University of Texas Bulletin

No. 2420: May 22, 1924

The Texas Mathematics Teachers' Bulletin

Volume IX, No. 3

PUBLISHED BY
THE UNIVERSITY OF TEXAS
AUSTIN

Publications of the University of Texas

Publications Committee:

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UNIVERSITY OF TEXAS PRESS, AUSTIN

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PUBLISHED BY THE UNIVERSITY FOUR TIMES A MONTH, AND ENTERED AS SECOND-CLASS MATTER AT THE POSTOFFICE AT AUSTIN, TEXAS, UNDER THE ACT OF AUGUST 24, 1912

The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

Mirabeau B. Lamar

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The Texas Mathematics Teachers' Bulletin

Volume IX, No. 3

Edited by

MARY E. DECHERD
Adjunct Professor of Pure Mathematics,

and

JESSIE M. JACOBS MULLER
Instructor in Pure Mathematics

This Bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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MAKING MATHEMATICS INTERESTING

BY J. M. BLEDSOE, PROFESSOR OF MATHEMATICS, EAST TEXAS STATE TEACHERS COLLEGE

During the past quarter century, there has been great interest manifested in the reconstruction of our program of secondary education. There has been an unusual awakening to the need of a reconstruction, not only in our course of study, but to the necessity of a revision in the subject matter contained in the textbooks of the different departments and the methods of presenting the materials to students in search of training for the duties of actual life.

It has been brought to the attention of even the strongest teachers of our oldest and best organized subjects in the curriculum that the last word had not been said in even these departments, and the fact has been clearly emphasized that the teacher or the subject that stops growing or fails to keep pace with the advancing age will soon be required to surrender to one more capable of meeting the demands of the hour.

There is no subject or department in the whole school curriculum which has received a stronger and more aggressive challenge to justify its place in the program of education than the subject of mathematics; and I feel free to say that there is no other subject which has proven itself more capable of justifying its right to existence, nor more resourceful in adjusting itself to the growing needs of the new and advancing age. Especially has this been true in America.

A recent report of the American Mathematical Society says: "Before 1873 the number of Americans who had ever made serious contributions to mathematics could be counted on the fingers of one hand. European scholars did not undervalue American mathematics; there was no American mathematics. Between 1873 and 1894 there was marked progress . . . Since 1894 America has risen from a position of negligible importance in the eyes of the mathematicians of the world to one where her contributions to the science and her facilities for the most advanced instruction, are comparable with the best that Europe can show."

It has long been a question of more or less serious concern whether mathematics can be mastered by every child. I was asked the question whether I believed there were certain individuals who could not learn mathematics. My answer was then, as it has ever been, Yes. And it is my honest opinion that when we find an individual who positively cannot learn mathematics, we will find one who cannot learn anything else which requires reasoning power. When I find an individual who claims to be fine in certain subjects, yet disclaims any ability to learn mathematics, I feel that at least one of two things is true, maybe both; viz., that such individual is deceived in thinking that he has mastered the favorite subjects as he claims, or that a fair and honest chance has never been given to the subject of mathematics. I have my first student to find who cannot learn mathematics, yet who makes satisfactory progress in other subjects which require reasoning power.

There is a great difference between having the ability and willingness to do a thing, and in being unwilling to try to master the task, either on account of a disgust for it caused by an unfavorable first impression, or an unfavorable attitude towards it as a result of being led to believe the mastery of the task is unnecessary and a waste of time. There is no such thing as a natural dislike for certain subjects. When a student dislikes a subject, it is the result of a bad start, usually the fault of a weak teacher.

It has probably been true all along that pupils in the schools sought to satisfy the requirements for promotion and graduation by enrolling in as many subjects as possible which they were led to believe required the least amount of effort; yet it seems that this tendency is not only increasing, but is actually being encouraged by many who claim to be endowed with sufficient wisdom and the power of prophecy to determine in advance just the amount and kind of training needed by each individual pupil. With this impression of their superior judgment and forethought, they diligently seek to convince every pupil that all his hopes of the future will be "bound in shallows and in miseries" if he fails to follow the path of least resistance, and does not choose those things only which tickle his fancy and refuses to select such subjects only as require the least amount of strenuous effort.

Now, it so happens that mathematics is not, nor can it be made an easy subject. Its greatest element of usefulness would disappear if this fact were changed. While it requires hard and honest toil to master the subject, yet it can be made intensely interesting if properly organized and presented. When it is handled in the right manner, all the drudgery may become a real satisfaction and a pleasure. After all, the most dreaded task is not the one which requires the hardest effort. The greatest joy comes as the result of successfully solving the most difficult undertaking. Such an ideal of earnest endeavor and worthy accomplishment may be instilled into the minds of the pupils by the efficient teacher who is inspired and guided by such ideals, provided his efforts are not smothered in an atmosphere of negative influences. And to my way of thinking, we need more of this kind of teaching.

It is impossible to suggest any plan which would make successful teaching certain for the weak, incompetent teacher. Dr. Porter of the University of Texas says: "The poorest method may be used successfully in the hands of a good teacher; the best method in the world will prove a failure in the hands of a poor teacher."

While this is true, there are certain suggestions which if carried out, may contribute to the successful teaching of mathematics, and help in making the subject interesting. Plenty of time should be given in assisting the pupils to gain a clear understanding of the fundamental definitions and axioms. These should be repeated, reviewed and illustrated until all the pupils become thoroughly familiar with every useful fact and principle, and able to make ready application of these facts and principles to the solution of simple problems of familiar situations. It is not enough to understand a principle merely, but it should be made so familiar to the student that he will be able to make it serve his everyday needs.

We should emphasize the need, and clearly show the students how to read a problem or theorem, and how to analyze its parts. Too often the inability to solve a problem is due to the failure of the pupil to comprehend the language in which it is stated. A successful teacher of mathematics must be good in English, and no better opportunity is found to emphasize the importance of a good knowledge of the mother tongue than is found in the teaching of mathematics. For the student to clearly understand the exact meaning of a problem, he must be able to grasp it as a whole, then to see what is given, what is required, and to intelligently correlate the two in a systematic effort to find a solution.

One of the most effective means of creating an interest in the study of mathematics is the use of illustrations and data familiar to the students being taught. Severe criticism of textbooks has been made on the ground that the types of problems contained in them were not suited to the needs of students, and that too much formalism was given instead of problems of a more concrete nature and involving facts and principles better suited to the capacities and experiences of the pupils. This criticism has been well founded and has resulted in a new type of textbook in every branch of the entire field of mathematics. To my way of thinking there is no good reason why problems suited to the localities in which the pupils reside should not be used, at least in elementary and secondary mathematics. Of course, to develop the ability to do abstract thinking should be the goal of all mathematical instruction, but this can be accomplished just as rapidly and more surely by concrete methods which do not discourage the learner. The natural process of development enables the child to crawl before walking.

Within recent years there have sprung up in Europe and America movements which bid fair to exert a far-reaching and beneficial influence on the teaching of mathematics. The dominating thought which has actuated such movements has been a fuller consideration of the laws of development of the child mind. More and more is the fact coming to be recognized that the keynote of all effective instruction is interest on the part of the learner. Professor A. H. Sage says in *School Science and Mathematics:* "There are only two ways to make things stick in the minds of the pupils; one is to repeat the thing without variation until it becomes a habit of mind . . . The other is to present the thing with such interest to the learner that his whole being responds to the act of accepting and adopting it, and with such intensity that with only one or possibly two presentations of the thing it is indelibly fixed in the mind of the pupil. This is the necessary method in all rational subjects."

Since the publication of the works of Dr. Perry, which was the beginning of the Perry Movement, in the teaching of mathematics, many of the leading educators are recognizing that mathematics would be much more successfully taught if treated as a laboratory science, at least that portion of the subject presented in the public school. Why should the teacher of mathematics not be provided with instruments for making accurate measurements in determining number ideas? He should be provided with such instruments as ruler and compass, balance, steelyards, pendulums, levers, pulleys, wedges, screws, barometer, thermometer, coördinate blackboard, surveyor's instruments, and other necessary apparatus for giving concrete illustrations in problem solving. Such helps would not only assist in arousing a healthy interest in the subject, but would make the work more easily comprehended by the pupils.

Dr. Moore in his presidential address before the American Mathematical Society said: "As a pure mathematician I hold as a most important suggestion of the English movement, the suggestion of Perry's just cited, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing, and graphical methods generally, in continuous relations with problems of physics, chemistry, engineering, etc., it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by laboratory process, and, on the other hand, by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics."

Again, it is necessary to give careful and individual attention to the students if the best results are to be attained in the teaching of mathematics. It is probably true, as has been asserted by Professor Young, that if as much individual instruction were given to the students of mathematics as is given to students of music, there would be just as few failures to make passing grades in mathematics as in music. I have never been able to see that it is any indication of efficiency for an instructor to fail a large per cent of his pupils; on the other hand, it may be an indication of weakness or inexcusable neglect of duty. It is an easy matter for some pupils of the class to become indifferent to the discussion of a problem or theorem, especially if such pupils have only a weak and superficial knowledge of the preceding principles which form the necessary background for a clear understanding of the proposition in hand. It is also important that the student be made to feel at home in the class, and that his honest efforts are appreciated by the teacher, and that his discoveries are valuable not only for his own progress, but a distinct contribution to the class.

But far and beyond all other considerations in making mathematics interesting and its effective presentation possible, is the proper preparation of the teacher. It is too generally true that this subject is being taught in the public schools by people who know very little about the subject and care less. Superintendents search diligently for specialized degree graduates for other departments, and assign to them classes in mathematics during their off periods. It is no wonder that our business men have such little respect for the mathematics given in the public schools, when they find so many high-school graduates unable to perform with any degree of certainty the simplest problems of the business world.

It is an erroneous notion to think that mathematics is an easy subject to teach. It is one of the most difficult subjects to teach properly. And it is not enough for the teacher to have simply studied arithmetic merely if she expects to become an efficient teacher of that subject. The same common sense should be applied when selecting teachers of mathematics that is followed in selecting teachers of other subjects. It is considered a pretty sound pedagogical axiom that the teacher should be at least four years in advance of the students to be taught. This would necessarily mean that the teacher of arithmetic in the grades be at least a high-school graduate with thorough courses in algebra, plane geometry, business arithmetic, and probably work in mechanical drawing, descriptive geometry, and a fair knowledge of the history of mathematics. The teacher of high-school mathematics should have majored in the subject for at least four college years, which would mean strong courses in advanced algebra, trigonometry, solid geometry, analytic geometry, calculus, the history and pedagogy of mathematics, and a fair knowledge of such allied subjects as physics, chemistry, statistical methods, and the elements of mechanics.

Dr. David Eugene Smith says that trigonometry, analytic geometry and projective geometry are the three subjects essential to a fair knowledge of elementary geometry. He further says: "There is a healthy and growing feeling in America that teachers of secondary mathematics need a more thorough training in the subject matter, even at the expense of some of the theory of education which they now have." The teacher who possesses a broad and thorough training in the subject matter of mathematics which necessarily implies a fair knowledge of the allied subjects in which the principles of mathematics are applied, also possesses a keen interest in and a strong appreciation of the subject which we do not find in the teacher who has attained only a limited and superficial knowledge of the subject.

The best type of teaching is done by one who loves his subject. The highest regard for any subject is held by the one who has found the broadest and most thorough training in the subject. The teacher who does not possess a real interest in her subject is not likely to arouse much interest in her pupils. Of course, some students will, by reason of their own will-power and by dint of their own effort, go ahead and learn the subject in spite of their pretended teacher; but I am discussing the proper qualification of a competent teacher of mathematics. And I want to assert with just as much emphasis as I can that *the greatest need in the mathematics teaching in our public schools today is a supply of teachers who possess a sufficient knowledge and appreciation of the subject to enable them to lead the pupils into a thorough mastery of the facts and principles, and render them skillful in applying these principles to the solution of the every-day problems of life.*

UNITS OF ANGULAR MEASURE

BY ALBERT A. BENNETT

That trigonometry is naturally and rightfully a branch of pure mathematics is not to be questioned. Trigonometric identities, De Moivre's theorem, series expansions of trigonometric functions, and other topics taken up in a treatise on the subject are beyond question worthy branches of mathematical analysis. But trigonometry in the character of many of its problems bears evidence of its close contact with surveying, navigation, astronomy, and mechanics, and leads eventually in a natural manner to those problems involving expansion by Fourier series which figure so brilliantly in numerous investigations of mathematical physics.

The terms, sine, tangent, and secant, suggest by their etymology their historical origin, and are of course more recent than the study of trigonometry itself. On the other hand the custom of taking the initial line from which angles are to be measured as the horizontal ray extending to the right from the origin, and measuring angles taken counterclockwise as positive, is reminiscent of the practice of using the north as the fundamental geographical direction, and tracing the paths of the sun or stars from their place of rising in the east to their setting in the west. For the apparent paths of the circumpolar stars, more than the half plane is required, the origin being the polar star—that is when there was such a star, various stars having shared this honor in turn during historic times.

Although no historical sense is required to understand the definitions and proofs of trigonometry, any more than an ability to use surveying instruments is required for the solution of triangles, some appreciation of how it came about that the units we employ today were ever originated would seem to be desirable. For this reason the present paper offers remarks of an historical nature, provided that the term "historical" be allowed the wide interpretation in which dates, authorities, and sources are not mentioned.

Introductory Remarks on the Ancient Calendar

Primitive man was undoubtedly most interested in the sensations of the instant. He was a child of nature, and nature lives always in the *now*. But experience of the past as a guide to the future affected and moulded his consciousness long before there was anything like a hint of inherited social culture. In fact this conscious directive memory he shared in common with all higher animals. Without assuming that he speculated unduly as to cause and effect, we can be sure that he must have noticed the regularity of some of the more strking natural phenomena. The inevitable alternation of night and day must have been one of the most obvious of the facts of nature. With no clear quantitative idea of time, the savage surely noted the distinction between darkness and daylight, morning and evening. But some sense of the duration of time was probably one of the endowments of man from the most primitive epochs, and even the correlation of space, time and physical endurance which is reflected in such a phrase as "a day's journey," must have been a relatively early acquirement. To mention an extreme case, no matter how the birds, insects or flowers might react after a temporary obliteration of the sun, it is hardly imaginable that men at any period of their existence could have failed to notice with astonishment a total eclipse if occurring in clear weather near noonday.

When language and traditional lore added to the inheritance of each generation a cultural quota in excess of the purely animal attributes that the child inherits with its body, those phenomena of change for which an immediate anthropological explanation was not forthcoming, could not fail to take on mystical or religious coloring. Man interpreted his associates in terms of himself, and his associates included not only his family, his fellow tribesmen, and such dangerous and hated strangers as might cross his path, but also the beasts which he hunted for their flesh or their pelts, and birds, fish, insects, flowers, trees, mountains, streams, winds, in fact the whole complex bit of nature in which he lived. The brilliant glory of the sun, its extreme distance, beyond the farthest mountains, its unwavering regularity, its mysterious disappearance under ground, these could not help but acquire new significance not shared by such more comprehensible phenomena as the migration of birds or beasts, the hibernation of many forms of complex life, the waving of the trees in the wind, the growth of plants and animals and the thousands of things for which the savage found analogies in his own life even although their explanation was not even guessed at. Nothing is more natural than that man should have regarded in all times and all places, the day as a natural unit of time. One who has not studied the growth of religious beliefs might be astonished at the divine characteristics with which the sun is endowed by early worshipers, and all of the mystical allegory that became entwined about a physical object so aloof from us as the center of our solar system, yet whether or not one feels inclined to do reverence to this one powerful source of terrestrial energy, no one will question the appropriateness of the day as a commercial and domestic measure of time.

The next most noticeable object in the sky is of course the moon, which although of more gentle mien, was much more mysterious to our ancestors. The slow and steady regularity of its phases and their wholly unearthly character could not have escaped the attention of prehistoric man. For intervals of more than a couple of days, when counting was cumbersome, and social records nonexistent, no simpler method of keeping track of. passing time could be conceived than the simple celestial calendar of the moon, which not only measures the completed intervals of single lunations, but shows to all who will look the approximate fraction of time since the last fulness. Such a sentiment as "Ere another moon shall wax and wane, I shall return to you again," was not confined to romantic lovers (who by the way did not exist in primitive races), but might voice the considered judgment of the traveler.

The adjustment between the sun's day and the moon's period, is not easily made. The period of about twentynine and one-half days is not well adapted to any commercial usage. We find therefore on the one hand, the practical adoption of the nominal month of thirty full days, somewhat in excess to be sure of the true lunar month, but not a bad correspondence (while the day itself, measured from noon to noon, is in winter a shorter portion of the astronomical year than the corresponding "day" in summer, so that the exact number of days in a lunar month, when the days are so measured is variable). We find also on the other hand an emphasis, particularly among oriental peoples, upon the phases of quarter period, and the resulting development of the seven-day week. The noticeable discrepancy between the interval of our weeks and a complete lunation was not as serious an obstacle as the mathematical purist might expect, and the number, seven, acquired mystical connotation that no slight maladjustment with the heavens could impair.

It might be hard for some of us to put ourselves back in the time when every event, every mark, had a deeper and more mysterious significance than would appear on the surface and afforded to the soothsayers fresh evidence of the hidden workings of a magical spirit world. We shall mention only a few of the most familiar peculiar attributes of numbers which in the mystical formulas of ancient priestcraft early impressed the wonder-loving wizards. One should not be surprised that in the study of the sequence of natural numbers and the unceasing search for cabalistic relations, special individual significance attached to each of the simpler numbers. We have first, 1, the symbol of unity and individuality, the first and original number, the generator of the entire sequence, the divisor of every integer, the starting point of the sequence that has no other end. Then comes, 2, the mark of duality, of balance, of alternation, the unique even prime, the symbol of bilateral symmetry, of the pairing of the sexes, of the branching tree, of ambiguity and doubt, of double-faced deceit, and also of even handed justice. From these by addition comes the number three, the first odd prime, and the symbol of so few living creatures, the number of the family (father, mother and children) or of sun, moon and stars, or the heavens, the earth, and the waters under the earth, as a whole celestial and universal, embracing masculine, feminine and neuter, whence even a three-legged stool is something of an oracular tripod. Then comes 4, the second power of 2, the sum of 2 and 2, the product of 2 by 2, the number of the square, of four-footed creatures, the number of the four directions, of the four seasons, of the four phases of the moon, of the four-cornered earth and of all things, inferentially, that are essentially earthly. By contrast, the relative infrequence of the number, 3, merely emphasizes its claim to celestial honors, as being the number of the mystical world of spirits, enveloping and pervading all, and yet so seldom openly encountered. Then 5, the uneven sum of 2 and 3, the number of a man's hand and of the mystical star-shaped figure of the pentagram, beautiful but hard to make, as the brier-rose with its five petals is beautiful but hard to pluck. Then 6, the product and also the sum of 1, 2 and 3, the rounded number whose square ends in itself, and so the number of the serpent biting its own tail, the number of infinity, the number of the simplest subdivision of the circle with compasses, the number of the lily, a factor of the number of lunations in a year and of days in a month. Then 7, the number of stars in the Great Bear, of days in the week, of "planets" in the sky, the sum of 3 and 4, and thus properly typifying the contact of the spirit forces with the earth. These "planets" in their apparent and reputed relative distances from the earth were listed by most ancient peoples in the order, Moon, Mecury, Venus, The Sun, Mars, Jupiter, and Saturn, and by any one of several methods could be regarded as giving a sequence indicated on the famous Heptachord of mythology and expressible by Sun, Moon, Mars, Mercury, Jupiter, Venus, Saturn, whence naturally the corresponding names of the days of the week, employed in nearly all Moon-worshiping civilizations. One should note, however, that the Grëeks observed decades of ten days, thus having a simple decimal notation, and one which fitted well with the month of thirty days. The Romans also did not observe the week, but were satisfied by the complicated method of counting from Calends, Ides and Nones, and that backward. Then comes 8, the third power of 2, the number of successive bisections, then 9 similar to it, the number of successive trisections, the second power of 3. Then comes 10, the number of a man's fingers, the natural unit of practical counting, the triangular number, sum of 1, 2, 3, and 4. Then comes 11, the incomplete and insignificant number. Then comes 12, the number divisible by 1, 2, 3, 4, 6, and having relatively prime to it, only, 5, 7, 11, among those less than it. It is the number of months in the year. Then comes 13, the number of fatal excess, the awkward prime. And so the numbers continued. In the special tribal traditions the numbers took on still further local color, and we have for the Hebrew ritual, the special Sabbath, the seven-branched candlestick, the twelve tribes of Israel. All these and many more of Christian or pagan origin permeated the medieval belief in magic.

The year which might appear to be one of the most natural measures of time may not have appealed to our early ancestors as anything so specific as the day or the lunation. In clear weather it is possible to identify the instant that the first edge of the sun's disk comes into view at sunrise, and the instant that it finally disappears in the west. The time of a quarter-moon can be easily identified to within a day, but the year might seem much more variable. In Egypt the year very naturally commenced with the rising of the Nile, in the case of other peoples, perhaps with the first leaving of the trees, with the disappearance of the last trace of snow, with the coming of the swallows, or the running of the fish, or with some other obvious and generally appreciated event which year after year marked the return of spring. But the length of the year so computed might be anywhere from 11 to 13 lunations. The fact that the sun lies low over the horizon in winter and blazes high overhead in summer was of course noticed, but it would not give to a primitive people an obvious quantitative measure of time. Some scholars have urged that in all early records of prehistoric traditions, the word which came to mean "year," must have been originally intended not for the sun's period but for the moon's so that records of extremely long lives that are to be found in the folk-tales of all such peoples are to be interpreted with a dividing factor of about $12\frac{1}{2}$. It is at least of interest to see what this gives for the lifetime of the antediluvian patriarchs whose life and deeds are chronicled in the Book of Genesis. There we read that Adam lived nine hundred and thirty years, Seth, nine hundred and twelve years, Enos, nine hundred and five years, Cainan, nine hundred and ten years, Mahalaleel, eight hundred ninety and five years, Jared, nine hundred sixty and two years, Enoch, three hundred sixty and five years, Methuselah, nine hundred sixty and nine years, Lamech, seven hundred seventy and seven years, Noah, nine hundred and fifty years. These divided by $12\frac{1}{2}$ give respectively, to the nearest whole year, 74, 73, 72, 73, 72, 77, 29, 77, 62, 76, which remind one of the "three score years and ten, or if by reason of strength they be four score, yet is their strength labor and sorrow."

The Degree

However slow or devious may have been the development of astronomical knowledge, there is no question that among some of the oriental people a knowledge of the path of the ecliptic, and naming of the signs of the Zodiac, antedated written historical records. The length of the year was known to the priesthood of these peoples certainly to within a few hours. It was a matter invested in the greatest mysticism, and attended with magic and supernatural auguries in every feature. It is therefore hardly surprising that the number adopted for its symbol should be one of completeness, one exactly divisible by the religious number, 12, and by the number of commerce, 10. The official year further had to be adjusted to the official month of thirty days, and the fact that an occasional correction was required by the insertion of an intercalary month was a matter of no serious moment and was consistent with the complete dependence of the people upon the priestcraft for the prediction of all astronomical phenomena. One can thus understand how in the systematic study of astronomical observations that had been carried on by the Babylonians for milleniums before the Christian Era, it seemed natural to divide the circle into 360 degrees, corresponding to the daily progress of the sun through the heaven of fixed stars. As has been remarked this progress is not uniform, and the discrepancy had been noted even in these prehistoric times, so that the average change from midnight to midnight in the summer is much nearer one part in three hundred and sixty than one might at first assume. For practical convenience in draftsmanship moreover, the nearest simple division to that theoretically required necessitates the use of a number easily divisible by simple numbers like 2, 3, 4, 5, 6, 8, 9, 10, and 12, and no number near 365 or 366 compares in this regard to 360.

The Right Angle

The Egyptians and other early teachers of geometry were greatly interested in the right triangle and its mystical and practical advantages. The fact that the three consecutive numbers, 3, 4, 5, representing respectively, mystical wisdom, the earth, and a man's hand, and whose sum is the familiar number, 12, should serve to enable man by science to measure the earth, was too remarkable a coincidence to have escaped deep speculation. A closed chain of twelve links or an endless rope of twelve evenly spaced knots, thus served when stretched at the properly chosen points as a visual embodiment of the right triangle. When the land along the Nile was worth the whole of the Sahara desert, and the fertilizing flood had washed away the landmarks of yesteryear, the priests with their records of old titles, and their mystical surveying ropes would descend to the fresh bank and lay off the land again in uniform strips. The elaborate masonry of temple and palace, the allotment of burial tracts in the City of the Dead, the construction of public works of all sorts required methods of mechanical draftsmanship involving the repeated delineation of perpendiculars. It is then but natural that the right angle came to be regarded by these architects as the important and fundamental angle. From Egypt, the Greeks imported the custom of measuring all angles in terms of this artistic and professional unit, the right angle.

For trigonometrical purposes the right angle is eminently unsuited, since all the usual angles must be expressed in terms of proper or improper fractions of it. The handling of fractions in the older mathematical notations was a matter of serious difficulty and fractions were avoided whenever this was possible. Of course our decimal fractional notation was unheard of, but even the rules for the addition and subtraction of ordinary fractions were a matter of great complication in most of these systems owing to what would appear to us to be narrow prejudice. Even in some of the well developed mathematical systems, only fractions with numerator, unity, were directly handled, in other systems only certain special denominators were admitted, so that the addition and subtraction of fractions were performed by reference to a carefully worked out table of approximate equivalents. One is not surprised therefore, to note that although Ptolemy the Great, was most immediately exposed to the Greek mathematical traditions taught at Alexandria, and may have been also directly influenced by the ancient native Egyptian lore, yet as an astronomer and familiar with the practice of the Babylonians, he adopted the notation of degrees and the equipartition of the circle into three hundred and sixty parts. The fact that this corresponds merely to a further even division of a right angle makes it possible for one to regard this as no real break with Greek custom, but merely the introduction of a new auxiliary unit, in the same manner that we may state the price of a cheap article in cents although the official unit of coinage for this country is the dollar.

The Grade

Other subdivisions of a right angle might naturally suggest themselves to one who is approaching the question for the first time or who is imbued with a revolutionary spirit of change. The degree is a fairly convenient unit for most purposes, and subdivisions of it into minutes and seconds according to the favorite sexagesimal notation of the Babylonians were naturally effected when needed. One of the outcomes of the French Revolution was a desire to resystematize all human conventions. Decimal coinage was established in France, the entire existing table of weights and measures was overthrown for a decimal one, the present Metric System. An attempt was made to return to the Greek usage of decades in place of weeks but religious influences soon prevailed and in revulsion against many of the excesses of the Terror and of the subsequent days, the week was re-established to the general satisfaction of all. The decimal craze was naturally extended to apply to angles. The right angle was divided into one hundred equal intervals called grades, these into centesimal minutes, and these again into centesimal seconds. The grade so introduced did not differ seriously from the degree for rough approximations of small angles and made trigonometrical computations somewhat simpler. Most histories of the subject remark that this system was immediately discarded, and has only historical interest. Such is not the case, however. The French army has since Napoleonic times employed the grade (Fr. *grad*) in its engineering work except when this in turn has been displaced by the *mil*. Tables, charts, and instruments graduated in this system have been in constant use for a century. When our army coöperated with the French in the World War, it was necessary for our soldiers also, in so far as they came in contact with French units using this system, to use it also. It is seldom even hinted at, however, in astronomical work which has followed the traditional lead of Ptolemy in the general notation if not in the theory of the science.

The Sixty-Degree Angle

It would appear natural that the simplest triangle to construct with compasses, the equilateral triangle, and for which furthermore, the three angles are equal, would have contributed its angle as a unit of geometrical measurement. It is not only the easiest angle to construct but the fact that it fits exactly six times about a point, and many other simple facts give it special prominence. It is the subject matter of the first proposition of the first book of Euclid's *Elements.* In a certain sense it is the angle adopted by the Babylonians. As has been remarked, the Babylonians used a sexagesimal system of numeration. There were sixty seconds in a minute and sixty minutes in a degree, the angle of sixty degrees, or one-sixth of a circle might be thought of as their next larger unit. This angle is also the basis of the Italian *mil* that will be mentioned presently. However in a general way, the fact that we do not even have a specific name for this angle is evidence enough that it has not been given the historical prominence that the degree and the right angle have always had.

The Point

Another regular subdivision of the right angle and one concerning which mathematical books are singularly silent is that which goes by the ambiguous title, the *point.* The circumference is divided into thirty-two equal divisions, the angle of one-eighth of a right angle being then called the point. This is one of the easiest methods of obtaining an angle, when once a pair of mutually perpendicular diameters have been drawn, only bisection being employed. Fractions of a point when required, are measured in the same spirit, in halves, quarters, eighths, and so forth. The reason that this subdivision is so seldom mentioned is probably because it is so extensively identified with the compass and is so seldom used for accurate measurement. The notation used with a compass is self-consistent but hardly simple to an average layman who would usually be hard put to it to "box the compass" in approved nautical fashion. Another method of subdividing the point, and one well adapted to careful work leads to the French *mil,* which will be mentioned shortly.

The Radian

A less naive choice of a unit of angular measure is afforded in the radian. It partakes with the right angle of a certain inevitableness in its motivation but it has no astronomical or constructional excuse for its selection. The radian is the angle subtended at the center of a circle by a circular arc whose length is equal to the radius. A short discussion suffices to show that if this formulation of a definition be accepted as having a meaning in the case of a given circle, then the angle so defined is indeed definite and independent of the particular circle selected. A much more serious question is raised by the postulated comparison of the length of a circular arc with that of a rectilinear segment. This comparison will be accepted by most laymen without serious scruples, but is abhorrent to Euclidean purists. It can only be justified by a careful investigation of what is meant by length of an arc at least in this simple case, and by a general reliance upon the methods and concepts of infinitesimal analysis. The radian, even after its definition is accepted, is found to be incommensurable with a right angle and hence with all of the other angles hitherto mentioned as units, and so is extremely awkward for many types of commercial computations. It has been adopted universally throughout the branches of higher mathematics, because of the simplification of the differential formulas involving the use of an angle that result from its introduction. It is thus from the psychological viewpoint on a par with e, the celebrated base of Naperian logarithms.

The True Mil

There are two reasons for the introduction of the true mil into mathematical work. These reasons are not entirely unrelated but are easily distinguished from each other.

We have in the first place the practical demand for a simple subdivision of the radian for use in trigonometric tables. It seems not unnatural therefore to introduce the true mil defined as the thousandth part of a radian, and thus avoid in large part the necessity of the introduction of a decimal point in three-place or four-place tables so far as the angles are concerned. This call for the introduction of the true mil is not an urgent one since radian tables are very little used and are seldom given in expanded form.

In the second place, the true mil is introduced to simplify the arithmetical computations incident to field work in comparing the angle of vision under which an object is seen, its width at right angles to the line of vision, and its distance. Such a comparison has a meaning only when the object is at a distance relatively great in comparison to its apparent width, so that the difference between arc and chord is negligible. In military observation, particularly in settled regions, there are numerous objects of military interest whose approximate width or height is known, or which by moving at known speeds determine points at terminable distances from each other. For example the height of a man standing, or of a cavalryman on horseback, the distance from the top of one window in a house to the top of the window in the next story above, the distance between telegraph poles along a railroad track, the width of an artillery battery with guns of a given type, the length of a freight car or the height of it, the length of column formed by a complete company of infantry and the rate at which it marches. All of these provide an intelligent and trained observer with an estimate of actual width or height of some landmark in the distance, proper discount being made for the foreshortening incident to not being perpendicular to the line of sight. The actual angle subtended at the eye is measured, one method being by the aid of field glasses with a scale of true mils marked in one of the lenses. Given the angle in mils and the projected width the distance in the same units as employed for the width is then inferred at once by multiplying the width by a thousand and dividing by the number of mils subtended at the eye.

It is clear that for this use of the true mil only small angles are concerned, and only approximate conclusions are desired or possible. Hence any unit angle not differing greatly from a mil will be satisfactory if for other reasons such substitution offers practical advantages. In order to retain a single system of angular measurement throughout, it is more convenient to replace this true mil by a unit easily commensurable with the right angle. It is therefore customary in army work to mention the true mil if at all only in passing and replace its consideration by that of a unit also known as a mil, and differing from it but slightly and equally suited for comparison of distances, widths, and angles, to the accuracy expected, while being an aliquot part of the total angle about a point.

The Italian Mil

The Italian army uses as the approximate mil, one six-thousandth part of a circumference. This is equivalent to replacing the radian by the angle of sixty degrees, one thousand of these mils giving the angle of sixty degrees. Thus a rectilinear segment whose extremities are both at one thousand units from the observer, will subtend approximately one mil if one unit in length, and exactly one thousand mils if one thousand units in length.

The French Mil

The French army uses as the approximate mil, one part in sixty-four hundred of the circumference, and during the World War this unit was extensively employed by all the Allies. This is nearer the true mil in value, the true mil being between the two, being in fact one part in a little more than 6283 of the circumference. This unit is more easily estimated by bisection of the right angle than the Italian mil, although the term in its fuller sense seems here largely meaningless, since one thousand of these mils has no longer any special significance. Its relation to the directions of the compass has been mentioned.

Other Divisions

Other divisions of the circle could easily be made and others are even familiar to us. The sundial suggests a division into twenty-four equal parts analogous to the division of the day into twenty-four hours, and the modern clock, although suggested by the sundial, marks the circle with twelve equal subdivisions, and did so in most cases even before the introduction of a minute hand made a factor of sixty desirable. Even the division of a circle into degrees cannot be effected by Euclidean methods any more than it is possible by such methods to lay off a radian, so that the possibility of Euclidean construction alone should not prevent us from using a seventh of the circumference as unit nor is it an encouragement in using a seventeenth part. It seems very doubtful whether there will ever be any other essentially independent unit used for measuring circular arcs and their subtended angles. Of course one computer may read his angles in degrees, minutes and seconds, another in degrees, minutes and decimal parts of a minute, or even as is done in some recent tables, in degrees and decimal parts of a degree, but such questions are far from the purpose of this paper.

THE PLACE OF MATHEMATICS*

PAUL M. BATCHELDER, UNIVERSITY OF TEXAS

I overheard somebody remark at the close of the first lecture of this series that if a man did everything he was advised to in these eight lectures it would take three lifetimes to become educated. Of course education is a matter of degree, and we cannot be specialists in every department of knowledge. I fully agree with what Mr. Leon said about the value of learning a foreign language, but I believe that much of this value can be obtained without completely mastering the language. A minimum of two numbered courses in some foreign language is required for the Bachelor of Arts degree at the University of Texas, and the three or four years' study which this usually implies is sufficient to give the student considerable insight into the life and thought of the people who speak the language, and if his instructor insists on his translations being made into good English he gets much valuable training in the use of his own language at the same time. Similarly, in mathematics, I am not going to advise every person who aspires to a liberal education to study calculus or quarternions or the theory of functions of a complex variable, but rather to explain why three years of algebra and geometry are required for entrance into the University, and another year of mathematics for the Bachelor of Arts degree.

Mathematics has held a high and honored position in the curriculum of higher education for over two thousand years. Plato considered it an essential part of the education of free men, and geometry was the most important subject studied in the famous philosophical schools founded by Pythagoras and his disciples. Geometry as an abstract

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*This is the second of a series of lectures on the general subject "The Educated Man" delivered at the University of Texas by various members of the faculty. The first lecture, "A Study of Language," was given by Mr. H. J. Leon of the department of Classical Languages.

science owes its origin to the ancient Greeks, whose keen minds were fascinated by its rigorous logic, and the high degree to which they developed the subject is evidenced by the fact that the geometry taught in our high schools is taken with only slight changes from the famous treatise of Euclid, which dates from the third century before Christ. During the Dark Ages the study of mathematics was carried on by the Arabs, and it became again an important subject in the schools of Europe after the great intellectual awakening in the fourteenth and fifteenth centuries known as the Renaissance, and has maintained its place down to our own times.

During the past century, however, knowledge has been increasing at an enormous rate; not only have old sciences like mathematics and astronomy greatly extended their bounds, but new ones have arisen, like geology and economics, and claimed an equal right to a place in the education of youth, a claim which has in many cases prevailed; with the result that some of the older subjects are being crowded out of the curriculum. Greek is almost gone, and Latin is studied by a rapidly decreasing proportion of our students; mathematics is under fire in certain quarters as the next of the traditional subjects to be abolished. Is any subject safe from the attack of alleged educational experts, or will the whole present curriculum be scrapped in favor of new and more "practical" subjects? In this critical age no subject will or should be allowed to maintain its place simply because it has been taught for centuries, but must compete with the newer branches of knowledge on an equal basis and demonstrate its value in the life of today. Nevertheless, I think it is safe to predict that much will survive. The present grows out of the past, and it is impossible to understand the present without knowing something of what has gone before, so an educated man must have some knowledge of history. We live in the midst of a physical universe, an infinite realm of matter and force and life, of which our very bodies form a part, and some intelligent adjustment to this material world is essential for mere existence even. An ignorant man may never have heard of Isaac Newton or of the law of gravitation, but he must at least learn that it is not wise to step off the edge of a high precipice. One of the great aims of education is to fit the student for his environment, and in such a process the natural sciences cannot be wholly neglected. Moreover, we live in a social world; every individual is a member of a family or other small group, which has relations with larger groups living in the same town or county or state or nation. In order to adapt himself to this social environment the student must acquire some knowledge of the social sciences. Again, every man in the course of his life meets many problems which demand logical thinking for their solution, and consequently his education must provide some training in clear and logical thought. The subject which above all others is fitted to give this training is mathematics.

It is impossible to frame an entirely satisfactory definition of mathematics, one which will mark it off clearly from logic on the one hand and the natural sciences on the other. The rapid progress of the subject in modern times has rendered obsolete such former definitions as "the science of quantity" and "the science of space and number," and most modern definitions of mathematics are based on its logical character. Bertrand Russell has defined it as "the class of all propositions of the form 'p implies q'," and Benjamin Pierce as "the science which draws necessary conclusions." These are both somewhat too broad, in that they include all exact reasoning, but they do emphasize the most characteristic feature of mathematics. Russell has given another definition, that mathematics is "the subject in which we never know what we are talking about, nor whether what we are saying is true," a statement which many students will heartily agree with. In spite of its paradoxical form, it is very instructive when suitably interpreted. Mathematicians do not know what they are talking about because they use symbols, such as x and y, which can represent many different things, and because in defining concepts in terms of simpler ones a starting-point is necessary, so that a few fundamental concepts, such as point and length, must be left undefined. Similarly, they do not know whether what they say is true or not, because all their theorems are based on certain axioms or assumptions, and the theorems are true only if the axioms are true.

One of the chief purposes of studying mathematics, then, is to train the mind in accurate and logical thinking. Some of the educational experts who wish to abolish Latin and mathematics and all other difficult subjects have denied the value of mental discipline, and asserted that learning to reason in mathematics does not improve the power of reasoning in other subjects. I am unable, however, to see any essential difference between mathematical reasoning and any other kind of reasoning; the mind goes through the same kind of process in both cases. The apparent difference arises from the fact that in mathematics we are dealing with simple and clearly defined concepts like planes and triangles and circles, about which our reasoning leads to definite and conclusive results; whereas in every-day life the problems which we reason about are too indefinite and involve too many unknown factors for us to reach certain conclusions. For example, suppose a merchant finds that his trade has increased to the point where his present store is not large enough, and he faces the problem of either building an addition to his store or selling it and moving to a new location. He can see some advantages in each plan, and some disadvantages; but he cannot by any process of rigorous deduction decide with absolute certainty which will be the more profitable; all he can do is to weigh the known advantages against the known disadvantages and decide which plan will *probably* be the better. Even so, his judgment is most likely to be correct if he reasons out logically the effects of all the conditions which are known to him, instead of jumping at conclusions or allowing himself to be influenced by irrelevant considerations.

Sloppy thinking is one of the great curses of this country, and perhaps of all democracies; it enables a skillful demagogue to secure great political influence, however full of fallacies his dogmas and arguments may be. In politics, and in some other departments of life as well, we need less appeal to prejudice and more hard thinking. Mathematics displays the most logical and convincing type of proof which the human mind is capable of, and every student should acquire some familiarity with and appreciation of it as a standard to which all reasoning should approximate as closely as the circumstances permit.

With regard to the general question of the disciplinary value of mathematics and other subjects, I want to point out that our whole system of education is based on the assumption that the student's mind is being trained and improved by the subjects he studies. If this is not the case, and if only the facts he retains are of value, then education is a colossal waste of time and money, for the student promptly forgets the great majority of the facts he learns as soon as he ceases to use them.

Another reason for studying mathematics is its immense practical value. The vital part which mathematics plays in the affairs of the world was brought home with great force to many people during the recent war, when the need of mathematical training was evident on every hand, not only in the engineering corps, but in other departments of the service as well, especially the artillery. It takes a lot of trigonometry to determine how to aim a big gun so as to make the projectile hit an object which may be several miles away on the further side of a mountain.

Less directly practical, perhaps, but none the less of great ultimate value to humanity, are the applications of mathematics to the various sciences. In the early stages of a science it is mainly descriptive. For example, the science of electricity began with the observation that a piece of amber when rubbed with wool acquires the property of attracting small objects like bits of paper. (The name "electricity," by the way, is derived from the Greek word for amber.) Later it was found that other substances can be electrified in the same way, that there are two kinds of electricity, which were called positive and negative, that lightning is the same as electricity, and that many interesting experiments can be performed with electric charges. Up to this point the known facts were so isolated and the relations between them so little understood that no detailed and systematic theory of the phenomena was possible. But when Coulomb made a series of careful quantitative experiments on the force which two charges exert on each other, and found that the force is directly proportional to the product of the charges and inversely proportional to the square of the distance between them, he had discovered an exact relationship which could be stated in mathematical form. As more and more facts were discovered, other exact relations appeared, and more and more mathematics was used to express them; until today large volumes are written on the theory of electricity, so bristling with mathematical formulas that no one but a specialist can read them.

All the sciences are going through much the same series of steps. The first stage is the collection of facts and arrangement of them in as orderly a form as possible. Some of the newer sciences, *e.g.*, economics and psychology, are still largely in this stage. Even in this preliminary work mathematics is frequently able to give some assistance; in particular, the branch of mathematics known as statistics enables large masses of data to be handled in a systematic fashion, and often brings out results which would otherwise escape attention. Also the use of graphical methods enables facts and relations to be presented in an easily comprehensible form. If an economist wishes to show how the price of cotton has varied over a series of years, the simplest way to do so is to draw a graph in which the years are plotted along a horizontal line and the price along a vertical line.

A warning should be issued at this point, however; some would-be scientists have allowed their enthusiasm to run away with their judgment, and have used mathematical methods in places where they are not appropriate. I have seen discussions in which elaborate mathematical machinery was employed to obtain conclusions from data that were so scanty or uncertain as to render accurate deductions impossible. Mathematics is not a magical machine into which one can pour any kind of facts or near-facts and by turning a crank draw out correct conclusions. Statistical methods merely enable the worker to obtain explicitly facts which were implicit in the data; if the data are uncertain or erroneous the conclusions will be uncertain or erroneous also. It is abuses of this sort which have led to the saying that figures don't lie but liars do figure, and to the famous classification of falsehoods into lies, what for the present purpose I will call profane lies, and statistics.

As a science develops, more and more exact relationships are discovered, and accurate deductions become possible, with the result that the language of mathematics is used in increasingly large measure. We usually think of biology as a descriptive science, yet the applications of mathematics in it have been so extensive that a new science known as biometry has grown up. Chemistry is another science which now finds much use for mathematics, in addition to having a symbolical language of its own analogous to that of mathematics. When we come to mature sciences like physics and astronomy, we find extensive collections of accurate facts bound together into a coherent system by logical reasoning, expressed largely in the language of mathematics. In fact, so many different branches of mathematics find applications in modern physics that it is becoming a serious burden to the physicist to learn all the mathematics he needs in order to master his own domain.

When I speak of physics and astronomy as mature sciences, that is not to be interpreted as meaning that they have reached the end of their career and become fixed and lifeless; on the contrary, neither science was ever more vitally active or making more rapid progress than at the present time. Their mathematical formulation is not a hindrance to their progress, but rather a help. By means of mathematics the consequences implied by a scientific theory can be worked out, and then tested experimentally; if the predicted results are verified, the theory is strengthened; if the predictions disagree with the experimental facts, the theory must be either modified or abandoned. A striking example of how a great discovery can be made by mathematical methods alone is furnished by the finding of the planet Neptune in 1846. After the discovery of

Uranus about sixty-five years earlier, it was observed that its motion was subject to certain small irregularities which could not be accounted for by the attractions of the other planets, and it occurred to two mathematicians independently, Adams in England and Leverrier in France, that the perturbations might be caused by an unknown planet. They both succeeded in devising methods for computing the size and position which the hypothetical planet must have in order to produce the observed irregularities, and when the astronomers turned their telescopes to the sky, they found the new planet within a degree of the predicted place.

In addition to its direct aid to the sciences, such as I have indicated, mathematics furnishes a standard or ideal toward which the other sciences work. I referred a few minutes ago to Benjamin Pierce's definition of mathematics as the science which draws necessary conclusions. The conclusions of the sciences are not always necessary, but sometimes merely probable; but it is the aim of the scientist to so coördinate his facts that the deductions from them shall be as definite and certain as possible, and in proportion as he succeeds in doing this his reasoning becomes mathematical.

Again, mathematics furnishes a clear and concise symbolical language in which the results of scientific investigations can be expressed. The immense value of a good symbolism can hardly be over emphasized. To take a simple example, if you will try to compute by the aid of Roman numerals alone the value of a bale of cotton weighing 487 pounds when cotton is selling at 32½ cents a pound, you will be thoroughly convinced of the great superiority of the Arabic system of numerals. The mere invention of a symbol for the number zero was a long step in advance. One reason why the progress made by the ancient Greeks was so much greater in geometry than in algebra was their clumsy system of representing numbers by the letters of the alphabet. To take another example, it is totally impossible for the human mind to form any conception of four-dimensional space. Yet the geometry of four-dimensional space can be studied mathematically without any great difficulty. The position of a point in three dimensions is determined by three numbers, called the coördinates of the point, and a point in four dimensions by four numbers, and what the mathematician knows about sets of three numbers can usually be extended directly to sets of four or even n numbers.

If we look back to the beginnings of human culture, when our ancestors were ceasing to be merely animals and beginning to be men, we shall find reason to believe that the invention of language was one of the greatest steps in their progress. Without language, only the simplest and most primitive ideas and emotions can be conveyed from one mind to another; with language, more complex and abstract thoughts can be communicated, and moreover they can be passed on from one generation to the next and thus become part of the permanent inheritance of the race. It is doubtful if any long chain of reasoning could be carried out without the aid of language; as soon as one link in the chain has been made, the words in which it is expressed hold it fast while the next link is being forged, and thus the mind proceeds step by step until the chain is complete.

The language of mathematics plays the same part in facilitating scientific reasoning that our ordinary language does in facilitating thought in general. It enables exact facts and relationships to be expressed in clear and concise fashion, and furnishes aid and comfort to the mind in carrying out step by step long and intricate deductions. The very symbolism appears at times to have a sort of momentum of its own, leading the investigator on to discoveries which he did not anticipate.

An idea which has remained barren for generations may be transformed and take on new life when it is subjected to mathematical treatment. Since the very beginnings of philosophy, the nature of space and time has been a frequent subject of speculation, and the doctrine has long been established that all motion is relative; yet it is only in our own time that by the genius of Einstein our concepts of space and time have been analyzed, and the doctrine of the relativity of motion built into a mathematical and physical theory which is giving us a far deeper insight into the nature of the material world about us than was previously possible.

In describing the relations between mathematics and the natural sciences, I do not wish to convey the impression that the benefits have been all on one side. Many of the most important branches of mathematics have arisen out of problems suggested by the sciences. If a physicist meets with a difficulty which requires mathematical treatment, for example if his work leads to a differential equation of a type which has never before been studied, he will usually either attack the problem himself, or else call it to the attention of the mathematicians, and if the latter find the problem interesting they are likely to develop and generalize it far beyond the point where the needs of the physicist are satisfied.

It is fortunate for the sciences, however, that not all the work of the mathematicians is of this directly practical character. Most mathematical theories have been built up for their own sake, and only long afterwards, if at all, found practical applications. The ancient Greeks studied the properties of the curves known as conic sections two thousand years before Kepler and Newton discovered that the paths followed by planets and comets as they revolve about the sun are conic sections, and Einstein could never have devised the general theory of relativity if certain abstract subjects like non-Euclidean geometry had not already been thoroughly investigated. In mathematics, and in the other sciences as well, the research worker is justified in seeking to extend the boundaries of knowledge in any direction, regardless of whether he can see any practical applications of his results or not. Some at least of the discoveries which pure science is making today will prove useful in the future, but the history of science warns us that we cannot predict which ones they will be.

Through the applications of the sciences to engineering and industry, mathematics has played an indirect but fundamentally important part in the development of modern civilization. It is difficult to imagine what our world would be like if everything that depends directly or indirectly on mathematics were to be removed; there would be no railroads or airships, no telegraph, telephone, or radio, no long bridges or tunnels or Panama canals, no great irrigation projects to make the desert blossom as the rose—in short, our whole civilization would be set back many centuries. Some progress would have been made, of course, by purely empirical methods, but such progress is necessarily slow and uncertain. The ancient Romans succeeded in building some great aqueducts to supply the city of Rome with water, but only by an appalling waste both of men and material. The first submarine cable across the Atlantic Ocean failed to work, and it was only after the defects had been remedied as a result of the mathematical investigations of Lord Kelvin that successful communication was established. The conquest and harnessing of the forces of nature which is so characteristic of the present age, while it is by no means the achievement of mathematics alone, would have been quite impossible without the contributions of the latter, which must be regarded as one of the most practical of the sciences. We can still say today what was said six hundred years ago by Roger Bacon, that remarkable forerunner of modern science in the darkness of medieval times, namely:

"Mathematics is the gate and key of the sciences . . . Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy."

In conclusion, I want to say a few words about the motives which actuate the pure mathematician, the seeker who pursues the subject for its own sake, regardless of its applications. Why did mathematics make such a strong appeal to many of the best minds among the ancient Greeks, and what is the attraction which has led certain men all through the ages to devote their lives to its study and advancement?

In part, the motive is one which is felt by all scientists:

an intense curiosity about the universe in which we live, and a love of truth which in some men almost reaches the fervor of religion. In mathematics we have a body of truth which, beginning with the geometric theorems discovered by the Greek philosophers, has now grown to such enormous proportions that no one man can hope to master it in all its details. The mass of facts has not become unmanageable, however, because they are parts of an organic whole. One of the striking features of the development of mathematics has been the increasing appearance of unity among the different branches. It has happened repeatedly that two different parts of the subject were developed independently and for a long time had no apparent connection with each other, and yet when the development was carried sufficiently far they were seen to be closely related, or even different aspects of the same thing. Unfortunately, most students do not penetrate deeply enough into mathematics to appreciate this tendency toward unity, although one of the subjects regularly taught in the freshman year in college, namely, analytic geometry, is a unifying bond between algebra and geometry, which are studied as separate topics in the high school. This union of different fields of mathematics has often led to great progress, since it enables problems in one field to be attacked by methods which were developed in the other. The famous problem of squaring the circle remained insoluble as long as it was attacked by purely geometric methods, but with the aid of algebraic processes the construction was definitely proved to be impossible, and the other two famous problems of antiquity, the trisection of an angle and the duplication of a cube, were settled in the same fashion.

The greatest appeal of pure mathematics, however, lies not in its truth or its unity, but in its harmony. Mathematics is not only a science, it is also an art. It has often been likened to music, and the analogy is a good one; there is much of common in the esthetic appeals which both make to their devotees. Love of beauty, and joy in creating beautiful things, belong to the mathematician no less than to the musician or artist or poet. Man is not content to live by bread alone, but demands the satisfaction of the deeper cravings of his soul. In the outer world, all is change and confusion. Men come, dwell a little while on the earth, and then depart, and their memory vanishes like a dream in the night. The wind bloweth where it listeth, and we cannot tell whence it cometh or whither it goeth. Change and decay appear on every hand, and fickle chance seems to rule supreme. In the midst of this turmoil and flood of events, the heart of man craves some firm rock to which it can cling, something of permanent and stable value; and what it seeks it finds. Firm above the flood of time stand the enduring treasures of religion, of art in its various forms, and of scientific truth. The mathematician is concerned primarily with the last of these, but in their highest manifestations they are closely akin. To one who knows the wonders of modern astronomy, the Psalmist speaks with words of ever deeper and more exalted meaning, "The heavens declare the glory of God, and the firmament showeth his handiwork."

A GEOMETRY CLASS PLUS A LITTLE "PEP"

By Ruth Cottingham, Tyler High School

Did you ever suddenly discover that the god Morpheus had crept into your room, and without your permission had waved his Magic Wand, and thus wafted your class into his mysterious realm? If you have not experienced this situation, you are indeed fortunate, and what I have to say will mean nothing to you.

In my attempt to trace to its source this "drowsiness" of the class, I came to the conclusion that the majority of us spend our class hour in one of the two following ways: (1) permitting the few who are mathematically endowed to do all of the reciting while the remainder of the class sits in subdued silence wondering how many minutes before the bell will ring; (2) explaining to one or two earnest pupils whose mathematical ability is below par, the others amusing themselves the best way possible.

Now, it seems to me that so long as we must teach in the same class pupils who differ to such a large extent in their abilities, our problem is to equalize the recitation. There may be many ways of doing this, and I have tried various methods, but the one that seems to bring the best results is the contest review, the plan of which I shall briefly relate: Divide the class into two well balanced sections, numbering each member of each section and seating according to numbers on opposite sides of the room. For example, No. 1 on "A" side is correspondingly seated to No. 1 on "B" side, etc. Give the class at least one day to unite their forces in preparation for the contest. I find this helpful because the best pupils in each section will utilize this time helping the weak pupils on their respective sides to overcome their difficulties. On the day the contest begins each individual comes to the class with a list of questions which he must be able to answer himself. This list may consist of good review questions, propositions, or problems of the pupil's own making. No. 1's on both sides exchange questions. Every member of each side is working on the problem given to No. 1 of his side. The time is limited, and when it is announced that the time is up, No. 1 on each side goes to the board and works the problem given him. If both problems are correct, each side gets ten points. If either fails to work the problem correctly, his side is given three chances to work it. If any one of the three works it correctly, his side is given five points; but if all three fail, then the problem returns to the one who gave it, and he makes ten points extra for his side. It is now time for No. 2's and so on. The pupils themselves must accept or reject the solution of the problem, the teacher being the score keeper and supreme judge. The reward I offer to the winning side is ten points added to the grade of each individual on the winning side. I let it be understood thoroughly that each person must put forth every effort to win, or else his prospective ten points will vanish. Those on the losing side who have worked faithfully I remember favorably when grading time comes.

I do not claim that the contest review is a panacea for all classroom ills, but the least it can do is to create an enthusiastic attitude on the part of the entire class, and according to my way of thinking, it is better than any written test that could be given. I hope that others will try the contest review and will find it as truly beneficial as I have found it.

It is with great pleasure that the editors acknowledge the receipt of the program of the "Mathematical Club" of the Teague High School sent by Miss Leila Weaver. It is an attractive booklet containing nine programs, every one of which is excellent.

Some of the subjects considered are "Examples for Finding Seats," "Primitive Arithmetic," "Development of Arithmetic," "The Geography of Arithmetic," "Names to Be Honored," "Percentage in the Business World," "Debate: the Rate of Interest Should Be the Same in All States of the United States," "Great Men Who Have Enjoyed Mathematics," "Debate: Algebra Should Be an Elective Course," "The Wide Use of Symbols," "The Geography of Algebra," "Relationship Between Variable Quantities," "The Graph and Astronomy," "The Geometry of House Keeping," "Trisecting the Angle," "The Geography of Geometry," "The Geometry of Architecture," "Where Algebra and Geometry Meet."

DR. MENDENHALL'S LEGACY TO TEACHERS

Anyone who has known much of Dr. Thomas C. Mendenhall knows that he was not a man whose mind was ever closed to new knowledge, even if it should require the recasting of opinions firmly held by him in the past, on the basis of the evidence which the past had made available. If certain educational tendencies of the present generation, then, have failed to win his approval, that fact deserves serious consideration in educational circles.

Dr. Mendenhall was to have delivered an address before the educators assembled at Ohio State University during the closing days of last week. On the day before his death he completed the manuscript of what he intended to say, and this manuscript was read before a general session of the conference by Dr. Edward Orton on Friday evening. He had entitled it "Inflation in Education," and the burden of his thought was that in the rapid expansion of subjects and courses of study under the elective system, the multiplication of "student activities" outside the curriculum, and the growth of intercollegiate athletic schedules, there has crept into our schools and colleges a serious depreciation of educational values, just as a country's currency goes down in value when inflated beyond the point of unquestioned ability to redeem it at par.

Among other features of the paper was a firm rejection of the view of a certain school of modern educational psychologists that there is no such thing as the general "mental discipline" which occupied so prominent a place in the educational philosophy of the past. This denial has been used with more vigor than evidence of careful consideration in the effort to break down belief in the special educational value of certain branches of study which have been urged because of their disciplinary effect on the pupil's mind, rather than their immediate relation to "getting a job." The rejection of the "mental discipline" plea has been based on limited laboratory experiment. Educators of the clear insight and long experience so admirably represented by

Dr. Mendenhall base their belief in the existence and high value of such discipline on the ample evidence which they have seen of its results in the later lives of thousands of students who have been under their observation. At the present time, the reaction on this question, we believe, is steadily toward the view expressed in his last utterance by Dr. Mendenhall.

Another point of importance in the address was its deprecation of a tendency in our normal schools to overstress "method," in comparison with the subject-matter to be taught. There can be little question, we think, that this tendency does exist; and perhaps from the very nature of the normal school, the danger of this loss of due proportion will always need to be carefully guarded against.

Still another strong feature was Dr. Mendenhall's protest against the modern educational tendency to seek "devices for smoothing, or rather avoiding, the rough places on that road which, for thousands of years, has been assumed to be the same for prince and pauper." All this takes away what he rightly describes as "the joy of victory over opposing forces and conditions." The entire paper will of course be printed somewhere, and it should be made generally accessible to students, teachers and parents alike.

[Editorial from the Columbus, Ohio, "Dispatch."]