DISCLAIMER:

This document does not meet the
current format guidelines of
the Graduate School at
The University of Texas at Austin.

It has been published for
informational use only.

The Dissertation Committee for Robin Tonra Neumayer
certifies that this is the approved version of the following dissertation:

Minimality and stability properties in Sobolev and isoperimetric inequalities

Committee:

Francesco Maggi, Supervisor

William Beckner

Luis Caffarelli

Alessio Figalli

David Jerison

Minimality and stability properties in Sobolev and isoperimetric inequalities

by

Robin Tonra Neumayer

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2017

To my parents.

Acknowledgments

With the deepest possible sincerity, I thank my advisors Alessio Figalli and Francesco Maggi. Their example and guidance have taught me so much throughout the past four years under their mentorship, and their patience, support, and generosity have been remarkable. I am immensely grateful for the many insights they have shared with me and it is an honor to have been their student.

I also thank my committee members: Bill Beckner, Luis Caffarelli, and David Jerison. It has been a true privilege to be part of the analysis group at UT Austin, where I have enjoyed many seminars, courses, and lively discussions; I am grateful to all the professors, postdocs, students, and visitors who contribute to its warm and dynamic environment. My visits to ETH, ICTP, and ENS Lyon have been incredible experiences for which I have Professors Figalli and Maggi to thank.

For many stimulating mathematical discussions over the years, I thank Dennis Kriventsov, Yash Jhaveri, Connor Mooney, Javier Morales, Cornelia Mihaila, Tim Carson, Hui Yu, Luis Duque, Xavi Ros-Oton, Rohit Jain, Pablo Stinga, Hector Chang, Maja Tasković, Sona Akopian, Matt Novack, Yunan Yang, Andrew Lee, Soledad Villar, Gian Paolo Leonardi, Giorgio Sarocco, Alessandro Carlotto, Matias Delgadino, Monica Torres, Glen Wheeler, Norisuke Ioku, Marco Carrocia, and Richard Voepel. I am indebted to Hector Chang and Mike Kelly for advice and inclusivity in my first year at UT that helped me become part of the analysis group from the very beginning. For being role models as well as friends, I thank Jen Berg and Laura Fredrickson.

My warm thanks go to Maria Girardi, whose undergraduate real analysis class made me want to be a mathematician (and an analyst) and whose influence played an enormous role in my pursuing graduate school.

With admiration and affection, I thank my brother Tony for being a friend, a source of support, and one of my favorite people in the world. Finally, I cannot possibly overstate my gratitude for my parents, Mary and Dennis. Their support throughout this journey, and every step that led up to it, has been unceasing.

Minimality and stability properties in Sobolev and isoperimetric inequalities

Publication No. __________________

Robin Tonra Neumayer, Ph.D.
The University of Texas at Austin, 2017

Supervisor: Francesco Maggi

This thesis addresses the characterization of minimizers in various Sobolev- and isoperimetric-type inequalities and the analysis of the corresponding stability phenomena. We first investigate a family of variational problems which arise in connection to a suitable interpolation between the classical Sobolev and Sobolev trace inequalities. We provide a full characterization of minimizers for each problem, in turn deriving a new family of sharp constrained Sobolev inequalities on the half-space. We then prove novel stability results for the Sobolev inequality on $\mathbb{R}^n$ and for the anisotropic isoperimetric inequality. Both of these results share the feature of being "strong-form" stability results, in the sense that the deficit in the inequality is shown to control the strongest possible distance to the family of equality cases.

Table of Contents

Chapter 1

Introduction

Functional and geometric inequalities, and particularly those of Sobolev and isoperimetric type, play a key role in a number of problems arising in the calculus of variations, partial differential equations, and geometry. A prototypical example is the classical Sobolev inequality on $\mathbb{R}^n$ for $n \geq 2$, which says that, for $1 \leq p < n$,

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \geq S\|u\|_{L^{p^\star}(\mathbb{R}^n)} \tag{1.0.1}$$

for any $u \in \dot{W}^{1,p}(\mathbb{R}^n)$.[1] Here, $p^\star = np/(n-p)$ and $S = S(n,p)$ denotes the optimal constant. Intimately related to Sobolev inequalities are isoperimetric inequalities, the most ubiquitous example of which is the Euclidean isoperimetric inequality: for $n \geq 2$, one has

$$P(E) \geq n|B|^{1/n}|E|^{1/n'}\,, \tag{1.0.2}$$

$n' = n/(n-1)$, with equality if and only if E is a dilation or translation of the unit ball B.

The main results of this thesis address two primary questions for certain Sobolev and isoperimetric inqualities:

[1] For $p = 1$, (1.0.1) holds for any $u \in BV(\mathbb{R}^n)$ with the left-hand side replaced by the total variation $|Du|(\mathbb{R}^n)$.

Problem 1: Is equality attained in the inequality? Can one characterize all extremals in the inequality?

An inequality with suitable scaling invariance can be equivalently be viewed variationally–for instance, (1.0.1) is equivalent to the minimization problem

$$\inf\left\{\|\nabla u\|_{L^p(\mathbb{R}^n)} : \|u\|_{L^{p^\star}(\mathbb{R}^n)} = 1\right\} = S\,. \tag{1.0.3}$$

With this perspective in mind, we identify the problem of characterizing extremals in the inequality with that of characterizing minimizers in the associated variational problem.

If both parts of Problem 1 are answered in the affirmative for a given inequality, a second natural question to ask is the following:

Problem 2: Suppose a function or set almost achieves equality in the inequality. Then, is it close, in a suitable sense, to an extremal function or set?

Problem 2 addresses the stability of the inequality, or, more precisely, the quantitative stability of the minimizers in the equivalent variational problem.

In the following two sections of the introduction, we briefly outline known stability and minimality properties for the Sobolev and isoperimetric inequalities considered in this thesis and present the main results that are proven in the subsequent chapters.

1.1 Sobolev inequalities

In the eighty years since the seminal papers [Sob36, Sob38] of S.L. Sobolev, Sobolev-type inequalities have been significantly refined and generalized and have become central tools in modern analysis. They are used, for instance, to address the solvability of certain boundary values problems and the structure of the spectra of elliptic operators (see [Maz85, Chaper 6]), and, in conjunction with energy inequalities, to prove various types of regularity results for elliptic and parabolic PDE (as in [DG57]). Determining the value of the sharp constants and characterizing the associated extremal functions in these inequalities often provides interesting geometric information. For example, the sharp constant and extremals in (1.0.1) played a crucial role in the solution of the Yamabe problem in conformal geometry, which asks whether every compact Riemannian manifold (M, g) admits a metric with constant scalar curvature that is conformal to g (see [Yam60, Tru68, Aub76a, Sch84] and the survey [LP87]). It was with this motivation that Aubin showed in [Aub76b], concurrently with Talenti in [Tal76], that equality in (1.0.1) for $1 < p < n$ is uniquely achieved by dilations, translations, and constant multiples of the function

$$U_S(x) = \frac{1}{(1 + |x|^{p'})^{(n-p)/p}} \,. \tag{1.1.1}$$

In other words, the $(n+2)$-dimensional family of extremals is given by

$$\mathcal{M} = \{c\, U_S(\lambda(x - x_0)) \ : \ c \in \mathbb{R},\, \lambda \in \mathbb{R}_+,\, x_0 \in \mathbb{R}^n\} \,. \tag{1.1.2}$$

Both proofs used symmetrization methods and an analysis of the Euler-Lagrange equation associated to the variational problem (1.0.3). A quite different proof of this characterization is given by tracing through the argument presented in [CENV04], which uses the Brenier-McCann theorem from the theory of mass transportation (see Section 2.2). A remarkable feature of this proof is that does not rely on geometric properties like symmetry and holds without modification for non-Euclidean norms on $\mathbb{R}^n$.

For $p = 1$, the Sobolev inequality is equivalent to the isoperimetric inequality (see [FF60, FR60, Maz60]), and accordingly, the extremal functions are translations, dilations, and constant multiples of the characteristic functions of the ball.

Related to the Yamabe problem is the question of whether every compact Riemannian manifold (M, g) with boundary admits a scalar flat metric conformal to g with constant mean curvature on the boundary. In this problem, considered in [Esc92a], the role of (1.0.1) is played by the Sobolev trace inequality on the half-space $H = \{x_1 > 0\} \subset \mathbb{R}^n$ for $n \geq 2$, which states that

$$\|\nabla u\|_{L^p(H)} \geq Q \, \|u\|_{L^{p^\sharp}(\partial H)} \tag{1.1.3}$$

for all $1 \leq p < n$. Here, $p^\sharp = (n-1)p/(n-p)$ and $Q = Q(n, p)$ is the optimal constant. Escobar showed in [Esc88] that when $p = 2$, all extremal functions in (1.1.3) are given by dilations, constant multiples, and translations by $x_0 \in H$ of the function $|x + \mathrm{e}_1|^{2-n}$. Beckner gave another proof in an unpublished note in 1987, later expanded into [Bec93]. Both proofs, though different in nature, crucially exploit the conformal invariance that is specific to the case $p = 2$. For the general case $1 < p < n$, it was not until [Naz06] that the function

$$U_Q = \frac{1}{|x + \mathrm{e}_1|^{(n-p)/(p-1)}} \tag{1.1.4}$$

and its invariant scalings $\{c\,U_Q(\lambda(x+x_0))\,:\,c\in\mathbb{R},\,\lambda\in\mathbb{R}_+,\,x_0\in H\}$ were shown to be extremals of (1.1.3), confirming a conjecture of Escobar in [Esc88].[2] The proof used a variant of the aforementioned optimal transport argument developed in [CENV04].

In Chapter 2, based on joint work with F. Maggi in [MN17], we prove a new one-parameter family of sharp constrained Sobolev inequalities which interpolate between the Sobolev inequality (1.0.1) and the Sobolev trace inequality (1.1.3), and characterize all extremal functions in each inequality. More specifically, we consider the following family of variational problems:

$$\Phi(T)=\inf\left\{\|\nabla u\|_{L^p(H)}\ :\ \|u\|_{L^{p^*}(H)}=1,\ \|u\|_{L^{p^\sharp}(\partial H)}=T\right\}\qquad T\geq 0\,. \tag{1.1.5}$$

We characterize minimizers in (1.1.5) for every $T>0$ and every $1\leq p<n$, and then use this information to provide a qualitative description of the behavior of the infimum value $\Phi(T)$ as a function of T. For $1<p<n$, the characterization result involves the following three families of functions:

Sobolev family: Let U_S be defined as in (1.1.1) and set, for every $t\in\mathbb{R}$,

$$U_{S,t}(x)=\frac{U_S(x-t\,\mathrm{e}_1)}{\|U_S(\mathrm{id}-t\,\mathrm{e}_1)\|_{L^{p^\star}(H)}}\qquad x\in H\,,$$

and

$$T_S(t)=\|U_{S,t}\|_{L^{p^\sharp}(\partial H)}\,,\qquad G_S(t)=\|\nabla U_{S,t}\|_{L^p(H)}\,.$$

[2] The uniqueness of this family of extremals was left open in [Naz06], but was shown in [MN17]; see Appendix B.

Escobar family: Letting U_Q be as in (1.1.4), we set for every $t < 0$

$$U_{Q,t}(x) = \frac{U_Q(x - t\,\mathrm{e}_1)}{\|U_Q(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^\star}(H)}} \qquad x \in H\,.$$

A simple computation shows that the trace and gradient norms of the $U_{Q,t}$ are independent of $t < 0$, and we set

$$\|U_{Q,t}\|_{L^{p^\sharp}(\partial H)} = T_Q\,, \qquad \|\nabla U_{Q,t}\|_{L^p(H)} = G_Q$$

for these constant values.

Beyond-Escobar family: We consider the function

$$U_B(x) = (|x|^{p'} - 1)^{(p-n)/p} \qquad |x| > 1\,,$$

and define, for every $t < -1$,

$$U_{B,t}(x) = \frac{U_B(x - t\,\mathrm{e}_1)}{\|U_B(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^\star}(H)}} \qquad x \in H\,.$$

Correspondingly, for every $t < -1$, we set

$$T_B(t) = \|U_{B,t}\|_{L^{p^\sharp}(\partial H)}\,, \qquad G_B(t) = \|\nabla U_{B,t}\|_{L^p(H)}\,.$$

Theorem 1.1.1 (Existence and Characterization of Minimizers)**.** *Let $n \geq 2$ and $p \in (1, n)$. For every $T \in (0, +\infty)$, a minimizer exists in the variational problem* (1.1.5) *and is unique up to dilations and translations orthogonal to* e_1*. More precisely:*

(i) for every $T \in (0, T_Q)$, there exists a unique $t \in \mathbb{R}$ such that

$$T = T_S(t)\,, \qquad \Phi(T) = G_S(t)\,,$$

and $U_{S,t}$ is the uniquely minimizer in (1.1.5) *up to dilations and translations orthogonal to* e_1*;*

(ii) *if* $T = T_Q$*, then, up to dilations and translations orthogonal to* e_1*,* $\{U_{Q,t} : t < 0\}$ *is the unique family of minimizers of* (1.1.5)*;*

(iii) *for every* $T \in (T_Q, +\infty)$ *there exists a unique* $t < -1$ *such that*

$$T = T_B(t)\,, \qquad \Phi(T) = G_B(t)\,, \tag{1.1.6}$$

and $U_{B,t}$ *is the unique minimizer of* (1.1.5) *up to dilations and translations orthogonal to* e_1*.*

As a consequence of Theorem 1.1.1, we obtain the sharp constants and characterization of extremals for the following family of constrained Sobolev inequalities: for any $0 < T < \infty$,

$$\|\nabla u\|_{L^p(H)} \geq \Phi(T)\|u\|_{L^{p^\star}(H)} \tag{1.1.7}$$

for all $u \in \dot{W}^{1,p}(H)$ with $\|u\|_{L^{p^\sharp}(\partial H)}/\|u\|_{L^{p^\star}(H)} = T$.

We also prove a qualitative description of $\Phi(T)$ as a function of T; see Theorem 2.1.2.

Carlen and Loss first considered the variational problem (1.1.5) for $p = 2$ in [CL94], where they characterize minimizers using their method of competing symmetries developed in [CL90b, CL90a, CL92]. Hence, Theorem 1.1.1 can be seen as a generalization of [CL94] from the case $p = 2$ to the full range $p \in (1, n)$. Their method, like the results of [Esc88] and [Bec93] characterizing extremals in (1.1.3), relies in an essential way on the conformal invariance that is present only in the case $p = 2$. In view of these considerations, we prove Theorem 1.1.1 with a mass transportation argument in the spirit of [CENV04].

Chapter 3 deals with the question of stability for the Sobolev inequality on $\mathbb{R}^n$, which was first raised Brezis and Lieb in [BL85]. To quantify how close a function is to achieving equality in (1.0.1), we define the *deficit* of a function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ in the Sobolev inequality by

$$\delta_S(u) = \frac{\|\nabla u\|^p_{L^p(\mathbb{R}^n)}}{S^p \|u\|^p_{L^{p^*}(\mathbb{R}^n)}} - 1.$$

Note that this nonnegative quantity vanishes if and only if $u \in \mathcal{M}$, with $\mathcal{M}$ as defined in (1.1.2). For an appropriately defined distance d of u to the family $\mathcal{M}$, we seek an inequality of the form

$$\delta_S(u) \geq \omega(d(u)), \tag{1.1.8}$$

where ω is a function such that $\omega(d(u)) \to 0^+$ as $d(u) \to 0^+$. Such an inequality can be viewed as a quantitative form of the Sobolev inequality with $\omega(d(u))$ serving as a remainder term in (1.0.1): after rearranging, (1.1.8) becomes

$$\|\nabla u\|^p_{L^p(\mathbb{R}^n)} \geq S^p \|u\|^p_{L^{p^\star}(\mathbb{R}^n)} (1 + \omega(d(u)))\,.$$

There are two natural distances to consider for $1 < p < n$:

$$\begin{aligned} \alpha_S(u) &= \inf_{U \in \mathcal{M}} \frac{\|u - U\|_{L^{p^\star}(\mathbb{R}^n)}}{\|u\|_{L^{p^\star}(\mathbb{R}^n)}} \qquad \text{and} \\ \beta_S(u) &= \inf_{U \in \mathcal{M}} \frac{\|\nabla u - \nabla U\|_{L^p(\mathbb{R}^n)}}{\|\nabla u\|_{L^p(\mathbb{R}^n)}}\,. \end{aligned} \tag{1.1.9}$$

Note that $\beta_S(u)$ controls $\alpha_S(u)$ and that $\beta_S(u)$ is the strongest notion of distance that one expects to control by the deficit.

Stability for (1.0.1) was first shown by Bianchi and Egnell in [BE91] in the case $p = 2$. They showed that $\delta_S(u)$ controls $\beta_S(u)^2$, providing a stability result that is optimal both in the strength of the distance and the rate of decay. At the core of their proof is an analysis of the second variation of the deficit through a spectral analysis of suitably weighted Laplace operator. Though these methods strongly exploit the Hilbertian structure of $\dot{W}^{1,2}(\mathbb{R}^n)$, we shall see in Chapter 3 that it is possible to extend these ideas even when $p \neq 2$.

For $p = 1$, following earlier results in [Cia06] and [FMP07], it was shown in [FMP13] that $\delta_S(u)$ controls the appropriate analogue[3] of $\beta_S(u)^2$ using rearrangement techniques and mass transportation theory. Again, this result is optimal both in the strength of the distance and the exponent of decay.

The general case $1 < p < n$ is more difficult. In [CFMP09], Cianchi, Fusco, Maggi, and Pratelli proved that the deficit controls $\alpha_S(u)$ with a non-sharp exponent, combining symmetrization techniques and a one-dimensional mass transportation argument. However, in view of [BE91] and [FMP13], one expects that the deficit should control a power of $\beta_S(u)$. In Chapter 3, we show that this is true for $p \geq 2$. More precisely, the main result of the chapter, based on joint work with A. Figalli in [FN], states the following:

Theorem 1.1.2. *Let $2 \leq p < n$. There exists a constant $C > 0$, depending only on p and n, such that for all $u \in \dot{W}^{1,p}(\mathbb{R}^n)$,*

$$\beta_S(u)^\zeta \leq C\delta_S(u)\,, \tag{1.1.10}$$

where $\zeta = p^*p\left(3 + 4p - \frac{3p+1}{n}\right)^2$.

---

[3] Since the extremals in (1.0.1) for $p = 1$ lie in the space $BV(\mathbb{R}^n)$ but not $\dot{W}^{1,1}(\mathbb{R}^n)$, the distance takes a slightly different form; see [FMP13].

A key idea behind Theorem 1.1.2 is to introduce a Hilbertian structure to $\dot{W}^{1,p}(\mathbb{R}^n)$ by defining a different weighted L^2-space for each $U \in \mathcal{M}$. In this way, we can analyze the second variation using a spectral gap argument in the spirit of [BE91], though the spectral analysis is somewhat delicate because we deal with a degenerate elliptic operator. This approach does not directly lead to (1.1.10), since when $p \neq 2$, there are certain terms in an expansion of the deficit that are in competition with the second variation. To overcome these difficulties, we develop an interpolation argument that makes use of the main result of [CFMP09]. We remark that ζ is likely not the optimal rate of decay in (1.1.10), which is conjectured to be $\max\{p, 2\}$; see [Fus15, Section 6].

The topic of stability for Sobolev-type inequalities has generated much interest in recent years. In addition to the aforementioned papers, results of this type have been addressed for the log-Sobolev inequality [IM14, BGRS14, FIL16], the higher order Sobolev inequality [GW10, BWW03], the fractional Sobolev inequality [CFW13], the Morrey-Sobolev inequality [Cia08] and the Gagliardo-Nirenberg-Sobolev inequality [CF13, DT13, Ruf14, Ngu]. Apart from their intrinsic interest, these results can be used to obtain quantitative rates of convergence for certain diffusion equations, as in [CF13, Ngu].

1.2 Isoperimetric inequalities

Many physical phenomena are governed by the minimization of energies related to surface area, so isoperimetric inequalities naturally come into play in a number of variational problems modeling these situations. In the description of systems of an anisotropic nature, such as equilibrium configurations for solid crystals (see [Wul01, Her51, Tay78]) and phase transitions (see [Gur85]), one must consider a generalization of the perimeter functional that is weighted to favor configurations where the boundary of a set faces certain directions. The anisotropic surface energy of a set $E \subset \mathbb{R}^n$ is defined by

$$\mathcal{F}(E) = \int_{\partial^* E} f(\nu_E(x))\, d\mathcal{H}^{n-1}(x)$$

for a convex positively 1-homogeneous function $f : \mathbb{R}^n \to [0, +\infty)$ that is positive on S^{n-1}. (Here, $\partial^* E$ is reduced boundary and the ν_E is the measure theoretic outer unit normal; see Section 4.2.1.) Just as the ball minimizes perimeter among sets at fixed volume, as expressed by (1.0.2), the surface energy is uniquely minimized among sets of a given volume by translations and dilations of the bounded convex set K known as the *Wulff shape* of $\mathcal{F}$ given by

$$K = \bigcap_{\nu \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot \nu < f(\nu)\}.$$

The minimality of the Wulff shape is expressed by the *Wulff inequality*:

$$\mathcal{F}(E) \geq n|K|^{1/n}|E|^{1/n'}, \tag{1.2.1}$$

with equality if and only if E is a translation or dilation of K. This was first shown in [Tay78] under certain assumptions, then in [Fon91, FM91, BM94]; see also [DP92, DGS92]. Observe that the isoperimetric inequality is the particular case of the Wulff inequality with $f(x) = |x|$.

In the setting of the isoperimetric inequalities, the question of stability dates back to the work of Bonnesen [Bon24] in the plane. To quantify how close a set is to achieving equality in (4.1.1), we define the deficit of a set E to be the scaling invariant quantity

$$\delta_f(E) = \frac{\mathcal{F}(E)}{n|K|^{1/n}|E|^{1/n'}} - 1\,.$$

A natural and well-studied distance of a set E to the family of extremals is the *asymmetry index*, $\alpha_f(E)$, defined by

$$\alpha_f(E) = \min_{y\in\mathbb{R}^n}\left\{\frac{|E\Delta(rK+y)|}{|E|}\ :\ |rK| = |E|\right\}, \tag{1.2.2}$$

where $E\Delta F = (E\setminus F)\cup(F\setminus E)$ is the symmetric difference of E and F. This L^1-type distance plays the role of the functional $\alpha_S(u)$ defined in (1.1.9). The quantitative isoperimetric inequality with respect to the asymmetry index was proven in sharp form by Fusco, Maggi, and Pratelli in [FMP08]. Using symmetrization techniques, they showed that if E is a set of finite perimeter with $0 < |E| < \infty$, then

$$\alpha_1(E)^2 \leq C(n)\delta_1(E). \tag{1.2.3}$$

Here and in the sequel, we use the notation δ_1 and α_1 for the deficit and asymmetry index corresponding to the perimeter. Before this full proof of (1.2.3) was given, several partial results were shown in [Fug89, Hal92, HHW91]. Another proof of (1.2.3) was given in [CL12], introducing a technique known as the selection principle, where a penalization technique and the regularity theory for almost-minimizers of perimeter reduce the problem to the case shown in [Fug89].

Stability of the Wulff inequality was first addressed in [EFT05], without the sharp exponent. Figalli, Maggi, and Pratelli later proved the sharp version in [FMP10] exploiting the mass transportation proof of (4.1.1) given in [BM94, MS86]. They showed that there exists a constant $C(n)$ such that

$$\alpha_f(E)^2 \leq C(n)\delta_f(E) \tag{1.2.4}$$

for any set of finite perimeter E with $0 < |E| < \infty$. In both (1.2.3) and (1.2.4), the power 2 is sharp.

In the aforementioned result of [Fug89], Fuglede proved (1.2.3) when ∂E is a small C^1 perturbation of ∂B. Within this class of sets, Fuglede's result is actually stronger: he showed that $\delta_1(E)$ controls a stronger distance, now known as the *oscillation index* $\beta_1(E)$, defined by

$$\beta_1(E) = \min_{y\in\mathbb{R}^n} \left\{ \left(|E|^{-1/n'} \int_{\partial^* E} 1 - \nu_E(x) \cdot \frac{x-y}{|x-y|}\, d\mathcal{H}^{n-1}(x) \right)^{1/2} \right\}, \tag{1.2.5}$$

which controls $\alpha_1(E)$ and is the analogue of $\beta_S(u)$ in this setting. In [FJ14], Fusco and Julin used a selection principle argument and the result of [Fug89] to improve (1.2.3) by showing

$$\alpha_1(E)^2 + \beta_1(E)^2 \leq C(n)\delta_1(E) \tag{1.2.6}$$

for any set of finite perimeter E with $0 < |E| < \infty$. Once again, the power 2 in (1.2.6) is sharp for both $\alpha_1(E)$ and $\beta_1(E)$.

The main result of Chapter 4, based on [Neu16], is a strong-form stability result for the Wulff inequality in the spirit of (1.2.6). Determining the appropriate analogue β_f of the oscillation index is actually a subtle point (see Sections 4.1 and 4.6). After defining it in Definition 4.1.2, we prove several stability results, which can be summarized in the following statement:

Theorem 1.2.1. *Fix $n \geq 2$ and let $\mathcal{F}$ be an anisotropic surface energy. There exist $C = C(n, f) > 0$ and $\alpha(n, f) > 0$ such that*

$$\alpha_f(E)^2 + \beta_f(E)^\alpha \leq C\delta_f(E) \tag{1.2.7}$$

for any set of finite perimeter E with $0 < |E| < \infty$.

There are two settings in which we obtain the sharp exponent $\alpha = 2$: when f is λ-elliptic, that is, f has sufficient regularity and convexity properties, or when $n = 2$ and f is crystalline, that is, the Wulff shape of K is a polygon. For an arbitrary surface tension f, we obtain the likely non-optimal exponent $\alpha = 4n/(n+1)$, but can prove the theorem with the constant C depending only on the dimension.

The proof of Theorem 1.2.1 uses a selection principle argument in the spirit of [CL12, FJ14], which allows us to reduce to the case of sets which are almost-minimizers of the anisotropic perimeter and are L^1-close to K. However, a key component of the selection principle is the regularity theory for almost-minimizers. For general anisotropies, we are missing this component. In the case of a crystalline surface tension in dimension 2, in lieu of the regularity theory, we use a rigidity result of Figalli and Maggi in [FM11] which lets us assume that E is a convex polygon with sides that align with those of K. For an arbitrary surface tension, density estimates are the strongest regularity property that one can hope to extract, and so, pairing these estimates with (1.2.4), we obtain the result with the non-sharp exponent.

When f is λ-elliptic, almost-minimizers of the corresponding surface energy $\mathcal{F}$ do enjoy strong regularity properties, so we may take ∂E to be a small C^1 perturbation of ∂K. We then prove the following analogue of Fuglede's result in the anisotropic case, which is interesting in its own right.

Proposition 1.2.2. *Let f be λ-elliptic with corresponding surface energy $\mathcal{F}$ and Wulff shape K. Let E be a set such that $|E| = |K|$ and $\operatorname{bar} E = \operatorname{bar} K$, where $\operatorname{bar} E = |E|^{-1} \int_E x\, dx$ denotes the barycenter of E. Suppose*

$$\partial E = \{x + u(x)\nu_K(x) : x \in \partial K\}$$

where $u : \partial K \to \mathbb{R}$ is in $C^1(\partial K)$. There exist C and ε_1 depending on f such that if $\|u\|_{C^1(\partial K)} \le \varepsilon_1$, then

$$\|u\|^2_{H^1(\partial K)} \le C\delta_f(E). \tag{1.2.8}$$

Fuglede proved Proposition 1.2.2 in the isotropic case using a spectral gap argument much in the sprit of [BE91], strongly exploiting the fact that the eigenvalues and eigenfunctions of the Laplacian on the sphere are explicitly known. At its core, the proof of Proposition 1.2.2 also relies on a spectral gap, but nothing explicit can be said about the spectrum of the elliptic differential operator on ∂K that plays the role of the Laplacian on ∂B. In the absence of explicit spectral information, we instead perform an implicit spectral analysis, using the main result of [FMP10] to establish the existence of an appropriately placed spectral gap.

The study of stability for isoperimetric type inequalities has seen an explosion of results in recent years. The literature is much too broad to account for here, so let us simply mention that analogous strong-form quantitative inequalities have recently been studied in several settings: in Gaussian space [Eld15, BBJ], on the sphere [BDF], and in hyperbolic n-space [BDS15]. We refer the reader to the recent survey paper [Fus15] for a rather complete overview of contemporary stability results for isoperimetric-type inequalities, and to [Oss79] for a survey of earlier results.

Chapter 2

A bridge between the Sobolev and Sobolev trace inequalities and beyond

2.1 Overview

2.1.1 A variational problem interpolating the Sobolev and Sobolev trace inequalities

In this chapter,[1] we illustrate a strong link between the Sobolev inequality on $\mathbb{R}^n$

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \geq S\,\|u\|_{L^{p^\star}(\mathbb{R}^n)} \qquad p^\star = \frac{np}{n-p}\,, \tag{2.1.1}$$

and the Sobolev trace inequality on the half-space $H = \{x_1 > 0\}$

$$\|\nabla u\|_{L^p(H)} \geq Q\,\|u\|_{L^{p^\sharp}(\partial H)} \qquad p^\sharp = \frac{(n-1)p}{n-p}\,, \tag{2.1.2}$$

where $n \geq 2$ and $p \in [1, n)$. These classical sharp inequalities both arise as particular cases of the variational problem $\Phi(T) = \Phi^{(p)}(T)$ defined by

$$\Phi(T) = \inf\Big\{\|\nabla u\|_{L^p(H)} : \|u\|_{L^{p^\star}(H)} = 1\,, \|u\|_{L^{p^\sharp}(\partial H)} = T\Big\} \qquad T \geq 0\,, \tag{2.1.3}$$

with $T = 0$ in the case of (2.1.1), and with $T = T_Q$ for a suitable $T_Q > 0$ in the case of (2.1.2). Our main result, Theorems 2.1.1 and 2.1.2, characterize the minimizers of

[1] This chapter is based on joint work with F. Maggi originally appearing in [MN17].

$\Phi(T)$ for every $T > 0$ and give a description of the behavior of $\Phi(T)$ as a function of T.

The cases $p = 2$ and $p = 1$ have interpretations in conformal geometry and in capillarity theory respectively. In particular, when $p = 2$, (2.1.3) amounts to minimizing a total curvature functional among conformally flat metrics on H – see (2.1.26) below. An interesting feature of this problem is that the corresponding minimizing geometries change their character from spherical (for $T \in (0, T_Q)$) to hyperbolic (for $T > T_Q$).

Let us start by setting our terminology and framework, focusing on the case $p \in (1, n)$. We work with locally summable functions $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ that are vanishing at infinity, that is, $|\{|u| > t\}| < \infty$ for every $t > 0$. If Du denotes the distributional gradient of u, then the minimization in (2.1.3) is over functions with $Du = \nabla u \, dx$ for $\nabla u \in L^p(H; \mathbb{R}^n)$. We recall from the introduction that equality holds in (2.1.1) if and only if there exist $\lambda > 0$ and $z \in \mathbb{R}^n$ such that

$$u(x) = \lambda^{(n-p)/p} \, U_S(\lambda(x - z)) \qquad \forall x \in \mathbb{R}^n \,, \tag{2.1.4}$$

where

$$U_S(x) = (1 + |x|^{p'})^{(p-n)/p} \qquad x \in \mathbb{R}^n \,. \tag{2.1.5}$$

(Here, as usual, $p' = p/(p-1)$.) We also reacall that equality holds in (2.1.2) if and only if there exist $\lambda > 0$ and $z \in \mathbb{R}^n$ with $z_1 < 0$ such that

$$u(x) = \lambda^{(n-p)/p} \, U_Q(\lambda(x - z)) \qquad \forall x \in H \,, \tag{2.1.6}$$

where U_Q is the fundamental solution of the p-Laplacian on $\mathbb{R}^n$:

$$U_Q(x) = |x|^{(p-n)/(p-1)}\,, \qquad x \in \mathbb{R}^n \setminus \{0\}\,. \tag{2.1.7}$$

Referring to the monograph [Maz85] for a broader picture on Sobolev-type inequalities, we now pass to the starting point of our analysis, which is the realization that (2.1.1) and (2.1.2) can be "embedded" in the family of variational problems (2.1.3). Indeed:

(a) The Sobolev inequality is essentially equivalent to the variational problem $\Phi(T)$ with the choice $T = 0$. Indeed, if $u = 0$ on ∂H, then by applying (2.1.1) to the zero extension of u outside of H, we find that $\Phi(0) \geq S$. Next, by considering an appropriate sequence of scalings as in (2.1.4) multiplied by smooth cutoff functions, we actually find that

$$\Phi(0) = S\,.$$

The characterization of equality cases in (2.1.1) implies that $\Phi(0)$ does not admit minimizers. However, a concentration-compactness argument shows that every minimizing sequence is asymptotically close to a sequence of optimal functions in the Sobolev inequality that is either concentrating at an interior point of H or whose peaks have distance from ∂H diverging to infinity. From this point of view, we consider the variational problem

$$S = \inf\left\{\|\nabla u\|_{L^p(\mathbb{R}^n)} : \|u\|_{L^{p^\star}(\mathbb{R}^n)} = 1\right\}$$

to be essentially equivalent to $\Phi(0)$.

(b) The Sobolev trace inequality boils down to the variational problem $\Phi(T)$ corresponding to $T = T_Q$ for the constant

$$T_Q = T_Q(n,p) = \frac{\|U_Q\|_{L^{p^\sharp}(\{x_1=1\})}}{\|U_Q\|_{L^{p^\star}(\{x_1>1\})}}\,. \tag{2.1.8}$$

Indeed, a simple scaling argument shows that, for every function $u(x)$ as in (2.1.6), one has

$$\frac{\|u\|_{L^{p^\sharp}(\partial H)}}{\|u\|_{L^{p^\star}(H)}} = T_Q$$

independently of the choices of λ and, more surprisingly, of z. Thus, by the definition of T_Q and the characterization of equality cases in (2.1.2), we have

$$\|u\|_{L^{p^\sharp}(\partial H)} = T_Q \qquad \text{for every } u \text{ optimal function in (2.1.2) with } \|u\|_{L^{p^\star}(H)} = 1\,.$$

As a consequence,

$$\Phi(T_Q) = Q\,,$$

and (the variational problem defined by) the Sobolev trace inequality is equivalent to (2.1.3) with $T = T_Q$.

2.1.2 What is known about $\Phi(T)$

As discussed in the introduction, a full characterization of $\Phi(T)$ in the important case $p = 2$ was already given by Carlen and Loss in [CL94]. The situation is quite different when $p \neq 2$. We now collect the information that, to the best of our knowledge, is all that is presently known about $\Phi(T)$. As we have just seen, $\Phi(0) = S$ by the Sobolev inequality, and we have a global linear lower bound

$$\Phi(T) \geq Q\,T \qquad \forall T \geq 0\,, \tag{2.1.9}$$

with equality if $T = T_Q$, thanks to the Sobolev trace inequality. Another piece of information comes from the validity of the *gradient domain inequality* (see [MV08, Section 7.2] for the terminology adopted here) on H:

$$\|\nabla u\|_{L^p(H)} \geq 2^{-1/n}\, S\, \|u\|_{L^{p^\star}(H)}\,, \tag{2.1.10}$$

with equality if and only if there exists $\lambda > 0$ such that

$$u(x) = \lambda^{(n-p)/p}\, U_S(\lambda\, x) \qquad \forall x \in \mathbb{R}^n\,.$$

The validity of (2.1.10), with equality cases, follows immediately by applying the Sobolev inequality (2.1.1) to the extension by reflection of u to $\mathbb{R}^n$. The gradient domain inequality implies that

$$\Phi(T) \geq 2^{-1/n}\, S\,, \qquad \forall T \geq 0 \tag{2.1.11}$$

with equality if and only if $T = T_0$ where

$$T_0 = \frac{\|U_S\|_{L^{p^\sharp}(\partial H)}}{\|U_S\|_{L^{p^\star}(H)}}\,.$$

As we will prove later on (see Proposition 2.3.2(i)),

$$T_0 < T_Q\,,$$

while clearly (by applying (2.1.10) to an optimal function for (2.1.2))

$$\Phi(T_0) = 2^{-1/n}\, S < Q\; = \Phi(T_Q)\,. \tag{2.1.12}$$

Next, we notice that, thanks to the divergence theorem and Hölder's inequality, for every non-negative u that is admissible in $\Phi(T)$, we have

$$\int_{\partial H} u^{p^\sharp} = \int_{\partial H} u^{p^\sharp}(-\mathrm{e}_1)\cdot \nu_H = p^\sharp \int_H u^{p^\sharp - 1}(-\nabla u)\cdot \mathrm{e}_1 < p^\sharp \|\nabla u\|_{L^p(H)}\, \|u\|_{L^{p^\star}(H)}^{p^\star/p'}$$

where Hölder's inequality must be strict (otherwise, u would just depend on x_1, and thus could not satisfy $u \in L^{p^\star}(H)$). As a consequence, we find that, with strict inequality,

$$\Phi(T) > \frac{T^{p^\sharp}}{p^\sharp} \qquad \forall T > 0 \,. \tag{2.1.13}$$

Finally, given any open connected Lipschitz set $\Omega \subset \mathbb{R}^n$, let us set

$$\Phi_\Omega(T) = \inf \Big\{ \|\nabla u\|_{L^p(\Omega)} : \|u\|_{L^{p^\star}(\Omega)} = 1 \,, \|u\|_{L^{p^\sharp}(\partial\Omega)} = T \Big\} \qquad T \ge 0 \,,$$

(so that $\Phi_H = \Phi$ by (2.1.3)), and define

$$\mathrm{ISO}\,(\Omega) = \frac{P(\Omega)}{|\Omega|^{(n-1)/n}} \,.$$

With this notation, the Euclidean isoperimetric inequality takes the form

$$\mathrm{ISO}\,(\Omega) \ge \mathrm{ISO}\,(B_1) \,, \tag{2.1.14}$$

with equality if and only if $\Omega = B_R(x) = \{y \in \mathbb{R}^n : |y - x| < R\}$ for some $x \in \mathbb{R}^n$ and $R > 0$. The following *trace-Sobolev comparison theorem* was proved in [MV05]:

$$\Phi_\Omega(T) \ge \Phi_{B_1}(T) \,, \qquad \forall T \in \Big[0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}\Big] \,, \tag{2.1.15}$$

with the additional information that: (i) if $0 < T \le \mathrm{ISO}\,(B_1)^{1/p^\sharp}$, $\Phi_\Omega(T) = \Phi_{B_1}(T)$, and $\Phi_\Omega(T)$ admits a minimizer, then Ω is a ball; (ii) Φ_{B_1} is strictly concave (and decreasing) on $[0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}]$. Notice that (2.1.15) cannot hold on a larger interval of Ts: indeed, $\Phi_{B_1}(T) = 0$ forces $T = \mathrm{ISO}\,(B_1)^{1/p^\sharp}$, and so if Ω is not a ball and thus $\mathrm{ISO}\,(\Omega) > \mathrm{ISO}\,(B_1)$, then

$$\Phi_{B_1}(\mathrm{ISO}\,(\Omega)^{1/p^\sharp}) > 0 = \Phi_\Omega(\mathrm{ISO}\,(\Omega)^{1/p^\sharp}) \,.$$

This said, we can apply (2.1.15) with $\Omega = H$ to obtain an additional lower bound on Φ on the interval $[0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}]$.

The constant lower bound given in (2.1.11) is actually stronger than the other three lower bounds for some values of T. Indeed, there exists $\delta > 0$ such that

$$\Phi(T_0) > \max\left\{1_{[0,\mathrm{ISO}\,(B_1)^{1/p^\sharp}]}(T)\Phi_{B_1}(T)\,, Q\,T\,, \frac{T^{p^\sharp}}{p^\sharp}\right\} \qquad \text{if } |T - T_0| < \delta\,. \tag{2.1.16}$$

By continuity, it suffices to check this assertion at $T = T_0$, and since (2.1.13) is strict for every $T > 0$, we only need to worry about (2.1.9) and (2.1.15). The fact that $\Phi(T_0) > \Phi_{B_1}(T_0)$ if $T_0 \le \mathrm{ISO}\,(B_1)^{1/p^\sharp}$ follows by property (i) after (2.1.15) and from the existence of a minimizer for $\Phi(T_0)$ shown in Theorem 2.1.1 below. At the same time, $\Phi(T_0) > Q\,T_0$, for otherwise, the explicit minimizer in $\Phi(T_0)$, that is the "half-Sobolev optimizer" $U_{S,0}$ (see (2.1.17) below), would be optimal in (2.1.2), contradicting the characterization of equality cases for (2.1.2) (which is already implicitly contained in [Naz06], and is rigorously established in here). This proves (2.1.16). We thus find the qualitative picture of the known lower bounds on $\Phi(T)$ depicted in Figure 2.1.

2.1.3 Main results

Our main result consists of characterizing minimizers in $\Phi(T)$ for every $T > 0$, and then using this knowledge to give a qualitative description of the behavior of $\Phi(T)$. Let us recall from three families of functions involved in the characterizaion:

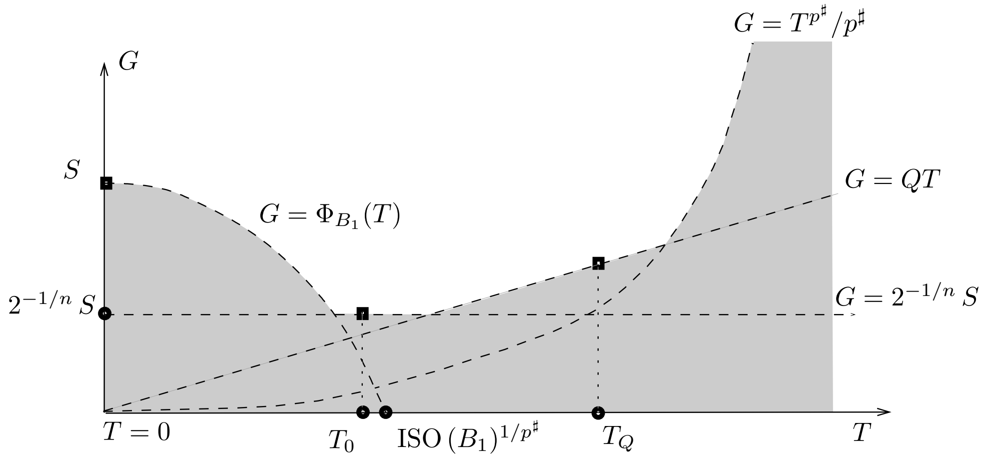

Figure 2.1: A qualitative picture of the known lower bounds on $\Phi(T)$. The picture gives sharp information only for three values of T, namely 0, T_0, and T_Q, which are depicted by black squares.

Sobolev family: Let U_S be defined as in (2.1.5) and set, for every $t \in \mathbb{R}$,

$$U_{S,t}(x) = \frac{U_S(x - t\,\mathrm{e}_1)}{\|U_S(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^\star}(H)}} \qquad x \in H\,, \tag{2.1.17}$$

and

$$T_S(t) = \|U_{S,t}\|_{L^{p^\sharp}(\partial H)}\,, \qquad G_S(t) = \|\nabla U_{S,t}\|_{L^p(H)}\,. \tag{2.1.18}$$

Thus, $U_{S,t}$ is a translation of the optimal function U_S in the Sobolev inequality so that its maximum point lies at signed distance t from ∂H, normalized to have $L^{p^\star}$-norm in H equal to 1.

Escobar family: Letting U_Q be as in (2.1.7), we set for every $t < 0$

$$U_{Q,t}(x) = \frac{U_Q(x - t\,\mathrm{e}_1)}{\|U_Q(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^\star}(H)}} \qquad x \in H\,. \tag{2.1.19}$$

As noticed before, a simple computation (factoring out $|t|$ from $|x - t\,\mathrm{e}_1|$ and then changing variables $y = -x/t$) shows that the trace and gradient norms of the $U_{Q,t}$

are independent of $t < 0$, and we set

$$\|U_{Q,t}\|_{L^{p^\sharp}(\partial H)} = T_Q\,, \qquad \|\nabla U_{Q,t}\|_{L^p(H)} = G_Q \tag{2.1.20}$$

for these constant values. Each function $U_{Q,t}$ is thus obtained by centering the fundamental solution of the p-Laplacian outside of H, and then by normalizing its $L^{p^\star}$-norm in H.

Beyond-Escobar family: We consider the function

$$U_B(x) = (|x|^{p'} - 1)^{(p-n)/p} \qquad |x| > 1\,, \tag{2.1.21}$$

and define, for every $t < -1$,

$$U_{B,t}(x) = \frac{U_B(x - t\,\mathrm{e}_1)}{\|U_B(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^\star}(H)}} \qquad x \in H\,.$$

Correspondingly, for every $t < -1$, we set

$$T_B(t) = \|U_{B,t}\|_{L^{p^\sharp}(\partial H)}\,, \qquad\qquad G_B(t) = \|\nabla U_{B,t}\|_{L^p(H)}\,. \tag{2.1.22}$$

As the name of this family of functions suggests, we later prove that $T_B(t) > T_Q$ for every $t < -1$, so that $\{U_B(t)\}_{t<-1}$ enters the description of $\Phi(T)$ for $T > T_Q$. Notice that (2.1.21) defines a function on the complement of the unit ball. The function $U_{B,t}$ is thus obtained by centering this unit ball outside of H, at distance $|t|$ from ∂H, and the by normalizing its tail to have unit $L^{p^\star}$-norm in H.

Theorem 2.1.1 (Characterization of minimizers of $\Phi(T)$)**.** *If $n \geq 2$ and $p \in (1, n)$, then for every $T > 0$, there exists a minimizer in $\Phi(T)$ that is unique up to dilations and translations orthogonal to e_1. More precisely:*

(i) the function $T_S(t)$ is strictly decreasing on $\mathbb{R}$ with range $(0, T_Q)$ and with $T_S(0) = T_0 < T_Q$; in particular, for every $T \in (0, T_Q)$, there exists a unique $t \in \mathbb{R}$ such that

$$T = T_S(t) \qquad \Phi(T) = G_S(t) \tag{2.1.23}$$

and $U_{S,t}$ uniquely minimizes $\Phi(T)$ up to dilations and translations orthogonal to e_1;

(ii) if $T = T_Q$, then, up to dilations and translations orthogonal to e_1, $\{U_{Q,t} : t < 0\}$ is the unique family of minimizers of $\Phi(T_Q)$;

(iii) the function $T_B(t)$ is strictly increasing on $(-\infty, -1)$ with range $(T_Q, +\infty)$; in particular, for every $T > T_Q$ there exists a unique $t < -1$ such that

$$T = T_B(t) \qquad \Phi(T) = G_B(t) \tag{2.1.24}$$

and $U_{B,t}$ uniquely minimizes $\Phi(T)$ up to dilations and translations orthogonal to e_1.

Theorem 2.1.1 provides an implicit description of Φ on $[0, \infty)$, and extends the Carlen–Loss theorem [CL94] from the case $p = 2$ to the full range $p \in (1, n)$. Notice that an implicit description of Φ_{B_1} on the interval $[0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}]$ was obtained in [MV05], and was at the basis of the further results obtained therein. (No characterization of Φ_{B_1} for $T > \mathrm{ISO}\,(B_1)^{1/p^\sharp}$ seems to be known.) Starting from the characterization of Φ obtained in Theorem 2.1.1, we can obtain a quite complete picture of its properties, which is stated in the next result and illustrated in Figure 2.2.

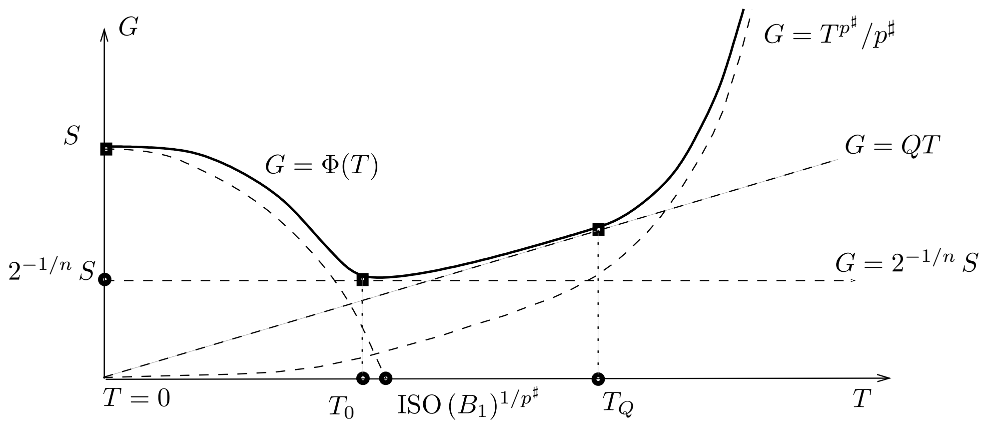

Figure 2.2: A qualitative picture of Theorem 2.1.2, which improves on the situation depicted in Figure 2.1. First, since in Theorem 2.1.1 we have proved that $\Phi(T)$ always admits minimizers, we are sure that $\Phi(T) > \Phi_{B_1}(T)$ for every $T \in [0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}]$, that is to say, the comparison theorem (2.1.15) is never optimal (but at $T = 0$). Notice also that the divergence theorem lower bound (2.1.13) turns out to be sharp, and is asymptotically saturated by the functions $U_{B,t}$ as $t \to 1^-$.

Theorem 2.1.2 (Properties of $\Phi(T)$). *If $n \geq 2$ and $p \in (1,n)$, then $\Phi(T)$ is differentiable on $(0,\infty)$, it is strictly decreasing on $(0,T_0)$ with $\Phi(0) = S$ and $\Phi(T_0) = 2^{-1/n}\,S$ and strictly increasing on (T_0,∞) with*

$$\Phi(T) = \frac{T^{p^\sharp}}{p^\sharp} + o(1) \qquad \text{as } T \to \infty\,. \tag{2.1.25}$$

Moreover, $\Phi(T)$ is strictly convex on $(T_0,+\infty)$, and there exists $T_* \in (0,T_0)$ such that $\Phi(T)$ is strictly concave on $(0,T_*)$.

We see from (2.1.25) that the lower bound (2.1.13) is saturated asymptotically as $T \to \infty$. A simple but interesting corollary of the characterization result obtained in Theorem 2.1.1 is the following comparison theorem, which is complementary to (2.1.15), the main result in [MV05].

Corollary 2.1.3 (Half-spaces have the best Sobolev inequalities). *If Ω is a non-empty open set with Lipschitz boundary on $\mathbb{R}^n$, then*

$$\Phi_\Omega(T) \le \Phi(T) \qquad \forall T \ge 0\,.$$

We now comment on the meaning of these theorems in the geometrically relevant cases $p = 2$ and $p = 1$.

2.1.4 The special case $p = 2$

In this case, which implicitly requires $n \ge 3$, (2.1.3) can be reformulated as a family of minimization problems *on conformally flat metrics on H*,

$$\Psi(P) = \inf\left\{ \int_H R_u \, d\mathrm{vol}_u + 2\,(n-1)\int_{\partial H} h_u \, d\sigma_u : \mathrm{vol}_u(H) = 1\,, P_u(H) = P \right\}, \tag{2.1.26}$$

for $P \ge 0$, which is related to the Yamabe problem on manifolds with boundary studied in the classical papers [Esc88, Esc92a, Esc92b]. Here, we view H as a conformally flat Riemannian manifold with boundary, endowed with the metric $u^{4/(n-2)}\,\delta$, where δ is the standard Euclidean metric. The volume and perimeter of a set $\Omega \subset H$ with respect to this metric are computed as

$$\mathrm{vol}_u(\Omega) = \int_\Omega u^{2^\star}\, dx, \qquad P_u(\Omega) = \int_{\partial\Omega} u^{2^\sharp}\, d\mathcal{H}^{n-1}\,, \tag{2.1.27}$$

while $R_u(x)$ and $h_u(x)$ stand, respectively, for the scalar curvature of $(H, u^{4/(n-2)}\,\delta)$ at $x \in H$, and the mean curvature of ∂H in $(H, u^{4/(n-2)}\,\delta)$ at $x \in \partial H$ computed with respect to the outer unit normal ν_H to H. Explicitly,

$$R_u = -\frac{4(n-1)}{n-2}\,\frac{\Delta u}{u^{(n+2)/(n-2)}}\,, \qquad h_u = -\frac{2}{n-2}\,\frac{1}{u^{n/(n-2)}}\,\frac{\partial u}{\partial x_1}\,. \tag{2.1.28}$$

An integration by parts thus gives

$$\begin{aligned}\int_H |\nabla u|^2 &= -\int_H u\,\Delta u - \int_{\partial H} u\,\frac{\partial u}{\partial x_1}\\ &= \frac{n-2}{4(n-1)}\int_H R_u\,d\mathrm{vol}_u + \frac{n-2}{2}\int_{\partial H} h_u\,d\sigma_u\,.\end{aligned}$$

In this way, we see the equivalence of the problems (2.1.3) when $p = 2$ and (2.1.26) through the identities

$$\Phi^{(2)}(T) = \Big(\frac{n-2}{4(n-1)}\Big)^{1/2}\,\Psi(T^{2^\sharp})^{1/2} \qquad \Psi(P) = \frac{4(n-1)}{n-2}\,\Phi^{(2)}(P^{1/2^\sharp})^2\,,$$

A standard argument shows that if u is a positive minimizer for $\Phi(T)$ (with a generic $p \in (1, n)$), then there exist $\lambda, \sigma \in \mathbb{R}$ such that

$$\begin{cases} -\Delta_p u = \lambda u^{p^\star - 1} & \text{in } H\\ -|\nabla u|^{p-2}\partial_{x_1} u = \sigma u^{p^\sharp - 1} & \text{on } \partial H\,. \end{cases}$$

This basic fact, applied with $p = 2$, implies that every minimizer in the variational problem (2.1.26) is a conformally flat metric on H with constant *scalar* curvature and with boundary of constant mean curvature. By [CL94, Theorem 3.1], or with an alternative proof, by Theorem 2.1.1 with $p = 2$, every minimizer actually has constant *sectional* curvature. Indeed, as a by-product of the characterization of minimizers of $\{\Phi(T)\}_{T\geq 0}$, we deduce that, as P increases from 0 to $P_Q = T_Q^{2^\sharp}$, minimizing metrics in (2.1.26) correspond to spherical caps of decreasing radii rescaled to unit volume. Their sectional curvature will be constant and positive along the way, while the constant mean curvature of the boundaries will initially be negative and then change sign in correspondence to hemispheres ($P = P_0 = T_0^{2^\sharp}$). Then, as P increases from P_Q to $+\infty$, minimizing metrics in (2.1.26) correspond to suitable sections of the hyperbolic space, all with constant negative sectional curvature and constant positive mean curvature of the boundary. Thus, we have a transition from spherical to hyperbolic geometry along minimizing metrics in (2.1.26). These results are summarized in the following statement:

Theorem 2.1.4 (Theorem 3.1 in [CL94] or Theorem 2.1.1 with $p = 2$)**.** *For each $P > 0$, a minimizing conformal metric g_P exists in* (2.1.26) *and is given, uniquely up to dilations and translations orthogonal to* e_1*, by*

$$\begin{array}{lll} U_{S,t}^{4/(n-2)}\,\delta & \textit{for some } t \in \mathbb{R} & \textit{if } P \in (0, P_Q)\,, \\ U_{Q,t}^{4/(n-2)}\,\delta & \textit{for any } t < 0 & \textit{if } P = P_Q\,, \\ U_{B,t}^{4/(n-2)}\,\delta & \textit{for some } t < -1 & \textit{if } P \in (P_Q, \infty)\,. \end{array}$$

For $P \in (0, P_Q)$, (H, g_P) is isometric to a spherical cap (Σ, g_0) with the standard metric induced by the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$ whose radius is determined by P; consequently, it has constant positive sectional curvature. The mean curvature of ∂H is constant and negative for $0 < P < P_0 = T_0^{2^\sharp}$ and is constant and positive for $P_0 < P < P_sQ$.

For $P = P_Q$, (H, g_P) has zero sectional curvature and constant positive mean curvature of ∂H.

For $P \in (P_Q, \infty)$, (H, g_P) has constant negative sectional curvature and is therefore a model for hyperbolic space. The mean curvature of ∂H is constant and positive.

2.1.5 The special case $p = 1$

In this case, the minimization in (2.1.3) takes place in the class of those $u \in L^1_{\rm loc}(H)$, vanishing at infinity, and whose distributional gradient Du is a measure on H with finite total variation, $|Du|(H) < \infty$. We thus consider the problems

$$\Phi(T) = \inf\Big\{|Du|(H) : \|u\|_{L^{n/(n-1)}(H)} = 1\,, \|u\|_{L^1(\partial H)} = T\Big\} \qquad T \ge 0\,. \quad (2.1.29)$$

In the restricted class of characteristic functions $u = 1_X$ for $X \subset H$, this is the relative isoperimetric problem in H with an additional constraint (aside from the unit volume constraint) on the contact region between the boundary of X and the boundary of H. In the notation of distributional perimeters, this restricted problem takes the form

$$\Phi_{\rm sets}(T) = \inf\Big\{P(X;H) : X \subset H\,, |X| = 1\,, P(X;\partial H) = T\Big\} \qquad T \ge 0\,, \quad (2.1.30)$$

where $P(X;A) = \mathcal{H}^{n-1}(A \cap \partial X)$ whenever X is an open set with Lipschitz boundary. The unique minimizers in (2.1.30) are obtained by intersecting H with balls (of suitable radius and centered at suitable distance from ∂H); see, e.g., [Mag12, Theorem 19.15], which also describes the relevance of (2.1.30) in capillarity theory. In the original problem (2.1.29), one obtains scaled versions of the characteristic functions of these sets as minimizers; precisely, u is a minimizer in (2.1.29) if and only if $u(x) = \lambda^{n-1}\, 1_X(\lambda\, x)$ for some $\lambda > 0$ and X a minimizer in (2.1.30). When $T = 0$, (2.1.30) is simply the Euclidean isoperimetric problem, and (2.1.29) is the Sobolev inequality on functions of bounded variation. Notice that the Sobolev trace inequality, in the case $p = 1$, takes the simple form

$$|Du|(H) \ge \|u\|_{L^1(\partial H)} \quad (2.1.31)$$

or, in more geometric terms, that is, for $u = 1_X$ with $X \subset H$,

$$P(X; H) \geq P(X; \partial H)\,.$$

Along the lines of (2.1.13), this follows by simply applying the divergence theorem on X to the constant vector field $T(x) = \mathrm{e}_1$ to get

$$0 = \int_X \mathrm{div}\,(\mathrm{e}_1) = \int_{H\cap\partial X} \nu_X \cdot \mathrm{e}_1 + \int_{\partial H\cap\partial X} (-\mathrm{e}_1)\cdot \mathrm{e}_1 < P(X;H) - P(X;\partial H)$$

where the inequality is strict as soon as $|X| > 0$. The proof of (2.1.31) is analogous, and in particular, there is no nontrivial equality case in (2.1.31). In the case $p = 1$, Theorems 2.1.1 and Theorem 2.1.2 take the following form.

Theorem 2.1.5. *For every $n \geq 2$ and $T > 0$ there exists a minimizer in* (2.1.29), *which is given, uniquely up to dilations and translations orthogonal to e_1, by*

$$U_{S,t}(x) = \frac{1_{B_1}(x - t\,\mathrm{e}_1)}{\|1_{B_1}(\cdot - t\,\mathrm{e}_1)\|_{L^{n'}(H)}} \qquad x \in H$$

for some $t \in (-1, 1)$. The function $\Phi(T)$ defined by (2.1.29) *is a smooth function of $T > 0$ given by the parametric curve*

$$\Phi(T_S(t)) = G_S(t) \qquad -1 < t < -1\,,$$

where $T_S(t) = \|U_{S,t}\|_{L^1(\partial H)}$ and $G_S(t) = |DU_{S,t}|(H)$. If we set $T_0 = T_S(0)$, then $\Phi(T)$ is strictly decreasing on $(0, T_0)$ and strictly increasing on (T_0, ∞), with $\Phi(0) = \mathrm{ISO}\,(B_1)$ and $\Phi(T_0) = 2^{-1/n}\mathrm{ISO}\,(B_1)$. Moreover, Φ is strictly convex on (T_0, ∞), there exists $T_* \in (0, T_0)$ such that $\Phi(T)$ is strictly concave on $(0, T_*)$, and $\Phi(T) = T + o(1)$ as $T \to \infty$.

We note that in the case $p = 1$, we have a single minimizing family, corresponding to the Sobolev family of the case $p \in (1, n)$, but no Escobar or beyond-Escobar families. This is a reflection of the fact that

$$\lim_{p \to 1^+} T_Q(n, p) = \infty \, ,$$

proven in Proposition 2.3.4 below. This fact indicates that no analogues of the Escobar or beyond-Escobar families exist for $p = 1$. In the same vein, one notices that the Φ curve asymptotically has the same slope (equal to 1) as the (limit position as $p \to 1^+$ of the) Sobolev trace line.

2.1.6 Organization of the chapter

In Section 2.2, we use a mass transportation argument to prove a family of inequalities which will serve as a key tool for proving the main results. In Section 2.3, we prove Theorems 2.1.1, 2.1.2, and 2.1.5. Finally, in Appendix A, we address some technical points related to the mass transportation argument.

2.2 Mass transportation argument

The starting point of our analysis is the mass transportation proof of the Sobolev inequality from [CENV04]. This argument, whose origin can be traced back to [Kno57, MS86], was exploited in [MV05] to prove a parameterized "mother family" of trace Sobolev inequalities on arbitrary Lipschitz domains, leading to the sharp comparison theorem stated in (2.1.15). In [Naz06], this method of proof is adapted to obtain the sharp Sobolev trace inequality for every $p \in (1, n)$. It is important to mention that, as already shown in [CENV04] (see also [AGK04, MV08, Ngu15]), this optimal transportation argument can also be applied to a very interesting special family of Gagliardo–Nirenberg inequalities, having some Faber-Krahn and log-Sobolev inequalities as limit cases.

At the core of this paper is a new iteration of this by-now-classical mass transportation argument. This iteration lies in between the ones of [MV05] and [Naz06]. In Theorem 2.2.1 we implement the same trick introduced in [Naz06], namely subtracting a unit vector from the Brenier map, but with the seemingly harmless addition of an intensity parameter t. (To be precise, the argument in [Naz06] corresponds to the choice $t = -1$ in the proof of Theorem 2.2.1.) This simple expedient leads to a new parameterized "mother family" of Sobolev trace–type inequalities on the half-space, whose equality cases (see Theorem 2.2.3 below) are given by the functions $U_{S,t}$, $U_{Q,t}$ and $U_{B,t}$ introduced in (2.1.17), (2.1.19) and (2.1.21). This means that each inequality in the mother family provides a sharp trace-Sobolev bound, which thus agrees with $\Phi(T)$ for a specific value of T depending on t. By adopting the same point of view of [MV05], where the Φ-function of the ball was computed for a special range of T, in Section 2.3 we exploit this implicit description of $\Phi(T)$ in order to prove Theorem 2.1.1.

Let us now recall some facts from the theory of optimal transportation. Given a (Borel regular) probability measure μ on $\mathbb{R}^n$ and a Borel measurable map $T : \mathbb{R}^n \to \mathbb{R}^n$, the *push-forward of* μ *through* T is the probability measure defined by

$$T\#\mu(A) = \mu(T^{-1}(A)) \qquad \forall A \subset \mathbb{R}^n.$$

As a consequence of this definition, for every Borel measurable function $\xi : \mathbb{R}^n \to [0, \infty]$ we have

$$\int_{\mathbb{R}^n} \xi \, dT\#\mu = \int_{\mathbb{R}^n} \xi \circ T \, d\mu \,. \tag{2.2.1}$$

If $F\,dx$ and $G\,dx$ are absolutely continuous probability measures on $\mathbb{R}^n$, then the Brenier-McCann theorem (see [Bre91, McC97] or [Vil03, Cor. 2.30]) ensures the existence of a lower semicontinuous convex function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$(\nabla\varphi)\#F\,dx = G\,dx \,. \tag{2.2.2}$$

By convexity, φ is differentiable a.e. on the open convex set Ω defined as the interior of $\{\varphi < \infty\}$, its gradient satisfies

$$\nabla\varphi \in (BV \cap L^\infty)_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \,,$$

and $F\,dx$ is concentrated on Ω with

$$\mathrm{spt}(G\,dx) = \overline{\nabla\varphi(\mathrm{spt}(F\,dx))} \,, \tag{2.2.3}$$

thanks to (2.2.2). The map $T = \nabla\varphi$ is called the *Brenier map* between $F\,dx$ and $G\,dx$, and, as shown in [McC97] (cf. [Vil03, Theorem 4.8]), it satisfies the Monge-Ampere equation

$$F(x) = G(\nabla\varphi(x)) \det\nabla^2\varphi(x) \qquad \text{a.e. on } \mathrm{spt}(F\,dx) \,. \tag{2.2.4}$$

Notice that the distributional gradient DT of T is an $n \times n$-symmetric tensor valued Radon measure on Ω. In (2.2.4) we have set $\nabla^2\varphi = \nabla T$ where $DT = \nabla T\,dx + D^sT$ is the decomposition of DT with respect to the Lebesgue measure on Ω. Notice that

$\nabla T\,dx \le DT$ on Ω, and thus, setting $\operatorname{div} T = \operatorname{tr}(\nabla T)$ and denoting by $\operatorname{Div} T$ the distributional divergence of T, we have

$$\operatorname{div}\, T\,dx \le \operatorname{Div}\, T \qquad \text{as measures on } \Omega\,.$$

Since $\nabla T(x)$ is positive semidefinite, by the arithmetic-geometric mean inequality,

$$(\det\nabla^2\varphi(x))^{1/n} = (\det\nabla T(x))^{1/n} \le \frac{\operatorname{div} T(x)}{n} \qquad \text{for a.e. } x \in \Omega\,,$$

we finally conclude that

$$(\det\nabla^2\varphi)^{1/n}\,dx \le \frac{\operatorname{Div} T}{n} \qquad \text{as measures on } \Omega\,. \tag{2.2.5}$$

Theorem 2.2.1. *If $n \ge 2$, $p \in [1,n)$, and f and g are non-negative functions in $L^1_{\mathrm{loc}}(H)$, vanishing at infinity, with*

$$\begin{cases} \int_H |\nabla f|^p < \infty \text{ and } \int_H |x|^{p'} g^{p^\star} < \infty \text{ if } p > 1 \\ |Df|(H) < \infty \text{ and } \operatorname{spt} g \subset\subset \overline{H} \text{ if } p = 1 \\ \|f\|_{L^{p^\star}(H)} = \|g\|_{L^{p^\star}(H)} = 1 \end{cases} \tag{2.2.6}$$

then for every $t \in \mathbb{R}$, we have

$$n\int_H g^{p^\sharp}\,dx \le p^\sharp \|\nabla f\|_{L^p(H)} Y(t,g) + t\int_{\partial H} f^{p^\sharp}\,d\mathcal{H}^{n-1} \tag{2.2.7}$$

where we let

$$Y(t,g) = \begin{cases} \Big(\int_H g^{p^\star}|x - t\,\mathrm{e}_1|^{p'}\,dx\Big)^{1/p'} & \text{if } p > 1\,, \\ \sup\{|x - t\,\mathrm{e}_1| : x \in \operatorname{spt}(g)\} & \text{if } p = 1\,, \end{cases} \tag{2.2.8}$$

and where $\|\nabla f\|_{L^p(H)}$ is replaced by $|Df|(H)$ when $p = 1$.

Remark 2.2.2. Let us first recall that the assumption that f is vanishing at infinity means that $|\{f > t\}| < \infty$ for every $t > 0$. Next we notice that, by (2.1.2), (2.2.6) implies $f \in L^{p^\sharp}(\partial H)$, so that the multiplication by a possibly negative t on the right-hand side of (2.2.7) is of no concern. Finally, we notice that (2.2.7) implies that $g \in L^{p^\sharp}(H)$, but this fact can be more directly deduced by means of Hölder's inequality from the assumptions on g stated in (2.2.6).

Proof. Arguing by approximation, it suffices to prove (2.2.7) when $f \in C^1_c(\overline{H})$ (that is, f admits an extension in $C^1_c(\mathbb{R}^n)$). Let us set $F = 1_H\, f^{p^\star}$ and $G = 1_H\, g^{p^\star}$ and consider the Brenier map $\nabla\varphi$ between the probability measures $F\,dx$ and $G\,dx$. In this way, $T = \nabla\varphi \in (BV \cap L^\infty)_{\rm loc}(\Omega;\mathbb{R}^n)$ with Ω defined as above and $F\,dx$ is concentrated on Ω. By (2.2.2), (2.2.1) (applied with $\xi = 1_{\{G>0\}}\, G^{-1/n}$), (2.2.4) and (2.2.5) respectively, we have

$$
\begin{aligned}
\int_H g^{p^\sharp} = \int_{\mathbb{R}^n} G^{1-1/n} &= \int_{\mathbb{R}^n} G(\nabla\varphi)^{-1/n} F \\
&= \int_{\mathbb{R}^n} (\det\nabla^2\varphi)^{1/n} F^{1-1/n} \le \frac{1}{n}\int_{\mathbb{R}^n} F^{1-1/n}\, d(\operatorname{div} T)\,.
\end{aligned}
\tag{2.2.9}
$$

We subtract the divergence-free vector field $t\,\mathrm{e}_1$ from T,

$$
\int_{\mathbb{R}^n} F^{1-1/n}\, d(\operatorname{Div} T) = \int_H f^{p^\sharp}\, d(\operatorname{Div} S)\,, \qquad S = T - t\,\mathrm{e}_1\,,
$$

where $S \in (BV \cap L^\infty)_{\rm loc}(\Omega;\mathbb{R}^n)$. By the trace theorem for BV functions (see e.g. [EG92, Theorem 1, p.177]), S has a trace $S \in L^1_{\rm loc}(\Omega \cap \partial H)$ such that

$$
\int_H \psi\, d(\operatorname{Div} S) = -\int_H \nabla\psi \cdot S - \int_{\partial H} \psi\,(S\cdot \mathrm{e}_1)\,, \qquad \forall \psi \in C^1_c(\Omega \cap \overline{H})\,.
$$

We now use the assumption that $f \in C^1_c(\overline{H})$, along with the fact that $F\,dx$ is concentrated on Ω, to apply this identity with $\psi = f^{p^\sharp}$. In this way, we find

$$\int_H f^{p^\sharp}\, d(\operatorname{div} S) = -p^\sharp \int_H f^{p^\sharp-1}\nabla f \cdot S\,dx - \int_{\partial H} f^{p^\sharp} S\cdot \mathrm{e}_1 d\mathcal{H}^{n-1}\,.$$

Since $\overline{T(\mathrm{spt}(F\,dx))} = \mathrm{spt}(G\,dx) \subset \overline{H}$, by standard properties of the trace operator we have $S(x)\cdot(-\mathrm{e}_1) \le t$ for $\mathcal{H}^{n-1}$-a.e. on $x \in \mathrm{spt}(f)\cap \partial H$. So, in summary,

$$n\int_H g^{p^\sharp} \le -p^\sharp \int_H f^{p^\sharp-1}\nabla f\cdot (T - t\,\mathrm{e}_1) + t\int_{\partial H} f^{p^\sharp}\, d\mathcal{H}^{n-1}\,. \tag{2.2.10}$$

Finally, we bound the first term on the right hand side of (2.2.10). In the case that $p \in (1,n)$, by using Hölder's inequality and the transport condition (2.2.1) we find

$$\begin{aligned}-p^\sharp \int_H f^{p^\sharp-1}\nabla f\cdot (T-t\,\mathrm{e}_1) &\le p^\sharp \|\nabla f\|_{L^p(H)}\Big(\int_H f^{p^\star}|T(x)-t\,\mathrm{e}_1|^{p'}\,dx\Big)^{1/p'}\\ &= p^\sharp \|\nabla f\|_{L^p(H)}\Big(\int_H g^{p^\star}|x-t\,\mathrm{e}_1|^{p'}\,dx\Big)^{1/p'}.\end{aligned} \tag{2.2.11}$$

Combining this with (2.2.10) implies (2.2.7). In the case $p=1$, in place of Hölder's inequality, we simply use (2.2.3) and the fact that $p^\sharp = 1$ to bound the left-hand side of (2.2.11) by $Y(t,g)\,|Df|(H)$. □

In order to analyze the mother family of inequalities of Theorem 2.2.1 we will need a characterization of the corresponding equality cases, which involves the functions $U_{S,t}$, $U_{Q,t}$ and $U_{B,t}$ previously introduced in (2.1.17), (2.1.19) and (2.1.21). Following [CENV04], given two non-negative measurable functions f and g, we call f a *dilation-translation image* of g if there exist $C>0, \lambda \ne 0$, and $x_0 \in \mathbb{R}^n$ such that $f(x) = Cg(\lambda(x-x_0))$. Since (2.2.7) is not invariant with respect to translations in the e_1 direction, we distinguish that f is a *dilation-translation image* of g *orthogonal to* e_1 if

f is a dilation-translation image of g with $x_0 \cdot \mathrm{e}_1 = 0$. If $\int_H f^{p^\star}\,dx = \int_H g^{p^\star}\,dx$ and f is a dilation-translation image of g orthogonal to e_1, then C must be equal to $\lambda^{(n-p)/n}$, and the Brenier map pushing forward $f^{p^\star}\,dx$ onto $g^{p^\star}\,dx$ satisfies $\nabla\varphi = \lambda(\mathrm{Id} - x_0)$ with $x_0 \cdot \mathrm{e}_1 = 0$. With this terminology at hand, we state the required characterization theorem:

Theorem 2.2.3. *Under the same assumptions of Theorem 2.2.1, suppose that*

$$n\int_H g^{p^\sharp}\,dx = p^\sharp \|\nabla f\|_{L^p(H)} Y(t,g) + t\int_{\partial H} f^{p^\sharp}\,d\mathcal{H}^{n-1}\,, \qquad \int_{\partial H} f^{p^\sharp} > 0\,, \qquad (2.2.12)$$

where $|Df|(H)$ replaces $\|\nabla f\|_{L^p(H)}$ when $p = 1$.

If $p \in (1,n)$, then (2.2.12) *holds for $t \geq 0$ if and only if f and g are both dilation-translation images orthogonal to* e_1 *of $U_{S,t}$; and for $t < 0$ if and only if f and g are both dilation-translation images orthogonal to* e_1 *of either $U_{S,t}$, $U_{Q,t}$, or $U_{B,t}$.*

If $p = 1$, then (2.2.12) *can hold only for $t \in (-1,1)$. For such t,* (2.2.12) *holds if and only if f and g are dilation-translation images orthogonal to* e_1 *of $U_{S,t}$.*

It is easily verified that the aforementioned functions are equality cases of (2.2.12). The uniqueness Theorem 2.2.3 is a technical variant of a similar argument from [CENV04], we postpone its discussion to Appendix A.

2.3 Study of the variational problem $\Phi(T)$

By Theorem 2.2.3, if equality is achieved in the mother inequality (2.2.7) by a triple (t,f,g) with $\int_{\partial H} f^{p^\sharp} > 0$, then we have

$$f = g = U_{S,t} \qquad \text{or} \qquad f = g = U_{Q,t} \qquad \text{or} \qquad f = g = U_{B,t}$$

(with the second and third possibilities only when $t < 0$ or $t < -1$ respectively). The same scaling argument used in (2.1.20) shows that $Y(t, U_{Q,t}) = |t|\, Y_Q$, where we let $Y_Q = Y(-1, U_{Q,-1})$ and $Y(t,g)$ be as defined in (2.2.8). Therefore, recalling the notation of (2.1.20), equality in (2.2.7) for the Escobar family implies that

$$n \int_H U_{Q,t}^{p^\sharp}\, dx = -tp^\sharp G_Q Y_Q + t\, T_Q^{p^\sharp} \qquad \forall t < 0\,. \tag{2.3.1}$$

Similarly, let us define the functions

$$Y_S(t) = Y(t, U_{S,t}) \qquad \text{and} \qquad Y_B(t) = Y(t, U_{B,t})\,.$$

Then, recalling the definitions in (2.1.18) and (2.1.22), equality in (2.2.7) for the Sobolev and beyond-Escobar families implies the identities

$$\begin{aligned} n \int_H U_{S,t}^{p^\sharp}\, dx &= p^\sharp G_S(t) Y_S(t) + t\, T_S(t)^{p^\sharp} \qquad && \forall t \in \mathbb{R}\,, \\ n \int_H U_{B,t}^{p^\sharp}\, dx &= p^\sharp G_B(t) Y_B(t) + t\, T_B(t)^{p^\sharp} \qquad && \forall t < -1\,. \end{aligned} \tag{2.3.2}$$

From (2.3.1) and (2.3.2), Theorems 2.2.1 and 2.2.3 yield the following corollary.

Corollary 2.3.1. *If $h \in L^1_{\mathrm{loc}}(H)$ is a non-negative function vanishing at infinity with $\nabla h \in L^p(H;\mathbb{R}^n)$ and $\|h\|_{L^{p^\star}(H)} = 1$, then,*

$$p^\sharp Y_S(t) G_S(t) + t\, T_S(t)^{p^\sharp} \le p^\sharp Y_S(t) \|\nabla h\|_{L^p(H)} + t\, \|h\|^{p^\sharp}_{L^{p^\sharp}(\partial H)} \qquad \forall t \in \mathbb{R}\,, \tag{2.3.3}$$

$$p^\sharp Y_B(t) G_B(t) + t\, T_B(t)^{p^\sharp} \le p^\sharp Y_B(t) \|\nabla h\|_{L^p(H)} + t\, \|h\|^{p^\sharp}_{L^{p^\sharp}(\partial H)} \qquad \forall t < -1\,, \tag{2.3.4}$$

$$p^\sharp Y_Q G_Q - T_Q^{p^\sharp} \le p^\sharp Y_Q \|\nabla h\|_{L^p(H)} - \|h\|^{p^\sharp}_{L^{p^\sharp}(\partial H)}. \tag{2.3.5}$$

Furthermore, equality in (2.3.3) *(resp.* (2.3.4), (2.3.5)*) is attained if and only if h is a dilation-translation image orthogonal to* e_1 *of $U_{S,t}$ (resp. $U_{B,t}$, $U_{Q,t}$). Particularly,*

$$\|h\|_{L^{p^\sharp}(\partial H)} = T_S(t) \quad \Longrightarrow \quad G_S(t) \le \|\nabla h\|_{L^p(H)}\,;$$

$$\|h\|_{L^{p^\sharp}(\partial H)} = T_B(t) \quad \Longrightarrow \quad G_B(t) \le \|\nabla h\|_{L^p(H)}\,;$$
$$\|h\|_{L^{p^\sharp}(\partial H)} = T_Q \quad \Longrightarrow \quad G_Q \le \|\nabla h\|_{L^p(H)}\,,$$

and the following identities hold

$$\begin{aligned} \Phi(T_S(t)) &= G_S(t) \quad \forall t \in \mathbb{R}\,, \\ \Phi(T_B(t)) &= G_B(t) \quad \forall t < 0\,, \\ \Phi(T_Q) &= G_Q\,. \end{aligned} \tag{2.3.6}$$

Next, we prove some properties of the Sobolev and beyond-Escobar families.

Proposition 2.3.2. *The following properties hold:*

(i) T_S *is strictly decreasing on* $\mathbb{R}$ *with range* $(0, T_Q)$*, and* $T_S(0) = T_0 < T_Q$*;*

(ii) G_S *is strictly increasing on* $[0,\infty)$ *with range* $[2^{-1/n}S, G_Q)$*, and is strictly decreasing on* $(-\infty, 0)$ *with range* $(2^{-1/n}S, G_Q)$*;*

(iii) $T_B(t)$ *is strictly increasing for* $t < -1$ *with range* (T_Q, ∞)*;*

(iv) $G_B(t)$ *is strictly increasing for* $t < -1$ *with range* (G_Q, ∞)*.*

Proof. Step 1: Monotonicity of $T_S(t)$ *and* $T_B(t)$*.* Fix $t_1, t_2 \in \mathbb{R}$ and suppose $T_S(t_1) = T_S(t_2) = T$. Then, (2.3.3) implies that

$$\begin{aligned} &p^\sharp Y_S(t_1) G_S(t_1) + t_1 T^{p^\sharp} \le p^\sharp Y_S(t_1) G_S(t_2) + t_1 T^{p^\sharp}, \quad \text{thus} \quad G_S(t_1) \le G_S(t_2), \quad \text{and} \\ &p^\sharp Y_S(t_2) G_S(t_2) + t_2 T^{p^\sharp} \le p^\sharp Y_S(t_2) G_S(t_1) + t_2 T^{p^\sharp}, \quad \text{thus} \quad G_S(t_2) \le G_S(t_1). \end{aligned}$$

That is, $G_S(t_1) = G_S(t_2) = G$. Hence, U_{S,t_2} attains equality in (2.3.3) with $t = t_1$. Uniqueness in (2.3.3) then implies that $t_1 = t_2$. We conclude that $T_S(t)$ is injective, and, as $T_S(t)$ is continuous, it is strictly monotone for $t \in \mathbb{R}$. The identical argument using (2.3.4) shows that T_B is strictly monotone for all $t < -1$.

Step 2: Piecewise monotonicity of $G_S(t)$ and $G_B(t)$. Fix $t_1, t_2 \geq 0$ and suppose that $G_S(t_1) = G_S(t_2) = G$. Then, (2.3.3) implies that

$$p^\sharp Y_S(t_1)G + t_1 T_S(t_1)^{p^\sharp} \leq p^\sharp Y_S(t_1)G + t_1 T_S(t_2)^{p^\sharp}, \quad \text{thus} \quad T_S(t_1) \leq T_S(t_2), \quad \text{and}$$
$$p^\sharp Y_S(t_2)G + t_2 T_S(t_2)^{p^\sharp} \leq p^\sharp Y_S(t_2)G + t_2 T_S(t_1)^{p^\sharp}, \quad \text{thus} \quad T_S(t_2) \leq T_S(t_1).$$

Since $T_S(t)$ is injective, we conclude that $t_1 = t_2$. Thus, $G_S(t)$ is strictly monotone for $t \geq 0$. The analogous argument shows that $G_S(t)$ is strictly monotone for $t < 0$ and that $G_B(t)$ is strictly monotone for $t < -1$.

Step 3: Limit values of $T_S(t)$ and $G_S(t)$. As $U_{S,t}$ is a renormalized translation of the optimal function U_S in (2.1.1), centered at $t\,\mathrm{e}_1$, it is clear that $T_S(t) \to 0$ and $G_S(t) \to S$ as $t \to \infty$. To compute the limit as $t \to -\infty$, let us set

$$\gamma_t(x) = (1 + |x - t\,\mathrm{e}_1|^{p'})^{-1} = |t|^{-p'}\,(|t|^{-p'} + |y + \mathrm{e}_1|^{p'})^{-1}$$

for $t < 0$ and $y = -x/t$. With this notation,

$$T_S(t) = \frac{\big(\int_{\partial H} \gamma_t^{n-1}\, d\mathcal{H}^{n-1}\big)^{1/p^\sharp}}{\big(\int_H \gamma_t^n\, dx\big)^{1/p^\star}}, \quad G_S(t) = \frac{(n-p)\big(\int_H \gamma_t^n |x - t\,\mathrm{e}_1|^{p'}\, dx\big)^{1/p}}{(p-1)\big(\int_H \gamma_t^n\, dx\big)^{1/p^\star}}. \quad (2.3.7)$$

Now, suppose $t < 0$ and let $\sigma = -(n-p)/(p-1)$. After factoring out $-t$ and changing variables, we find that

$$\int_{\partial H} \gamma_t^{n-1}\, d\mathcal{H}^{n-1} = |t|^{-p'(n-1)+(n-1)} \int_{\partial H} (|t|^{-p'} + |y + \mathrm{e}_1|^{p'})^{-(n-1)}\, d\mathcal{H}^{n-1}_y\,,$$

$$\int_H \gamma_t^n \, dx = |t|^{-p'n+n} \int_H (|t|^{-p'} + |y + \mathrm{e}_1|^{p'})^{-n} \, dy \,,$$
$$\int_H \gamma_t^n |x - t\,\mathrm{e}_1|^{p'} \, dx = |t|^{-p'n+p'+n} \int_H (|t|^{-p'} + |y + \mathrm{e}_1|^{p'})^{-n} |y + \mathrm{e}_1|^{p'} \, dy \,.$$

Since

$$\frac{-p'(n-1) + (n-1)}{p^\sharp} - \frac{-p'n + n}{p^\star} = 0 \qquad\qquad \frac{-p'n + p' + n}{p} + \frac{p'n - n}{p^\star} = 0 \,,$$

we find that, setting

$$\bar\gamma_t(y) = (|t|^{-p'} + |y + \mathrm{e}_1|^{p'})^{-1} \qquad y \in H \,,$$

we have

$$T_S(t) = \frac{\Big(\int_{\partial H} \bar\gamma_t^{n-1}\Big)^{1/p^\sharp}}{\Big(\int_H \bar\gamma_t^n\Big)^{1/p^\star}} \qquad G_S(t) = \frac{(n-p)\big(\int_H \bar\gamma_t^n |y + \mathrm{e}_1|^{p'} \, dy\big)^{1/p}}{(p-1)\big(\int_H \bar\gamma_t^n\big)^{1/p^\star}}$$

By monotone convergence, we thus find that

$$\lim_{t \to -\infty} T_S(t) = \frac{\|U_Q(\cdot + \mathrm{e}_1)\|_{L^{p^\sharp}(\partial H)}}{\|U_Q(\cdot + \mathrm{e}_1)\|_{L^{p^\star}(H)}} = T_Q \,, \quad \lim_{t \to -\infty} G_S(t) = \frac{\|\nabla U_Q(\cdot + \mathrm{e}_1)\|_{L^p(H)}}{\|U_Q(\cdot + \mathrm{e}_1)\|_{L^{p^\star}(H)}} = G_Q \,,$$

as claimed. Having shown that T_S is smooth and injective on $\mathbb{R}$ with $T_S(+\infty) = 0$ and $T_S(-\infty) = T_Q > 0$, we deduce that T_S is strictly decreasing on $\mathbb{R}$ with range $(0, T_Q)$. Since $T_0 = T_S(0) < T_S(-\infty) = T_Q$, we have completed the proof of statement (i). Similarly, the first part of (ii) follows since $G_S(0) = 2^{-1/n} S < S = G_S(+\infty)$ and G_S is smooth and injective on $[0, \infty)$. Similarly, the injectivity of G_S on $(-\infty, 0)$ together with the fact that by (2.1.10) (recall (2.1.12)) $G_S(0) = 2^{-1/n}\, S < Q = G_Q = G_S(-\infty)$ implies that G_S is strictly decreasing on $(-\infty, 0)$ with range $(2^{-1/n} S, Q)$. This proves statement (ii).

Step 4: Limit values of $T_B(t)$ and $G_B(t)$. With an argument identical to that given for T_S and G_S, we establish that $T_B(t) \to T_Q$ and $G_B(t) \to G_Q$ as $t \to -\infty$. To compute the limit as $t \to -1^+$, we first notice that, for every $t < -1$ and setting $\varepsilon = |t| - 1$,

$$\int_{\partial H} U_B(x - t\,\mathrm{e}_1)^{p^\sharp} \geq \int_{B_{|t|+\varepsilon}(t\mathrm{e}_1)\cap\partial H} \frac{d\mathcal{H}^{n-1}}{(|x - t\,\mathrm{e}_1|^{p'} - 1)^{n-1}}\,.$$

Since $B_{|t|+\varepsilon}(t\mathrm{e}_1) \cap \partial H$ is a $(n-1)$-dimensional disk of radius $\sqrt{(|t|+\varepsilon)^2 - t^2} = \sqrt{2\varepsilon\,|t| + \varepsilon^2} \geq c\sqrt{\varepsilon}$, and since $|x - t\,\mathrm{e}_1|^{p'} - 1 \leq (|t| + \varepsilon)^{p'} - 1 \leq C\,\varepsilon$ for constants c and C depending on n and p only, we find that

$$\int_{\partial H} U_B(x - t\,\mathrm{e}_1)^{p^\sharp} \geq \frac{c}{\varepsilon^{(n-1)/2}} = \frac{c}{|t+1|^{(n-1)/2}}\,. \tag{2.3.8}$$

At the same time, we have

$$\int_H U_B(x - t\,\mathrm{e}_1)^{p^\star} = \int_H (|x - t\,\mathrm{e}_1|^{p'} - 1)^{-n}\,dx = \int_{-t}^{\infty} (r^{p'} - 1)^{-n}\,\mathcal{H}^{n-1}\big(H \cap \partial B_r(-t\,\mathrm{e}_1)\big)\,dr$$

where, thanks to the coarea formula,

$$\mathcal{H}^{n-1}\big(H \cap \partial B_r(-t\,\mathrm{e}_1)\big) = c(n)\,r^{n-1}\int_{-t/r}^{1} (1 - s^2)^{(n-3)/2}\,ds\,.$$

Since $1 \leq (1+s)^{(n-3)/2} \leq C(n)$ for $s \in (-t/r, 1)$ and

$$r^{n-1}\int_{-t/r}^{1} (1 - s)^{(n-3)/2}\,ds = C(n)\,r^{n-1}\,(1 + t/r)^{(n-1)/2} = C\,r^{(n-1)/2}\,(r + t)^{(n-1)/2}\,,$$

we conclude that

$$c(n) \leq \frac{\mathcal{H}^{n-1}\big(H \cap \partial B_r(-t\,\mathrm{e}_1)\big)}{r^{(n-1)/2}\,(r+t)^{(n-1)/2}} \leq C(n)\,, \qquad \forall r \in (-t, \infty)\,. \tag{2.3.9}$$

Hence, by $p > 1$, and provided t is close enough to -1

$$
\begin{aligned}
\int_H (|x - t\,e_1|^{p'} - 1)^{-n}\,dx &\le C \int_2^\infty \frac{r^{(n-1)/2}\,(r+t)^{(n-1)/2}}{(r^{p'}-1)^n}\,dr \\
&+ C \int_{-t}^2 \frac{r^{(n-1)/2}\,(r+t)^{(n-1)/2}}{(r^{p'}-1)^n}\,dr \\
&\le C \int_2^\infty \frac{r^{n-1}}{r^{np'}}\,dr + C \int_{-t}^2 \frac{dr}{(r-1)^{n-(n-1)/2}} \\
&\le C\left(1 + |t+1|^{-(n-1)/2}\right) \le C\,|t+1|^{-(n-1)/2}\,.
\end{aligned}
$$

(We also notice that, by (2.3.9), one also has an analogous estimate from below, that is

$$
\int_H (|x - t\,e_1|^{p'} - 1)^{-n}\,dx \ge c\,|t+1|^{-(n-1)/2} \qquad \text{for } |t+1| \text{ small enough} \tag{2.3.10}
$$

as well as

$$
\int_H (|x - t\,e_1|^{p'} - 1)^{-(n-1)}\,dx \le C\,|t+1|^{-(n-3)/2} \qquad \text{for } |t+1| \text{ small enough}\,. \tag{2.3.11}
$$

Both estimates will be used in the last step of the proof of Theorem 2.1.2.) By combining this last estimate with (2.3.8) we find that

$$
T_B(t) \ge c\left(|t+1|^{-(n-1)/2}\right)^{1/p^\sharp - 1/p^\star} = c\,|t+1|^{-1/2p^\star}\,, \tag{2.3.12}
$$

for every t close enough to -1, where $c = c(n,p) > 0$. This proves that $T_B(t) \to +\infty$ as $t \to -1$. Analogously, again with $\varepsilon = |t+1|$,

$$
\begin{aligned}
\int_H |\nabla U_B(x - t\,e_1)|^p &\ge c \int_{H \cap B_{|t|+\varepsilon}(t\,e_1)} (|x - t\,e_1|^{p'} - 1)^{-n}\,|x - t\,e_1|^{p'}\,dx \\
&= c \int_{-t}^{-t+\varepsilon} \frac{(r^2 - t^2)^{(n-1)/2} r^{\,p'}}{(r^{p'}-1)^n}\,dr
\end{aligned}
$$

so that, setting $r = |t| + s\,|t+1|$, noticing that $r^2 - t^2 \geq c\,s\,|t+1|$ and $1 \leq r^{p'} \leq 1 + C\,|t+1|$, we get

$$\int_H |\nabla U_B(x - t\,\mathrm{e}_1)|^p \quad \geq \quad |t+1|^{(n-1)/2} \int_0^1 \frac{s^{(n-1)/2}|t+1|\,ds}{|t+1|^n} \geq \frac{c}{|t+1|^{(n-1)/2}}\,.$$

Hence,

$$G_B(t) \geq c\left(|t+1|^{-(n-1)/2}\right)^{(1/p)-(1/p^\star)} = c\,|t+1|^{-(n-1)/2n}\,, \tag{2.3.13}$$

and

$$\lim_{p \to -1^+} G_B(t) = \infty\,.$$

(We also notice, again for future use in the proof of Theorem 2.1.2, that together with (2.3.13) we also have

$$G_B(t) \leq C(n)\,|t+1|^{-(n-1)/2n}\,, \tag{2.3.14}$$

provided t is close enough to -1.) Statements (iii) and (iv) follow immediately. □

Proof of Theorem 2.1.1. Immediate from Theorem 2.2.3 and Proposition 2.3.2. □

We now turn to the quantitative study of $\Phi(T)$. Let us recall that, by a classical variational argument, if u is a minimizer in $\Phi(T)$, then there exists constants λ and σ such that

$$\begin{cases} -\Delta_p u = \lambda |u|^{p^\star-2} u & \text{in } H \\ -|\nabla u|^{p-2}\partial_{x_1} u = \sigma |u|^{p^\sharp-2} u & \text{on } \partial H\,. \end{cases} \tag{2.3.15}$$

Observe that the existence of constants λ and σ satisfying (2.3.15) follows by direct computation using our characterization of minimizers. Moreover, we know that non-negative minimizers are positive, so that there is no need for the absolute values in (2.3.15).

Lemma 2.3.3. *Let $n \geq 2$ and $1 \leq p < n$. Fix $T \in (0, \infty)$ and let λ and σ be the Lagrange multipliers appearing in* (2.3.15) *corresponding to a minimizer u in the variational problem $\Phi(T)$. Then, the following identities hold:*

$$\Phi(T)^p = \lambda + \sigma\, T^{p^\sharp}\,, \qquad \Phi'(T) = \frac{p^\sharp T^{p^\sharp - 1}}{\Phi(T)^{p-1}}\,\sigma\,. \tag{2.3.16}$$

Proof. The first identity follows from an integration by parts and (2.3.15), so we focus on the second. Since $T > 0$ implies $\int_{\partial H} u^{p^\sharp} > 0$, there must be a function $\varphi \in C_c^\infty(\partial H)$ such that

$$\int_{\partial H} u^{p^\sharp - 1} \varphi \, d\mathcal{H}^{n-1} = 1\,. \tag{2.3.17}$$

Similarly, there exists $\xi \in C_c^\infty(H)$ such that

$$\int_H u^{p^\star - 1}\, \xi = 1\,.$$

Let ψ be any function $\psi \in C_c^\infty(\overline{H})$ with $\psi = \varphi$ on ∂H, and extend φ to H by setting

$$\varphi = \psi - \Big(\int_H u^{p^\star - 1}\, \psi \Big)\, \xi\,.$$

Then $\varphi \in C_c^\infty(\overline{H})$ and

$$\int_H u^{p^\star - 1} \varphi = 0\,. \tag{2.3.18}$$

Now define a function $f : \mathbb{R}^2 \to [0, \infty)$ by setting

$$f(\varepsilon, \delta) = -1 + \int_H |u + \varepsilon\varphi + \delta\xi|^{p^\star} \qquad (\varepsilon, \delta) \in \mathbb{R}^2\,.$$

Since $u > 0$ on $\overline{H}$, there exists a neighborhood $\mathcal{U}$ of $(\varepsilon, \delta) = (0, 0)$ such that $u + \varepsilon\varphi + \delta\xi > 0$ on $\overline{H}$ for every $(\varepsilon, \delta) \in \mathcal{U}$. Correspondingly, by (2.3.18)

$$f \in C^1(\mathcal{U}) \qquad f(0,0) \qquad \frac{\partial f}{\partial \delta}(0,0) = p^\star \int_H u^{p^\star - 1} \xi = 1\,,$$

and thus there exists $\varepsilon_0 > 0$ and $g : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ such that$(\varepsilon, g(\varepsilon)) \in \mathcal{U}$ and $f(\varepsilon, g(\varepsilon)) = 0$ for every $|\varepsilon| < \varepsilon_0$. In particular,

$$v_\varepsilon = u + \varepsilon\varphi + \gamma(\varepsilon)\,\xi \in C^\infty(\overline{H}; (0,\infty)) \qquad \int_H v_\varepsilon^{p^\star} = 1\,, \qquad \forall |\varepsilon| < \varepsilon_0\,.$$

By (2.3.17),

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\partial H} \frac{v_\varepsilon^{p^\sharp}}{p^\sharp}\, d\mathcal{H}^{n-1} = \int_{\partial H} u^{p^\sharp - 1}\,\varphi = 1\,, \tag{2.3.19}$$

so that the function $\tau(\varepsilon) = \|v_\varepsilon\|_{L^{p^\sharp}(\partial H)}$ satisfies $\tau(0) = T$ and is strictly increasing on $(-\varepsilon_0, \varepsilon_0)$, up to possibly decreasing the value of ε_0. If we set $\Gamma(\varepsilon) = \int_H |\nabla v_\varepsilon|^p$, then, by construction, $\Phi(\tau(\varepsilon))^p \le \Gamma(\varepsilon)$ for every $|\varepsilon| < \varepsilon_0$, with equality at $\varepsilon = 0$, and thus

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Phi(\tau(\varepsilon))^p = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Gamma(\varepsilon). \tag{2.3.20}$$

We compute that

$$\frac{1}{p}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Gamma(\varepsilon) = \int_H |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx = -\int_H \Delta_p u\varphi - \int_{\partial H} |\nabla u|^{p-2}\partial_{x_1} u\varphi\, d\mathcal{H}^{n-1}. \tag{2.3.21}$$

From (2.3.15), $-\Delta_p u\varphi = \lambda u^{p^\star - 1}$, and so the first term on the right-hand side of (2.3.21) is equal to zero. Then, from (2.3.15) and (2.3.19), the right-hand side of (2.3.21) is equal to σ, and thus that of (2.3.20) to $p\,\sigma$. Since, again by (2.3.17),

$$\tau'(0) = \frac{T^{1-p^\sharp}}{p^\sharp}\,,$$

we conclude from (2.3.19) that

$$\Phi(T)^{p-1}\Phi'(T)\frac{T^{1-p^\sharp}}{p^\sharp} = \sigma\,,$$

thus completing the proof of the lemma. □

We now prove Theorem 2.1.2, Corollary 2.1.3 and Theorem 2.1.5.

Proof of Theorem 2.1.2. Step 1: Differentiability and monotonicity. By Proposition 2.3.2, we know that $\Phi(T_B(t)) = G_B(t)$ for every $t \in (-\infty, -1)$, where T_B is smooth and strictly increasing on $(-\infty, -1)$ with range (T_Q, ∞) and $G_B(t)$ is smooth and strictly increasing on $(-\infty, -1)$ with range (G_Q, ∞). Thus, Φ is smooth on (T_Q, ∞) with $\Phi'(T) = G_B'(t)/T_B'(t) > 0$ for $T = T_B(t)$. This shows that Φ is smooth and strictly increasing on (T_Q, ∞). One can compute that

$$\lim_{T \to T_Q^+} \Phi'(T) = \lim_{t \to -\infty} \frac{G_B'(t)}{T_B'(t)} = \frac{G_Q}{T_Q} = Q \, .$$

Similarly, $\Phi(T_S(t)) = G_S(t)$ for every $t \in \mathbb{R}$ where T_S is strictly decreasing on $\mathbb{R}$ with range $(0, T_Q)$, and $T_S(0) = T_0 < T_Q$, and where G_S is strictly increasing on $[0, \infty)$ with range $[2^{-1/n} S, S)$, and is strictly decreasing on $(-\infty, 0)$ with range $(2^{-1/n} S, Q)$. So, Φ is smooth on $(0, T_Q)$, and $\Phi'(T) = G_S'(t)/T_S'(t) > 0$ for $T = T_S(t)$. Hence, Φ is strictly decreasing on $(0, T_0)$ and strictly increasing on (T_0, T_Q), and one computes

$$\lim_{T \to T_Q^-} \Phi'(T) = \lim_{t \to -\infty} \frac{G_S'(t)}{T_S'(t)} = \frac{G_Q}{T_Q} = Q \, .$$

Therefore, Φ is differentiable at $T = T_Q$, and thus on $(0, \infty)$.

Step 2: Concavity of Φ. Next, we use Lemma 2.3.3 to show that Φ is concave for T sufficiently small. By (2.3.16), we find that for every $t \in \mathbb{R}$,

$$\frac{d}{dT} \Phi(T_S(t)) = p^\sharp \frac{T_S(t)^{p^\sharp - 1}}{G_S(t)^{p-1}} \, \sigma(U_S(t)) \, , \tag{2.3.22}$$

where $\sigma(U_S(t))$ denotes the boundary Lagrange multiplier of $U_S(t)$. By combining (2.1.17) and (2.3.15), we see that

$$\sigma(U_S(t)) = -c(n,p) \, t \, \|U_S\|_{L^{p^\star}(\{x_1 > t\})}^{p(p-1)/(n-p)} \, ,$$

for a positive constant $c(n,p)$. Thus,

$$\frac{d}{dT}\Phi(T_S(t)) = -c(n,p)\,t\,\frac{\|U_S\|_{L^{p^\sharp}(\{x_1=t\})}^{p^\sharp-1}}{\|\nabla U_S\|_{L^p(\{x_1>t\})}^{p-1}}\,,$$

so, differentiating in t (recall that Φ is smooth $(0,T_Q)$), we find that

$$\frac{d^2}{dT^2}\Phi(T_S(t))\,T_S'(t) = -c(n,p)\frac{d}{dt}\left(t\,\frac{\|U_S\|_{L^{p^\sharp}(\{x_1=t\})}^{p^\sharp-1}}{\|\nabla U_S\|_{L^p(\{x_1>t\})}^{p-1}}\right).$$

Since $T_S'(t)<0$ for every $t\in\mathbb{R}$, we conclude that $\Phi(T)$ is going to be concave on any interval $J=\{T_S(t):t\in J'\}$ corresponding to an interval $J'\subset\mathbb{R}$ such that

$$\frac{d}{dt}\log\left(t\,\frac{\|U_S\|_{L^{p^\sharp}(\{x_1=t\})}^{p^\sharp-1}}{\|\nabla U_S\|_{L^p(\{x_1>t\})}^{p-1}}\right)<0\,,\qquad\forall t\in J'\,. \tag{2.3.23}$$

For the sake of brevity, set

$$h(t)=\int_{\{x_1=t\}}U_S^{p^\sharp}=\int_{\partial H}(1+|x-t\,\mathrm{e}_1|^{p'})^{-(n-1)}\,d\mathcal{H}^{n-1}\,.$$

We are thus looking for an interval J' such that

$$\frac{1}{t}+\frac{p^\sharp-1}{p^\sharp}\,\frac{h'(t)}{h(t)}-\frac{p-1}{p}\,\frac{\frac{d}{dt}\int_{\{x_1>t\}}|\nabla U_S|^p}{\int_{\{x_1>t\}}|\nabla U_S|^p}<0\qquad\forall t\in J'\,.$$

Since $\int_{\{x_1>t\}}|\nabla U_S|^p$ is trivially increasing in t, it suffices to find an interval J' such that

$$\frac{1}{t}+\frac{n(p-1)}{p(n-1)}\,\frac{h'(t)}{h(t)}<0\qquad\forall t\in J'\,.$$

If $t>0$, then factoring and changing variables, we find that

$$h(t)=t^{-(n-1)/(p-1)}\int_{\partial H}(t^{-p'}+|x-\mathrm{e}_1|^{p'})^{-(n-1)}\,d\mathcal{H}^{n-1}\,.$$

Therefore, we compute

$$\frac{h'(t)}{h(t)} = -\frac{n-1}{(p-1)t} + \frac{p'(n-1)\int_{\partial H}(t^{-p'}+|x+\mathrm{e}_1|^{p'})^{-n}\,d\mathcal{H}^{n-1}}{t^{p'+1}\int_{\partial H}(t^{-p'}+|x+\mathrm{e}_1|^{p'})^{-(n-1)}\,d\mathcal{H}^{n-1}}\,,$$

where trivially $t^{-p'}+|x+\mathrm{e}_1|^{p'}>1$ for $x\in\partial H$, and thus

$$\int_{\partial H}(t^{-p'}+|x+\mathrm{e}_1|^{p'})^{-n}\,d\mathcal{H}^{n-1} < \int_{\partial H}(t^{-p'}+|x+\mathrm{e}_1|^{p'})^{-(n-1)}\,d\mathcal{H}^{n-1}\,.$$

We have thus proved that for every $t>0$,

$$\frac{h'(t)}{h(t)} \le \frac{1}{t}\,\frac{n-1}{p-1}\Big(-1+\frac{p}{t^{p'}}\Big)\,,$$

so that

$$\frac{1}{t}+\frac{n(p-1)}{p(n-1)}\,\frac{h'(t)}{h(t)} \le \frac{1}{t}+\frac{1}{t}\,\frac{n}{p}\Big(-1+\frac{p}{t^{p'}}\Big)\,.$$

This last quantity is negative for $t>(p^\star)^{1/p'}$. Thus, (3.4.6) holds with the choice $J'=(-\infty,(p^\star)^{1/p'})$ and correspondingly $\Phi(T)$ is strictly concave on $(0,T_*)$ provided we set

$$T_* = T_S((p^\star)^{1/p'})\,.$$

Step 3: Convexity of Φ. By (2.3.4) we have that, for every $t<-1$,

$$\Phi(T) \ge G_B(t) + t\,\frac{T_B(t)^{p^\sharp}-T^{p^\sharp}}{p^\sharp\,Y_B(t)} \qquad \forall T>0\,, \tag{2.3.24}$$

with equality if and only if $T=T_B(t)$. If we denote by $\Psi_t(T)$ the right-hand side of (2.3.24), this shows that

$$\Phi(T) = \sup_{t<-1}\Psi_t(T) \qquad \forall T\in\big\{T_B(t):t<-1\big\}=(T_Q,\infty)\,.$$

Since each $\Psi_t(T)$ is convex as a function of T (recall that t is negative), this proves that $\Phi(T)$ is convex on (T_Q,∞). We can perform the same argument based on (2.3.3), as soon as the parameter $t\in\mathbb{R}$ describing the Sobolev family is negative. This proves the convexity of $\Phi(T)$ over the interval

$$\big\{T_S(t):t<0\big\}=(T_0,T_Q)\,.$$

Since $\Phi(T)$ is convex on (T_0,T_Q) and on (T_Q,∞), with $\Phi(T)\ge Q\,T$ for every $T\ge 0$ and $\Phi(T_Q)=Q\,T_Q$, we conclude that $\Phi(T)$ is convex on (T_0,∞).

Step 4: Asymptotic growth of Φ. First, we claim that

$$\lim_{T\to\infty}\frac{p^\sharp\,\Phi(T)}{T^{p^\sharp}}=1\,.$$

Having in mind (2.1.13), and taking into account that $T_B(t)\to+\infty$ as $t\to-1$, it suffices to show that

$$\lim_{t\to-1}\frac{p^\sharp G_B(t)}{T_B(t)^{p^\sharp}}=1\,.$$

To prove this, we notice that the identity

$$n\int_H U_{B,t}^{p^\sharp}=p^\sharp\,G_B(t)\,Y_B(t)+t\,T_B(t)^{p^\sharp}\qquad\forall t<-1\,,\tag{2.3.25}$$

allows us to write

$$p^\sharp\,\frac{G_B(t)}{T_B(t)^{p^\sharp}}=-\frac{t}{Y_B(t)}+\frac{n\int_H U_{B,t}^{p^\sharp}}{Y_B(t)\,T_B(t)^{p^\sharp}}\,.$$

It will thus be enough to prove

$$\lim_{t\to-1}Y_B(t)=1\qquad\lim_{t\to-1}\int_H U_{B,t}^{p^\sharp}=0\,.\tag{2.3.26}$$

To this end, we first notice that by (2.3.10) and (2.3.11)

$$Y_B(t)^{p'} - 1 = \frac{\int_H (|x - t\,\mathrm{e}_1|^{p'} - 1)^{-(n-1)}}{\int_H (|x - t\,\mathrm{e}_1|^{p'} - 1)^{-n}} \le C(n)\,\frac{|t+1|^{-(n-3)/2}}{|t+1|^{-(n-1)/2}} = C(n)\,|t+1|\,,$$

while

$$\int_H U_{B,t}^{p^\sharp} = \frac{\int_H (|x - t\,\mathrm{e}_1|^{p'} - 1)^{-(n-1)}}{\big(\int_H (|x - t\,\mathrm{e}_1|^{p'} - 1)^{-n}\big)^{(n-1)/n}} \le C(n)\,\frac{|t+1|^{-(n-3)/2}}{|t+1|^{-(n-1)^2/2n}} = C(n)\,|t+1|^{(n+1)/2n}$$

so that (2.3.26) is proven. Now, to prove that

$$\lim_{T\to\infty} \Phi(T) - \frac{T^{p^\sharp}}{p^\sharp} = 0\,,$$

we simply notice that, again by (2.3.25),

$$p^\sharp G_B(t) - T_B(t)^{p^\sharp} = p^\sharp\, G_B(t)\Big(1 + \frac{Y_B(t)}{t}\Big) - \frac{n}{t}\int_H U_{B,t}^{p^\sharp}\,.$$

Since $|t + Y_B(t)| \le |t+1| + |1 - Y_B(t)| \le C(n)\,|t+1|$, thanks to (2.3.14) we have

$$G_B(t)\Big|1 + \frac{Y_B(t)}{t}\Big| \le C(n)\,|t+1|^{1-(n-1)/2n} = C(n)\,|t+1|^{(n+1)/2n} \to 0\,.$$

This completes the proof of Theorem 2.1.2. □

Proof of Corollary 2.1.3. Since Ω is a set of locally finite perimeter in $\mathbb{R}^n$ [Mag12, Example 12.6], there exists $x_0 \in \partial\Omega$ such that, up to a rotation,

$$\begin{aligned} \Omega_r &\to H && \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n), \\ \mathcal{H}^{n-1}\llcorner\partial\Omega_r &\overset{*}{\rightharpoonup} \mathcal{H}^{n-1}\llcorner\partial H && \text{as Radon measures on } \mathbb{R}^n. \end{aligned} \tag{2.3.27}$$

where we have set $\Omega_r = (\Omega - x_0)/r$, $r > 0$. Precisely, every x_0 in the reduced boundary of Ω satisfies (2.3.27) up to a rotation, see e.g. [Mag12, Theorem 15.5].

We now define a function w_T depending on T as follows. If $T \in (0, T_Q)$, setting $t = T_S^{-1}(T)$, we let

$$w_T(x) = U_{S,t}(x) \qquad \forall x \in \mathbb{R}^n \,;$$

if $T = T_Q$, we set $t = -1$ and let

$$w_T(x) = U_{Q,t}(x) \qquad \forall x \in \mathbb{R}^n \setminus \{\mathrm{e}_1\} \,;$$

finally, if $T > T_Q$, then, setting $t = T_B^{-1}(T) < -1$, we let

$$w_T(x) = U_{B,t}(x) \qquad \forall x \in \mathbb{R}^n \setminus \overline{B_1(t\,\mathrm{e}_1)} \,.$$

Notice that in each case, there exists a compact set K_T with $K_T \cap H = \emptyset$ such that $w_T \in L^{p^\star}(\mathbb{R}^n \setminus U)$ and $\nabla w_T \in L^p(\mathbb{R}^n \setminus U)$ for every open neighborhood U of K_T. In particular, for $\varepsilon > 0$ small enough depending on T, we have $\{x_1 > -\varepsilon\} \cap K_T = \emptyset$. We pick $\zeta \in C^\infty(\mathbb{R}^n)$ such that $\zeta = 1$ on $\{x_1 > -\varepsilon\}$ and $\zeta = 0$ on K_T, and define $v_T = \zeta\, w_T$ on the whole $\mathbb{R}^n$. Then

$$v_T \in L^{p^\star}(\mathbb{R}^n) \qquad \nabla v_T \in L^p(\mathbb{R}^n) \qquad v_T = w_T \text{ on } H \,. \tag{2.3.28}$$

Next, we fix $R > 0$ and consider $\psi_R \in C_c^\infty(B_{2R}; [0,1])$ with $\psi_R = 1$ on B_R. Finally, for each $r > 0$, we define

$$u_r(x) = r^{1-n/p}\,(\psi_R\, v_T)\Big(\frac{x - x_0}{r}\Big) \qquad x \in \Omega \,.$$

By (2.3.27), (2.3.28) and $\psi_R \in C_c^\infty(B_{2R}; [0,1])$ we can exploit dominated convergence to find that

$$\int_\Omega u_r^{p^\star} = \int_{\mathbb{R}^n} 1_{\Omega_r} (\psi_R v_T)^{p^\star} \to \int_H (\psi_R v_T)^{p^\star} = \int_H (\psi_R w_T)^{p^\star}$$

$$\int_\Omega |\nabla u_r|^p \to \int_H |\nabla(\psi_R\, w_T)|^p\,,$$

as $r \to 0^+$. Similarly, since $(\psi_R\, v_T)^{p^\sharp} \in C^0_c(\mathbb{R}^n)$, by (2.3.27) we have

$$\int_{\partial\Omega} u_r^{p^\sharp}\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n} (\psi_R\, v_T)^{p^\sharp}\, d(\mathcal{H}^{n-1} \llcorner \partial\Omega_r) \to \int_{\partial H} (\psi_R\, v_T)^{p^\sharp}\, d\mathcal{H}^{n-1} = \int_{\partial H} (\psi_R\, w_T)^{p^\sharp}\, d\mathcal{H}^{n-1}$$

as $r \to 0^+$. Since

$$\int_H w_T^{p^\star} = 1\,, \qquad \int_{\partial H} w_T^{p^\sharp} = T^{p^\sharp}\,, \qquad \int_H |\nabla w_T|^p = \Phi(T)^p\,,$$

for every $\delta > 0$ there exists r small enough and R large enough such that

$$\left|\int_\Omega u_r^{p^\star} - 1\right| + \left|\int_\Omega |\nabla u_r|^p - \Phi(T)^p\right| + \left|\int_{\partial\Omega} u_r^{p^\sharp} - T^{p^\sharp}\right| < \delta\,.$$

In particular, we can find $\{\xi_r\}_{r>0} \subset C^\infty_c(\Omega)$ such that

$$\frac{\|u_r + \xi_r\|_{L^{p^\sharp}(\partial\Omega)}}{\|u_r + \xi_r\|_{L^{p^\star}(\Omega)}} = \frac{\|u_r\|_{L^{p^\sharp}(\partial\Omega)}}{\|u_r + \xi_r\|_{L^{p^\star}(\Omega)}} = T \qquad \forall r > 0\,,$$

and $\|\xi_r\|_{L^{p^\star}(\Omega)} \to 0$ and $\|\nabla\xi_r\|_{L^p(\Omega)} \to 0$ as $r \to 0^+$. Then, for r sufficiently small,

$$\Phi_\Omega(T) \le \frac{\|\nabla u_r + \nabla\xi_r\|_{L^p(\Omega)}}{\|u_r + \xi_r\|_{L^{p^\star}(\Omega)}} \le (1 + C\,\delta)\,\Phi(T)$$

for a constant $C = C(n,p)$. □

Proof of Theorem 2.1.5. With the same reasoning as given in Corollary 2.3.1, we find that

$$Y_S(t)G_S(t) + tT_S(T) \le Y_S(t)|Dh|(H) + t\|h\|_{L^1(\partial H)}$$

for any $t \in (-1,1)$ and any non-negative h, vanishing at infinity, with $|Dh|(H) < \infty$, and with equality if and only if h is a dilation translation image of $U_{S,t}$ orthogonal to e_1. In particular, if additionally $\|h\|_{L^1(\partial H)} = T_S(t)$, then

$$G_S(t) \le |Dh|(H).$$

From this, we deduce that

$$\Phi(T_S(t)) = G_S(t)$$

for $t \in (-1,1)$. The same arguments given in the proof of Proposition 2.3.2 imply that $T_S(t)$ is a strictly decreasing function with range $[0,\infty)$, and that $G_S(t)$ is strictly increasing for $t > 0$ with range $(2^{-1/n}S, S)$ and is strictly decreasing for $t < 0$ with range $(2^{-1/n}S, \infty)$. Finally, the same proof as that of Theorem 2.1.2 shows that $\Phi(T)$ is a smooth function of T that is decreasing for $T \in (0, T_0)$ and concave for $T \in (0, T_*)$ for some $0 < T_* < T_0$ and increasing and convex for $T \in (T_0, \infty)$. Finally, to show that $\Phi(T) = T + o(1)$ as $T \to \infty$, we will equivalently show that $G_S(t) = T_S(t) + o(1)$ as $t \to -1$. Indeed, since $Y(t, U_{S,t}) = 1$ for all $-1 < t < 1$ when $p = 1$, (2.3.2) implies that

$$G_S(t) = n \int_H U_{S,t} - t\, T_S(t) = T_S(t) + n \int_H U_{S,t} - (t+1)\, T_S(t).$$

Note that

$$\int_H U_{S,t} = \frac{|B_1(t\,\mathrm{e}_1) \cap H|}{|B_1(t\,\mathrm{e}_1) \cap H|^{(n-1)/n}} = |B_1(t\,\mathrm{e}_1) \cap H|^{1/n} = o(1)$$

as $t \to -1$. Furthermore, since

$$T(t) = \frac{\omega_{n-1}(1-t^2)^{(n-1)/2}}{|B_1(t\,\mathrm{e}_1) \cap H|^{(n-1)/n}},$$

and we easily estimate that

$$|B_1(t\,\mathrm{e}_1) \cap H| = \omega_{n-1} \int_{-t}^{1} (1-s^2)^{(n-1)/2}\, ds \geq c \int_{-t}^{1} (1-s)^{(n-1)/2} \geq C|1+t|^{(n+1)/2}$$

for $t < 0$, we see that

$$|t+1| T(t) \leq |t+1|^{1-(n+1)/2n} = o(1).$$

Hence, $G_S(t) = T_S(t) + o(1)$ as $t \to -1$ and the proof is complete. □

We conclude with the following proposition, which was mentioned after the statement of Theorem 2.1.5.

Proposition 2.3.4. *For every $n \geq 2$, one has $T_Q(n,p) \to +\infty$ as $p \to 1^+$.*

Proof. As a first step, we explicitly compute

$$T_Q^{p^\sharp} = C\left(\frac{\Gamma\left(\frac{n-1}{2(p-1)}\right)}{\Gamma\left(\frac{(n-1)p}{2(p-1)}\right)}\right)\Bigg/\left((p-1)\frac{\Gamma\left(\frac{n+p-1}{2(p-1)}\right)}{\Gamma\left(\frac{np}{2(p-1)}\right)}\right)^{(n-1)/n}, \tag{2.3.29}$$

where, here and throughout the proof, C denotes a constant depending only on n, whose value may change at each instance. Indeed,

$$\int_{\partial H} |x+\mathrm{e}_1|^{-(n-1)p'}\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^{n-1}} (|z|^2+1)^{-(n-1)p'/2}\, dz = C\int_0^\infty (r^2+1)^{-(n-1)p'/2} r^{n-2}\, dr.$$

Making the change of variables $s = 1/(r^2+1)$, the right-hand side becomes

$$\begin{aligned} C\int_0^1 s^{[(n-1)p'/2]-2}\,(1/s-1)^{(n-3)/2}\, ds &= C\int_0^1 s^{[(n-1)/2(p-1)]-1}(1-s)^{(n-3)/2}\, ds \\ &= C\,\mathcal{B}\Big(\frac{n-1}{2(p-1)},\frac{n-1}{2}\Big) = C\,\Gamma\Big(\frac{n-1}{2(p-1)}\Big)\Big/\Gamma\Big(\frac{(n-1)p}{2(p-1)}\Big). \end{aligned}$$

To express the term in the denominator of T_Q, the coarea formula implies that

$$\int_H |x+\mathrm{e}_1|^{-np'}\, dx = \int_1^\infty r^{-np'+(n-1)} \int_{1/r}^1 (1-s^2)^{(n-3)/2}\, ds\, dr.$$

By Fubini's Theorem, the right-hand side is equal to

$$\int_0^1 (1-s^2)^{(n-3)/2} \int_{1/s}^\infty r^{-[n/(p-1)]-1}\, dr\, ds = \frac{p-1}{n}\int_0^1 (1-s^2)^{(n-3)/2} s^{n/(p-1)}\, ds\,.$$

With the change of variables $\rho = s^2$, this is equal to

$$\frac{p-1}{2n}\int_0^1 (1-\rho)^{(n-3)/2}\rho^{[n/2(p-1)]-1/2}\, d\rho = \frac{p-1}{2n}\,\mathcal{B}\Big(\frac{n-1}{2},\frac{n}{2(p-1)}+\frac{1}{2}\Big)$$

$$= C(p-1)\,\Gamma\Big(\frac{n+p-1}{2(p-1)}\Big)\Big/\Gamma\Big(\frac{np}{2(p-1)}\Big).$$

This proves (2.3.29). By taking the logarithm of $T_Q^{p^\sharp}/C$, we find that

$$\begin{aligned}\log(T_Q^{p^\sharp}/C) &= \log\Gamma\Big(\frac{n-1}{2(p-1)}\Big) - \log\Gamma\Big(\frac{p(n-1)}{2(p-1)}\Big)\\ &\quad - \frac{n-1}{n}\left[\log(p-1) + \log\Gamma\Big(\frac{n+p-1}{2(p-1)}\Big) - \log\Gamma\Big(\frac{np}{2(p-1)}\Big)\right].\end{aligned} \tag{2.3.30}$$

By Stirling's approximation, $\log\Gamma(z)$ asymptotically behaves like $z\log(z)$ as $z \to \infty$. Hence, in the limit $p \to 1^-$ the first two terms on the right-hand side of (2.3.30), behave like

$$\begin{aligned}&\frac{n-1}{2(p-1)}\log\Big(\frac{n-1}{2(p-1)}\Big) - \frac{p(n-1)}{2(p-1)}\log\Big(\frac{p(n-1)}{2(p-1)}\Big)\\ &\qquad = -\frac{p(n-1)}{2(p-1)}\log(p) + \frac{(n-1)}{2}\log(p-1) + C.\end{aligned}$$

On the other hand, the term in brackets on the right-hand side of (2.3.30) behaves like

$$\begin{aligned}&\log(p-1) + \frac{n+p-1}{2(p-1)}\log\Big(\frac{n+p-1}{2(p-1)}\Big) - \frac{np}{2(p-1)}\log\Big(\frac{np}{2(p-1)}\Big)\\ &= \log(p-1)\Big(\frac{n+1}{2}\Big) + \frac{n+p-1}{2(p-1)}\log(n+p-1) - \frac{np\log(n)}{2(p-1)} - \frac{np}{2(p-1)}\log(p) + C.\end{aligned}$$

So, the full right-hand side of (2.3.30) asymptotically behaves like

$$-\frac{n-1}{2n}\log(p-1) + \frac{n-1}{2n(p-1)}\left[-(n+p-1)\log(n+p-1) + np\log(n)\right] + C.$$

Since $\log(n+p-1) = \log(n) + (p-1)/n + o(p-1)$, this quantity is bounded above and below (with appropriate choices of C) by

$$-\frac{n-1}{2n}\log(p-1) + \frac{n-1}{2n(p-1)}\left[-(n+p-1)\Big(\log(n) + \frac{p-1}{n}\Big) + np\log(n)\right] + C$$

$$= -\frac{n-1}{2n}\log(p-1) + \frac{(n-1)}{2n}\left[(n-1)\log(n) - \frac{n+p-1}{n}\right] + C\,.$$

The second term is bounded above and below by dimensional constants, while the first term goes to $+\infty$ as $p \to 1^+$. $\square$

Chapter 3

Strong-form stability for the Sobolev inequality on $\mathbb{R}^n$: the case $p \geq 2$

3.1 Overview

In this chapter,[1] we prove strong-form stability for the Sobolev inequality

$$\|\nabla u\|_{L^p} \geq S\|u\|_{L^{p^*}} \tag{3.1.1}$$

in the case $p \geq 2$. All integrals and function spaces in this chapter will be over $\mathbb{R}^n$, so we omit the domain of integration when no confusion arises. Furthermore, throughout the chapter, we assume that $2 \leq p < n$. Recall that equality is attained in (3.1.1) if and only if u belongs to the $(n+2)$-dimensional manifold of extremal functions

$$\mathcal{M} = \{c\, U_{\lambda,y} : c \in \mathbb{R},\ \lambda \in \mathbb{R}_+,\ y \in \mathbb{R}^n\}\ , \tag{3.1.2}$$

where $cU_{\lambda,y}(x) = c\lambda^{n/p^*}U_1(\lambda(x-y))$ and

$$U_1(x) = \frac{\kappa_0}{(1+|x|^{p'})^{(n-p)/p}}, \tag{3.1.3}$$

Here, κ_0 is chosen so that $\|v_1\|_{L^{p^*}} = 1$, and so $\|cv_{\lambda,y}\|_{L^{p^*}} = c$. In the introduction, we introduced the scaling invariant Sobolev deficit $\delta_S(u)$; for simplicity we will now use

[1] This chapter is based on joint work with A. Figalli originally appearing in [FN].

the p-homogeneous deficit $\delta(u)$ defined by

$$\delta(u) = S^p \|u\|_{L^{p^\star}}^p \delta_S(u) = \|\nabla u\|_{L^p}^p - S^p \|u\|_{L^{p^*}}^p .$$

Our main result is the following theorem:

Theorem 3.1.1. *Let $2 \leq p < n$. There exists a constant $C > 0$, depending only on p and n, such that for all $u \in \dot{W}^{1,p}$,*

$$\|\nabla u - \nabla U\|_{L^p}^p \leq C\,\delta(u) + C\|u\|_{L^{p^*}}^{p-1} \|u - U\|_{L^{p^*}} \tag{3.1.4}$$

for some $U \in \mathcal{M}$.

By combining Theorem 3.1.1 and the main result of [CFMP09] (see Theorem 3.4.5), we deduce the following corollary, proving the desired stability at the level of gradients:

Corollary 3.1.2. *Let $2 \leq p < n$. There exists a constant $C > 0$, depending only on p and n, such that for all $u \in \dot{W}^{1,p}$,*

$$\left(\frac{\|\nabla u - \nabla v\|_{L^p}}{\|\nabla u\|_{L^p}}\right)^\zeta \leq C \frac{\delta(u)}{\|\nabla u\|_{L^p}^p} \tag{3.1.5}$$

for some $U \in \mathcal{M}$, where $\zeta = p^* p \left(3 + 4p - \frac{3p+1}{n}\right)^2$.

3.1.1 Theorem 3.1.1: idea of the proof

As a starting point to prove stability of (3.1.1) at the level of gradients, one would like to follow the argument used to prove the analogous result in [BE91]. However, this approach turns out to be sufficient only in certain cases, and additional ideas are needed to conclude the proof. Indeed, a Taylor expansion of the deficit $\delta(u)$ and a spectral gap for the linearized problem allow us to show that the second variation is strictly positive, but in general we cannot absorb the higher order terms. Let us provide a few more details to see to what extent this approach works, where it breaks down, and how we get around it.

3.1.1.1 The expansion approach.

Let us sketch how an argument following [BE91] would go. In order to introduce a Hilbert space structure to our problem, we define a weighted L^2-distance of a function $u \in \dot{W}^{1,p}$ to $\mathcal{M}$. To this end, for each $U \in \mathcal{M}$, we define

$$A_U(x) := (p-2)|\nabla U|^{p-2}\hat{r} \otimes \hat{r} + |\nabla U|^{p-2}\mathrm{Id}, \qquad \hat{r} = \frac{x-y}{|x-y|}, \tag{3.1.6}$$

where $(a \otimes b)c := (a \cdot c)b$. Then, with the notation $A_U[a,a] := a^T A_U a$ for $a \in \mathbb{R}^n$, we define

$$\begin{aligned} \mathrm{d}(u,\mathcal{M}) &= \inf\Big\{\Big(\int A_U[\nabla u - \nabla U, \nabla u - \nabla U]\Big)^{1/2} : \ U \in \mathcal{M}, \ \|U\|_{L^{p^*}} = \|u\|_{L^{p^*}}\Big\} \\ &= \inf\Big\{\Big(\int A_{cU_{\lambda,y}}[\nabla u - \nabla cU_{\lambda,y}, \nabla u - \nabla cU_{\lambda,y}]\Big)^{1/2} : \ \lambda \in \mathbb{R}_+, \ y \in \mathbb{R}^n, \ c = \|u\|_{L^{p^*}}\Big\}. \end{aligned} \tag{3.1.7}$$

Note that

$$\int A_U[\nabla u - \nabla U, \nabla u - \nabla U] = \int |\nabla U|^{p-2}|\nabla u - \nabla U|^2 + (p-2)\int |\nabla U|^{p-2}|\partial_r u - \partial_r U|^2.$$

A few remarks about this definition are in order.

Remark 3.1.3. The motivation to define $\mathrm{d}(u,\mathcal{M})$ in this way instead of, for instance,

$$\inf\Big\{\Big(\int |\nabla U|^{p-2}|\nabla u - \nabla U|^2\Big)^{1/2} : \ U \in \mathcal{M}, \ \|U\|_{L^{p^*}} = \|u\|_{L^{p^*}}\Big\},$$

will become apparent in Section 3.2. This choice, however, is only technical, as

$$\int |\nabla U|^{p-2}|\nabla u - \nabla U|^2 \leq \int A_U[\nabla u - \nabla U, \nabla u - \nabla U] \leq (p-1)\int |\nabla U|^{p-2}|\nabla u - \nabla U|^2.$$

Remark 3.1.4. One could alternatively define the distance in (3.1.7) without the constraint $c = \|u\|_{L^{p^*}}$, instead also taking the infimum over the parameter c. Up to adding a small positivity constraint to ensure that the infimum is not attained at $U = 0$, this definition works, but ultimately the current presentation is more straightforward.

Remark 3.1.5. The distance $\mathrm{d}(u, \mathcal{M})$ has homogeneity $p/2$, that is, $\mathrm{d}(cu, \mathcal{M}) = c^{p/2}\mathrm{d}(u, \mathcal{M})$.

In Proposition 3.3.1(1), we show that there exists $\delta_0 = \delta_0(n, p) > 0$ such that if

$$\delta(u) \leq \delta_0 \|\nabla u\|_{L^p}^p, \tag{3.1.8}$$

then the infimum in $\mathrm{d}(u, \mathcal{M})$ is attained. Given a function $u \in \dot{W}^{1,p}$ satisfying (3.1.8), let $U \in \mathcal{M}$ attain the infimum in (3.1.7) and define

$$\varphi = \frac{u - U}{\|\nabla(u - U)\|_{L^p}},$$

so that $u = U + \varepsilon\varphi$ with $\varepsilon = \|\nabla(u - U)\|_{L^p}$ and $\int |\nabla \varphi|^p = 1$. Since U is a minimum of δ, the Taylor expansion of the deficit of u at U vanishes at the zeroth and first order. Thus, the expansion leaves us with

$$\delta(u) = \varepsilon^2 p \int A_U[\nabla\varphi, \nabla\varphi] - \varepsilon^2 S^p p(p^* - 1)\int |U|^{p^*-2}|\varphi|^2 + o(\varepsilon^2). \tag{3.1.9}$$

Since U is a projection of u into $\mathcal{M}$, $\varepsilon\varphi$ is orthogonal (in an appropriate sense) to the tangent space of $\mathcal{M}$ at U, which coincides with the span the first two eigenspaces of an appropriate weighted linearized p-Laplacian. A gap in the spectrum in this operator allows us to show that

$$c\,\mathrm{d}(u,\mathcal{M})^2 = c\,\varepsilon^2 \int A_U[\nabla\varphi,\nabla\varphi] \le \varepsilon^2 p \int A_U[\nabla\varphi,\nabla\varphi] - \varepsilon^2 S^p p(p^*-1) \int |U|^{p^*-2}|\varphi|^2$$

for a positive constant $c = c(n,p)$. Together with (3.1.9), this implies

$$\mathrm{d}(u,\mathcal{M})^2 + o(\varepsilon^2) \le C\delta(u).$$

Now, if the term $o(\varepsilon^2)$ could be absorbed into $\mathrm{d}(u,\mathcal{M})^2$, then we could use the estimate (3.1.11) below to obtain

$$\int |\nabla u - \nabla U|^p \le C\delta(u),$$

which would conclude the proof.

3.1.1.2 Where the expansion approach falls short.

The problem arises exactly when trying to absorb the term $o(\varepsilon^2)$. Indeed, recalling that $\varepsilon = \|\nabla(u-U)\|_{L^p}$, we are asking whether

$$o(\|\nabla u - \nabla U\|_{L^p}^2) \ll \mathrm{d}(u,\mathcal{M})^2 \approx \int |\nabla U|^{p-2}|\nabla u - \nabla U|^2$$

(recall Remark 3.1.3), and unfortunately this is false in general. Notice that this problem never arises in [BE91] for the case $p = 2$, as the above inequality reduces to

$$o(\|\nabla u - \nabla U\|_{L^2}^2) \ll \|\nabla u - \nabla U\|_{L^2}^2,$$

which is clearly true.

3.1.1.3 The solution.

A Taylor expansion of the deficit will not suffice to prove Theorem 3.1.1 as we cannot hope to absorb the higher order terms. Instead, for a function $u \in \dot{W}^{1,p}$, we give two different expansions, each of which gives a lower bound on the deficit, by splitting the terms between the second order term and the pth order term. Pairing this with an analysis of the second variation, we obtain the following:

Proposition 3.1.6. *There exist constants $\mathbf{c}_1, \mathbf{C}_2$, and $\mathbf{C}_3$, depending only on p and n, such that the following holds. Let $u \in \dot{W}^{1,p}$ be a function satisfying* (3.1.8) *and let $U \in \mathcal{M}$ be a function where the infimum of the distance* (3.1.7) *is attained. Then*

$$\mathbf{c}_1 \, \mathrm{d}(u, \mathcal{M})^2 - \mathbf{C}_2 \int |\nabla u - \nabla U|^p \leq \delta(u), \tag{3.1.10}$$

$$-\mathbf{C}_3 \, \mathrm{d}(u, \mathcal{M})^2 + \frac{1}{4} \int |\nabla u - \nabla U|^p \leq \delta(u). \tag{3.1.11}$$

Individually, both inequalities are quite weak. However, as shown in Corollary 3.3.3, they allow us to prove Theorem 3.1.1 (in fact, the stronger statement $\int |\nabla u - \nabla U|^p \leq \delta(u)$) for the set of functions u such that

$$\begin{aligned} \mathrm{d}(u, \mathcal{M})^2 = \int A_U[\nabla u - \nabla U, \nabla u - \nabla U] &\ll \int |\nabla u - \nabla U|^p \\ \text{or} \\ \mathrm{d}(u, \mathcal{M})^2 = \int A_U[\nabla u - \nabla U, \nabla u - \nabla U] &\gg \int |\nabla u - \nabla U|^p. \end{aligned} \tag{3.1.12}$$

We are then left to consider the middle regime, where

$$\int A_U[\nabla u - \nabla U, \nabla u - \nabla U] \approx \int |\nabla u - \nabla U|^p.$$

We handle this case as follows. Let $u_t := (1 - t)u + tU$ be the linear interpolation between u and U. Choosing t_* small enough, u_{t_*} falls in the second regime in (3.1.12), so Theorem 3.1.1 holds for u_{t_*}. We then must relate the deficit and distance of u_{t_*} to those of u. While relating the distances is straightforward, it is not clear for the deficits whether the estimate $\delta(u_{t_*}) \leq C\delta(u)$ holds. Still, we can show that

$$\delta(u_{t_*}) \leq C\delta(u) + C\|U\|_{L^{p^*}}^{p-1}\|u - U\|_{L^{p^*}},$$

which allows us to conclude the proof. It is this point in the proof that introduces that term $\|u-U\|_{L^{p^*}}$ in Theorem 3.1.1, and for this reason we rely on the main theorem of [CFMP09] to prove Corollary 3.1.2. We note that the application of [CFMP09] is not straightforward, since the function U which attains the minimum in our setting is a priori different from the one considered there (see Section 3.4 for more details).

3.1.2 Outline of the chapter

In Section 3.2, we introduce the operator $\mathcal{L}_U$ that appears in the second variation of the deficit and prove some facts about the spectrum of this operator. We also prove some elementary but crucial inequalities in Lemma 3.2.2 and provide orthogonality constraints that arise from taking the infimum in (3.1.7). In Section 3.3, we prove Proposition 3.1.6. In Section 3.4, we prove Theorem 3.1.1 and Corollary 3.1.2. In Section 3.5, we show that $\mathcal{L}_U$ has a discrete spectrum and justify the use of Sturm-Liouville theory in the proof of Proposition 3.2.1.

3.2 Preliminaries

3.2.1 The tangent space of $\mathcal{M}$ and the operator $\mathcal{L}_U$

The set $\mathcal{M}$ of extremal functions defined in (3.1.2) is an $(n+2)$-dimensional manifold which is smooth except at $0 \in \mathcal{M}$. For a nonzero $U = c_0 U_{\lambda_0, y_0} \in \mathcal{M}$, the tangent space is computed to be

$$T_U\mathcal{M} = \operatorname{span}\{U,\, \partial_\lambda U,\, \partial_{y^1} U, \dots,\, \partial_{y^n} U\},$$

where y^i denotes the ith component of y and $\partial_\lambda U = \partial_\lambda|_{\lambda=\lambda_0} U$, $\partial_{y^i} U = \partial_{y^i}|_{y^i = y_0^i} U$.

Since the functions $U = U_{\lambda_0, y_0}$ minimize $u \mapsto \delta(u)$ and have $\|U_{\lambda_0,y_0}\|_{L^{p^*}} = 1$, by computing the Euler-Lagrange equation one discovers that

$$-\Delta_p U = S^p U^{p^*-1}, \tag{3.2.1}$$

where the p-Laplacian Δ_p is defined by $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2} \nabla w)$. Hence, differentiating (3.2.1) with respect to y^i or λ, we see that

$$-\operatorname{div}(A_U(x)\nabla w) = (p^*-1) S^p U^{p^*-2} w, \qquad w \in \operatorname{span}\{\partial_\lambda U,\, \partial_{y^1} U, \dots,\, \partial_{y^n} U\}, \tag{3.2.2}$$

where $A_U(x)$ is as defined in (3.1.6). This motivates us to consider the weighted operator

$$\mathcal{L}_U w = -\operatorname{div}(A_U(x)\nabla w) U^{2-p^*} \tag{3.2.3}$$

on the space $L^2(U^{p^*-2})$, where, for a measurable weight $\omega : \mathbb{R}^n \to \mathbb{R}$, we let

$$\|w\|_{L^2(\omega)} = \left(\int_{\mathbb{R}^n} |w|^2 \omega \right)^{1/2}, \qquad L^2(\omega) = \{ w : \mathbb{R}^n \to \mathbb{R} : \|w\|_{L^2(\omega)} < \infty \}.$$

Proposition 3.2.1. *The operator $\mathcal{L}_U$ has a discrete spectrum $\{\alpha_i\}_{i=1}^{\infty}$, with $0 < \alpha_i < \alpha_{i+1}$ for all i, and*

$$\alpha_1 = (p-1)S^p, \qquad H_1 = \operatorname{span}\{U\}, \tag{3.2.4}$$

$$\alpha_2 = (p^*-1)S^p, \qquad H_2 = \operatorname{span}\{\partial_\lambda U,\, \partial_{y^1} U, \dots,\, \partial_{y^n} U\}, \tag{3.2.5}$$

where H_i denotes the eigenspace corresponding to α_i.

Proposition 3.2.1 implies that

$$T_U \mathcal{M} = \operatorname{span}\{H_1 \cup H_2\} \tag{3.2.6}$$

and that

$$\alpha_3 = \inf\left\{ \frac{\langle \mathcal{L}_U w, w\rangle}{\langle w, w\rangle} = \frac{\int A_U[\nabla w, \nabla w]}{\int U^{p^*-2} w^2} : \quad w \perp \operatorname{span}\{H_1 \cup H_2\} \right\}. \tag{3.2.7}$$

Here, orthogonality is with respect to the inner product defined by

$$\langle w_1, w_2 \rangle = \int U^{p^*-2}\, w_1\, w_2. \tag{3.2.8}$$

Proof of Proposition 3.2.1. A scaling argument shows that the eigenvalues of $\mathcal{L}_U$ are invariant under changes of λ and y, so it suffices to consider the operator $\mathcal{L} = \mathcal{L}_U$ for $U = U_{0,1}$. We let $A = A_{U_{0,1}}$. The discreteness of the spectrum of $\mathcal{L}_U$ is standard after establishing the right compact embedding theorem; we show the compact embedding in Corollary 3.5.2 and give details confirming the discrete spectrum in Corollary 3.5.3.

One easily verifies that U is an eigenfunction of $\mathcal{L}$ with eigenvalue $(p-1)S^p$ and that $\partial_\lambda U$ and $\partial_{y^i} U$ are eigenfunctions with eigenvalue $(p^*-1)S^p$, using (3.2.1) and (3.2.2) repectively. Furthermore, $U > 0$, so $\alpha_1 = (p-1)S^p$ is the *first* eigenvalue, which is simple, so (3.2.4) holds.

To prove (3.2.5), we must show that $\alpha_2 = (p^*-1)S^p$ is the *second* eigenvalue and verify that there are no other eigenfunctions in H_2. Both of these facts follow from separation of variables and Sturm-Liouville theory. Indeed, an eigenfunction φ of $\mathcal{L}$ satisfies

$$\operatorname{div}(A(x)\nabla\varphi) + \alpha U^{p^*-2}\varphi = 0. \tag{3.2.9}$$

Assume that φ takes the form $\varphi(x) = Y(\theta)f(r)$, where $Y : \mathbb{S}^{n-1} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. In polar coordinates,

$$\begin{aligned} \operatorname{div}(A(x)\nabla\varphi) = (p-1)|\nabla U|^{p-2}\partial_{rr}\varphi + \frac{(p-1)(n-1)}{r}|\nabla U|^{p-2}\partial_r\varphi \\ + \frac{1}{r^2}|\nabla U|^{p-2}\sum_{j=1}^{n-1}\partial_{\theta_j\theta_j}\varphi + (p-1)(p-2)|\nabla U|^{p-4}\partial_r U\,\partial_{rr}U\,\partial_r\varphi \end{aligned} \tag{3.2.10}$$

(this computation is given in Appendix B for the convenience of the reader). As U is radially symmetric, that is, $U(x) = w(|x|)$, we introduce the slight abuse of notation by letting $U(r)$ also denote the radial component: $U(r) = w(r)$, so $U'(r) = \partial_r U$ and $U''(r) = \partial_{rr}U$. From (3.2.10), we see that (3.2.9) takes the form

$$\begin{aligned} 0 = (p-1)|U'|^{p-2}f''(r)Y(\theta) + \frac{(p-1)(n-1)}{r}|U'|^{p-2}f'(r)Y(\theta) \\ + \frac{1}{r^2}|U'|^{p-2}f(r)\Delta_{\mathbb{S}^{n-1}}Y(\theta) + (p-1)(p-2)|U'|^{p-4}U'U''f'(r)Y(\theta) + \alpha U^{p^*-2}f(r)Y(\theta), \end{aligned}$$

which yields the system

$$0 = \Delta_{\mathbb{S}^{n-1}}Y(\theta) + \mu Y(\theta) \qquad \text{on } \mathbb{S}^{n-1}, \tag{3.2.11}$$

$$\begin{aligned} 0 = (p-1)|U'|^{p-2}f'' + \frac{(p-1)(n-1)}{r}|U'|^{p-2}f' - \frac{\mu}{r^2}|U'|^{p-2}f \\ +(p-1)(p-2)|U'|^{p-4}U'U''f' + \alpha U^{p^*-2}f \end{aligned} \quad \text{on } [0,\infty). \quad (3.2.12)$$

The eigenvalues and eigenfunctions of (3.2.11) are explicitly known; these are the spherical harmonics. The first two eigenvalues are $\mu_1 = 0$ and $\mu_2 = n-1$.

Taking $\mu = \mu_1 = 0$ in (3.2.12), we claim that:

- $\alpha_1^1 = (p-1)S^p$ and the corresponding eigenspace is span $\{U\}$;
- $\alpha_1^2 = (p^*-1)S^p$ with the corresponding eigenspace span $\{\partial_\lambda U\}$.

Indeed, Sturm-Liouville theory ensures that each eigenspace is one-dimensional, and that the ith eigenfunction has $i-1$ interior zeros. Hence, since U (resp. $\partial_\lambda U$) solves (3.2.12) with $\mu = 0$ and $\alpha = (p-1)S^p$ (resp. $\alpha = (p^*-1)S^p$), having no zeros (resp. one zero) it must be the first (resp. second) eigenfunction.

For $\mu_2 = n-1$, the eigenspace for (3.2.11) is n dimensional with n eigenfunctions giving the spherical components of $\partial_{y^i}U$, for $i = 1,\dots,n$. The corresponding equation in (3.2.12) gives $\alpha_2^1 = (p^*-1)S^p$. As the first eigenvalue of (3.2.12) with $\mu = \mu_2$, α_2^1 is simple.

The eigenvalues are strictly increasing, so this shows that $\alpha_1^3 > (p^*-1)S^p$ and $\alpha_2^2 > (p^*-1)S^p$, concluding the proof. □

The application of Sturm-Liouville theory in the proof above is not immediately justified because ours is a singular Sturm-Liouville problem. The proof of Sturm-Liouville theory in our setting, that is, that each eigenspace is one-dimensional and that the ith eigenfunction has $i-1$ interior zeros, is shown in Section 3.5.

3.2.2 Some useful inequalities

The following lemma contains four elementary inequalities that will yield bounds on the deficit in lieu of a Taylor expansion, allowing us to circumvent the issues with higher order terms presented in the chapter overview.

Lemma 3.2.2. *Let $x, y \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$. The following inequalities hold.*

For all $\kappa > 0$, there exists a constant $\mathbf{C} = \mathbf{C}(p, n, \kappa)$ such that

$$\begin{aligned} |x+y|^p \geq & |x|^p + p|x|^{p-2}x \cdot y \\ & + (1-\kappa)\Big(\frac{p}{2}\,|x|^{p-2}|y|^2 + \frac{p(p-2)}{2}\,|x|^{p-4}(x\cdot y)^2\Big) - \mathbf{C}|y|^p. \end{aligned} \tag{3.2.13}$$

For all $\kappa > 0$, there exists $\mathbf{C} = \mathbf{C}(p, \kappa)$ such that

$$|a+b|^{p^*} \leq |a|^{p^*} + p^*|a|^{p^*-2}ab + \Big(\frac{p^*(p^*-1)}{2} + \kappa\Big)|a|^{p^*-2}|b|^2 + \mathbf{C}|b|^{p^*}. \tag{3.2.14}$$

There exists $\mathbf{C} = \mathbf{C}(p, n)$ such that

$$|x+y|^p \geq |x|^p + p|x|^{p-2}x \cdot y - \mathbf{C}|x|^{p-2}|y|^2 + \frac{|y|^p}{2}. \tag{3.2.15}$$

There exists $\mathbf{C} = \mathbf{C}(p)$ such that

$$|a+b|^{p^*} \leq |a|^{p^*} + p^*|a|^{p^*-2}ab + \mathbf{C}|a|^{p^*-2}|b|^2 + 2|b|^{p^*}. \tag{3.2.16}$$

Proof of Lemma 3.2.2. We only give the proof of (3.2.13), as the proofs of (3.2.14)–(3.2.16) are analogous. Observe that if p is an even integer or p^* is an integer, these inequalities follow (with explicit constants) from a binomial expansion and splitting the intermediate terms between the second order and pth or p^*th order terms using Young's inequality.

Suppose (3.2.13) fails. Then there exists $\kappa > 0$, $\{C_j\} \subset \mathbb{R}$ such that $C_j \to \infty$, and $\{x_j\}$, $\{y_j\} \subset \mathbb{R}^n$ such that

$$\begin{aligned} |x_j + y_j|^p - |x_j|^p &< p\,|x_j|^{p-2} x_j \cdot y_j \\ &\quad + (1-\kappa)\left(\frac{p}{2}|x_j|^{p-2}|y_j|^2 + \frac{p(p-2)}{2}|x_j|^{p-4}(x_j \cdot y_j)^2\right) - C_j |y_j|^p \,. \end{aligned}$$

If $x_j = 0$, we immediately get a contradiction. Otherwise, we divide by $|x_j|^p$ to obtain

$$\frac{|x_j + y_j|^p}{|x_j|^p} - 1 < p\,\frac{x_j \cdot y_j}{|x_j|^2} + (1-\kappa)\frac{p}{2}\left(\frac{|y_j|^2}{|x_j|^2} + (p-2)\frac{(x_j \cdot y_j)^2}{|x_j|^4}\right) - C_j \frac{|y_j|^p}{|x_j|^p}\,. \quad (3.2.17)$$

The left-hand side is bounded below by -1, so in order for (3.2.17) to hold, $|y_j|/|x_j|$ converges to 0 at a sufficiently fast rate. In this case, $|y_j|$ is much smaller that $|x_j|$, so a Taylor expansion reveals that the left-hand side behaves like

$$p\,\frac{x_j \cdot y_j}{|x_j|^2} + \frac{p}{2}\frac{|y_j|^2}{|x_j|^2} + \frac{p(p-2)}{2}\frac{(x_j \cdot y_j)^2}{|x_j|^4} + o\Big(\frac{|y_j|^2}{|x_j|^2}\Big)\,,$$

which is larger than the right-hand side, contradicting (3.2.17). □

With the same proof, one can show (3.2.14) with the opposite sign: For all $\kappa > 0$, there exists $\mathbf{C} = \mathbf{C}(p, \kappa)$ such that

$$|a+b|^{p^*} \geq |a|^{p^*} + p^*|a|^{p^*-2}ab - \Big(\frac{p^*(p^*-1)}{2} + \kappa\Big)|a|^{p^*-2}|b|^2 - \mathbf{C}|b|^{p^*}.$$

Applying this and (3.2.14) to functions U and $U + \psi$ with $\int |U|^{p^*} = \int |U+\psi|^{p^*}$, one obtains

$$\Big|\int |U|^{p^*-2}U\psi\Big| \leq \Big(\frac{p^*(p^*-1)}{2} + \kappa\Big)\int |U|^{p^*-2}|\psi|^2 + \mathbf{C}\int |\psi|^{p^*}. \quad (3.2.18)$$

3.2.3 Orthogonality constraints for $u - U$

Given a function $u \in \dot{W}^{1,p}$ satisfying (3.1.8), suppose the infimum in (3.1.7) is attained at $U = c_0 U_{\lambda_0, y_0}$. Then

$$\int |u|^{p^*} = \int |U|^{p^*} = c_0^{p^*}, \tag{3.2.19}$$

and the energy

$$\mathcal{E}(\lambda, y) = \int A_{c_0 U_{\lambda,y}} [\nabla u - c_0 \nabla U_{\lambda,y}, \nabla u - c_0 \nabla U_{\lambda,y}],$$

has a critical point at (λ_0, y_0):

$$\begin{aligned} 0 &= \partial_\lambda|_{\lambda=\lambda_0} \int A_{c_0 U_{\lambda,y}} [\nabla u - c_0 \nabla U_{\lambda,y}, \nabla u - c_0 \nabla U_{\lambda,y}], \\ 0 &= \partial_{y^i}|_{y^i = y_0^i} \int A_{c_0 U_{\lambda,y}} [\nabla u - c_0 \nabla U_{\lambda,y}, \nabla u - c_0 \nabla U_{\lambda,y}]. \end{aligned} \tag{3.2.20}$$

Let $u = U + \varepsilon\varphi$ with φ scaled such that $\int |\nabla\varphi|^p = 1$. By (3.2.19) and (3.2.18), we have

$$\left|\varepsilon \int |U|^{p^*-2} U\varphi\right| \le \varepsilon^2 \frac{p^* - 1 + \kappa}{2} \int |U|^{p^*-2} |\varphi|^2 + \mathbf{C}\varepsilon^{p^*} \int |\varphi|^{p^*} \tag{3.2.21}$$

for any $\kappa > 0$, with $\mathbf{C} = \mathbf{C}(p, n, \kappa)$. Computing the derivatives in (3.2.20) yields

$$\begin{aligned} \varepsilon \int A_U[\nabla \partial_\lambda U, \nabla\varphi] = \varepsilon^2 C \Big\{ &\int |\nabla\varphi|^2 |\nabla U|^{p-4} \nabla U \cdot \nabla \partial_\lambda U \\ &+ (p-2) \int |\nabla\varphi|^2 |\nabla U|^{p-4} \partial_r U \, \partial_{r\lambda} U \Big\} \end{aligned} \tag{3.2.22}$$

and

$$\begin{aligned} \varepsilon \int A_U[\nabla \partial_{y^i} U, \nabla\varphi] = \varepsilon^2 C \Big\{ &\int |\nabla\varphi|^2 |\nabla U|^{p-4} \nabla U \cdot \nabla \partial_{y^i} U \\ &+ (p-2) \int |\nabla\varphi|^2 |\nabla U|^{p-4} \partial_r U \, \partial_{r y^i} U \\ &+ 2 \int |\nabla U|^{p-2} \partial_r \varphi \nabla\varphi \cdot \partial_{y^i} \hat{r} \Big\}, \end{aligned} \tag{3.2.23}$$

where $\hat{r}$ is as in (3.1.6) and $C = (p-2)/2$. At the same time, multiplying (3.2.2) by $\varepsilon\varphi$ and integrating by parts implies that

$$S^p(p^*-1)\varepsilon \int |U|^{p^*-2}\partial_\lambda U\,\varphi = \varepsilon \int A_U[\nabla\partial_\lambda U, \nabla\varphi]\,,$$
$$S^p(p^*-1)\varepsilon \int |U|^{p^*-2}\partial_{y^i} U\,\varphi = \varepsilon \int A_U[\nabla\partial_{y^i} U, \nabla\varphi]\,.$$

Combining this with (3.2.22) and (3.2.23) and letting $C_1 = (p-2)/2(p^*-1)S^p$, we have the following "almost orthogonality" constraints:

$$\varepsilon \int |U|^{p^*-2}\partial_\lambda U\,\varphi = \varepsilon^2 C_1\Big\{\int |\nabla\varphi|^2|\nabla U|^{p-4}\nabla U\cdot\nabla\partial_\lambda U + (p-2)\int |\nabla\varphi|^2|\nabla U|^{p-4}\partial_r U\,\partial_{r\lambda}U\Big\}\,, \tag{3.2.24}$$

$$\varepsilon \int |U|^{p^*-2}\partial_{y^i} U\,\varphi = \varepsilon^2 C_1\Big\{\int |\nabla\varphi|^2|\nabla U|^{p-4}\nabla U\cdot\nabla\partial_{y^i} U + (p-2)\int |\nabla\varphi|^2|\nabla U|^{p-4}\partial_r U\,\partial_{ry^i}U + 2\int |\nabla U|^{p-2}\partial_r\varphi\nabla\varphi\cdot\partial_{y^i}\hat{r}\Big\}\,. \tag{3.2.25}$$

The conditions (3.2.24), (3.2.25), and (3.2.21) show that φ is "almost orthogonal" to $T_U\mathcal{M}$ with respect to the inner product given in (3.2.8). Indeed, dividing through by ε, the inner product of φ with each basis element of $T_U M$ appears on the left-hand side of (3.2.24), (3.2.25), and (3.2.21), while the right-hand side is $O(\varepsilon)$. As a result of (3.2.6) and φ being almost orthogonal to $T_U\mathcal{M}$, we show that φ satisfies a Poincaré-type inequality (3.3.13), which is an essential point in the proof of Proposition 3.1.6.

Remark 3.2.3. In [BE91], the analogous constraints give orthogonality rather than almost orthogonality; this is easily seen here, as taking $p = 2$ makes the right-hand sides of (3.2.24) and (3.2.25) vanish.

3.3 Two expansions of the deficit and their consequences

We prove Proposition 3.1.6 combining an analysis of the second variation and the inequalities of Lemma 3.2.2. As a consequence (Corollary 3.3.3), we show that, up to removing the assumption (3.1.8), Theorem 3.1.1 holds for the two regimes described in (3.1.12).

To prove Proposition 3.1.6, we will need two facts. First, we want to know that the infimum in (3.1.7) is attained, so that we can express u as $u = U + \varepsilon\varphi$ where $\int |\nabla\varphi|^p = 1$ and φ satisfies (3.2.24), (3.2.25), and (3.2.21). Second, it will be important to know that if δ_0 in (3.1.8) is small enough, then ε is small as well. For this reason, we first prove the following:

Proposition 3.3.1. *The following two claims hold.*

1. *There exists $\delta_0 = \delta_0(n,p) > 0$ such that if*

$$\delta(u) \le \delta_0 \|\nabla u\|_{L^p}^p, \tag{3.3.1}$$

   then the infimum in (3.1.7) is attained. In other words, there exists some $U \in \mathcal{M}$ with $\int |U|^{p^*} = \int |u|^{p^*}$ such that

$$\int A_U[\nabla u - \nabla U, \nabla u - \nabla U] = \mathrm{d}(u,\mathcal{M})^2.$$

2. *For all $\varepsilon_0 > 0$, there exists $\delta_0 = \delta_0(n,p,\varepsilon_0) > 0$ such that if $u \in \dot{W}^{1,p}$ satisfies (3.3.1), then*

$$\varepsilon := \|\nabla u - \nabla U\|_{L^p} < \varepsilon_0$$

   where $U \in \mathcal{M}$ is a function that attains the infimum in (3.1.7).

Proof. We begin by showing the following fact, which will be used in the proofs of both parts of the proposition: for all $\gamma > 0$, there exists $\delta_0 = \delta_0(n, p, \gamma) > 0$ such that if $\delta(u) \leq \delta_0 \|\nabla u\|_{L^p}^p$, then

$$\inf\{\|\nabla u - \nabla U\|_{L^p} : U \in \mathcal{M}\} \leq \gamma \|\nabla u\|_{L^p}. \tag{3.3.2}$$

Otherwise, for some $\gamma > 0$, there exists a sequence $\{u_k\} \subset \dot{W}^{1,p}$ such that $\|\nabla u_k\|_{L^p} = 1$ and $\delta(u_k) \to 0$ while

$$\inf\{\|\nabla u_k - \nabla U\|_{L^p} : U \in \mathcal{M}\} > \gamma.$$

A concentration compactness argument as in [Lio85, Str84] ensures that there exist sequences $\{\lambda_k\}$ and $\{y_k\}$ such that, up to a subsequence, $\lambda_k^{n/p^*} u_k(\lambda_k(x - y_k))$ converges strongly in $\dot{W}^{1,p}$ to some $\bar{U} \in \mathcal{M}$. Since

$$\gamma < \left\| \nabla u_k - \nabla \left[\lambda_k^{-n/p^*} \bar{U}\left(\frac{\cdot}{\lambda_k} + y_k \right) \right] \right\|_{L^p} = \left\| \nabla \left[\lambda_k^{n/p^*} u_k(\lambda_k(\cdot - y_k)) \right] - \nabla \bar{U} \right\|_{L^p} \to 0$$

this gives a contradiction for k sufficiently large, hence (3.3.2) holds.

Proof of (1). Suppose u satisfies (3.3.1), with δ_0 to be determined in the proof. Up to multiplication by a constant, we may assume that $\|u\|_{L^{p^*}} = 1$. By the claim above, we may take δ_0 small enough so that (3.3.2) holds for γ as small as needed.

The infimum on the left-hand side of (3.3.2) is attained. Indeed, let $\{U_k\}$ be a minimizing sequence with $U_k = c_k U_{\lambda_k, y_k}$. The sequences $\{c_k\}$, $\{\lambda_k\}$, $\{1/\lambda_k\}$, and $\{y_k\}$ are bounded: if $\lambda_k \to \infty$ or $\lambda_k \to 0$, then for k large enough there will be little cancellation in the term $|\nabla u - \nabla U_k|^p$, so that

$$\int |\nabla u - \nabla U_k|^p \geq \frac{1}{2} \int |\nabla u|^p,$$

contradicting (3.3.2). The analogous argument holds if $|y_k| \to \infty$ or $|c_k| \to \infty$. Thus $\{c_k\}$, $\{\lambda_k\}$, $\{1/\lambda_k\}$, and $\{y_k\}$ are bounded and so, up to a subsequence, $(c_k, \lambda_k, y_k) \to (c_0, \lambda_0, y_0)$ for some $(c_0, \lambda_0, y_0) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^n$. Since the functions $cU_{\lambda,y}$ are smooth, decay nicely, and depend smoothly on the parameters, we deduce that $U_k \to c_0 U_{\lambda_0,y_0} = \tilde{U}$ in $\dot{W}^{1,p}$ (actually, they also converge in C^k for any k), hence $\tilde{U}$ attains the infimum.

To show that the infimum is attained in (3.1.7), we obtain an upper bound on the distance by using $\bar{U} = \tilde{U}/\|\tilde{U}\|_{L^{p^*}}$ as a competitor. Indeed, recalling Remark 3.1.3, it follows from Hölder's inequality that

$$\mathrm{d}(u, \mathcal{M})^2 \leq (p-1) \int \left|\nabla \bar{U}\right|^{p-2} \left|\nabla u - \nabla \bar{U}\right|^2 \leq (p-1) S^{(p-2)/p} \|\nabla u - \nabla \bar{U}\|_{L^p}^{2/p}.$$

Notice that, since $\|u\|_{L^{p^*}} = 1$, it follows by (3.3.1) that $\|\nabla u\|_{L^p} \leq 2S^p$ provided $\delta_0 \leq 1/2$. Hence, since

$$\left|\|\bar{U}\|_{L^{p^*}} - 1\right| \leq \|\bar{U} - u\|_{L^{p^*}} \leq S^{-p} \|\nabla \bar{U} - \nabla u\|_{L^p},$$

it follows by (3.3.2) and the triangle inequality that $\|\nabla u - \nabla \bar{U}\|_{L^p} \leq C(n,p)\,\gamma$, therefore

$$\mathrm{d}(u, \mathcal{M})^2 \leq C(n,p)\,\gamma^{2/p}. \tag{3.3.3}$$

Hence, if $\{U_k\}$ is a minimizing sequence for (3.1.7) with $U_k = U_{\lambda_k,y_k}$ (so that $\int |U_k|^{p^*} = \int |u|^{p^*} = 1$), the analogous argument as above shows that if either of the sequences $\{\lambda_k\}$, $\{1/\lambda_k\}$, or $\{y_k\}$ are unbounded, then

$$\mathrm{d}(u, \mathcal{M})^2 \geq \frac{1}{2},$$

contradicting (3.3.3) for γ sufficiently small. This implies that $U_k \to U_{\lambda_0,y_0}$ in $\dot{W}^{1,p}$, and by continuity U_{λ_0,y_0} attains the infimum in (3.1.7).

Proof of (2). We have shown that (3.3.2) holds for δ_0 sufficiently small. Therefore, we need only to show that, up to further decreasing δ_0, there exists $C = C(p,n)$ such that

$$\|\nabla u - \nabla U_0\|_{L^p} \leq C \inf\{\|\nabla u - \nabla U\|_{L^p} : U \in \mathcal{M}\},$$

where $U_0 \in \mathcal{M}$ is the function where the infimum is attained in (3.1.7).

Suppose for the sake of contradiction that there exists a sequence $\{u_j\}$ such that $\delta(u_j) \to 0$ and $\|\nabla u_j\|_{L^p} = 1$ but

$$\int |\nabla u_j - \nabla U_j|^p \geq j \int |\nabla u_j - \nabla \bar{U}_j|^p, \tag{3.3.4}$$

where $U_j, \bar{U}_j \in \mathcal{M}$ are such that

$$\int A_{U_j}[\nabla u_j - \nabla U_j, \nabla u_j - \nabla U_j] = \mathrm{d}(u_j, \mathcal{M})^2$$

and

$$\int |\nabla u_j - \nabla \bar{U}_j|^p = \inf\Big\{ \int |\nabla u_j - \nabla U_j|^p \, : \, U \in \mathcal{M}\Big\}.$$

Since $\delta(u_j) \to 0$, the same concentration compactness argument as above implies that there exist sequences $\{\lambda_j\}$ and $\{y_j\}$ such that, up to a subsequence, $\lambda_j^{n/p^*} u_j(\lambda_j(x-y_j))$ converges in $\dot{W}^{1,p}$ to some $U \in \mathcal{M}$ with $\|\nabla U\|_{L^p} = 1$. By an argument analogous to that in part (1), we determine that $U_j \to U$ in C^k and $\bar{U}_j \to U$ in C^k for any k. Let

$$\phi_j = \frac{u_j - U_j}{\|\nabla u_j - \nabla U_j\|_{L^p}} \qquad \text{and} \qquad \bar{\phi}_j = \frac{u_j - \bar{U}_j}{\|\nabla u_j - \nabla U_j\|_{L^p}}.$$

Then (3.3.4) implies that

$$1 = \int |\nabla \phi_j|^p \geq j \int |\nabla \bar{\phi}_j|^p. \tag{3.3.5}$$

In particular, $\nabla \bar{\phi}_j \to 0$ in L^p. Now define

$$\psi_j = \phi_j - \bar{\phi}_j = \frac{\bar{U}_j - U_j}{\|\nabla u_j - \nabla U_j\|_{L^p}}.$$

For any $\eta > 0$, (3.3.5) implies that $1 - \eta \leq \|\nabla \psi_j\|_{L^p} \leq 1 + \eta$ for j large enough. In particular, $\{\nabla \psi_j\}$ is bounded in L^p and so $\nabla \psi_j \rightharpoonup \nabla \psi$ in L^p for some $\psi \in \dot{W}^{1,p}$.

We now consider the finite dimensional manifold $\bar{\mathcal{M}} := \{U - \bar{U} : U, \bar{U} \in \mathcal{M}\}$. Since $U_j, \bar{U}_j \to U$, the sequences $\{\lambda_j\}, \{1/\lambda_j\}, \{y_j\}, \{\bar{\lambda}_j\}, \{1/\bar{\lambda}_j\}$ and $\{\bar{y}_j\}$ are contained in some compact set, and thus all norms of $\bar{U}_j - U_j$ are equivalent: for any norm $|||\cdot|||$ on $\bar{\mathcal{M}}$ there exists $\mu > 0$ such that

$$\mu \|\nabla \bar{U}_j - \nabla U_j\|_{L^p} \leq \left|\left|\left|\nabla \bar{U}_j - \nabla U_j\right|\right|\right| \leq \frac{1}{\mu} \|\nabla \bar{U}_j - \nabla U_j\|_{L^p}. \tag{3.3.6}$$

Dividing (3.3.6) by $\|\nabla u_j - \nabla U_j\|_{L^p}$ gives

$$\mu(1 - \eta) \leq \mu \|\nabla \psi_j\|_{L^p} \leq |||\nabla \psi_j||| \leq \frac{1}{\mu} \|\nabla \psi_i\|_{L^p} \leq \frac{1 + \eta}{\mu}. \tag{3.3.7}$$

Taking the norm $|||\cdot||| = \|\cdot\|_{C^k}$, the upper bound in (3.3.7) and the Arzelà-Ascoli theorem imply that ψ_j converges, up to a subsequence, to ψ in C^k. The lower bound in (3.3.7) implies that $\|\psi\|_{C^k} \neq 0$.

To get a contradiction, we use the minimality of U_j for $\mathrm{d}(u_j, \mathcal{M})$ to obtain

$$\begin{aligned}
&\int |\nabla \bar{U}_j|^{p-2}|\nabla \bar{\phi}_j|^2 + (p-2)\int |\nabla \bar{U}_j|^{p-2}|\partial_r \bar{\phi}_j|^2 \\
&\quad \geq \int |\nabla U_j|^{p-2}|\nabla \phi_j|^2 + (p-2)\int |\nabla U_j|^{p-2}|\partial_r \phi_j|^2 \\
&\quad = \int |\nabla U_j|^{p-2}|\nabla \bar{\phi}_j|^2 + 2\int |\nabla U_j|^{p-2}\nabla \bar{\phi}_j \cdot \nabla \psi_j + \int |\nabla U_j|^{p-2}|\nabla \psi_j|^2 \\
&\quad + (p-2)\left(\int |\nabla U_j|^{p-2}|\partial_r \bar{\phi}_j|^2 + 2\int |\nabla U_j|^{p-2}\partial_r \bar{\phi}_j \partial_r \psi_j + \int |\nabla U_j|^{p-2}|\partial \psi_j|^2\right).
\end{aligned}$$

Since

$$\int |\nabla \bar{U}_j|^{p-2}|\nabla \bar{\phi}_j|^2 - \int |\nabla U_j|^{p-2}|\nabla \bar{\phi}_j|^2 \to 0$$

and

$$\int |\nabla \bar{U}_j|^{p-2}|\partial_r \bar{\phi}_j|^2 - \int |\nabla U_j|^{p-2}|\partial_r \bar{\phi}_j|^2 \to 0,$$

the above inequality implies that

$$\begin{aligned}
0 &\geq 2\lim_{j\to\infty}\int |\nabla U_j|^{p-2}\nabla \bar{\phi}_j \cdot \nabla \psi_j + \lim_{j\to\infty}\int |\nabla U_j|^{p-2}|\nabla \psi_j|^2 \\
&\quad + (p-2)\left(2\lim_{j\to\infty}\int |\nabla U_j|^{p-2}\partial_r \bar{\phi}_j \partial_r \psi_j + \lim_{j\to\infty}\int |\nabla U_j|^{p-2}|\partial \psi_j|^2\right).
\end{aligned} \tag{3.3.8}$$

However, since $\nabla \bar{\phi}_j \to 0$ in L^p,

$$\lim_{j\to\infty}\int |\nabla U_j|^{p-2}\nabla \bar{\phi}_j \cdot \nabla \psi_j = 0 \quad \text{and} \quad \lim_{j\to\infty}\int |\nabla U_j|^{p-2}\partial_r \bar{\phi}_j \partial_r \psi_j = 0.$$

In addition, the terms

$$\int |\nabla U_j|^{p-2}|\nabla \psi_j|^2 \quad \text{and} \int |\nabla U_j|^{p-2}|\partial_r \psi_j|^2$$

converge to something strictly positive, as $\psi_j \to \psi \not\equiv 0$ and $U_j \to U$ with $\nabla U(x) \neq 0$ for all $x \neq 0$. This contradicts (3.3.8) and concludes the proof. □

The following Poincaré inequality will be used in the proof of Proposition 3.1.6:

Lemma 3.3.2. *There exists a constant* $C > 0$ *such that, for all* $\varphi \in \dot{W}^{1,p}$ *and* $U \in \mathcal{M}$,

$$\int |U|^{p^*-2}|\varphi|^2 \leq C \int |\nabla U|^{p-2}|\nabla\varphi|^2 \,. \tag{3.3.9}$$

Proof. Let $U \in \mathcal{M}$ and $\varphi \in C_0^\infty$. As U is a local minimum of the functional δ,

$$\begin{aligned} 0 \leq \frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \delta(U+\varepsilon\varphi) = p\int |\nabla U|^{p-2}|\nabla\varphi|^2 + p(p-2)\int |\nabla U|^{p-2}|\partial_r\varphi|^2 \\ - S^p p\Big(\Big(\frac{p}{p^*}-1\Big)\Big(\int |U|^{p^*}\Big)^{\frac{p}{p^*}-2}\Big(\int |U|^{p^*-2}U\,\varphi\Big)^2 \\ + (p^*-1)\Big(\int |U|^{p^*}\Big)^{\frac{p^*}{p}-1}\int |U|^{p^*-2}\varphi^2\Big). \end{aligned}$$

Noting that

$$\int |\nabla U|^{p-2}|\partial_r\varphi|^2 \leq \int |\nabla U|^{p-2}|\nabla\varphi|^2 \,, \qquad \Big(\int |U|^{p^*}\Big)^{p/p^*-2}\Big(\int |U|^{p^*-2}U\,\varphi\Big)^2 \geq 0,$$

this implies that

$$0 \leq p(p-1)\int |\nabla U|^{p-2}|\nabla\varphi|^2 - S^p p(p^*-1)\Big(\int |U|^{p^*}\Big)^{p^*/p-1}\int |U|^{p^*-2}\varphi^2.$$

Thus (3.3.9) holds for $\varphi \in C_0^\infty$, and for $\varphi \in \dot{W}^{1,p}$ by approximation. □

We now prove Proposition 3.1.6.

Proof of Proposition 3.1.6. First of all, thanks to (3.1.8), we can apply Proposition 3.3.1(1) to ensure that some $U = c_0 U_{\lambda_0,y_0} \in \mathcal{M}$ attains the infimum in (3.1.7). Expressing u as $u = U + \varepsilon\varphi$ where $\int |\nabla\varphi|^p = 1$, it follows from Proposition 3.3.1(2) and the discussion in Section 3.2.3 that ε can be assumed to be as small as desired (provided δ_0 is chosen small enough) and that φ satisfies (3.2.24), (3.2.25), and (3.2.21).

Note that, since all terms in (3.1.10) and (3.1.11) are p-homogeneous, without loss of generality we may take $c_0 = 1$.

Proof of (3.1.10). The inequalities (3.2.13) and (3.2.14) are used to expand the gradient term and the function term in $\delta(u)$ respectively.

From (3.2.13) and for $\kappa = \kappa(p, n) > 0$ to be chosen at the end of the proof, we have

$$\begin{aligned}\int |\nabla u|^p \geq \int |\nabla U|^p + \varepsilon p \int |\nabla U|^{p-2}\nabla U \cdot \nabla \varphi \qquad (3.3.10)\\ + \frac{\varepsilon^2 p(1-\kappa)}{2}\Big(\int |\nabla U|^{p-2}|\nabla\varphi|^2 + (p-2)\int |\nabla U|^{p-2}|\partial_r \varphi|^2\Big) - \varepsilon^p\, \mathbf{C}\int |\nabla \varphi|^p.\end{aligned}$$

Note that the second order term is precisely $\frac{\varepsilon^2 p}{2}(1-\kappa)\int A_U[\nabla\varphi, \nabla\varphi]$. Similarly, (3.2.14) gives

$$\int |u|^{p^*} \leq 1 + \varepsilon p^* \int U^{p^*-1}\varphi + \varepsilon^2\Big(\frac{p^*(p^*-1)}{2} + \frac{p^*\kappa}{2S^p}\Big)\int U^{p^*-2}\varphi^2 + \mathbf{C}\varepsilon^{p^*}\int |\varphi|^{p^*}. \qquad (3.3.11)$$

From the identity (3.2.1), the first order term in (3.3.11) is equal to

$$\varepsilon p^* \int U^{p^*-1}\varphi = \varepsilon p^* S^{-p}\int |\nabla U|^{p-2}\nabla U \cdot \nabla\varphi. \qquad (3.3.12)$$

Using (3.3.12) and recalling that $(p^*-1)S^p = \alpha_2$ (see (3.2.5)), (3.3.11) becomes

$$\int |u|^{p^*} \leq 1 + \frac{\varepsilon p^*}{S^p}\int |\nabla U|^{p-2}\nabla U\cdot\nabla\varphi + \frac{\varepsilon^2 p^*(\alpha_2+\kappa)}{2S^p}\int U^{p^*-2}\varphi^2 + \mathbf{C}\varepsilon^{p^*},$$

The following estimate holds, and is shown below:

$$\varepsilon^2\int U^{p^*-2}\varphi^2 \leq (1+2\kappa)\frac{\varepsilon^2}{\alpha_3}\int A_U[\nabla\varphi,\nabla\varphi] + \mathbf{C}\varepsilon^p, \qquad (3.3.13)$$

Philosophically, (3.3.13) follows from a spectral gap analysis, using (3.2.7) and the fact that (3.2.24), (3.2.25), and (3.2.21) imply that φ is "almost orthogonal" to H_1 and H_2.

As ε may be taken as small as needed, using (3.3.13) we have

$$\int |u|^{p^*} \leq 1 + \frac{p^*}{S^p}\left(\varepsilon \int |\nabla U|^{p-2}\nabla U \cdot \nabla\varphi + \frac{\varepsilon^2(\alpha_2+\kappa)(1+2\kappa)}{2\alpha_3}\int A_U[\nabla\varphi,\nabla\varphi] + \mathbf{C}\varepsilon^p\right).$$

The function $z \mapsto |z|^{p/p^*}$ is concave, so $\|u\|_{L^{p^*}}^p \leq 1 + \frac{p}{p^*}(\int |u|^{p^*} - 1)$:

$$S^p\|u\|_{L^{p^*}}^p \leq S^p + p\left(\varepsilon \int |\nabla U|^{p-2}\nabla U \cdot \nabla\varphi + \frac{\varepsilon^2(\alpha_2+\kappa)(1+2\kappa)}{2\alpha_3}\int A_U[\nabla\varphi,\nabla\varphi] + \mathbf{C}\varepsilon^p\right). \tag{3.3.14}$$

Subtracting (3.3.14) from (3.3.10) gives

$$\delta(u) \geq \frac{\varepsilon^2 p}{2}\left(1-\kappa-\frac{(\alpha_2+\kappa)(1+2\kappa)}{\alpha_3}\right)\int A[\nabla\varphi,\nabla\varphi] - \mathbf{C}\varepsilon^p.$$

Since $1-\frac{\alpha_2}{\alpha_3} > 0$, we may choose κ sufficiently small so that $1-\kappa-\frac{(\alpha_2+\kappa)(1+2\kappa)}{\alpha_3} > 0$. To conclude the proof of (3.1.10), we need only to prove (3.3.13).

Proof of (3.3.13). If φ were orthogonal to $T_U\mathcal{M}$ instead of almost orthogonal, that is, if the right-hand sides of (3.2.24), (3.2.25), and (3.2.21) were equal to zero, then (3.3.13) would be an immediate consequence of (3.2.7). Therefore, the proof involves showing that the error in the orthogonality relations is truly higher order, in the sense that it can be absorbed in the other terms.

Up to rescaling u and U, we may assume that $\lambda_0 = 1$ and $y_0 = 0$. We recall the inner product $\langle w, y\rangle$ defined in (3.2.8) which gives rise to the norm

$$\|w\| = \left(\int |U|^{p^*-2}w^2\right)^{1/2}.$$

As in Section 3.2, we let H_i denote the eigenspace of $\mathcal{L}_U$ in $L^2(U^{p^*-2})$ corresponding to eigenvalue α_i, so $H_i = \mathrm{span}\,\{Y_{i,j}\}_{j=1}^{N(i)}$, where $Y_{i,j}$ is an eigenfunction with eigenvalue α_i with $\|Y_{i,j}\| = 1$. We express $\varepsilon\varphi$ in the basis of eigenfunctions:

$$\varepsilon\varphi = \sum_{i=1}^{\infty}\sum_{j=1}^{N(i)} \beta_{i,j} Y_{i,j} \qquad \text{where} \qquad \beta_{i,j} := \varepsilon \int |U|^{p^*-2}\varphi Y_{i,j}.$$

We let $\varepsilon\tilde{\varphi}$ be the truncation of $\varepsilon\varphi$:

$$\varepsilon\tilde{\varphi} = \varepsilon\varphi - \sum_{i=1}^{2}\sum_{j=1}^{N(i)} \beta_{i,j} Y_{i,j},$$

so that $\tilde{\varphi}$ is orthogonal to $\mathrm{span}\,\{H_1 \cup H_2\}$ and, introducing the shorthand $\beta_i^2 := \sum_{j=1}^{N(i)} \beta_{i,j}^2$,

$$\int |U|^{p^*-2}(\varepsilon\varphi)^2 = \int |U|^{p^*-2}(\varepsilon\tilde{\varphi})^2 + \beta_1^2 + \beta_2^2. \tag{3.3.15}$$

Applying (3.2.7) to $\tilde{\varphi}$ implies that

$$\int |U|^{p^*-2}(\varepsilon\tilde{\varphi})^2 \leq \frac{\varepsilon^2}{\alpha_3}\langle \mathcal{L}_U\tilde{\varphi}, \tilde{\varphi}\rangle,$$

which combined with (3.3.15) gives

$$\begin{aligned}\int |U|^{p^*-2}(\varepsilon\varphi)^2 &\leq \frac{\varepsilon^2}{\alpha_3}\langle \mathcal{L}_U\tilde{\varphi}, \tilde{\varphi}\rangle + \beta_1^2 + \beta_2^2 \\ &= \frac{1}{\alpha_3}\sum_{i=3}^{\infty} \alpha_i\beta_i^2 + \beta_1^2 + \beta_2^2 \\ &\leq \frac{\varepsilon^2}{\alpha_3}\langle \mathcal{L}_U\varphi, \varphi\rangle + \Big(1 - \frac{\alpha_1}{\alpha_3}\Big)(\beta_1^2 + \beta_2^2).\end{aligned} \tag{3.3.16}$$

We thus need to estimate $\beta_1^2 + \beta_2^2$. The constraint (3.2.21) implies

$$\beta_1^2 \leq \Big(\varepsilon^2 \, \frac{p^* - 1 + \kappa}{2} \int |U|^{p^*-2}|\varphi|^2 + \mathbf{C}\varepsilon^{p^*} \int |\varphi|^{p^*}\Big)^2$$

$$\leq \mathbf{C}\varepsilon^4\Big(\int |U|^{p^*-2}|\varphi|^2\Big)^2 + \mathbf{C}\varepsilon^{2p^*}\Big(\int |\varphi|^{p^*}\Big)^2.$$

By (3.3.9), $\int |U|^{p^*-2}|\varphi|^2 \leq \int \nabla U|^{p-2}|\nabla\varphi|^2$. Furthermore, both $\int |\nabla U|^{p-2}|\nabla\varphi|^2$ and $\int |\varphi|^{p^*}$ are universally bounded, so for ε sufficiently small depending only on p and n and κ,

$$\beta_1^2 \leq \frac{\kappa\varepsilon^2}{\alpha_3}\Big(\int |\nabla U|^{p-2}|\nabla\varphi|^2 + (p-2)\int |\nabla U|^{p-2}|\partial_r\varphi|^2\Big) + \mathbf{C}\varepsilon^p. \tag{3.3.17}$$

For $\beta_{2,1}^2$, we notice that Hölder's inequality and (3.2.24) imply

$$\begin{aligned}\beta_{2,1}^2 &\leq \Big(C_{p,n}\varepsilon^2\int |\nabla U|^{p-3}|\nabla\varphi|^2\frac{|\nabla\partial_\lambda U|}{\|\partial_\lambda U\|}\Big)^2\\ &\leq C_{p,n}\frac{\int |\nabla U|^{p-2}|\nabla\partial_\lambda U|^2}{\|\partial_\lambda U\|^2}\int |\nabla U|^{p-4}|\varepsilon\nabla\varphi|^4 = C_{p,n}\varepsilon^4\int |\nabla U|^{p-4}|\nabla\varphi|^4,\end{aligned} \tag{3.3.18}$$

where the final equality follows because the term $\int |\nabla U|^{p-2}|\nabla\partial_\lambda U|^2/\|\partial_\lambda U\|^2$ is bounded (in fact, it is bounded by α_2). Then, using Young's inequality, we get

$$\beta_{2,1}^2 \leq \frac{\varepsilon^2\kappa}{(n+1)\alpha_3}\Big(\int |\nabla U|^{p-2}|\nabla\varphi|^2 + (p-2)\int |\nabla U|^{p-2}|\partial_r\varphi|^2\Big) + C_{\kappa,p}\varepsilon^p\int |\nabla\varphi|^p.$$

The analogous argument using (3.2.25) implies that

$$\beta_{2,j}^2 \leq C_{p,n}\varepsilon^4\int |\nabla U|^{p-4}|\nabla\varphi|^4 + C_{p,n}\varepsilon^4\left(\int |\nabla U|^{p-2}\partial_r\varphi\nabla\varphi\cdot\frac{\partial_{y^i}\hat{r}}{\|\partial_y U\|}\right)^2. \tag{3.3.19}$$

for $j = 2,\dots,n+1$. For the second term in (3.3.19), Hölder's inequality implies that

$$\left(\int |\nabla U|^{p-2}\partial_r\varphi\nabla\varphi\cdot\frac{\partial_{y^i}\hat{r}}{\|\partial_y U\|}\right)^2 \leq \int |\nabla U|^{p-4}|\nabla\varphi|^4\int |\nabla U|^p\frac{|\partial_{y^i}\hat{r}|^2}{\|\partial_{y^i}U\|^2}.$$

Since

$$\partial_{y^i}\hat{r} = \frac{x^i x}{|x|^3}, \qquad |\partial_{y^i}\hat{r}| \leq \frac{1}{|x|},$$

we find that $\int |\nabla U|^p \frac{|\partial_{y^i}\hat{r}|^2}{\|\partial_{y^i}U\|^2}$ converges, so (3.3.19) implies that

$$\beta_{2,j}^2 \le C_{p,n}\varepsilon^4 \int |\nabla U|^{p-4}|\nabla\varphi|^4.$$

Then using Young's inequality just as in (3.3.18), we find that

$$\beta_{2,j}^2 \le \frac{\varepsilon^2\kappa}{(n+1)\alpha_3}\Big(\int |\nabla U|^{p-2}|\nabla\varphi|^2 + (p-2)\int |\nabla U|^{p-2}|\partial_r\varphi|^2\Big) + C_{\kappa,p}\varepsilon^p \int |\nabla\varphi|^p,$$

and thus

$$\beta_2^2 \le \frac{\varepsilon^2\kappa}{\alpha_3}\Big(\int |\nabla U|^{p-2}|\nabla\varphi|^2 + (p-2)\int |\nabla U|^{p-2}|\partial_r\varphi|^2\Big) + C_{\kappa,p}\varepsilon^p. \tag{3.3.20}$$

Together (3.3.16), (3.3.17), and (3.3.20) imply (3.3.13), as desired.

Proof of (3.1.11). The proof of (3.1.11) is similar to, but simpler than, the proof of (3.1.10), as no spectral gap or analysis of the second variation is needed. The principle of the expansion is the same, but now we use (3.2.15) and (3.2.16) to expand the deficit.

From (3.2.15), we have

$$\int |\nabla u|^p \ge \int |\nabla U|^p + p\varepsilon \int |\nabla U|^{p-2}\nabla U\cdot\nabla\varphi - \mathbf{C}\,\varepsilon^2 \int |\nabla U|^{p-2}|\nabla\varphi|^2 + \frac{\varepsilon^p}{2}\int |\nabla\varphi|^p. \tag{3.3.21}$$

Similarly, (3.2.16) implies

$$\int |u|^{p^*} \le 1 + \varepsilon p^* \int U^{p^*-1}\varphi + \mathbf{C}\,\varepsilon^2 \int U^{p^*-2}\varphi^2 + 2\varepsilon^{p^*}\int |\varphi|^{p^*}. \tag{3.3.22}$$

As before, the identity (3.2.1) implies (3.3.12), so (3.3.22) becomes

$$\int |u|^{p^*} \le 1 + \varepsilon p^* S^{-p} \int |\nabla U|^{p-2}\nabla U\cdot\nabla\varphi + \mathbf{C}\,\varepsilon^2 \int U^{p^*-2}\varphi^2 + 2\varepsilon^{p^*}\int |\varphi|^{p^*}.$$

By the Poincaré inequality (3.3.9),

$$\int |u|^{p^*} \leq 1 + \varepsilon p^* S^{-p} \int |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi + \mathbf{C}\varepsilon^2 \int |\nabla U|^{p-2} |\nabla \varphi|^2 + 2\varepsilon^{p^*}.$$

As in (3.3.14), the concavity of $z \mapsto |z|^{p/p^*}$ yields

$$S^p \|u\|_{L^{p^*}}^p \leq S^p + \varepsilon p \int |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi + \mathbf{C}\varepsilon^2 \int |\nabla U|^{p-2} |\nabla \varphi|^2 + \mathbf{C}\varepsilon^{p^*}. \quad (3.3.23)$$

Subtracting (3.3.23) from (3.3.21) gives

$$\begin{aligned} \delta(u) &\geq -\mathbf{C}\,\varepsilon^2 \int |\nabla U|^{p-2} |\nabla \varphi|^2 + \frac{\varepsilon^p}{2} - \mathbf{C}\varepsilon^{p^*} \\ &\geq -\mathbf{C}\mathrm{d}(u, M)^2 + \frac{\varepsilon^p}{4}. \end{aligned}$$

The final inequality follows from Remark 3.1.3 and once more taking ε is as small as needed. This concludes the proof of (3.1.11). □

Corollary 3.3.3. *Suppose $u \in \dot{W}^{1,p}$ is a function satisfying* (3.1.8) *and $U \in \mathcal{M}$ is a function where the infimum in* (3.1.7) *is attained. There exist constants $\mathbf{C}_*, \mathbf{c}_*$ and c, depending on n and p only, such that if*

$$\mathbf{C}_* \leq \frac{\int A_U[\nabla u - \nabla U, \nabla u - \nabla U]}{\int |\nabla u - \nabla U|^p} \quad \text{or} \quad \mathbf{c}_* \geq \frac{\int A_U[\nabla u - \nabla U, \nabla u - \nabla U]}{\int |\nabla u - \nabla U|^p}, \quad (3.3.24)$$

then

$$c \int |\nabla u - \nabla U|^p \leq \delta(u).$$

Proof. Let $\mathbf{C}_* = \frac{2\mathbf{C}_2}{\mathbf{c}_1}$ and let $\mathbf{c}_* = \frac{1}{8\mathbf{C}_3}$ where $\mathbf{c}_1, \mathbf{C}_2$ and $\mathbf{C}_3$ are as defined in Proposition 3.1.6. First suppose that u satisfies the first condition in (3.3.24). Then in

(3.1.10), we may absorb the term $\mathbf{C}_2 \int |\nabla u - \nabla U|^p$ into the term $\mathbf{c}_1 \mathrm{d}(u, \mathcal{M})^2$, giving us

$$\frac{\mathbf{c}_1}{2} \mathrm{d}(u, \mathcal{M})^2 \leq \delta(u).$$

Given this control, we may bootstrap using (3.1.11) to gain control of the stronger distance:

$$\frac{1}{4} \int |\nabla u - \nabla U|^p \leq \delta(u) + \mathbf{C}_3 \, \mathrm{d}(u, \mathcal{M})^2 \leq \mathbf{C}\delta(u).$$

Similarly, if u satisfies the second condition in (3.3.24), then we may absorb the term $\mathbf{C}_3 \, \mathrm{d}(u, \mathcal{M})^2$ into the term $\frac{1}{4} \int |\nabla u - \nabla U|^p$ in (3.1.11), giving us

$$\frac{1}{8} \int |\nabla u - \nabla U|^p \leq \delta(u).$$

□

3.4 Proof of the main result

Corollary 3.3.3 implies Theorem 3.1.1 for the functions $u \in \dot{W}^{1,p}$ that satisfy (3.1.8) and that lie in one of the two regimes described in (3.1.12). Therefore, to prove Theorem 3.1.1, it remains to understand the case when the terms $\int A_U[\nabla u - \nabla U, \nabla u - \nabla U]$ and $\int |\nabla u - \nabla U|^p$ are comparable and to remove the assumption (3.1.8). The following proposition accomplishes the first.

Proposition 3.4.1. *Let $u \in \dot{W}^{1,p}$ be a function satisfying* (3.1.8), *and let $U \in \mathcal{M}$ be a function where the infimum in* (3.1.7) *is attained. If*

$$\mathbf{c}_* \leq \frac{\int A_U[\nabla u - \nabla U, \nabla u - \nabla U]}{\int |\nabla u - \nabla U|^p} \leq \mathbf{C}_*, \tag{3.4.1}$$

where $\mathbf{c}_*$ *and* $\mathbf{C}_*$ *are the constants from the Corollary 3.3.3, then*

$$\int |\nabla u - \nabla U|^p \le C\delta(u) + C\|U\|_{L^{p^*}}^{p-1}\|u - U\|_{L^{p^*}} \tag{3.4.2}$$

for a constant C *depending only on* p *and* n*.*

Proof. Suppose u lies in the regime (3.4.1). Then we consider the linear interpolation $u_t := tu + (1-t)U$ and notice that

$$\frac{\int A_U[\nabla u_t - \nabla U, \nabla u_t - \nabla U]}{\int |\nabla u_t - \nabla U|^p} = \frac{t^2 \int A_U[\nabla u - \nabla U, \nabla u - \nabla U]}{t^p \int |\nabla u - \nabla U|^p} \ge t^{2-p}\mathbf{c}_*.$$

Hence, there exists t_* sufficiently small, depending only on p and n, such that $t_*^{2-p}\mathbf{c}_* > \mathbf{C}_*$.

We claim that we may apply Corollary 3.3.3 to u_{t_*}. This is not immediate because U may not attain the infimum in (3.1.7) for u_{t_*}. However, each step of the proof holds if we expand u_{t_*} around U. Indeed, keeping the previous notation of $u - U = \varepsilon\varphi$ with $\int |\nabla\varphi|^p = 1$, we have $u_{t_*} - U = t_*\varepsilon\varphi$. so the orthogonality constraints in (3.2.24), (3.2.25), and (3.2.21) still hold for u_{t_*} and U by simply multiplying through by t_* (this changes the constants by a factor of t_* but this does not affect the proof). Furthermore, (3.1.8) is used in the proofs of Proposition 3.1.6 and (3.3.13) to ensure that ε is a small as needed to absorb terms. Since $t_* < 1$, if ε is sufficiently small then so is $t_*\varepsilon$. With these two things in mind, every step in the proof of Proposition 3.1.6, and therefore Corollary 3.3.3 goes through for u_{t_*}.

Corollary 3.3.3 then implies that

$$t_*^p \int |\nabla u - \nabla U|^p = \int |\nabla u_{t_*} - \nabla U|^p \le C\delta(u_{t_*}).$$

Therefore, (3.4.2) follows if we can show

$$\delta(u_{t_*}) \leq C\delta(u) + C\|U\|_{L^{p^*}}^{p-1}\|u-U\|_{L^{p^*}}. \tag{3.4.3}$$

In the direction of (3.4.3), by convexity and recalling that $\|\nabla U\|_{L^p} = S\|U\|_{L^{p^*}} = S\|u\|_{L^{p^*}}$, we have

$$\begin{aligned}
\delta(u_{t_*}) &= \int |t_*\nabla u + (1-t_*)\nabla U|^p - S^p\,\|t_* u + (1-t_*)U\|_{L^{p^*}}^p \\
&\leq t_* \int |\nabla u|^p + (1-t_*)\int |\nabla U|^p - S^p\|t_* u + (1-t_*)U\|_{L^{p^*}}^p \\
&= t_*\,\delta(u) + S^p\left(\|U\|_{L^{p^*}}^p - \|t_* u + (1-t_*)U\|_{L^{p^*}}^p\right).
\end{aligned} \tag{3.4.4}$$

Also, by the triangle inequality,

$$\|t_*(u-U) + U\|_{L^{p^*}}^p \geq (\|U\|_{L^{p^*}} - \|t_*(u-U)\|_{L^{p^*}})^p,$$

and by the convexity of the function $f(z) = |z|^p$, $f(z+y) \geq f(z) + f'(z)y$, and so

$$(\|U\|_{L^{p^*}} - \|t_*(u-U)\|_{L^{p^*}})^p \geq \|U\|_{L^{p^*}} - p\|U\|_{L^{p^*}}^{p-1}\|u-U\|_{L^{p^*}}.$$

These two inequalities imply that

$$\|U\|_{L^{p^*}}^p - \|t_* u + (1-t_*)U\|_{L^{p^*}}^p \leq p\|U\|_{L^{p^*}}^{p-1}\|u-U\|_{L^{p^*}}.$$

Combining this with (3.4.4) yields (3.4.3), concluding the proof. □

From here, the proof of Theorem 3.1.1 follows easily:

Proof of Theorem 3.1.1. Together, Corollary 3.3.3 and Proposition 3.4.1 imply the following: there exists some constant C such that if $u \in \dot{W}^{1,p}$ satisfies (3.1.8), then there is some $U \in \mathcal{M}$ such that

$$\int |\nabla u - \nabla U|^p \leq C\delta(u) + C\|U\|_{L^{p^*}}^{p^*-1}\|u-U\|_{L^{p^*}}.$$

Therefore, we need only to remove the assumption (3.1.8) in order to complete the proof of Theorem 3.1.1. However, in the case where (3.1.8) fails, then trivially,

$$\inf\{\|\nabla u - \nabla U\|_{L^p}^p : U \in \mathcal{M}\} \leq \|\nabla u\|_{L^p}^p \leq \frac{1}{\delta_0}\delta(u).$$

Choosing the constant to be sufficiently large, Theorem 3.1.1 is proven. $\square$

We now prove Corollary 3.1.2 using the main result from [CFMP09], which we recall here:

Theorem 3.4.2 (Cianchi, Fusco, Maggi, Pratelli, [CFMP09])**.** *There exists C such that*

$$\lambda(u)^{\zeta'}\|u\|_{L^{p^*}} \leq C(\|\nabla u\|_{L^p} - S\|u\|_{L^{p^*}}), \tag{3.4.5}$$

where $\lambda(u) = \inf\left\{\|u - U\|_{L^{p^*}}^{p^*}/\|u\|_{L^{p^*}}^{p^*} : U \in \mathcal{M}, \int |U|^{p^*} = \int |u|^{p^*}\right\}$ *and* $\zeta' = p^*\left(3 + 4p - \frac{3p+1}{n}\right)^2$.

Proof of Corollary 3.1.2. As before, if (3.1.8) does not hold, then Corollary 3.1.2 holds trivially by simply choosing the constant to be sufficiently large. Now suppose $u \in \dot{W}^{1,p}$ satisfies (3.1.8). There are two obstructions to an immediate application of Theorem 3.4.2. The first is the fact that the deficit in (3.4.5) is defined as $\|\nabla u\|_{L^p} - S\|u\|_{L^{p^*}}$, while in our setting it is defined as $\|\nabla u\|_{L^p}^p - S^p\|u\|_{L^{p^*}}^p$. However, this is easy to fix. Indeed, using the elementary inequality

$$a^p - b^p \geq a - b \qquad \forall\ a \geq b \geq 1,$$

we let $a = \|\nabla u\|_{L^p}/S\|u\|_{L^{p^*}}$ and $b = 1$ to get

$$\frac{\|\nabla u\|_{L^p} - S\|u\|_{L^{p^*}}}{S\|u\|_{L^{p^*}}} \leq \frac{\|\nabla u\|_{L^p}^p - S^p\|u\|_{L^{p^*}}^p}{S^p\|u\|_{L^{p^*}}^p} \leq \frac{1}{1-\delta_0}\frac{\|\nabla u\|_{L^p}^p - S^p\|u\|_{L^{p^*}}^p}{\|\nabla u\|_{L^p}^p},$$

where the last inequality follows from (3.1.8). Therefore, up to increasing the constant, (3.4.5) implies that

$$\lambda(u)^{\zeta'} \leq C\frac{\delta(u)}{\|\nabla u\|_{L^p}^p}. \tag{3.4.6}$$

The second obstruction to applying Theorem 3.4.2 is the fact that (3.4.5) holds for the *infimum* in $\lambda(u)$, while we must control $\|u - U\|_{L^{p^*}}$ for U attaining the infimum in (3.1.7). To solve this issue it is sufficient to show that there exists some constant $C = C(n,p)$ such that

$$\int |\bar{U} - u|^{p^*} \leq C \inf\left\{\|u - U\|_{L^{p^*}}^{p^*} : U \in \mathcal{M},\ \int |U|^{p^*} = \int |u|^{p^*}\right\}$$

where $\bar{U}$ attains the infimum in (3.1.7). The proof of this fact is nearly identical (with the obvious adaptations) to that of part (2) of Proposition 3.3.1, with the only nontrivial difference being that one must integrate by parts to show that the analogue of first term in (3.3.8) goes to zero.

Therefore, (3.4.5) implies

$$\left(\frac{\|u - U\|_{L^{p^*}}}{\|u\|_{L^{p^*}}}\right)^{\zeta'} \leq C\frac{\delta(u)}{\|\nabla u\|_{L^p}}$$

where $U \in \mathcal{M}$ attains the infimum in (3.1.7). Paired with Theorem 3.1.1, this proves Corollary 3.1.2 with $\zeta = \zeta' p$. □

3.5 Spectral Properties of $\mathcal{L}_U$

In this section, we give the proofs of the compact embedding theorem and Sturm-Liouville theory that were postponed in the proof of Proposition 3.2.1. As in Proposition 3.2.1, by scaling, it suffices to consider the operator $\mathcal{L} = \mathcal{L}_U$ where $U = U_{0,1}$.

3.5.1 The discrete spectrum of $\mathcal{L}$

Given two measurable functions $\omega_0, \omega_1 : \Omega \to \mathbb{R}$, let

$$W^{1,2}(\Omega, \omega_0, \omega_1) := \{g : \|g\|_{W^{1,2}(\Omega,\omega_0,\omega_1)} < \infty\},$$

where $\|\cdot\|_{W^{1,2}(\Omega,\omega_0,\omega_1)}$ is the norm defined by

$$\|g\|_{W^{1,2}(\Omega,\omega_0,\omega_1)} = \left(\int_\Omega g^2\omega_0 + \int_\Omega |\nabla g|^2\omega_1\right)^{1/2}. \tag{3.5.1}$$

The space $W_0^{1,2}(\Omega, \omega_0, \omega_1)$ is defined as the completion of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,2}(\Omega,\omega_0,\omega_1)}$. The following compact embedding result was shown in [Opi88]:

Theorem 3.5.1 (Opic, [Opi88]). *Let $Z = W_0^{1,2}(\mathbb{R}^n, \omega_0, \omega_1)$ and suppose*

$$\omega_i \in L^1_{\mathrm{loc}} \quad \text{and} \quad \omega_i^{-1/2} \in L^{2^*}_{\mathrm{loc}}, \tag{3.5.2}$$

$i = 0, 1$. *If there are local compact embeddings*

$$W^{1,2}(B_k, \omega_0, \omega_1) \subset\subset L^2(B_k, \omega_0), \;\; k \in \mathbb{N}, \tag{3.5.3}$$

where $B_k = \{x : |x| < k\}$, and if

$$\lim_{k\to\infty} \sup\left\{\|u\|_{L^2(\mathbb{R}^n\setminus B_k,\omega_0)} : u \in Z, \;\; \|u\|_Z \le 1\right\} = 0, \tag{3.5.4}$$

then Z embeds compactly in $L^2(\mathbb{R}^n, \omega_0)$.

We apply Theorem 3.5.1 to show that the space

$$X = W_0^{1,2}(\mathbb{R}^n, U^{p^*-2}, |\nabla U|^{p-2}), \tag{3.5.5}$$

embeds compactly into $L^2(\mathbb{R}^n, U^{p^*-2})$.

Corollary 3.5.2. *The compact embedding* $X \subset\subset L^2(\mathbb{R}^n, U^{p^*-2})$ *holds, with* X *as in* (3.5.5).

Proof. Let us verify that Theorem 3.5.1 may be applied in our setting, taking

$$\omega_0 = U^{p^*-2}, \qquad \omega_1 = |\nabla U|^{p-2}.$$

In other words, we must show that (3.5.2)–(3.5.4) are satisfied. A simple computation verifies (3.5.2). To show (3.5.3), we fix $\delta > 0$ small (the smallness depending only on n and p) and show the three inclusions below:

$$W^{1,2}(B_r, \omega_0, \omega_1) \overset{(1)}{\subset} W^{1,2(n+\delta)/(n+2)}(B_r) \overset{(2)}{\subset\subset} L^2(B_r) \overset{(3)}{\subset} L^2(B_r, \omega_0).$$

Since $(2n/(2+n))^* = 2$, the Rellich-Kondrachov compact embedding theorem implies (2), while the inclusion (3) holds simply because $U^{p^*-2} \geq c_{n,p,r}$ for $x \in B_r$. In the direction of showing (1), we use this fact and Hölder's inequality to obtain

$$\left(\int_{B_r} |u|^{2(n+\delta)/(n+2)}\right)^{(n+2)/(n+\delta)} \leq |B_r|^{(2-\delta)/(n+\delta)} \int_{B_r} |u|^2 \leq C_{n,p,r} \int_{B_r} |U|^{p^*-2}|u|^2. \tag{3.5.6}$$

Furthermore, since

$$|\nabla U|^{p-2} = C(1+|x|^{p'})^{-n(p-2)/p}|x|^{(p-2)/(p-1)} \geq c_{n,p,r}|x|^{(p-2)/(p-1)} \quad \text{for } x \in B_r,$$

Hölder's inequality implies that

$$\begin{aligned}\left(\int_{B_r} |\nabla u|^{2(n+\delta)/(n+2)}\right)^{(n+2)/(n+\delta)} &\leq \left(\int_{B_r} |x|^{(p-2)/(p-1)}|\nabla u|^2\right)\left(\int_{B_r} |x|^{-\beta}\right)^{(2-\delta)/(n+\delta)} \\ &\leq C_{n,p,r}\int_{B_r} |\nabla U|^{p-2}|\nabla u|^2,\end{aligned} \tag{3.5.7}$$

where $\beta = \left(\frac{p-2}{p-1}\right)\left(\frac{n+\delta}{n+2}\right)\left(\frac{n+2}{2-\delta}\right)$. Then the inclusion (1) follows from (3.5.6) and (3.5.7), and thus (3.5.3) is verified.

To show (3.5.4), let u_k be a function almost attaining the supremum in (3.5.4), in other words, for a fixed $\eta > 0$, let u_k be such that $u_k \in X$, $\|u_k\|_X \le 1$, and

$$\sup\left\{\|u\|_{L^2(\mathbb{R}^n \setminus B_k, \omega_0)} : u \in X,\ \|u\|_X \le 1\right\} \le \|u_k\|_{L^2(\mathbb{R}^n \setminus B_k, \omega_0)} + \eta.$$

By mollifying u and multiplying by a smooth cutoff $\eta \in C_0^\infty(\mathbb{R}^n \setminus B_k)$, we may assume without loss of generality that $u_k \in C_0^\infty(\mathbb{R}^n \setminus B_k)$. Recalling that $U = U_1$ with U_1 as in (3.1.3), we have

$$\int_{\mathbb{R}^n \setminus B_k} U^{p^*-2} u_k^2 = \int_{\mathbb{R}^n \setminus B_k} \kappa_0 (1+|x|^{p'})^{-(p^*-2)(n-p)/p} u_k^2 \le 2\kappa_0 \int_{\mathbb{R}^n \setminus B_k} |x|^{-(p^*-2)(n-p)/(p-1)} u_k^2 \tag{3.5.8}$$

for $k \ge 2$. We use Hardy's inequality in the form

$$\int_{\mathbb{R}^n} |x|^s u^2 \le C \int_{\mathbb{R}^n} |x|^{s+2} |\nabla u|^2 \tag{3.5.9}$$

for $u \in C_0^\infty(\mathbb{R}^n)$ (see, for instance, [Zyg02]). Applying (3.5.9) to the right-hand side of (3.5.8) implies

$$\int_{\mathbb{R}^n \setminus B_k} |x|^{-(p^*-2)(n-p)/(p-1)} u_k^2 \le C \int_{\mathbb{R}^n \setminus B_k} |x|^{-(p^*-2)(n-p)/(p-1)+2} |\nabla u_k|^2 \tag{3.5.10}$$

and (3.5.8) and (3.5.10) combined give

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_k} U^{p^*-2} u_k^2 &\le C \int_{\mathbb{R}^n \setminus B_k} |x|^{-(p^*-2)(n-p)/(p-1)+2} |\nabla u_k|^2 \\ &= C \int_{\mathbb{R}^n \setminus B_k} |x|^{-p'} |x|^{-(p-2)(n-1)/(p-1)} |\nabla u_k|^2 \end{aligned}$$

$$\leq Ck^{-p'} \int_{\mathbb{R}^n \setminus B_k} |\nabla U|^{p-2} |\nabla u_k|^2,$$

where the final inequality follows because

$$|\nabla U|^{p-2} \geq C|x|^{-(p-2)(n-1)/(p-1)} \quad \text{for } x \in \mathbb{R}^n \setminus B_1.$$

Thus

$$\int_{\mathbb{R}^n \setminus B_k} U^{p^*-2} u_k^2 \leq Ck^{-p'} \|u_k\|_X,$$

and (3.5.4) is proven. □

Thanks to the compact embedding $X \subset\subset L^2(\mathbb{R}^n, \omega_0)$, we can now prove the following important fact:

Corollary 3.5.3. *The operator $\mathcal{L}$ has a discrete spectrum $\{\alpha_i\}_{i=1}^{\infty}$.*

Proof. We show that the operator $\mathcal{L}^{-1} : L^2(U^{p^*-2}) \to L^2(U^{p^*-2})$ is bounded, compact, and self-adjoint. From there, one applies the spectral theorem (see for instance [Eva98]) to deduce that $\mathcal{L}^{-1}$ has a discrete spectrum, hence so does $\mathcal{L}$.

Approximating by functions in $C_0^\infty(\mathbb{R}^n)$, the Poincaré inequality (3.3.9) holds for all functions $\varphi \in X$, with X as defined in (3.5.5). Thanks to this fact, the existence and uniqueness of solutions to $\mathcal{L}u = f$ for $f \in L^2(U^{p^*-2})$ follow from the Direct Method, so the operator $\mathcal{L}^{-1}$ is well defined.

Self-adjointness is immediate. From (3.3.9) and Hölder's inequality, we have

$$c\|u\|_X^2 \leq \int |\nabla U|^{p-2} |\nabla u|^2 \leq \int A[\nabla u, \nabla u] \leq \|u\|_X \|\mathcal{L}u\|_{L^2(U^{p^*-2})}.$$

This proves that $\mathcal{L}^{-1}$ is bounded from $L^2(U^{p^*-2})$ to $L^2(U^{p^*-2})$, and by Corollary 3.5.2 we see that $\mathcal{L}^{-1}$ is a compact operator. □

3.5.2 Sturm-Liouville theory

Multiplying by the integrating factor r^{n-1}, the ordinary differential equation (3.2.12) takes the form of the Sturm-Liouville eigenvalue problem

$$Lf + \alpha f = 0 \quad \text{on} \quad [0, \infty), \tag{3.5.11}$$

where

$$Lf = \frac{1}{w}[(Pf')' - Qf]$$

with

$$\begin{aligned} P(r) &= (p-1)|U'|^{p-2} r^{n-1}, \\ Q(r) &= \mu r^{n-3} |U'|^{p-2}, \\ w(r) &= U^{p^*-2} r^{n-1}. \end{aligned} \tag{3.5.12}$$

This is a *singular* Sturm-Liouville problem; first of all, our domain is unbounded, and second of all, the equation is degenerate because $U'(0) = 0$. Nonetheless, we show that Sturm-Liouville theory holds for this singular problem.

Lemma 3.5.4 (Sturm-Liouville Theory)**.** *The following properties hold for the singular Sturm-Liouville eigenvalue problem* (3.5.11)*:*

1. *If f_1 and f_2 are two eigenfunctions corresponding to the eigenvalue α, then $f_1 = cf_2$. In other words, each eigenspace of L is one-dimensional.*

2. *The ith eigenfunction of L has $i - 1$ interior zeros.*

Note that L has a discrete spectrum because $\mathcal{L}$ does (Corollary 3.5.3), and that eigenfunctions f of L live in the space

$$Y = W_0^{1,2}\big([0, \infty), U^{p^*-2} r^{n-1}, |U'|^{p-2} r^{n-1}\big),$$

using the notation introduced at the beginning of Section 3.5.1. In any ball B_R around zero, the operator $\mathcal{L}$ is degenerate elliptic with the matrix A bounded by an A_2-Muckenhoupt weight, so eigenfunctions of $\mathcal{L}$ are Hölder continuous; see [FKS82, Gut89]. Therefore, eigenfunctions of L are Hölder continuous on $[0, \infty)$.

Remark 3.5.5. The function $P(r)$ as defined in (3.5.12) has the following behavior:

$$P(r) \approx r^{(p-2)(p-1)+n-1} \quad \text{in } [0,1],$$
$$P(r) \approx r^{(n-1)/(p-1)} \quad \text{as } r \to \infty.$$

In particular, the weight $|U'|^{p-2}r^{n-1} \approx r^{(n-1)/(p-1)}$ goes to infinity as $r \to \infty$, which implies that $\int_1^\infty |f'|^2 dr < \infty$ for any $f \in Y$.

In order to prove Lemma 3.5.4, we first prove the following lemma, which describes the asymptotic decay of solutions of (3.5.11).

Lemma 3.5.6. *Suppose $f \in Y$ is a solution of* (3.5.11)*. Then, for any $0 < \beta < \frac{n-p}{p-1}$, there exist C and r_0 such that*

$$|f(r)| \leq Cr^{-\beta} \quad \textit{and} \quad |f'(r)| \leq Cr^{-\beta-1}$$

for $r \geq r_0$.

Proof. Step 1: Qualitative Decay of f. For any function $f \in Y$, $f(r) \to 0$ as $r \to \infty$. Indeed, near infinity, $|U'|^{p-2}r^{p-1}$ behaves like Cr^γ where $\gamma := \frac{n-1}{p-1} > 1$. Then for any r, s large enough with $r < s$,

$$|f(r) - f(s)| \leq \int_r^\infty |f'(t)|dt \leq \Big(\int_r^\infty f'(t)^2 t^\gamma dt\Big)^{1/2}\Big(\int_r^\infty t^{-\gamma} dt\Big)^{1/2} \qquad (3.5.13)$$

by Hölder's inequality. As both integrals on the right-hand side of (3.5.13) converge, for any $\varepsilon > 0$, we may take r large enough such that the right-hand side is bounded by ε, so the limit of $f(r)$ as $r \to \infty$ exists.

We claim that this limit must be equal to zero. Indeed, since Y is obtained as a completion of C_0^∞, if we apply (3.5.13) to a sequence $f_k \in C_0^\infty([0,\infty))$ converging in Y to f and we let $s \to \infty$, we get

$$|f_k(r)| \leq \Big(\int_r^\infty f_k'(t)^2 t^\gamma dt\Big)^{1/2}\Big(\int_r^\infty t^{-\gamma} dt\Big)^{1/2},$$

thus, by letting $k \to \infty$,

$$|f(r)| \leq \Big(\int_r^\infty f'(t)^2 t^\gamma dt\Big)^{1/2}\Big(\int_r^\infty t^{-\gamma} dt\Big)^{1/2}.$$

Since the right-hand side tends to zero as $r \to \infty$, this proves the claim.

Step 2: Qualitative Decay of f'. For $r > 0$, (3.5.11) can be written as

$$L'f := f'' + af' + bf = 0 \tag{3.5.14}$$

where

$$a = \frac{P'}{P} \quad \text{and} \quad b = \frac{-Q + w\alpha}{P}.$$

Fixing $\varepsilon > 0$, an explicit computation shows that there exists r_0 large enough such that

$$\frac{(1-\varepsilon)(n-1)}{p-1}\frac{1}{r} \leq a \leq \frac{(1+\varepsilon)(n-1)}{p-1}\frac{1}{r}$$

and

$$-\frac{\mu}{p-1}\frac{1}{r^2} + \frac{(1-\varepsilon)c_{p,n}\alpha}{r^{(3p-2)/(p-1)}} \leq b \leq -\frac{\mu}{p-1}\frac{1}{r^2} + \frac{(1+\varepsilon)c_{p,n}\alpha}{r^{(3p-2)/(p-1)}}$$

for $r \geq r_0$, where $c_{n,p}$ is a positive constant depending only on n and p. Asymptotically, therefore, our equation behaves like

$$f'' + \frac{n-1}{p-1}\frac{f'}{r} + \Big(\frac{c_{p,n}\alpha}{r^{p'}} - \frac{\mu}{p-1}\Big)\frac{f}{r^2} = 0.$$

If f is a solution of (3.5.11), then squaring (3.5.14) on $[r_0, \infty)$, we obtain

$$|f''|^2 \leq 2\left(\Big(\frac{n-1}{p-1} + \varepsilon\Big)\frac{f'}{r}\right)^2 + 2\left(\Big(\frac{(1+\varepsilon)c_{p,n}\alpha}{r^{p'}} + \frac{\mu}{p-1}\Big)\frac{f}{r^2}\right)^2 \leq C(|f|^2 + |f'|^2).$$

Integrating on $[R, R+1]$ for $R \geq r_0$ implies

$$\int_R^{R+1} |f''|^2 \leq C\int_R^{R+1} |f'|^2 + C\int_R^{R+1} |f|^2.$$

Step 1 and Remark 3.5.5 ensure that both terms on the right-hand side go to zero. Applying Morrey's embedding to $f'\eta_R$, where η_R is a smooth cutoff equal to 1 in $[R, R+1]$, we determine that $\|f'\|_{L^\infty([R,R+1])} \to 0$ as $R \to \infty$, proving that $f'(r) \to 0$ as $r \to \infty$.

Step 3: Quantitative Decay of f and f'. Standard arguments (see for instance [CH89, VI.6]) show that, also in our case, the ith eigenfunction f of L has at most $i-1$ interior zeros; in particular, $f(r)$ does not change sign for r sufficiently large. Without loss of generality, we assume that eventually $f \geq 0$.

Taking r_0 as in Step 2 and applying the operator L' defined in (3.5.14) to the function $g = Cr^{-\beta} + c$, $c > 0$, for $r \geq r_0$ gives

$$L'g \leq C\beta(\beta+1)r^{-\beta-2} - \frac{(1-\varepsilon)(n-1)}{p-1}C\beta r^{-\beta-2} + \Big(\frac{(1+\varepsilon)c_{p,n}\alpha}{r^{(3p-2)/(p-1)}} - \frac{\mu}{p-1}\Big)(Cr^{-\beta-2} + c)$$

$$\leq Cr^{-\beta-2}\Big(\beta(\beta+1) - \frac{(1-\varepsilon)(n-1)}{p-1}\beta + \frac{(1+\varepsilon)c_{p,n}\alpha}{r^{p'}}\Big) + \frac{(1+\varepsilon)c_{p,n}\alpha}{r^{(3p-2)/(p-1)}}c.$$

For any $0 < \beta < (n-p)/(p-1)$, r_0 may be taken large enough (and therefore ε small enough) such that

$$L'g < 0 \quad \text{on} \quad [r_0, \infty),$$

so g is a supersolution of the equation on this interval.

Choosing $C = f(r_0)r_0^{\beta}$ and $c > 0$, then $(g-f)(r_0) > 0$ and $(g-f)(r) \to c > 0$ as $r \to \infty$. Since $L'(g-f) < 0$, we claim that $g - f > 0$ on (r_0, ∞). Indeed, otherwise, $g - f$ would have a negative minimum at some $r \in (r_0, \infty)$, implying that

$$(g-f)(r) \leq 0, \quad (g-f)'(r) = 0, \quad \text{and} \quad (g-f)''(r) \geq 0,$$

forcing $L'(g-f) \geq 0$, a contradiction. This proves that $0 \leq f \leq g$ on $[r_0, \infty)$, and since $c > 0$ was arbitrary, we determine that $f \leq Cr^{-\beta}$ on $[r_0, \infty)$.

We now derive bounds on f': by the fundamental theorem of calculus and using (3.5.14) and the bound on f for $r \geq r_0$, we get

$$|f'(r)| = \Big|\int_r^\infty f''\Big| \leq \frac{C}{r}\Big|\int_r^\infty f'\Big| + C\Big|\int_r^\infty t^{-\beta-2}\Big| \leq \frac{C}{r}|f(r)| + \frac{C}{\beta+2}r^{-\beta-1} \leq Cr^{-\beta-1}.$$

□

With these asymptotic decay estimates in hand, we are ready to prove Lemma 3.5.4.

Proof of Lemma 3.5.4. We begin with the following remark about uniqueness of solutions. If f_1 and f_2 are two solutions of (3.5.11) and

$$f_1(r_0) = f_2(r_0), \quad f_1'(r_0) = f_2'(r_0)$$

for some $r_0 > 0$, then $f_1 = f_2$ on $[0, \infty)$. Indeed, for $r > 0$, we may express our equation as in (3.5.14). As a and b are continuous on $(0, \infty)$, the standard proof of uniqueness for (non-degenerate) second order ODE holds. Once $f_1 = f_2$ on $(0, \infty)$, they are also equal at $r = 0$ by continuity.

Proof of (1). Suppose α is an eigenvalue of L with f_1 and f_2 satisfying (3.5.11). In view of the uniqueness remark, if there exists $r_0 > 0$ and some linear combination f of f_1 and f_2 such that $f(r_0) = f'(r_0) = 0$, then f is constantly zero and f_1 and f_2 are linearly dependent. Let

$$W(r) = W(f_1, f_2)(r) := \det \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} (r)$$

denote the Wronskian of f_1 and f_2. This is well defined for $r > 0$ (since f_1 and f_2 are C^2 there) and a standard computation shows that $(PW)' = 0$ on $(0, \infty)$: indeed, since $W' = f_1 f_2'' - f_2 f_1''$, we get

$$(PW)' = PW' + P'W = P(f_1 f_2'' - f_2 f_1'') + P'(f_1 f_2' - f_2 f_1'),$$

and by adding and subtracting the term $(\alpha w - Q) f_1 f_2$ it follows that

$$(PW)' = f_1 \left(P f_2'' + P' f_2' + (\alpha w - Q) f_2\right) - f_2 \left(P f_1'' + P' f_2' + (\alpha w - Q) f_1\right) = 0.$$

Thus PW is constant on $(0,\infty)$. We now show that that PW is continuous up to $r=0$ and that $(PW)(0)=0$. Indeed, (3.5.11) implies that

$$(Pf_i')' = (Q-\alpha w)f_i$$

for $i=1,2$. The right-hand side is continuous, so $(Pf_i')'$ is continuous, from which it follows easily that PW is also continuous on $[0,\infty)$.

To show that $(PW)(0)=0$, we first prove that $(Pf_i')(0)=0$. Indeed, let $c_i := (Pf_i')(0)$. If $c_i \neq 0$, then keeping in mind Remark 3.5.5,

$$f_i'(r) \approx \frac{c_i}{P(r)} \approx \frac{c_i}{r^{(p-2)/(p-1)+n-1}} \qquad \text{for } r \ll 1, \tag{3.5.15}$$

therefore

$$\int_0^R |U'|^{p-2}|f'|^2 r^{n-1} dr \gtrsim \int_0^R r^{(p-2)/(p-1)+n-1}|f'|^2 dr \gtrsim \int_0^R \frac{dr}{r^{(p-2)/(p-1)+n-1}} = +\infty,$$

contradicting the fact that $f \in Y$. Hence, we conclude that $\lim_{r\to 0}(Pf_i')(r) = 0$, and using this fact we obtain

$$(PW)(0) = \lim_{r\to 0}(Pf_1'f_2 - Pf_2'f_1) = \lim_{r\to 0}(Pf_1')\lim_{r\to 0} f_2 - \lim_{r\to 0}(Pf_2')\lim_{r\to 0} f_1 = 0.$$

Therefore $(PW)(r)=0$ for all $r \in [0,\infty)$. Since $P(r)>0$ for $r>0$, we determine that $W(r)=0$ for all $r>0$. In particular, given $r_0 \in (0,\infty)$, there exist c_1, c_2 such that $c_1^2+c_2^2 \neq 0$ and

$$c_1 f_1(r_0) + c_2 f_2(r_0) = 0,$$
$$c_1 f_1'(r_0) + c_2 f_2'(r_0) = 0.$$

Then $f := c_1 f_1 + c_2 f_2$ solves (3.5.11) and $f(r_0) = f'(r_0) = 0$. By uniqueness, $f \equiv 0$ for all $t \in (0, \infty)$, and so $f_1 = cf_2$.

Proof of (2). Thanks to our preliminary estimates on the behavior of f_i at infinity, the following is an adaptation of the standard argument in, for example, [CH89, VI.6].

Suppose that f_1 and f_2 are eigenfunctions of L corresponding to eigenvalues α_1 and α_2 respectively, with $\alpha_1 < \alpha_2$, that is,

$$(Pf_i')' - Qf_i + \alpha_i w f_i = 0.$$

Our first claim is that between any two consecutive zeros of f_1 is a zero of f_2, including zeros at infinity. Note that

$$\begin{aligned}(PW)' &= P[f_1 f_2'' - f_2 f_1''] + P'[f_1 f_2' - f_2 f_1'] \\ &= f_1[(Pf_2')' + (\alpha_2 - Q)f_2] - f_2[(Pf_1')' + (\alpha_1 w - Q)f_1] + (\alpha_1 - \alpha_2)wf_1 f_2 \\ &= (\alpha_1 - \alpha_2)wf_1 f_2. \end{aligned} \tag{3.5.16}$$

Suppose that f_1 has consecutive zeros at r_1 and r_2, and suppose for the sake of contradiction that f_2 has no zeros in the interval (r_1, r_2). With no loss of generality, we may assume that f_1 and f_2 are both nonnegative in $[r_1, r_2]$.

Case 1: Suppose that $r_2 < \infty$. Then integrating (3.5.16) from r_1 to r_2 implies

$$\begin{aligned} 0 &> (\alpha_1 - \alpha_2)\int_{r_1}^{r_2} wf_1 f_2 = (PW)(r_2) - (PW)(r_1) \\ &= P(r_2)[f_1(r_2)f_2'(r_2) - f_1'(r_2)f_2(r_2)] - P(r_1)[f_1(r_1)f_2'(r_1) - f_1'(r_1)f_2(r_1)] \\ &= -P(r_2)f_1'(r_2)f_2(r_2) + P(r_1)f_1'(r_1)f_2(r_1). \end{aligned}$$

The function f_1 is positive on (r_1, r_2), so $f_1'(r_1) \geq 0$ and $f_1'(r_2) \leq 0$. Also, since $f_1(r_1) = f_1(r_2) = 0$ we cannot have $f_1'(r_1) = 0$ or $f_1'(r_2) = 0$, as otherwise f_1 would vanish identically. Furthermore, f_2 is nonnegative on $[r_1, r_2]$, so we conclude that the right-hand side is nonnegative, giving us a contradiction.

Case 2: Suppose that $r_2 = \infty$. Again integrating the identity (3.5.16) from r_1 to ∞, we obtain

$$\begin{aligned} 0 > (\alpha_1 - \alpha_2) \int_{r_1}^{\infty} w f_1 f_2 &= \lim_{r\to\infty} (PW)(r) - (PW)(r_1) \\ &= \lim_{r\to\infty} [P(r)(f_1(r)f_2'(r) - f_1'(r)f_2(r))] - P(r_1)(f_1(r_1)f_2'(r_1) - f_1'(r_1)f_2(r_1)). \end{aligned} \tag{3.5.17}$$

We notice that Lemma 3.5.6 implies that

$$\lim_{r\to\infty} [P(r)(f_1(r)f_2'(r) - f_1'(r)f_2(r))] = 0.$$

Indeed, taking $\frac{n-p}{2(p-1)} < \beta < \frac{n-p}{p-1}$,

$$|f_1'f_2 - f_1f_2'| \leq |f_1'||f_2| + |f_1||f_2'| \leq Cr^{-2\beta-1},$$

and, recalling Remark 3.5.5,

$$P(r) \leq Cr^{(n-1)/(p-1)},$$

implying that

$$P\,|f_1'f_2 - f_1f_2'| \leq Cr^{\gamma} \to 0,$$

where $\gamma = -2\beta - 1 + \frac{n-1}{p-1} < 0$. Then (3.5.17) becomes

$$0 > -P(r_1)f_1'(r_1)f_2(r_1).$$

Since $f_1'(r_1) > 0$ and $f_2(r_1) \geq 0$ (see the argument in Case 1), this gives us a contradiction.

We now claim that f_2 has a zero in the interval $[0, r_1)$, where r_1 is the first zero of f_1. Again, we assume for the sake of contradiction that f_2 has no zero in this interval and that, without loss of generality, f_1 and f_2 are nonnegative in $[0, r_1]$. Integrating (3.5.16) implies

$$0 > (\alpha_1 - \alpha_2) \int_0^{r_1} w f_1 f_2 = PW(r_1) - PW(0). \tag{3.5.18}$$

The same computation as in the proof of Part (1) of this lemma implies that $(PW)(0) = 0$, so (3.5.18) becomes

$$0 > -P(r_1) f_1'(r_1) f_2(r_1),$$

once more giving us a contradiction.

The first eigenfunction of an operator is always positive in the interior of the domain, so the second eigenfunction of L must have at least one interior zero by orthogonality. Thus the claims above imply that the ith eigenfunction has at least $i-1$ interior zeros. On the other hand, as mentioned in the proof of Lemma 3.5.6, the standard theory also implies that the ith eigenfunction has at most $i-1$ interior zeros, and the proof is complete. □

Chapter 4

Strong form stability for the Wulff inequality

4.1 Overview

In this chapter,[1] we prove a strong form stability result for the Wulff inequality

$$\mathcal{F}(E) \geq n|K|^{1/n}|E|^{1/n'} . \tag{4.1.1}$$

Let us recall from the introduction that the (anisotropic) *surface energy* of a set of finite perimeter $E \subset \mathbb{R}^n$ is defined by

$$\mathcal{F}(E) = \int_{\partial^* E} f(\nu_E(x))\, d\mathcal{H}^{n-1}(x)$$

given a convex positively 1-homogeneous *surface tension* $f : \mathbb{R}^n \to [0, +\infty)$ that is positive on S^{n-1}. We also recall that equality is attained in (4.1.1) if and only if E is a translation or dilation of the Wulff shape

$$K = \bigcap_{\nu \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot \nu < f(\nu)\} .$$

Given a surface tension f, the *gauge function* $f_* : \mathbb{R}^n \to [0, +\infty)$ is defined by

$$f_*(x) = \sup\{x \cdot \nu : f(\nu) \leq 1\}.$$

The gauge function provides another characterization of the Wulff shape: $K = \{x : f_*(x) < 1\}$.

[1]This chapter is based on work originally appearing in [Neu16].

4.1.1 Statements of the main theorems

We prove a *strong form* of the quantitative Wulff inequality along the lines of (1.2.6), improving (1.2.4) by adding a term to the left hand side that quantifies the oscillation of $\partial^* E$ with respect to ∂K. We define the *anisotropic oscillation index* by

$$\beta_f(E) = \min_{y\in\mathbb{R}^n}\left(\frac{1}{n|K|^{1/n}|E|^{1/n'}}\int_{\partial^* E} f(\nu_E(x)) - \nu_E(x)\cdot\frac{x-y}{f_*(x-y)}\,d\mathcal{H}^{n-1}(x)\right)^{1/2}. \tag{4.1.2}$$

The following theorem is a strong form of the quantitative Wulff inequality that holds for an arbitrary surface tension.

Theorem 4.1.1. *There exists a constant C depending only on n such that*

$$\alpha_f(E)^2 + \beta_f(E)^{4n/(n+1)} \le C\delta_f(E) \tag{4.1.3}$$

for every set of finite perimeter E with $0 < |E| < \infty$.

As in (1.2.4), the constant is independent of f. We expect that, as in (1.2.6), the sharp exponent for $\beta_f(E)$ in (4.1.3) should be 2. With additional assumptions on the surface tension f, we prove the stability inequality in sharp form for two special cases.

Definition 4.1.2. *A surface tension f is λ-elliptic, $\lambda > 0$, if $f \in C^2(\mathbb{R}^n \setminus \{0\})$ and*

$$(\nabla^2 f(\nu)\tau)\cdot\tau \ge \frac{\lambda}{|\nu|}\left|\tau - \left(\tau\cdot\frac{\nu}{|\nu|}\right)\frac{\nu}{|\nu|}\right|^2$$

for $\nu, \tau \in \mathbb{R}^n$ with $\nu \ne 0$.

This is a uniform ellipticity assumption for $\nabla^2 f(\nu)$ in the tangential directions to ν. If f is λ-elliptic, then the corresponding Wulff shape K is of class C^2 and uniformly convex (see [Sch13], page 111). When $\mathcal{F}$ is a surface energy corresponding to a λ-elliptic surface tension, the following sharp result holds. The constant depends on m_f and M_f, a pair of constants defined in (4.2.2) that describe how much f stretches and shrinks unit-length vectors.

Theorem 4.1.3. *Suppose f is a λ-elliptic surface tension with corresponding surface energy $\mathcal{F}$. There exists a constant C depending on $n, \lambda, m_f/M_f$, and $\|\nabla^2 f\|_{C^0(\partial K)}$ such that*

$$\alpha_f(E)^2 + \beta_f(E)^2 \leq C\delta_f(E) \tag{4.1.4}$$

for any set of finite perimeter E with $0 < |E| < \infty$.

The second case where we obtain the strong form quantitative Wulff inequality with the sharp power is the case of a crystalline surface tension.

Definition 4.1.4. *A surface tension f is crystalline if it is the maximum of finitely many linear functions, in other words, if there exists a finite set $\{x_j\}_{j=1}^N \subset \mathbb{R}^n \setminus \{0\}, N \in \mathbb{N}$, such that*

$$f(\nu) = \max_{1\leq j\leq N}\{x_j \cdot \nu\} \qquad \text{for all } \nu \in S^{n-1}.$$

If f is a crystalline surface tension, then the corresponding Wulff shape K is a convex polyhedron. In dimension two, when f is a crystalline surface tension, we prove the following sharp quantitative Wulff inequality.

Theorem 4.1.5. *Let $n = 2$ and suppose f is a crystalline surface tension with corresponding surface energy $\mathcal{F}$. There exists a constant C depending on f such that*

$$\alpha_f(E)^2 + \beta_f(E)^2 \leq C\delta_f(E)$$

for any set of finite perimeter E with $0 < |E| < \infty$.

Some remarks about the definition of the anisotropic oscillation index β_f in (4.1.2) are in order. The oscillation index $\beta_1(E)$ in (1.2.5) measures oscillation of the reduced boundary of a set E with respect to the boundary of the ball. Indeed, the quantity $\beta_1(E)$ is the integral over $\partial^* E$ of the Cauchy-Schwarz deficit $1 - \frac{x}{|x|} \cdot \nu_E(x)$, which quantifies in a Euclidean sense how closely $\nu_E(x)$ aligns with $\frac{x}{|x|}$.

To understand (4.1.2), we remark that f and f_* satisfy a Cauchy-Schwarz-type inequality called the Fenchel inequality, which states that

$$\nu_E(x) \cdot \frac{x}{f_*(x)} \leq f(\nu_E(x)). \tag{4.1.5}$$

Just as $\beta_1(E)$ in (1.2.5) quantifies the overall Cauchy-Schwarz deficit between $\frac{x}{|x|}$ and $\nu_E(x)$, the term $\beta_f(E)$ is an integral along $\partial^* E$ of the deficit in the Fenchel inequality. In Section 4.2.2, we show that $f(\nu_E(x)) = y \cdot \nu_E(x)$ for $y \in \partial K$ if and only if $\nu_E(x)$ is normal to a supporting hyperplane of K at y. In this way, $\beta_f(E)$ quantifies how much normal vectors of E align with corresponding normal vectors of K, and thus provides a measure of the oscillation of the reduced boundary of E with respect to the boundary of K. Note that in the case f constantly equal to one, β_f agrees with β_1.

It is not immediately clear that (4.1.2) is the appropriate analogue of (1.2.5) in the anisotropic case. Noting that $x \mapsto (x-y)/f_*(x-y)$ is the radial projection of $\mathbb{R}^n \setminus \{0\}$ onto $\partial K + y$, one may initially want to consider the term

$$\beta_f^*(E) = \min_{y \in R^n} \left(\frac{1}{2n|K|^{1/n}|E|^{1/n'}} \int_{\partial^* E} \left| \nu_E(x) - \nu_K\Big(\frac{x-y}{f_*(x-y)}\Big) \right|^2 d\mathcal{H}^{n-1}(x) \right)^{1/2}$$

$$= \min_{y \in R^n} \left(\frac{1}{n|K|^{1/n}|E|^{1/n'}} \int_{\partial^* E} 1 - \nu_E(x) \cdot \nu_K\Big(\frac{x-y}{f_*(x-y)}\Big)\, d\mathcal{H}^{n-1}(x) \right)^{1/2}. \tag{4.1.6}$$

In Section 4.6, however, we show that such a term does not admit any stability result for general f. Indeed, in Example 4.6.1, we construct a sequence of crystalline surface tensions that show that there does not exist a power σ such that

$$\beta_f^*(E)^\sigma \leq C(n, f)\delta_f(E) \tag{4.1.7}$$

for all sets E of finite perimeter with $0 < |E| < \infty$ and for all $\mathcal{F}$. Furthermore, Example 4.6.2 shows that even if we restrict our attention to surface energies which are γ-λ convex, a weaker notion of λ-ellipticity introduced in Definition 4.1.6, an inequality of the form (4.1.7) cannot hold with an exponent less than $\sigma = 4$. The examples in Section 4.6 illustrate the fact that, in the anisotropic case, measuring the alignment of normal vectors in a Euclidean sense is not suitable for obtaining a stability inequality for general f; it is essential to account for the anisotropy in this measurement. The anisotropic oscillation index $\beta_f(E)$ in (4.1.2) does exactly this.

In the positive direction, when the surface tension f is γ-λ convex, $\beta_f^*(E)$ is controlled by $\beta_f(E)$. As one expects from Example 4.6.2, the exponent in this bound depends on the γ-λ convexity of f. We now define γ-λ convexity.

Definition 4.1.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative, convex, positively one-homogeneous function. Then we say that f is γ-λ convex for $\gamma \geq 0, \lambda > 0$ if*

$$f(\nu + \tau) + f(\nu - \tau) - 2f(\nu) \geq \frac{\lambda}{|\nu|} \left| \tau - \Big(\tau \cdot \frac{\nu}{|\nu|}\Big)\frac{\nu}{|\nu|} \right|^{2+\gamma} \tag{4.1.8}$$

for all $\nu, \tau \in \mathbb{R}^n$ such that $\nu \neq 0$.

Dividing (4.1.8) by τ^2, the left hand side gives a second difference quotient of f. While λ-ellipticity assumes that $f \in C^2(\mathbb{R}^n \setminus \{0\})$ and that its second derivatives in directions τ that are orthogonal to ν are bounded from below, γ-λ convexity only assumes that the second difference quotients in these directions have a bound from below that degenerates as τ goes to 0. Of course, a 0-λ convex surface tension f with $f \in C^2(\mathbb{R}^n \setminus \{0\})$ is λ-elliptic. The ℓ^p norms $f_p(x) = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \in (1,\infty)$ are examples of γ-λ convex surface tensions; see Section 4.6. For a γ-λ convex surface tension f, the following theorem shows that β_f controls β_f^*.

Theorem 4.1.7. *Let f be a γ-λ convex surface tension. Then there exists a constant C depending on $\gamma, \lambda,$ and m_f/M_f such that*

$$\beta_f^*(E)^{(2+\gamma)/2} \le C \left(\frac{P(E)}{n|K|^{1/n}|E|^{1/n'}} \right)^{\gamma/4} \beta_f(E).$$

for any set of finite perimeter E with $0 < |E| < \infty$.

As in Theorem 4.1.3, the constant depends on m_f and M_f which are defined in (4.2.2). As an immediate consequence of Theorem 4.1.7, Theorem 4.1.1, and Theorem 4.1.3, we have the following result.

Corollary 4.1.8. *If f is a γ-λ convex surface tension, then there exists a constant C depending on $n, \gamma, \lambda,$ and m_f/M_f such that*

$$\alpha_f(E)^2 + \beta_f^*(E)^\sigma \le C \left(\frac{P(E)}{n|K|^{1/n}|E|^{1/n'}} \right)^{\gamma n/(n+1)} \delta_f(E)$$

for any set of finite perimeter E with $0 < |E| < \infty$, where $\sigma = 2n(2+\gamma)/(n+1)$.

If f is a λ-elliptic surface tension, then there exists a constant C depending on

$n, \gamma, \lambda, m_f/M_f$, and $\|\nabla^2 f\|_{C^0(\partial K)}$ such that

$$\alpha_f(E)^2 + \beta_f^*(E)^2 \leq C\delta_f(E)$$

for any set of finite perimeter E with $0 < |E| < \infty$.

4.1.2 Discussion of the proofs

At the core of the proof of (1.2.6) are a selection principle argument, the regularity theory of almost-minimizers of perimeter, and an analysis of the second variation of perimeter. Indeed, with a selection principle argument in the spirit of the proof of (1.2.3) by Cicalese and Leonardi in [CL12], Fusco and Julin reduce to a sequence $\{F_j\}$ such that each F_j is a (Λ, r_0)-minimizer of perimeter (Definition 4.4.4) and $F_j \to B$ in L^1. Then, by the standard regularity theory, each set F_j has boundary given by a small C^1 perturbation of the boundary of the ball. This case is handled by a theorem of Fuglede in [Fug89], which says the following: Let E be a *nearly spherical set*, i.e., a set with barycenter $\operatorname{bar} E = |E|^{-1}\int_E x\,dx$ at the origin such that $|E| = |B|$ and

$$\partial E = \{x + u(x)x : x \in \partial B\}$$

for $u : \partial B \to \mathbb{R}$ with $u \in C^1(\partial B)$. There exist C and ε depending on n such that if $\|u\|_{C^1(\partial B)} \leq \varepsilon$, then

$$\|u\|^2_{H^1(\partial B)} \leq C\delta_1(E). \tag{4.1.9}$$

The proof of (4.1.9) makes explicit use of spherical harmonics to provide a lower bound for the second variation of perimeter. It is then easily shown that $\alpha_1(E) + \beta_1(E) \leq C\|u\|_{H^1(\partial B)}$, and therefore (4.1.9) implies (1.2.6) in the case of nearly spherical sets.

Indeed, $\alpha_1(E) \leq C\beta_1(E)$ as shown in Proposition 4.2.4, and in the case of nearly spherical sets, the oscillation index β_1 is essentially an L^2 distance of gradients: if $w(x) = x + u(x)x$, then

$$\nu_E(w(x)) = \frac{x(1+u(x)) + \nabla u(x)}{\sqrt{(1+u)^2 + |\nabla u|^2}},$$

where the ∇u is the tangential gradient of u. Then

$$\begin{aligned} n|K|\beta_1(E)^2 &\leq \int_{\partial E} 1 - \nu_E(w) \cdot \frac{w}{|w|}\, d\mathcal{H}^{n-1} \\ &= \int_{\partial B} \sqrt{(1+u)^2 + |\nabla u|^2} - (1+u)\, d\mathcal{H}^{n-1} \\ &= \int_{\partial B} \frac{1}{2}|\nabla u|^2 + O(|\nabla u|^2)\, d\mathcal{H}^{n-1} \leq \|u\|^2_{H^1(\partial B)}. \end{aligned}$$

In each of Theorems 4.1.1, 4.1.3, and 4.1.5, at least one of the three key ingredients of the proof of Fusco and Julin is missing. The proof of Theorem 4.1.1 uses a selection principle to reduce to a sequence of (Λ, r_0)-minimizers of $\mathcal{F}$ converging in L^1 to K. However, for an arbitrary surface tension, uniform density estimates (Lemma 4.3.3) are the strongest regularity property that one can hope to extract. We pair these estimates with (1.2.4) to obtain the result.

The proof of Theorem 4.1.3 follows a strategy similar to that of the proof of (1.2.6) in [FJ14]. If f is a λ-elliptic surface tension, then (Λ, r_0)-minimizers of the corresponding surface energy $\mathcal{F}$ enjoy strong regularity properties. Using a selection principle argument and the regularity theory, we reduce to the case where ∂E is a small C^1 perturbation of ∂K. The difficulty arises, however, in showing the following analogue of Fuglede's result (4.1.9) in the setting of the anisotropic surface energy.

Proposition 4.1.9. *Let f be a λ-elliptic surface tension with corresponding surface energy $\mathcal{F}$ and Wulff shape K. Let E be a set such that $|E| = |K|$ and* $\operatorname{bar} E = \operatorname{bar} K$, *where* $\operatorname{bar} E = |E|^{-1}\int_E x\,dx$ *denotes the barycenter of E. Suppose*

$$\partial E = \{x + u(x)\nu_K(x) : x \in \partial K\}$$

where $u : \partial K \to \mathbb{R}$ is in $C^1(\partial K)$. There exist C and ε_1 depending on n, λ, and m_f/M_f such that if $\|u\|_{C^1(\partial K)} \leq \varepsilon_1$, then

$$\|u\|^2_{H^1(\partial K)} \leq C\delta_f(E). \tag{4.1.10}$$

Again, m_f and M_f are defined in (4.2.2). To prove (4.1.9), Fuglede shows that, due to the volume and barycenter constraints respectively, the function u is orthogonal to the first and second eigenspaces of the Laplace operator on the sphere. This implies that, thanks to a gap in the spectrum of this operator, functions satisfying these constraints satisfy an improved Poincaré inequality. Fuglede's reasoning uses that fact that the eigenvalues and eigenfunctions of the Laplacian on the sphere are *explicitly known.*

The analogous operator on ∂K arising in the second variation of $\mathcal{F}$ also has a discrete spectrum, but one cannot expect to understand its spectrum explicitly. Instead, to prove (4.1.9), we exploit (1.2.4) in order to obtain an improved Poincaré inequality for functions $u \in H^1(\partial K)$ satisfying the volume and barycenter constraints.

Then, as in the isotropic case, one shows that $\alpha_f(E) + \beta_f(E) \leq C\|u\|_{H^1(\partial K)}$ for a constant $C = C(n, \|\nabla^2 f\|_{C^0(\partial K)})$, and therefore (4.1.10) implies (4.1.4) for small C^1 perturbations. Indeed, Proposition 4.2.4 implies that $\alpha_f(E) \leq C(n)\beta_f(E)$, while

$\beta_f(E) \leq C\|u\|_{H^1(\partial K)}$ by a Taylor expansion and a change of coordinates. The computation is postponed until (4.4.16) as it relies on notation introduced in Section 4.4.

The proof of Theorem 4.1.5 also uses a selection principle-type argument to reduce to a sequence of almost-minimizers of $\mathcal{F}$ converging in L^1 to the Wulff shape. In this case, a rigidity result of Figalli and Maggi in [FM11] allows us reduce to the case where E is a convex polygon whose set of normal vectors is equal to the set of normal vectors of K. From here, an explicit computation (Proposition 4.5.1) shows the result.

4.1.3 Organization of the chapter

In Section 4.2, we introduce some necessary preliminaries for our main objects of study. Section 4.3 is dedicated to the proof of Theorem 4.1.1, while in Sections 4.4 and 4.5 we prove Theorems 4.1.3 and 4.1.5 respectively. In Section 4.6, we consider the term $\beta^*_f(E)$ defined in (4.1.6), providing two examples that show that one cannot expect stability with a power independent of the regularity of f and proving Theorem 4.1.7.

4.2 Preliminaries

Let us introduce a few key properties about sets of finite perimeter, the anisotropic surface energy, and the anisotropic oscillation index β_f.

4.2.1 Sets of finite perimeter

Given an $\mathbb{R}^n$-valued Borel measure μ on $\mathbb{R}^n$, the *total variation* $|\mu|$ of μ on a Borel set E is defined by

$$|\mu|(E) = \sup \left\{ \sum_{j \in \mathbb{N}} |\mu(E_j)| : \; E_j \cap E_i = \emptyset, \; \bigcup_{j \in \mathbb{N}} E_j \subset E \right\}.$$

A measurable set $E \subset \mathbb{R}^n$ is said to be a *set of finite perimeter* if the distributional gradient $\mathrm{D}1_E$ of the characteristic function of E is an $\mathbb{R}^n$-valued Borel measure on $\mathbb{R}^n$ with $|\mathrm{D}1_E|(\mathbb{R}^n) < \infty$.

For a set of finite perimeter E, the *reduced boundary* $\partial^* E$ is the set of points $x \in \mathbb{R}^n$ such that $|\mathrm{D}1_E|(B_r(x)) > 0$ for all $r > 0$ and

$$\lim_{r \to 0^+} \frac{\mathrm{D}1_E(B_r(x))}{|\mathrm{D}1_E|(B_r(x))} \quad \text{exists and belongs to } S^{n-1}. \tag{4.2.1}$$

If $x \in \partial^* E$, then we let $-\nu_E$ denote the limit in (4.2.1). We then call $\nu_E : \partial^* E \to S^{n-1}$ the *measure theoretic outer unit normal to E*. Up to modifying E on a set of Lebesgue measure zero, one may assume that the topological boundary ∂E is the closure of the reduced boundary $\partial^* E$. For the remainder of the chapter, we make this assumption.

4.2.2 The surface tension and the gauge function

Throughout the chapter, we let

$$m_f = \inf_{\nu \in S^{n-1}} f(\nu), \qquad M_f = \sup_{\nu \in S^{n-1}} f(\nu). \tag{4.2.2}$$

It follows that

$$\frac{1}{M_f} = \inf_{x \in S^{n-1}} f_*(x), \qquad \frac{1}{m_f} = \sup_{x \in S^{n-1}} f_*(x).$$

One easily shows that $f(\nu) = \sup\{x \cdot \nu : x \in K\}$ and $f_*(x) = \inf\{\lambda : \frac{x}{\lambda} \in K\}$. This also implies that $B_{m_f} \subset K \subset B_{M_f}$, and so if $|K| = 1$, then $m_f^n|B| \leq 1 \leq M_f^n|B|$. As mentioned in the chapter overview, f and f_* satisfy the Fenchel inequality (4.1.5) for all $x, \nu \in \mathbb{R}^n$. We may characterize the equality cases in the Fenchel inequality: for any ν, $x \cdot \nu = f_*(x)f(\nu)$ if and only if ν is normal to a supporting hyperplane of K at the point $\frac{x}{f_*(x)} \in \partial K$. Indeed, ν is normal to a supporting hyperplane of K at $x \in \partial K$ if and only if $\nu \cdot (y - x) \leq 0$ (so $\nu \cdot y \leq \nu \cdot x$) for all $y \in K$. This holds if and only if $\nu \cdot x = \sup\{y \cdot \nu : y \in K\} = f(\nu)$. In particular, if $x \in \partial^* K$, then $f_*(x) = 1$ and

$$f(\nu_K(x)) = x \cdot \nu_K(x). \tag{4.2.3}$$

We may compute the gradient of f_* at points of differentiability using the Fenchel inequality. The gauge function f_* is differentiable at $x_0 \in \mathbb{R}^n$ if there is a unique supporting hyperplane to K at $\frac{x_0}{f_*(x_0)} \in \partial K$. For such an x_0, let $\nu_0 = \nu_K(\frac{x}{f_*(x)}) \in \mathbb{R}^n$ be normal to the supporting hyperplane to K at $\frac{x_0}{f_*(x_0)}$, so $\frac{x_0}{f_*(x_0)} \cdot \nu_0 = f(\nu_0)$ by (4.2.3). We define the Fenchel deficit functional by $G(x) = f(\nu_0)f_*(x) - x \cdot \nu_0$. By the Fenchel inequality, $G(x) \geq 0$ for all x and $G(x_0) = 0$, so G has a local minimum at x_0 and thus

$$0 = \nabla G(x_0) = f(\nu_0)\nabla f_*(x_0) - \nu_0.$$

Rearranging, we obtain $\nabla f_*(x_0) = \frac{\nu_0}{f(\nu_0)}$. The 1-homogeneity of f then implies that

$$f(\nabla f_*(x)) = 1. \tag{4.2.4}$$

Furthermore, this implies that

$$x \cdot \nabla f_*(x) = x \cdot \nu_K\Big(\frac{x}{f_*(x)}\Big) = f_*(x) \tag{4.2.5}$$

(alternatively, this follows from Euler's identity for homogeneous functions). An analogous argument ensures that

$$\nabla f(\nu_K(x)) = x \tag{4.2.6}$$

for $x \in \partial^* K$. Furthermore, using (4.2.5), we compute

$$\operatorname{div} \frac{x}{f_*(x)} = \frac{n-1}{f_*(x)}\,. \tag{4.2.7}$$

4.2.3 Properties of α_f, β_f, and γ_f

Using the divergence theorem, by approximation and the dominated convergence theorem, and (4.2.7), we find that for any $y \in \mathbb{R}^n$,

$$\int_{\partial^* E} \frac{x-y}{f_*(x-y)} \cdot \nu_E(x)\, d\mathcal{H}^{n-1} = (n-1)\int_E \frac{dx}{f_*(x-y)}.$$

We may then write

$$\beta_f(E)^2 = \frac{\mathcal{F}(E) - (n-1)\gamma_f(E)}{n|K|^{1/n}|E|^{1/n'}}, \tag{4.2.8}$$

where $\gamma_f(E)$ is defined by

$$\gamma_f(E) = \sup_{y\in\mathbb{R}^n} \int_E \frac{dx}{f_*(x-y)}. \tag{4.2.9}$$

The supremum in (4.2.9) is attained, though perhaps not uniquely. If $y \in \mathbb{R}^n$ is a point such that

$$\gamma_f(E) = \int_E \frac{dx}{f_*(x-y)},$$

then we call y a *center of* E, and we denote by y_E a generic center of E. The Wulff shape K has unique center $y_K = 0$. Indeed, take any $y \in \mathbb{R}^n$, $y \neq 0$, and recall that $K = \{f_*(x) < 1\}$. Then

$$\begin{aligned}\int_K \frac{dx}{f_*(x)} - \int_K \frac{dx}{f_*(x-y)} &= \int_K \frac{dx}{f_*(x)} - \int_{K+y} \frac{dx}{f_*(x)} \\ &= \int_{K\setminus(K+y)} \frac{dx}{f_*(x)} - \int_{(K+y)\setminus K} \frac{dx}{f_*(x)} > \int_{K\setminus(K+y)} 1dx - \int_{(K+y)\setminus K} 1dx = 0.\end{aligned}$$

A similar argument verifies that if $|E| = |K|$, then

$$\gamma_f(E) \leq \gamma_f(K). \tag{4.2.10}$$

Moreover, $(n-1)\gamma_f(K) = \mathcal{F}(K) = n|K|$.

The following continuity properties of $\mathcal{F}$ and γ_f will be useful.

Proposition 4.2.1. *Suppose that* $\{E_j\}$ *is a sequence of sets converging in* L^1 *to a set* E*, and suppose that* $\{f^j\}$ *is a sequence of surface tensions converging locally uniformly to* f*, with corresponding surface energies* $\{\mathcal{F}_j\}$ *and* $\mathcal{F}$*.*

(1) The following lower semicontinuity property holds:

$$\mathcal{F}(E) \leq \liminf_{j\to\infty} \mathcal{F}_j(E_j).$$

(2) The function γ_f *defined in (4.2.9) is Hölder continuous with respect to* L^1 *convergence of sets with Hölder exponent equal to* $1/n'$*. In particular,*

$$|\gamma_f(E) - \gamma_f(F)| \leq \frac{n|K|}{n-1}|E\Delta F|^{1/n'}$$

for any two sets of finite perimeter $E, F \subset \mathbb{R}^n$. Moreover,

$$\lim_{j\to\infty} \gamma_{f^j}(E_j) = \gamma_f(E).$$

Proof. Proof of (1)*:* From the divergence theorem and the characterization $f(\nu) = \sup\{x \cdot \nu : f_*(x) \leq 1\}$, one finds that the surface energy of a set E is the anisotropic total variation of its characteristic function 1_E:

$$\mathcal{F}_j(E_j) = TV_{f^j}(1_{E_j}) := \sup\left\{ \int_{E_j} \operatorname{div} T\, dx \;\Big|\; T \in C_c^1(\mathbb{R}^n, \mathbb{R}^n),\; f_*^j(T) \leq 1 \right\}. \quad (4.2.11)$$

Let $T \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ be a vector field such that $f_*(T) \leq 1$ for all $x \in \mathbb{R}^n$. Then,

$$\int_E \operatorname{div} T\, dx = \lim_{j\to\infty} \int_{E_j} \operatorname{div} T\, dx = \lim_{j\to\infty} \|f_*^j(T)\|_{L^\infty(\mathbb{R}^n)} \int_{E_j} \operatorname{div} S_j\, dx \leq \liminf_{j\to\infty} \mathcal{F}_j(E_j),$$

where we take $S_j = T/\|f_*^j(T)\|_{L^\infty(\mathbb{R}^n)}$. Taking the supremum over all $T \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ with $f_*(T) \leq 1$, we obtain the result.

Proof of (2)*:* By (4.2.9),

$$\gamma_f(E) - \gamma_f(F) \leq \int_E \frac{dx}{f_*(x - y_E)} - \int_F \frac{dx}{f_*(x - y_E)} \leq \int_{E\Delta F} \frac{dx}{f_*(x - y_E)}.$$

Letting r be such that $|rK| = |E\Delta F|$ and recalling (4.2.10), we have

$$\begin{aligned} \int_{E\Delta F} \frac{dx}{f_*(x - y_E)} &\leq \int_{rK} \frac{dx}{f_*(x)} = \gamma_f(rK) \\ &= \frac{\mathcal{F}(rK)}{n-1} = \frac{n|K|r^{n-1}}{n-1} = \frac{n|K|}{n-1}|E\Delta F|^{1/n'}. \end{aligned} \quad (4.2.12)$$

Thus $\gamma_f(E) - \gamma_f(F) \leq \frac{n|K|}{n-1}|E\Delta F|^{1/n'}$. The analogous argument holds for $\gamma_f(F) - \gamma_f(E)$, implying the Hölder continuity of γ_f.

For the second equation, we note that if $f^j \to f$ locally uniformly, then $f_*^j \to f_*$ locally uniformly and $M_{f^j} \to M_f$. The triangle inequality gives

$$|\gamma_{f^j}(E_j) - \gamma_f(E)| \leq |\gamma_{f^j}(E_j) - \gamma_f(E_j)| + |\gamma_f(E_j) - \gamma_f(E)|.$$

The second term goes to zero by the Hölder continuity that we have just shown. To bound the first term, let y_{E_j} be a center of E_j with respect to the surface energy $\mathcal{F}_j$. If $\gamma_{f^j}(E_j) \geq \gamma_f(E_j)$, then

$$\begin{aligned} 0 \leq \gamma_{f^j}(E_j) - \gamma_f(E_j) &\leq \int_{E_j} \frac{1}{f_*^j(x - y_{E_j})} - \frac{1}{f_*(x - y_{E_j})} dx = \int_{E_j + y_{E_j}} \frac{1}{f_*^j(x)} - \frac{1}{f_*(x)} dx \\ &= \int_{\mathbb{R}^n} 1_{(E_j + y_{E_j}) \setminus B_\varepsilon(0)} \left(\frac{1}{f_*^j(x)} - \frac{1}{f_*(x)} \right) dx + \int_{B_\varepsilon(0)} \frac{1}{f_*^j(x)} - \frac{1}{f_*(x)} dx. \end{aligned}$$

For $\varepsilon > 0$ fixed, the first integral goes to zero as $j \to \infty$. For the second integral, we have

$$\int_{B_\varepsilon(0)} \frac{1}{f_*^j(x)} + \frac{1}{f_*(x)} \, dx \leq \int_{B_\varepsilon(0)} \frac{M_{f^j} + M_f}{|x|} \, dx \leq C\varepsilon^{n-1}.$$

Taking $\varepsilon \to 0$, we conclude that $\gamma_{f^j}(E_j) - \gamma_f(E_j) \to 0$ as $j \to \infty$. The case where $\gamma_{f^j}(E_j) \leq \gamma_f(E_j)$ is analogous. □

Remark 4.2.2. With sequences as in the hypothesis of Proposition 4.2.1 above, β_f has the following lower semicontinuity property:

$$\beta_f(E) \leq \liminf_{j \to \infty} \beta_{f^j}(E_j).$$

This follows immediately from parts (1) and (2) of Proposition 4.2.1 and the decomposition in (4.2.8).

Lemma 4.2.3. *For every $\varepsilon > 0$, there exists $\eta > 0$ such that if $|F \Delta K| \leq \eta$, then $|y_F| < \varepsilon$ for any center y_F of F.*

Proof. Suppose $|K\Delta F_j| \to 0$. By the triangle inequality,

$$\int_K \frac{dx}{f_*(x)} \leq \left| \int_K \frac{dx}{f_*(x)} - \int_{F_j} \frac{dx}{f_*(x - y_{F_j})} \right| + \left| \int_{F_j} \frac{dx}{f_*(x - y_{F_j})} - \int_K \frac{dx}{f_*(x - y_{F_j})} \right| + \int_K \frac{dx}{f_*(x - y_{F_j})}.$$

By (4.2.12), the first two terms on the right hand side go to zero as $j \to \infty$, implying that

$$\int_K \frac{dx}{f_*(x)} \leq \lim_{j\to\infty} \int_K \frac{dx}{f_*(x - y_{F_j})}.$$

Because K has unique center $y_K = 0$, we conclude that $|y_{F_j}| \to 0$. □

We now introduce the relative surface energy and the anisotropic coarea formula. Given an open set A and a set of finite perimeter E, the anisotropic surface energy of E *relative to* A is defined by

$$\mathcal{F}(E; A) = \int_{\partial^* E \cap A} f(\nu_E(x))\, d\mathcal{H}^{n-1}(x).$$

For a Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$ and an open set E, the anisotropic coarea formula states that

$$\int_E f(-\nabla u(x))\, dx = \int_0^\infty \mathcal{F}(\{u > r\}; E)\, dr.$$

The anisotropic coarea formula is proved in the same way as the coarea formula (see, for instance, [Mag12, Theorem 13.1]), replacing the Euclidean norm with f and f_* and using (4.2.11). When u is bounded by a constant C on E, then applying the anisotropic coarea formula to $w = C - u$ yields

$$\int_E f(\nabla u(x))\, dx = \int_E f(-\nabla w(x))\, dx = \int_0^C \mathcal{F}(\{C - u > r\}; E)\, dt$$

$$= \int_0^C \mathcal{F}(\{u < C - r\}; E)\, dr = \int_0^C \mathcal{F}(\{u < r\}; E)\, dr$$

Moreover, approximating by simple functions, we may produce a weighted version:

$$\int_E f(\nabla u(x)) g(f_*(x))\, dx = \int_0^\infty \mathcal{F}(\{u < r\}; E) g(r)\, dr$$

whenever $g : \mathbb{R} \to [0, \infty]$ is a Borel function. We will frequently use this weighted version with $u(x) = f_*(x)$, E a bounded set, and $g(r) = \frac{1}{r}$, which, using (4.2.4), gives

$$\int_E \frac{dx}{f_*(x)} = \int_0^\infty \frac{\mathcal{F}(\{f_*(x) < r\}; E)}{r}\, dr = \int_0^\infty \frac{\mathcal{F}(rK; E)}{r}\, dr. \tag{4.2.13}$$

We conclude this section with the following Poincaré-type inequality, which shows that $\beta_f(E)$ controls $\alpha_f(E)$ for all sets of finite perimeter E.

Proposition 4.2.4. *There exists a constant $C(n)$ such that if E is a set of finite perimeter with $0 < |E| < \infty$, then*

$$\alpha_f(E) + \delta_f(E)^{1/2} \leq C(n) \beta_f(E). \tag{4.2.14}$$

Proof. We follow the proof of the analogous result for the perimeter in [FJ14]. Due to the scaling and translation invariance of α_f, β_f, and δ_f, we may assume that $|E| = |K| = 1$ and that E has center zero. We have

$$\gamma_f(K) - \gamma_f(E) = \int_K \frac{dx}{f_*(x)} - \int_E \frac{dx}{f_*(x)} = \int_{K \setminus E} \frac{dx}{f_*(x)} - \int_{E \setminus K} \frac{dx}{f_*(x)}.$$

Therefore, adding and subtracting $\mathcal{F}(K)/n = (n-1)\gamma_f(K)/n$ in (4.2.8), we have

$$\beta_f(E)^2 = \delta_f(E) + \frac{n-1}{n} \left(\int_{K \setminus E} \frac{dx}{f_*(x)} - \int_{E \setminus K} \frac{dx}{f_*(x)} \right).$$

We want to bound the final two integrals from below by $\alpha_f(E)^2$. To this end, we let $a := |E \setminus K| = |K \setminus E|$ and define the K-annuli $A_{R,1} = K_R \setminus K$ and $A_{1,r} = K \setminus K_r$, where $R > 1 > r$ are chosen such that $|A_{R,1}| = |A_{1,r}| = a$. In particular, $R = (1+a)^{1/n}$ and $r = (1-a)^{1/n}$. By (4.2.10) and (4.2.13),

$$\int_{K\setminus E} \frac{dx}{f_*(x)} \geq \int_{A_{1,r}} \frac{dx}{f_*(x)} = \int_r^1 \frac{\mathcal{F}(sK)}{s}\,ds = \int_r^1 ns^{n-2}\,ds = \frac{n}{n-1}[1 - r^{n-1}]$$

and

$$\int_{E\setminus K} \frac{dx}{f_*(x)} \leq \int_{A_{R,1}} \frac{dx}{f_*(x)} = \int_1^R \frac{\mathcal{F}(sK)}{s}\,ds = \int_1^R ns^{n-2}\,ds = \frac{n}{n-1}[R^{n-1} - 1].$$

Subtracting the second from the first, we have

$$\frac{n-1}{n}\left(\int_{K\setminus E} \frac{dx}{f_*(x)} - \int_{E\setminus K} \frac{dx}{f_*(x)}\right) \geq 2 - r^{n-1} - R^{n-1}.$$

The function $g(t) = (1+t)^{1/n'}$ is function is strictly concave, with $\frac{1}{2}(g(t) + g(s)) \leq g(\frac{t}{2} + \frac{s}{2}) - C|t-s|^2$, and therefore $2 - [(1+a)^{1/n'} + (1-a)^{1/n'}] \geq 8C|a|^2$. Thus

$$\begin{aligned}
\beta_f(E)^2 &\geq \delta_f(E) + [2 - (1-a)^{1/n'} - (1+a)^{1/n'}] \\
&\geq \delta_f(E) + 8C|a|^2 = \delta_f + 2C\left(|E \setminus K| + |K \setminus E|\right)^2 \\
&= \delta_f(E) + 2C|K\Delta E|^2 \geq \delta_f(E) + 2C\alpha_f(E)^2.
\end{aligned}$$

□

4.3 General surface tensions

In this section, we prove Theorem 4.1.1. We begin by introducing a few lemmas that are needed the proof. The first allows us to reduce the problem to sets contained in some fixed ball.

Lemma 4.3.1. *There exist constants $R_0 > 0$ and $C > 0$ depending only on n and M_f such that, given a set of finite perimeter E with $|E| = |K|$, we may find a set E' such that $|E'| = |K|$, $E' \subset B_{R_0}$, and*

$$\beta_f(E)^2 \leq \beta_f(E')^2 + C\delta_f(E), \qquad \delta_f(E') \leq C\delta_f(E). \tag{4.3.1}$$

Proof. A simple adaptation of the proof of [Mag08, Theorem 4.1] ensures that we may find constants δ_0, C_0, C_1, and $\tilde{R}_0$ depending on n and M_f such that $C_0\delta_0 < 1/2$ and the following holds: if $\delta_f(E) \leq \delta_0$, then there exists a set $\tilde{E} \subset E$ such that $\tilde{E} \subset B_{\tilde{R}_0}$ and

$$|\tilde{E}| \geq |K|(1 - C_1\delta_f(E)), \qquad \mathcal{F}(\tilde{E}) \leq \mathcal{F}(E) + C_0\delta_f(E)|E|^{1/n'}. \tag{4.3.2}$$

If $\delta_f(E) > \delta_0$, then

$$\beta_f^2(E) \leq \frac{\mathcal{F}(E)}{n|K|} = \delta_f(E) + 1 \leq \frac{1+\delta_0}{\delta_0}\delta_f(E).$$

Simply taking $E' = K$, we have $\delta_f(E') \leq \delta_f(E)$ and $\beta_f(E)^2 \leq \frac{1+\delta_0}{\delta_0}\delta_f(E)$, proving (4.3.1).

On the other hand, if $\delta_f(E) \leq \delta_0$, let $E' = r\tilde{E}$ with $r \geq 1$ such that $|E'| = |r\tilde{E}| = |E|$. By (4.2.8),

$$\begin{aligned}\beta_f(E)^2 - \beta_f(E')^2 &= \frac{\mathcal{F}(E) - \mathcal{F}(E')}{n|K|} + \frac{n-1}{n|K|}\left(\gamma_f(E') - \gamma_f(E)\right) \\ &\leq \delta_f(E) + \frac{n-1}{n|K|}\left(r^{n-1}\gamma_f(\tilde{E}) - \gamma_f(E)\right).\end{aligned} \tag{4.3.3}$$

Since $\tilde{E} \subset E$, $\gamma_f(\tilde{E}) \leq \gamma_f(E)$, which implies that

$$\frac{n-1}{n|K|}\left(r^{n-1}\gamma_f(\tilde{E}) - \gamma_f(E)\right) \leq \frac{n-1}{n|K|}(r^{n-1} - 1)\gamma_f(E).$$

By (4.2.10) and the fact that $\gamma_f(K) = n|K|/(n-1)$,

$$\frac{n-1}{n|K|}(r^{n-1}-1)\gamma_f(E) \leq \frac{n-1}{n|K|}(r^{n-1}-1)\gamma_f(K) = r^{n-1}-1,$$

and since $r \geq 1$,

$$r^{n-1} - 1 \leq r^n - 1 = \frac{|E| - |\tilde{E}|}{|\tilde{E}|}.$$

The first part of (4.3.2) implies that

$$\frac{|E| - |\tilde{E}|}{|\tilde{E}|} \leq \frac{C_1\delta_f(E)}{1 - C_1\delta_f(E)} \leq \frac{C_1}{1 - C_1\delta_0}\delta_f(E).$$

We have therefore shown that

$$\frac{n-1}{n|K|}\left(r^{n-1}\gamma_f(\tilde{E}) - \gamma_f(E)\right) \leq \frac{C_1}{1 - C_1\delta_0}\delta_f(E);$$

this together with (4.3.3) concludes the proof of the first claim in (4.3.1).

In the direction of the second claim in (4.3.1), the first and second parts of (4.3.2) respectively imply that

$$\mathcal{F}(E') = r^{n-1}\mathcal{F}(\tilde{E}) \leq \frac{\mathcal{F}(\tilde{E})}{(1 - C_1\delta_f(E))^{1/n'}} \leq \frac{\mathcal{F}(E) + C_0\delta_f(E)|E|^{1/n'}}{(1 - C_1\delta_f(E))^{1/n'}}.$$

A Taylor expansion in $\delta_f(E)$ of the right hand side shows that

$$\begin{aligned}\mathcal{F}(E') &\leq \mathcal{F}(E) + C_0\delta_f(E)|E|^{1/n'} + \frac{n-1}{n}C_1\delta_f(E)\mathcal{F}(E) + O(\delta_f(E)^2) \\ &\leq \mathcal{F}(E) + C\delta_f(E)\mathcal{F}(E)\end{aligned}$$

for δ_0 chosen sufficiently small. Thus

$$\delta_f(E') = \frac{\mathcal{F}(E') - \mathcal{F}(K)}{n|K|} \leq \frac{\mathcal{F}(E') - \mathcal{F}(E)}{n|K|} \leq \frac{C\mathcal{F}(E)\delta_f(E)}{n|K|} \leq C\delta_f(E),$$

since $\mathcal{F}(E) \leq \mathcal{F}(K) + n|K|\delta_0$. Finally, since $\tilde{E} \subset B_{\tilde{R}_0}$ and $E' = r\tilde{E}$ with $r \leq 1/(1 - C_1\delta_0)^{1/n}$, we have $E' \subset B_{R_0}$ for $R_0 = r\tilde{R}_0$. $\square$

Let us now consider the functional

$$Q(E) = \mathcal{F}(E) + \frac{|K|m_f}{8M_f}\left|\beta_f(E)^2 - \varepsilon^2\right| + \Lambda\big||E| - |K|\big|, \tag{4.3.4}$$

with $0 < \varepsilon < 1$ and $\Lambda > 0$.

Lemma 4.3.2. *A minimizer exists for the problem*

$$\min\{Q(E) \,:\, E \subset B_{R_0}\}$$

for $\Lambda > 4n$ and $\varepsilon > 0$ sufficiently small. Moreover, any minimizer F satisfies

$$|F| \geq \frac{|K|}{2}, \qquad \mathcal{F}(F) \leq 2n|K|. \tag{4.3.5}$$

Proof. Let $\overline{Q} = \inf\{Q(E) : E \subset B_{R_0}\}$, and let $\{F_j\}$ be a sequence such that $Q(F_j) \to \overline{Q}$. Since $F_j \subset B_{R_0}$ and $\mathcal{F}(F_j) < 2\overline{Q}$ for j large enough, up to a subsequence, $F_j \to F$ in L^1 for some $F \subset B_{R_0}$. The lower semicontinuity of $\mathcal{F}$ (Proposition 4.2.1(1)) ensures that $\mathcal{F}(F) < \infty$.

We first show that $|F| \geq \frac{|K|}{2}$. For any $\eta > 0$, $Q(F_j) \leq \overline{Q} + |K|\eta$ for j sufficiently large. Furthermore, $\overline{Q} \leq Q(K) = \mathcal{F}(K) + \frac{\varepsilon^2|K|m_f}{8M_f}$, so

$$\big||F_j| - |K|\big| \leq \frac{1}{\Lambda}\left(\mathcal{F}(K) + |K|\eta + \frac{\varepsilon^2|K|m_f}{8M_f}\right) = \frac{|K|}{\Lambda}\left(n + \eta + \frac{\varepsilon^2 m_f}{8M_f}\right) \leq \frac{|K|}{2}$$

for ε and η sufficiently small. Therefore $|F_j| \geq \frac{|K|}{2}$, implying that $|F| \geq \frac{|K|}{2}$ as well.

We now show that $\liminf Q(F_j) \geq Q(F)$, so F is a minimizer. Recalling (4.2.8), we have

$$Q(F_j) = \mathcal{F}(F_j) + \frac{|K|m_f}{8M_f}\left|\frac{\mathcal{F}(F_j) - (n-1)\gamma_f(F_j)}{n|K|^{1/n}|F_j|^{1/n'}} - \varepsilon^2\right| + \Lambda\big||F_j| - |K|\big|$$

$$
\begin{aligned}
&\geq \mathcal{F}(F_j) + \frac{|K|m_f}{8M_f}\left|\frac{\mathcal{F}(F) - (n-1)\gamma_f(F_j)}{n|K|^{1/n}|F_j|^{1/n'}} - \varepsilon^2\right| \\
&\quad - \frac{|K|m_f}{8M_f}\left|\frac{\mathcal{F}(F_j) - \mathcal{F}(F)}{n|K|^{1/n}|F_j|^{1/n'}}\right| + \Lambda\big||F_j| - |K|\big|.
\end{aligned}
$$

Let $a = \liminf \mathcal{F}(F_j)$. Up to a subsequence, we may take this limit infimum to be a limit. By the lower semicontinuity of $\mathcal{F}$, $a \geq \mathcal{F}(F)$. Furthermore, γ_f is continuous by Proposition 4.2.1(2), so

$$
\begin{aligned}
\liminf_{j\to\infty} Q(F_j) &\geq Q(F) + (a - \mathcal{F}(F)) - \frac{|K|^{1/n'}m_f}{8n|F|^{1/n'}M_f}\,|a - \mathcal{F}(F)| \\
&= Q(F) + (a - \mathcal{F}(F))\left(1 - \frac{|K|^{1/n'}m_f}{8n|F|^{1/n'}M_f}\right) \\
&\geq Q(F) + (a - \mathcal{F}(F))\left(1 - \frac{2^{1/n'}m_f}{8nM_f}\right) \geq Q(F).
\end{aligned}
$$

Finally, $\varepsilon < 1$ and therefore $\mathcal{F}(F) \leq Q(F) \leq Q(K) \leq 2n|K|$. □

The following lemma shows that a minimizer of (4.3.4) satisfies uniform density estimates.

Lemma 4.3.3 (Density Estimates)**.** *Suppose F is a minimizer of $Q(E)$ as defined in (4.3.4) among all sets $E \subset B_{R_0}$. Then there exist $r_0 > 0$ depending on n, Λ, and $|K|$ and $0 < c_0 < 1/2$ depending on n and Λ such that for any $x \in \partial^* F$ and for any $r < r_0$,*

$$
\frac{c_0 m_f^n}{M_f^n}\,\omega_n r^n \leq |B_r(x) \cap F| \leq \left(1 - \frac{c_0 m_f^n}{M_f^n}\right)\omega_n r^n. \tag{4.3.6}
$$

Proof. We follow the standard argument for proving uniform density estimates for minimizers of perimeter functionals; see, for example, [Mag12, Theorem 16.14]. The only difficulty arises when handling the term $\frac{|K|m_f}{8M_f}|\beta_f(E)^2 - \varepsilon^2|$ in $Q(E)$, as it scales like the surface energy.

For any $x_0 \in \partial^* F$, let $r < r_0$, where r_0 is to be chosen later in the proof and r is chosen such that

$$\mathcal{H}^{n-1}(\partial^* F \cap \partial B_r(x_0)) = 0. \tag{4.3.7}$$

This holds for almost every $r > 0$. Note that if (4.3.6) holds for almost every $r < r_0$, then it must hold for all $r < r_0$ by continuity; it is therefore enough to consider r such that (4.3.7) holds. Let $G = F \setminus B_r(x_0)$. For simplicity, we will use the notation B_r for $B_r(x_0)$. Because F minimizes Q,

$$\begin{aligned}\mathcal{F}(F) + \frac{|K|m_f}{8M_f}\left|\beta_f(F)^2 - \varepsilon^2\right| + \Lambda\big||F| - |K|\big| \\ \leq \mathcal{F}(G) + \frac{|K|m_f}{8M_f}\left|\beta_f(G)^2 - \varepsilon^2\right| + \Lambda\big||G| - |K|\big|,\end{aligned}$$

and so rearranging and using the triangle inequality, we have

$$\mathcal{F}(F) \leq \mathcal{F}(G) + \frac{|K|m_f}{8M_f}\left|\beta_f(F)^2 - \beta_f(G)^2\right| + \Lambda|F \cap B_r|.$$

We subtract $\mathcal{F}(F; \mathbb{R}^n \setminus B_r)$ from both sides; this is the portion of the surface energy where $\partial^* F$ and $\partial^* G$ agree. We obtain

$$\mathcal{F}(F; B_r) \leq \int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1} + \frac{|K|m_f}{8M_f}\left|\beta_f(F)^2 - \beta_f(G)^2\right| + \Lambda|F \cap B_r|. \tag{4.3.8}$$

Indeed, this holds because (4.3.7) implies that

$$\mathcal{F}(G) = \mathcal{F}(F; \mathbb{R}^n \setminus B_r) + \int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1}.$$

We must control the term $\frac{|K|m_f}{8M_f}\left|\beta_f(F)^2 - \beta_f(G)^2\right|$ and require a sharper bound than the one obtained using Hölder continuity of γ_f shown in Proposition 4.2.1(2). Indeed, we must show that the only contributions of this term are perimeter terms that match those in (4.3.8) and terms that scale like the volume and thus behave as higher order perturbations. We have

$$
\begin{aligned}
|\beta_f(F)^2-\beta_f(G)^2| &= \frac{1}{n|K|^{1/n}}\left|\frac{\mathcal{F}(F)-(n-1)\gamma_f(F)}{|F|^{1/n'}}-\frac{\mathcal{F}(G)-(n-1)\gamma_f(G)}{|G|^{1/n'}}\right| \\
&\le \frac{2\mathcal{F}(F)}{n|K|^{1/n}}\big||F|^{-1/n'}-|G|^{-1/n'}\big|+\frac{|\mathcal{F}(F)-\mathcal{F}(G)|+(n-1)\,|\gamma_f(F)-\gamma_f(G)|}{n|K|^{1/n}|G|^{1/n'}}.
\end{aligned}
$$

The function $v(z)=1-(1-z)^{1/n'}$ is convex and increasing with $v(1)=1$, hence $v(z)\le z$ for $z\in[0,1]$. Thus, as $|G|=|F|-|F\cap B_r|$,

$$
\big||F|^{-1/n'}-|G|^{-1/n'}\big| = |G|^{-1/n'}\left(1-\left(1-\frac{|F\cap B_r|}{|F|}\right)^{1/n'}\right)\le\frac{|F\cap B_r|}{|G|^{1/n'}|F|}. \tag{4.3.9}
$$

Since $2|F|\ge|K|$ by (4.3.5), $4|G|\ge|K|$ for r_0 sufficiently small depending on n, so the right hand side of (4.3.9) is bounded by $8|K|^{-1-1/n'}|F\cap B_r|$. The coefficient $\frac{2\mathcal{F}(F)}{n|K|^{1/n}}$ is bounded by $4|K|^{1/n'}$ thanks to (4.3.5), so

$$
\frac{2\mathcal{F}(F)}{n|K|^{1/n}}\big||F|^{-1/n'}-|G|^{-1/n'}\big|\le 32|K|^{-1}|F\cap B_r|. \tag{4.3.10}
$$

Therefore, by (4.3.10) and again using the facts that $4|G|\ge|K|$, $2|F|\ge|K|$, and $m_f/M_f\le 1$, we have shown that

$$
\begin{aligned}
&\frac{|K|m_f}{8M_f}|\beta_f(F)^2-\beta_f(G)^2| \\
&\qquad\le 4|F\cap B_r|+\frac{|\mathcal{F}(F)-\mathcal{F}(G)|}{2n}+\frac{m_f}{M_f}\frac{n-1}{2n}|\gamma_f(F)-\gamma_f(G)|\,.
\end{aligned}
\tag{4.3.11}
$$

For the term $|\mathcal{F}(F)-\mathcal{F}(G)|$, using (4.3.7), we have

$$
\begin{aligned}
|\mathcal{F}(F)-\mathcal{F}(G)| &= \left|\int_{\partial^*F}f(\nu_F)\,d\mathcal{H}^{n-1}-\int_{\partial^*G}f(\nu_G)\,d\mathcal{H}^{n-1}\right| \\
&\le \mathcal{F}(F;B_r)+\int_{\partial B_r\cap F}f(\nu_{B_r})\,d\mathcal{H}^{n-1}\,,
\end{aligned}
\tag{4.3.12}
$$

using (4.3.7) and the fact that $\partial^* F$ and $\partial^* G$ agree outside of B_r. Similarly, for the term $|\gamma_f(F) - \gamma_f(G)|$, when $\gamma_f(F) \geq \gamma_f(G)$, thanks to (4.3.7) we have

$$\begin{aligned}\gamma_f(F) - \gamma_f(G) &\leq \int_{\partial^* F} \frac{(x - y_F)\cdot \nu_F(x)}{f_*(x-y_F)}\, d\mathcal{H}^{n-1} - \int_{\partial^* G} \frac{(x - y_F)\cdot \nu_G(x)}{f_*(x-y_F)}\, d\mathcal{H}^{n-1} \\ &\leq \frac{M_f}{m_f}\Big(\mathcal{F}(F;B_r) + \int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1}\Big).\end{aligned}$$

The analogous inequality holds when $\gamma_f(G) \geq \gamma_f(F)$, so

$$|\gamma_f(F) - \gamma_f(G)| \leq \frac{M_f}{m_f}\Big(\mathcal{F}(F;B_r) + \int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1}\Big). \tag{4.3.13}$$

Combining (4.3.11), (4.3.12), and (4.3.13), we have shown

$$\frac{|K| m_f}{8 M_f}\left|\beta_f(F)^2 - \beta_f(G)^2\right| \leq 4|F\cap B_r| + \frac{1}{2}\Big(\mathcal{F}(F;B_r) + \int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1}\Big). \tag{4.3.14}$$

Combining (4.3.8) and (4.3.14) and rearranging, we have

$$\frac{1}{2}\mathcal{F}(F;B_r) \leq \frac{3}{2}\int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1} + (4+\Lambda)\,|F\cap B_r|\,.$$

Proceeding in the standard way, we add the term $\frac{1}{2}\int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1}$ to both sides, which gives

$$\frac{1}{2}\mathcal{F}(F\cap B_r) \leq 2\int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1} + (4+\Lambda)\,|F\cap B_r|\,.$$

By the Wulff inequality, $\mathcal{F}(F\cap B_r) \geq n|K|^{1/n}|F\cap B_r|^{1/n'}$, and for r_0 small enough depending on n, Λ, and $|K|$, we may absorb the last term on the right hand side to obtain

$$\frac{n|K|^{1/n}|F\cap B_r|^{1/n'}}{4} \leq 2\int_{\partial B_r \cap F} f(\nu_{B_r})\, d\mathcal{H}^{n-1}. \tag{4.3.15}$$

Let $u(r) = |F \cap B_r|$, and thus $u'(r) = \mathcal{H}^{n-1}(\partial B_r \cap F)$, so the right hand side above is bounded by $2M_f u'(r)$. Furthermore, $|K|^{1/n} \geq m_f$, so (4.3.15) yields the differential inequality

$$\frac{nm_f}{8M_f} \leq u'(r)u(r)^{-1/n'} = n(u^{1/n})'.$$

Integrating these quantities over the interval $[0, r]$, we get

$$\frac{m_f r}{8M_f} \leq u(r)^{1/n} = |B_r \cap F|^{1/n},$$

and taking the power n of both sides yields the lower density estimate. The upper density estimate is obtained by applying an analogous argument, using $G = F \cup B_r(x_0)$ as a comparison set for $x_0 \in \partial^* F$ and $r < r_0$ satisfying (4.3.7). □

The following lemma is a classical argument showing that a set that is close to K in L^1 and satisfies uniform density estimates is close to K in an L^∞ sense.

Lemma 4.3.4. *Suppose that F satisfies uniform density estimates as in* (4.3.6)*. Then there exists C depending on m_f/M_f, n, and Λ such that*

$$\mathrm{hd}(\partial F, \partial K)^n \leq C|F \Delta K|,$$

where $\mathrm{hd}(\cdot,\cdot)$ *is the Hausdorff distance between sets. In particular, for any $\eta > 0$, there exists $\varepsilon > 0$ such that if $|F\Delta K| < \varepsilon$, then $K_{1-\eta} \subset F \subset K_{1+\eta}$, where $K_a = aK$.*

Proof. Let $d = \mathrm{hd}(\partial F, \partial K)$. Then there is some $x \in \partial F$ such that either $B_d(x)$ is contained entirely in the complement of K or $B_d(x)$ is entirely contained in K. If the first holds, then the lower density estimate in (4.3.6) implies that

$$|F\Delta K| \geq |F \cap B_d(x)| \geq \frac{c_0 m_f^n}{M_f^n} d^n,$$

while if the second holds, then the upper density estimate in (4.3.6) implies that

$$|F\Delta K| \geq |B_d(x) \setminus F| \geq \frac{c_0 m_f^n}{M_f^n} d^n.$$

□

We will make use of the following form of the Wulff inequality without a volume constraint.

Lemma 4.3.5. *Let $R_0 > diam(K)$ and $\Lambda > n$. Up to translation, the Wulff shape K is the unique minimizer of the functional*

$$\mathcal{F}(F) + \Lambda\big||F| - |K|\big|$$

among all sets $F \subset B_{R_0}$.

Proof. Let E be a minimizer of $\mathcal{F}(F)+\Lambda\big||F|-|K|\big|$ among all sets of finite perimeter $F \subset B_{R_0}$; this functional is lower semicontinuous so such a set exists. Comparing with K, we find that

$$\mathcal{F}(E) + \Lambda\big||E| - |K|\big| \leq \mathcal{F}(K) = n|K|. \tag{4.3.16}$$

The Wulff inequality implies that $|E| \leq |K|$, and so $\mathcal{F}(E) \geq n|E|^{1/n'}|K|^{1/n} \geq n|E|$. Thus (4.3.16) implies that $\Lambda\left(|K| - |E|\right) \leq n\left(|K| - |E|\right)$. Since $\Lambda > n$, it follows that $|E| = |K|$. It follows that E must be a translation of K, the unique (up to translation) equality case in the Wulff inequality. □

We are now ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. By (1.2.4), we need only to show that there exists a constant $C = C(n)$ such that

$$\beta_f(E)^{4n/(n+1)} \leq C\delta_f(E), \tag{4.3.17}$$

for any set of finite perimeter E with $0 < |E| < \infty$. By Lemma 4.3.1, it suffices to consider sets contained in B_{R_0}. Let us introduce the set

$$X_N = \left\{ f \; : \; \frac{M_f}{m_f} \leq N \right\}$$

for $N \geq 1$, recalling M_f and m_f defined in (4.2.2). In Steps 1–4, we prove that, for every $N \geq 1$, there exists a constant $C = C(n, N)$ such that (4.3.17) holds for any surface energy $\mathcal{F}$ corresponding to a surface tension $f \in X_N$. In Step 5, we remove the dependence of the constant on N.

Step 1: Set-up.

Suppose for the sake of contradiction that (4.3.17) is false for some N. We may then find a sequence of sets $\{E_j\}$ with $E_j \subset B_{R_0}$ and a sequence of surface energies $\{\mathcal{F}_j\}$, each $\mathcal{F}_j$ with corresponding surface tension $f^j \in X_N$, Wulff shape K_j, and support function f_*^j, such that the following holds:

$$\begin{aligned} |E_j| &= |K_j| = 1, \\ \mathcal{F}_j(E_j) - \mathcal{F}_j(K_j) &\to 0, \\ \mathcal{F}_j(E_j) &< \mathcal{F}_j(K_j) + c_1 \beta_{f^j}(E_j)^{4n/(n+1)}, \end{aligned} \tag{4.3.18}$$

where $c_1 = c_1(N, n)$ is a constant to be chosen later in the proof.

Each f^j is in X_N and is normalized to make $|K_j| = 1$ implying that $\{f^j\}$ is locally uniformly bounded above, and hence, by convexity, locally uniformly Lipschitz. By the Arzelà-Ascoli theorem, up to a subsequence, $f^j \to f^\infty$ locally uniformly. The uniform convergence ensures that this limit function f^∞ is a surface tension in X_N. We denote the corresponding surface energy by $\mathcal{F}_\infty$, Wulff shape by K_∞ and support function by f_*^∞. Note that $|K_\infty| = 1$.

There exists $c(N)$ such that $\mathcal{F}_j(E) \geq c(N)P(E)$ for any set of finite perimeter E, again thanks to $f^j \in X_N$ and $|K_j| = 1$. Then, since $\mathcal{F}_j(E_j) \to n$ (as $\mathcal{F}_j(K_j) = n$), the perimeters are uniformly bounded. Furthermore, $E_j \subset B_{R_0}$, so up to a subsequence, $E_j \to E_\infty$ in L^1 with $|E_\infty| = 1$.

Proposition 4.2.1(1) implies that $\mathcal{F}_\infty(E_\infty) \leq \lim \mathcal{F}_j(E_j) = n$, so by the Wulff inequality, $E_\infty = K_\infty$ up to translation. Furthermore, Proposition 4.2.1(2) then ensures that $\lim \gamma_{f^j}(E_j) = \gamma_{f^\infty}(K_\infty) = \frac{n}{n-1}$, and therefore, by (4.2.8),

$$\lim_{j\to\infty} \beta_{f^j}(E_j)^2 = \lim_{j\to\infty} \frac{1}{n}\left(\mathcal{F}_j(E_j) - (n-1)\gamma_{f^j}(E_j)\right) = 0.$$

Step 2: Replace each E_j with a minimizer F_j.

As in [FJ14], the idea is to replace each E_j with a set F_j for which we can say more about the regularity. We let $\varepsilon_j = \beta_{f^j}(E_j)$ and let F_j be a minimizer to the problem

$$\min\left\{Q_j(F) = \mathcal{F}_j(F) + \frac{m_{f^j}}{8M_{f^j}}|\beta_{f^j}(F)^2 - \varepsilon_j^2| + \Lambda\big||F| - 1\big| \;:\; F \subset B_{R_0}\right\}$$

for a fixed $\Lambda > 4n$. Lemma 4.3.2 ensures that such a minimizer exists. As before, $\mathcal{F}_j(F_j) \geq c(N)P(F_j)$. Pairing this with (4.3.5) provides a uniform bound on $P(F_j)$, so by compactness, $F_j \to F_\infty$ in L^1 up to a subsequence for some $F_\infty \subset B_{R_0}$.

For each j, we use the fact that F_j minimizes Q_j, choosing E_j as a comparison set. This, combined with (4.3.18) and Lemma 4.3.5, yields

$$\begin{aligned}\mathcal{F}_j(F_j) + \frac{1}{8N}|\beta_{f^j}(F_j)^2 - \varepsilon_j^2| + \Lambda\big||F_j| - 1\big| &\le Q_j(F_j) \le \mathcal{F}_j(E_j) \\ \le \mathcal{F}_j(K_j) + c_1\varepsilon_j^{4n/(n+1)} &\le \mathcal{F}_j(F_j) + \Lambda\big||F_j| - 1\big| + c_1\varepsilon_j^{4n/(n+1)}. \qquad (4.3.19)\end{aligned}$$

It follows that $\frac{1}{8N}\big|\beta_{f^j}(F_j)^2 - \varepsilon_j^2\big| \le c_1\varepsilon_j^{4n/(n+1)}$, immediately implying that $\beta_{f^j}(F_j) \to 0$. Moreover, rearranging and using the fact that $\varepsilon_j \to 0$ and $\frac{4n}{n+1} > 2$, we have

$$\frac{\varepsilon_j^2}{2^{(n+1)/2n}} \le \varepsilon_j^2 - 8Nc_1\varepsilon_j^{4n/(n+1)} \le \beta_{f^j}(F_j)^2,$$

where the exponent $(n+1)/2n$ is chosen so that, taking the power $2n/(n+1)$, we obtain

$$\varepsilon_j^{4n/(n+1)} \le 2\beta_{f^j}(F_j)^{4n/(n+1)}. \qquad (4.3.20)$$

In the last inequality in (4.3.19), if we replace F_j with arbitrary set of finite perimeter $E \subset B_{R_0}$, then we obtain

$$\mathcal{F}_j(F_j) + \Lambda\big||F_j| - 1\big| \le \mathcal{F}_j(E) + \Lambda\big||E| - 1\big| + c_1\varepsilon_j^{4n/(n+1)},$$

again using Lemma 4.3.5. Taking the limit inferior as $j \to \infty$, this implies that F_∞ is a minimizer of the problem

$$\min\left\{\mathcal{F}_\infty(F) + \Lambda||F| - 1| \ : \ F \subset B_{R_0}\right\},$$

and so $F_\infty = K_\infty$ up to a translation by Lemma 4.3.5. With no loss of generality, we translate each F_j such that $\inf\{|(F_j + z)\Delta K_\infty| : z \in \mathbb{R}^n\} = |F_j \Delta K_\infty|$.

Step 3: For j sufficiently large, $\frac{1}{2}K_j \subset F_j \subset 2K_j$ and $|F_j| = 1$.

Lemma 4.3.3 implies that each F_j satisfies uniform density estimates, and thus for

j sufficiently large, Lemma 4.3.4 ensures that $\frac{1}{2}K_j \subset F_j \subset 2K_j$, as $|K_j \Delta F_j| \leq |K_j \Delta K_\infty| + |K_\infty \Delta F_j|$ and both terms on the right hand side go to zero.

Let $r_j > 0$ be such that $|r_j F_j| = 1$. We may take $r_j F_j$ as a comparison set for F_j; $r_j \leq 2$ by Lemma 4.3.2, so $r_j F_j \subset 4K_j \subset B_{R_0}$ as long as $R_0 > 4M_f > CN$, the second inequality following from $|K_j| = 1$. Since β_{f^j} is invariant under scaling, $Q_j(F_j) \leq Q_j(r_j F_j)$ yields

$$\mathcal{F}_j(F_j) + \Lambda|1 - |F_j|| \leq r_j^{n-1}\mathcal{F}_j(F_j). \tag{4.3.21}$$

This immediately implies that $r_j \geq 1$ for all j, in other words, $|F_j| \leq 1$. Furthermore, $r_j \to 1$ because $F_j \to K_\infty$ in L^1 and $|K_\infty| = 1$. Suppose that, for some subsequence, $r_j > 1$. Then, using $|F_j| = 1/r_j^n$, (4.3.21) implies

$$\Lambda \leq \left(\frac{r_j^n(r_j^{n-1} - 1)}{r_j^n - 1}\right)\mathcal{F}_j(F_j). \tag{4.3.22}$$

For any $0 < \eta < \frac{1}{n}$ and for j sufficiently large, the right hand side is bounded by $(1-\eta)\mathcal{F}_j(F_j)$, as $\lim\limits_{r \to 1^+} \frac{r^n(r^{n-1}-1)}{r^n - 1} = \frac{n-1}{n}$. Furthermore, $\mathcal{F}_j(F_j) \leq n + \varepsilon_j^2$ since $Q_j(F_j) \leq Q_j(K_j)$, so (4.3.22) implies that

$$\Lambda \leq (1-\eta)\mathcal{F}_j(F_j) \leq (1-\eta)\left(n + \varepsilon_j^2\right) \leq n$$

for j sufficiently large. Since $n < \Lambda$, we reach a contradiction, concluding that $|F_j| = 1$ for j sufficiently large.

Step 4: Derive a contradiction to (4.3.18)*.*

We will show that $\beta_{f^j}(F_j)^{4n/(n+1)} \leq C\delta_{f^j}(F_j)$, which in turn will be used to contradict

(4.3.18). Adding and subtracting the term $\mathcal{F}_j(K_j)/n = (n-1)\gamma_{f^j}(K_j)/n$ to (4.2.8), we have

$$\beta_{f^j}(F_j)^2 \leq \frac{\mathcal{F}_j(F_j)}{n} - \frac{n-1}{n}\int_{F_j}\frac{dx}{f_*^j(x)} = \delta_{f^j}(F_j) + \frac{n-1}{n}\left(\int_{K_j}\frac{dx}{f_*^j(x)} - \int_{F_j}\frac{dx}{f_*^j(x)}\right)$$
$$= \delta_{f^j}(F_j) + \frac{n-1}{n}\left(\int_{F_j\setminus K_j} 1 - \frac{1}{f_*^j(x)}\;dx + \int_{K_j\setminus F_j}\frac{1}{f_*^j(x)} - 1\;dx\right).$$

We now control the last term in terms of $\delta_{f^j}(F_j)$. Note the following: since $\frac{1}{2}K_j \subset F_j \subset 2K_j$, the last term above is bounded by $C|F_j\Delta K_j| \leq \delta_{f^j}(F_j)^{1/2}$. This could establish (4.3.17) with the exponent 4. However, with the following argument, we obtain the improved exponent $4n/(n+1)$.

As noted before, Lemma 4.3.3 implies that each F_j satisfies uniform density estimates (4.3.6) with $m_{f^j}/M_{f^j} \geq 1/N$. The lower density estimate provides information about how far $f_*^j(x)$ can deviate from 1 for $x \in F_j \setminus K_j$, thus bounding the first integrand. Indeed, arguing as in the proof of Lemma 4.3.4, for any $x \in F_j \setminus K_j$, let $d = f_*^j(x) - 1$. The intersection $K_j \cap B_d(x)$ is empty by the definition of f_*^j, and thus $F_j \cap B_d(x) \subset F_j \setminus K_j$. Therefore, for $x \in \partial^* F_j \setminus K_j$,

$$\frac{c_0 d^n}{N^n} \leq |B_d(x) \cap F_j| \leq |F_j \Delta K_j| \leq C\delta_{f^j}(F_j)^{1/2}$$

by the lower density estimate in (4.3.6) and the quantitative Wulff inequality as in (1.2.4). In fact, this bound holds for any $x \in F_j \setminus K_j$; since F_j is bounded, for any $x \in F_j \setminus K_j$, there is some $y \in \partial^* F_j \setminus K_j$ such that $f_*^j(x) \leq f_j^*(y)$. Therefore, $f_*^j(x) - 1 \leq C\delta_{f^j}(F_j)^{1/2n}$ for all $x \in F_j \setminus K_j$, and so

$$\int_{F_j\setminus K_j} 1 - \frac{1}{f_*^j(x)} dx \leq \int_{F_j\setminus K_j} f_*(x) - 1\;dx \leq \int_{F_j\setminus K_j} C\delta_{f^j}(F_j)^{1/2n}\;dx$$
$$= C|F_j\Delta K_j|\delta_{f^j}(F_j)^{1/2n} \leq C\delta_{f^j}(F_j)^{1/2+1/2n}, \tag{4.3.23}$$

where $C = C(N, n)$ and the final inequality uses (1.2.4) once more. The analogous argument using the upper density estimate in (4.3.6), paired with the fact that eventually $\frac{1}{2}K_j \subset F_j$, provides an upper bound for the size of $1 - f_*^j(x)$ for $x \in K_j \setminus F_j$, giving

$$\int_{K_j \setminus F_j} \frac{1}{f_*^j(x)} - 1 \, dx \leq 2 \int_{K_j \setminus F_j} 1 - f_*^j(x) \, dx \leq C \delta_{f^j}(F_j)^{1/2+1/2n}. \tag{4.3.24}$$

Combining (4.3.23) and (4.3.24), we conclude that

$$\beta_{f^j}(F_j)^{4n/(n+1)} \leq C_1 \delta_{f^j}(F_j) \tag{4.3.25}$$

where $C_1 = C_1(N, n)$.

We now use the minimality of F_j, comparing against E_j, along with (4.3.18) and (4.3.20) to obtain

$$\mathcal{F}_j(F_j) \leq \mathcal{F}_j(E_j) \leq \mathcal{F}_j(K_j) + c_1 \varepsilon_j^{4n/(n+1)} \leq \mathcal{F}_j(K_j) + 2c_1 \beta_{f^j}(F_j)^{4n/(n+1)}.$$

By (4.3.20), $\beta_{f^j}(F_j)$ is positive, so by choosing $c_1 < n/2C_1$, this contradicts (4.3.25), thus proving (4.3.17) for the class X_N with the constant C depending on n and N.

Step 5: Remove the dependence on N of the constant in (4.3.17).

We argue as in [FMP10]. We will use the following notation: $\mathcal{F}_K$ is the surface energy with Wulff shape K, surface tension f^K, and support function f_*^K. We use δ_K, β_K, and γ_K to denote $\delta_{f^K}, \beta_{f^K}$, and γ_{f^K} respectively.

By John's Lemma ([Joh48, Theorem III]), for any convex set $K \subset \mathbb{R}^n$, there exists an affine transformation L such that $\det L > 0$ and $B_1 \subset L(K) \subset B_n$. This implies that $M_{L(K)}/m_{L(K)} \leq n$ and so $f^{L(K)} \in X_n$. Our goal is therefore to show that $\beta_K(E)$ and $\delta_K(E)$ are invariant under affine transformations. Indeed, once we verify that $\beta_K(E) = \beta_{L(K)}(L(E))$ and $\delta_K(E) = \delta_{L(K)}(L(E))$, we have

$$\beta_K(E)^{4n/(n+1)} = \beta_{L(K)}(L(E))^{4n/(n+1)} \leq C(n)\delta_{L(K)}(L(E)) = C(n)\delta_K(E),$$

and (4.3.17) is proven with a constant depending only on n.

Suppose E is a smooth, open, bounded set. Then

$$\mathcal{F}_K(E) = \lim_{\varepsilon \to 0} \frac{|E + \varepsilon K| - |E|}{\varepsilon};$$

this is shown by applying the anisotropic coarea formula to the function

$$d^K(x, \partial E) := \begin{cases} \inf\{f_*(x-y) : y \in \partial E\} & \text{if } x \in E^c \\ -\inf\{f_*(x-y) : y \in \partial E\} & \text{if } x \in E \end{cases}$$

and noting that $(E + \varepsilon K) \setminus E = \{x : 0 \leq d^K(x, \partial E) < \varepsilon\}$.

Since L is affine, $|L(E + \varepsilon K)| - |L(E)| = \det L\,(|E + \varepsilon K| - |E|)$, and so

$$\mathcal{F}_K(E) = \lim_{\varepsilon \to 0} \frac{|L(E + \varepsilon K)| - |L(E)|}{\varepsilon \det L} = \frac{\mathcal{F}_{L(K)}(L(E))}{\det L}.$$

Since $|E| = |L(E)|/\det L$, we have

$$\delta_K(E) = \frac{\mathcal{F}_K(E)}{n|K|^{1/n}|E|^{1/n'}} - 1 = \frac{\mathcal{F}_{L(K)}(L(E))}{n|L(K)|^{1/n}|L(E)|^{1/n'}} - 1 = \delta_{L(K)}(L(E)),$$

and thus $\delta_K(E)$ is invariant. Similarly,

$$\begin{aligned} f_*^K(L^{-1}z - y) &= \inf\left\{\lambda \ : \ \frac{L^{-1}(z) - y}{\lambda} \in K\right\} \\ &= \inf\left\{\lambda \ : \ \frac{z - L(y)}{\lambda} \in L(K)\right\} = f_*^{L(K)}\left(z - L(y)\right), \end{aligned}$$

and thus

$$\int_E \frac{dx}{f_*^K(x-y)} = \int_{L(E)} \frac{dz}{f_*^K\,(L^{-1}(z)-y)\det L} = \int_{L(E)} \frac{dz}{f_*^{L(K)}\,(z-L(y))\det L}.$$

Taking the supremum over $y \in \mathbb{R}^n$ of both sides, we have

$$\gamma_K(E) = \frac{\gamma_{L(K)}(L(E))}{\det L}.$$

From (4.2.8),

$$\beta_K(E) = \left(\frac{\mathcal{F}_K(E) - (n-1)\gamma_K(E)}{n|K|^{1/n}|E|^{1/n'}}\right)^{1/2}.$$

We have just shown that, for the denominator,

$$\left(\frac{1}{n|K|^{1/n}|E|^{1/n'}}\right)^{1/2} = \left(\frac{\det L}{n|L(K)|^{1/n}|L(E)|^{1/n'}}\right)^{1/2},$$

and for the numerator,

$$\Big(\mathcal{F}_K(E) - (n-1)\gamma_K(E)\Big)^{1/2} = \left(\frac{\mathcal{F}_{L(K)}(L(E)) - (n-1)\gamma_{L(K)}(L(E))}{\det L}\right)^{1/2}.$$

The term $\det L$ cancels, yielding

$$\beta_K(E) = \left(\frac{\mathcal{F}_{L(K)}(L(E)) - (n-1)\gamma_{L(K)}}{n|L(K)|^{1/n}|L(E)|^{1/n'}}\right)^{1/2} = \beta_{L(K)}(L(E)),$$

showing that $\beta_K(E)$ too is invariant. □

4.4 Elliptic surface tensions

In this section, we prove Theorem 4.1.3. This proof closely follows the proof of (1.2.6) in [FJ14]. Using a selection principle argument and the regularity theory for (Λ, r_0)-minimizers of $\mathcal{F}$, we reduce to the case of sets that are small C^1 perturbations of the

Wulff shape K. In [FJ14], this argument brings Fusco and Julin to the case of nearly spherical sets, at which point they call upon (4.1.9), where Fuglede proved precisely this case in [Fug89].

We therefore prove in Proposition 4.1.9 an analogue of (4.1.9) in the case of the anisotropic surface energy $\mathcal{F}$ when f is a λ-elliptic surface tension. The following lemma shows that if E is a small C^1 perturbation of the Wulff shape K with $|E| = |K|$, then the Taylor expansion of the surface energy vanishes at first order and takes the form (4.4.1). We then use the quantitative Wulff inequality as in (1.2.4) and the barycenter constraint along with (4.4.1) to prove Proposition 4.1.9.

Lemma 4.4.1. *Suppose that $\mathcal{F}$ is a surface energy corresponding to a λ-elliptic surface tension f, and E is a set such that $|E| = |K|$ and*

$$\partial E = \{x + u(x)\nu_K(x) \,:\, x \in \partial K\}$$

where $u : \partial K \to \mathbb{R}$ and $\|u\|_{C^1(\partial K)} = \varepsilon$. There exists $\varepsilon_0 > 0$ depending on λ and n such that if $\varepsilon < \varepsilon_0$,

$$\mathcal{F}(E) = \mathcal{F}(K) + \frac{1}{2}\int_{\partial K} (\nabla u)^{\mathrm{T}} \nabla^2 f(\nu_K) \nabla u - H_K u^2 \, d\mathcal{H}^{n-1} + \varepsilon\, O(\|u\|^2_{H^1(\partial K)}), \quad (4.4.1)$$

where H_K is the mean curvature of K and all derivatives are restricted to the tangential directions.

Remark 4.4.2. The second fundamental form A_K of K satisfies

$$\nabla^2 f(\nu_K(x)) A_K(x) = \mathrm{Id}_{\mathrm{T_x \partial K}} \quad \text{for all } \mathrm{x} \in \partial\mathrm{K}.$$

Therefore, $H_K = \mathrm{tr}(A_K)$ is equal to $\mathrm{tr}(\nabla^2 f\, A_K^2)$ and thus (4.4.1) agrees with, for example, [CVDM04, Corollary 4.2].

Proof of Lemma 4.4.1. For a point $x \in \partial K$, let $\{\tau_1, \dots, \tau_{n-1}\}$ be normalized eigenvectors of $\nabla \nu_K$, where each τ_i corresponds to the eigenvalue λ_i. This set is an orthonormal basis for $T_x K$, and thus $\{\tau_1, \dots, \tau_{n-1}, \nu_K\}$ is an orthonormal basis for $\mathbb{R}^n$. A basis for $T_{x+u\nu_K} E$ is given by the set $\{g_1, \dots, g_{n-1}\}$, where, adopting the notation $u_i = \partial_{\tau_i} u$,

$$g_i = \partial_{\tau_i}[x + u\nu_K] = (1 + \lambda_i u)\tau_i + u_i \nu_K.$$

We make the standard identification of an $(n-1)$-vector with a vector in $\mathbb{R}^n$ in the following way. The norm of an $(n-1)$-vector $v_1 \wedge \cdots \wedge v_{n-1}$ is given by $|v_1 \wedge \ldots \wedge v_{n-1}| = |\det(v_1, \ldots, v_{n-1})|$. If $|v_1 \wedge \ldots \wedge v_{n-1}| \neq 0$, then the vectors $v_1, \ldots, v_{n-1}$ are linearly independent and we may consider the $n-1$ dimensional hyperplane Π spanned by $v_1, \cdots, v_{n-1}$. Letting ν be a normal vector to Π, we make the identification

$$v_1 \wedge \ldots \wedge v_{n-1} = \pm |v_1 \wedge \ldots \wedge v_{n-1}|\, \nu,$$

where the sign is chosen such that $\det(v_1, \ldots, v_{n-1}, \pm\nu) > 0$. In particular, we make the identifications

$$\tau_1 \wedge \ldots \wedge \tau_{n-1} = \nu_K, \qquad \frac{g_1 \wedge \ldots \wedge g_{n-1}}{|g_1 \wedge \ldots \wedge g_{n-1}|} = \nu_E, \quad \text{and} \quad \tau_1 \wedge \ldots \wedge \nu_K \wedge \ldots \wedge \tau_{n-1} = -\tau_i.$$

The sign is negative in the third identification because

$$\begin{aligned} \det(\tau_1, \ldots, \nu_K, \ldots, \tau_{n-1}, -\tau_i) &= -\det(\tau_1, \ldots, -\tau_i, \ldots, \tau_{n-1}, \nu_K) \\ &= \det(\tau_1, \ldots, \tau_i, \ldots, \tau_{n-1}, \nu_K) = 1. \end{aligned}$$

We let $w := g_1 \wedge \ldots \wedge g_{n-1}$, and so

$$w = [(1 + \lambda_1 u)\tau_i + u_1 \nu_K] \wedge \ldots \wedge [(1 + \lambda_{n-1} u)\tau_{n-1} + u_{n-1}\nu_K]$$

$$= \prod_{i=1}^{n-1}(1+\lambda_i u)\nu_K - \sum_{i=1}^{n-1} u_i \prod_{i\neq j}(1+\lambda_j u)\tau_i$$

$$= \Big[1 + H_K u + \sum_{i<j} \lambda_i\lambda_j u^2\Big]\nu_K - \sum_{i=1}^{n-1} u_i\Big[1+\sum_{j\neq i}\lambda_j u\Big]\tau_i + \varepsilon\, O(|u|^2 + |\nabla u|^2). \quad (4.4.2)$$

In order to show (4.4.1), the volume constraint is used to show that the first order terms in the Taylor expansion of the surface tension vanish. We achieve this by expanding the volume in two different ways. First, the divergence theorem implies that

$$n|E| = \int_{\partial E} x\cdot\nu_E\, d\mathcal{H}^{n-1} = \int_{\partial K}(x+u\,\nu_K)\cdot\frac{w}{|w|}|w|\, d\mathcal{H}^{n-1} = \int_{\partial K}(x+u\,\nu_K)\cdot w\, d\mathcal{H}^{n-1}.$$

Adding and subtracting $\nu_K = \tau_1\wedge\ldots\wedge\tau_{n-1}$, and using (4.4.2) and the fact that $\nu_K\cdot\tau_i = 0$, we have

$$n|E| = \int_{\partial K} x\cdot\nu_K\, d\mathcal{H}^{n-1} + \int_{\partial K} u + x\cdot(w-\nu_K) + H_K u^2\, d\mathcal{H}^{n-1} + \varepsilon\, O(\|u\|^2_{H^1(\partial K)}).$$

Since $\int_{\partial K} x\cdot\nu_K\, d\mathcal{H}^{n-1} = n|K|$, the volume constraint $|E| = |K|$ implies that

$$\int_{\partial K} x\cdot(w-\nu_K)\, d\mathcal{H}^{n-1} = -\int_{\partial K} u + H_K u^2\, d\mathcal{H}^{n-1} + \varepsilon\, O(\|u\|^2_{H^1(\partial K)}). \quad (4.4.3)$$

Now we expand the volume in a different way. Because f is a λ-elliptic surface tension, the Wulff shape K is C^2 with mean curvature depending on λ and n. Therefore, there exists $t_0 = t_0(\lambda, n) > 0$ such that the neighborhood

$$D = \{x + t\nu_K(x) : x\in\partial K, t\in(-t_0, t_0)\}$$

satisfies the following property: for each $y\in D$, there is a unique projection $\pi : D\to\partial K$ such that $\pi(y) = x$ if and only if $y = x + t\nu_K(x)$ for some $t\in(-t_0,t_0)$. In this way, we extend the normal vector field ν_K to a vector field N_K defined on D by letting $N_K : D \to \mathbb{R}^n$ be defined by $N_K(y) = \nu_K(\pi(y))$. We also extend u to be defined on D by letting $u(y) = u(\pi(y))$ for all $y \in D$. Therefore, if $\varepsilon_0 < t_0$, ∂E may be realized as the time $t = 1$ image of ∂K under the flow defined by

$$\frac{d}{dt}\psi_t(x) = uN_K(\psi_t(x)), \qquad \psi_0(x) = x.$$

Such a flow is given by $\psi_t(x) = x + tuN_K$, and so $\nabla\psi_t(x) = \mathrm{Id} + tA$ where $A = \nabla(uN_K)$. An adaptation of the proof of [Mag12, Lemma 17.4] gives

$$J\psi_t = 1 + t\ \mathrm{tr}(A) + \frac{t^2}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)) + \varepsilon\, O(|u|^2 + |\nabla u|^2). \tag{4.4.4}$$

Integrating by parts, it is easily verified that

$$\begin{aligned}\int_K \mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\, dx \\ &= \int_K \mathrm{div}\,(uN_K\, \mathrm{div}\,(uN_K))\, dx - \int_{\partial K} \sum_{i,j=1}^n (uN_K)^{(i)} \partial_i (uN_K)^{(j)} \nu_K^{(j)}\, d\mathcal{H}^{n-1} \\ &= \int_K \mathrm{div}(uN_K\, \mathrm{div}\,(uN_K))\, dx - \int_{\partial K} u\nabla u \cdot \nu_K\, d\mathcal{H}^{n-1}.\end{aligned}$$

The second equality is clear by choosing the basis $\tau_1, \ldots, \tau_{n-1}, \tau_n$, where $\tau_n = \nu_K$. Furthermore, the divergence theorem implies that

$$\int_K \mathrm{div}\,(uN_K\, \mathrm{div}\,(uN_K))\, dx = \int_{\partial K} u\ \mathrm{div}\,(uN_K)\, d\mathcal{H}^{n-1} = \int_{\partial K} u\nabla u \cdot \nu_K + H_K u^2\, d\mathcal{H}^{n-1},$$

so that

$$\int_K \mathrm{tr}(A)^2 - \mathrm{tr}(A^2)\, dx = \int_{\partial K} H_K u^2\, d\mathcal{H}^{n-1}.$$

With this and (4.4.4) in hand, we have the following expansion of the volume:

$$|\psi_t(K)| = \int_K J\psi_t\, dx = |K| + t\int_{\partial K} u\, d\mathcal{H}^{n-1} + \frac{t^2}{2}\int_{\partial K} H_K u^2\, d\mathcal{H}^{n-1} + t^3\varepsilon\, O(\|u\|^2_{H^1(\partial K)}).$$

Therefore, the volume constraint $|K| = |E| = |\psi_1(K)|$ implies that

$$\int_{\partial K} u \, d\mathcal{H}^{n-1} = -\frac{1}{2}\int_{\partial K} H_K u^2 \, d\mathcal{H}^{n-1} + \varepsilon \, O(\|u\|^2_{H^1(\partial K)}). \tag{4.4.5}$$

Combining (4.4.3) and (4.4.5), we conclude that

$$\int_{\partial K} x \cdot (w - \nu_K) \, d\mathcal{H}^{n-1} = -\frac{1}{2}\int_{\partial K} u^2 H_K \, d\mathcal{H}^{n-1} + \varepsilon \, O(\|u\|^2_{H^1(\partial K)}). \tag{4.4.6}$$

We now proceed with a Taylor expansion of the surface energy of E:

$$\begin{aligned}
\mathcal{F}(E) &= \int_{\partial^* E} f(\nu_E) \, d\mathcal{H}^{n-1} = \int_{\partial K} f\Big(\frac{w}{|w|}\Big)|w| \, d\mathcal{H}^{n-1} = \int_{\partial K} f(w) \, d\mathcal{H}^{n-1} \\
&= \int_{\partial K} f(\nu_K) \, d\mathcal{H}^{n-1} + \int_{\partial K} \nabla f(\nu_K) \cdot (w - \nu_K) \, d\mathcal{H}^{n-1} \\
&\quad + \frac{1}{2}\int_{\partial K} [w - \nu_K]^{\mathrm{T}} \nabla^2 f(\nu_K)[w - \nu_K] \, d\mathcal{H}^{n-1} + \varepsilon \, O(\|u\|^2_{H^1(\partial K)}),
\end{aligned}$$

so, recalling that $\nabla f(\nu_K(x)) = x$ by (4.2.6),

$$\begin{aligned}
\mathcal{F}(E) = \mathcal{F}(K) &+ \int_{\partial K} x \cdot (w - \nu_K) \, d\mathcal{H}^{n-1} \\
&+ \frac{1}{2}\int_{\partial K} \sum_{i,j=1}^{n-1} u_i u_j (\tau_i^{\mathrm{T}} \nabla^2 f(\nu_K)\tau_j) \, d\mathcal{H}^{n-1} + \varepsilon \, O(\|u\|^2_{H^1(\partial K)}) \, .
\end{aligned}$$

Applying (4.4.6) yields (4.4.1), completing the proof. □

We now prove Proposition 4.1.9, using (4.4.1) as a major tool.

Proof of Proposition 4.1.9. Suppose E is a set as in the hypothesis of the proposition, i.e., $|E| = |K|$, $\operatorname{bar} E = \operatorname{bar} K$, and

$$\partial E = \{x + u(x)\nu_K(x) : x \in \partial K\},$$

where $u : \partial K \to \mathbb{R}$ is a function such that $u \in C^1(\partial K)$ and $\|u\|_{C^1(\partial K)} = \varepsilon \le \varepsilon_1$ with ε_1 to be fixed during the proof. Up to multiplying f by a constant, which changes λ by the same factor and leaves m_f/M_f unchanged, we may assume that $|K| = 1$. Let

$$B(u) = \frac{1}{2}\int_{\partial K} (\nabla u)^{\mathrm{T}} \nabla^2 f(\nu_K) \nabla u \, d\mathcal{H}^{n-1} - \frac{1}{2}\int_{\partial K} H_K u^2 \, d\mathcal{H}^{n-1},$$

so that, by (4.4.1),

$$\delta_f(E) = \frac{1}{n} B(u) + \varepsilon\, O(\|u\|^2_{H^1(\partial K)}) \tag{4.4.7}$$

as long as $\varepsilon_1 \le \varepsilon_0$ for ε_0 from Lemma 4.4.1.

Step 1: There exists $C = C(n, \lambda, m_f/M_f)$ such that, for ε_1 small enough depending on m_f/M_f and λ,

$$\Big(\int_{\partial K} |u| \, d\mathcal{H}^{n-1} \Big)^2 \le C \delta_f(E). \tag{4.4.8}$$

Step 1(a): There exists $C = C(n, m_f/M_f)$ such that, for $\varepsilon_1 = \varepsilon_1(m_f/M_f)$ small enough,

$$|E \Delta K| \le C \delta_f(E)^{1/2}. \tag{4.4.9}$$

The quantitative Wulff inequality in the form (1.2.4) states that $|E\Delta(K + x_0)| \le C(n)\delta_f(E)^{1/2}$ for some $x_0 \in \mathbb{R}^n$, so by the triangle inequality,

$$|E\Delta K| \le C(n)\delta_f(E)^{1/2} + |(K + x_0)\Delta K|. \tag{4.4.10}$$

It therefore suffices to show that $|(K + x_0)\Delta K| \le C\delta_f(E)^{1/2}$. By [Mag12, Lemma 17.9],

$$|K\Delta(K + x_0)| \le 2|x_0| P(K) \le \frac{2n}{m_f}|x_0|. \tag{4.4.11}$$

Furthermore, the barycenter constraint $\operatorname{bar} E = \operatorname{bar} K$ implies that

$$x_0 = \int_K x_0\,dx = \int_E x\,dx - \int_K x - x_0\,dx = \int_E x\,dx - \int_{K+x_0} x\,dx.$$

For ε_1 small enough depending on M_f/m_f, $E, K + x_0 \subset B_{2M_f}$, a fact that is verified geometrically since $|x_0| \to 0$ as $\varepsilon \to 0$ and thus $|x_0|$ may be taken as small as needed. Therefore,

$$|x_0| = \left| \int_E x\,dx - \int_{K+x_0} x\,dx \right| \leq 2M_f|E\Delta(K + x_0)| \leq M_f C(n)\delta_f(E)^{1/2},$$

where the second inequality comes from (1.2.4). This, (4.4.11), and (4.4.10) prove (4.4.9).

Step 1(b): For ε_1 sufficiently small depending on λ and n,

$$\int_{\partial K} |u|\,d\mathcal{H}^{n-1} \leq 2|E\Delta K|. \tag{4.4.12}$$

Let $d_K(x) = \operatorname{dist}(x, \partial K)$. As in the proof of Lemma 4.4.1, there exists $t_0 = t_0(\lambda, n)$ such that for all $t < t_0$, $\{d_K = t\} = \{x + t\nu_K(x)\}$. Take $\varepsilon_1 < t_0$ and let $G_t = \{d_K = t\} \cap (E \setminus K)$. Then

$$\begin{aligned} E \setminus K &= \{x + t\nu_K \,:\, x \in \{x \in \partial K : u(x) > 0\},\, t \in (0, u(x))\}, \\ G_t &= \{x + t\nu_K \,:\, x \in \{x \in \partial K : u(x) > t\}\}. \end{aligned}$$

The coarea formula and the area formula imply that

$$|E \setminus K| = \int_{E\setminus K} |\nabla d_K|\,dx = \int_0^\infty dt \int_{G_t} d\mathcal{H}^{n-1} = \int_0^\infty dt \int_{\{u>t\}} J(\mathrm{Id} + t\nu_K)\,d\mathcal{H}^{n-1},$$

so

$$|E \setminus K| \geq \frac{1}{2}\int_0^\infty dt \int_{\{u>t\}} d\mathcal{H}^{n-1} = \frac{1}{2}\int_0^\infty |\{u>t\}|\, dt = \frac{1}{2}\int_{\partial K} u^+ \, d\mathcal{H}^{n-1}.$$

The analogous argument yields $|K \setminus E| \geq \frac{1}{2}\int_{\partial K} u^- \, d\mathcal{H}^{n-1}$, and (4.4.12) is shown. Combining (4.4.9) and (4.4.12) implies (4.4.8).

Step 2: There exists $C = C(n, \lambda, m_f/M_f)$ such that, for $\varepsilon_1 = \varepsilon_1(n, \lambda, m_f/M_f)$ small enough,

$$\|u\|^2_{H^1(\partial K)} \leq C\delta_f(u). \tag{4.4.13}$$

The λ-ellipticity of f implies

$$\begin{aligned}\int_{\partial K} |\nabla u|^2 \, d\mathcal{H}^{n-1} &\leq \frac{1}{\lambda}\int_{\partial K} (\nabla u)^{\mathrm{T}} \nabla^2 f(\nu_K)(\nabla u)\, d\mathcal{H}^{n-1} \\ &= \frac{1}{\lambda}\Big(2B(u) + \int_{\partial K} H_K |u|^2 \, d\mathcal{H}^{n-1}\Big).\end{aligned}$$

The Wulff shape K is bounded and C^2, so H_K is bounded by a constant $C = C(n, \lambda)$. Therefore,

$$\int_{\partial K} |\nabla u|^2 \, d\mathcal{H}^{n-1} \leq \frac{2}{\lambda} B(u) + C\int_{\partial K} |u|^2 \, d\mathcal{H}^{n-1}. \tag{4.4.14}$$

As pointed out in [DPM14, proof of Theorem 4], from the Sobolev inequality on ∂K ([Sim83, Section 18]), one may produce a version of Nash's inequality on ∂K that takes the form

$$\int_{\partial K} |u|^2 \, d\mathcal{H}^{n-1} \leq c\eta^{(n+2)/n}\int_{\partial K} |\nabla u|^2 \, d\mathcal{H}^{n-1} + \frac{c}{\eta^{(n+2)/2}}\Big(\int_{\partial K} |u| \, d\mathcal{H}^{n-1}\Big)^2 \tag{4.4.15}$$

for all $\eta > 0$, Here, c is a constant depending on H_K (and therefore on λ and n) and M_f/m_f. We pair (4.4.15) with (4.4.14) and (4.4.8) to obtain

$$\int_{\partial K} |\nabla u|^2 \, d\mathcal{H}^{n-1} \leq \frac{2}{\lambda}B(u) + C\eta^{(n+2)/n}\int_{\partial K} |\nabla u|^2 \, d\mathcal{H}^{n-1} + \frac{C}{\eta^{(n+2)/2}}\delta_f(E).$$

For η small enough, we absorb the middle term into the left hand side. Then, recalling (4.4.7), we have

$$\frac{1}{2}\int_{\partial K} |\nabla u|^2 \, d\mathcal{H}^{n-1} \leq C\delta_f(E) + \varepsilon\, O\big(\|u\|^2_{H^1(\partial K)}\big).$$

Combining this estimate with (4.4.15) and (4.4.8), we find that $\int_{\partial K} |u|^2 \, d\mathcal{H}^{n-1}$ is also bounded by $C\delta_f(E) + \varepsilon\, O\big(\|u\|^2_{H^1(\partial K)}\big)$. Therefore,

$$\|u\|^2_{H^1(\partial K)} \leq C\delta_f(E) + \varepsilon\, O\big(\|u\|^2_{H^1(\partial K)}\big).$$

Finally, taking ε_1 small enough, we absorb the second term on the right, proving (4.4.13). □

We now show that if $\partial E = \{x + u\nu_K : x \in \partial K\}$ with $\|u\|_{C^1(\partial K)}$ small, then $\beta_f(E)$ is controlled by $\|u\|_{H^1(\partial K)}$. With the notation from the proof of Lemma 4.4.1,

$$n|K|\beta_f(E)^2 \leq \int_{\partial E} f(\nu_E) - \frac{x}{f_*(x)} \cdot \nu_E \, d\mathcal{H}^{n-1} = \int_{\partial K} f(w) - x \cdot w \, d\mathcal{H}^{n-1}.$$

From the expansion of $\mathcal{F}$ in the proof of Lemma 4.4.1 and the fact that $x \cdot \nu_K = f(\nu_K)$ by (4.2.3), the right hand side is equal to

$$\frac{1}{2}\int_{\partial K} (\nabla u)^{\mathrm{T}} \nabla^2 f(\nu_K) \nabla u \, d\mathcal{H}^{n-1} + \varepsilon\, O(\|u\|^2_{H^1(\partial K)}) \leq C\|u\|^2_{H^1(\partial K)} + \varepsilon\, O(\|u\|^2_{H^1(\partial K)}),$$

where $C = \|\nabla^2 f\|_{C^0(\partial K)}$. For ε sufficiently small, we absorb the term $\varepsilon\, O(\|u\|^2_{H^1(\partial K)})$ and have

$$\beta_f(E)^2 \leq \frac{C}{n|K|}\|u\|^2_{H^1(\partial K)}. \tag{4.4.16}$$

Remark 4.4.3. This is the first point at which we use the upper bound on the Hessian of f. In other words, Proposition 4.1.9 still holds for surface tensions $f \in$

$C^{1,1}(\mathbb{R}^n \setminus \{0\})$ that satisfy the lower bound on the Hessian in the definition of λ-ellipticity.

Next, we prove Theorem 4.1.3, for which we need the following definition.

Definition 4.4.4. *A set of finite perimeter E is a (Λ, r_0)-minimizer of $\mathcal{F}$, for some $0 \leq \Lambda < \infty$ and $r_0 > 0$, if*

$$\mathcal{F}(E; B(x,r)) \leq \mathcal{F}(F; B(x,r)) + \Lambda |E \Delta F|$$

for $E\Delta F \subset\subset B(x,r)$ and $r < r_0$.

Proof of Theorem 4.1.3. Proposition 4.2.4 implies that the proof reduces to showing

$$\beta_f(E)^2 \leq C\delta_f(E). \tag{4.4.17}$$

where $C = C(n, \lambda, \|\nabla^2 f\|_{C^0(\partial K)}, m_f/M_f)$. Suppose for contradiction that (4.4.17) fails. There exists a sequence $\{E_j\}$ such that $|E_j| = |K|$ for all j, $\delta_f(E_j) \to 0$, and

$$\mathcal{F}(E_j) \leq \mathcal{F}(K) + c_2 \beta_f(E_j)^2 \tag{4.4.18}$$

for c_2 to be chosen at the end of this proof. Arguing as in the proof of Theorem 4.1.1, we determine that, up to a subsequence, $\{E_j\}$ converges in L^1 to a translation of K. As in the proof of Theorem 4.1.1 (and as in [FJ14]), we replace the sequence $\{E_j\}$ with a new sequence $\{F_j\}$, where each F_j is a minimizer of the problem

$$\min \left\{ Q_j(E) = \mathcal{F}(E) + \frac{|K| m_f}{8M_f} \left|\beta_f(E)^2 - \varepsilon_j^2\right| + \Lambda \big||E| - |K|\big| \ : \ E \subset B_{R_0} \right\}$$

with $\varepsilon_j = \beta_f(E_j)$; existence for this problem is shown in Lemma 4.3.2. Continuing as in the proof of Theorem 4.1.1, we determine that

$$\varepsilon_j^2 \leq 2\beta_f(F_j)^2, \tag{4.4.19}$$

that up to a subsequence and translation, $F_j \to K$ in L^1, and that $|F_j| = |K|$ for j sufficiently large. By Lemma 4.3.3, each F_j satisfies uniform density estimates, and so by Lemma 4.3.4, for any $\eta > 0$, we may choose j sufficiently large such that $K_{1-\eta} \subset F_j \subset K_{1+\eta}$.

Arguing as in [FJ14], we show that F_j is a (Λ, r_0)-minimizer of $\mathcal{F}$ for j large enough, where Λ and r_0 are uniform in j. Let G such that $G \Delta F_j \subset\subset B_r(x_0)$ for $x_0 \in F_j$ and for $r < r_0$, where r_0 is to be fixed during the proof. For any $\eta > 0$, if $B_r(x_0) \subset K_{1-\eta}$, then trivially $\mathcal{F}(G) \geq \mathcal{F}(F_j)$. If $B_r(x_0) \not\subset K_{1-\eta}$, then for η sufficiently small, Lemma 4.2.3 implies that $|y_{F_j}| \leq 1/4$ and $|y_G| \leq 1/4$. Furthermore, by choosing η and r_0 sufficiently small, we may take $B_r(x_0) \cap K_{1/2} = \emptyset$. The minimality of F_j implies $Q(F_j) \leq Q(G)$; after rearranging and applying the triangle inequality, this implies that

$$\mathcal{F}(F_j) \leq \mathcal{F}(G) + \Lambda |F_j \Delta G| + \frac{|K| m_f}{8 M_f} \left| \beta_f(G)^2 - \beta_f(F_j)^2 \right|. \tag{4.4.20}$$

As in (4.3.11) in the proof of Lemma 4.3.3,

$$\frac{|K| m_f}{8 M_f} \left| \beta_f(F)^2 - \beta_f(G)^2 \right| \leq \frac{|\mathcal{F}(F_j) - \mathcal{F}(G)|}{2} + \frac{|\gamma_f(F_j) - \gamma_f(G)|}{2} + 4|F_j \Delta G|$$

for r_0 small enough depending on n. If $\mathcal{F}(F_j) \leq \mathcal{F}(G)$, then the (Λ, r_0)-minimizer condition is automatically satisfied. Otherwise, subtracting $\frac{1}{2}\mathcal{F}(F_j)$ from both sides of (4.4.20) and renormalizing, we have

$$\mathcal{F}(F_j) \leq \mathcal{F}(G) + |\gamma_f(G) - \gamma_f(F_j)| + (8 + 2\Lambda)|F_j \Delta G|. \tag{4.4.21}$$

To control $|\gamma_f(G) - \gamma_f(F_j)|$, we need something sharper than the Hölder modulus of continuity of γ_f given in Proposition 4.2.1(2). Indeed, γ_f is Lipschitz continuous for sets whose intersection contains a ball around their centers:

$$\gamma_f(F_j) - \gamma_f(G) \le \int_{F_j} \frac{dx}{f_*(x - y_{F_j})} - \int_G \frac{dx}{f_*(x - y_{F_j})} = \int_{F_j \Delta G} \frac{dx}{f_*(x - y_{F_j})},$$

and analogously,

$$\gamma_f(G) - \gamma_f(F_j) \le \int_{F_j \Delta G} \frac{dx}{f_*(x - y_G)}.$$

Since $B_r \cap K_{1/2} = \emptyset$, $|y_{F_j}| \le 1/4$, and $|y_G| \le 1/4$, we know that $1/f_*(x - y_{F_j}) \ge 4/m_f$ and $1/f_*(x - y_G) \ge 4/m_f$ for any $x \in F_j \Delta G$, implying that

$$|\gamma_f(F_j) - \gamma_f(G)| \le \frac{4}{m_f} |F_j \Delta G|.$$

Therefore, (4.4.21) becomes

$$\mathcal{F}(F_j) \le \mathcal{F}(G) + \Lambda_0 \left| F_j \Delta G \right|, \tag{4.4.22}$$

where $\Lambda_0 = 8 + 2\Lambda + 4/m_f$, and so F_j is a (Λ_0, r_0)-minimizer for j large enough.

We now exploit some regularity theorems for sets F_j that are (Λ, r_0)-minimizers that converge in L^1 to a C^2 set. First, let us introduce a bit of notation. For $x \in \mathbb{R}^n$, $r > 0$, and $\nu \in S^{n-1}$, we define

$$\mathbf{C}_\nu(x, r) = \{y \in \mathbb{R}^n : |p_\nu(y - x)| < r, |q_\nu(y - x) < r\},$$
$$\mathbf{D}_\nu(x, r) = \{y \in \mathbb{R}^n : |p_\nu(y - x)| < r, |q_\nu(y - x)| = 0\},$$

where $q_\nu(y) = y \cdot \nu$ and $p_\nu(y) = y - (y \cdot \nu)y$. We then define the *cylindrical excess* of E at x in direction ν at scale r to be

$$\mathbf{exc}(E, x, r, \nu) = \frac{1}{r^{n-1}} \int_{\mathbf{C}_\nu(x,r) \cap \partial^* E} \frac{|\nu_E - \nu|^2}{2} \, d\mathcal{H}^{n-1}$$

The following regularity theorem for almost minimizers of an elliptic integrand is the translation in the language of sets of finite perimeter of a classical result in the theory of currents, see [Alm66, SSA77, Bom82, DS02]. For a closer statement to ours, see Lemma 3.1 in [DPM15].

Theorem 4.4.5. *Let f be a λ-elliptic surface tension with corresponding surface energy $\mathcal{F}$. Suppose E is a (Λ, r_0)-minimizer of $\mathcal{F}$. For all $\alpha < 1$ there exist constants ε and C_1 depending on n, λ and α such that if*

$$\mathbf{exc}(E, x, r, \nu) + \Lambda r < \varepsilon$$

then there exists $u \in C^{1,\alpha}(\boldsymbol{D}_\nu(x, r))$ with $u(x) = 0$ such that

$$\begin{aligned} \boldsymbol{C}_\nu(x, r/2) \cap \partial^* E &= (\mathrm{Id} + u\nu)(\mathbf{D}_\nu(x, r/2)), \\ \|u\|_{C^0(\mathbf{D}_\nu(x_0, r/2))} &< C_1 r\, \mathbf{exc}(E, x, r, \nu)^{1/(2n-2)}, \\ \|\nabla u\|_{C^0(\mathbf{D}_\nu(x_0, r/2))} &< C_1\, \mathbf{exc}(E, x, r, \nu)^{1/(2n-2)}, \\ \text{and} \qquad r^\alpha [\nabla u]_{C^{0,\alpha}(\mathbf{D}_\nu(x, r/2))} &< C_1\, \mathbf{exc}(E, x, r, \nu)^{1/2}. \end{aligned}$$

Applying Theorem 4.4.5 as in [CL12], we come to prove the following statement.

Theorem 4.4.6. *Let f be λ-elliptic with corresponding surface energy $\mathcal{F}$ and let $\{E_j\}$ be a sequence of (Λ, r_0)-minimizers such that $E_j \to E$ in L^1, with $\partial E \in C^2$. Then there exist functions $\psi_j \in C^1(\partial E)$ such that*

$$\partial E_j = (\mathrm{Id} + \psi_j \nu_E)(\partial E),$$

and $\|\psi_j\|_{C^1(\partial E)} \to 0$.

Theorem 4.4.6 implies that we may express ∂F_j as

$$\partial F_j = \{x + \psi_j \nu_K : x \in \partial K\},$$

where $\|\psi_j\|_{C^1(\partial K)} \to 0$. Moreover, $\operatorname{bar} F_j = \operatorname{bar} K$ and $|F_j| = |K|$, so Proposition 4.1.9 and (4.4.16) imply that

$$C\delta_f(F_j) \geq \|\psi_j\|^2_{H^1(\partial K)} \geq c\beta_f(F_j)^2. \tag{4.4.23}$$

On the other hand, F_j minimizes Q_j, so choosing E_j as a comparison set and using (4.4.18) and (4.4.19), we have

$$\mathcal{F}(F_j) \leq \mathcal{F}(E_j) \leq \mathcal{F}(K) + c_2\varepsilon_j^2 \leq \mathcal{F}(K) + 2c_2\beta_f(F_j)^2.$$

By (4.4.19), $\beta_f(F_j) > 0$,. Then, using (4.4.23) and choosing c_2 sufficiently small, we reach a contradiction. □

4.5 Crystalline surface tensions in dimension 2

In this section, we prove Theorem 4.1.5. As in the previous section, we begin by showing the result in a special case, and then use a selection principle argument paired with specific regularity properties to reduce to this case.

Let $n = 2$ and suppose that f is a crystalline surface tension as defined in Definition 4.1.4, with $\mathcal{F}$ the corresponding anisotropic surface energy. The corresponding Wulff shape $K \subset \mathbb{R}^2$ is a convex polygon with normal vectors $\{\nu_i\}_{i=1}^N$. Let us fix some notation to describe K, illustrated in Figure 1. Denote by s_i the side of K with normal vector ν_i, choosing the indices such that s_i is adjacent to s_{i+1} and s_{i-1}. Let

$\theta_i \in (0, \pi)$ be the angle between s_i and s_{i+1}, adopting the convention that $s_{n+1} = s_1$. Let H_i be the distance from the origin to the side s_i. By construction,

$$f(\nu_i) = H_i. \tag{4.5.1}$$

We say that a set $E \subset \mathbb{R}^2$ is *parallel to* K if E is an open convex polygon with $\{\nu_E\} = \{\nu_i\}_{i=1}^N$, that is, $\nu_E(x) \in \{\nu_i\}_{i=1}^N$ for all $x \in \partial^* E$, and for each $i \in \{1, \ldots, N\}$, there exists $x \in \partial^* E$ with $\nu_E(x) = \nu_i$. For a set E that is parallel to K, we denote by σ_i the side of E with normal vector ν_i, and h_i the distance between the origin and σ_i; again see Figure 1. We define $\varepsilon_i = h_i - H_i$. Notice that ε_i has a sign, with $\varepsilon_i \geq 0$ when $\text{dist}(0, s_i) \leq \text{dist}(0, \sigma_i)$ and $\varepsilon_i \leq 0$ when $\text{dist}(0, s_i) \geq \text{dist}(0, \sigma_i)$. For simplicity of notation, we let $|s| = \mathcal{H}^1(s)$ for any line segment s.

The following proposition proves strong form stability for sets E that are parallel to K such that $|E| = |K|$ and $|E \Delta K| = \inf\{|E \Delta (K + y)| : y \in \mathbb{R}^2\}$. Then, by a selection principle-type argument and a rigidity result, we will reduce to this case.

Proposition 4.5.1. *Let $E \subset \mathbb{R}^2$ be parallel to K such that $|E| = |K|$ and $|E \Delta K| = \inf\{|E \Delta (K + y)| : y \in \mathbb{R}^2\}$. Then there exists a constant C depending on f such that*

$$\beta_f(E)^2 \leq C \delta_f(E).$$

Proof. Let E be as in the hypothesis of the proposition. By (4.5.1), we have

$$\mathcal{F}(E) = \sum_{i=1}^N H_i |\sigma_i|, \qquad \mathcal{F}(K) = \sum_{i=1}^N H_i |s_i|, \qquad |E| = \sum_{i=1}^N \frac{h_i |\sigma_i|}{2}, \qquad |K| = \sum_{i=1}^N \frac{H_i |s_i|}{2}.$$

Recalling that $\varepsilon_i = h_i - H_i$, we may express the volume constraint $|E| = |K|$ as

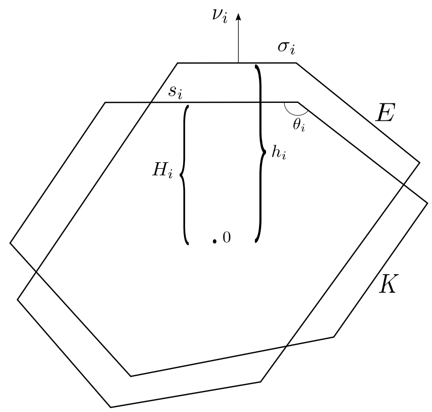

Figure 4.1: Notation used for K and a parallel set E.

$$\sum_{i=1}^{N} \frac{H_i|s_i|}{2} = |K| = |E| = \sum_{i=1}^{N} \frac{H_i|\sigma_i|}{2} + \sum_{i=1}^{N} \frac{\varepsilon_i|\sigma_i|}{2}.$$

Furthermore,

$$2|K|\delta_f(E) = \mathcal{F}(E) - \mathcal{F}(K) = \sum_{i=1}^{N} H_i(|\sigma_i| - |s_i|) = -\sum_{i=1}^{N} \varepsilon_i|\sigma_i|. \tag{4.5.2}$$

Note that $\sum_{i=1}^{N} |\varepsilon_i| \leq C|E\Delta K|$ for some constant $C = C(f)$, and so by (1.2.4),

$$\Big(\sum_{i=1}^{N} |\varepsilon_i|\Big)^2 \leq C\delta_f(E), \tag{4.5.3}$$

and in particular, $|\varepsilon_i|^2 \leq C\delta_f(E)$ for each i.

Step 1: We use (4.2.8) and add and subtract $\frac{\mathcal{F}(K)}{2|K|} = \frac{\gamma_f(K)}{2|K|}$ to obtain

$$\beta_f(E)^2 \leq \frac{1}{2|K|}\Big(\mathcal{F}(E) - \int_E \frac{dx}{f_*(x)}\Big) = \delta_f(E) + \frac{1}{2|K|}\Big(\int_{K\setminus E} \frac{dx}{f_*(x)} - \int_{E\setminus K} \frac{dx}{f_*(x)}\Big).$$

Thus we need only to control the term $A - B$ linearly by the deficit, where

$$A = \int_{K\setminus E} \frac{dx}{f_*(x)}, \qquad B = \int_{E\setminus K} \frac{dx}{f_*(x)}.$$

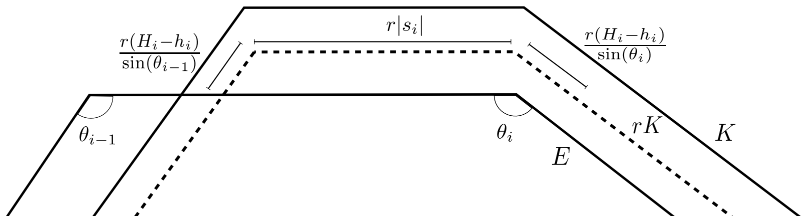

Figure 4.2: The surface energy of rK relative to $K \setminus E$ is bounded by the right hand side of (4.5.5).

To bound the term $A - B$ from above, we bound A from above and bound B from below. Our main tool is the anisotropic coarea formula in the form given in (4.2.13).

First, we consider the term A, where (4.2.13) yields

$$A = \int_{K \setminus E} \frac{dx}{f_*(x)} = \int_0^\infty \frac{\mathcal{F}(rK; K \setminus E)}{r} dr = \int_0^1 \frac{\mathcal{F}(rK; K \setminus E)}{r} dr. \tag{4.5.4}$$

We introduce the notation

$$I^- = \{i \in \{1, \dots N\} : \varepsilon_i < 0\}, \qquad I^+ = \{1, \dots N\} \setminus I^-.$$

From (4.5.4), we obtain an upper bound on A by integrating over r, for each $i \in I^-$, the part of the perimeter of rK that lies between σ_i and s_i. This means that for each r, we pick up the part of $\partial^*(rK)$ that is parallel to σ_i and s_i, as well as part of the adjacent sides:

$$\mathcal{F}(rK; K \setminus E) \leq \sum_{I^-} \left[H_i r |s_i| + H_{i-1} \frac{(rH_i - h_i)}{\sin(\theta_{i-1})} + H_{i+1} \frac{(rH_i - h_i)}{\sin(\theta_i)} \right]; \tag{4.5.5}$$

see Figure 2 and recall (4.5.1). This and (4.5.4) imply that

$$A \leq \sum_{I^-} \int_{h_i/H_i}^1 \left[H_i r |s_i| + H_{i-1} \frac{(rH_i - h_i)}{\sin(\theta_{i-1})} + H_{i+1} \frac{(rH_i - h_i)}{\sin(\theta_i)} \right] \frac{dr}{r}, \tag{4.5.6}$$

Now we add and subtract the term $\int_{h_i/H_i}^1 H_i|\sigma_i|\frac{dr}{r}$. The idea is that $H_i|\sigma_i|$ gives a rough estimate of the term in brackets on the right hand side of (4.5.6). Indeed, for each r, the part of $\partial^*(rK)$ between σ_i and s_i has length roughly equal to $H_i|\sigma_i|$. We will see that this estimate is not too rough; the error can be controlled by the deficit. Thus we rewrite (4.5.6) as

$$A \leq \sum_{i\in I^-}\int_{h_i/H_i}^1 \frac{H_i|\sigma_i|}{r}\,dr$$
$$+\sum_{i\in I^-}\int_{h_i/H_i}^1 H_i|s_i| + \left[H_i - \frac{h_i}{r}\right]\left(\frac{H_{i-1}}{\sin(\theta_{i-1})}+\frac{H_{i+1}}{\sin(\theta_i)}\right) - \frac{H_i|\sigma_i|}{r}\,dr.$$

Noting that $H_i/\sin(\theta_j) \leq C = C(f)$ for each i, j, the right hand side is bounded by $A_1 + A_2$, where

$$A_1 = \sum_{i\in I^-}\int_{h_i/H_i}^1 \frac{H_i|\sigma_i|}{r}dr, \qquad A_2 = \sum_{i\in I^-}\int_{h_i/H_i}^1 H_i|s_i| + C\left[H_i - \frac{h_i}{r}\right] - \frac{H_i|\sigma_i|}{r}dr.$$

The term A_2 is the error term that we will show is controlled by the deficit in Step 2.

First, we perform an analogous computation for B, and show how, once the error terms are taken care of, the proof is complete. Again, by (4.2.13), we have

$$B = \int_{E\setminus K}\frac{dx}{f_*(x)} = \int_0^\infty \frac{\mathcal{F}(rK; E\setminus K)}{r}\,dr = \int_1^\infty \frac{\mathcal{F}(rK; E\setminus K)}{r}\,dr.$$

To bound B from below, we integrate, for each $i \in I^+$, only the part of $\partial^*(rK)$ that is parallel to s_i and σ_i and lies between s_i and σ_i . We call this segment $\ell_i^r := E\setminus K \cap \{e_i + rx_i\}$, where e_i is the vector parallel to the sides σ_i and s_i, $x_i \in s_i$, and $r \in [1, h_i/H_i]$.

Thus, letting s_i^r be the side of rK parallel to s_i and recalling (4.5.1), we have

$$\int_1^\infty \frac{\mathcal{F}(rK; E\setminus K)}{r}\,dr \geq \sum_{i\in I^+}\int_1^{h_i/H_i}\frac{H_i|s_i^r\cap \ell_i^r|}{r}\,dr.$$

Once again, a rough estimate for $H_i|s_i^r \cap \ell_i^r|$ is given by $H_i|\sigma_i|$. We will again show that this estimate is not too rough, specifically, that the error between these integrals is controlled by the deficit. So we continue:

$$B \geq \sum_{i\in I^+} \int_1^{h_i/H_i} \frac{H_i|\sigma_i|}{r}\, dr + \sum_{i\in I^+} \int_1^{h_i/H_i} \frac{H_i|s_i^r \cap \ell_i^r|}{r} - \frac{H_i|\sigma_i|}{r} dr = B_1 + B_2,$$

where

$$B_1 = \sum_{i\in I^+} \int_1^{h_i/H_i} \frac{H_i|\sigma_i|}{r}\, dr, \qquad B_2 = \sum_{i\in I^+} \int_1^{h_i/H_i} \frac{H_i|s_i^r \cap \ell_i^r|}{r} - \frac{H_i|\sigma_i|}{r}\, dr.$$

Like A_2, B_2 is an error term that we will show is controlled by the deficit in Step 2.

Before bounding $|A_2|$ and $|B_2|$ by the deficit, let us see how this will conclude the proof. As we saw, $\beta_f(E)^2 \leq \delta_f(E) + \frac{1}{2|K|}(A - B)$. Recalling that $h_i = H_i + \varepsilon_i$,

$$\begin{aligned}
A - B &= \sum_{i\in I^-} \int_{h_i/H_i}^{1} \frac{H_i|\sigma_i|}{r} dr - \sum_{i\in I^+} \int_1^{h_i/H_i} \frac{H_i|\sigma_i|}{r} dr + A_2 - B_2 \\
&= -\sum_{i\in I^-} H_i|\sigma_i| \log\left(\frac{h_i}{H_i}\right) - \sum_{i\in I^+} H_i|\sigma_i| \log\left(\frac{h_i}{H_i}\right) + A_2 - B_2 \\
&= -\sum_{i=1}^{N} H_i|\sigma_i| \Big(\frac{\varepsilon_i}{H_i} + O(\varepsilon_i^2)\Big) + A_2 - B_2 = -\sum_{i=1}^{N} \varepsilon_i|\sigma_i| + \sum_{i=1}^{N} O(\varepsilon_i^2) + A_2 - B_2.
\end{aligned}$$

The first term is precisely equal to $2|K|\delta_f(E)$ by (4.5.2), while $\sum_i O(\varepsilon_i^2) \leq C\delta_f(E)$ by (4.5.3). Therefore, once we show that $|A_2|$ and $|B_2|$ are controlled linearly by the deficit, our proof is complete.

Step 2: In this step we bound the error terms. We show that $|A_2| \leq C\delta_f(E)$; the proof that $|B_2| \leq C\delta_f(E)$ is analogous. The main idea for estimating the integral A_2

is to show that the contribution of the adjacent sides is small, and then estimate the rest of integrand slice by slice. Recalling A_2, the triangle inequality gives

$$|A_2| \leq \left| \sum_{i\in I^-} \int_{h_i/H_i}^{1} \frac{H_i}{r}(r|s_i| - |\sigma_i|)dr \right| + C \sum_{i\in I^-} \left| \int_{h_i/H_i}^{1} \left[H_i - \frac{h_i}{r} \right] dr \right|. \tag{4.5.7}$$

The second term in (4.5.7) corresponds to the contribution of adjacent sides. By $h_i = H_i + \varepsilon_i$,

$$\begin{aligned} &C \sum_{i\in I^-} \left| \int_{h_i/H_i}^{1} \left[H_i - \frac{h_i}{r} \right] dr \right| = C \sum_{i\in I^-} \left| (H_i - h_i) + h_i \log\left(\frac{h_i}{H_i} \right) \right| \\ =&C \sum_{i\in I^-} \left| -\varepsilon_i + h_i \frac{\varepsilon_i}{H_i} + O(\varepsilon_i^2) \right| = C \sum_{i\in I^-} \left| \frac{\varepsilon_i^2}{H_i} + O(\varepsilon_i^2) \right| = C \sum_{I^-} O(\varepsilon_i^2) \leq C\delta_f(E). \end{aligned}$$

To bound the first term in (4.5.7), we will show that $\big| r|s_i| - |\sigma_i| \big| \leq C \max\{|\varepsilon_{i-1}|\}$ for $r \in [h_i/H_i, 1]$, where the constant C depends on f, and then obtain our bound by integrating. To this end, we rotate our coordinates such that $\nu_i = e_2$, so the side s_i has endpoints (a, H_i) and (b, H_i) for some $a < b$. We compute explicitly the endpoints of σ_i; it has, respectively, left and right endpoints

$$\left(a + \tan(\theta_{i-1} - \pi/2)\, \varepsilon_i - \frac{\varepsilon_{i-1}}{\sin(\theta_{i-1})},\ h_i \right) \qquad \text{and} \qquad \left(b - \tan(\theta_i - \pi/2)\, \varepsilon_i + \frac{\varepsilon_{i+1}}{\sin(\theta_i)},\ h_i \right).$$

Thus

$$|\sigma_i| = \left| b - \tan(\theta_i - \pi/2)\, \varepsilon_i + \frac{\varepsilon_{i+1}}{\sin(\theta_i)} - \left(a + \tan(\sigma_{i-1} - \pi/2)\, \varepsilon_i - \frac{\varepsilon_{i-1}}{\sin(\theta_{i-1})} \right) \right|.$$

and so

$$||\sigma_i| - |b-a|| \leq C(|\varepsilon_i| + |\varepsilon_{i+1}| + |\varepsilon_{i-1}|),$$

where C depends on f. Therefore, recalling that $|b - a| = |s_i|$,

$$\Big| r|s_i| - |\sigma_i| \Big| \leq (1-r)|s_i| + C(|\varepsilon_i| + |\varepsilon_{i+1}| + |\varepsilon_{i-1}|) \leq \frac{|\varepsilon_i|}{H_i}|s_i| + C\max\{|\varepsilon_j|\} \leq C\max\{|\varepsilon_j|\}.$$

Given this estimate on slices, we integrate over r:

$$\sum_{i\in I^-}\int_{h_i/H_i}^{1}\frac{H_i}{r}\left(r|s_i|-|\sigma_i|\right)dr \le C\max\{|\varepsilon_j|\}\sum_{i\in I^-}\int_{h_i/H_i}^{1}\frac{H_i}{r}dr$$
$$= C\max\{|\varepsilon_j|\}\sum_{i\in I^-}H_i\Big|\log\left(\frac{h_i}{H_i}\right)\Big|$$
$$= C\max\{|\varepsilon_j|\}\sum_{i\in I^-}(\varepsilon_i+O(\varepsilon_i^2)) = O(\max\{|\varepsilon_j|^2\}) \le C(\mathcal{F})\delta_f(E),$$

where the last inequality follows from (4.5.3). □

We prove Theorem 4.1.5 after introducing the following definition that we will need in the proof.

Definition 4.5.2. *A set E is a volume constrained (ε,η_0)-minimizer of $\mathcal{F}$ if*

$$\mathcal{F}(E)\le\mathcal{F}(F)+\varepsilon|E\Delta F|$$

for all F such that $|E|=|F|$ and $(1-\eta_0)E\subset F\subset(1+\eta_0)E$.

Proof of Theorem 4.1.5. By Proposition 4.2.4, we need only to show that there exists some C depending on f such that

$$\beta_f(E)^2\le C\delta_f(E). \tag{4.5.8}$$

for all sets E of finite perimeter with $0<|E|<\infty$. Suppose for contradiction that (4.5.8) does not hold. There exists a sequence $\{E_j\}$ such that $|E_j|=|K|$, $\delta_f(E_j)\to 0$, and

$$\mathcal{F}(E_j)\le\mathcal{F}(K)+c_3\beta_f(E_j)^2 \tag{4.5.9}$$

for c_3 to be chosen at the end of this proof. By an argument identical to the one given in the proof of Theorem 4.1.3, we obtain a new sequence $\{F_j\}$ with $F_j \subset B_{R_0}$ for all j such that the following properties hold:

- each F_j is a minimizer of $Q_j(E) = \mathcal{F}(E) + \frac{|K|m_f}{8M_f}|\beta_f(E)^2 - \varepsilon_j^2| + \Lambda\big||E| - |K|\big|$ among all sets $E \subset B_{R_0}$, where $\varepsilon_j = \beta_f(E_j)$;
- F_j converges in L^1 to a translation of K;
- $|F_j| = |K|$ for j sufficiently large;
- the following lower bound holds for $\beta_f(F_j)$:

$$\varepsilon_j^2 \le 2\beta_f(F_j)^2. \tag{4.5.10}$$

Translate each F_j such that $|F_j \Delta K| = \inf\{|F_j \Delta (K+y)| : y \in \mathbb{R}^2\}$. We claim that for all $\varepsilon > 0$, there exists $\eta_0 > 0$ such that F_j is a volume constrained (ε, η_0)-minimizer of $\mathcal{F}$ (Definition 4.5.2) for j large enough. Indeed, fix $\varepsilon > 0$ and let $\eta_1 = c_1\varepsilon$, where $c_1 = c_1(f)$ will be chosen later. By Lemma 4.2.3, there exists η_2 such that if $(1-\eta_2)K \subset E \subset (1+\eta_2)K$, then $|y_E| < \eta_1$. Let $\eta_0 = \min\{\eta_1, \eta_2\}/2$.

By Lemma 4.3.3, each F_j satisfies uniform density estimates, and so Lemma 4.3.4 implies that, for j large, $(1-\eta_0)K \subset F_j \subset (1+\eta_0)K$ and thus $|y_{F_j}| < \eta_1$. Let E be such that $|E| = |F_j|$ and $(1-\eta_0)F_j \subset E \subset (1+\eta_0)F_j$. Then $|y_E| < \eta_1$ and

$$(1-\eta_1)K \subset F_j \subset (1+\eta_1)K, \qquad (1-\eta_1)K \subset E \subset (1+\eta_1)K.$$

Because F_j minimizes Q_j,

$$\mathcal{F}(F_j) + \frac{|K|m_f}{4M_f}|\beta_f(F_j)^2 - \varepsilon_j^2| \le \mathcal{F}(E) + \frac{|K|m_f}{4M_f}|\beta_f(E)^2 - \varepsilon_j^2|$$

and so by the triangle inequality and since $m_f \le M_f$,

$$\mathcal{F}(F_j) \le \mathcal{F}(E) + \frac{|K|}{4}|\beta_f(E)^2 - \beta_f(F_j)^2|.$$

If $\mathcal{F}(F_j) \le \mathcal{F}(E)$, then the volume constrained minimality condition holds trivially. Otherwise, with a bound as in (4.3.11), we have

$$\mathcal{F}(F_j) \le \mathcal{F}(E) + \frac{\mathcal{F}(F_j) - \mathcal{F}(E)}{2} + \frac{|\gamma_f(E) - \gamma_f(F_j)|}{2}.$$

and so

$$\mathcal{F}(F_j) \le \mathcal{F}(E) + |\gamma_f(E) - \gamma_f(F_j)|.$$

As in the proof of Theorem 4.1.3, the Hölder modulus of continuity for γ_f shown in Proposition 4.2.1(2) does not provide a sharp enough bound on the term $|\gamma_f(E) - \gamma_f(F_j)|$; we must show that γ_f is Lipschitz when the centers of E and F_j are bounded away from their symmetric difference. In this case, we must be more careful and show that the Lipschitz constant is small when $|E| = |F|$ and E and F_j are L^∞ close. If $\gamma_f(E) \ge \gamma_f(F_j)$, then using (4.2.9), we have

$$\begin{aligned}\gamma_f(E) - \gamma_f(F_j) &\le \int_E \frac{dx}{f_*(x - y_E)} - \int_{F_j} \frac{dx}{f_*(x - y_E)} \\ &= \int_{E \setminus F_j} \frac{dx}{f_*(x - y_E)} - \int_{F_j \setminus E} \frac{dx}{f_*(x - y_E)}.\end{aligned}$$

One easily shows from the definition that for any $x, y \in \mathbb{R}^n$,

$$f_*(x) - \frac{1}{m_f}|y| \le f_*(x - y) \le f_*(x) + \frac{1}{m_f}|y|.$$

Therefore, since $(1 - \eta_1)K \subset E \Delta F_j \subset (1 + \eta_1)K$ and $|y_E| \le \eta_1$,

$$1 - \eta_1(1 + 1/m_f) \le f_*(x - y_E) \le 1 + \eta_1(1 + 1/m_f)$$

for $x \in E\Delta F_j$, implying that

$$\gamma_f(E) - \gamma_f(F_j) \leq \int_{E \setminus F_j} \frac{dx}{1 - \eta_1(1 + 1/m_f)} - \int_{F_j \setminus E} \frac{dx}{1 + \eta_1(1 + 1/m_f)} \leq C\eta_1 |E\Delta F_j|.$$

where $C = 1 + 1/m_f$. The analogous argument holds if $\gamma_f(E) \leq \gamma_f(F_j)$, and so

$$\mathcal{F}(F_j) \leq \mathcal{F}(E) + C\eta_1 |E\Delta F_j|.$$

Letting $c_1 = 1/C$, we conclude that F_j is a volume constrained (ε, η_0)-minimizer of surface energy, and for j large enough, $(1 - \eta_0/2)K \subset F_j \subset (1 + \eta_0/2)K$ by Lemma 4.3.4. Therefore, Theorem 4.5.3 below implies that, for j sufficiently large, F_j is a convex polygon with $\nu_{F_j}(x) \in \{v_i\}_{i=1}^N$ for $\mathcal{H}^1$-a.e. $x \in \partial F_j$. Moreover, for any η, $(1 - \eta)K \subset F_j \subset (1 + \eta)K$ for j large enough, so actually $\{v_{F_j}\} = \{v_i\}_{i=1}^N$ for j sufficiently large. In other words, for j large enough, F_j is parallel to K, so Proposition 4.5.1 implies that

$$\beta_f(F_j)^2 \leq C_1 \delta_f(F_j), \tag{4.5.11}$$

where C_1 depends on f. On the other hand, F_j minimizes Q_j, so comparing against E_j and using (4.5.9) and (4.5.10) implies

$$\mathcal{F}(F_j) \leq \mathcal{F}(E_j) \leq \mathcal{F}(K) + c_3\varepsilon_j^2 \leq \mathcal{F}(K) + 2c_3\beta_f(F_j)^2.$$

By (4.5.10), $\beta_f(F_j) > 0$, so choosing c_3 small enough such that $c_3 < |K|/C_1$, we reach a contradiction. □

Theorem 4.5.3 (Figalli, Maggi, Theorem 7 of [FM11])**.** *Let $n = 2$ and let f be a crystalline surface tension. There exists a constant ε_0 such that if, for some $\eta > 0$*

and some $0 < \varepsilon < \varepsilon_0$, $(1-\eta/2)K \subset E \subset (1+\eta/2)K$ *and* E *is a volume constrained* (ε, η)*-minimizer, then* E *is a convex polygon with*

$$\nu_E(x) \in \{v_i\}_{i=1}^N \qquad \textit{for } \mathcal{H}^1\textit{-a.e. } x \in \partial E.$$

Remark 4.5.4. In [FM11, Theorem 7], Figalli and Maggi assume that E is a volume constrained $(\varepsilon, 3)$-minimizer (and actually, their notion of $(\varepsilon, 3)$-minimality is slightly stronger than ours). However, by adding the additional assumption that $(1-\eta/2)K \subset E \subset (1+\eta/2)K$, it suffices to take E to be a volume constrained-(ε, η) minimizer (with the definition given here) with η as small as needed. Indeed, if $(1-\eta/2)K \subset E \subset (1+\eta/2)K$, then $(1-\eta)E \subset \mathrm{co}(E) \subset (1+\eta)E$ where $\mathrm{co}(E)$ is the convex hull of E. Then, in the proof of [FM11, Theorem 7], the only sets F used as comparison sets are such that $|E| = |F|$ and $(1-\eta)E \subset F \subset (1+\eta)E$.

4.6 An alternative definition of the oscillation index

The oscillation index $\beta_f(E)$ is the natural way to quantify the oscillation of the boundary of a set E relative to the Wulff shape K for a given surface energy $\mathcal{F}$, as it admits the stability inequality (4.1.3) with a power that is independent of f. One may wonder if it would be suitable to quantify the oscillation of E by looking at the Euclidean distance between normal vectors of E and corresponding normal vectors of K. While such a quantity may be useful in some settings, in this section we show that it does not admit a stability result with a power independent of f. This section examines the term $\beta_f^*(E)$ defined in (4.1.6) and gives two examples showing a failure of stability. We then give a relation between β_f and β_f^* for γ-λ convex surface tensions. As a consequence of Theorem 4.1.1, this implies a stability result for β^*_f, though, as the examples show, there is a necessary dependence on the γ-λ convexity of f.

The following example illustrates that there does not exist a power σ such that

$$\beta^*_f(E)^\sigma \leq C(n, f)\delta_f(E) \tag{4.6.1}$$

for all sets E of finite perimeter with $0 < |E| < \infty$ and for all surface energies $\mathcal{F}$.

Example 4.6.1. In dimension $n = 2$, we construct a sequence of Wulff shapes K_θ (equivalently, a sequence of surface tensions f_θ and surface energies $\mathcal{F}_\theta$) and a sequence of sets E_θ such that $\delta_\theta(E_\theta) \to 0$ but $\beta^*_\theta(E_\theta) \to \infty$ as $\theta \to 0$. We use the notation $\delta_\theta = \delta_{f_\theta}$ and $\beta^*_\theta = \beta^*_{f_\theta}$.

We let K_θ be a unit area rhombus where one pair of opposing vertices has angle $\theta < \frac{\pi}{4}$ and the other has angle $\frac{\pi}{2} - \theta$. The length of each side of K_θ is proportional to $\theta^{-1/2}$. Let $L = \theta^{-1/4}$. We then construct the sets E_θ by cutting away a triangle with a zigzag base and with height L from both corners of K_θ with vertex of angle θ (see Figure 3). We choose the zigzag so that each edge in the zigzag is parallel to one of the adjacent edges of K_θ. By taking each segment in the zigzag to be as small as we wish, we may make the area of each of the two zigzag triangles arbitrarily close to the area of the triangle with a straight base, which is

$$A = L^2 \tan(\theta/2) = \theta^{-1/2} \tan(\theta/2) \approx \theta^{1/2},$$

as this triangle has base $2L\tan(\theta/2)$. Both of the other two sides of the triangle have length $m = L/\cos(\theta/2)$.

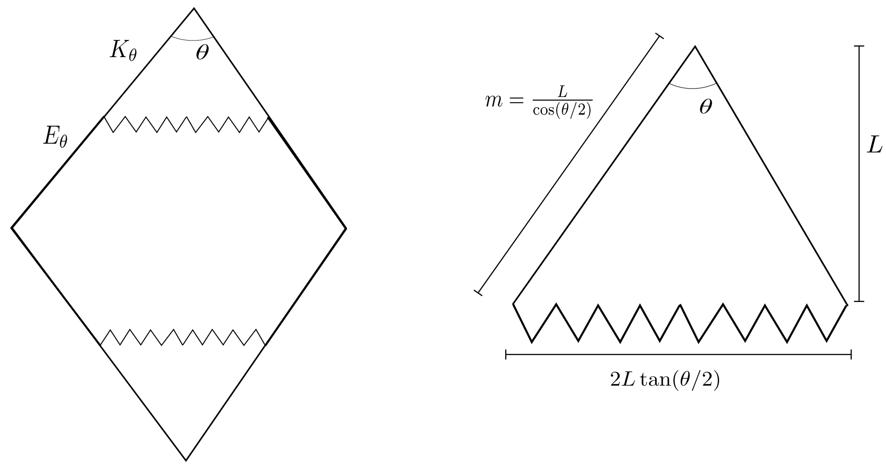

Figure 4.3: The sets E_θ are formed by cutting away a zigzag triangle from the top and bottom of K_θ and have $\delta_\theta(E_\theta) \to 0$ but $\beta^*_\theta(E_\theta) \to \infty$ as $\theta \to 0$.

Let us now compute the deficit δ_θ and the Euclidean oscillation index β^*_θ of E_θ. By construction, $\mathcal{F}_\theta(E_\theta) = \mathcal{F}_\theta(K_\theta) = 2$, and therefore

$$\delta_\theta(E_\theta) = \frac{2}{2(1-A)^{1/2}} - 1 = \frac{1}{(1-A)^{1/2}} - 1 = \theta^{1/2} + o(\theta^{1/2}).$$

To compute $\beta^*_\theta(E_\theta)^2$, we cannot characterize the point y for which the minimum in (4.1.6) is attained in general. However, something may be said for an n-symmetric set, i.e., a set E that for which there exist n orthogonal hyperplanes such that E is invariant under reflection with respect to each of them. The intersection of these orthogonal hyperplanes is called the *center of symmetry of* E. Indeed, a slight variation in the proof of [Mag08, Lemma 5.2] shows that

$$3\beta^*_\theta(E) \geq \left(\frac{1}{n|K|^{1/n}|E|^{1/n'}} \int_{\partial^* E} 1 - \nu_E(x) \cdot \nu_K\Big(\frac{x-z}{f_*(x-z)}\Big)\, d\mathcal{H}^{n-1}(x) \right)^{1/2}. \quad (4.6.2)$$

where z is the center of symmetry of E. By construction, E_θ is a 2-symmetric set with center of symmetry 0, so

$$\begin{aligned} 9\beta_\theta^*(E_\theta)^2 &\geq \frac{1}{2(1-A)^{1/2}} \int_Z 1 - \nu_{E_\theta}(x) \cdot \nu_{K_\theta}\Big(\frac{x}{f_*(x)}\Big)\, d\mathcal{H}^1 \\ &\geq \frac{1}{2} \int_Z 1 - \nu_{E_\theta}(x) \cdot \nu_{K_\theta}\Big(\frac{x}{f_*(x)}\Big)\, d\mathcal{H}^1, \end{aligned}$$

where Z denotes the union of the two zigzags. By construction, $\mathcal{H}^1(Z)$ is exactly equal to $\mathcal{H}^1(\partial K_\theta \setminus \partial E_\theta) = 4m$. Moreover, because the edges of E_θ are parallel to those of K_θ, we find that

$$1 - \nu_{E_\theta}(x) \cdot \nu_{K_\theta}\Big(\frac{x}{f_*(x)}\Big) = \begin{cases} 0 & x \in Z_1 \\ 1 - \cos(\pi - \theta) & x \in Z_2 \end{cases}$$

where Z_1 is the set of $x \in Z$ where $\nu_{E_\theta}(x)$ is equal to $\nu_{K_\theta}(\frac{x}{f_*(x)})$ and Z_2 is the set of $x \in Z$ where $\nu_{E_\theta}(x)$ is equal to the normal vector to the other side of K_θ. Moreover, we have constructed E_θ so that $\mathcal{H}^1(Z_1) = \mathcal{H}^1(Z_2) = 2m$. Thus, as $\theta < \frac{\pi}{4}$,

$$\beta_\theta^*(E_\theta)^2 \geq \frac{1}{2} \int_{Z_2} 1 - \cos(\pi - \theta)\; d\mathcal{H}^1 \geq \frac{\mathcal{H}^1(Z_2)}{2} = m = 1/(\theta^{1/4}\cos(\theta/2)) \to \infty$$

as $\theta \to 0$. Therefore, for any exponent σ, the inequality (4.6.1) fails to hold; we may choose θ sufficiently small such that E_θ is a counterexample.

The next example shows that, even if we restrict our attention to surface energies that are γ-λ convex (Definition 4.1.6), an inequality of the form in (4.6.1) cannot hold with an exponent smaller than $\sigma = 4$. The example is presented in dimension $n = 2$ for convenience, though the analogous example in higher dimension also holds.

Example 4.6.2. Fix $p > 2$ and define the surface tension $f_p(x) = (|x_1|^p + |x_2|^p)^{1/p}$ to be the ℓ^p norm in $\mathbb{R}^2$. We show below that f_p is a γ-λ convex surface tension. Hölder's inequality ensures that the support function f_* is given by f_q, in the notation above,

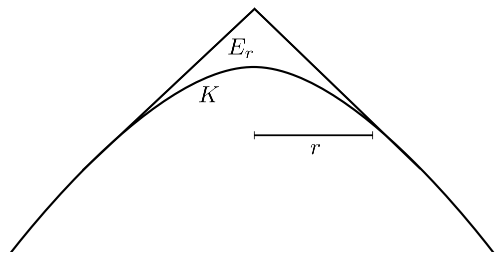

Figure 4.4: The sets E_r formed by replacing the top and bottom of the ℓ^q unit ball with a cone show that (4.6.1) cannot hold for $\sigma < 4$.

where q is the Hölder conjugate of p. The Wulff shape $K = \{f_q(x) < 1\}$ is therefore the ℓ^q unit ball. We let $\mathcal{F}_p$ denote the surface energy corresponding to the surface tension f_p.

We build a sequence of sets $\{E_r\}$ depending on p such that, for any $\sigma < 4$, we may choose p large enough so that $\delta_p(E_r)/\beta^*_p(E_r)^\sigma \to 0$ as $r \to 0$. Here we use the notation $\beta^*_p(E) = \beta^*_{f_p}(E)$ and $\delta_p(E) = \delta_{f_p}(E)$. We may locally parameterize K near $(0,1)$ as the subgraph of the function $v_q(x_1) = (1 - |x_1|^q)^{1/q}$. Thus $v_q'(x_1) = -|x_1|^{q-2}x_1/(1 - |x_1|^q)^{1/p}$ and

$$\nu_K((x_1,\ v_q(x_1))) = \frac{\left(\frac{|x_1|^{q-2}x_1}{(1-|x_1|^q)^{1/p}},\ 1\right)}{\sqrt{1 + \frac{|x_1|^{2q-2}}{(1-|x_1|^q)^{2/p}}}} = \frac{(|x_1|^{q-2}x_1 + O(|x_1|^{2q-1}),\ 1)}{\sqrt{1 + |x_1|^{2q-2} + O(|x_1|^{3q-2})}} \tag{4.6.3}$$

The sets E_r are formed by replacing the top and bottom of K with cones. More precisely, let $\mathbf{C}_r = (-r,r) \times \mathbb{R}$. We form E_r by replacing $\partial K \cap \mathbf{C}_r$ with the graphs of w and $-w$, where $w_1 : (-r,r) \to \mathbb{R}$ is defined by $w(x_1) = -r^{q-1}|x_1|/(1 - r^q)^{1/p} + C_0$. Here, the constant $C_0 = (1-r^q)^{1/q} + r^q/(1 - r^q)^{1/p}$ is chosen so that $w(r) = v_q(r)$ and $w(-r) = v_q(-r)$. For $x_1 \in (-r,r)$ for $r < 1$, we have $w'(x_1) = -r^{q-1}\mathrm{sgn}(x_1)/(1 -$

$r^q)^{1/p}$ and

$$\nu_E\left((x_1,\ w(x_1))\right) = \frac{\left(\operatorname{sgn}(x_1)\frac{r^{q-1}}{(1-r^q)^{1/p}},\ 1\right)}{\sqrt{1+\frac{r^{2q-2}}{(1-r^q)^{2/p}}}} = \frac{(\operatorname{sgn}(x_1)r^{q-1}+O(r^{2q-1}),\ 1)}{\sqrt{1+r^{2q-2}+O(r^{3q-2})}}. \tag{4.6.4}$$

Now, $\mathcal{F}_p(E_r) = \mathcal{F}_p(K) + \mathcal{F}_p(E_r;\mathbf{C}_r) - \mathcal{F}_p(K;\mathbf{C}_r)$, so

$$\begin{aligned}\mathcal{F}_p(E_r) - \mathcal{F}_p(K) &= \int_{-r}^{r}\Big(\frac{r^q}{1-r^q}+1\Big)^{1/p} - \Big(\frac{|x_1|^q}{1-|x_1|^q}+1\Big)^{1/p}dx_1\\ &= \frac{1}{p}\int_{-r}^{r} r^q - |x_1|^q + O(r^{2q})\ dx_1 = Cr^{q+1} + o(r^{q+1}).\end{aligned}$$

The graph of w lies above the graph of v_q for all $|x_1| < r$, so $|E_r| > |K|$. This implies that

$$\delta_p(E_r) \le \frac{\mathcal{F}_p(E_r) - \mathcal{F}_p(K)}{2|K|} = Cr^{q+1} + o(r^{q+1}).$$

Next we compute $\beta_p^*(E_r)$ in several steps. As in Example 4.6.1, E_r is a 2-symmetric set with center of symmetry 0, thus it is enough to compute the right hand side of (4.6.2). First, the Taylor expansions in (4.6.3) and (4.6.4) imply that, for $x \in \mathbf{C}_r \cap \partial^* E$, $\nu_E(x)\cdot\nu_K\big(\frac{x}{f_*(x)}\big)$ is given by

$$\begin{aligned}&\frac{(|x_1|^{q-2}x_1 + O(|x_1|^{2q-1}),\ 1)}{\sqrt{1+|x_1|^{2q-2}+O(|x_1|^{3q-2})}}\cdot\frac{(\operatorname{sgn}(x_1)r^{q-1}+O(r^{2q-1}),\ 1)}{\sqrt{1+r^{2q-2}+O(r^{3q-2})}}\\ &\qquad= \frac{1+|x_1|^{q-1}r^{q-1}+O(r^{3q-2})}{\sqrt{(1+|x_1|^{2q-2}+r^{2q-2}+O(r^{4q-4}))}}\\ &\qquad= 1+|x_1|^{q-1}r^{q-1} - \frac{1}{2}(|x_1|^{2q-2}+r^{2q-2}) + O(r^{3q-2})\\ &\qquad= 1-\frac{1}{2}(|x_1|^{q-1}-r^{q-1})^2 + O(r^{3q-2}).\end{aligned}$$

For $x \in \partial^* E \setminus \mathbf{C}_r$, $\nu_E(x)\cdot\nu_K\big(\frac{x}{f_*(x)}\big) = 0$. Hence,

$$\Big(\frac{1}{2}\int_{\partial^* E}\Big|\nu_E(x) - \nu_K\Big(\frac{x}{f_*(x)}\Big)\Big|^2 d\mathcal{H}^1\Big)^{1/2} = \Big(\int_{\partial^* E\cap\mathbf{C}_r} 1 - \nu_E\cdot\nu_K\big(\frac{x}{f_*(x)}\big)\, d\mathcal{H}^1\Big)^{1/2}$$

$$= \Big(\int_{-r}^{r} \frac{1}{2}(|x_1|^{q-1} - r^{q-1})^2 \sqrt{1 + r^{2q-2} + O(r^{3q-1})} + O(r^{3q-2})\, dx_1 \Big)^{1/2}$$
$$= \mathcal{B}9g(\int_{-r}^{r} \frac{1}{2}(|x_1|^{q-1} - r^{q-1})^2 + O(r^{3q-2})\, dx_1 \Big)^{1/2} = Cr^{q-1/2} + o(r^{q-1/2}).$$

Furthermore, $|E| = |K| + o(1)$, so $\sqrt{2}|K|^{-1/4}|E|^{-1/4} = \sqrt{2}|K|^{-1/2} + o(1)$, and so

$$\beta_p^*(E_r) = \frac{1}{2|K|^{1/4}|E|^{1/4}} \Big(\int_{\partial^* E} \Big| \nu_E(x) - \nu_K\Big(\frac{x}{f_*(x)}\Big)\Big|^2 d\mathcal{H}^1 \Big)^{1/2} = Cr^{q-1/2} + o(r^{q-1/2}).$$

Therefore,

$$\frac{\delta_p(E_r)}{\beta_p^*(E_r)^\sigma} \approx \frac{r^{(q+1)}}{r^{\sigma(q-1/2)}} = r^{q+1-\sigma q+\sigma/2}.$$

This quantity goes to 0 as r goes to zero if and only if $q + 1 - \sigma q + \sigma/2 > 0$, or, equivalently, if and only if $\frac{2+\sigma}{2(\sigma-1)} > q$. For any $\sigma < 4$ we may find $1 < q < \frac{2+\sigma}{2(\sigma-1)}$. Therefore, for any $\sigma < 4$, there exists a γ-λ convex surface tension f such that a bound of the form $\delta_f(E) \geq C\beta_f^*(E)^\sigma$ fails.

When f is γ-λ convex (recall Definition 4.1.6), we can control $\beta_f^*(E)$ by $\beta_f(E)$. As one expects after the previous example, the exponent in this bound depends on the γ-λ convexity of $\mathcal{F}$. Indeed, this is the content of Theorem 4.1.7. First, we show that the ℓ_p norms f_p as defined in the previous example are γ-λ convex for each $p \in (1, \infty)$. In the case where $1 < p \leq 2$, f_p is actually *uniformly* convex in tangential directions, so it is γ-λ convex with $\gamma = 0$. Indeed, $f_p(\nu+\tau) = f_p(\nu)+\nabla f_p(\nu)\tau+\frac{1}{2}\int_0^1 \nabla^2 f_p(\nu+s\tau)[\tau,\tau]ds$, and thus

$$f_p(\nu + \tau) + f_p(\nu - \tau) - 2f_p(\nu) = \frac{1}{2}\int_{-1}^{1} \nabla^2 f_p(\nu + s\tau)[\tau, \tau]ds.$$

We can bound the integrand from below pointwise. We compute

$$\partial_{ii} f_p(\nu) = (p-1)\Big(\frac{|\nu_i|^{p-2}}{f_p(\nu)^{p-1}} - \frac{|\nu_i|^{2p-2}}{f_p(\nu)^{2p-1}}\Big), \qquad \partial_{ij} f_p(\nu) = (1-p)\frac{|\nu_i|^{p-2}\nu_i|\nu_j|^{p-2}\nu_j}{f_p(\nu)^{2p-1}}.$$

Therefore, if $f_p(\nu) = 1$, then

$$\nabla^2 f_p(\nu) = (p-1)\sum_{i=1}^n |\nu_i|^{p-2} e_i \otimes e_i - (p-1)\sum_{i,j=1}^n |\nu_i|^{p-2}\nu_i |\nu_j|^{p-2}\nu_j e_i \otimes e_j$$

and so

$$\nabla^2 f_p(\nu)[\tau,\tau] = (p-1)\sum_{i=1}^n |\nu_i|^{p-2}\tau_i^2 - (p-1)\Big(\sum_{i=1}^n |\nu_i|^{p-2}\nu_i\tau_i\Big)^2.$$

It is enough to consider τ such that τ is tangent to $K_p = \{f_p < 1\}$ at ν, as f_p is positive 1-homogeneous and the span of ν and $T_\nu K_p$ is all of $\mathbb{R}^n$. Observe that $\nabla f_p(\nu) = \sum_{i=1}^n |\nu_i|^{p-2}\nu_i e_i$; this is verified by the fact that the support function of f_p is f_q, and that $\nabla f_p(\nu) = \frac{x}{f_q(x)}$ such that $\frac{x}{f_q(x)} \cdot \nu = f_p(\nu) = 1$. Thus τ is tangent to K_p at ν if and only if $\tau \cdot \nabla f_p(\nu) = \sum_{i=1}^n |\nu_i|^{p-2}\nu_i\tau_i = 0$. Therefore, for such τ,

$$\nabla^2 f_p(\nu)[\tau,\tau] = (p-1)\sum_{i=1}^n |\nu_i|^{p-2}\tau_i^2 \geq (p-1)|\tau|^2.$$

In the case where $p \geq 2$, we use Clarkson's inequality, which states that for $p \geq 2$,

$$f_p\Big(\frac{x+y}{2}\Big)^p + f_p\Big(\frac{x-y}{2}\Big)^p \leq \frac{f_p(x)^p}{2} + \frac{f_p(y)^p}{2}.$$

For ν such that $f_p(\nu) = 1$ and τ tangent to K_p at ν with $f_p(\tau) = 1$, Clarkson's inequality with $x = \nu + \varepsilon\tau$ and $y = \nu - \varepsilon\tau$ implies

$$2\varepsilon^p \leq f_p(\nu+\varepsilon\tau)^p + f_p(\nu-\varepsilon\tau) - 2.$$

This is almost the condition we need, except we have f_p^p instead of f_p for the terms on the right hand side. Note that both $f_p(\nu+\varepsilon\tau)$ and $f_p(\nu-\varepsilon\tau)$ are greater than 1, as moving in the tangent direction to $K_p = \{f_p < 1\}$ increases f_p. The function z^p is convex with derivative pz^{p-1}, so $z^p \leq 2^{p-1}pz + (2^{p-1}p - 1)$ for all $z \in [1, 2]$. Applying this to $z_1 = f_p(\nu + \varepsilon\tau)$ and $z_2 = f_p(\nu - \varepsilon\tau)$ yields

$$2\varepsilon^p \leq 2^{p-1}pf_p(\nu + \varepsilon\tau)^p + 2^{p-1}pf_p(\nu - \varepsilon\tau)^p - 2(2^{p-1}p).$$

Thus f_p is γ-λ convex with $\gamma = p - 2$ and $\lambda = 1/(2^{p-2}p)$.

The following lemma about γ-λ convexity condition will be used in the proof of Theorem 4.1.7.

Lemma 4.6.3. *Assume that f is γ-λ convex. Then for all $\nu, \tau \in \mathbb{R}^n$ such that $\nu \neq 0$,*

$$f(\nu + \tau) \geq \frac{\lambda}{2^{2+\gamma}|\nu|}\left|\tau - \left(\tau \cdot \frac{\nu}{|\nu|}\right)\frac{\nu}{|\nu|}\right|^{2+\gamma} + f(\nu) + \nabla f(\nu) \cdot \tau, \tag{4.6.5}$$

Proof. Note that if f is γ-λ convex, then f is convex. To see that (4.6.5) holds for given ν_0 and τ_0, we let $\tilde{f}(\nu) = f(\nu) - f(\nu_0) - \nabla f(\nu_0) \cdot (\nu - \nu_0)$. At the midpoint $\nu_0 + \frac{\tau_0}{2}$, the γ-λ convexity condition gives us the following:

$$\tilde{f}(\nu_0) + \tilde{f}(\nu_0 + \tau_0) - 2\tilde{f}(\nu_0 + \frac{\tau_0}{2}) \geq \frac{\lambda}{|\nu_0|}\left|\frac{\tau_0}{2} - \left(\frac{\tau_0}{2} \cdot \frac{\nu_0}{|\nu_0|}\right)\frac{\nu_0}{|\nu_0|}\right|^{2+\gamma}.$$

Convexity implies that $\tilde{f}(\nu_0 + \frac{\tau_0}{2}) \geq 0$, and $\tilde{f}(\nu_0) = 0$ by definition of $\tilde{f}$, implying (4.6.5). □

Finally, we prove Theorem 4.1.7.

Proof of Theorem 4.1.7. The quantity $\beta_f^*(E)$ measures the overall size of the Cauchy-Schwarz deficit on the boundary of E, while $\beta_f(E)$ measures the overall deficit in the Fenchel inequality. Our aim is to obtain a pointwise bound of the Cauchy-Schwarz deficit functional by the Fenchel deficit functional, and then integrate over the reduced boundary of E. Without loss of generality, we may assume that $|E| = |K| = 1$ and E has center zero in the sense defined in Section 4.2.3.

We fix $x \in \partial^* E$ and consider the Fenchel deficit functional $G(\nu) = f(\nu) - \nu \cdot \frac{x}{f_*(x)}$, which possesses the properties that $G(\nu) \geq 0$ and $G(\nu) = 0$ if and only if $\nu = c \nabla f_*(x)$ for some $c > 0$.

Let $w = \frac{\nabla f_*(x)}{|\nabla f_*(x)|} = \nu_K(\frac{x}{f_*(x)})$. Lemma 4.6.3, with $\nu = w$ and $\tau = \nu_E - w$, implies that

$$f(\nu_E) \geq \frac{\lambda}{2^{2+\gamma}} |(\nu_E - w) - ((\nu_E - w) \cdot w)w|^{2+\gamma} + f(w) + \nabla f(w) \cdot (\nu_E - w).$$

Therefore, since $\nabla f(w) = \frac{x}{f_*(x)}$ and $f(w) = \nabla f(w) \cdot w$,

$$\begin{aligned} G(\nu_E) \geq \frac{\lambda}{2^{2+\gamma}} |(\nu_E - w) - ((\nu_E - w) \cdot w)w|^{2+\gamma} &= \frac{\lambda(1 - (\nu_E \cdot w)^2)^{(2+\gamma)/2}}{2^{2+\gamma}} \\ &= \frac{\lambda((1 - \nu_E \cdot w)(1 + \nu_E \cdot w))^{(2+\gamma)/2}}{2^{2+\gamma}}. \end{aligned}$$

We want to show that there exists some c_1 such that

$$G(\nu_E) \geq c_1 (1 - \nu_E \cdot w)^{(2+\gamma)/2}. \tag{4.6.6}$$

When $w \cdot \nu_E \geq -c_0$ for some fixed $0 < c_0 < 1$, then $G(\nu_E) \geq \frac{\lambda}{2^{2+\gamma}}(1 - c_0)^{(2+\gamma)/2}(1 - \nu_E \cdot w)^{(2+\gamma)/2}$ and (4.6.6) holds. On the other hand, when $w \cdot \nu_E < -c_0$ for c_0 small, we expect that $\frac{x}{f_*(x)} \cdot \nu_E$ must also be small and so $G(\nu_E)$ is not too small. Indeed,

$$m_f \leq f(w) = \frac{x}{f_*(x)} \cdot w = \frac{|x|}{f_*(x)} \cos(\theta_1) \leq M_f \cos(\theta_1),$$

where θ_1 is the angle between w and $\frac{x}{f_*(x)}$. Similarly,

$$-c_0 \geq \nu_E \cdot w = \cos(\theta_2),$$

where θ_1 is the angle between w and ν_E. Noting that $0 < m_f/M_f < 1$, and so $\cos^{-1}(m_f/M_f) \in (0, \pi/2)$, we let $\theta_0 = 2\cos^{-1}(m_f/M_f) + \varepsilon$, where $\varepsilon > 0$ is chosen small enough so that $\theta_0 < \pi$. Letting $c_0 = -\cos(\theta_0)$, we deduce that $\theta_1 \le \cos^{-1}(m_f/M_f)$ and $\theta_2 \ge \theta_0$. Then

$$\frac{x}{f_*(x)} \cdot \nu_E \le \frac{|x|}{f_*(x)} \cos(\theta_2 - \theta_1) \le M_f \cos\left(\cos^{-1}(m_f/M_f) + \varepsilon\right) \le m_f - M_f c_\varepsilon,$$

for a constant $c_\varepsilon > 0$. Since $f(\nu_E) \ge m_f$, we have $G(\nu_E) \ge M_f c_\varepsilon$, implying (4.6.6) because $1 - \nu_E \cdot w \le 2$.

Hölder's inequality and (4.6.6) imply

$$\begin{aligned}\int_{\partial^* E} 1 - \nu_E \cdot w \, d\mathcal{H}^{n-1} &\le \mathcal{H}^{n-1}(\partial^* E)^{\gamma/(2+\gamma)} \Big(\int_{\partial^* E} (1 - \nu_E \cdot w)^{(2+\gamma)/2} d\mathcal{H}^{n-1} \Big)^{2/(2+\gamma)} \\ &= c_1^{-2/(2+\gamma)} P(E)^{\gamma/(2+\gamma)} \Big(\int_{\partial^* E} c_1 (1 - \nu_E \cdot w)^{(2+\gamma)/2} d\mathcal{H}^{n-1} \Big)^{2/(2+\gamma)} \\ &\le c_1^{-2/(2+\gamma)} P(E)^{\gamma/(2+\gamma)} \Big(\int_{\partial^* E} G(\nu_E) d\mathcal{H}^{n-1} \Big)^{2/(2+\gamma)}.\end{aligned}$$

Dividing by $n|K|^{1/n}|E|^{1/n'}$ and taking the square root, we obtain

$$\beta_f^*(E) \le c_1^{-1/(2+\gamma)} \Big(\frac{P(E)}{n|K|^{1/n}|E|^{1/n'}} \Big)^{\gamma/2(2+\gamma)} \beta_f(E)^{2/(2+\gamma)}.$$

□

Appendices

Appendix A

Uniqueness of minimizers of $\Phi(T)$

In this appendix, we prove Theorem 2.2.3, which aims to characterize the equality cases in Theorem 2.2.1. The main step is to prove the validity of (2.2.10) (see the proof of Theorem 2.2.1) without the assumption that $f \in C^1_c(\overline{H})$. This is the content of the following lemma, whose proof resembles [CENV04, Theorem 7].

Lemma A.0.1. *If $n \geq 2$, $p \in [1,n)$, and f and g are non-negative functions in $L^1_{\rm loc}(H)$, vanishing at infinity, with*

$$\begin{cases} \int_H |\nabla f|^p < \infty \text{ and } \int_H |x|^{p'} g^{p^\star} < \infty \text{ if } p > 1 \\ |Df|(H) < \infty \text{ and } \operatorname{spt} g \subset\subset \overline{H} \text{ if } p = 1 \\ \|f\|_{L^{p^\star}(H)} = \|g\|_{L^{p^\star}(H)} = 1 \end{cases} \tag{A.0.1}$$

then (2.2.10) *holds for every $t \in \mathbb{R}$, that is*

$$n \int_H g^{p^\sharp} \leq -p^\sharp \int_H f^{p^\sharp - 1} \nabla f \cdot (T - t\, e_1) + t \int_{\partial H} f^{p^\sharp}, \qquad \forall t \in \mathbb{R}. \tag{A.0.2}$$

Here $T = \nabla\varphi$ is the Brenier map from $f^{p^\star}\, dx$ and $g^{p^\star} dx$.

Proof. We let Ω be the interior of $\{\varphi < \infty\}$, and recall that $T \in (BV \cap L^\infty)_{\rm loc}(\Omega; \mathbb{R}^n)$ with $F\, dx$ concentrated on $H \cap \Omega$. We notice that in the proof of Theorem 2.2.1, see (2.2.9), the identity

$$\int_H g^{p^\sharp} = \int_H (\det \nabla^2 \varphi)^{1/n} f^{p^\sharp}, \tag{A.0.3}$$

was established without exploiting the additional assumption $f \in C^1_c(\overline{H})$. Thus (A.0.3) also holds in the present setting.

We first let $p \in (1,n)$. By a translation orthogonal to e_1, we may assume that $0 \in \Omega$. For $\varepsilon > 0$ let $\eta_\varepsilon \in C^\infty_c(B_{2/\varepsilon};[0,1])$ with $\eta_\varepsilon = 1$ on $B_{1/\varepsilon}$ and $\eta_\varepsilon \uparrow 1$ pointwise on $\mathbb{R}^n$ as $\varepsilon \to 0^+$, and set

$$f_\varepsilon(x) = \min\Big\{ f\Big(\frac{x}{1-\varepsilon}\Big), f(x)\eta_\varepsilon(x)\Big\} 1_{H_\varepsilon}(x), \qquad x \in H\,,$$

where $H_\varepsilon = \{x_1 > \varepsilon\}$. By density of $C^0_c(H)$ into $L^{p^\star}(H)$ we see that $f\circ((1-\varepsilon)^{-1}\mathrm{Id}) \to f$ in $L^{p^\star}(H)$ as $\varepsilon \to 0^+$, so that $f_\varepsilon \to f$ in $L^{p^\star}(H)$. Analogously, $\nabla[f\circ((1-\varepsilon)^{-1}\mathrm{Id})] \to \nabla f$ in $L^p(H)$ as $\varepsilon \to 0^+$. If we choose $\eta_\varepsilon(x) = \eta(\varepsilon\, x)$ for some fixed $\eta \in C^1_c(B_2;[0,1])$ with $\eta = 1$ on B_1, then we find

$$\int_H |f\nabla\eta_\varepsilon|^p \le \Big(\int_{\mathbb{R}^n\setminus B_{1/\varepsilon}} f^{p^\star}\Big)^{p/p^\star} \Big(\int_{B_2} |\nabla\eta|^n\Big)^{p/n} \to 0 \qquad \text{as } \varepsilon \to 0^+\,,$$

and thus $\nabla(f\,\eta_\varepsilon) \to \nabla f$ in $L^p(H)$. Finally, $\int_{H\setminus H_\varepsilon} |\nabla f|^p \to 0$ as $\varepsilon \to 0^+$, so that

$$\begin{cases} f_\varepsilon \to f \text{ in } L^{p^\star}(H) \text{ and a.e. on } H \\ 1_{H_\varepsilon}\nabla f_\varepsilon \to \nabla f \text{ in } L^p(H) \end{cases} \qquad \text{as } \varepsilon \to 0^+\,. \tag{A.0.4}$$

Moreover, as $0 \in \Omega$ and $f = 0$ a.e. on Ω^c, there exists an open set $\Omega_\varepsilon \mathbf{c} \Omega$ such that $\mathrm{spt}(f_\varepsilon)\mathbf{c}\Omega_\varepsilon$. We can thus find $\{f_{\varepsilon,k}\}_{k\in\mathbb{N}} \subset C^1_c(\Omega_\varepsilon \cap \overline{H_\varepsilon})$ such that

$$\begin{cases} f_{\varepsilon,k} \to f_\varepsilon \text{ in } L^{p^\star}(H_\varepsilon) \text{ and a.e. on } H_\varepsilon \\ \nabla f_{\varepsilon,k} \to \nabla f_\varepsilon \text{ in } L^p(H_\varepsilon) \end{cases} \qquad \text{as } k \to \infty\,. \tag{A.0.5}$$

Since $f_{\varepsilon,k} \in C^1_c(\overline{H_\varepsilon})$, arguing as in Theorem 2.2.1 we find that

$$n\int_{H_\varepsilon} (\det\nabla^2\varphi)^{1/n} f_{\varepsilon,k}^{p^\sharp} \le -p^\sharp \int_{H_\varepsilon} f_{\varepsilon,k}^{p^\sharp-1}\nabla f_{\varepsilon,k}\cdot S dx + t\int_{\partial H_\varepsilon} f_{\varepsilon,k}^{p^\sharp}\, d\mathcal{H}^{n-1} \tag{A.0.6}$$

where $S = T - t\,\mathrm{e}_1 \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^n)$. Since S is bounded on Ω_ε, where the $f_{\varepsilon,k}$ are uniformly supported in, and since $p^\sharp - 1 = p^\star/p'$, by (A.0.5) we find

$$\lim_{k\to\infty}\int_{H_\varepsilon} f_{\varepsilon,k}^{p^\sharp-1}\nabla f_{\varepsilon,k}\cdot S dx = \int_{H_\varepsilon} f_\varepsilon^{p^\sharp-1}\nabla f_\varepsilon\cdot S dx\,.$$

Moreover, by the trace inequality

$$\|u\|_{L^{p^\sharp}(\partial A)} \le C(A)\left(\|\nabla u\|_{L^p(A)} + \|u\|_{L^1(A)}\right),$$

which is valid whenever A is an open bounded Lipschitz set (see, for example, [MV05]), and again by the uniform support property, (A.0.5) implies

$$\lim_{k\to\infty}\int_{\partial H_\varepsilon} f_{\varepsilon,k}^{p^\sharp}\,d\mathcal{H}^{n-1} = \int_{\partial H_\varepsilon} f_\varepsilon^{p^\sharp}\,d\mathcal{H}^{n-1}\,.$$

Hence, by pointwise convergence and Fatou's lemma, (A.0.6) implies

$$n\int_{H_\varepsilon}(\det\nabla^2\varphi)^{1/n} f_\varepsilon^{p^\sharp} \le -p^\sharp\int_{H_\varepsilon} f_\varepsilon^{p^\sharp-1}\nabla f_\varepsilon\cdot S + t\int_{\partial H_\varepsilon} f_\varepsilon^{p^\sharp}\,d\mathcal{H}^{n-1}\,. \tag{A.0.7}$$

In order to take the limit $\varepsilon\to 0^+$ in (A.0.7), we first notice that $f_\varepsilon \le f$ everywhere on H. Hence, by (2.2.1) and (A.0.1), we find

$$\int_H |f_\varepsilon^{p^\sharp-1}\,S|^{p'} \le \int_H f^{p^\star}\,|S|^{p'} = \int_H g^{p^\star}\,|x - t\,e_1|^{p'} < \infty\,.$$

Since $f_\varepsilon \to f$ a.e. on H, it must be $f_\varepsilon^{p^\sharp-1}S \rightharpoonup f^{p^\sharp-1}S$ in $L^{p'}(H)$ as $\varepsilon\to 0^+$. By combining this last fact with the strong convergence $1_H\,\nabla f_\varepsilon \to \nabla f$ in $L^p(H)$, we conclude that

$$\int_{H_\varepsilon} f_\varepsilon^{p^\sharp-1}\nabla f_\varepsilon\cdot S dx = \int_H f_\varepsilon^{p^\sharp-1}\nabla f_\varepsilon\cdot S \to \int_H f^{p^\sharp-1}\nabla f\cdot S \tag{A.0.8}$$

as $\varepsilon \to 0^+$. Next, let us set $h_\varepsilon(x) = f_\varepsilon(x + \varepsilon\,\mathrm{e}_1)$ for $x \in H$, so that $1_{H_\varepsilon}\nabla f_\varepsilon \to \nabla f$ in $L^p(H)$ and the density of $C^0_c(H)$ in $L^p(H)$ gives us $\nabla h_\varepsilon \to \nabla f$ in $L^p(H)$. By applying (2.1.2) to $h_\varepsilon - f$ we find that $h_\varepsilon \to f$ in $L^{p^\sharp}(H)$, which clearly implies

$$\lim_{\varepsilon \to 0^+} \int_{\partial H_\varepsilon} f_\varepsilon^{p^\sharp}\, d\mathcal{H}^{n-1} = \int_{\partial H} f^{p^\sharp}\, d\mathcal{H}^{n-1}\,.$$

By combining this last fact with (A.0.8) with the fact that $1_{H_\varepsilon} f_\varepsilon^{p^\sharp} \to 1_H\, f^{p^\sharp}$ a.e. on $\mathbb{R}^n$ and with Fatou's lemma, we deduce from (A.0.7) that

$$n\int_H (\det\nabla^2\varphi)^{1/n} f^{p^\sharp} \le -p^\sharp \int_H f^{p^\sharp-1}\nabla f\cdot S + t\int_{\partial H} f^{p^\sharp}\, d\mathcal{H}^{n-1}\,.$$

Combining this inequality with (A.0.3), we complete the proof of the lemma in the case $p \in (1, n)$.

We now consider the case $p = 1$. We now have $|Df|(H) < \infty$ and $\operatorname{spt} g$ bounded. Thanks to the latter property, by arguing as in [MV05, pg. 96] we can assume that $S = T - t\,e_1 \in (BV_{\mathrm{loc}} \cap L^\infty)(H;\mathbb{R}^n)$. Setting $f_k = 1_{B_k}\min\{f, k\}$, $k \in \mathbb{N}$, then $f_k\, S \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and by the divergence theorem

$$\operatorname{div}(f_k\, S)(H) = \int_{\partial H} f_k\, S\cdot(-\mathrm{e}_1) = \int_{\partial H} f_k\, T\cdot(-\mathrm{e}_1) + t\int_{\partial H} f \le t\int_{\partial H} f\,.$$

If we identify f_k and S with their precise representatives, we have

$$\operatorname{div}(f_k\, S)(H) = \int_H f_k\, d(\operatorname{div} S) + \int_H S\cdot Df_k$$

where, of course,

$$\int_H f_k\, d(\operatorname{div} S) = \int_H f_k\, d(\operatorname{div} T) \ge n\int_H f_k\,(\det\nabla^2\varphi)^{1/n}\,.$$

We have thus proved

$$n\int_H f_k\,(\det\nabla^2\varphi)^{1/n}\le-\int_H S\cdot Df_k+t\int_{\partial H}f_k\,d\mathcal{H}^{n-1}\,. \tag{A.0.9}$$

By monotone convergence $\int_{\partial H} f_k\to\int_{\partial H} f$, while (2.2.3) and the boundedness of spt$g$ imply the existence of $R>0$ such that $|S|\le R$ on $\mathrm{spt}(Df)$, and thus

$$\Big|\int_H S\cdot Df_k-\int_H S\cdot Df_k\Big|\le R\,|Df|\Big(H\setminus(B_k\cup\{f<k\}^{(1)}\Big)$$

where $E^{(1)}$ denotes the set of density points of a Borel set $E\subset\mathbb{R}^n$ and we have used $D(1_E\,f)(K)=Df(E^{(1)}\cap K)$ for every $K\subset\mathbb{R}^n$. Since $|Df|(H)<\infty$, letting $k\to\infty$ and finally exploiting Fatou's lemma we deduce from (A.0.9)

$$-\int_H S\cdot Df+t\int_{\partial H}f\,d\mathcal{H}^{n-1}\ge n\int_H f\,(\det\nabla^2\varphi)^{1/n}=n\int_H g\,,$$

where in the last inequality we have used (A.0.3). The proof is complete. □

Proof of Theorem 2.2.3. Let us consider two functions f and g as in Lemma A.0.1 such that, for some $t\in\mathbb{R}$,

$$n\int_H g^{p^\sharp}=p^\sharp\|\nabla f\|_{L^p(H)}\,Y(t,g)+t\int_{\partial H}f^{p^\sharp}\qquad\text{with}\qquad\int_{\partial H}f^{p^\sharp}>0\,. \tag{A.0.10}$$

where $|Df|(H)$ replaces $\|\nabla f\|_{L^p(H)}$ if $p=1$. By arguing as in the proof of [CENV04, Proposition 6] in the case $p\in(1,n)$, and as in [FMP10, Theorem A.1] if $p=1$, we find that $T(x)=\nabla\varphi(x)=\lambda(x-x_0)$ for some $\lambda>0$ and $x_0\in\mathbb{R}^n$.

We claim that $x_0\cdot\mathrm{e}_1=0$. Keeping the proof of Lemma A.0.1 in mind, (A.0.10) implies that

$$\lim_{\varepsilon\to0^+}\lim_{k\to\infty}\int_{\partial H_\varepsilon}(T\cdot\mathrm{e}_1)\,f_{\varepsilon,k}^{p^\sharp}\,d\mathcal{H}^{n-1}=0\,,$$

where $T = \lambda(x - x_0)$ gives

$$\int_{\partial H_\varepsilon} (T \cdot \mathrm{e}_1)\, f_{\varepsilon,k}^{p^\sharp} = \lambda(\varepsilon - x_0 \cdot \mathrm{e}_1) \int_{\partial H_\varepsilon} f_{\varepsilon,k}^{p^\sharp} \,.$$

Since we have proved that

$$\lim_{\varepsilon \to 0^+} \lim_{k \to \infty} \int_{\partial H_\varepsilon} f_{\varepsilon,k}^{p^\sharp}\, d\mathcal{H}^{n-1} = \int_{\partial H} f^{p^\sharp}\, d\mathcal{H}^{n-1} \,,$$

where the latter quantity is assumed positive, we conclude that $x_0 \cdot \mathrm{e}_1 = 0$, as claimed. Up to a translation and up to apply an $L^{p^\star}$-norm preserving dilation to f, we can now assume that $x_0 = 0$ and $\lambda = 1$, that is $T(x) = x$.

We first consider the case $p \in (1, n)$. By combining (A.0.2) and (A.0.10) we find that we have an equality case in the Hölder's inequality $\int_H A \cdot B\, dx \le \|A\|_{L^p(H)}\, \|B\|_{L^{p'}(H)}$ with

$$A = -\nabla f \qquad B = f^{p^\sharp - 1}(x - t\,\mathrm{e}_1)\,.$$

In particular, there exist Borel functions $v : H \to \mathbb{R}^n$ and $a, b : H \to [0, \infty)$ such that $A = a\,v$, $B = b\,v$, and $a = c\,b^{1/(p-1)}$ for some constant $c > 0$. Hence, if we set $r = |x - t\,\mathrm{e}_1|$ and $v = (x - t\,\mathrm{e}_1)/r$, there exists a Borel function $u : [0, \infty) \to [0, \infty)$ such that

$$f(x) = u(r) \qquad -\nabla f(x) = -u'(r) \frac{x - t\,\mathrm{e}_1}{|x - t\,\mathrm{e}_1|}\,,$$

and the above conditions hold with $a = -u'(r)$ and $b = r u(r)^{p^\sharp - 1}$. In particular,

$$-u'(r) = c\,(r u(r)^{p^\sharp - 1})^{1/(p-1)} \qquad \text{for a.e. } r > 0\,,$$

and consequently, for some $c_1 > 0$ and $c_2 \in \mathbb{R}$

$$u(r) = (c_1 r^{p'} + c_2)_+^{-n/p^\star} \qquad \forall r > 0\,,$$

where $x_+ = \max\{x, 0\}$. In terms of f, this means that

$$f(x) = (c_1|x - t\mathrm{e}_1|^{p'} + c_2)_+^{-n/p^\star} \qquad \forall x \in H\,.$$

The cases where c_2 is positive, zero, and negative correspond, respectively, to f being a dilation-translation image of U_S, U_E, and U_B. If $t > 0$, the finiteness of the $L^{p^\star}(H)$-norm of f excludes the possibilities that f is a dilation-translation image orthogonal to e_1 of U_E and U_B.

Let us now consider the case $p = 1$. Recall that we have already set $T(x) = x$, so that $f = g$ and the combination of (A.0.2) and (A.0.10) gives

$$-\int_H (x - t\,\mathrm{e}_1) \cdot Df = \|\cdot - t\,\mathrm{e}_1\|_{L^\infty(\mathrm{spt}(Df))}|Df|(H)\,, \tag{A.0.11}$$

that is

$$-Df = \frac{x - t\,\mathrm{e}_1}{|x - t\,\mathrm{e}_1|}|Df| \qquad \text{as measures on } H\,.$$

By [Mag12, Exercise 15.19], there exists $\mu > 0$ such that $f = c\,1_{H \cap B_\mu(t\,\mathrm{e}_1)}$. This completes the proof. □

Appendix B

The operator $\mathcal{L}_U$ in polar coordinates

In this section we prove the polar coordinates form of the operator $\operatorname{div}(A(x)\nabla\varphi)$ given in (3.2.10).

Proof of (3.2.10). We will use the following classical relations:

$$\partial_r \hat{r} = 0 \qquad \partial_r \hat{\theta}_i = 0, \qquad \partial_{\theta_i} \hat{r} = \hat{\theta}_i, \qquad \partial_{\theta_i} \hat{\theta}_i = -\hat{r}, \qquad \partial_{\theta_j} \hat{\theta}_i = 0 \quad \text{for } i \neq j.$$

The chain rule implies that

$$\operatorname{div}(A(x)\nabla\varphi) = \operatorname{tr}(A(x)\nabla^2\varphi) + \operatorname{tr}(\nabla A(x)\nabla\varphi). \tag{B.0.1}$$

We compute the two terms on the right-hand side of (B.0.1) separately. For the first, we begin by computing the Hessian of φ in polar coordinates, starting from

$$\nabla\varphi = \partial_r\varphi\,\hat{r} + \frac{1}{r}\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat{\theta}_j, \tag{B.0.2}$$

We have

$$\begin{aligned}
\nabla^2\varphi &= \partial_r\Big(\partial_r\varphi\,\hat{r} + \frac{1}{r}\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat{\theta}_j\Big)\hat{r} + \frac{1}{r}\sum_{i=1}^{n-1}\partial_{\theta_i}\Big(\partial_r\varphi\,\hat{r} + \frac{1}{r}\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat{\theta}_j\Big)\hat{\theta}_i \\
&= \partial_{rr}\varphi\,\hat{r}\otimes\hat{r} - \frac{1}{r^2}\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat{\theta}_j\otimes\hat{r} + \frac{1}{r}\sum_{j=1}^{n-1}\partial_{\theta_j r}\varphi\,\hat{\theta}_j\otimes\hat{r} + \frac{1}{r}\sum_{i=1}^{n-1}\partial_{\theta_i r}\varphi\,\hat{r}\otimes\hat{\theta}_i
\end{aligned}$$

$$+\frac{1}{r}\sum_{i=1}^{n-1}\partial_r\varphi\,\theta_i\otimes\theta_i+\frac{1}{r^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\partial_{\theta_i\theta_j}\varphi\,\hat\theta_j\otimes\hat\theta_i-\frac{1}{r^2}\sum_{i=1}^{n-1}\partial_{\theta_i}\varphi\,\hat r\otimes\hat\theta_i\,.$$

In order to compute $A(x)\nabla^2\varphi$, we note that

$$(\hat r\otimes\hat r)(\hat r\otimes\hat r)=\hat r\otimes\hat r,\qquad(\hat r\otimes\hat r)(\hat\theta_j\otimes\hat\theta_i)=0,$$

$$(\hat r\otimes\hat r)(\hat r\otimes\hat\theta_i)=0,\qquad(\hat r\otimes\hat r)(\hat\theta_i\otimes\hat r)=\hat\theta_i\otimes\hat r.$$

Thus we have

$$\begin{aligned}&A(x)\nabla^2\varphi=(p-2)|\nabla U|^{p-2}\hat r\otimes\hat r(\nabla^2\varphi)+|\nabla U|^{p-2}\mathrm{Id}(\nabla^2\varphi)\\&=(p-2)|\nabla U|^{p-2}\Big[\partial_{rr}\varphi\,\hat r\otimes\hat r-\frac{1}{r^2}\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat\theta_j\otimes\hat r+\frac{1}{r}\sum_{j=1}^{n-1}\partial_{\theta_j r}\varphi\,\hat\theta_j\otimes\hat r\Big]\\&+|\nabla U|^{p-2}\Big[\partial_{rr}\varphi\,\hat r\otimes\hat r-\frac{1}{r^2}\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat\theta_j\otimes\hat r+\frac{1}{r}\sum_{j=1}^{n-1}\partial_{\theta_j r}\varphi\,\hat\theta_j\otimes\hat r+\frac{1}{r}\sum_{i=1}^{n-1}\partial_{\theta_i r}\varphi\,\hat r\otimes\hat\theta_i\\&+\frac{1}{r}\sum_{i=1}^{n-1}\partial_r\varphi\,\theta_i\otimes\theta_i+\frac{1}{r^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\partial_{\theta_i\theta_j}\varphi\,\hat\theta_j\otimes\hat\theta_i-\frac{1}{r^2}\sum_{i=1}^{n-1}\partial_{\theta_i}\varphi\,\hat r\otimes\hat\theta_i\Big],\end{aligned}$$

and the first term in (B.0.1) is

$$\mathrm{tr}(A(x)\nabla^2\varphi)=(p-1)|\nabla U|^{p-2}\partial_{rr}\varphi+\frac{n-1}{r}|\nabla U|^{p-2}\partial_r\varphi+\frac{1}{r^2}|\nabla U|^{p-2}\sum_{i=1}^{n-1}\partial_{\theta_i\theta_i}\varphi.\tag{B.0.3}$$

Now we compute the second term in (B.0.1), starting by computing $\nabla A(x)$. We reintroduce the slight abuse of notation by letting $U(r)=U(x)$, so $U'=\partial_r U$, $U''=\partial_{rr}U$. Note that $\partial_\theta\mathrm{Id}=\partial_r\mathrm{Id}=0$, thus

$$\begin{aligned}\nabla A(x)&=\partial_r A(x)\otimes\hat r+\frac{1}{r}\sum_{j=1}^{n-1}\partial_{\theta_j}A(x)\otimes\hat\theta_j\\&=(p-2)^2|U'|^{p-4}U'\,U''\,\hat r\otimes\hat r\otimes\hat r+(p-2)|U'|^{p-4}U'\,U''\,\mathrm{Id}\otimes\hat r\end{aligned}$$

$$+\frac{p-2}{r}\sum_{j=1}^{n-1}\Big[|U'|^{p-2}\hat\theta_j\otimes\hat r\otimes\hat\theta_j+|U'|^{p-2}\hat r\otimes\hat\theta_j\otimes\hat\theta_j\Big].$$

Recalling (B.0.2), we then have

$$\begin{aligned}\nabla A(x)\nabla\varphi &= (p-2)^2|U'|^{p-4}U'\,U''\,\partial_r\varphi(\hat r\otimes\hat r\otimes\hat r)\hat r+(p-2)|U'|^{p-4}U'\,U''\,\partial_r\varphi(\mathrm{Id}\otimes\hat r)\hat r\\ &\quad+\frac{p-2}{r}\sum_{j=1}^{n-1}\Big[|U'|^{p-2}\partial_r\varphi(\hat\theta_j\otimes\hat r\otimes\hat\theta_j)\hat r+|U'|^{p-2}\partial_r\varphi(\hat r\otimes\hat\theta_j\otimes\hat\theta_j)\hat r\Big]\\ &\quad+\frac{1}{r}\sum_{i=1}^{n-1}\Big[(p-2)^2|U'|^{p-4}U'U''\partial_{\theta_i}\varphi(\hat r\otimes\hat r\otimes\hat r)\hat\theta_i+(p-2)|U'|^{p-4}U'U''\partial_{\theta_i}\varphi(\mathrm{Id}\otimes\hat r)\hat\theta_i\Big]\\ &\quad+\frac{p-2}{r^2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\Big[|U'|^{p-2}\partial_{\theta_i}\varphi(\hat\theta_j\otimes\hat r\otimes\hat\theta_j)\hat\theta_i+|U'|^{p-2}\partial_{\theta_i}\varphi(\hat r\otimes\hat\theta_j\otimes\hat\theta_j)\hat\theta_i\Big],\end{aligned}$$

where we used that $(a\otimes b\otimes c)d=(a\cdot d)b\otimes c$. Writing out these terms gives

$$\begin{aligned}\nabla A(x)\nabla\varphi &= (p-1)(p-2)|U'|^{p-4}U'U''\partial_r\varphi\,\hat r\otimes\hat r+\frac{p-2}{r}|U'|^{p-2}\sum_{j=1}^{n-1}\partial_r\varphi\,\hat\theta_j\otimes\hat\theta_j\\ &\quad+\frac{p-2}{r}|U'|^{p-4}U'U''\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat\theta_j\otimes\hat r+\frac{p-2}{r^2}|U'|^{p-2}\sum_{j=1}^{n-1}\partial_{\theta_j}\varphi\,\hat r\otimes\hat\theta_j,\end{aligned}$$

thus the second term in (B.0.1) is

$$\mathrm{tr}(\nabla A(x)\nabla\varphi)=(p-1)(p-2)|\nabla U|^{p-4}\partial_r U\,\partial_{rr}U\,\partial_r\varphi+\frac{(n-1)(p-2)}{r}|\nabla U|^{p-2}\partial_r\varphi. \tag{B.0.4}$$

Combining (B.0.3) and (B.0.4), (B.0.1) implies that

$$\begin{aligned}\mathrm{div}(A(x)\nabla\varphi) &= (p-1)|\nabla U|^{p-2}\partial_{rr}\varphi+\frac{(p-1)(n-1)}{r}|\nabla U|^{p-2}\partial_r\varphi\\ &\quad+\frac{1}{r^2}|\nabla U|^{p-2}\sum_{j=1}^{n-1}\partial_{\theta_j\theta_j}\varphi+(p-1)(p-2)|\nabla U|^{p-4}\partial_r U\,\partial_{rr}U\,\partial_r\varphi,\end{aligned}$$

as desired. □

Bibliography

[AGK04] M. Agueh, N. Ghoussoub, and X. Kang. Geometric inequalities via a general comparison principle for interacting gases. *Geom. Funct. Anal.*, 14(1):215–244, 2004.

[Alm66] F. J. Almgren, Jr. Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. Math.*, 84(2):277–292, 1966.

[Aub76a] T. Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9)*, 55(3):269–296, 1976.

[Aub76b] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. *J. Differential Geom.*, 11(4):573–598, 1976.

[BBJ] M. Barchiesi, A. Brancolini, and V. Julin. Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality. *To appear in Ann. Probab.*

[BDF] V. Bögelein, F. Duzaar, and N. Fusco. A quantitative isoperimetric inequality on the sphere. *To appear in Adv. Calc. Var.*

[BDS15] V. Bögelein, F. Duzaar, and C. Scheven. A sharp quantitative isoperimetric inequality in hyperbolic n-space. *Calc. Var. Partial Differential*

Equations, 54(4):3967–4017, 2015.

[BE91] G. Bianchi and H. Egnell. A note on the Sobolev inequality. *J. Funct. Anal.*, 100(1):18–24, 1991.

[Bec93] W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242, 1993.

[BGRS14] S. G. Bobkov, N. Gozlan, C. Roberto, and P.-M. Samson. Bounds on the deficit in the logarithmic Sobolev inequality. *J. Funct. Anal.*, 267(11):4110–4138, 2014.

[BL85] H. Brezis and E. H. Lieb. Sobolev inequalities with remainder terms. *J. Funct. Anal.*, 62(1):73–86, 1985.

[BM94] J. E. Brothers and F. Morgan. The isoperimetric theorem for general integrands. *Mich. Math. J.*, 41(3):419–431, 1994.

[Bom82] E. Bombieri. Regularity theory for almost minimal currents. *Arch. Ration. Mech. Anal.*, 7(7):99–130, 1982.

[Bon24] T. Bonnesen. Über die isoperimetrische defizit ebener figuren. *Math. Ann.*, 91:252–268, 1924.

[Bre91] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

[BWW03] T. Bartsch, T. Weth, and M. Willem. A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator. *Calc. Var. Partial Differential Equations*, 18(3):253–268, 2003.

[CENV04] D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math.*, 182(2):307–332, 2004.

[CF13] E. A. Carlen and A. Figalli. Stability for a GNS inequality and the log-HLS inequality, with application to the critical mass Keller-Segel equation. *Duke Math. J.*, 162(3):579–625, 2013.

[CFMP09] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli. The sharp Sobolev inequality in quantitative form. *J. Eur. Math. Soc. (JEMS)*, 11(5):1105–1139, 2009.

[CFW13] S. Chen, R. L. Frank, and T. Weth. Remainder terms in the fractional Sobolev inequality. *Indiana Univ. Math. J.*, 62(4):1381–1397, 2013.

[CH89] R. Courant and D. Hilbert. *Methods of Mathematical Physics. Vol. I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989.

[Cia06] A. Cianchi. A quantitative Sobolev inequality in BV. *J. Funct. Anal.*, 237(2):466–481, 2006.

[Cia08] A. Cianchi. Sharp Morrey-Sobolev inequalities and the distance from extremals. *Trans. Amer. Math. Soc.*, 360(8):4335–4347, 2008.

[CL90a] E. A. Carlen and M. Loss. Competing symmetries of some functionals arising in mathematical physics. In *Stochastic processes, physics and* *geometry (Ascona and Locarno, 1988)*, pages 277–288. World Sci. Publ., Teaneck, NJ, 1990.

[CL90b] E. A. Carlen and M. Loss. Extremals of functionals with competing symmetries. *J. Funct. Anal.*, 88(2):437–456, 1990.

[CL92] E. A. Carlen and M. Loss. Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n. *Geom. Funct. Anal.*, 2(1):90–104, 1992.

[CL94] E. A. Carlen and M. Loss. On the minimization of symmetric functionals. *Rev. Math. Phys.*, 6(5A):1011–1032, 1994.

[CL12] M. Cicalese and G. P. Leonardi. A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Ration. Mech. Anal.*, 206(2):617–643, 2012.

[CVDM04] U. Clarenz and H. Von Der Mosel. On surfaces of prescribed F-mean curvature. *Pacific J. Math.*, 213(1):15–36, 2004.

[DG57] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[DGS92] B. Dacorogna, W. Gangbo, and N. Subía. Sur une généralisation de l'inégalité de Wirtinger. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9(1):29–50, 1992.

[DP92] B. Dacorogna and C. E. Pfister. Wulff theorem and best constant in Sobolev inequality. *J. Math. Pures Appl.*, 71(2):97–118, 1992.

[DPM14] G. De Philippis and F. Maggi. Sharp stability inequalities for the Plateau problem. *J. Differential Geom.*, 96(3):399–456, 2014.

[DPM15] G. De Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.*, 216(2):473–568, 2015.

[DS02] F. Duzaar and K. Steffen. Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. *J. Reine Angew. Math.*, 546:73–138, 2002.

[DT13] J. Dolbeault and G. Toscani. Improved interpolation inequalities, relative entropy and fast diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(5):917–934, 2013.

[EFT05] L. Esposito, N. Fusco, and C. Trombetti. A quantitative version of the isoperimetric inequality: the anisotropic case. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(4):619–651, 2005.

[EG92] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[Eld15] R. Eldan. A two-sided estimate for the Gaussian noise stability deficit. *Invent. Math.*, 201(2):561–624, 2015.

[Esc88] J. F. Escobar. Sharp constant in a Sobolev trace inequality. *Indiana Univ. Math. J.*, 37(3):687–698, 1988.

[Esc92a] J. F. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)*, 136(1):1–50, 1992.

[Esc92b] J. F. Escobar. The Yamabe problem on manifolds with boundary. *J. Differential Geom.*, 35(1):21–84, 1992.

[Eva98] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[FF60] H. Federer and W. H. Fleming. Normal and intregral currents. *Ann. Math.*, 72:458–520, 1960.

[FIL16] M. Fathi, E. Indrei, and M. Ledoux. Quantitative logarithmic Sobolev inequalities and stability estimates. *Discrete Contin. Dyn. Syst.*, 36(12):6835–6853, 2016.

[FJ14] N. Fusco and V. Julin. A strong form of the quantitative isoperimetric inequality. *Calc. Var. Partial Differential Equations*, 50(3-4):925–937, 2014.

[FKS82] E. B. Fabes, C. E. Kenig, and R. P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.

[FM91] I. Fonseca and S. Müller. A uniqueness proof for the Wulff theorem. *Proc. R. Soc. Edinb. Sect. A*, 119:125–136, 1991.

[FM11] A. Figalli and F. Maggi. On the shape of liquid drops and crystals in the small mass regime. *Arch. Ration. Mech. Anal.*, 201(1):143–207, 2011.

[FMP07] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative Sobolev inequality for functions of bounded variation. *J. Funct. Anal.*, 244(1):315–341, 2007.

[FMP08] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. Math.*, 168:941–980, 2008.

[FMP10] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.

[FMP13] A. Figalli, F. Maggi, and A. Pratelli. Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation. *Adv. Math.*, 242:80–101, 2013.

[FN] A. Figalli and R. Neumayer. Gradient stability for the Sobolev inequality: the case $p \geq 2$. *To appear in J. Eur. Math. Soc. (JEMS).*

[Fon91] I. Fonseca. The Wulff theorem revisited. *Proc. Roy. Soc. London Ser. A*, 432(1884):125–145, 1991.

[FR60] W. H. Fleming and R. Rishel. An integral formula for total gradient variation. *Arch. Math. (Basel)*, 11:218–222, 1960.

[Fug89] B. Fuglede. Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbb{R}^n$. *Trans. Amer. Math. Soc.*, 314:619–638, 1989.

[Fus15] N. Fusco. The quantitative isoperimetric inequality and related topics. *Bull. Math. Sci.*, 5(3):517–607, 2015.

[Gur85] M. E. Gurtin. On a theory of phase transitions with interfacial energy. *Arch. Ration. Mech. Anal.*, 87(3):187–212, 1985.

[Gut89] C. E. Gutiérrez. Harnack's inequality for degenerate Schrödinger operators. *Trans. Amer. Math. Soc.*, 312(1):403–419, 1989.

[GW10] F. Gazzola and T. Weth. Remainder terms in a higher order Sobolev inequality. *Arch. Math. (Basel)*, 95(4):381–388, 2010.

[Hal92] R. R. Hall. A quantitative isoperimetric inequality in n-dimensional space. *J. Reine Angew. Math.*, 428:161–176, 1992.

[Her51] C. Herring. Some theorems on the free energies of crystal surfaces. *Phys. Rev.*, 82:87–93, 1951.

[HHW91] R. R. Hall, W. K. Hayman, and A. W. Weitsman. On asymmetry and capacity. *J. d'Analyse Math.*, 56:87–123, 1991.

[IM14] E. Indrei and D. Marcon. A quantitative log-Sobolev inequality for a two parameter family of functions. *Int. Math. Res. Not.*, (20):5563–5580, 2014.

[Joh48] F. John. Extremum Problems with Inequalities as Subsidiary Conditions.

In *Studies and Essays: Courant Anniversary Volume*, pages 187–204. Wiley-Interscience, New York, 1948.

[Kno57] H. Knothe. Contributions to the theory of convex bodies. *Mich. Math. J.*, 4:39–52, 1957.

[Lio85] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[LP87] J. M. Lee and T. H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[Mag08] F. Maggi. Some methods for studying stability in isoperimetric type problems. *Bull. Amer. Math. Soc.*, 45(3):367–408, 2008.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2012.

[Maz60] V. Maz'ya. Classes of regions and imbedding theorems for function spaces. *Dokl. Akad. Nauk. SSSR*, 133:527–530, 1960.

[Maz85] V. G. Maz'ja. *Sobolev spaces.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[McC97] R. J. McCann. A convexity principle for interacting gases. *Adv. Math.*, 128(1):153–179, 1997.

[MN17] F. Maggi and R. Neumayer. A bridge between the Sobolev and Escobar inequalities and beyond. *J. Funct. Anal.*, 273(6):2070–2106, 2017.

[MS86] V. D. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces. With an appendix by M. Gromov.* Number 1200 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986.

[MV05] F. Maggi and C. Villani. Balls have the worst best Sobolev inequalities. *J. Geom. Anal.*, 15(1):83–121, 2005.

[MV08] F. Maggi and C. Villani. Balls have the worst best Sobolev inequalities. II. Variants and extensions. *Calc. Var. Partial Differential Equations*, 31(1):47–74, 2008.

[Naz06] B. Nazaret. Best constant in Sobolev trace inequalities on the half-space. *Nonlinear Anal.*, 65(10):1977–1985, 2006.

[Neu16] R. Neumayer. A strong form of the quantitative Wulff inequality. *SIAM J. Math. Anal.*, 48(3):1727–1772, 2016.

[Ngu] V.H. Nguyen. The sharp Gagliardo–Nirenberg–Sobolev inequality in quantitative form. Preprint: arXiv:1702.01039.

[Ngu15] V. H. Nguyen. Sharp weighted Sobolev and Gagliardo-Nirenberg inequalities on half-spaces via mass transport and consequences. *Proc. Lond. Math. Soc. (3)*, 111(1):127–148, 2015.

[Opi88] B. Opic. Compact imbedding of weighted Sobolev space defined on an unbounded domain. I. *Časopis Pěst. Mat.*, 113(1):60–73, 1988.

[Oss79] R. Osserman. Bonnesen-style isoperimetric inequalities. *Amer. Math. Monthly*, 86(1):1–29, 1979.

[Ruf14] B. Ruffini. Stability theorems for Gagliardo-Nirenberg-Sobolev inequalities: a reduction principle to the radial case. *Rev. Mat. Complut.*, 27(2):509–539, 2014.

[Sch84] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.

[Sch13] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151. Cambridge University Press, 2013.

[Sim83] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[Sob36] S. L. Sobolev. On some estimates relating to families of functions having derivatives that are square integrable. *Dokl. Akad. Nauk SSSR*, 1:267–270, 1936.

[Sob38] S. L. Sobolev. On theorem in functional analysis. *Sb. Math.*, 4:471–497, 1938. English translation: Am. Math. Soc., 34:39–68(2), 1963.

[SSA77] R. Schoen, L. Simon, and F.J. Almgren, Jr. Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. *Acta Mathematica*, 139(1):217–265, 1977.

[Str84] M. Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187(4):511–517, 1984.

[Tal76] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.

[Tay78] J. E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc.*, 84(4):568–588, 07 1978.

[Tru68] N. S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)*, 22:265–274, 1968.

[Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Wul01] G. Wulff. Zur frage der geschwindigkeit des wachsturms und der auflösungder kristall- flächen. *Z. Kristallogr.*, 34:449–530, 1901.

[Yam60] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.

[Zyg02] A. Zygmund. *Trigonometric Series. Vol. I, II.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2002.