop}}$, $Q \subseteq (Q^\circ)^\circ$. If $Q \subseteq S$, then $S^\circ \subseteq Q^\circ$.*

Definition 3.3.3. A polytope Q is called *self-polar* if $Q = (Q^\circ)^\circ$.

Lemma 3.3.4. *Q° is self-polar. Thus, Q being self-polar is equivalent to Q being the polar polytope of some set.*

Proof. The first statement of Lemma 3.3.2 immediately gives us $P^\circ \subseteq ((P^\circ)^\circ)^\circ$. It also gives us $P \subseteq (P^\circ)^\circ$, and then the second statement gives us $((P^\circ)^\circ)^\circ \subseteq P^\circ$. $\square$

Proposition 3.3.5. *A set $Q \subset U^{\mathrm{trop}}$ is strongly convex if and only if it is an intersection of sets of the form $\{\langle \cdot, v\rangle \geq a_v\}$.*

Proof. $\mathbf{Conv}(Q)$ is by definition an intersection of sets of this form, with $a_v := \inf_{q\in Q}\langle q, v\rangle$. So Q being convex implies it has this form.

Conversely, suppose $Q = \bigcap_{v\in I}\{q \in U^{\mathrm{trop}} | \langle q, v\rangle \geq a_v \in \mathbb{R}\}$. If Q is not convex, then there is some $x \notin Q$ such that for every $v \in V^{\mathrm{trop}}$, $\langle x, v\rangle \geq \langle q_v, v\rangle$ for some $q_v \in Q$ (since Q is closed, the infimum in the definition of $\mathbf{Conv}(Q)$ is obtained for some $q_v \in Q$). But this implies x is in each of the sets in the intersection defining Q, hence in Q. $\square$

Corollary 3.3.6. *Self-polar polytopes are exactly the strongly convex polytopes containing the origin in their interiors, which are the same as ordinary convex polytopes with the origin in their interiors.*

Proof. Polar polytopes by definition have the form given in Proposition 3.3.5. So self-polar polytopes are convex. It is easy to see that they contain 0 in their interiors.

Conversely, strongly convex polytopes with 0 in their interiors have the form given in Proposition 3.3.5 with each $a_v < 0$. Thus, by multiplying the v's by positive scalars, we can assume each a_v equals -1. The form from Proposition 3.3.5 is then the definition for a polar polytope.

For the last statement, sets of the form $\{\langle \cdot, v\rangle \geq a_v\}$ with $a_v < 0$ are exactly the zero-sides of straight lines in U^{trop}, and ordinary convex polytopes (i.e., those which are convex with respect to the canonical integral linear structure on U^{trop}) with the origin in their interiors are the intersections of such sets. □

Recall our notation $Q_\varphi = \{q \in U^{\mathrm{trop}} | \varphi(q) \geq -1\}$, for φ a piecewise linear function on U^{trop}. We use the analogous notation in V^{trop}.

Proposition 3.3.7. *If $\varphi : V^{\mathrm{trop}} \to \mathbb{R}$ is tropical, then Q_φ is self-polar. If φ is integral piecewise-linear and Q_φ is self-polar and bounded, then φ tropical.*

Proof. First suppose that φ is tropical. Lemma 3.3.2 gives us $Q \subseteq (Q^\circ)^\circ$. On the other hand, Theorem 3.2.24 tells us that there is some regular function f on V with $f^{\mathrm{trop}} = \varphi$. We can write $f = \sum_{q\in S} a_q \vartheta_q$, $a_q \neq 0$ for some finite set $S \subset U^{\mathrm{trop}}(\mathbb{Z})$. Since $f^{\mathrm{trop}}(v) = \min_{q\in S}\langle q, v\rangle$ and $v \in Q$ if and only if $f^{\mathrm{trop}}(v) \geq -1$, this means that $S \subseteq Q^\circ$. Now, $v \in (Q^\circ)^\circ$ means that $\langle q, v\rangle \geq -1$ for all $q \in Q^\circ$, hence all $q \in S$, and this implies that $f^{\mathrm{trop}}(v) \geq -1$. This means that $v \in Q$, as desired.

On the other hand, Q_φ being strongly convex and having the form $\{\varphi \geq -1\}$ for φ integral piecewise-linear means that it has the form $\bigcap_{q\in S}\{\langle q, \cdot\rangle \geq -1\}$ for some finite set $S \subset U^{\mathrm{trop}}(\mathbb{Z})$. Let $f = \sum_{q \in S} \vartheta_q$. Then $Q_\varphi = Q_{f^{\mathrm{trop}}}$. Q_φ bounded implies that $\varphi < 0$ everywhere on U_0^{trop}, so Q_φ determines φ. Hence, $\varphi = f^{\mathrm{trop}}$. □

Recall that in the usual vector space situation, a polytope being convex means that any line segment with endpoints in the polytope is entirely contained in the polytope. The following theorem generalizes that characterization.

Theorem 3.3.8. *If a set $Q \subseteq U^{\mathrm{trop}}$ is strongly convex, then every broken line segment with endpoints in Q is contained entirely within Q. Conversely, if Q is a rational polytope containing every broken line segment[9] with endpoints in Q, then Q is strongly convex.*

Proof. Suppose Q is strongly convex. So Q is an intersection of sets of the form $\{\langle \cdot, v\rangle \geq a_v \in \mathbb{R}\}$. Let γ be a segment of a broken line with endpoints in Q. We know that each $\langle \cdot, v\rangle$ is convex along γ, so if we give U^{trop} a linear structure in which γ is straight, then the usual notion of convexity tells us that indeed $\gamma \subset \{\langle \cdot, v\rangle \geq a_v \in \mathbb{R}\}$. Thus, $\gamma \subset Q$.

Now suppose that Q is a rational polytope and that every broken line with endpoints in Q is contained entirely within Q. Assume that Q is two-dimensional (the lower dimensional cases are easier). We claim that the boundary of Q is a finite union of closed sets Γ each of which satisfies $\langle \Gamma, v_\Gamma\rangle = a_\Gamma \in \mathbb{Q}$ for some $v_\Gamma \in V^{\mathrm{trop}}(\mathbb{Z})$ such

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[9] Here we must include broken lines through the origin, by which we mean limits of sequence of broken lines whch are all equivalent to eachother in the sense of §3.1.4. Alternatively, in addition to the usual broken lines, we allow sets of the form $\langle \cdot, v\rangle = 0$ for $v \in V^{\mathrm{trop}}(\mathbb{Z})$.

that $\langle q, v_\Gamma \rangle \geq a_\Gamma$ for all $q \in Q$. This implies that $Q = \bigcap_\Gamma \{q \in U^{\mathrm{trop}} | \langle q, v_\Gamma \rangle \geq a_\Gamma\}$, which by Proposition 3.3.5 means that Q is strongly convex.

It is not hard to see that each point of the boundary is contained in a closed interval Γ (of length > 0) which can be extended to a fiber $\widetilde{\Gamma} = \{\langle \cdot, v_\Gamma \rangle = a_v\}$ for some $v \in V^{\mathrm{trop}}(\mathbb{Z})$, $a_v \in \mathbb{Q}$, satisfying $\langle q, v \rangle > a_v$ for some $q \in Q$. Suppose there is also a $q' \in Q$ such that $\langle q', v \rangle = a'_v < a_v$. Since Q is connected, we may assume $a'_v = a_v - \epsilon$ for any sufficiently small $\epsilon > 0$. We may also assume q' is a rational point. Let p be a point in the interior of Γ. If $\widetilde{\Gamma}$ is a straight line, then it is clear that rotating it slightly about p will give a straight line connecting p to q' which is not contained in Q in between, a contradiction. If $\widetilde{\Gamma}$ is not straight, then there is some seed with respect to which it is straight, and here we can preform a similar rotation. This proves the claim. □

Remark 3.3.9. We note that rather than checking the above condition for every broken line, it suffices to check for broken lines which are rational fibers of $\langle \cdot, v \rangle$ for some $v \in V^{\mathrm{trop}}(\mathbb{Z})$. Such broken lines are either straight in U^{trop} or are contained in the cluster complex and are straight with respect to some seed structure. So it is not necessary to understand the entire scattering diagram to understand strong convexity. Similarly for convexity of functions along broken lines.

Examples 3.3.10.

- Let $p \in U^{\mathrm{trop}}$ be the self-intersection point of some straight line L that wraps once. Then $\mathbf{Conv}(p) = Z(L)$. Since a point is convex with respect to every seed, this shows that a polytope being strongly convex is stronger than being convex with respect to every seed.

- In the cubic surface case, the convex hull of a point $q \in U^{\mathrm{trop}}(\mathbb{Z})$ is the line segment connecting 0 to q. This illustrates the need for considering L_q^0.

3.3.1.1 Line Bundles and Polytopes

Let $W = \sum a_i \mathfrak{D}_{v_i}$ be a $\mathbb{Q}$-divisor in a compactification $(Z, \mathfrak{D})$ of V, with $\mathfrak{D}_{v_i}$ being the divisor corresponding to some primitive $v_i \in V^{\mathrm{trop}}(\mathbb{Z})$, and $a_i \in \mathbb{Q}$. Recall that φ_W denotes the piecewise-linear function on V^{trop} which takes the value a_i at v_i and is linear off the rays generated by the v_i's. Let $Q_W := Q_{-\varphi_W} = \{v \in V^{\mathrm{trop}} | -\varphi_W(v) \geq -1\}$. We note that if φ_{-W} is non-positive (i.e., if W is effective), then Q_W is the convex hull of the points $\frac{1}{a_i}v_i$ (since $0 \in Q_W$, convex and strongly convex are equivalent).

Definition 3.3.11. $Q_W^\vee :=$ **Conv**$\{q \in U^{\mathrm{trop}} | \langle q, v_i \rangle \geq -a_i \text{ for all } i\}$. That is, $Q_W^\vee$ is the Newton polytope of a generic section of $\mathcal{O}(W)$.

Note that this actually depends on W, not just on the polytope Q_W as the notation suggests. It follows easily from the definitions that:

Lemma 3.3.12. *If W is integral, then $q \in Q_W^\vee \cap U^{\mathrm{trop}}(\mathbb{Z})$ if and only if $\vartheta_q \in \Gamma(Z, \mathcal{O}(W))$. Thus, as a vector space, $\Gamma(Z, \mathcal{O}(W)) = \bigoplus_{q \in Q_W^\vee \cap U^{\mathrm{trop}}(\mathbb{Z})} \Bbbk \vartheta_q$. If W is effective, then $Q_W^\vee = Q_W^\circ$. In general, $Q_W^\vee \subseteq Q_W^\circ$.*

Proposition 3.3.13. *The strongly convex integral (resp. rational) polytopes are exactly those of the form $Q_W^\vee$ for some divisor (resp. some $\mathbb{Q}$-divisor) W.*

Proof. This follows immediately from the definition of $Q_W^\vee$ and Proposition 3.3.5. $\square$

Lemma 3.3.14. *Let f be a regular function on V. Let W_f be negative the boundary divisor corresponding to f^{trop}. Then* $\mathrm{Newt}(f) = Q^{\vee}_{W_f}$.

Proof. Once again, this follows easily from the definitions. $\square$

Using our descriptions of the fibers of val_{v_i} from Corollary 3.2.12 and Proposition 3.2.13, we can easily describe $Q^{\vee}_W$ explicitely. In particular:

Proposition 3.3.15. *Use w_U to identify U^{trop} with V^{trop}. Assume that $W = \sum a_i \mathfrak{D}_{v_i}$ is strictly effective (so each $a_i > 0$). Then:*

$$Q^{\vee}_W = Q^{\circ}_W = \bigcap_i \overline{Z(L^{a_i}_{v_i})}.$$

This is analogous to the toric picture of a "normal polytope," except that using the wedge form in place of the dot product results in "parallel polytopes." This description was previously observed in [GHK].

May other facts about polytopes from the toric world generalize to our situation with virtually no change. For example:

Proposition 3.3.16. *Let $W = \sum a_i \mathfrak{D}_{v_i}$ be an integral Weil divisor, and let F_{v_i} be the (possibly empty) set $Q^{\vee}_W \cap \{q \in U^{\mathrm{trop}} | \langle q, v_i\rangle = -a_i\}$. Let d_i be one less than the number of lattice points on F_{v_i}. If $d_i \geq 1$, then $d_i = W \cdot D_i$.*

Proof. On any affine open subset containing part of $\mathfrak{D}_{v_i}$, the global sections of $\mathcal{O}_W$ whose restrictions to $\mathfrak{D}_{v_i}$ are not 0 correspond to the lattice points on F_{v_i}. Thus, $\mathcal{O}_W|_{D_i} \cong \mathcal{O}_{\mathbb{P}^1}(d_i)$, which has degree d_i. So $W \cdot D_i = d_i$ by [Har77], Lemma V.1.3. $\square$

Corollary 3.3.17. *Any strongly convex polytope* Q *in* U^{trop} *is* $Q_W^\vee$ *for some not necessarily effective* $\mathfrak{D}$*-ample divisor* W *in some compactification of* V*.* W *is effective if and only if* Q *contains the origin.*

Proof. We can write $Q = \bigcap_{v\in S}\{\langle \cdot, v\rangle \geq -a_v\}$ with S minimal. Then $W = \sum_{v\in S} a_v \mathfrak{D}_v$, where if $v = |v|v'$ with v' primitive, then $\mathfrak{D}_v$ denotes $|v|\mathfrak{D}_{v'}$. The $\mathfrak{D}$-ampleness follows from the fact that since S is minimal, F_v contains at least two integer points. $\square$

[GHK] describes the corresponding maps to projective space in the cases where the $\mathfrak{D}$-ample divisor W is effective. We do not need W to be effective because we do not require the origin to be in the interior of the stongly convex polytope. However, $\mathfrak{D}$-ample divisors which are not effective are typically not ample on V, even if V is generic.

Example 3.3.18. For U the affine cubic surface, U^{trop} contains a reflexive polytope which includes four integer points. This shows that any surface whose tropicalization is U^{trop} must be a degree 3 del Pezzo surface, i.e., the cubic surface.

3.3.2 Dual Cones

Let σ be a cone in a fan Σ corresponding to some compactification $(Z, \mathfrak{D})$ of V.

Definition 3.3.19. The dual cone to σ is $\sigma^\vee := \{q \in U^{\mathrm{trop}} | \langle q, v\rangle \geq 0 \text{ for all } v \in \sigma\}$.

Let $(Z_\sigma, \mathfrak{D}_\sigma)$ be the partial compactification of V which includes the non-nodal points of $\mathfrak{D}_\rho$ for each boundary ray ρ of σ, along with the point $\mathfrak{D}_\rho \cap \mathfrak{D}_{\rho'}$ if σ is two-dimensional. From the definitions, we have:

Lemma 3.3.20. *$q \in \sigma^\vee \cap U^{\mathrm{trop}}(\mathbb{Z})$ if and only if the global regular function ϑ_q on V extends to a regular function on $(Z_\sigma, \mathfrak{D}_\sigma)$.*

Corollary 3.3.21. *Let A_σ be the subalgebra of $\Gamma(V, \mathcal{O}_V)$ generated by the ϑ_q's with $q \in \sigma^\vee$. If $\sigma^\vee$ is two-dimensional, then $Z_\sigma = \operatorname{Spec} A_\sigma$.*

3.3.3 Tropical Multiplication and Minkowski Sums

The theta function multiplication formula in Theorem 3.1.6 is quite complicated. However, tropicalization allows us to at least see which theta functions might have nonzero coefficients in a product $\vartheta_{q_1}\vartheta_{q_2}$. If $f = \sum c_q \vartheta_q$ is a regular funciton, then $c_q = 0$ unless $q \in \mathrm{Newt}(f)$, and if q is a vertex[10] of f, then $c_q \neq 0$. We would therefore like to describe $\mathrm{Newt}(\vartheta_{q_1}\vartheta_{q_2})$.

Definition 3.3.22. The *Minkowski sum* of two strongly convex polytopes $\mathrm{Newt}(f)$ and $\mathrm{Newt}(g)$ is $\mathrm{Newt}(f) + \mathrm{Newt}(g) := \mathrm{Newt}(fg)$.

Of course, since $(fg)^{\mathrm{trop}} = f^{\mathrm{trop}} + g^{\mathrm{trop}}$, we have that $\mathrm{Newt}(fg) = \{x \in U^{\mathrm{trop}} | \langle x, v\rangle \geq f^{\mathrm{trop}}(v) + g^{\mathrm{trop}}(v) \text{ for all } v \in V^{\mathrm{trop}}\}$. This is enough to tell us that:

Proposition 3.3.23. *For any $k \in \mathbb{Z}_{\geq 0}$, $\mathrm{Newt}(f^k) = k\,\mathrm{Newt}(f) := \{ku | u \in \mathrm{Newt}(f)\}$.*

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[10] We can write $\mathrm{Newt}(f) = \mathbf{Conv}(S)$ for some set $S \subset U^{\mathrm{trop}}(\mathbb{Z})$. By a vertex, we mean a point $q \in S$ such that $\mathbf{Conv}(S \setminus \{q\}) \neq \mathbf{Conv}(S)$. For example, suppose $\mathrm{Newt}(f)$ is two-dimensional and is given by $\cap_{v\in I}\{\langle \cdot, v\rangle \geq a_v\}$ with I being minimal in the sense that removing some v would result in the intersection being a larger set. Then r being a vertex means that it is a point on the boundary where $\{\langle \cdot, v\rangle = a_v\}$ intersects $\{\langle \cdot, v'\rangle = a_{v'}\}$ for some points v, v' in I. In case I has only one element, r can be a self-intersection point of $\{\langle \cdot, v\rangle = a_v\}$. We do not, however, include points that only look like vertices because they are kinks in some broken line (after all, such points no longer look like vertices when viewed with respect to some seed).

Finding a nice formula for Newt(fg) in general is a bit more complicated. We will use a different approach, and will assume that f^{trop} and g^{trop} are both non-positive (i.e., their Newton polytopes contain 0).

Recall that $\beta_{v_1,\ldots,v_k}$ denotes a piecewise-linear funciton which bends along ρ_{v_i} with bending parameter $|v_i|$. We rely on the following lemma:

Lemma 3.3.24. *Assume we are not in one of the I_k cases of §2.3.4.1 (so β_v is unique for each $v \in V^{\mathrm{trop}}$). Suppose that $f = \beta_{v_1,\ldots,v_k}$ is a tropical function, and assume that $v_1, \ldots, v_k \in U^{\mathrm{trop}}(\mathbb{Z})$ are cyclically ordered according to the orientation of V^{trop}. Let $+_i$ denote addition as defined on the complement of the interior of σ_{v_{i-1},v_i}. Assume that $f(u) < 0$ for u in the interior of $\sigma_i := \sigma_{v_{i-1},v_i}$. Then*

$$\beta_{v_1,\ldots,v_k}|_{\sigma_i} = \beta_{v_1 +_i \cdots +_i v_k}|_{\sigma_i}.$$

Consequently, if $f \leq 0$ everywhere and is 0 along at most a single ray, then

$$\sum_{i=1}^{k} \beta_{v_i} = \min_{i=1}^{k} \beta_{v_1 +_i \cdots +_i v_k}.$$

Proof. If $v_1 +_i \ldots +_i v_k \in U^{\mathrm{trop}} \setminus \sigma_{v_{i-1},v_i}$, then the first claim follows immediately from our analysis in §2.2.4.2, even without the assumption that $f(u) < 0$.

However, if $v_1 +_i \ldots +_i v_k$ is not in $U^{\mathrm{trop}} \setminus \sigma_{v_{i-1},v_i}$, it means that there is no convex piecewise-linear function on U^{trop} bending along a single ray whose restriction to $\sigma_{i-1,i}$ agrees with f. But since f is tropical, we know from Theorem 3.2.24 that it can be written as a minimum of tropical theta functions, and our analysis in §3.2.6.1 shows that the negative part of any tropical theta function is equal to some convex β_q. Thus, this case must not occur.

The last statement follows immediately once we note that $\sum_{i=1}^k \beta_{v_i} = \beta_{v_1,\ldots,v_k}$.

$\square$

Recall the function $b : U^{\mathrm{trop}} \to U^{\mathrm{trop}}$ of Lemma 3.2.16 which takes q to the bend of $\langle q, \cdot\rangle$, viewed as a function on U^{trop} using the identification w_U. That is, $\beta_{w_U \circ b(q)} = \langle q, \cdot\rangle$.

Theorem 3.3.25 (Tropical Multiplication Formula)**.** *Assume we are not in one of the I_k $(k \neq 0)$ cases. Let $q_1, \ldots, q_s \in U^{\mathrm{trop}}(\mathbb{Z})$ be cyclically ordered, and let $+_i$ denote addition on the complement of $\sigma_i := \sigma_{q_{i-1},q_i}$. Suppose $\left(\prod_{k=1}^s \vartheta_{q_k}\right)^{\mathrm{trop}}(u) < 0$ for all $u \in \sigma_i$. Then*

$$\left(\prod_{k=1}^{s} \vartheta_{q_k}\right)^{\mathrm{trop}}\Bigg|_{\sigma_i} = \vartheta^{\mathrm{trop}}_{q_1 +_i \cdots +_i q_s}\Big|_{\sigma_i}.$$

Consequently, if $\left(\prod_{i=1}^s \vartheta_{q_i}\right)^{\mathrm{trop}} \leq 0$ everywhere and is 0 along at most a single ray, then

$$\left(\prod_{i=1}^{s} \vartheta_{q_i}\right)^{\mathrm{trop}} = \left(\sum_{i=1}^{s} \vartheta_{q_1 +_i \cdots +_i q_n}\right)^{\mathrm{trop}}.$$

Proof. Combining Lemmas 3.2.16 and 3.2.26, we have that the map $w_U \circ b : U^{\mathrm{trop}} \to V^{\mathrm{trop}}$ is linear. By definition, $\beta_{w_U \circ b(q)}$ agrees with the function $\langle q, \cdot\rangle$ on V^{trop} whenever both are non-positive. Thus

$$\begin{aligned}
\left(\prod_{k=1}^{s} \vartheta_{q_i}\right)^{\mathrm{trop}}\Bigg|_{\sigma_i} &= \sum_{k=1}^{s} \beta_{w_U \circ b(q_k)}\Bigg|_{\sigma_i} \\
&= \beta_{w_U \circ b(q_1) +_i \cdots +_i w_U \circ b(q_n)}\Big|_{\sigma_i} \\
&= \beta_{w_U \circ b(q_1 +_i \cdots +_i q_n)}\Big|_{\sigma_i} \\
&= \vartheta^{\mathrm{trop}}_{q_1 +_i \cdots +_i q_n}\Big|_{\sigma_i},
\end{aligned}$$

as desired. In the second line above, $+_i$ means that addition is taken as defined on the complement of the interior of $\sigma_{w_U \circ b(q_i), w_U \circ b(q_{i-1})}$ (the order-reversal coming from the reversed orientation of V^{trop}).

The second claim follows immediately. $\square$

I expect the theorem to also hold for the I_k cases, but I have not checked this.

Remark 3.3.26. Note that the above lemma and theorem still hold if we replace the cone σ_i with some subcone $\sigma_i' \subset \sigma_i$ on which the tropical function is negative, even if the tropical function is positive somewhere on σ_i.

Theorem 3.3.27. *Let $Q_1, \ldots, Q_s$ be strongly convex integral polytopes such that $Q_1 + \ldots + Q_s$ contains the origin (which in particular is the case if all the Q_k's contain the origin). Let $\rho_1, \ldots, \rho_m$ be a collection of rays in U^{trop} not intersecting the vertices of the Q_k's such that no two non-equal vertices from different Q_k's lie in the same component of $U^{\mathrm{trop}} \setminus \bigcup_{i=1}^m \rho_i$. Then*

$$Q_1 + \ldots + Q_s = \mathbf{Conv}\left(\bigcup_{i=1}^m (Q_1 +_i \ldots +_i Q_s)\right),$$

where $+_i$ denotes addition on the complement of ρ_i, and $Q_1 +_i \ldots +_i Q_s := \{q_1 +_i \ldots +_i q_s \in U^{\mathrm{trop}} | q_k \in Q_k\}$.[11]

Proof. Let $q_{k,j}$ denote the vertices of Q_k, each of which is integral. Since each Q_k is the convex hull of its vertices, we can say $Q_1 + \ldots + Q_s$ is the convex hull of the points $q \in U^{\mathrm{trop}}(\mathbb{Z})$ whose corresponding theta functions appear in the expansion of some

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[11] If some $q_1 +_i \ldots +_i q_s$ is not defined in U^{trop}, we simply do not include it in the set.

$\prod_{k=1}^{s} \vartheta_{q_{k,j_k}}$. It suffices to consider the q's for which $\vartheta_q^{\mathrm{trop}}\big|_\sigma = \prod_{k=1}^{s}\left(\vartheta_{q_{k,j_k}}\right)^{\mathrm{trop}}\Big|_\sigma$ on some cone $\sigma \subset V^{\mathrm{trop}}$.

We do not need to worry about when these tropical funcitons are positive on σ, since we assumed the Minkowski sum contains the origin (implying that for some choice of j_k's the function will be negative on σ). By breaking σ up into a union of smaller cones, we may assume that σ contains none of the q_{k,j_k}'s in its interior. Then for some ρ_i, addition of the q_{k,j_k}'s on the complement of ρ_i is the same as on the complement of σ. Thus, when $\vartheta_q^{\mathrm{trop}}\big|_\sigma$ is negative, we have from Theorem 3.3.25 that $q = q_{1,j_1} +_i \ldots +_i q_{s,j_s}$. The claim follows. □

3.4 Integral Formulas

For this section, let $\Bbbk = \mathbb{C}$. Recall that since V is log Calabi-Yau like U, it has a holomorphic volume form Ω with log poles along the boundary $\mathfrak{D}$ of any maximal boundary compactification $(Z, \mathfrak{D})$. [GHK] defines a class $\gamma \in H_2(V, \mathbb{Z})$ as follows. Take any nonsingular $(Z, \mathfrak{D} = \mathfrak{D}_1 + \ldots + \mathfrak{D}_n)$ as above. Then γ is the class of a torus $0 < |z_i| = |z_{i+1}| = \epsilon \ll 1$, where z_i and z_{i+1} are local coordinates for Z in a neighborhood of $p = \mathfrak{D}_i \cap \mathfrak{D}_{i+1}$ such that $\mathfrak{D}_i$ is locally given by $z_i = 0$.

Lemma 3.4.1. *The class γ is canonical (it does not depend on our choice of compactification or vertex p). This remains true even if we remove from Z a curve C which intersects only one boundary divisor.*

Proof. Suppose we have two different choices of compactification of V. Then we apply the following argument to a common toric blowup of the two:

Recall that each toric model $(Z,\mathfrak{D}) \to (\overline{Z},\overline{\mathfrak{D}})$ (i.e., each seed) gives us a torus $T = (\mathbb{C}^*)^2$ in V, equal to the complement of the exceptional divisors in V. In fact, the complement of the images of the exceptional divisors in $\overline{Z}$ can be identified with a subvariety of Z. It is well-known that there is a "moment map" from $\overline{Z}$ to a polygon Q in $M_{\mathbb{R}}$ with $\mathfrak{D}$ mapping to the boundary of the polygon and with fibers over the k-dimensional faces being k-dimensional tori in the k-strata of $\overline{Z}$. So each $p_i = \mathfrak{D}_i \cap \mathfrak{D}_{i+1}$ maps to a vertex $\overline{p_i}$ of Q. z_i and z_{i+1} can be chosen so that γ is a fiber of the moment map over a point very close to p. Since all the fibers are homologous, the first claim follows from taking fibers near different vertices.

Suppose we remove a curve C intersecting, say, $\mathfrak{D}_i$. Let $\overline{C}$ denote the closure in $\overline{Z}$ of $C \cap T$. Then the image of $\overline{C}$ under the moment map only intersects the edge F_i which is the image of $\mathfrak{D}_i$. So even on the complement of the image of $\overline{C}$, there is a path in Q between any two of Q's vertices, showing that the claim still holds. $\square$

See [GHK] for a slightly different proof of the first statement of the lemma.

Remark 3.4.2. Conjecturally, γ is the homology class of a fiber of an SYZ fibration of V over V^{trop}. At the very least, if we factor the singularity in V^{trop} into focus-focus singularities which are still contained in some convex polytope Q, then V admits a Largangian fibration over the interior of Q. See [Sym03] for the details. This fibration can be used for an alternative proof of the lemma.

Assume Ω is normailized[12] so that $\int_\gamma \Omega = 1$. Following [GHK], we define a

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[12] Recall that if we take the cyclic ordering of $D = D_1 + \ldots + D_n$ as part of our data, then we can use this to orient U^{trop}, and V^{trop} gets the opposite orientation. This can be used to orient γ (by

function $Tr : \mathcal{O}_V(V) \to \mathbb{C}$,

$$Tr(f) := \int_\gamma f\Omega.$$

[GHK] shows that $Tr(f)$ is equal to the coefficient of $\vartheta_0 = 1$ in the unique expression of f as a linear combination of theta functions. We will now describe how to modify this to give the coefficients of the other theta functions.

For $q \in U^{\text{trop}}(\mathbb{Z})$, define $Tr_q : \mathcal{O}_V(V)_{\vartheta_q} \to \mathbb{C}$ by

$$Tr_q(f) := \int_\gamma f\vartheta_q^{-1}\Omega$$

Lemma 3.4.3. *Tr_q is well-defined.*

Proof. Since ϑ_q^{-1} is only regular on $V \setminus Z(\vartheta_q)$, it is not immediately clear from Stokes' theorem that this definition is independent of our choice of p for defining γ. If $\vartheta_q^{\text{trop}} \leq 0$ everywhere, then our description of tropical theta functions shows that the zero set $V(\vartheta_q)$ intersects only one boundary divisor, so the well-definedness follows from Lemma 3.4.1. If $\vartheta_q^{\text{trop}}$ is positive somewhere, then q is in the cluster complex, and so there is some open torus T in V on which ϑ_q is a monomial and therefore has no zeroes. The claim then follows from Lemma 3.4.1 applied to T. $\square$

Lemma 3.4.4. *Let $q, r \in U^{\text{trop}}(\mathbb{Z})$, and suppose that $r \notin \mathbf{Conv}(q) \setminus \{q\}$. Then $Tr_r(\vartheta_q) = \delta_{q,r}$.*

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ordering z_i and z_{i+1}). Alternatively, we can take the sign of Ω as part of our data and say that γ is oriented to make $\int_\gamma \Omega > 0$.

Proof. If $r = q$, then the claim is obvious. Otherwise, $r \notin \mathbf{Conv}(q)$, so there is some primitive $v \in V^{\text{trop}}(\mathbb{Z})$ such that $\langle r, v\rangle < \langle q, v\rangle$. Then $\text{val}_{\mathfrak{D}_v}(\vartheta_q\vartheta_r^{-1}) > 0$. Since Ω only has a simple pole along $\mathfrak{D}_v$, $\vartheta_q\vartheta_r^{-1}\Omega$ is generically regular along $\mathfrak{D}_v$. If we view γ as the class of an S^1 bundle over a loop γ' in $\mathfrak{D}_v$, then the claim follows from the Residue Theorem:

$$\int_\gamma \vartheta_q\vartheta_r^{-1}\Omega = \int_{\gamma'} \text{Res}_{\mathfrak{D}_v}\left(\vartheta_q\vartheta_r^{-1}\Omega\right) = \int_{\gamma'} 0 = 0.$$

□

Theorem 3.4.5. *Let $f = \sum_q c_q\vartheta_q$ be a function on V. Suppose that at least one of the following hold:*

- *r is not in the convex hull of any point $q \in \text{Newt}(f) \cap U^{\text{trop}}(\mathbb{Z})$ with $q \neq r$. In particular, this includes cases where r is a vertex of* Newt(f)*, as well as cases where r is in the complement of* Newt(f).
- *$r \in U^{\text{trop}}(\mathbb{Z})$ is in the cluster complex (i.e., $r = 0$ or $\langle r, v\rangle > 0$ for some v).*

Then $c_r = Tr_r(f)$. In particular, if every point of Newt$(f) \cap U^{\text{trop}}(\mathbb{Z})$ *which is not a vertex is in the cluster complex, then*

$$f = \sum_{r\in U^{\text{trop}}(\mathbb{Z})} Tr_r(f)\vartheta_r. \tag{3.7}$$

Proof. If r is not in the convex hull of any point in $\text{Newt}(f) \cap (U^{\text{trop}}(\mathbb{Z}) \setminus \{r\})$, then the claim follows immediately from Lemma 3.4.4.

Suppose that $r \neq 0$ is in the cluster complex. We can refine our fan Σ from the construction of $\mathcal{V}$ so that there is some cone $\sigma \ni r$ which has no scattering rays on its interior. Then there is a torus $T_\sigma \cong (\mathbb{C}^*)^2$ in V corresponding to σ on which ϑ_r is just the restriction of the monomial $z^{\widetilde{\varphi}(r)}$, which we may view as a constant times z^r. Let Γ be a broken line in σ with attached monomial $z^{\widetilde{\varphi}(r)}$. By flowing backwards (in the r direction) along Γ, we see that Γ does not hit any scattering walls, hence does not bend. So $z^{\widetilde{\varphi}(r)}$ must have been the initial monomial attached to Γ. Hence, ϑ_r is the only theta function whose expansion in terms of monomials in T_σ contains a z^r term. Since $\int_\gamma z^q z^{-r}\Omega = \delta_{q,r}$ always holds (a standard fact about tori, and also a corollary of Lemma 3.4.4), the claim follows. The $r = 0$ case was proven in [GHK11]. $\square$

Remark 3.4.6. We note that Equation 3.7 resembles the formula for the Fourier series expansion of a function on a compact torus. Indeed, in the case that V is a toric variety, applying this theorem to monomials and restricting to the orbits of the torus action recovers the usual formula for (finite) Fourier expansions.

Remark 3.4.7. Suppose that $\mathrm{Newt}(f) \cap U^{\mathrm{trop}}(\mathbb{Z})$ contains points which are neither vertices nor in the cluster complex. We can still use Tr_q with various q to get all the coefficients in the theta function expansion for f as follows: we first use the theorem to get the coefficients for the vertices $\{q_1, \ldots, q_s\}$ of $\mathrm{Newt}(f)$. We then subtract the contributions of these theta functions to get $\widetilde{f} := f - \sum_{i=1}^{s} Tr_{q_i}(f)\vartheta_{q_i}$. $\mathrm{Newt}(\widetilde{f})$ is now smaller than $\mathrm{Newt}(f)$ (it is contained in the convex hull of $\mathrm{Newt}(f) \cap U^{\mathrm{trop}}(\mathbb{Z}) \setminus \{q_1, \ldots, q_s\}$), so we have a new set of vertices and can apply the process again. Repeating this will eventually yield all the coefficients.

3.4.1 Theta Functions up to Linear Equivalence

Consider $(Z, \mathfrak{D})$, $V = Z \setminus \mathfrak{D}$, as usual. Recall that if f is a regular function on V, then f determines a divisor $(f) = \mathfrak{D}(f) + V(f)$, where $\mathfrak{D}(f) := \sum f^{\mathrm{trop}}(v_i)\mathfrak{D}_{v_i}$, and $V(f)$ is the divisor of interior zeroes of f. Knowing $V(f)$ of course determines f up to scalar multiplication, and we see that knowing f^{trop} is sufficient for determining the linear equivalence class $|\mathfrak{D}(f)| = |-V(f)|$. The global sections of the corresponding line bundle are the funcitons of the form $\sum_{q \in \mathrm{Newt}(f) \cap U^{\mathrm{trop}}(\mathbb{Z})} a_q \vartheta_q$. In particular, the dimension of the linear system is one less than the number of integer points in $\mathrm{Newt}(f)$.

Examples 3.4.8.

- If $\mathrm{Newt}(f)$ is just a single point $q \in U^{\mathrm{trop}}(\mathbb{Z})$, then f is uniquely determined up to scaling. Of course, in this case, q is in the cluster complex, and we have already seen an explicit description of such funcitons.
- If $\mathrm{Newt}(\vartheta_q) \cap U^{\mathrm{trop}}(\mathbb{Z})$ is contained entirely in the cluster complex except for the point q, then we can identify ϑ_q as the unique (up to scaling) nonzero global section f of $|\mathfrak{D}(\vartheta_q)|$ such that $\mathrm{Tr}_r(f) = 0$ for all $r \in \mathrm{Newt}(\vartheta_q) \cap U^{\mathrm{trop}}(\mathbb{Z}) \setminus \{q\}$.
- One can show that for any U^{trop} with at least some lines wrapping, there is some q with $\mathbf{Conv}(q) \cap U^{\mathrm{trop}}(\mathbb{Z}) = \{q, 0\}$. Then ϑ_q is uniquely determined by $\vartheta_q^{\mathrm{trop}}$ and the fact that $Tr_0(\vartheta_q) = 0$. For example, in the cubic surface case, any primitive q satisfies this condition.

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This dissertation was typeset with $\LaTeX$[†] by the author.

[†] $\LaTeX$ is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's $\TeX$ Program.