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Regularity Estimates for some Free Boundary Problems of Obstacle-Type

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Regularity Estimates for some Free Boundary Problems of Obstacle-Type

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For my Family.

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"The job is almost finished, the goal almost attained, everything possible seems to have been achieved, every difficulty overcomed - and yet the quality is just not there. The work needs more finish, perhaps further research. In that moment of weariness and self-satisfaction, the temptation is greatest to give up, not to strive for the peak of quality. That's the realm of the last inch - here the work is very, very complex but it's also particularly valuable because it's done with the most perfect means. The rule of the last inch is simply this - not to leave it undone. And not to put it off -because otherwise your mind loses touch with that realm. And not to mind how much time you spend on it, because the aim is not to finish the job quickly, but to reach perfection."

— A. Solzhenitsyn, *The First Circle*

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Regularity Estimates for some Free Boundary Problems of Obstacle-Type

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We study regularity estimates for solutions to implicit constraint obstacle problems and penalized boundary obstacle problems. We first prove regularity estimates for the solution and the free boundary in the classical stochastic impulse control problem. We show that the free boundary $\partial\{u < \varphi_u\}$, where φ_u is the implicit constraint obstacle, can be decomposed into a union of regular points, singular points, and degenerate points with corresponding regularity and measure theoretic estimates. We then turn to generalizing our analysis to the fully nonlinear problem with obstacles admitting a general modulus of semiconvexity $\omega(r)$. We prove that solutions to the fully nonlinear stochastic impulse control problem are $C^{\omega(r)}$ up to $C^{1,1}$. Finally we turn our attention to study both nonuniform and uniform estimates for the penalized boundary obstacle problem, $\Delta^{1/2}u^{\epsilon} = \beta_{\epsilon}(u^{\epsilon})$. We obtain sharp estimates for the solution in both the nonuniform and uniform theory.

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Chapter 1

Introduction

> *"For the unity of field theory lies in its techniques of analysis, the mathematical tools it uses to obtain answers."*
> — P. Morse, H. Feshbach,
> *Methods of Theoretical Physics*

Mathematical models play an important role in the study of scientific and engineering phenomena. A common feature of many mathematical models is that they necessarily lead to equations admitting an ellipticity condition. A particular class of models we consider are elliptic and parabolic free boundary problems. Free boundary problems have been used widely as mathematical models in the context of fluid dynamics, elasticity, shape optimization and optimal stopping time problems. Moreover, their mathematical treatment has led to important developments and connections with other areas of modern mathematics such as minimal surface theory, optimal transportation theory, calculus of variations, geometric measure theory, harmonic analysis, geometric analysis, and stochastic processes. For a nice overview of past and recent applications of free boundary theory we refer to [11], [52], [59], [28].

In this research work, we are particularly interested in investigating mathematical questions related to existence, uniqueness, asymptotic behavior, and regularity. The following thesis is devoted to two classes of free boundary problems of obstacle-type. The first class of problems studies regularity estimates for obstacle problems admitting a nonlocal obstacle. The second class of problems considers uniform regularity estimates for singularly perturbed lower dimensional obstacle problems.

1.1 Regularity Estimates for Free Boundary Problems

A free boundary problem can be described as a boundary value problem for either an evolving or stationary physical system, where some of the unknowns or their derivatives change their behavior discontinuously at some particular values. More precisely the goal is to find an unknown pair (u, Ω), where the function u solves some equation outside $\partial\Omega$ (Free Boundary), and some global conditions prescribe the behavior of u across $\partial\Omega$. Some examples of free boundary problems are:

a. The solid-liquid interphase, when a material undergoes a phase transtion.
b. The boundary between the exercise and continuation region for a financial instrument such as options.
c. The transition from elastic to plastic behavior when stress goes through a critical value.

As in the theory of elliptic partial differential equations one first attempts to find weak solutions to the problem by variational methods or methods stemming from nonlinear PDE theory i.e. maximum principles, viscosity solutions. The next step is to study regularity estimates for the pair $(u, \partial\Omega)$. Many of the applications mentioned above fall into one of two classes of free boundary problems:

1. Obstacle-Type Free Boundary Problems
2. Bernoulli-Type Free Boundary Problems or problems with transmission conditions across the free boundary.

The obstacle problem is considered a one-phase problem, where the solution u is non-trivial on only one side of the interface. A nice visualization of the obstacle problem is of an elastic membrane being pressed against a solid surface due to an external force. Here we assume that the membrane is given by the graph of the function u. The free bounadry is the separating curve between the contact and noncontact region. The obstacle problem can be studied as an energy minimization problem for the membrane under the constraint that it must lie above the solid surface. The existence and uniqueness theory for this problem can be obtained through a study of variational inequalities (See Appendix C) or the theory of sets of finite perimeter.

Bernoulli-Type Free Boundary Problems often appear as two-phase problems. In the stationary problem, the solution u, may represent temperature such as in flame propogation problems, taking both positive and negative values. Along the free boundary, or the zero level surface of u, here representing the edge of the flame, u_ν, the normal derivative satisfies a jump condition expressing the dynamics of the process. The general idea is to build classical solutions so that the separating surface $\partial\Omega$ is regular, u is regular up to $\partial\Omega$ from both phases, and the free boundary conditions expressed in terms of u_ν are satisfied pointwise. This is usually done by integrating the transition condition to give a weak formulation to the problem through variational techniques expressed as conservation laws or by supersolution methods.

We point out that our interest in subsequent chapters is to study one phase free boundary problems. The general methodology to study regularity estimates for obstacle-type free boundary problems are inspired by related techniques in the theory of minimal surfaces. We briefly discuss the main steps in studying existence, uniqueness and regularity estimates for minimal surfaces. As before we split the problem into two distinct steps:

Step 1: Find a family of surfaces, for which a general notion of area can be defined.

One first shows that this family is closed under a limiting process and that the area is semicontinuous with respect to this process. Recall a set Ω has finite perimeter if for any smooth vector field v with $\sup_{x\in\Omega}|v| \leq 1$,

$$\left|\int_\Omega \nabla \cdot v\right| \leq C_0.$$

The best constant C_0 is called the perimeter of $\partial\Omega$. Heuristically,

$$\left|\int_\Omega \nabla \cdot v\right| = \left|\int_{\partial\Omega} v \cdot \nu\right| \leq \left|\text{Area}(\partial\Omega)\right|.$$

Moreover the perimeter functional is shown to be lower-semicontinuous and hence the direct methods of calculus of variations gives an existence proof to the problem of finding a set with minimum perimeter.

Step 2: Show that $\partial\Omega$ is a smooth hypersurface outside of an unavoidable singular set.

One technique to study the regularity theory of minimal surfaces is to look at invariance properties of minimal surfaces, in particular invariance under rigid motions and dilations as well as monotonicity formulas. Formally, if S is an area minimizing surface in $\mathbb{R}^{n+1}$ through 0, then,

$$E(r, S, z) = \frac{\text{Area}(S \cap B_r(z))}{r^n}$$

is a monotonically increasing quantity in r. This monotonicity implies that $E(r)$ has a limit as r tends to 0 or ∞. If we blow up a minimal surface then,

$$E(0, S, z) = \lim_{r\to 0} E(rt, S, z) = \lim_{r\to 0} E(t, S_r, z) = E(t, S_0, 0)$$

a quantity that is constant in r. Here $S_r = \{x : z+rx \in S\}$ denotes the scaling and the blow-up is denoted by S_0. The next step is to classify the minimizing cones S_0. Studying various alternatives, one tries to deduce the regualrity for S near 0.

We conclude this section by making some remarks about monotonicity formulas and their usefulness for regularity theory. Heuristically they measure a sort of radial entropy for solutions to diffusive problems, that increases as we diverge from a fixed point. Some known examples of monotone quantities of interest in elliptic free boundary theory are the Almgren Frequency Functional for harmonic functions, the average of subharmonic functions, Weiss-Type Monotonicity Formulas, Monneau-Type Monotonicity Formulas and the Alt-Caffarelli-Friedman Monotonicity Formula. In a later section, we will use monotonicity formulas to understand optimal regularity estimates for solutions in the lower dimensional obstacle problem. The general idea is that as you scale into the origin, the blow-up solution is less complex and oscillatory. Hence in a neighborhood of the origin we obtain a kind of structural control on the growth rate of the local solution. There are some interesting connections between monotonicity formulas and entropy in a physical system. The interested reader should see [8], [30]. Moreover a very thorough explanation of monotonicity formulas in the context of free boundary problems can be found in [51], [21].

1.2 Summary of Thesis

We now give a summary of the main chapters of the thesis. We begin in the second chapter by reviewing some of the main results in the regularity theory for free boundary problems of obstacle-type. In particular we discuss classical obstacle problems, lower dimensional obstacle problems and their singular perturbations.

In the third chapter we consider an implicit constraint obstacle problem arising in impulse control theory. Stochastic impulse control problems ([11], [45], [46], [32]) are control problems that fall between classical diffusion control and optimal stopping problems. In these problems the controller is allowed to instantaneously move the state process by a certain amount every time the state exits the non-intervention region. This allows for the controlled process to have sample paths with jumps. There is an enormous literature studying stochastic impulse control models and many of these models have found a wide range of applications in electrical engineering, mechanical engineering, quantum engineering, robotics, image processing, and mathematical finance. Some classical references are [45], [32], [11]. A key operator in stochastic impulse control problems is the intervention operator,

$$\mathrm{M}u(x) = \inf_{\xi \geq 0}(u(x+\xi)+1). \tag{1.1}$$

The operator represents the value of the control policy that consists of taking the best immediate action in state x and behaving optimally afterwards. Since it is not always the case that the optimal control requires intervention at $t = 0$, this leads to the quasi-variational inequality,

$$u(x) \leq \mathrm{M}u(x) \quad \forall x \in \mathbb{R}^n. \tag{1.2}$$

From the analytic perspective one obtains an obstacle problem where the obstacle depends implicitly and nonlocally on the solution. More precisely we can consider the classical stochastic impulse control problem as a boundary value problem,

$$\begin{cases} Lu \leq f(x) & \forall x \in \Omega. \\ u(x) \leq \mathrm{M}u(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{1.3}$$

Here we let, $Lu \equiv -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$ with suitable regularity assumptions on the data and,

$$\mathrm{M}u(x) = 1 + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \bar{\Omega}}} (u(x+\xi)). \tag{1.4}$$

In this chapter, we present a new proof for the sharp $C^{1,1}_{loc}(\Omega)$ estimate for the solution to (1.3). We point out that the sharp $C^{1,1}_{loc}$ estimate has been previously obtained (see [13], [14]). As a corollary of our proof we also obtain a direct proof of the fact that the nonlocal obstacle, $\mathrm{M}u(x)$ is $C^{1,1}_{loc}$ on the contact set $\{u = \mathrm{M}u\}$. Since the obstacle depends on the solution, the strategy is to improve the regulariity of the solution and use it to improve the regularity of the obstacle. We start by first proving continuity of the solution and then proceeding to prove a semiconcavity estimate for the obstacle. In the following section we use the semiconcavity of the obstacle and the superhamonicity of the solution to produce the $C^{1,1}$ estimate.

In the last section we study regularity estimates for the free boundary $\partial\{u < \mathrm{M}u\}$. We first observe that the set of free boundary points can be structured according to where $\inf u(x+\xi)$ is realized. If the infimum is realized in the interor of the positive cone then we conclude that the obstacle is locally constant in a neighborhood of a free boundary point. This gives us regularity estimates of the free boundary at regular points and singular points as defined in the classical obstacle problem. If the infimum is realized on the boundary of the cone then under the assumption that f is analytic we conclude that the free boundary is contained in a finite collection of C^∞ submanifolds. In particular we prove the following theorem,

Theorem 1. *Consider the classical stochastic impulse control problem*

$$\begin{cases} Lu \leq f(x) & \forall x \in \Omega. \\ u(x) \leq \mathrm{M}u(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{1.5}$$

Assume that all coefficients in L are analytic, f is analytic and $f(x) \leq f(x+\xi)\ \ \forall \xi \geq 0$. Then it follows that, $\partial\{u < \mathrm{M}u\} = \Gamma^r(u) \cup \Gamma^s(u) \cup \Gamma^d(u)$ where,

1. $\forall x_0 \in \Gamma^r(u)$ there exists some appropriate system of coordinates in which the coincidence set $\{u = \mathrm{M}u\}$ is a subgraph $\{x_n \leq g(x_1, \ldots, x_{n-1})\}$ in a neighborhood of x_0 and the function g is analytic.

2. $\forall x_0 \in \Gamma^s(u)$*,* x_0 *is either isolated or locally contained in a* C^1 *submanifold.*

3. $\Gamma^d(u) \subset \Sigma(u)$ *where* $\Sigma(u)$ *is a finite collection of* C^∞ *submanifolds.*

In the fourth chapter we consider a fully nonlinear problem. We consider $F(D^2u)$, a fully nonlinear uniformly elliptic operator. We assume that the operator is either convex or concave in the hessian variable. We define $\varphi_u(x)$ to be a semiconvex function with a general modulus of semiconvexity $\omega(r)$. We consider the following boundary value problem.

$$\begin{cases} F(D^2u) \leq 0 & \forall x \in \Omega. \\ u(x) \geq \varphi_u(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{1.6}$$

The following are our main results in this chapter,

Theorem 2. *Consider the fully nonlinear obstacle problem with obstacle* φ_u*, admitting a modulus of semiconvexity,* $\omega(r)$*. Then the solution* u *has a modulus of continuity* $\omega(r)$ *up to* $C^{1,1}(\Omega)$*.*

As an application we apply our result to obtain a sharp estimate for the solution to the following fully nonlinear stochastic impulse control problems,

Theorem 3. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with a* $C^{2,\alpha}$ *boundary* $\partial\Omega$*. Define*

$$\mathrm{M}u(x) = \varphi(x) + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \bar{\Omega}}} (u(x+\xi)). \tag{1.7}$$

Here $\varphi(x)$ is $\omega(r)$ semiconcave, strictly positive, bounded, and decreasing in the positive cone $\xi \geq 0$. Consider the solution to the following fully nonlinear stochastic impulse control problem,

$$\begin{cases} F(D^2u) \geq 0 & \forall x \in \Omega. \\ u(x) \leq \mathrm{M}u(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{1.8}$$

Then, the solution u has modulus of continuity $\omega(r)$ up to $C^{1,1}(\Omega)$.

We remark that as a corollary of this theorem we recover the sharp $C^{1,1}$ estimate for the classical stochastic impulse control problem.

We proceed in stages to prove the stated theorems. The main point of interest in the first theorem is to improve the modulus of continuity for the obstacle φ_u on the contact set $\{u = \varphi_u\}$. In particular the goal is to obtain a uniform modulus of continuity $\omega(r)$ for φ_u which we can then propogate to the solution u. The second theorem follows from the first theorem once we apply the semiconcavity estimates for the nonlocal obstacle $\mathrm{M}u(x)$ obtained in the previous chapter. Moreover we can extend the free boundary regularity from the classical implicit constraint problem under the assumption that the data is analytic and $\varphi(x) = 1$. Finally as an application of the previous results we consider a singularly perturbed fully nonlinear obstacle problem and show optimal decay rates for Hölder norm estimates.

In the fifth chapter we are interested in investigating estimates for the solution to the penalized boundary obstacle problem. Our interest in this problem is understanding both sharp uniform and nonuniform estimates in the penalizing parameter. We begin by first considering the nonuniform theory and characterize the optimal growth of solutions from the free boundary. Regarding uniform estimates, as in the theory for the boundary obstacle problem, by standard regularity theory it is enough to prove uniform estimates at the level of u_y^ϵ. A consequnce of our estimates is that we can prove uniform convergence of the penalized solution to the solution of the boundary obstacle problem. This follows from the sharp uniform estimates and the following observation. Assume for a uniform constant C,

$$|\frac{1}{\epsilon}(u^\epsilon)_-| = |u_y^\epsilon| \leq C.$$

This implies,

$$|(u^\epsilon)_-| \leq \epsilon C.$$

Letting $\epsilon \to 0$ we conclude that,

$$(u^0)_- \equiv 0.$$

Furthermore the uniform estimate from below on u_y^ϵ allows us to conclude that u_y^0 does not deteriorate on $\{u^0 = 0\}$. Hence we recover the solution to the boundary obstacle problem with zero obstacle. Since the sharp estimate for the limiting solution is known (see [4]), we aim to show that u_y^ϵ is uniformly $C^{1/2}$. The following is our main result,

Theorem 4. *Let u^ϵ be a solution to the penalized boundary obstacle problem. Then there exists a modulus of continuity $\omega : (0, \infty) \to (0, \infty)$ independent of ϵ, such that $\omega(\delta) = O(\delta^{1/2})$ as $\delta \to 0$ and $\forall x, y \in B_{r/2}$ and $\forall \epsilon > 0$,*

$$|u_y^\epsilon(x) - u_y^\epsilon(y)| \leq |x - y|^{1/2}. \tag{1.9}$$

We proceed in stages to prove the uniform estimates. We make the standing assumption that $u^\epsilon(0) = 0$, so in particular $u_y^\epsilon(0) = 0$. The idea is to first prove the semiconvexity of the solution in the tangential directions. An iteration argument will allow us to conclude a Hölder growth estimate for u_y^ϵ from the interface $\partial\{u^\epsilon > 0\}$. To obtain the sharp estimate, we study global solutions of the penalized problem. Global solutions are convex hence we are able to employ a monotonicty formula first proved in [4] to improve the growth estimate from the interface obtained in the preceding section. A scaling argument in the penalization parameter concludes the proof of the desired universal Hölder estimate. For the local problem we utilize a technical estimate to correct for semiconvexity and then an iterative application of the monotonicity formula improves the growth estimate of u_y^ϵ from the interface. To conclude the universal Hölder norm estimate we apply again the scaling arguments in the penalization parameter as considered for the global solutions.

Chapter 2

The Obstacle Problem

"The Procrustean tendency to force the physical situation to fit the requirements of a partial differential equation results in a field which is both more regular and more irregular then the 'actual' conditions. A solution of a differential equation is more smoothly continuous over most of space and time than is the corresponding physical situation, but it usually is also provided with a finite number of mathematical discontinuities which are considerably more 'sharp' than the actual condition exhibits."

— P. Morse, H. Feshbach,
Methods of Theoretical Physics

In this chapter we recall some of the key results in the theory of obstacle problems. We consider the classical obstacle problem, the thin obstacle problem and some singular perturbations.

2.1 The Classical Obstacle Problem

Consider a fixed horizontal wire with an attached membrane which is forced to lie above a fixed obstacle in the domain. The resulting geometry gives us a contact area between the membrane and the obstacle, a non contact region where the membrane is strictly above the obstacle and the boundary separating these regions. The goal is to understand the qualitative properties of the membrane and the geometric behavior of the boundary. As formulated, this is the classical obstacle problem. More precisely assume the membrane is the graph of a function

$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$$

where Ω is a smooth bounded domain and $u = g$ on the boundary $\partial\Omega$, representing the wire. Moreover let ψ represent the graph of the obstacle in the domain. Such a problem can be posed in the theory of variational inequlities and leads to an existence and uniquness theory for the problem (see Appendix C). For convenience we consider the problem with $w = u - \psi$. This allows one to utilize geometric methods such as scaling and blow-ups. Thus we define the coincidence set to be $\Lambda = \{w = 0\}$, the noncoincidence set to be $N = \{w > 0\}$, and the free boundary, $\Gamma = \partial\{u > 0\}$. This leads to some nice regularity properties which were first proven by J. Frehse,

Theorem 5. *(Frehse [31], Regularity of the Mnimizer) The solution to the obstacle problem satisfies,*

$$\begin{cases} \Delta w = 1 & \text{on } \{w > 0\} \cap \Omega \\ w \geq 0 & \text{on } \Omega \\ w \in C^{1,1}(\overline{\Omega}). \end{cases}$$

We remark that a solution w satisfying the above is considered to be a normalized solution. We also remark that on $\{w > 0\}$ one actually has on the right hand side,

$$\Delta w = \Delta(u - \psi) = -\Delta\psi.$$

What is needed to be a normalized solution is that $\Delta\psi \leq -C < 0$ and $\Delta\psi$ is dini continuous (see [6]).

With the above structural theorem, one is now in position to study the geometric properties of the free boundary. We point out that since $0 \in \Gamma$, $\|w\|_{C^{1,1}}$ is preserved under the family of dilations $w_\lambda = \frac{1}{\lambda^2} w(\lambda x)$. Hence one has enough compactness to consider the blow-up solution, i.e. $w_{\lambda_k} \to w_\infty$ as $\lambda_k \to \infty$. It clearly follows that w_∞ will be non-negative, $C^{1,1}$ and satisfy $\Delta w_\infty = 1$ on the set $w_\infty > 0$. What is not clear is that w_∞ does not identically vanish. In other words can one prove compactness of $\Gamma_k = \partial\{w_{\lambda_k} > 0\}$? Can one say that free boundaries converge to free boundaries? This is answered positively by the following non-degeneracy statement,

Lemma 1. *(Non-Degeneracy) If* $x_0 \in \overline{N}$*, then*

$$\sup_{B_r(x_0)} w \geq \frac{1}{2n} r^2.$$

The key point in this lemma is the strict concavity of the obstacle in a neighborhood of the free boundary point. The proof follows a straightforward application of the maximum principle. In particular we conclude that

$$\sup_{B_r(x_0)} w_\infty \geq \frac{1}{2n} r^2$$

and hence $0 \in \Gamma_\infty = \partial\{w_\infty > 0\}$.

The next step is to classify blow-up solutions. This is obtained in the following theorem first proven by Luis Caffarelli,

Theorem 6. *(Caffarelli [15], [20], Characterization of the Blow-up Limit) The blow-up limit is unique and depends only on the point x_0 on the free boundary. Either there exists a unit vector $\nu_{x_0} \in \mathbb{S}^{n-1}$ such that,*

$$w_\infty(x_0) = \frac{1}{2}\max(\langle x, \nu_{x_0}\rangle, 0)^2,$$

*and the point x_0 is called a **Regular Point**. Or w_∞ is a quadratic form. In particular,*

$$w_\infty(x_0) = \frac{1}{2}\langle x, A_{x_0} x\rangle$$

*where A_{x_0} is a symmetrtic $n \times n$ matrix such that $\operatorname{Tr} A_{x_0} = 1$ and x_0 is called a **Singular Point**.*

We remark that the theorem gives us a Liouville result which classifies the blowups as well as the uniqueness of the blowups. The first step to prove the classification theorem is to show that w_∞ is convex. More precisely,

Lemma 2. *There is a modulus of continuity $\omega(r)$ where $\omega(r)$ monotone, $\omega(0^+) = 0$, such that $D_{ii}w(x) \geq -\omega(|x|)$.*

The proof is a direct application of the Harnack Inequality and an iteration argument. It follows that the coincidence set $\{w_\infty = 0\}$ is a convex set. The next step is analogous to the minimal surface theory where we would like to rule out that Γ is not asymptotically a cone. Hence we perform another blow-up in a neighborhood of 0, and consider

$$(w_\infty)_\infty = \lim_{\lambda_k \to 0} (w_\infty)_{\lambda_k}.$$

It follows that $\{(w_\infty)_\infty = 0\}$ is either a point or a convex cone. If it is a point or a convex cone with empty interior then it follows that

$$w_\infty(x_0) = \frac{1}{2}\langle x, A_{x_0} x\rangle,$$

where A_{x_0} is a symmetrtic $n \times n$ matrix such that $\operatorname{Tr} A_{x_0} = 1$. On the other hand if $\{(w_\infty)_\infty = 0\}$ has nonempty interior, then,

$$w_\infty(x_0) = \frac{1}{2}\max(\langle x, \nu_{x_0}\rangle, 0)^2,$$

and $\partial\{w_\infty > 0\}$ is a Lipschitz surface. The proof follows from showing that ∇w_∞ locally around 0, is strictly monotone in a cone of directions centered around δ, a vector directed towards the interior of the contact set. Moreover it follows after a straightening of the boundary and an application of the Boundary Harnack Principle that in a neighborhood of 0 the level surfaces of w_∞ are in fact $C^{1,\alpha}$. We remark that it follows that points on the free boundary in the local picture which blowup to this profile are then unique.

What remains to be done is to carry the analysis in the global picture back to the local solution w. In 2−dimensions Schaeffer [58] showed that the free boundary can form a thin neck or a cusp point. The intuition behind these examples comes from considering continuous deformations with varying obstacles with different components. Such a construcion follows from the fact that a priori the coincidence set could be composed of an infinite number of components with accumulation points. Hence it is important to understand what assumptions in the local picture will guarantee convergence in the blow-up to a convex coincidence set with a non-empty interior. Here we introduce a measurement which gives us a dichotomoy for points on the free boundary of w,

Definition 1. *We define the thickness of the conincidence set* $\{w = 0\}$*, in a ball* $B_r(x_0)$ *by*

$$\delta_r(x_0) = \frac{1}{r} m.d.(\{w = 0\} \cap B_r(x_0)\},$$

where the minimum diameter (m.d.) of $\{w = 0\} \cap B_r(x_0)$ *is the infimum of distances between pairs of parallel hyperplanes which contains* $\{w = 0\} \cap B_r(x_0)$*.*

We now state a theorem regarding convergence of free boundary points in the local picture to the global picture. This allows us to define the **Regular Points** and **Singular Points** of the free boundary. The following theorem was first proven by Luis Caffarelli,

Theorem 7. *(Caffarelli [15]) Let* $0 \in \Gamma$*. There exists a modulus of continuity* $\omega(\rho)$ *such that for* $0 \in \Gamma$ *either,*
(a) 0 *is a* ***Singular Point*** *and* $m.d.(\{w = 0\} \cap B_\rho(x_0)\} \leq \rho\omega(\rho) \;\; \forall \rho \leq 1$ *or*
(b) 0 *is a* ***Regular Point*** *and* $\exists \rho_0$ *such that* $m.d.(\{w = 0\} \cap B_{\rho_0}(x_0)\} \geq \rho_0\omega(\rho_0)$ *and* $\forall \rho < \rho_0$*,* $m.d.(\{w = 0\} \cap B_\rho(x_0)\} \geq c\rho\omega(\rho_0)$*.*

The proof follows from compactness and properties for normalized solutions. Moreover it follows from a density estimate that the set of points satisfying (b) is an open set of the free boundary and at those points satisfying (a) the free boundary becomes cusp like. Now it remains to propogate the regularity of the free boundary in the global picture back to the local picture at regular points. This follows from the following theorem,

Theorem 8. *(Caffarelli [20]) Suppose w is a normalized solution. Then there exists a modulus of continuity $\omega(r)$ such that if for one value of r, say r_0, m.d.($\{w = 0\} \cap B_{r_0}(x_0)\} > r_0\omega(r_0)$, then in a r_0^2 neighborhood of the origin, the free boundary is a $C^{1,\alpha}$ surface $x_n = f(x')$ with*

$$\|f\|_{C^{1,\alpha}} \leq \frac{C(n)}{r_0}.$$

Subsequently one also has higher regularity of the free boundary at regular points,

Theorem 9. *(Kinderlehrer-Nirenberg [43], Isakov [35]) Let w is a normalized solution, $0 \in \Gamma$, and $\Delta w = f$. Then $f \in C^{m,\alpha}(B_1)$ implies Γ is a hypersurface of class $C^{m+1,\alpha}$ in some neighborhood of 0. Furthermore if f is analytic then near the origin Γ is analytic.*

Finally we can also consider the **Singular Points** on the free boundary. Recall that a **Singular Point** are those points for which $|(\{w = 0\} \cap B_r(x_0)\}| \subset S_{r\omega(r)}$ (a strip of width $r\omega(r)$) for every positive r. The unqiueness of the blow-up is a consequence of the Alt-Caffarelli-Friedman monotonicity formula [20] or a monotonicity formula due to Monneau [47]. It follows that one can control the convergence rate in a uniform fashion. In particular it is shown that the normalized solution has a Taylor expansion of order 2 at singular points,

Theorem 10. *(Caffarelli [20], Monneau [47]) If w is a normalized solution and 0 is a **Singular Point**, then*

$$w(x) = \frac{1}{2}\langle x, D^2 w(0) x\rangle + o(x^2).$$

It follows from an application of the ACF monotonicity formula or an application of the Whitney Extension Lemma combined with the implicit function theorem that one can prove a structural theorem for points on the singular set,

Theorem 11. *(Caffarelli [20], Monneau [47]) The map*

$$x_0 \to A_{x_0}$$

is continuous on the set of singular points on the free boundary. Moreover the set of singular points is a closed set in a C^1 $(n-1)$-dimensional manifold.

2.2 The Lower Dimensional Obstacle Problem

In this section we give a brief overview of a variant of the classical obstacle problem, where the obstacle is restricted to lie on a lower dimensional manifold $\mathcal{M}$. Such problems arise in the context of flow through semi-permeable membranes, elasticity, boundary control temperature, or heat conduction problems (see [29]). Let Ω be a domain in $\mathbb{R}^{n+1}$ divided into two parts

Ω^+ and Ω^- by $\mathcal{M}$. Let $\varphi : \mathcal{M} \to \mathbb{R}$ be a thin obstacle and g a given function on $\partial\Omega$ satisfying $g > \varphi$ on $\mathcal{M} \cap \partial\Omega$. The problem can be given a formulation in the theory of variational inequalities and is shown to satisfy,

$$\begin{cases} \Delta u \leq 0 & \Omega, \\ \Delta u = 0 & \Omega \setminus \mathcal{M}, \\ u \geq \varphi & \mathcal{M}, \\ u_{\nu^+} + u_{\nu^-} \leq 0 & \mathcal{M}, \\ (u-\varphi)(u_{\nu^+} + u_{\nu^-}) = 0 & \mathcal{M}. \end{cases}$$

Moreover if $\mathcal{M} = \mathbb{R}^n \times \{0\}$ then the problem is equivalent to,

$$\begin{cases} \Delta u \leq 0 & \Omega, \\ \Delta u = 0 & \Omega \setminus \mathcal{M}, \\ u \geq \varphi & \mathcal{M}, \\ u_{\nu^+} \leq 0 & \mathcal{M}, \\ (u-\varphi)u_{\nu^+} = 0 & \mathcal{M}. \end{cases}$$

We now consider the specific case when $\Omega = B_1$, the unit ball in $\mathbb{R}^{n+1}$. Furthermore we define $B_1' = B_1 \cap \{x_{n+1} = 0\}$. We fix $u = 0$ on $\partial B_1 \cap \{x_{n+1} > 0\}$ and $\varphi < 0$ on $\partial B_1'$. We remark that regularity estimates for the local problem in $\Omega = B_1^+$ can be deduced from regularity estimates in the global problem $\Omega = \mathbb{R}^n \times (0, \infty)$ by using radially symmetric cutoff functions. Also the converse is true.

We point out that the lower dimensional problem is strongly related to problems in fractional diffusion. To better understand this relationship we study the harmonic extension problem. Given u_0 a rapidly decaying function, let u be the unique solution to the Dirichlet problem,

$$\begin{cases} \Delta u = 0 & \mathbb{R}^n \times (0, \infty), \\ u(x,0) = u_0 & \mathbb{R}^n. \end{cases}$$

Consider the Dirichlet to Neumann map $T : u_0 \to -u_y(x,0)$. Since u_0 is smooth and u_y is harmonic we find,

$$T \circ T[u_0] = -\partial_y(-\partial_y u_y(x,0)) = u_{yy}(x,0) = -\Delta u_0.$$

Hence it follows that

$$T = (-\Delta)^{1/2},$$

where solutions to $(-\Delta)^{1/2}$ arise as minimizers of the functional

$$J(v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}}\, dxdy.$$

From this we are able to draw the following conclusions,

a. If u is a solution to the lower dimensional obstacle problem in $\mathbb{R}^n \times (0, \infty)$, then $u_0 = u(\cdot, 0)$ solves the obstace problem for $(-\Delta)^{1/2}$.
b. If we start with a solution u_0 to the obstacle problem for $(-\Delta)^{1/2}$ then its harmonic extension u solves the lower dimensional obstacle problem.

We consider again the lower dimensional obstacle problem in the local geometry. From the 2-dimensional problem it is strongly suggested that the solution has a saddle shape, with convex behavior in the tangential directions and concavity in the normal direction. The main questions become as in the classical obstacle problem, to study the regularity estimates for the solution u as well as the $(n-1)$-dimensional free boundary $\Gamma = \partial\{u(\cdot,0) > \varphi\}$. When studying the optimal regularity estimates for the solution, the best one hopes to obtain is $C^{1,1/2}$ on each side of $\mathcal{M}$. This has been obtained in [4], [25]. Moreover in recent work [10] the complete classification of free boundary points is achieved for concave obstacles as discussed in the previous section on the classical obstacle problem. The new idea is to prove a non-degeneracy statement at all points on the free boundary and show that solutions grow at least quadratically at every free boundary point. This work unifies many of the previous results related to the regularity of the free boundary (see [25], [55] [49]). To conclude this section we outline the major steps to show the optimal regularity estimate for the solution and the free boundary. We remark that many of these steps are closely followed and inspired by work in the classical obstacle problem.

a. Lipschitz Continuity and local $C^{1,\alpha}$ estimates of the solution for some $0 < \alpha < 1$ [12].

b. Optimal $C^{1,1/2}$ regularity for tangentially convex global solutions using a monotonicity formula [4].

c. Correction for semiconvexity and optimal $C^{1,1/2}$ regularity for local solutions [4].

d. Almgren's Frequency formula and another derivation of the optimal $C^{1,1/2}$ estimate [25].

e. Classification of asymptotic blow-up profiles around a free boundary point according to homogeneity of the blow-up [5].

f. Lipschitz Regularity of Free Boundary in a neighborhood of stable points (homogeneity of blow-up is $\frac{3}{2}$) [5] [25].

g. Boundary Harnack Principle and $C^{1,\alpha}$ regularity at stable points of the Free Boundary [5], [25].

h. Structure of the set of points on the Free Boundary with vanishing density. (Singular Set) [49].

Before closing this section we remark that all of the previous results have been generalized to the fractional obstacle problem where one studies the obstacle problem for $(-\Delta)^s$ for $0 < s < 1$. In fact many of the above results are derived in this more general framework. As before one has an extension formula analgous to the case $s = \frac{1}{2}$ (see [23]) and corresponding statements about optimal regularity of the solution and free boundary. For a complete reference to the fractional obstacle problem see [56].

2.3 Singularly Perturbed Free Boundary Problems

Problems in differential equations are often approximated by regularizing ones. To obtain information about the original problem, one tries to establish results for the regularizing solution which carry over in the limit. In the context of obstacle problems a classical technique in this area is the penalization method. The underlying idea to study perturbed solutions is that small perturbations for uniformly elliptic equations propogate in a quantifiable fashion (Harnack Inequalities and Maximum Principles), hence studying perturbed solutions can help to establish regularity estimates and stability estimates for the original solution. In free boundary problems the question to ask is how would a perturbation of order ϵ displace the free boundary? The problem can be formulated as a study of a one-parameter family of solutions of operators that degenerate along a level surface $\{u = 0\}$. In particular we ask the question, what are the regularity properties of the solution and level surfaces of the solution to an equation of the type,

$$\Delta u^\epsilon = \beta_\epsilon(u^\epsilon)?$$

Specifically we would like to consider those properties of the solution that are independent of the parameter ϵ, hence those properties which hold for the limiting solution u_0. Restricting our attention to variational solutions we consider an ϵ smoothing of minimizers with the following Euler-Lagrange Equation,

$$\Delta u_0 = \alpha[(u_0)^+]^{\alpha-1} \quad 0 < \alpha < 2.$$

We note that equations of this type are invariant under the rescaling

$$w_0(x) = \frac{1}{\lambda^{\frac{2}{2-\alpha}}} u_0(\lambda x).$$

It follows that the natural regularity for this problem is $C^{\frac{2}{2-\alpha}}$. We remark that in the case

$$\alpha \to 2$$

we formally recover the obstacle problem and in the case

$$\alpha \to 0$$

we formally recover the Alt-Caffarelli problem [2]. The intermediate cases were studied by Alt and Phillips [3]. Much work has also been done on the penalized problem (see [53] [54]). In the penalization problem one is interested in studying uniform estimates for the solution u^ϵ in the ϵ-strip, namely, measure theoretic estimates of $0 < u^\epsilon < \epsilon$, and its uniform speed of convergence to the limiting problem. We remark that there do exist some general technique to establish such uniform estimates [17]:

a. Establishing optimal regularity estimates and non-degeneracy estimates for the penalized solution. Such estimates are in principle constrained by the homogeneity of the limiting solution.
b. Measure estimates for the ϵ-strip. These should in the limit recover the Hausdorff measure estimates for the free boundary.
c. Lipschitz level surfaces are uniformly $C^{1,\alpha}$. One has to obtain a Haranck Inequality for level surfaces.
d. Flatness implies Lipschitz. In the case of penalizations for obstacle type problems we would like to establish that blow-ups for the penalization problem are close to the blow-ups of regular points in the limiting obstacle problem.

We take up singularly perturbed problems in chapter 5, when we consider the penalized boundary obstacle problem.

Chapter 3

The Classical Implicit Constraint Obstacle Problem

3.1 Basic Definitions and Assumptions

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^{2,\alpha}$ boundary $\partial\Omega$. Assume $c(x) \geq c_0 > 0$; a_{ij}, b_i, c, $\in C^{2+\alpha}(\bar{\Omega})$ for $0 < \alpha < 1$, and the matrix (a_{ij}) is positive definite for all $x \in \bar{\Omega}$. Furthermore let $f \in C^{\alpha}(\bar{\Omega})$. For any $\xi = (\xi_1, \ldots, \xi_n)$ we let $\xi \geq 0$ denote $\xi_i \geq 0 \ \forall i$. Consider,

$$Lu \equiv -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u. \tag{3.1}$$

Define the operator:

$$\mathrm{M}u(x) = 1 + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x+\xi). \tag{3.2}$$

We introduce the bilinear form $a(u,v)$ associated to our operator L,

$$a(u,v) = (Lu, v) \ \ \forall u, v \in C_0^{\infty}(\Omega). \tag{3.3}$$

Furthermore assume that our bilinear form is coercive,

$$a(u,u) \geq \gamma(\|u\|_{W^{1,2}(\Omega)})^2 \ \ \forall u \in W_0^{1,2}(\Omega), \ \ \gamma > 0. \tag{3.4}$$

We consider the quasi-variational inequality:

$$u \in W_0^{1,2}(\Omega) \;\; u \leq \mathrm{M}u,$$

$$a(u, v-u) \geq (f, v-u) \;\; \forall v \in W_0^{1,2}(\Omega) \;\; v \leq \mathrm{M}u. \tag{3.5}$$

We list a few properties of our operator $\mathrm{M}u$ that will be useful for the remaining parts of this chapter,

$$u_1(x) \leq u_2 \; a.e. \Rightarrow \mathrm{M}u_1(x) \leq \mathrm{M}u_2(x) \; a.e.$$

$$\mathrm{M} : L^\infty \to L^\infty.$$

$$\mathrm{M} : C(\bar{\Omega}) \to C(\bar{\Omega}).$$

Furthermore we assume that $f \geq -\frac{1}{c_0}$. This implies that the solution $\bar{u}$ to the variational equation $L\bar{u} = f$ in Ω, $\bar{u} \in H_0^1(\Omega)$ satisfies the property $\bar{u} \geq -1$. This in particular implies that the set of solutions to $v \in H_0^1(\Omega) \;\; v \leq \mathrm{M}\bar{u}$ is nonempty. Without loss of generality we assume that $\bar{u} < 1$.

3.2 Existence and Uniqueness Theory

We now proceed to prove the existence of a unique continuous solution to (3.5). We follow closely the proof in [38].

Lemma 3. *There exists a unique solution $u \in C(\Omega)$ of (3.5).*

Proof. From standard elliptic theory we know that there exists a unique solution $u_0 \in C(\Omega)$ of

$$\begin{cases} a(u,v) = (f, u-v) & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.6}$$

Since $\mathrm{M}u_0$ is continuous we know from the theory of variational inequalities that there exists a unique solution $u_1 \in C(\Omega)$ of

$$\begin{cases} a(u,v) \geq (f, u-v) & \forall x \in \Omega, \\ u \leq \mathrm{M}u_0 & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.7}$$

Moreover for $n = 2, 3, \ldots$ we obtain $u_n \in C(\Omega)$ satisfying,

$$\begin{cases} a(u,v) \geq (f, u-v) & \forall x \in \Omega, \\ u \leq \mathrm{M}u_{n-1} & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.8}$$

Since u_1 is a subsolution of (3.6), by the comparison principle (see Appendix C), we know that $u_1 \leq u_0$. We also know that -1 is a subsolution of (3.77), hence the comparison implies that $-1 \leq u_1$. Moreover it follows from the properties of $\mathrm{M}u$ that $0 \leq \mathrm{M}u_1 \leq \mathrm{M}u_0$. This implies in particular that u_2 is an admissable subsolution to (3.7). Arguing as before we see that $-1 \leq u_2 \leq u_1$. We can continue this process and obtain a sequence of functions

$$-1 \leq \ldots \leq u_n \leq \ldots \leq u_1 \leq u_0. \tag{3.9}$$

Now we look to prove an upper bound on the sequence. Consider $\mu \in (0,1)$ such that $\mu \|u_0\|_{C(\Omega)} \leq 1$. Assume there exists $\theta_n \in (0,1]$ such that $\forall n \in \mathbb{N}$,

$$u_n - u_{n+1} \leq \theta_n u_n. \tag{3.10}$$

We claim that this implies

$$u_{n+1} - u_{n+2} \leq \theta_n(1-\mu)u_{n+1}. \tag{3.11}$$

With this claim we are able to almost conclude the proof of the theorem. In particular the positivity of u_n implies that $u_1 - u_2 \leq u_2$. We can set $\theta_1 = 1$. Moreover from (3.11) it follows that $u_2 - u_3 \leq (1-\mu)u_2$. Hence $\theta_2 = (1-\mu)$. Therefore setting $\theta_n = (1-\mu)^{n-1}$ we find

$$u_{n+1} - u_{n+2} \leq (1-\mu)^n u_{n+1} \leq (1-\mu)^n \|u_0\|_{C(\Omega)}. \tag{3.12}$$

Combining (3.12) with (3.9) we see that there exists a function $u \in C(\Omega)$ such that $\|u_n - u\|_{C(\Omega)} \to 0$ as $n \to \infty$. Moreover from the estimate $\|\mathrm{M}u - \mathrm{M}v\|_{C(\Omega)} \leq \|u - v\|_{C(\Omega)}$ it follows that u is a solution to the classical stochastic impulse control problem. Hence we are reduced to proving (3.11) and establishing uniqueness of the solution. By the concavity of $\mathrm{M}u$ and (3.10) it follows,

$$\psi = (1-\theta_n)\mathrm{M}u_n + \theta_n \leq (1-\theta_n)\mathrm{M}u_n + \theta_n \mathrm{M}0 \leq \mathrm{M}(1-\theta_n u_n) \leq \mathrm{M}u_{n+1}. \tag{*}$$

We consider the continuous solutions to the following obstacle problems. Let $w \in C(\Omega)$ solve,

$$\begin{cases} a(u,v) \geq (f, u-v) & \forall x \in \Omega. \\ u \leq \psi & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.13}$$

Let $z \in C(\Omega)$ solve,

$$\begin{cases} a(u,v) \geq (f, u-v) & \forall x \in \Omega. \\ u \leq 1 & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.14}$$

From (*) and the comparision theorem for variational inequalities it follows that $w \leq u_{n+2}$. Moreover it follows that $\theta_n z$ solves,

$$\begin{cases} a(u,v) \geq (f, u-v) & \forall x \in \Omega. \\ u \leq \theta_n & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.15}$$

Observing that $\psi \geq \theta_n$, it follows from comparision that $\theta_n w \geq \theta_n z$. Next we observe that $(1-\theta_n)u_{n+1}$ is a subsolution and $(1-\theta_n)w$ is a solution of the following obstacle problem,

$$\begin{cases} a(u,v) \geq (f, u-v) & \forall x \in \Omega. \\ u \leq (1-\theta_n)\psi & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.16}$$

Hence we find, $(1-\theta_n)u_{n+1} \leq (1-\theta_n)w$. Putting this together we obtain,

$$(1-\theta_n)u_{n+1} + \theta_n z \leq (1-\theta_n)w + \theta_n w = w \leq u_{n+2}. \tag{**}$$

Recall that $\forall n$, $\mu u_{n+1} \leq 1$. This implies that μu_{n+1} is a subsolution of (3.14). So in particular, $\mu u_{n+1} \leq z$. Putting this into (**) we obtain our desired estimate (3.11),

$$u_{n+1} - u_{n+2} \leq \theta_n(1-\mu)u_{n+1}.$$

Finally to prove uniqueness, suppose u and $\bar{u}$ are distinct solutions. The positivity of the solution implies $u - \bar{u} \leq u$. Hence arguing as above we find

$u - \bar{u} \le (1-\mu)^n u$, for all $n \ge 0$. Letting $n \to \infty$ we find that $u - \bar{u} \le 0$. Interchanging u and $\bar{u}$ we conclude $u = \bar{u}$. □

3.3 Localization of the Obstacle and Semiconcavity Estimates

Using the improved regularity on the solution u, we now proceed to prove that the obstacle $\mathrm{M}u(x)$ is semi-concave with semiconcave with a linear modulus, i.e. $\omega(r) = Cr^2$. The strategy of the proof will follow the ideas presented in [13]. For the present argument we consider a more general obstacle.

Lemma 4. *Let $\varphi(x)$ be $\omega(r)$ semi-concave, strictly positive, bounded, and decreasing in the positive cone $\xi \ge 0$. Then the Obstacle*

$$\mathrm{M}u(x) = \varphi(x) + \inf_{\substack{\xi \ge 0 \\ x+\xi \in \Omega}} u(x+\xi)$$

is semi-concave with modulus of semi-concavity $\omega(r)$.

Proof. We consider two distinct cases:

1. $x_0 \in \{u = \mathrm{M}u\}$.
2. $x_0 \in \{u < \mathrm{M}u\}$.

Case 1: Fix $x_0 \in \{u = \mathrm{M}u\}$.

The proof in this case is based on characterizing the set where the infimum of u occurs and establishing that this set is uniformly contained in the non-contact region $\{u < \mathrm{M}u\}$. This is the content of the following claims. We define the following sets:

1. $\Sigma_{\geq x_0} = \{x_0 + \xi \ : \ \xi \geq 0\}$.

2. $\Sigma_{x_0} = \{\varphi(x_0) + u(x_0 + \xi) = \mathrm{M}u(x_0)\}$.

The following claim characterizes Σ_{x_0} as the set of points where u realizes its infimum.

Claim 1. *For every $y \in (\Sigma_{\geq x_0} \setminus \Sigma_{x_0})$ and for every $x \in \Sigma_{x_0}$, $u(x) \leq u(y)$.*

Proof. Fix $\bar{x} \in \Sigma_{x_0}$. Suppose by contradiction that $\exists x_1 \in \Sigma_{\geq x_0} \setminus \Sigma_{x_0}$ such that $u(x_1) < u(\bar{x})$. This implies the following,

$$\begin{aligned} \varphi(x_0) + u(x_1) &< \varphi(x_0) + u(\bar{x}) \\ &= \mathrm{M}u(x_0) = \varphi(x_0) + \inf_{\substack{\xi \geq 0 \\ x_0 + \xi \in \Omega}} u(x_0 + \xi). \end{aligned}$$

In particular we obtain,

$$u(x_1) < \inf_{\substack{\xi \geq 0 \\ x_0 + \xi \in \Omega}} u(x_0 + \xi).$$

This is a contradiction. □

We now prove that pointwise the elements of Σ_{x_0} are contained in the non-contact region, $\{u < \mathrm{M}u\}$.

Claim 2. *Suppose the solution to the Boundary Value Problem $L\bar{u} = f$ satisfies*

$$\bar{u} < \inf_{\partial\Omega} \varphi.$$

Then $\forall x \in \Sigma_{x_0}$ *it follows that* $u(x) < \mathrm{M}u(x)$.

Moreover in a neighborhood N_1 *of* x *we have* $u \in C^{1,1}(N_1)$.

Proof. We observe that the first statement ensures that $\Sigma_{x_0} \cap (\partial\Omega) = \varnothing$. Suppose $x_0 \in \Omega^\circ$, $x \in \partial\Omega$ and $x_0 \leq x$. Then we observe,

$$\begin{aligned}\mathrm{M}u(x_0) &= u(x_0)\\ &\leq \bar{u}(x_0) < \inf_{\partial\Omega} \varphi \leq \varphi(x) + u(x) \leq \varphi(x_0) + u(x).\end{aligned}$$

The last inequality follows from the monotonicity of $\varphi(x)$ in the cone. Hence in particular $\Sigma_{x_0} \cap (\partial\Omega) = \varnothing$.

Suppose now by contradiction that $\exists x \in \Sigma_{x_0}$ such that $u(x) = \mathrm{M}u(x)$. Then we have the following,

$$\begin{aligned}u(x_0) &= \mathrm{M}u(x_0)\\ &= \varphi(x_0) + u(x)\\ &= \varphi(x_0) + \mathrm{M}u(x) \geq \varphi(x_0) + \mathrm{M}u(x_0) > \mathrm{M}u(x_0)\end{aligned}$$

The last inequality follows from the strict positivity of the function φ. We observe that the inequality contradicts the obstacle constraint $u(x_0) \leq \mathrm{M}u(x_0)$. Hence we have reached our desired contradiction.

Finally the last statement of the claim follows from the continuity of u. The continuity of the solution implies that $\{u < \mathrm{M}u\}$ is an open set and thus in a small neighborhood N_1 of x, u satisfies the equation, $Lu = f$. We can therefore apply interior regularity estimates for the operator L to conclude. □

We now strenghten the previous claim to obtain a uniform neighborhood of Σ_{x_0} that is strictly contained in the non-contact region.

Claim 3. $\exists \delta_0 > 0$ *such that* $d\{\{u = \mathrm{M}u\}, \Sigma_{x_0}\} > \delta_0$.

Proof. Suppose by contradiction $\exists\{\delta_k\} \searrow 0$ and $\{x_k\} \subset \Sigma_{x_0}$, such that

$$d(x_k, \{u = \mathrm{M}u\}) \leq \delta_k \ \ \forall k.$$

By definition, $x_k \in \Sigma_{x_0}$, implies

$$\varphi(x_0) + u(x_k) = \mathrm{M}u(x_0) \ \ \forall k.$$

By the continuity of $u(x)$ this implies in particular that $\varphi(x_0) + u(\bar{x}) = \mathrm{M}u(x_0)$ for some $\bar{x} \in \{u = \mathrm{M}u\}$. On the other hand, $\varphi(x_0) + u(\bar{x}) = \mathrm{M}u(x_0)$ implies $\bar{x} \in \Sigma_{x_0}$. Hence from the previous claim we obtain,

$$\mathrm{M}u(\bar{x}) = u(\bar{x}) < \mathrm{M}u(\bar{x}).$$

This is our desired contradiction. □

We now state and prove a claim which allows us to localize the obstacle in the neighborhood of a contact point.

Claim 4. *For every* $x, \bar{x} \in \Omega$, $\exists \delta > 0$, *such that if* $|x - x_0| < \delta$, *and* $d(\bar{x}, \Sigma_{x_0}) > \delta$, *then* $u(x) < \varphi(x) + u(\bar{x})$. *Moreover, if* $x \in \{u = \mathrm{M}u\}$, *then* $\bar{x} \notin \Sigma_x$.

Proof. Suppose by contradiction that there exists a sequence of points $\{x_k\}$ and $\{\bar{x}_{k'}\}$ satisfying:

1. $|x_k - x_0| = \delta_k$.
2. $d(\bar{x}_{k'}, \Sigma_{x_0}) > \delta_{k'} > 0$.
3. $\{\delta_k\} \searrow 0$ and $\{\delta_{k'}\} \searrow 0$.
4. $u(x_k) \geq \varphi(x_k) + u(\bar{x}_{k'}) \;\; \forall k$ and $\forall k'$.

We observe that from the previous claim $\exists k_0, k_0'$, such that $\forall k \geq k_0$ we have the following chain of inequalities,

$$\begin{aligned} \mathrm{M}u(x_0 + \delta_k) &\leq \mathrm{M}u(\bar{x}_{k_0'}) \\ &\leq \varphi(\bar{x}_{k_0'}) + u(\bar{x}_{k_0'}) \\ &\leq \varphi(x_0 + \delta_k) + u(\bar{x}_{k_0'}) \\ &\leq u(x_0 + \delta_k) \leq \mathrm{M}u(x_0 + \delta_k). \end{aligned}$$

Thus the above inequalities are all equalities. This implies $\forall k \geq k_0$,

$$\mathrm{M}u(x_0 + \delta_k) = \varphi(x_0 + \delta_k) + u(\bar{x}_{k_0'}).$$

Letting $k \to \infty$ we obtain,

$$\mathrm{M}u(x_0) = \varphi(x_0) + u(\bar{x}_{k_0'}).$$

Which implies in particular that $\bar{x}_{k_0'} \in \Sigma_{x_0}$. This is our desired contradiction. $\square$

From the last claim we can redefine the obstacle for $V_\delta = \{|x - x_0| < \delta\}$. In particular by taking δ sufficiently small $\exists N_2$ neighborhood of Σ_{x_0} such that,

$$\mathrm{M}u(x) = \varphi(x) + \inf_{\substack{\xi \geq 0 \\ x+\xi \in N_2}} u(x+\xi).$$

For an even smaller δ,

$$\mathrm{M}u(x) = \varphi(x) + \inf_{\substack{\xi \geq 0 \\ x_0+\xi \in N_3}} u(x+\xi).$$

Where N_3 is such that,

$$V_\delta + N_3 - x_0 \subseteq N_1.$$

Here N_1 is the neighborhood obtained in Claim 3. In particular for $x \in V_\delta$ and $\xi \in N_3 - x_0$, we can bound the second incremental quotients,

$$\delta^2 u = u(x+h+\xi) + u(x-h+\xi) - 2u(x+\xi) \leq c|h|^2.$$

Moreover we know that for some $x + \bar{\xi}$ in N_1, we have,

$$\inf_{\substack{\xi \geq 0 \\ x+\xi \in N_1}} u(x+\xi) = u(x+\bar{\xi}).$$

Now we consider the second incremental quotients of the obstacle $Mu(x)$. By the semiconcavity of φ we obtain,

$$\begin{aligned}
\delta^2 \mathrm{M}u(x) &\leq \omega(h) + u(x+\bar{\xi}+h) + u(x+\bar{\xi}-h) - 2u(x+\bar{\xi}) \\
&\leq \omega(h) + c|h|^2 \\
&\leq C\omega(h).
\end{aligned}$$

Thus, in a neighborhood of a contact point, $\mathrm{M}u(x)$ is semiconcave with semiconcavity modulus $\omega(h)$.

Case 2: Fix $x \in \{u < \mathrm{M}u\}$. We argue as before by considering the second incremental quotients of the obstacle, $\delta^2\mathrm{M}u(x)$. We observe that the infimum of u in the positive cone, $\xi \geq 0$, must always be realized at a non-contact point. Suppose $\exists\ x + \xi_1 \in \{u = \mathrm{M}u\}$ satisfing,

$$\inf_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x+\xi) = u(x+\xi_1).$$

Then from **Case 1** there exists $\xi_2 \in \Sigma_{x+\xi_1} \subset \{u < \mathrm{M}u\}$ such that,

$$\inf_{\substack{\xi \geq 0 \\ x+\xi_1+\xi \in \Omega}} u(x+\xi_1+\xi) = u(x+\xi_1+\xi_2).$$

Since $\xi_1 + \xi_2 \geq 0$, we have found a positive vector admissable to

$$\inf_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x+\xi).$$

Furthermore, $u(x+\xi_1+\xi_2) \leq u(x+\xi_1)$. Hence we conclude that for a fixed $x \in \{u < \mathrm{M}u\}$, and for some $x+\bar{\xi}$ in $\{u < \mathrm{M}u\}$,

$$\inf_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x+\xi) = u(x+\bar{\xi}).$$

Moreover from Claim 4 we know that $x+\bar{\xi}$ is a uniform positive distance away from the contact set $\{u = \mathrm{M}u\}$. Hence there exists a uniform neighborhood N_0 of points around $x+\bar{\xi}$ where $\{u < \mathrm{M}u\}$. In a smaller neighborhood N_1, $u \in C^{1,1}(N_1)$. In particular for $x+\xi \in N_1$, we can bound again the second incremental quotients,

$$u(x+h+\xi) + u(x-h+\xi) - 2u(x+\xi) \leq c|h|^2.$$

Using once more the semiconcavity estimate on $\varphi(x)$ and for some $x+\bar{\xi}$ in N_1 we find,

$$\begin{aligned}
\delta^2 \mathrm{M}u(x) &\leq \omega(h) + u(x+\bar{\xi}+h) + u(x+\bar{\xi}-h) - 2u(x+\bar{\xi}) \\
&\leq \omega(h) + c|h|^2 \\
&\leq C\omega(h).
\end{aligned}$$

Thus, in a neighborhood of a non-contact point, $\mathrm{M}u(x)$ is semi-concave with semi-concavity modulus $\omega(h)$. □

Remark 1. *We point out that the existence and uniqueness of a continuous viscosity solution follows from taking a sequence of solutions to a regularized obstacle problem and proving convergence in* $C(\overline{\Omega})$*. We refer the reader to [48] for remarks in this direction as well as showing equivalence between various notions of solutions to the obstacle problem.*

3.4 Optimal $C^{1,1}$ Estimates for the Solution

In the previous section we proved that the unique bounded solution to the classical stochastic impulse control problem is continuous and that our implicit constraint obstacle is locally semi-concave. We now consider the sharp

$C^{1,1}$ estimate for the solution to the classical stochastic impulse control problem. We restrict to the case $L = \Delta$ and set $f = 0$. All of the arguments can be suitably modified for general L and nonzero f. We consider,

$$\begin{cases} \Delta u(x) \geq 0 & \forall x \in \Omega, \\ u(x) \leq \varphi_u(x) & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.17}$$

Here we have set,

$$\varphi_u(x) = \varphi + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \bar{\Omega}}} (u(x+\xi)),$$

where φ has a linear modulus of semiconcavity. We point out again that the sharp $C^{1,1}_{loc}$ estimate in the classical stochastic impulse control problem has been previously obtained (see [13], [14]). Our intent is to provide an alternate proof that is generalizable to the fully nonlinear problem considered in the next chapter.

Recall that a function v is semiconcave with semiconcavity modulus $\omega(r)$ if a vector $p \in \mathbb{R}^n$ belongs to $D^+v(x)$ if and only if $v(y) - v(x) - \langle p, y - x\rangle \leq \omega(|x-y|)$. Fix $x_0 \in \{u = \varphi_u\}$. Define the linear part of the obstacle, $L_{x_0}(x) = \varphi_u(x_0) + \langle p, x - x_0\rangle$. We consider

$$w(x) = u(x) - L_{x_0}(x).$$

We observe that in $B_r(x)$, $w(x)$ has a modulus of semiconcavity $\omega(r) = Cr^2$, i.e. $w(x) \leq Cr^2$. We now state our main lemma.

Lemma 5. *There exists universal constants $K, C > 0$, such that $\forall x \in B_{r/4}(x_0)$,*

$$-K \leq \Delta w \leq C. \tag{3.18}$$

Before proving this lemma we make a few observations. Fix $\Phi \in C_0^\infty(B_{\frac{r}{2}}(x_0))$. We recall the following fact from the theory of distributions: If u is a negative distribution in X with $u(\Phi) \leq 0$ for all non-negative $\Phi \in C_0^\infty(X)$, then u is a negative measure. In particular we have,

$$0 \geq \int_{B_{\frac{r}{2}}} \Phi \, d\mu = \int_{B_{\frac{r}{2}}} \Delta u \, \Phi. \tag{3.19}$$

We consider $\forall \rho < \frac{r}{2}$,

$$\frac{\mu(B_\rho(x_0))}{|B_\rho(x_0)|} = \frac{1}{\alpha(n)\rho^n} \int_{B_\rho} d\mu = \frac{1}{\alpha(n)\rho^n} \int_{B_\rho} \Delta u. \tag{3.20}$$

A straightforward application of the Gauss-Green Formula gives to us the following identity,

$$\frac{1}{\alpha(n)\rho^n} \int_{B_\rho} \Delta w = \frac{n}{\rho} \frac{d}{d\rho} \Psi(\rho). \tag{3.21}$$

Where $\Psi(\rho) = \frac{1}{n\alpha(n)\rho^{n-1}} \int_{\partial B_\rho} w$.

Claim 5. *Let $w = u - L_{x_0}$ be defined as before. Then for some universal constant $K(n) > 0$,*

$$\frac{n}{\rho} \frac{d}{d\rho} \Psi(\rho) \geq -K.$$

Proof. We expand the derivative and compute.

$$\frac{n}{\rho} \frac{d}{d\rho} \Psi(\rho) = \frac{n}{\rho} \frac{1-n}{n\alpha(n)\rho^n} \int_{\partial B_\rho(x_0)} w(y) dS(y) + \frac{n}{n\alpha(n)\rho^n} \frac{d}{d\rho} \int_{\partial B_\rho(x_0)} w(y) dS(y)$$

$$= \frac{n}{\rho}\frac{n-1}{n\alpha(n)\rho^n}\int_{\partial B_\rho(x_0)} -w(y)dS(y) + \frac{1}{\alpha(n)\rho^n}\frac{d}{d\rho}\rho^{n-1}\int_{\partial B_1(0)} w(x_0+\rho z)dS(z)$$

$$= \frac{n}{\rho}\frac{n-1}{n\alpha(n)\rho^n}\int_{\partial B_\rho(x_0)} -w(y)dS(y) + \frac{\rho^{n-2}(n-1)}{\alpha(n)\rho^n}\frac{\rho^{n-1}}{\rho^{n-1}}\int_{\partial B_1(0)} w(x_0+\rho z)dS(z)$$

$$+\frac{\rho^{n-1}}{\alpha(n)\rho^n}\frac{d}{d\rho}\int_{\partial B_1(0)} w(x_0+\rho z)dS(z)$$

$$= \frac{n}{\rho}\frac{n-1}{n\alpha(n)\rho^n}\int_{\partial B_\rho(x_0)} -w(y)dS(y) + \frac{(n-1)}{\alpha(n)\rho^{n+1}}\int_{\partial B_\rho(x_0)} w(y)dS(y)$$

$$+\frac{\rho^{n-1}}{\alpha(n)\rho^n}\frac{d}{d\rho}\int_{\partial B_1(0)} w(x_0+\rho z)dS(z)$$

By the modulus of semi-concavity on the ball we have,

$$\frac{n}{\rho}\frac{n-1}{n\alpha(n)\rho^n}\int_{\partial B_\rho(x_0)} -w(y)dS(y) \geq \frac{n(n-1)}{\alpha(n)n\rho^{n+1}}\left|\partial B_\rho(x_0)\right|(-C\rho^2) = -C(n^2-n).$$

By the mean value theorem for subharmonic functions we have,

$$\frac{(n-1)}{\alpha(n)\rho^{n+1}}\int_{\partial B_\rho(x_0)} w(y)dS(y) \geq \frac{(n-1)}{\alpha(n)\rho^{n+1}}\, w(x_0) = 0.$$

By the nondecreasing property for the average integral we have:

$$\frac{\rho^{n-1}}{\alpha(n)\rho^n}\frac{d}{d\rho}\int_{\partial B_1(0)} w(x_0+\rho z)dS(z) \geq 0.$$

Hence for $K = C(n^2-n)$ we obtain the desired estimate. □

Proof. (Lemma 5) From the claim we obtain the estimate,

$$\frac{1}{\alpha(n)\rho^n}\int_{B_\rho} \Delta w \geq -K.$$

Moreover from (3.19) and the semiconcavity estimate we obtain,

$$C \geq \frac{\mu(B_\rho(x_0))}{|B_\rho(x_0)|} \geq -K.$$

Letting $\rho \to 0$ we find $\forall x \in B_{\frac{r}{4}}(x_0)$,

$$C \geq \Delta u(x) \geq -K.$$

□

We now state and prove the sharp estimate for the solution.

Theorem 12. *Let u be a solution to the classical stochastic impulse control problem. Then,*

$$\|u\|_{C^{1,1}(B_{r/4})} \leq C \tag{3.22}$$

Proof. We recall some basic notions and definitions for convenience. For further details refer to [18]. We say that P is a parabaloid of opening M whenever,

$$P(x) = l_0 + l(x) \pm \frac{M}{2}|x|^2.$$

We define,

$$\overline{\Theta}(u, A)(x_0),$$

to be the infimum of all positive constants M for which there is a conex parabaloid of opening M that touches u from above at x_0 in A. Similarly one can define the infimum of all positive constants M for which there is a convex parabaloid of opening $-M$ that touches u from below at x_0 in A,

$$\underline{\Theta}(u, A)(x_0).$$

We further define,

$$\Theta(u,A)(x_0) = \sup\{\overline{\Theta}(u,A)(x_0), \underline{\Theta}(u,A)(x_0)\} \leq \infty.$$

As before we fix $x_0 \in \{u = \varphi_u\}$. We consider the second incremental quotients of u and Mu,

$$\Delta_h^2 u(x_0) = \frac{u(x_0+h) + u(x_0-h) - 2u(x_0)}{|h|^2}.$$
$$\Delta_h^2 \varphi_u(x_0) = \frac{\varphi_u(x_0+h) + \varphi_u(x_0-h) - 2\varphi_u(x_0)}{|h|^2}.$$

We make the following observations,

1. $\Delta_h^2 u(x_0) \leq \Delta_h^2 \varphi_u(x_0)$.
2. $0 \leq \overline{\Theta}(u,B_\rho)(x_0) = \overline{\Theta}(\varphi_u,B_\rho)(x_0) \leq C$.
3. $0 \leq \underline{\Theta}(u,B_\rho)(x_0) = \underline{\Theta}(\varphi_u,B_\rho)(x_0) \leq K$.

Putting the estimates together we obtain,

$$-K \leq -\underline{\Theta}(u,B_\rho)(x_0) \leq \Delta_h^2 u(x_0) \leq \Delta_h^2 \varphi_u(x_0) \leq \overline{\Theta}(Mu,B_\rho)(x_0) \leq C.$$

In particular $\forall x \in B_\rho$,

$$-K \leq -\underline{\Theta}(u,B_\rho)(x) \leq \Delta_h^2 u(x) \leq \overline{\Theta}(u,B_\rho)(x) \leq C.$$

This follows from choosing $\forall x \in B_\rho$, the lower parabaloid and upper parabaloid to be respectively,

$$P_1(y) = u(x) + \langle p_1, y-x \rangle - \frac{K}{2}|y|^2.$$

$$P_2(y) = u(x) + \langle p_2, y - x \rangle + \frac{C}{2}|y|^2.$$

We obtain,

$$\Theta(u, \epsilon) = \Theta(u, B_\rho \cap B_\epsilon(x))(x) \in L^\infty(B_\rho).$$

By Proposition 1.1 in [18] it follows,

$$\|D^2 u\|_{L^\infty(B_\rho)} \leq C.$$

□

3.5 Regularity Estimates for the Free Boundary

In this section we prove a structural theorem for the free boundary $\Gamma = \partial\{u < \mathrm{M}u\}$.

Theorem 13. *Consider the classical stochastic impulse control problem*

$$\begin{cases} \Delta u(x) \geq f(x) & \forall x \in \Omega, \\ u(x) \leq \mathrm{M}u(x) = 1 + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x + \xi) & \forall x \in \Omega, \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{3.23}$$

Moreover assume that f is analytic and $f(x) \leq f(x + \xi)$ $\forall \xi \geq 0$. Then it follows that, $\partial\{u < \mathrm{M}u\} = \Gamma^r(u) \cup \Gamma^s(u) \cup \Gamma^d(u)$ where,

1. $\forall x_0 \in \Gamma^r(u)$ there exists some appropriate system of coordinates in which the coincidence set $\{u = \mathrm{M}u\}$ is a subgraph $\{x_n \leq g(x_1, \ldots, x_{n-1})\}$ in a neighborhood of x_0 and the function g is analytic.

2. $\forall x_0 \in \Gamma^s(u)$, x_0 is either isolated or locally contained in a C^1 submanifold.

3. $\Gamma^d(u) \subset \Sigma(u)$ where $\Sigma(u)$ is a finite collection of C^∞ submanifolds.

Proof. Recall $\Sigma_x = \{1 + u(x+\xi) = \mathrm{M}u(x)\}$ and $\Sigma_{\geq x} = \{x + \xi \ : \ \xi \geq 0\}$. We define the following sets

1. $\Sigma^0_{\geq x} = \{\xi \in \Sigma_{\geq x} \mid \xi_i > 0 \ \forall i = 1, \ldots, n\}$.
2. $\partial_i \Sigma_{\geq x} = \{\xi \in \Sigma_{\geq x} \mid \xi_i > 0 \text{ and } \xi_k = 0 \ \ \forall k = 1, \ldots, i-1, i+1, \ldots, n\}$.
3. $\Sigma^0_x = \{\xi \in \Sigma_x \mid \xi \in \Sigma^0_{\geq x}\}$.
4. $\partial_i \Sigma_x = \{\xi \in \Sigma_x \mid \xi \in \partial_i \Sigma_{\geq x}\}$.

We note that

$$\Sigma_{\geq x} = \Sigma^0_{\geq x} \cup (\bigcup_i^n \partial_i \Sigma_{\geq x}),$$

$$\Sigma_x = \Sigma^0_x \cup (\bigcup_i^n \partial_i \Sigma_x).$$

Fix $x_0 \in \partial\{u < \mathrm{M}u\}$ and let ξ_0 be the positive vector such that,

$$\inf_{\substack{\xi \geq 0 \\ x_0 + \xi \in \Omega}} u(x_0 + \xi) = 1 + u(x_0 + \xi_0).$$

Case 1: $\xi_0 \in \Sigma^0_{x_0}$. Then it follows from Claim 4, that $\forall x \in B_{\frac{\delta}{2}}(x_0)$, $\xi_0 \in \Sigma^0_x$. In particular for a fixed constant C, $\mathrm{M}u = C$ in $B_{\frac{\delta}{2}}(x_0)$. Without loss of generality we take $C = 0$. Furthermore it follows that at a contact point x_0 we have the following chain of inequalities,

$$f(x_0) \leq \Delta u(x_0) \leq \Delta \mathrm{M}u(x_0) \leq f(x_0 + \xi_0).$$

In particular,

$$f(x_0) \leq \Delta u(x_0) \leq 0.$$

We make the following claim,

Claim 6. $f(x_0) < 0$.

Proof. Suppose by contradiction that $f(x_0) = 0$. By analyticity of f, it follows that $\Omega = \{f > 0\}$ satisfies an interior sphere condition. Hence, $\forall z \in \partial\Omega$ there exists, $y \in \Omega$ and open ball $B_r(y)$ such that $\overline{B_r(y)} \cap \overline{\Omega} = \{z\}$. In particular consider $z = x_0$ and $y = y_0$. Observe that $\forall x \in \overline{B_r(y_0)} \setminus \{x_0\}$, it follows that $w = u - \mathrm{M}u < 0$ and $\Delta w = \Delta u = f > 0$. Hence by the Hopf Boundary point lemma,

$$\frac{\partial w}{\partial \nu}(x_0) > 0.$$

But $w \in C^{1,1}(x_0)$. A contradiction. $\square$

From the claim it follows that in a small neighborhood $B_\eta(x_0)$, we can study the following problem,

$$\begin{cases} \Delta w(x) = f(x) < 0 & \forall x \in \{w < 0\} \cap B_\eta(x_0), \\ w(x) \leq 0 & \forall x \in B_\eta(x_0), \\ w \in C^{1,1} & \forall x \in \overline{B_\eta(x_0)} \end{cases} \tag{3.24}$$

Hence w is a normalized solution and the conclusion follows for,

Finally to conclude we define,

$$\Gamma^r(u) = \{x \in \Gamma \mid \Sigma_x^0 = \Sigma_x \text{ and } x \text{ is a } \textbf{Regular Point}\}.$$

$$\Gamma^s(u) = \{x \in \Gamma \mid \Sigma_x^0 = \Sigma_x \text{ and } x \text{ is a } \textbf{Singular Point}\}.$$

Case 2: $\xi_0 \in \partial_i \Sigma_{x_0}$. We consider the set

$$\Sigma(u) = \bigcup_i^n \{u_{x_i} = 0\} \times \mathbb{R}^{n-1}.$$

By analyticity of f it follows that $\{u_{x_i} = 0\}$ is a finite set $\forall i = 1, \ldots, n$. Hence $\Sigma(u)$ is a finite collection of hyperplanes $\{l_j\}_{j=1}^k \subset \mathbb{R}^n$. We define

$$\Gamma^d(u) = \{x \in \Gamma(u) \mid \exists \bar{\xi} \in \partial_i \Sigma_x\}.$$

Finally to conclude we observe,

$$\Gamma^d(u) \subset \Sigma(u).$$

□

Chapter 4

The Fully Nonlinear Implicit Constraint Obstacle Problem

In this chapter we prove estimates for fully nonlinear obstacle problems admitting obstacles with a general modulus of semiconvexity. We recall our main results in this direction.

Theorem 14. *We consider $F(D^2u)$, a fully nonlinear uniformly elliptic operator. We assume that the operator is either convex or concave in the hessian variable. We define $\varphi_u(x)$ to be a semiconvex function with a general modulus of semiconvexity $\omega(r)$. We consider the following boundary value problem.*

$$\begin{cases} F(D^2u) \leq 0 & \forall x \in \Omega. \\ u(x) \geq \varphi_u(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{4.1}$$

Then the solution u has a modulus of continuity $\omega(r)$ up to $C^{1,1}(\Omega)$.

As an application we apply our result to obtain a sharp estimate for the solution to the following fully nonlinear stochastic impulse control problem:

Theorem 15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^{2,\alpha}$ boundary $\partial\Omega$.*

Define

$$\mathrm{M}u(x) = \varphi(x) + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \bar{\Omega}}} (u(x+\xi)). \tag{4.2}$$

Here $\varphi(x)$ is $\omega(r)$ semi-concave, strictly positive, bounded, and decreasing in the positive cone $\xi \geq 0$. Consider the solution to the following fully nonlinear stochastic impulse control problem,

$$\begin{cases} F(D^2 u) \geq f & \forall x \in \Omega. \\ u(x) \leq \mathrm{M}u(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{4.3}$$

Then, the solution u has modulus of continuity $\omega(r)$ up to $C^{1,1}(\Omega)$.

In the rest of this chapter, we proceed in stages to prove the stated theorems. We remark that Theorem 15 follows directly from Theorem 14 and Lemma 4 in the previous section.

4.1 Lipschitz Estimates for the Solution

To obtain the optimal estimate we first prove initial regularity estimates which we hope to extend. We fix $f = 0$. All the proofs may be modified for a nonzero sufficiently regular f. We begin by first proving that solutions are indeed continuous.

Lemma 6. *Let u be a solution to (4) with semiconvex obstacle φ_u. Then $u \in C(\Omega)$.*

The lemma follows from a result due to Evans.

Lemma 7. *If u is continuous in $\{u = \varphi_u\}$, then u is continuous in Ω.*

Proof. The possibility of a discontinuity is limited to a point on the free boundary, $\partial\{u > \varphi_u\}$. Consider $x_0 \in \{u = \varphi_u\}$ and without loss of generality assume $u(x_0) = 0$. Suppose by contradiction, that there exists a sequence of points $\{x_k\}$ with the following properties:

1. $\{x_k\} \to x_0$.
2. $\forall k$, $x_k \in \{u > \varphi_u\}$.
3. $\mu = \lim_{x_k \to x_0} u(x_k) > u(x_0) = 0$.

By lower semicontinuity of u, we know that $\forall \delta > 0$ there exists a neighborhood of x_0 such that $u \geq -\delta$ for $\delta << \mu$. We consider

$$r_k = \text{dist}[x_k, \{u = \varphi_u\}].$$

For a large enough k we can ensure that:

1. $u(x) + \delta \geq 0$ in $B_{r_k}(x_k)$.
2. $u(x_k) + \delta \geq \frac{\mu}{2}$.

Moreover we know that $u(x) + \delta$ satisfies the equation in $B_{r_k}(x_k)$. By the Harnack Inequality we obtain,

$$\frac{\mu}{2} \leq u(x_k) + \delta \leq C \inf_{B_{\frac{r_k}{2}}(x_k)} (u + \delta).$$

This implies for some $C_0 > 0$ universal,

$$\inf_{B_{\frac{r_k}{2}}(x_k)} u \geq C_0 \mu.$$

Since u is also superharmonic, we know from the weak Harnack Inequality in $B_{4r_k}(y_k)$ that,

$$
\begin{aligned}
u(y_k) &\geq c \left(\fint_{B_{2r_k}(y_k)} u^p \right)^{\frac{1}{p}} \\
&= \frac{c}{|B_{2r_k}|^{1/p}} \left(\int_{B_{2r_k}(y_k) \setminus B_{\frac{r_k}{2}}(x_k)} u^p + \int_{B_{\frac{r_k}{2}}(x_k)} u^p \right)^{\frac{1}{p}} \\
&\geq \frac{c}{|B_{2r_k}|^{1/p}} \left(-(\delta)^p |B_{2r_k}| + (C_0 \mu)^p |B_{\frac{r_k}{2}}| \right)^{\frac{1}{p}} \\
&\geq C_1 \mu \quad \text{for } C_1 > 0.
\end{aligned}
$$

On the other hand $u(y_k) = \varphi_u(y_k)$ and $y_k \to x_0$. This implies in particular that,

1. $\varphi_u(y_k) \geq C_1 \mu$.
2. $\varphi(x_0) = u(x_0) = 0$.

This is our desired contradiction. □

Remark 2. *We observe that the conditions on the obstacle may be relaxed in the proof of this lemma. In fact continuity of the obstacle is sufficient.*

A generic semiconvex function with a general modulus of semiconvexity is known to be Lipschitz in the interior (See Appendix B). We extend the previous result to show that solutions to a fully nonlinear obstacle problem admitting obstacles with a Lipschitz modulus of continuity grow from the free boundary with a comparable rate.

Lemma 8. *Let u be a solution to (4.1) with semiconvex obstacle φ_u. Fix*

$0 \in \partial\{u > \varphi_u\}$. *Then*

$$\sup_{B_r(0)} u(x) \leq Cr.$$

Proof. Let $\gamma(r)$ denote the Lipschitz modulus of continuity for the obstacle φ_u in $B_r(0)$. The obstacle condition $u \geq \varphi_u$ implies in $B_r(0)$ that

$$u \geq \varphi_u(0) - \gamma(r).$$

Define

$$v(x) = u - (\varphi_u(0) - \gamma(r)).$$

We note that $F(D^2 v) = F(D^2 u) \leq 0$ and $F(D^2 v) = 0$ inside $\{u > \varphi_u\}$. We consider $x \in B_{r/4}(0) \cap \{u > \varphi_u\}$. Moreover we let y be the closest free boundary point to x. Let ρ be the distance of x to its closest free boundary point y. From the Weak Harnack Inequality it follows,

$$v(y) \geq C \left(\fint_{B_{2\rho(y)}} v^p \right)^{1/p}.$$

By the positivity of v and Harnack Inequality in $B_{\rho(x)}$, it follows that the right hand side,

$$\geq C \left(\frac{B_\rho(x)}{B_{2\rho(y)}} \fint_{B_{\rho(x)}} v^p \right)^{1/p} \geq Cv(x).$$

Recall that $|\varphi_u(0) - \varphi_u(y)| \leq \gamma(r)$. Hence,

$$0 \leq v(y) = \varphi_u(y) - \varphi_u(0) + \gamma(r) \leq 2\gamma(r).$$

Changing back to our solution u, we find,

$$0 \leq u(x) - (u(0) - \gamma(r)) \leq Cv(y) \leq C\gamma(r).$$

In particular,

$$u(x) - u(0) \leq C\gamma(r).$$

$\square$

4.2 Optimal $C^{\omega(r)}$ Estimates for the Solution

In the previous section we assumed that the obstacle had a uniform modulus of continutity. A priori for semi-concave functions you only know that the uniform modulus of continuity is Lipschitz. Our goal as in the classical case will be to study the interplay between the equation and the obstacle to improve regularity estimates for the obstacle on the contact set. We start this section by stating and proving a lemma in the particular case that our operator is the Laplacian. The motivating calculation will help us proceed to prove the desired estimate in the more general case. The content of the lemma says that for any given point $x_1 \in \{u(x) > \varphi_u(x)\}$, $\exists x_0 \in \{u(x) = \varphi_u(x)\}$ such that the solution grows at most by $\omega(2|x_1 - x_0|)$ where $\omega(|x_1 - x_0|)$ denotes the modulus of semiconvexity for the obstacle on the ball $B_{|x_1 - x_0|}(x_0)$. Since a lower estimate is available via the obstacle, what we aim to show is that around a fixed contact point the modulus of continuity of the solution is controlled by the modulus of semiconvexity of the obstacle.

Lemma 9. *Let $\varphi_u(x)$ be a semiconvex function with general modulus of semiconvexity $\omega(r)$. Consider the following obstacle problem:*

$$\begin{cases} \Delta u \leq 0 & \forall x \in \Omega. \\ u(x) \geq \varphi_u(x) & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega \end{cases} \tag{4.4}$$

Fix $x \in \{u(x) > \varphi_u(x)\}$ and define $L_{x_0}(x) = \varphi_u(x_0) + \langle p, x - x_0 \rangle$, the linear part of the obstacle at the point x_0. Then $\exists x_0 \in \{u(x) = \varphi_u(x)\}$ and $C(n) > 0$ such that $u(x) - L_{x_0}(x) \leq C(n)\omega(2|x - x_0|)$.

Proof. We fix $x_1 \in \{u(x) > \varphi_u(x)\}$. Let x_0 denote the closest point to x_1 in $\{u = \varphi_u\}$. We denote this distance by $\rho = |x_1 - x_0|$. Define $w(x) = u(x) - L_{x_0}(x)$. Using the mean value theorem for superharmonic functions in $B_{2\rho}(x_0)$ we have,

$$\begin{aligned} 0 = w(x_0) &\geq \frac{1}{\alpha(n)2^n\rho^n} \int_{B_{2\rho}(x_0)} w(y)\, dy \\ &= K(n) \int_{B_{2\rho}(x_0) \setminus B_\rho(x_1)} w(y)\, dy \; + K(n) \int_{B_\rho(x_1)} w(y)\, dy. \end{aligned}$$

Semiconvexity of $w(x)$ in $B_{2\rho}(x_0)$ and an application of the mean value theorem for harmonic functions in $B_\rho(x_1)$ implies,

$$\begin{aligned} &\geq K(n) \int_{B_{2\rho}(x_0) \setminus B_\rho(x_1)} -\omega(2|y - x_0|) + C_1(n)w(x_1) \\ &\geq -\tilde{C}(n)\omega(2\rho) + C_1(n)w(x_1). \end{aligned}$$

In particular we obtain the desired bound,

$$w(x_1) \leq C(n)\omega(2\rho).$$

□

We now look to generalize the previous argument in the fully nonlinear setting. In the preceding proof the lower bound on the obstacle was transferred to the solution at the contact point. Moreover we were able to renormalize the solution by subtracting off a linear part. We will also need a generalization of the mean value theorem that was used to connect pointwise information with information about the measure Δu.

We consider again (4.1). For clarity we set $\omega(r) = \bar{C}r^2$ for some positive constant $\bar{C} > 0$. The arguments presented below can be trivially modified for the general semiconvex modulus by an appropriate rescaling. We make a remark in this direction towards the end of this section.

Lemma 10. *Let $x_1 \in \{u > \varphi_u\}$. Then $\exists x_0 \in \{u = \varphi_u\}$ such that for $w(x) = u(x) - L_{x_0}(x)$, where $L_{x_0}(x) = \varphi_u(x_0) + \langle p, x - x_0\rangle$ denotes the linear part of the obstacle at the point x_0, and a universal constant $K(n) > 0$,*

$$w(x_1) \leq K(n)|x_1 - x_0|^2.$$

Proof. Fix $x_1 \in \{u > \varphi_u\}$. Let x_0 be the closest point to x_1 in $\{u = \varphi_u\}$. We denote this distance by $\rho = |x_1 - x_0|$. By the modulus of semiconvexity of the obstacle we know that $w(x) \geq -16\bar{C}\rho^2$ on $B_{4\rho}(x_0)$. The idea of the proof is to zoom out to scale 1 and prove that the solution is bounded by a universal constant and then rescale back to obtain the desired bound. Consider the transformation $y = \frac{x-x_0}{\rho}$ and the scaled solution,

$$v(y) = \frac{w(\rho y + x_0)}{16\bar{C}\rho^2} + 1. \tag{4.5}$$

We note that $v(y)$ is a non-negative supersolution on $B_4(0)$ with

$$\inf_{B_4(0)} v(y) \leq 1.$$

Moreover $v(y)$ is a solution in $B_1(y_1)$, since x_0 is the closest point in the contact set to x_1. By the interior Harnack Inequality,

$$v(y_1) \leq \sup_{B_{\frac{1}{2}}(y_1)} v(y) \leq C \inf_{B_{\frac{1}{2}}(y_1)} v(y).$$

We also know from the weak L^ϵ estimate for supersolutions that for universal constants d, ϵ,

$$|\{v \geq t\} \cap B_2(0)| \leq dt^{-\epsilon} \quad \forall t > 0.$$

We observe that $B_{\frac{1}{2}}(y_1) \subseteq B_2(0)$. Hence we can choose $t = t_0$ such that

$$t_0 = \left(\frac{\delta d}{|B_{\frac{1}{2}(y_1)}|} \right)^{\frac{1}{\epsilon}},$$

for $\delta > 0$. It follows that,

$$|\{v \leq t\} \cap B_{\frac{1}{2}}(y_1)| > \delta |B_{\frac{1}{2}}(y_1)| > 0.$$

Hence there exists a universal constant C such that,

$$v(y_1) \leq C.$$

This implies from (4.5),

$$\frac{w(\rho y_1 + x_0)}{4\bar{C}\rho^2} \leq C.$$

Rescaling back we find,

$$w(x_1) \leq K|x_1 - x_0|^2.$$

$\square$

We now prove a lemma that controls the oscillation of the solution between two arbitrary points on the contact set $\{u = \varphi_u\}$. As a corollary which we state after the proof, we improve the modulus of continuity for the obstacle φ_u on the contact set $\{u = \varphi_u\}$.

Lemma 11. *Let $x_1 \in \{u = \varphi_u\}$ and $x_0 \in \{u = \varphi_u\}$. Then for $w(x) = u(x) - L_{x_0}(x)$, where $L_{x_0}(x) = \varphi_u(x_0) + \langle p, x - x_0 \rangle$ denotes the linear part of the obstacle at the point x_0, and $K(n) > 0$ a universal constant,*

$$w(x_1) \leq K(n)|x_1 - x_0|^2.$$

Proof. Assume by contradiction that for an arbitrary large constant $K > 0$

$$w > K|x_1 - x_0|^2. \tag{4.6}$$

As before we denote the distance between the points by $\rho = |x_1 - x_0|$. We begin with a claim.

Claim 7. $\exists$ Half Ball $HB_\rho(x_1)$ such that $\forall x \in HB_\rho(x_1),\ w(x) \geq \frac{K}{2}\rho^2$.

Proof. We define $\varphi_w = \varphi_u - L_{x_0}$, where as before $L_{x_0} = \varphi_u(x_0) + \langle p, x - x_0 \rangle$ for $p \in D^+\varphi_u(x_0)$ the superdifferential of φ_u at the point x_0. We make the following observations:

1. $w \geq \varphi_w \quad \forall x \in B_{2\rho}(x_0)$.

2. $\varphi_w(x_1) = \varphi_u(x_1) - \varphi_u(x_0) + \langle p, x - x_1 \rangle - \langle p, x - x_0 \rangle \quad \forall x \in B_{2\rho}(x_0)$.

In particular,

$$w(x) \geq \varphi_w(x_1) + \varphi_u(x) - \varphi_u(x_1) - \langle p, x - x_1 \rangle.$$

Now consider $d \in D^+\varphi_u(x_1)$ and observe that $w(x_1) = \varphi_w(x_1)$. This produces the following inequality,

$$w(x) \geq w(x_1) + \varphi_u(x) - \varphi_u(x_1) - \langle d, x - x_1 \rangle - \langle p - d, x - x_1 \rangle.$$

By semiconvexity on $B_\rho(x_1)$, (4.6), and fixing $x \in HB_\rho(x_1) = \{x \in B_\rho(x_1) \mid \langle p - d, x - x_1 \rangle \leq 0\}$, we have,

$$w(x) \geq K\rho^2 - \bar{C}\rho^2.$$

We can choose K large enough so that we obtain,

$$w(x) \geq \frac{K}{2}\rho^2.$$

This is our desired half ball. $\square$

We now consider again the dilated solution $v(y)$ from the previous lemma (4.5). By the weak Harnack Inequality for supersolutions, $\exists C > 0$ universal and $\epsilon > 0$ such that,

$$\int_{B_4(0)} |v(x)|^{\epsilon/2} \leq \left(C \inf_{B_2(0)} v(x) \right)^{\frac{2}{\epsilon}} \leq (Cv(0))^{\frac{2}{\epsilon}} = C.$$

From our previous claim we obtain

$$0 < |\{v(y) > \frac{K}{32\bar{C}}\} \cap B_1(y_1)|.$$

Here $\bar{C}$ is our semiconvexity constant from before. We now have the following chain of inequalities,

$$\begin{aligned}
&0 < |\{v(y) > \frac{K}{32\bar{C}}\} \cap B_1(y_1)| \, (\frac{K}{32\bar{C}})^{\epsilon/2} \\
&= \int_{\{v(x) > \frac{K}{32\bar{C}}\} \cap B_1(y_1)} (\frac{K}{32\bar{C}})^{\epsilon/2} \\
&\leq \int_{\{v(x) > \frac{K}{32\bar{C}}\} \cap B_1(y_1)} |v(x)|^{\epsilon/2} \\
&\leq \int_{B_4(0)} |v(x)|^{\epsilon/2} \leq \left(C \inf_{B_2(0)} v(x)\right)^{\frac{2}{\epsilon}} \leq (Cv(0))^{\frac{2}{\epsilon}} = C.
\end{aligned}$$

For K large enough we obtain a contradiction. Hence for a universal constant $K(n) > 0$,

$$w(x_1) \leq K(n)|x_1 - x_0|^2.$$

□

Remark 3. *Assume our obstacle is semiconvex on $B_r(x)$ with modulus of semiconvexity $\omega(r)$. We can translate our solution to the origin and scale by the modulus of semiconvexity of the obstacle. In particular, set ρ to be the distance between our fixed points.*

$$v(y) = \frac{w(\rho y + x_0)}{\omega(4\rho)} + 1. \tag{4.7}$$

One can check that we get similar estimates in terms of the modulus of semiconvexity $\omega(\rho)$.

Remark 4. *A corollary of the previous lemma is that on the contact set* $\{u = \varphi_u\}$ *the obstacle* φ_u *has a modulus of continuity* $\omega(r)$*. In particular,*

$$\|\varphi_u\|_{C^{\omega(r)}_{loc}(\{u=\varphi_u\})} \leq C. \tag{4.8}$$

We can now state and prove a sharp estimate for our solutions.

Theorem 16. *Consider the boundary value problem (4.1) with semiconvex obstacle* φ_u *admitting a modulus of semiconvexity,* $\omega(r)$*. Then the solution* u *has modulus of continuity* $\omega(r)$ *up to* $C^{1,1}(\Omega)$*. In particuluar,*

$$\|u\|_{C^{\omega(r)}(\Omega)} \leq C. \tag{4.9}$$

Proof. To prove this theorem we consider three distinct cases.

Case 1: $x_1 \in \{u > \varphi_u\}$, $x_0 \in \{u = \varphi_u\}$.

Choose the closest point in the contact set to x_1 and call it $\bar{x}_1$. Then we apply Lemma 10 to obtain the correct oscillation estimate up to the free boundary. Then an application of Lemma 11 gives us the correct oscillation estimate between two contact points. Finally we use the triangle inequality to conclude.

Case 2: $x_1, x_0 \in \{u = \varphi_u\}$.

This is the content of Lemma 11.

Case 3: $x_1, x_0 \in \{u > \varphi_u\}$.

We distinguish two different subcases.

Case 3a: $\max\{d(x_1, \{u = \varphi_u\}), d(x_0, \{u = \varphi_u\})\} \geq 4|x_1 - x_2|$.

Suppose $\max\{d(x_1, \{u = \varphi_u\}), d(x_2, \{u = \varphi_u\})\} = \rho$. Without loss of generality we assume that the maximum distance is realized at the point x_1. We observe that $B_{|x_1 - x_0|}(x_1) \subseteq B_{\frac{\rho}{2}}(x_1)$, and we consider $w = u - L_{x_1}$, where L_{x_1} denotes the linear part of the solution at x_1. By an application of the Harnack Inequality we obtain,

$$\sup_{B_\rho(x_1)} w \leq C \inf_{B_{\rho/2}(x_1)} w \leq Cw(x_2) \leq C\omega(\rho).$$

Moreover we also appeal to the interior estimates for solutions to our fully nonlinear convex or concave operator, $F(D^2u) = 0$,

$$\|w - w(x_1)\|_{C^{\omega(\rho)}(B_{\frac{\rho}{2}}(x_1))} \leq \frac{K}{\omega(\rho)} \|w - w(x_1)\|_{L^\infty(B_\rho(x_1))}.$$

Hence,

$$\|w - w(x_1)\|_{C^{\omega(\rho)}(B_{\frac{\rho}{2}}(x_1))} \leq C.$$

Case3b: $\max\{d(x_1, \{u = \varphi_u\}), d(x_2, \{u = \varphi_u\})\} < 4|x_1 - x_2|$

In this case one considers $\rho_1 = d(x_1, \{u = \varphi_u\})$ and $\rho_0 = d(x_0, \{u = \varphi_u\})$. Let $\bar{x}_1$ be the closest contact point to x_1 and $\bar{x}_0$ the closest contact point to x_0. We can apply Lemma 10 to obtain the desired oscillation estimate for each point up to the free boundary. We then apply Lemma 11 to control the oscillation between two contact points. Finally we apply the triangle inequality to conclude. □

Finally as in the classical case, assuming analytic data and $f(x) \leq f(x + \xi)\ \ \forall \xi \geq 0$, as well as concavity of $F(\cdot)$ in the hessian variable, it follows from an application of a nonlinear version of the Hopf Boundary Point Lemma [7] and the results of [44] that we obtain the following structural theorem for the free boundary,

Theorem 17. *Given the Fully Nonlinear Stochastic Impulse Control Problem*

$$\begin{cases} F(D^2u) \geq f & \forall x \in \Omega. \\ u(x) \leq \mathrm{M}u(x) = 1 + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x+\xi). & \forall x \in \Omega. \\ u = 0 & \forall x \in \partial\Omega. \end{cases} \tag{4.10}$$

It follows that, $\partial\{u < \mathrm{M}u\} = \Gamma^1(u) \cup \Gamma^2(u)$ where,

1. $\forall x_0 \in \Gamma^1(u)$ satisfying a uniform thickness condition on the coincidence set $\{u = \mathrm{M}u\}$, there exists some appropriate system of coordinates in which the coincidence set is a subgraph $\{x_n \leq g(x_1, \ldots, x_{n-1})\}$ in a neighborhood of x_0 and the function g is analytic.

2. $\Gamma^2(u) \subset \Sigma(u)$ where $\Sigma(u)$ is a finite collection of C^∞ submanifolds.

Remark 5. *We point out that the above theorem holds for the more general implicit constraint obstacle*

$$\mathrm{M}u = h(x) + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} u(x+\xi)$$

where the regularity of $\Gamma^1(u)$ corresponds to the regularity of $h(x)$.

4.3 Applications to a Penalized Problem

In this section we study a penalized fully nonlinear obstacle problem. The goal is to obtain optimal uniform estimates in the penalizing paramter ϵ.

For this section we fix the modulus of semiconvexity to be linear, i.e. $\omega(r) = cr^2$. We point out that the followng can be suitably modified for a general modulus of semiconvexity. The idea to obtain the optimal estimate is to use the interplay between semiconvexity of the obstacle and the superharmonicity of the equation as before.

Lemma 12. *Consider the fully nonlinear penalized obstacle problem with obstacle φ_u, admitting a modulus of semiconvexity, $\omega(r) = Cr^2$ and a suitably defined class of penalizations β_ϵ,*

$$\begin{cases} F(D^2u) = \beta_\epsilon(u - \varphi_u) & \Omega, \\ u = 0 & \partial\Omega, \\ \varphi_u < 0 & \partial\Omega. \end{cases} \tag{4.11}$$

Then the solution u has a modulus of continuity $\omega(r)$ up to $C^{1,\alpha}(\Omega)$ $\forall \alpha < 1$ independent of the penalizing parameter ϵ.

Proof. Let $\rho(x)$ be a function in $C^\infty(\mathbb{R}^n)$ with support in the unit ball, such that $\rho \geq 0$ and $\int_{\mathbb{R}^n} \rho = 1$. Define for any $\delta > 0$,

$$\rho_\delta(x) = \delta^{-n}\rho(\frac{x}{\delta}).$$

Consider the mollifier

$$J_\delta[\varphi_u](x) = \int_\Omega \rho_\delta(x - y)\varphi_u(y)\, dy.$$

Recall that φ_u semi-convex with a linear modulus implies that for any $\xi \in C_0^\infty(\Omega_0)$, $\xi \geq 0$, where $\Omega_0 \subset \Omega$ is an open set, it holds that for any directional derivative, $\frac{\partial}{\partial \eta}$ and some constant $C > 0$ independent of δ,

$$\int_\Omega \varphi_u \frac{\partial^2 \xi}{\partial \eta^2} \geq -C.$$

Taking $\xi = \rho_\delta$, it follows that pointwise in Ω,

$$\frac{\partial^2 J_\delta[\varphi_u]}{\partial \eta^2} \geq -C.$$

We consider, $\varphi_u^\delta = J_\delta[\varphi_u + \frac{C}{2}|x|^2] - \frac{C}{2}|x|^2$. It follows that

$$|D\varphi_u^\delta| \leq C.$$

$$\frac{\partial^2 \varphi_u^\delta}{\partial \eta^2} \geq -C.$$

$$\varphi_u^\delta \to \varphi_u \text{ uniformly in } \Omega \text{ as } \delta \to 0.$$

Define $\beta_\epsilon(t) \in C^\infty$ for $0 < \epsilon < 1$ and C a constant independent of ϵ, such that,

1. $\beta_\epsilon'(t) > 0$.
2. $\beta_\epsilon(t) \to 0$ if $t > 0, \epsilon \to 0$.
3. $\beta_\epsilon(t) \to -\infty$ if $t < 0, \epsilon \to 0$.
4. $\beta_\epsilon(t) \leq C$
5. $\beta_\epsilon''(t) \leq 0$.

Consider the penalized problem,

$$\begin{cases} F(D^2u) - \beta_\epsilon(u - \varphi_u^\epsilon) = 0 & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{4.12}$$

Define for $N > 0$,

$$\beta_{\epsilon,N}(t) = \max\{\min\{\beta_\epsilon, N\}, -N\}.$$

Consider the problem,

$$\begin{cases} F(D^2u) - \beta_{\epsilon,N}(u - \varphi_u^\epsilon) = 0 & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{4.13}$$

It follows from $W^{2,p}$ theory for fully nonlinear equations that for each $v \in L^p(\Omega) \cap C^0(\bar{\Omega})$ $(1 < p < \infty)$, there exists a unique solution $w \in W^{2,p}(\Omega) \cap C^0(\bar{\Omega})$ solving,

$$\begin{cases} F(D^2w) - \beta_{\epsilon,N}(v - \varphi_u^\epsilon) = 0 & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.14}$$

and for $\bar{C}$ independent of v,

$$\|w\|_{W^{2,p}} \leq \bar{C}.$$

Define the solution map T such that $Tv = w$. Notice that T maps $B_{\bar{C}}(0) \subset L^p(\Omega)$ into itself and is compact. Hence by Schauder's fixed-point theorem, it follows, that there exists u such that $Tu = u$. In particular, we have found a solution to (4.12). Moreover $\beta_{\epsilon,N}(u - \varphi_u^\epsilon) \in C^{0,\alpha}$. Hence by Evans-Krylov $\|u\|_{C^{2,\alpha}} \leq C(\epsilon)$. We now estimate $\zeta = \beta_{\epsilon,N}(u - \varphi_u^\epsilon)$. By definition we know that $\beta_{\epsilon,N}(u - \varphi_u^\epsilon) \leq C$ for a constant C independent of N, ϵ. Let x_0 be the minimum point of ζ. Without loss of generality we assume,

$$\mu = \zeta(x_0), \quad \mu \leq 0, \quad \mu < \beta_\epsilon(0).$$

It follows that $x_0 \notin \partial\Omega$. If not, then,

$$\mu = \zeta(x_0) = \beta_{\epsilon,N}(-\varphi_u^\epsilon) \geq \beta_{\epsilon,N}(0) \geq \beta_\epsilon(0).$$

A contradiction. On the other hand if $x_0 \in \Omega$, then $\beta'_\epsilon(t) \geq 0$ implies that,

$$\min_{\Omega}(u - \varphi_u^\epsilon) = u - \varphi_u^\epsilon(x_0) < 0.$$

Moreover it follows that $D^2(u-\varphi_u^\epsilon)(x_0) \geq 0$. Hence $F(D^2(u-\varphi_u^\epsilon)(x_0)) \geq 0$. By Ellipticity and the semiconvexity estimate it follows that,

$$\begin{aligned} \beta_{\epsilon,N}(u - \varphi_u^\epsilon)(x_0) &= F(D^2 u_{\epsilon,N} - D^2\varphi_u^\epsilon + D^2\varphi_u^\epsilon) \\ &\geq F(D^2 u_{\epsilon,N} - D^2\varphi_u^\epsilon) + \lambda\|D^2(\varphi_u^\epsilon)^+\| - \Lambda\|D^2(\varphi_u^\epsilon)^-\| \\ &\geq -C. \end{aligned}$$

In particular, $|\beta_{\epsilon,N}(u-\varphi_u^\epsilon)| \leq C$ for a constant C independent of ϵ and N. Furthermore $|F(D^2u)| \leq C$. It follows from elliptic estimates,

$$\|u\|_{W^{2,p}} \leq C.$$

Hence for N large enough u is a solution for the penalized problem (4.10). $\square$

We now prove the optimal estimate as before

Theorem 18. *Consider the solution to the fully nonlinear penalized obstacle problem with obstacle φ_u, admitting a modulus of semiconvexity, $\omega(r) = Cr^2$ and a suitably defined class of penalizations β_ϵ. Moreover assume that $F(D^2u)$ is convex in the Hessian variable. Then the solution u is $C^{1,1}$ independent of ϵ.*

Proof. Consider the penalization problem

$$\begin{cases} F(D^2u) = \beta_\epsilon(u - \varphi_u) & \Omega, \\ u = 0 & \partial\Omega. \\ \varphi_u < 0 & \partial\Omega. \end{cases} \tag{4.15}$$

We aim to bound $\inf u_{\tau\tau}$ from below. The following computation continues to hold for viscosity solutions by using incremental quotients and recalling that second order incremental quotients are supersolutions of a convex equation. We fix a directional derivative τ and differentiate the penalization identity to obtain,

$$F_{ij,kl}(D^2u)(D_{ij}u_\tau)(D_{kl}u_\tau) + F_{ij}(D^2u)(D_{ij}u_{\tau\tau}) =$$

$$\beta_\epsilon''(u-\varphi_u)(u-\varphi_u)_\tau^2 + \beta_\epsilon'(u-\varphi_u)(u-\varphi_u)_{\tau\tau}.$$

By convexity of the operator and the structural conditions on the penalization family $\beta_\epsilon(t)$ it follows that

$$F_{ij}(D^2u))(D_{ij}u_{\tau\tau}) \leq \beta_\epsilon'(u-\varphi_u)(u-\varphi_u)_{\tau\tau}.$$

Suppose the minimum point of $u_{\tau\tau}$ is in the interior of the domain then, since $\beta'(t) > 0$, we find $(u-\varphi_u)_{\tau\tau} \geq 0$. In particular, $u_{\tau\tau} \geq -C$. Suppose now that the minimum point of $u_{\tau\tau}$ is realized on the boundary of the domain. We differentiate the equation with respect to x_τ for $\tau \in \{1, \ldots, n-1\}$ and obtain,

$$F_{ij}(D^2u)D_{ij}u_\tau = \beta_\epsilon'(u-\varphi_u)(u-\varphi_u)_\tau.$$

Recall $\varphi_u < 0$ on $\partial\Omega$. Hence for a fixed $\epsilon_0 > 0$ it follows that $\varphi_u \leq u + \epsilon_0$ in $\{x \in \bar{\Omega} \mid d(x, \partial\Omega) \leq \frac{\epsilon_0}{2}\}$. Morevover by the uniform continuity of $u^\epsilon \to u$ on $\bar{\Omega}$, there exists a small ϵ_1, such that $\varphi_u \leq u + \frac{\epsilon_0}{2}$, $|\beta'| < \epsilon_0$, and $|\beta''| < \epsilon_0$ in $\{x \in \bar{\Omega} \mid d(x, \partial\Omega) \leq \frac{\epsilon_0}{2}\}$ for $0 < \epsilon < \epsilon_1$. Hence it follows from the boundary Hölder estimates for linear non-divergence form equations,

$$\|u_{\tau n}\|_{L^\infty(\partial B^+(\frac{\epsilon_0}{4}))} \leq C.$$

Moreover by uniform ellipticity we can use the equation to solve for u_{nn} in terms of β and u_{kl} for $k \in (1, \ldots, n-1)$ and $l \in (1, \ldots, n)$. Hence we obtain after straightening the boundary,

$$\|D^2 u\|_{L^\infty(\partial\Omega)} \leq C.$$

Hence it follows that the solution is semiconvex with a linear modulus. Moreover $F(D^2 u) \leq 0$. Hence an application of Lemma 11 proves that u has a uniform $C^{1,1}$ estimate. □

Remark 6. *We point out that the above arguments give us a straightforward proof for $C^{1,1}$ estimates when the operator is convex. The previous section was based on $C^{1,\alpha}$ estimates for Fully Nonlinear equations hence did not have a restriction on the sign of the operator.*

Remark 7. *Previous computation and estimates can be generalized to Viscosity Solutions of convex operators (see [18]).*

Finally, as an application of the uniform estimates, we prove how the $C^{2,\alpha}$ estimate for the penalized problem decays in the penalizing parameter.

Corollary 1. *Consider the fully nonlinear penalized obstacle problem with obstacle φ_u, admitting a modulus of semi-convexity, $\omega(r) = Cr^2$ and a suitably defined class of penalizations β_ϵ,*

$$\begin{cases} F(D^2 u) = \beta_\epsilon(u - \varphi_u) & \Omega, \\ u = 0 & \partial\Omega. \\ \varphi_u < 0 & \partial\Omega. \end{cases} \tag{4.16}$$

Moreover assume that $F(\cdot)$ is convex in the hessian variable. Then for a constant C independent of ϵ,

$$\|u\|_{C^{2,\alpha}} \leq C\epsilon^{-\alpha}.$$

Proof. It is well known that the penalization problem converges to the obstacle problem independent of the choice of penalizing family. Hence we fix a penalizing family,

$$\beta_\epsilon(t) = \begin{cases} \frac{t}{\epsilon^2} & t < 0. \\ 0 & t \geq 0. \end{cases} \tag{4.17}$$

Also we fix $\varphi_u = 0$. We consider the scaled function,

$$v^\epsilon(x) = \frac{1}{\epsilon^2}u^\epsilon(\epsilon x).$$

We note that

$$F(D^2v^\epsilon) = F(D^2(\frac{1}{\epsilon^2}u^\epsilon(\epsilon x))) = F(D^2u^\epsilon(\epsilon x)) = \frac{1}{\epsilon^2}u^\epsilon(\epsilon x) = v^\epsilon(x).$$

Hence we obtain for a constant C independent of ϵ,

$$\|v^\epsilon\|_{C^{2,\alpha}} \leq C.$$

It follows,

$$\begin{aligned} |D^2u^\epsilon(x) - D^2u^\epsilon(y)| &= |\epsilon^2 D^2v^\epsilon(\frac{x}{\epsilon}) - \epsilon^2 D^2v^\epsilon(\frac{y}{\epsilon})| \\ &= |D^2v^\epsilon(\frac{x}{\epsilon}) - D^2v^\epsilon(\frac{y}{\epsilon})| \\ &\leq C|\frac{x}{\epsilon} - \frac{y}{\epsilon}|^\alpha \\ &\leq C\epsilon^{-\alpha}|x-y|^\alpha. \end{aligned}$$

□

Chapter 5

The Penalized Boundary Obstacle Problem

5.1 A Mathematical Model for Homogenization

We consider a homogenization problem modeling diffusion through a semi-permeable membrane. In this model the transport of the molecules through the membrane is possible only across some given channels and in a fixed direction. More precisely:

Given a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and a subset T_δ of $\mathbb{R}^n$, consider the solution u^δ to the following obstacle-type problem

$$\begin{cases} u^\delta(x) \geq \phi(x) & \forall x \in T_\delta. \\ (-\Delta)^s u^\delta \geq 0 & \forall x \in \mathbb{R}^n. \\ (-\Delta)^s u^\delta = 0 & \forall x \in \mathbb{R}^n \setminus T_\delta. \\ (-\Delta)^s u^\delta = 0 & \forall x \in T_\delta \text{ and } u^\delta > \phi(x). \end{cases}$$

Here $(-\Delta)^s$ denotes the fractional Laplace operator of order $s \in (0, 1)$. We think of the domain $\mathbb{R}^n$ as being perforated with holes and the obstacle, ϕ, supported on the set T_δ. Here T_δ is a union of small sets $S_\delta(k)$ that are periodically distributed. Here $S_\delta(k)$ remains periodically distributed but is allowed to take random shapes and sizes. In this case we introduce a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and assume $\forall \omega \in \Omega$, $\forall \delta > 0$, there exists some subset

$S_\delta(k,\omega) \subset B_\delta(\delta k)$ where $B_\delta(\delta k)$ denotes the ball of radius δ centered at δk. We then define $T_\delta = \bigcup_{k\in\mathbb{Z}^n} S_\delta(k,\omega)$.

Restricting the problem to an open subset $D \subset \mathbb{R}^{n+1}_+$, and assuming the capacity of $S_\delta(k,\omega) = \delta^n \gamma(k,\omega) \leq \delta^n \bar{\gamma}$ where $\gamma(k,\omega)$ is a stationary ergodic process and $\bar{\gamma} > 0$, it follows that the solution to the above system converges $W^{1,2}(D, \|y\|^a dxdy)$-weak and almost surely with resepct to $\omega \in \Omega$ to the minimizer of the the penalized energy functional,

$$E_\epsilon(u) = \frac{1}{2}\int_D |y|^a |\nabla u|^2 \; dxdy + \frac{1}{2\epsilon}\int_\Sigma (u-\phi)_-^2 \; dx. \tag{5.1}$$

Here $\Sigma = D \cap \{y = 0\}$, $0 < \frac{1}{\epsilon} < C(\bar{\gamma})$, and $a = 1-2s$. We refer to ([24]) for the relevant details. We also point out that it follows from Proposition 2.8 in [55] that there exists a constant $C(\epsilon)$ such that for $s \in (0,1)$ and $s \neq 1/2$,

$$\|u\|_{C^{1,2s}} \leq C(\epsilon).$$

In this chapter our interest is to study the critical case $s = \frac{1}{2}$. We are interested in studying optimal estimates for the solution u_ϵ to (5.1) wth $a = 0$.

5.2 Optimal Nonuniform Estimates $s = \frac{1}{2}$

Recently in [9], [50] it is shown that solutions to the penalized functional are $C^{1,\alpha}$. Here we show that generically this is the best you can do. In particular assuming $\phi = 0$, $D = B_1(0)$, $a = 0$, $B_r' = \mathbb{R}^{n-1} \cap B_r$, and given a function $\varphi \in C^{2,\alpha}\left(\overline{B_1}\right)$, we consider the following problem,

$$\begin{cases} \Delta u = 0 & \text{in } B_1^+, \\ u_y = u^- & \text{on } B_r', \\ u = \varphi(x) & \text{on } (\partial B_1)^+, \end{cases} \tag{5.2}$$

where $(\partial B_1)^+$ denote the set $\partial B_1^+ \setminus \{y = 0\}$. Our result is the following,

Theorem 19. *If $\nabla u(0) \neq 0$, then $u \notin C^{1,1}(0)$.*

Proof. Assume by contradiction that $u \in C^{1,1}(0)$ and $\nabla u(0) \neq 0$. By the extension theorem [23], and the semigroup property of $-(-\Delta u)^s$, it follows that,

$$-(-\Delta)^{1/2}u^- = -(-\Delta)^{1/2} \circ -(-\Delta)^{1/2}u = \Delta u \in L^\infty.$$

By [9], [50] we know that $u \in C^{1,\alpha}(0)$ hence u has a unique differential, namely $P_1 = \nabla u(0)$. Without loss of generality it follows that $P_1 = \nabla u(0)$ is also a superdifferential for u^- (if not consider u^+). Trivally, $P_2 = 0$ is another subdifferential for u^-. Moreover it follows by the $C^{1,1}$ assumption that $(u^-)_{\tau\tau}(0) \leq C$ for any directional derivative τ. Consider for $P_1 \neq P_2$,

$$\varphi(x) = [u(0) + \min\{P_1 \cdot x, P_2 \cdot x\} + \frac{C}{2}|x|^2]\chi_{B_1(0)}.$$

It follows from a straightforward computation that,

$$-(-\Delta)^{1/2}\varphi(0) = -\infty.$$

Moreover $u^-(x) \leq \varphi(x)$ with equality at $x = 0$. Hence by the comparison principle,

$$-(-\Delta)^{1/2}u^-(0) \leq -(-\Delta)^{1/2}\varphi(0) = -\infty.$$

But we showed above that $-(-\Delta)^{1/2}u^{-}(0) \geq -C$. A contradiction. $\square$

We point out that the above proof relied crucially on the fact that the gradient of the function does not vanish at the origin. We now consider the case where $\nabla u(0) = 0$. We proceed to show that in this case $u \in C^{1,1}(0)$. We do so by showing monotonicity of the Almgren Frequency Functional.

Lemma 13. *Let u^ϵ be the solution to the penalized boundary obstacle problem in B_1 with the following properties:*

1. $\Delta u^\epsilon = 0$ in $B_1 \setminus \{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}$.

2. $u^\epsilon(0) = 0$

3. $u^\epsilon(x,0)u^\epsilon_\nu(x,0) \leq 0 \ \forall x \in B'_1$.

Define

$$\Phi(r; u^\epsilon) = r\frac{\int_{B_r} |\nabla u^\epsilon|^2}{\int_{\partial B_r} (u^\epsilon)^2} = r\frac{V(r; u^\epsilon)}{H(r; u^\epsilon)}. \tag{5.3}$$

Then $\forall r \in (0,1)$,

(i) $\Phi(r; u^\epsilon) < +\infty$

(ii) $\Phi(r; u^\epsilon)$ is monotone increasing in r.

Moreover define

$$0 \leq \lim_{r \to 0^+} \Phi(r; u^\epsilon) = \mu < \infty.$$

Then, $\Phi(r) = \mu$ identically in $(0,1)$ if and only if u has homogeneity μ and $\int_{y=0} uu_\nu = 0$.

Proof. We start by computing,

$$\log \Phi(r; u^\epsilon) = \log r + \log V(r; u^\epsilon) - \log H(r; u^\epsilon).$$

Differentiating this identify gives us,

$$\frac{d(\log \Phi(r; u^\epsilon))}{dr} = \frac{1}{r} + \frac{V'(r; u^\epsilon)}{V(r; u^\epsilon)} - \frac{H'(r; u^\epsilon)}{H(r; u^\epsilon)}. \tag{5.4}$$

We are reduced to showing that

$$\frac{d(\log \Phi(r; u^\epsilon))}{dr} \geq 0.$$

We have that,

$$V'(r; u^\epsilon) = \int_{\partial B_r} |\nabla u^\epsilon|^2.$$

$$H'(r; u^\epsilon) = \frac{n}{r} H(r; u^\epsilon) + 2 \left[\int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu + \int_{\{y=0\} \cap \partial B_r} u^\epsilon u^\epsilon_\nu \right].$$

Moreover we know that,

$$\int_{B_r} |\nabla u^\epsilon|^2 = \frac{1}{2} \int_{B_r} \Delta (u^\epsilon)^2 = \int_{\{y=0\} \cap B_r} u^\epsilon u^\epsilon_\nu + \int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu.$$

We consider the following vector field,

$$h(x) = div[x |\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon) \nabla u^\epsilon].$$

Since $\Delta u^\epsilon = 0$ in $B_r \setminus \{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\}$, we find $h(x) = (n-1)|\nabla u^\epsilon|^2$ in $B_r \setminus \{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\}$. This implies in particular by the divergence theorem,

$$(n-1) \int_{B_r} |\nabla u^\epsilon|^2 = \int_{B_r} h$$

$$= \int_{\partial B_r \setminus \{y=0\}} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu$$

$$+ \int_{\{y=0\} \cap \partial B_r} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu.$$

We consider each term separately.

$$\int_{\partial B_r \setminus \{y=0\}} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu = r \int_{\partial B_r \setminus \{y=0\}} |\nabla u^\epsilon|^2$$

$$-2r \int_{\partial B_r \setminus \{y=0\}} (u^\epsilon_\nu)^2.$$

Moreover,

$$\int_{\{y=0\} \cap \partial B_r} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu =$$

$$\int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu$$

$$+ \int_{\{u^\epsilon_y = 0\} \cap \partial B_r} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu.$$

Since we are on the hyperplane it follows that,

$$\int_{\{u^\epsilon_y = 0\} \cap \partial B_r} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu = 0.$$

$$\int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} [x|\nabla u^\epsilon|^2 - 2(x \cdot \nabla u^\epsilon)\nabla u^\epsilon] \cdot \nu = -2 \int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} (x \cdot \nabla u^\epsilon) u^\epsilon_\nu.$$

Hence we obtain,

$$(n-1) \int_{B_r} |\nabla u^\epsilon|^2 = r \int_{\partial B_r \setminus \{y=0\}} |\nabla u^\epsilon|^2 - 2r \int_{\partial B_r \setminus \{y=0\}} (u^\epsilon_\nu)^2$$

$$-2 \int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} (x \cdot \nabla u^\epsilon) u^\epsilon_\nu.$$

Which in particular give us after rearrangement,

$$\int_{\partial B_r \setminus \{y=0\}} |\nabla u^\epsilon|^2 = \frac{n-1}{r} \int_{B_r} |\nabla u^\epsilon|^2 + 2 \int_{\partial B_r \setminus \{y=0\}} (u^\epsilon_\nu)^2$$

$$+ \frac{2}{r} \int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} (x \cdot \nabla u^\epsilon) u^\epsilon_\nu.$$

Plugging back into (5.4) we find,

$$\frac{d(\log \Phi(r; u^\epsilon))}{dr} =$$

$$\frac{1}{r} + \frac{\frac{n-1}{r} \int_{B_r} |\nabla u^\epsilon|^2 + 2 \int_{\partial B_r \setminus \{y=0\}} (u^\epsilon_\nu)^2 + \frac{2}{r} \int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} (x \cdot \nabla u^\epsilon) u^\epsilon_\nu}{\int_{B_r} |\nabla u^\epsilon|^2}$$

$$- \frac{\frac{n}{r} H(r; u^\epsilon) + 2[\int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu + \int_{\{y=0\} \cap \partial B_r} u^\epsilon u^\epsilon_\nu]}{\int_{\partial B_r} (u^\epsilon)^2}$$

Which reduces to,

$$= \frac{2 \int_{\partial B_r \setminus \{y=0\}} (u^\epsilon_\nu)^2}{\int_{B_r} |\nabla u^\epsilon|^2} - \frac{2 \int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu}{\int_{\partial B_r} (u^\epsilon)^2} - \frac{2 \int_{\{y=0\} \cap \partial B_r} u^\epsilon u^\epsilon_\nu}{\int_{\partial B_r} (u^\epsilon)^2}$$

$$+ \frac{\frac{2}{r} \int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} (x \cdot \nabla u^\epsilon) u^\epsilon_\nu}{\int_{B_r} |\nabla u^\epsilon|^2}.$$

We note the following inequalities:

1. $\int_{\{y=0\} \cap B_r} u^\epsilon u^\epsilon_\nu \leq 0$.
2. $0 \leq \int_{B_r} |\nabla u^\epsilon|^2 = \int_{\{y=0\} \cap B_r} u^\epsilon u^\epsilon_\nu + \int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu \leq \int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu$.
3. $\int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu \leq (\int_{\partial B_r} (u^\epsilon)^2)^{1/2} (\int_{\partial B_r \setminus \{y=0\}} (u^\epsilon_\nu)^2)^{1/2}$.

Continuing the inequality we obtain,

$$\geq \frac{2 \int_{\partial B_r \setminus \{y=0\}} (u^\epsilon_\nu)^2}{\int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu} - \frac{2 \int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu}{\int_{\partial B_r} (u^\epsilon)^2} + \frac{\frac{2}{r} \int_{\{u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon\} \cap \partial B_r} (x \cdot \nabla u^\epsilon) u^\epsilon_\nu}{\int_{B_r} |\nabla u^\epsilon|^2}$$

$$\geq \frac{2(\int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu)^2}{\int_{\partial B_r}(u^\epsilon)^2 \int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu} - \frac{2\int_{\partial B_r \setminus \{y=0\}} u^\epsilon u^\epsilon_\nu}{\int_{\partial B_r}(u^\epsilon)^2} + \frac{\frac{2}{r}\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(x\cdot \nabla u^\epsilon)u^\epsilon_\nu}{\int_{B_r}|\nabla u^\epsilon|^2}$$

$$\geq \frac{\frac{2}{r}\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(x\cdot \nabla u^\epsilon)u^\epsilon_\nu}{\int_{B_r}|\nabla u^\epsilon|^2}.$$

Recalling the penalization and applying an integration by parts we get,

$$\frac{2}{r}\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(x\cdot \nabla u^\epsilon)u^\epsilon_\nu = \frac{-1}{r\epsilon}\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r} x\cdot \nabla (u^\epsilon)^2$$

$$= \frac{-1}{r\epsilon}\left[r\int_{\partial\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(u^\epsilon)^2 + r\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(u^\epsilon)^2 - n\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(u^\epsilon)^2\right]$$

$$= \frac{n}{r\epsilon}\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(u^\epsilon)^2 - \frac{1}{\epsilon}\int_{\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}\cap \partial B_r}(u^\epsilon)^2$$

The last equatlity follows from the fact that $u^\epsilon = 0$ on $\partial\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}$. Moreover $n \geq 1$ and $r \leq 1$ implies,

$$\frac{d(\log \Phi(r; u^\epsilon))}{dr} \geq 0.$$

Our desired estimate. Equality follows when u is proportional to u_ν on ∂B_r $\forall 0 < r < 1$, and $\int_{\{y=0\}} uu_\nu = 0$. When both are satisfied then, by the radial formula for the Laplace operator and unique continuation it follows that $u = |x|^\mu g(\theta)$ where $\theta \in \partial B_1$.

$\square$

We supress the ϵ in what follows. Define

$$\varphi(r) = \varphi(r; u) = \fint_{\partial B_r^+} u^2.$$

We remark that $\Phi(r; u) = \frac{r}{2}\frac{d}{dr}\log \varphi(r; u)$. We now state some corrolaries that follow from the monotonicity of the Almgren Frequency Functional.

Corollary 2. *Let* $0 \leq \lim_{r\to 0^+} \Phi(r;u) = \mu < \infty$. *Then,*

(a) The function $r \to r^{-2\mu}\varphi(r)$ *is nondecreasing for* $0 < r < 1$. *In particular,*

$$\varphi(r) \leq r^{2\mu}\varphi(1) \leq r^{2\mu} \sup_{B_1} |u|.$$

(b) Let $0 < r < 1$. $\forall \delta > 0$, $\exists r_0(\delta) > 0$ *such that* $\forall r, R \leq r_0(\delta)$,

$$\varphi(R) \leq \left(\frac{R}{r}\right)^{2(\mu+\delta)} \varphi(r).$$

Proof. For (a): We compute,

$$\varphi'(r) = \frac{d}{dr} \fint_{\partial B_r} u^2 = \frac{C(n)}{r^n} \int_{\partial B_r} u^2 + 2 \fint_{\partial B_r} u u_\nu$$

Hence,

$$\begin{aligned}
\frac{d}{dr}(r^{-2\mu}\varphi(r)) &= -2\mu r^{-2\mu-1}\varphi(r) + r^{-2\mu}\left(\frac{C(n)}{r^n}\int_{\partial B_r} u^2 + 2\fint_{\partial B_r} u u_\nu\right) \\
&\geq -2\mu r^{-2\mu-1}\varphi(r) + 2r^{-2\mu}\fint_{\partial B_r} u u_\nu \\
&= -2\mu r^{-2\mu-1}\varphi(r) + 2r^{-2\mu}\frac{1}{|\partial B_r|}\int_{B_r} |\nabla u|^2 \\
&= \frac{-2\mu r^{-2\mu-1}}{|\partial B_r|}\left(r\int_{B_r} |\nabla u|^2 - \mu \int_{\partial B_r} u^2\right) \\
&\geq 0.
\end{aligned}$$

For (b) Let $r_0(\delta)$ be such that $\Phi(r;u) \leq \mu + \delta$ for $r, R \leq r_0$. Then it follows that,

$$\Phi(r;u) = \frac{r}{2}\frac{d}{dr}\log\varphi(r;u) \leq \mu + \delta.$$

To conclude we integrate the inequality over (r, R). □

We now conclude that the growth rate of the solution to the penalized problem are constrained by the homogeneity of the blow up solutions.

Corollary 3. *Let u solve the penalized problem. Then $\forall x \in B_{r/2}$,*

$$|u(x)| \leq r^{\mu} \sup_{B_1} |u|.$$

Proof. We note that u_+^2 is a positive subharmonic function in the domain. Hence,

$$(u^+)^2 \leq \fint_{\partial B_r} (u^+)^2 \leq r^{2\mu} \sup_{B_1} |(u^+)^2|.$$

A similar estimate holds for $(u^-)^2$. □

The final step to obtain the optimal regularity estimate is to study blow up sequences around a free boundary point. We define

$$v_r(x) = \frac{u(rx)}{[\varphi(u;r)]^{1/2}}.$$

We note that $\|v_r\|_{L^2(\partial B_1)} = 1$. We obvserve,

$$\int_{B_R} |\nabla v_r|^2 = \frac{\Phi(v_r;R)}{R} \int_{\partial B_R} v_r^2 = |\partial B_1| R^{n-1} \Phi(u;rR) \frac{\varphi(rR;u)}{\varphi(r;u)}.$$

For a fixed $R > 1$ and every r such that $rR \leq r_0(\delta)$,

$$\int_{B_R} |\nabla v_r|^2 \leq |\partial B_1| (\mu + \delta) R^{n-1+2(\mu+\delta)}.$$

Hence $\{v_r\}$ are equibounded in H^1_{loc} and by the $C^{1,\alpha}$ estimate they are also bounded in $C^{1,\alpha}_{loc}$. Thus there exists a uniformly convergent subsequence on every compact subset of $\mathbb{R}^n$ such that,

$$v_j \to v^*, \quad \nabla v_j \to \nabla v^*.$$

Note: $\|v_r\|_{L^2(\partial B_1)} = 1$, implies that the blow-up is nontrivial. Moreover,

$$[u(rx)]_y = ru_y(rx) = \frac{r}{\epsilon}u_-(rx).$$

Letting $r \to 0$, we find that v^* satisfies,

$$\begin{cases} \Delta v^* = 0 & \text{in } B_1^+. \\ v_y^* = 0 & \text{on } B_r'. \end{cases} \tag{5.5}$$

Also as $r_j \to 0$,

$$\Phi(r_j, u) = \Phi(1, v_j) \to \Phi(1, v^*) = \mu.$$

Hence v^* is homogenous of degree μ. We can evenly reflect v^* in the entire domain and consider the solution in B_1. Hence it follows from [5] that v^* is a quadratic polynomial. In particular $\mu = 2$. From Corollary 3 it follows that,

Theorem 20. *Let u be a solution to the penalized problem with $\nabla u(0) = 0$. Then $\forall x \in B_{r/2}$,*

$$|u(x)| \le r^2 \sup_{B_1} |u|.$$

Remark 8. *We point out that the analysis of the free boundary $\partial\{u > 0\}$ is carried out in recent work [1] for the general fractional problem. In that paper the author obtains monotonicity for a perturbed frequency functional and discusses the Hausdorff dimension of the singular set, defined to be the set of points where the gradient vanishes. It is left open as an interesting problem to study higher regularity of the free boundary at points of nonvanishing gradient.*

5.3 Preliminary Uniform Estimates $s = \frac{1}{2}$

It is the purpose of this section to study uniform estimates of minimizers to $E_\epsilon(u)$,

$$E_\epsilon(u) = \frac{1}{2}\int_D |y|^a |\nabla u|^2 \, dxdy + \frac{1}{2\epsilon}\int_\Sigma (u-\phi)_-^2 \, dx. \tag{5.6}$$

The functional $E_\epsilon(u)$ can be thought of as a family of functionals paramaterized by ϵ. From the point of view of the limiting obstacle problem, the penalizing term accounts for the obstacle constraint in the boundary obstacle problem. The idea is that the family of functionals contstructed in this way behave like $E(u)$ when $u \geq \phi$ and penalizes the function when $u < \phi$. The strength of the penalization increases as ϵ decreases.

We let u^ϵ deonte the solution to the penalized boundary obstacle problem. In particular assuming $\phi = 0$, $D = B_1(0)$, $a = 0$, $B_r^{'} = \mathbb{R}^{n-1} \cap B_r$, and given a function $\varphi \in C^{2,\alpha}\left(\overline{B_1}\right)$ strictly positive on $\partial B_1^+ \cap \{y = 0\}$, we consider the following penalized problem,

$$\begin{cases} \Delta u^\epsilon = 0 & \text{in } B_1^+, \\ u_y^\epsilon = \beta_\epsilon(u^\epsilon) & \text{on } B_r', \\ u^\epsilon = \varphi(x) & \text{on } (\partial B_1)^+, \end{cases} \tag{5.7}$$

where $(\partial B_1)^+$ denotes as before the set $\partial B_1^+ \setminus \{y = 0\}$. Motivated by the random homogenization problem, we define the following family of penalization functions:

Definition 2. *For $\epsilon > 0$, a family of functions $\beta_\epsilon(t)$ is an admissible penalization if it satisfies the following:*

1. $\forall \epsilon > 0$, $\beta_\epsilon(t)$ *is uniformly Lipschitz for* $-\infty < t < \infty$.

2. $\forall \epsilon > 0$, $\beta_\epsilon(t) \leq 0$.

3. $\forall \epsilon > 0$ *and* $\forall t \geq 0$, $\beta_\epsilon(t) = 0$.

4. $\beta_\epsilon'(t) \geq 0$.

5. $\beta_\epsilon''(t) \leq 0$.

Remark 9. *We point out a scaling property of the class of penalizing functions. If* $\beta_1(t)$ *satisfies the conditions of the definition, then* $\forall \epsilon > 0$, $\beta_\epsilon(t) = \beta_1(t/\epsilon)$ *is an admissible family of penalizations. In general if* $\beta_\epsilon(t)$ *is an element of an admissable family of penalizations, then the function* $\beta(t) = \beta_\epsilon(\sigma t)$ *is an element of the same admissable family corresponding to the parameter* $\frac{\epsilon}{\sigma}$. *We point out that a similar class of penalization functionals was considered in [53], [54].*

Without loss of generality we consider

$$\beta_\epsilon(t) = \begin{cases} \frac{t}{\epsilon} & t < 0. \\ 0 & t \geq 0. \end{cases} \tag{5.8}$$

We start by noting that we can perfom an even reflection in the y variable and consider the problem posed on the entire domain B_1, where u^ϵ is harmonic in the upper and lower half spaces and $u^\epsilon = \varphi$ on ∂B_1. When proving estimates it will suffice to consider only one of the half spaces. For covenience we study estimates in B_1^+.

Lemma 14. *Let* u^ϵ *be the solution to the penalized boundary obstacle problem. Then,*

$$\|u^\epsilon\|_{L^\infty(B_1)} \leq \|\varphi\|_{L^\infty(\partial B_1)}. \tag{5.9}$$

Proof. Since $u^\epsilon_y(x,0) = \beta_\epsilon(u^\epsilon) \le 0$, by the maximum principle it follows that,

$$\inf_{x\in B_1^+} u^\epsilon \ge \inf_{x\in\partial B_1^+} \varphi.$$

Suppose now by contradiction that,

$$\sup_{x\in B_1^+} u^\epsilon > \sup_{x\in\partial B_1^+} \varphi.$$

Then by the Hopf Lemma, this must be obtained at some point $(x_0, 0)$ where $u^\epsilon_y(x_0, 0) < 0$. But these are exactly the set of points where $u^\epsilon(x, 0) \le 0$. Since we are assuming that $\varphi(x, 0) > 0$ we have our desired contradiction. By reflection we obtain a similar estimate in B_1^-. □

The next result shows that the normal derivative is uniformly bounded.

Lemma 15. *Let u^ϵ be the solution to the penalized boundary obstacle problem. Then,*

$$\|u^\epsilon_y\|_{L^\infty(B_1)} \le C. \tag{5.10}$$

Proof. We consider the following auxillary problem,

$$\begin{cases} \Delta h = 0 & \text{in } \ B_1 \setminus \{y=0\}, \\ h = \min u^\epsilon & \text{in } \ B_1', \\ h = -M & \text{on } \ \partial B_1. \end{cases}$$

Here we let $-M < \inf_{x\in\partial B_1} \varphi$. Since $\Delta h = 0$ and $h = \inf u^\epsilon$ on B_1', we know by the comparison principle that $u^\epsilon \ge h$ everywhere. Furthermore at the minimum point $(x_0, 0)$ of u^ϵ on B_1', we know that,

$$u^\epsilon_y(x_0, 0) \ge h_y(x_0, 0).$$

From harmonic estimates it follows that for a universal constant C,

$$h_y(x_0, 0) \geq -C.$$

Moreover using the boundary condition $u^\epsilon_y = \frac{1}{\epsilon} u^\epsilon$, we see that,

$$u^\epsilon_y(x_0, 0) = \min_{B'_1} u^\epsilon_y(x, 0).$$

On the other hand $u^\epsilon_y = \beta_\epsilon(u^\epsilon) \leq 0$. Hence this proves that,

$$-C \leq u^\epsilon_y(x, 0) \leq 0 \ \ \forall x \in B'_1.$$

Finally, noting that $\Delta u^\epsilon_y = 0$ in the interior of the domain an application of the maximium princple propogates the estimate inside. That is,

$$\|u^\epsilon_y\|_{L^\infty(B_1)} \leq C.$$

□

Before proving the tangential semiconvexity of the solution we state and prove a result that restricts our penalization to the interior of the domain. More specifically, by the positivity of φ on $\partial B_1 \cap \{y = 0\}$, we know that there exists a neighborhood $\mathbf{N}(\partial B_1)$ of $\partial B_1 \cap \{y = 0\} \subset \partial B_1$ where $\varphi > 0$. Our next lemma helps us propogate this information into the interior.

Lemma 16. *$\exists \delta_0 > 0$, such that $\forall x \in B'_1 \setminus B'_{1-\delta_0}$, $u^\epsilon(x, 0) > 0$. In particular, in the annlular region $B'_1 \setminus B'_{1-\delta_0}$, $\beta_\epsilon(u^\epsilon) = 0$.*

Proof. Let $(x, y) \in \mathbf{N}(\partial B_1)$. Then by the uniform boundedness of u^ϵ_y we see that,

$$\varphi(x, y) - u^\epsilon(x, 0) = \int_0^y u^\epsilon_y(\bar{x}, s)\, ds \leq Cy.$$

Hence for $y \leq y_0$ where y_0 small enough,

$$0 < \varphi(x, y) - Cy \leq u^\epsilon(x, 0).$$

To conclude, we define

$$\delta_0 = distance\, \{\ \{(x, 0) \mid (x, y) \in \partial B_1 \text{ where } y > y_0\ \}, \partial B_1 \cap \{y = 0\}\ \} > 0. \tag{5.11}$$

□

Using the previous lemma we now prove that solutions are semi-convex in the tangential directions.

Lemma 17. *Let u^ϵ be the solution to the penalized boundary obstacle problem. Then for any direction τ parallel to $\mathbb{R}^{n-1}$,*

$$\inf_{B_{1-\delta_0}} u^\epsilon_{\tau\tau} \geq -C_0. \tag{5.12}$$

Here δ_0 is from the previous lemma, and C_0 is a constant independent of ϵ.

Proof. We consider the tangential second incremental quotients for our solution u^ϵ and u^ϵ_y at a point x. Specifically, $\forall \delta > 0$,

$$u^\epsilon_{\tau\tau,\delta}(x) = \frac{u^\epsilon(x + \delta\tau) + u^\epsilon(x - \delta\tau) - 2u^\epsilon(x)}{\delta^2}.$$

$$(u_y^\epsilon)_{\tau\tau,\delta}(x) = \frac{u_y^\epsilon(x+\delta\tau) + u_y^\epsilon(x-\delta\tau) - 2u_y^\epsilon(x)}{\delta^2}.$$

We point out that for every $x \in \{u_y^\epsilon = \frac{1}{\epsilon}u^\epsilon\}$ we have the inequality,

$$\frac{1}{\epsilon}u^\epsilon_{\tau\tau,\delta}(x) \geq (u_y^\epsilon)_{\tau\tau,\delta}(x). \tag{5.13}$$

This follows from the observation that for every $x \in \{u_y^\epsilon = \frac{1}{\epsilon}u^\epsilon\}$, $x \pm \delta\tau$ could lie outside of the set $\{u_y^\epsilon = \frac{1}{\epsilon}u^\epsilon\}$. Outside of this set $u^\epsilon > 0$ and $u_y^\epsilon = 0$. In particular we always have the inequality, $\frac{1}{\epsilon}u^\epsilon(x \pm \delta\tau) \geq u_y^\epsilon(x \pm \delta\tau)$. The following claim characterizes where $u^\epsilon_{\tau\tau,\delta}(x)$ achieves its minimum point.

Claim: Let $(x_0, 0) \in B_1'$ be such that

$$u^\epsilon_{\tau\tau,\delta}(x_0) = \min_{B_1'} \; u^\epsilon_{\tau\tau,\delta}(x).$$

Then, $x_0 \in \{u^\epsilon > 0\}$.

Proof. Suppose that the minimum point $(x_0, 0)$ for $u^\epsilon_{\tau\tau,\delta}(x)$ is not realized on the set $\{u^\epsilon > 0\}$. Then for some $x_0 \in \{u_y^\epsilon = \frac{1}{\epsilon}u^\epsilon\}$,

$$u^\epsilon_{\tau\tau,\delta}(x_0) = \min_{B_1'} u^\epsilon_{\tau\tau,\delta}(x).$$

Recalling (5.13) and using Hopf's Lemma we see that,

$$\frac{1}{\epsilon}u^\epsilon_{\tau\tau,\delta}(x_0) \geq (u_y^\epsilon)_{\tau\tau,\delta}(x_0) > 0.$$

In particular $u^\epsilon_{\tau\tau,\delta}$ cannot achieve a negative minimum on the set $\{u_y^\epsilon = \frac{1}{\epsilon}u^\epsilon\}$. This is our desired contradiction. Hence the minimum points of $u^\epsilon_{\tau\tau,\delta}(x)$ must be achieved on the set $\{u^\epsilon > 0\}$ as desired. $\square$

We observe that for every $x \in \{u^\epsilon > 0\}$,

$$\Delta(u^\epsilon_{\tau\tau,\delta}(x)) \leq 0. \tag{5.14}$$

Since $\forall x \in \{u^\epsilon > 0\}$, it follows that $\Delta u^\epsilon(x) = 0$, a direct computation shows

$$\begin{aligned}\Delta(u^\epsilon_{\tau\tau,\delta}(x)) &= \frac{\Delta u^\epsilon(x+\delta\tau) + \Delta u^\epsilon(x-\delta\tau)}{\delta^2} \\ &= \frac{(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial \nu})(x+\delta\tau)\mathcal{H}^n + (\frac{\partial u}{\partial y} + \frac{\partial u}{\partial \nu})(x-\delta\tau)\mathcal{H}^n}{\delta^2} \leq 0.\end{aligned}$$

Here $\mathcal{H}^n$ denotes the n-dimensional Hausdorff Measure and $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial y}$ is the outward pointing normal. Thus it follows that in distribution $u^\epsilon_{\tau\tau,\delta}(x_0)$ is superharmonic in B_1. In particular by the minimum principle for superharmonic functions we know that for some $x_1 \in \partial B_{1-\delta_0}$, and for δ_0 defined before (5.11),

$$u^\epsilon_{\tau\tau,\delta}(x_0) \geq \min_{\partial B_{1-\delta_0}} u^\epsilon_{\tau\tau,\delta}(x) = u^\epsilon_{\tau\tau,\delta}(x_1).$$

From standard harmonic estimates it follows that there exists a constant C_0 universal such that,

$$\|D^2 u^\epsilon\|_{L^\infty(B_{\frac{\delta_0}{2}}(x_1))} \leq C_0.$$

Shrinking the neighborhood slightly we find that $\forall\ 0 < \delta < \frac{\delta_0}{4}$,

$$\|u^\epsilon_{\tau\tau,\delta}\|_{L^\infty(B_{\frac{\delta_0}{4}}(x_1))} \leq C_0.$$

In particular $u^\epsilon_{\tau\tau,\delta}(x_1) \geq -C_0$. By the minimum principle for superharmonic functions we can propogate the estimate into the interior. That is,

$\forall\ 0 < \delta < \frac{\delta_0}{4}$ and $\forall x \in B_{1-\delta_0}$,

$$u^\epsilon_{\tau\tau,\delta}(x) \geq -C_0.$$

Finally letting $\delta \to 0$ we obtain our desired estimate,

$$u^\epsilon_{\tau\tau} \geq -C_0.$$

□

Remark 10. *The semi-convexity in the tangential direction implies for any tangential direction τ that $\|u^\epsilon_\tau\|_{L^\infty(B_{1-\delta_0})} \leq C$. Combining this fact with the previous L^∞ estimate for u^ϵ_y we know that our solution u^ϵ is uniformly Lipschitz continuous in $B_{1-\delta_0}$. In particular $\|\nabla u^\epsilon\|_{L^\infty(B_{1-\delta_0})} \leq C$.*

Remark 11. *We observe that, semi-convexity in the tangential directions implies by the equation semi-concavity in the y-direction. In particular,*

$$0 \geq \Delta u^\epsilon = \sum_{i=1}^{n-1} u^\epsilon_{\tau\tau} + u^\epsilon_{yy} \geq -(n-1)C_0 + u^\epsilon_{yy}$$

So in particular,

$$\sup_{B^+_{1-\delta_0}} u^\epsilon_{yy} \leq (n-1)C_0 \tag{5.15}$$

We conclude this section by stating a corollary that follows directly from the previous lemma. In particular we point out that we already have some control on the solution u^ϵ from above.

Corollary 4. *Let u^ϵ be a solution to the penalized boundary obstacle problem. Then for some universal constant C in $B^+_{1-\delta_0}$,*

(a) $u^\epsilon(x',y) - Cy^2$ is concave in y, and $u^\epsilon(x',y) + |x'|^2$ is convex in x'.

(b) $u^\epsilon_y(x',t) - u^\epsilon_y(x',s) \leq C(t-s)$. $(t > s)$

(c) $u^\epsilon(x',t) - u^\epsilon(x',0) \leq Ct^2$.

(d) If $u^\epsilon(x,t) \geq h$ then in the half ball

$$HB'_\rho = \{z \; : \; |z-x| \leq \rho, \;\; \langle z-x, \nabla_x u^\epsilon\rangle \geq 0\},$$

$$u^\epsilon(z,t) \geq h - C\rho^2.$$

Proof. a) This is a restatatement of the semiconvexity estimate in the tangential directions and the semiconcavity estimate in the normal direction.

b) We have that $u^\epsilon_y - 2Cy$ is decreasing therefore,

$$u^\epsilon_y(x,t) - 2Ct \leq u^\epsilon_y(x,s) - 2Cs.$$

c) Integrating the inequality $u^\epsilon_y \leq Cs$ from 0 to t gives the desired inequality.

d) From convexity we know,

$$u^\epsilon(z,t) + C|z|^2 \geq u^\epsilon(x,t) + C|x|^2 + \langle z-x, \nabla_x u(x,t) + 2Cx\rangle.$$

Hence, if $u^\epsilon(x,t) \geq h$ in $HB'_\rho(x)$, then,

$$u^\epsilon(z,t) \geq u^\epsilon(x,t) - C|x-z|^2 \geq h - C\rho^2.$$

□

5.4 Uniform $C^{1,\alpha}$ Growth from the $\epsilon-$Level Set

In this section we prove an estimate of technical interest. We show that u_y^ϵ has a uniform $C^{1,\alpha}$ growth from the set $\partial\{u^\epsilon > 0\}$. We point out that our argument is very close to the argument presented in ([27]). We repeat the main steps of the argument to check that all estimates are independent of ϵ. The idea is to use the semi-concavity estimate and an iteration argument to obtain the desired Hölder growth.

Lemma 18. *Let u^ϵ be our solution in Γ_1 and let $0 \in \partial\{u^\epsilon > 0\}$. Then there exists two constants $K_1 > 0$ and $\mu \in (0,1)$ such that*

$$\inf_{\Gamma_{4^{-k}}} u_y^\epsilon \geq -K_1 \mu^k.$$

Proof. We proceed to prove the lemma using mathematical induction.

Case k =1: The base case follows from the uniform estimates obtained previously on u_y^ϵ (Lemma 15).

Induction Step: Assume, for some constants $K_1 > 0$ and $\mu \in (0,1)$ to be chosen later, the result is true for some $k = k_0$, i.e.,

$$\inf_{\Gamma_{4^{-k_0}}} u_y^\lambda \geq -K_1 \mu^{k_0}. \tag{5.16}$$

We start by renormalizing the solution inside Γ_1. We define,

$$\bar{u}^\lambda(x,y) = \frac{1}{K_1}\left(\frac{4}{\mu}\right)^{k_0} u^\lambda\left(\frac{x}{4^{k_0}}, \frac{y}{4^{k_0}}\right). \tag{5.17}$$

We obtain the following scaled estimates,

$$\begin{aligned} &(i) \;\; \inf_{\Gamma_1} \bar{u}_y^\lambda \geq -1. \\ &(ii) \;\; \bar{u}_{yy}^\lambda \leq \frac{(n-1)C_0}{K_1(4\mu)^{k_0}}. \end{aligned} \tag{5.18}$$

Recall that $(n-1)C_0$ is the semi-concavity estimate found before (5.15). Fix $L = \bar{C}C_0$ for $\bar{C} >> 1$ and define

$$w^\epsilon(x,y) = \bar{u}^\epsilon(x,y) - \frac{L}{K_1(4\mu)^{k_0}}\left[|x|^2 - (n-1)y^2\right]. \qquad (5.19)$$

We make the following observations about $w^\epsilon(x,y)$:

1. w^ϵ is harmonic in the interior of $\Gamma_{1/2}$.
2. w^ϵ is strictly negative on the set $\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}$.
3. w^ϵ approaches 0 at the origin.
4. By Hopf's Lemma w^ϵ obtains its non-negative maximum on $\partial[\Gamma_{1/2} \setminus \{y = 0\}]$.

We consider two distinct cases:

Case 1: The maximum of w^ϵ is attained on $\partial\Gamma_{1/2} \cap \{y = \frac{1}{2\sqrt{2n}}\}$.

This implies that there exists $x_0 \in B'_{1/2}$ such that

$$\bar{u}^\epsilon\left(x_0, \frac{1}{2\sqrt{2n}}\right) \geq -C\frac{L}{K_1(4\mu)^{k_0}}.$$

Using part (d) of Corollary 1, we observe that there exists an (n-1) dimensional half ball $HB'_{1/2}\left(x_0, \frac{1}{2\sqrt{2n}}\right)$ such that

$$\bar{u}^\epsilon\left(x, \frac{1}{2\sqrt{2n}}\right) \geq -\frac{C}{2}\frac{L}{K_1(4\mu)^{k_0}} \quad \forall x \in HB'_{1/2}\left(x_0, \frac{1}{2\sqrt{2n}}\right).$$

Recalling the definition of the penalization, $\beta_\epsilon(u^\epsilon)$, and by the semi-concavity estimate for $\bar{u}^\epsilon$ (5.18), a Taylor expansion on the set $\{u^\epsilon_y = \frac{1}{\epsilon}u^\epsilon\}$, gives us the following inequality,

$$u^{\epsilon}(x,y) \leq u^{\lambda}(x,y) - u^{\epsilon}(x,0) \leq u^{\epsilon}_y(x,0) \cdot y + (n-1)C_0 y^2. \tag{5.20}$$

Moreover we obtain the following estimate,

$$\bar{u}^{\epsilon}_y(x,0) \geq -C\frac{L}{K_1(4\mu)^{k_0}} \quad \forall x \in HB'_{1/2}(x_0,0). \tag{5.21}$$

Case 2: The maximum is attained on $\partial\Gamma_{1/2} \setminus \{y = \frac{1}{2\sqrt{2n}}\}$.

Let (x'_0, y'_0) be the maximum point. Since this point is on the lateral side of the cylinder, we have that $|x'_0|^2 \geq 2(n-1)|y_0|^2$. This provides for us the following estimate,

$$\bar{u}^{\epsilon}(x'_0, y'_0) \geq \frac{L}{K_1(4\mu)^{k_0}}.$$

As before we know that there exists an (n-1) dimensional half ball $HB'_{1/2}(x'_0, y'_0)$ such that,

$$\bar{u}^{\epsilon}(x, y'_0) \geq \frac{L}{2K_1(4\mu)^{k_0}} \quad \forall x \in HB'_{1/2}(x'_0, y'_0).$$

Again recalling the definition of the penalization $\beta_{\epsilon}(u^{\epsilon})$, and using (5.20) we obtain,

$$\bar{u}^{\epsilon}_y(x,0) \geq 0 \quad \forall x \in HB'_{1/2}(x'_0, 0). \tag{5.22}$$

Thus in both cases we reach the conclusion that there exists $C_1 > 0$ and a point $\bar{x} \in B'_{1/2}$ such that

$$\bar{u}^{\epsilon}_y(x,0) \geq -C_1\frac{L}{K_1(4\mu)^{k_0}} \quad \forall x \in HB'_{1/2}(\bar{x},0).$$

Furthermore if we choose K_1 and μ satisfying, $K_1 > 2C_1$ and $\mu \geq \frac{1}{4}$ then we have

$$\bar{u}_y^\epsilon(x,0) > -\frac{1}{2}.$$

Recalling again (5.18), we use the following result for harmonic functions which is a consequence of the Poisson Representation Formula. It gives us pointwise information from a measure estimate.

Remark 12. *Let $v \leq 0$, $\Delta v = 0$ in $B_1'(x_0) \times (0,1)$, and continuous in $B_1'(x_0) \times [0,1]$. Assume $v(x,0) \geq -1/2$ in $B_\delta'(x^*)$ for some $B_\delta(x^*) \subset B_1(x_0)$. Then,*

$$v(x,y) \geq -\eta(\delta) \ \ \textit{in } \overline{B_{1/2}'(x_0)} \times [1/4, 3/4]. \tag{5.23}$$

Using (5.23) we obtain the existence of a constant $\eta < 1$ such that,

$$\bar{u}_y^\epsilon\left(x, \frac{1}{4\sqrt{2n}}\right) > -\eta \quad \forall x \in B_{1/4}'.$$

Once more applying the semi-concavity estimate we obtain for a K_1 sufficiently large,

$$\bar{u}_y^\epsilon(x,y) > -\eta - \frac{(n-1)C_0}{K_1(4\mu)^{k_0}} =: -\mu > -1.$$

Rescaling back we find for $k = k_0 + 1$ our desired inequality,

$$\inf_{\Gamma_{4-k}} u_y^\epsilon \geq -K_1\mu^k.$$

$\square$

Remark 13. *We observe that the conclusion of this lemma implies the existence of an* $0 < \alpha < 1$ *such that* $\forall r \leq 1 - \delta_0$,

$$\sup_{\Gamma_r} |u_y^\epsilon| \leq C r^\alpha. \tag{5.24}$$

In particular for $x \in \{u_y^\epsilon = \frac{1}{\epsilon} u^\epsilon\}$,

$$u_y^\epsilon(x) \leq d^\alpha(\partial\{u^\epsilon > 0\}). \tag{5.25}$$

$$u_y^\epsilon(\epsilon x) \leq \epsilon^\alpha d^\alpha(\partial\{u^\epsilon > 0\}). \tag{5.26}$$

5.5 Uniform $C^{1,1/2}$ Estimate for Global Penalized Solutions

In this section we restrict our attention to global solutions of the penalized boundary obstacle problem. More specifically we are interested in solutions that are tangentially convex, i.e. $u_{\tau\tau}^\epsilon \geq 0$. We remark that this implies that the set $\{(x,0) : u_y^\epsilon(x,0) < 0\}$ is a convex set. We will prove the uniform $C^{1,1/2}$ estimate for this class of solutions. Our result relies on a monotonicity formula and the first eigenvalue of the following problem,

Theorem 21. *Let* ∇_θ *denote the surface gradient on the unit sphere* ∂B_1. *Consider,*

$$\lambda_0 = \inf_{\substack{w \in H^{1/2}(\partial B_1^+) \\ w=0 \text{ on } (\partial B_1'(x,0))^-}} \frac{\int_{\partial B_1^+} |\nabla_\theta w|^2 \, dS}{\int_{\partial B_1^+} |w|^2 \, dS},$$

where $\partial B_1'(x,0)^- = \{x' = (x'', x_{n-1}) \in B_1' \mid x_{n-1} < 0\}$. *Then,*

$$\lambda_0 = \frac{2n-1}{4}. \tag{5.27}$$

We do not prove this theorem here but refer to [4] where it is proven in detail. We turn our attention instead to proving a monotonicity result which is crucial in the sequel to prove the sharp estimate.

Lemma 19. *Let* w *be any continuous function on* $\overline{B_r^+}$ *with the following properties:*

1. $\Delta w = 0$ *in* B_r^+.

2. $w(0) = 0$.

3. $w(x,0) \leq 0$ *and* $w(x,0)w_\nu(x,0) \leq 0$ $\forall x \in B_r'$.

4. $\{x \in B_r' \mid w(x,0) < 0\}$ *is nonempty and convex.*

Define

$$\varphi(r) = \frac{1}{r}\int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-1}}. \tag{5.28}$$

Then $\forall r \in (0,R)$,

(i) $\varphi(r) < +\infty$

(ii) $\varphi(r)$ *is monotone increasing in* r.

Proof. Harmonicity of w in the interior gives to us the following identity,

$$\Delta w^2 = 2w\Delta w + 2|\nabla w|^2 = 2|\nabla w|^2.$$

This allows us to rewrite the integrand as,

$$\varphi(r) = \frac{1}{2r}\int_{B_r^+} \frac{\Delta w^2}{|x|^{n-1}}.$$

It will be sufficient to prove the monotonicity of $\varphi(r)$ since $\varphi(1) < +\infty$. Differentiating $\varphi(r)$ we obtain,

$$\varphi'(r) = \frac{-1}{2r^2}\int_{B_r^+} \frac{\Delta w^2}{|x|^{n-1}} + \frac{1}{r^n}\int_{\partial B_r^+} |\nabla w|^2. \qquad (*)$$

Expanding out the first term gives us,

$$\frac{1}{2r^2}\int_{B_r^+} \frac{\Delta w^2}{|x|^{n-1}} = \frac{1}{r^{n+1}}\int_{(\partial B_r)^+} w w_\nu + \frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds$$
$$-\frac{1}{2r^2}\int_{B_r^+} \nabla w^2 \cdot \nabla\left(\frac{1}{|x|^{n-1}}\right) ds.$$

Recalling that $w(0) = 0$, the last term in this expansion can be further expanded to get,

$$-\frac{1}{2r^2}\int_{B_r^+} \nabla w^2 \cdot \nabla\left(\frac{1}{|x|^{n-1}}\right) ds = \frac{n-1}{2r^{n+2}}\int_{(\partial B_r)^+} w^2 ds$$
$$-\frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} w^2 \left(\frac{1}{|x|^{n-1}}\right) \cdot \nu \, ds.$$

We observe that the second term in this expansion is zero, hence we obtain,

$$-\frac{1}{2r^2}\int_{B_r^+} \nabla w^2 \cdot \nabla\left(\frac{1}{|x|^{n-1}}\right) ds = \frac{n-1}{2r^{n+2}}\int_{(\partial B_r)^+} w^2 ds.$$

Putting the above together we obtain,

$$\frac{1}{2r^2}\int_{B_r^+} \frac{\Delta w^2}{|x|^{n-1}} = \frac{1}{r^{n+1}}\int_{(\partial B_r)^+} w w_\nu + \frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds$$

$$+\frac{n-1}{2r^{n+2}}\int_{(\partial B_r)^+} w^2 ds.$$

An application of Cauchy-Schwarz to the first term allows us to continue the inequality,

$$\leq \left(\frac{1}{2r^{n+2}}\int_{(\partial B_r)^+} w^2 ds\right)^{1/2}\left(\frac{2}{r^n}\int_{(\partial B_r)^+} w_\nu^2 ds\right)^{1/2}$$
$$+\frac{n-1}{2r^{n+2}}\int_{(\partial B_r)^+} w^2 ds + \frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds.$$

Moreover the positivity of the integrands allows us to integrate over the larger spatial domain ∂B_r^+. In particular we have,

$$\leq \left(\frac{1}{2r^{n+2}}\int_{\partial B_r^+} w^2 ds\right)^{1/2}\left(\frac{2}{r^n}\int_{\partial B_r^+} w_\nu^2 ds\right)^{1/2} + \frac{n-1}{2r^{n+2}}\int_{\partial B_r^+} w^2 ds$$
$$+\frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds.$$

Rewriting the spatial gradient in terms of the surface gradient we obtain,

$$\int_{\partial B_r^+} |\nabla w|^2 = \int_{\partial B_r^+} |\nabla_\theta w|^2 + \int_{\partial B_r^+} w_\nu^2.$$

Putting this back into (*) we obtain,

$$\varphi'(r) \geq -\frac{2n-1}{4r^{n+2}}\int_{\partial B_r^+} w^2 ds - \frac{1}{r^n}\int_{\partial B_r^+} w_\nu^2 ds - \frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds$$
$$+\frac{1}{r^n}\int_{\partial B_r^+} |\nabla_\theta w|^2 + \frac{1}{r^n}\int_{\partial B_r^+} w_\nu^2.$$

After cancellation we are reduced to,

$$\varphi'(r) \geq -\frac{2n-1}{4r^{n+2}}\int_{\partial B_r^+} w^2 ds + \frac{1}{r^n}\int_{\partial B_r^+} |\nabla_\theta w|^2 - \frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds.$$

Since we are assuming $\{x \in B'_r \mid w(x,0) < 0\}$ is nonempty and convex, this implies that w vanishes on at least $(\partial B'_1)^-$, and hence is admissable to the eigenvalue problem (Theorem 21). This implies in particular that

$$\frac{\int_{\partial B_1^+} |\nabla_\theta w|^2 \, dS}{\int_{\partial B_1^+} |w|^2 \, dS} \geq \frac{2n-1}{4}.$$

We are thus reduced to studying the positivity of the corrective term,

$$\varphi'(r) \geq -\frac{1}{2r^2} \int_{\{y=0\} \cap \overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds.$$

Finally using the assumption that $w(x,0)w_\nu(x,0) \leq 0$ implies that,

$$-\frac{1}{2r^2} \int_{\{y=0\} \cap \bar{B}_r} \frac{2ww_\nu}{|x|^{n-1}} ds \geq 0.$$

Thus we conclude,

$$\varphi'(r) \geq 0 \text{ for any } 0 < r \leq R.$$

In particular we have shown,

$$\varphi(r) \leq \varphi(R) \text{ for any } 0 < r \leq R.$$

□

We now use the monotonicity of $\varphi(r)$ to conclude the sharp estimate for global solutions to the penalized boundary obstacle problem.

Theorem 22. *Let u^ϵ be a global solution to the penalized boundary obstacle problem. Then there exists a modulus of continuity $\omega : (0, \infty) \to (0, \infty)$*

independent of ϵ*, such that* $\omega(\delta) = O(\delta^{1/2})$ *as* $\delta \to 0$ *and* $\forall x, y \in B_{r/2}$ *and* $\forall \epsilon > 0$*,*

$$|u_y^\epsilon(x) - u_y^\epsilon(y)| \leq |x - y|^{1/2}. \tag{5.29}$$

Proof. We begin by setting $w = u_y^\epsilon$. Observe that w satisfies the assumptions of the previous lemma. We thus obtain,

$$\frac{1}{r^n} \int_{B_r^+} |\nabla w|^2 \leq \frac{1}{r} \int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-1}} \leq \varphi(1/2).$$

Since w vanishes on half of the ball in B_r', the Poincare Inequality implies that,

$$\fint_{B_r^+} w^2 \leq Cr^2 \fint_{B_r^+} |\nabla w|^2 \leq C_0 r.$$

Moreover since w^2 is subharmonic across $\{y = 0\}$ an application of the mean value theorem produces the estimate,

$$w^2|_{B_{r/2}^+} \leq \fint_{B_r^+} w^2 \leq Cr.$$

In particular we have obtained,

$$\sup_{B_{r/2}} |u_y^\epsilon| \leq Cr^{1/2}.$$

Since $u_y^\epsilon = 0$ in the region $\{u^\epsilon > 0\}$ and we have proved uniform $C^{1/2}$ estimates for u_y^ϵ on $\partial\{u^\epsilon > 0\}$, it is sufficient by standard regularity theory to prove the estimate when approaching $\partial\{u^\epsilon > 0\}$ from inside the set $\{u_y^\epsilon = \frac{1}{\epsilon} u^\epsilon\}$. We let $d_F(x)$ denote the distance of x to $\partial\{u^\epsilon > 0\}$, and $d(x, y)$ denote the distance between two arbitrary points x and y. We start by fixing two points, x and $y \in \{u_y^\epsilon = \frac{1}{\epsilon}u^\epsilon\}$. We consider two distinct cases.

Case 1:

$$\bar{d}_F := \max\{d_F(x), d_F(y)\} \leq 4d\,(x, y)\,.$$

Let us set $\bar{x}, \bar{y} \in \partial\{u^\epsilon > 0\}$ such that $|x-\bar{x}| = d_F(x)$ and $|y-\bar{y}| = d_F(y)$. Then we have the following estimate,

$$|u_y^\epsilon(x) - u_y^\epsilon(y)| \leq \sup_{\overline{B^+}_{4|x-y|}(\bar{x})} |u_y^\epsilon| + \sup_{\overline{B^+}_{4|x-y|}(\bar{y})} |u_y^\epsilon| \leq C|x-y|^{1/2}.$$

Case 2:

$$\bar{d}_F := \max\{d_F(x), d_F(y)\} \geq 4d\,(x, y)\,.$$

In this case we consider two interior points that are far from the interface. It is shown above that, $u_y^\epsilon(x) \leq C d_F^{1/2}(x)$. Define the following function,

$$v^\epsilon(x) = \frac{1}{\epsilon^{3/2}} u^\epsilon(\epsilon x). \tag{5.30}$$

We point out that v^ϵ and v_y^ϵ are of the same order. In particular,

$$v_y^\epsilon(x) = \frac{1}{\epsilon^{1/2}} u_y^\epsilon(\epsilon x) = \frac{1}{\epsilon^{1/2}} \cdot \frac{1}{\epsilon} u^\epsilon(\epsilon x) = \frac{1}{\epsilon^{3/2}} u^\epsilon(\epsilon x) = v^\epsilon(x).$$

Moreover we know from (22) that,

$$u_y^\epsilon(\epsilon x) \leq C\epsilon^{1/2} d_F^{1/2}(x).$$

This provides for us the following estimate,

$$v^\epsilon(x) = v_y^\epsilon(x) \le \frac{1}{\epsilon^{1/2}} \cdot C\epsilon^{1/2} d_F^{1/2}(x) = C d_F^{1/2}(x). \tag{5.31}$$

We consider interior estimates for boundary value problems with the Robin boundary condition, $v_y^\epsilon(x) = v^\epsilon(x)$. Since v^ϵ is of lower order, we inherit the Hölder regularity estimate for the Dirichlet problem. In particular we have the following estimate for a constant C independent of ϵ,

$$\|v_y^\epsilon\|_{C^{1/2}(B_{R/2}(x))} \le \frac{C}{R^{1/2}} \|v^\epsilon\|_{L^\infty(B_R(x))}. \tag{5.32}$$

Fix $R = \frac{d_F(x)}{\epsilon}$. Plugging (5.31) into (5.32) we obtain,

$$\|v_y^\epsilon\|_{C^{1/2}(B_{R/2}(x/\epsilon))} \le \frac{C\epsilon^{1/2}}{d_F^{1/2}(x)} \|v^\epsilon\|_{L^\infty(B_R(x/\epsilon))} \le \frac{C\epsilon^{1/2}}{d_F^{1/2}(x)} \cdot \frac{d_F^{1/2}(x)}{\epsilon^{1/2}} = C. \tag{5.33}$$

Applying the estimate obtained in (5.33), it follows from (5.30),

$$\begin{aligned} |u_y^\epsilon(x) - u_y^\epsilon(y)| &= |\epsilon^{1/2} v_y^\epsilon(x/\epsilon) - \epsilon^{1/2} v_y^\epsilon(y/\epsilon)| \\ &= \epsilon^{1/2} |v_y^\epsilon(x/\epsilon) - v_y^\epsilon(y/\epsilon)| \\ &\le C\epsilon^{1/2} |\frac{x}{\epsilon} - \frac{y}{\epsilon}|^{1/2} \\ &= C|x-y|^{1/2}. \end{aligned}$$

Our desired estimate.

□

5.6 Uniform $C^{1,1/2}$ Estimate for General Penalized Solutions

In this section we prove the sharp estimate for general solutions to the penalized boundary obstacle problem. First we prove a lemma that quantifies the fact that general solutions are tangentially almost convex. The proof is identical to the one presented in [27]. We present it here for completeness.

Lemma 20. *Let $C > 0$ and $\alpha \in (0, 1/2]$ be as in Remark 12. Let C_0 be the semiconvexity constant (5.12). Set $\delta_\alpha = \frac{1}{4}(\frac{\alpha}{\alpha+1} - \frac{\alpha}{2})$. Then there exists $r_0 = r_0(\alpha, C, C_0) > 0$ such that the convex hull of the set $\{x \in B_r' \,:\, u_y^\epsilon < -r^{\alpha+\delta_\alpha}\}$ does not contain the origin for $r \leq r_0$.*

Proof. Consider $(x', 0) \in \{u_y^\epsilon < -r^{\alpha+\delta_\alpha}\}$. Utilizing (16) we obtain,

$$u^\epsilon(x', h) \leq -r^{\alpha+\delta_\alpha} h + \frac{(n-1)C_0}{2} h^2.$$

Recalling the $C^{1,\alpha}$ estimate for u^ϵ we also know,

$$u^\epsilon(0, h) = u^\epsilon(0, h) - u^\epsilon(0, 0) \geq -Ch^{1+\alpha}.$$

Assume by contradiction that the convex hull of the set $\{x \in B_r' : u_y^\epsilon < -r^{\alpha+\delta_\alpha}\}$ contains the origin. We know from the semi-convexity estimate that $\forall x \in \{u_y^\epsilon < -r^{\alpha+\delta_\alpha}\}$,

$$u^\epsilon(0, h) \leq u^\epsilon(x, h) + C_0 h^2.$$

Combining the previous three estimates we see that for all $r, h \in (0, 1)$,

$$Ch^{1+\alpha} \geq r^{\alpha+\delta_\alpha} h - \frac{(n-1)C_0}{2} h^2 - C_0 h^2.$$

To contradict this inequality we choose $h = h(r)$ in such a way that for r sufficiently small,

$$h^2 << r^2 << h^{1+\alpha} << r^{\alpha+\delta_\alpha} h.$$

We set $h = r^{1+2\delta_\alpha/\alpha}$ and $\delta_\alpha < \frac{1}{2}(\frac{\alpha}{\alpha+1} - \frac{\alpha}{2})$. This is our desired contradiction. □

We now study the monotonicity formula as applied to general solutions.

Lemma 21. *Let $\delta_\alpha > 0$ be as in the previous lemma and u^ϵ the solution to the penalized boundary obstacle problem. Define $v^\epsilon = u^\epsilon + \frac{(n-1)C_0}{2}x^2 - \frac{(n-1)C_0}{2}y^2$ where $(n-1)C_0$ is the semiconcavity constant of u^ϵ. Furthermore set $w = v^\epsilon_y$ and $\varphi(r)$ as before. Then there exists a universal constant C such that ,*
(i) $2\alpha + \delta_\alpha > 1 \implies \varphi(r) \leq C$.
(ii) $2\alpha + \delta_\alpha < 1 \implies \varphi(r) \leq Cr^{2\alpha+\delta_\alpha-1}$.

Proof. Since $\Delta w = 0$ in the interior we can proceed as before to obtain the identity,

$$\Delta w^2 = 2w\Delta w + 2|\nabla w|^2 = 2|\nabla w|^2.$$

Differentiating φ we obtain as before,

$$\varphi'(r) \geq -\frac{2n-1}{4r^{n+2}}\int_{\partial B_r^+} w^2 ds + \frac{1}{r^n}\int_{\partial B_r^+} |\nabla_\theta w|^2 - \frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}}ds.$$

We first consider the corrective term,

$$-\frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}}ds.$$

Notice that for our choice of w,

1. $w|_{\{y=0\}} = u^\epsilon_y(x,0) \leq 0$.
2. $w_\nu = -(u^\epsilon_y)_y = -(u^\epsilon_{yy} - (n-1)C_0) \geq 0$.

In particular,

$$w(x,0)w_\nu(x,0) \leq 0.$$

This implies as before,

$$-\frac{1}{2r^2}\int_{\{y=0\}\cap\overline{B_r}} \frac{2ww_\nu}{|x|^{n-1}} ds \geq 0.$$

Thus we can drop the corrective term and consider the following inequality,

$$\varphi'(r) \geq -\frac{2n-1}{4r^{n+2}}\int_{\partial B_r^+} w^2 ds + \frac{1}{r^n}\int_{\partial B_r^+} |\nabla_\theta w|^2.$$

To account for the semi-convexity we introduce the truncated function,

$$w_t = \begin{cases} w + r^{\alpha+\delta_\alpha} & w < r^{\alpha+\delta_\alpha}. \\ 0 & \text{otherwise.} \end{cases} \tag{5.34}$$

We make the following observations about w_t:

1. $|w_t| \leq |w| \leq Cr^\alpha + Cr \leq \bar{C}r^\alpha$.
2. $|w - w_t| \leq r^{\alpha+\delta}$.
3. $\int_{\partial B_r^+} |\nabla_\theta w_t|^2 \leq \int_{\partial B_r^+} |\nabla_\theta w|^2$.

Hence we have the following estimate,

$$\varphi'(r) \geq -\frac{2n-1}{4r^{n+2}}\int_{\partial B_r^+} [(w - w_t) + w_t]^2 ds + \frac{1}{r^n}\int_{\partial B_r^+} |\nabla_\theta w_t|^2.$$

Using the previous lemma we see that w_t is admissable for the eigenvalue problem (Theorem 21). Hence,

$$\varphi'(r) \geq -\frac{2n-1}{4r^{n+2}}\int_{\partial B_r^+} [(w - w_t)^2 + 2w_t(w - w_t)] ds.$$

Using the growth estimates for w_t we have in particular,

$$\varphi'(r) \geq -Cr^{2\alpha+\delta-2}.$$

After integrating the inequality we find,

$$\varphi(1) - \varphi(r) = \int_r^1 \varphi'(r) \geq \int_r^1 -Cr^{2\alpha+\delta-2} = \frac{-C}{2\alpha+\delta-1}[1 - r^{2\alpha+\delta-1}].$$

This implies in particular,

$$\varphi(r) \leq \varphi(1) + \frac{C}{2\alpha+\delta-1} - \frac{C}{2\alpha+\delta-1} r^{2\alpha+\delta-1}.$$

□

With this lemma in hand we can now state and prove our sharp estimate for the solution to the penalized boundary obstacle problem.

Theorem 23. *Let u^ϵ be a solution to the penalized boundary obstacle problem. Then there exists a modulus of continuity $\omega : (0, \infty) \to (0, \infty)$ independent of ϵ, such that $\omega(\delta) = O(\delta^{1/2})$ as $\delta \to 0$ and $\forall x, y \in B_{r/2}$ and $\forall \epsilon > 0$,*

$$|u_y^\epsilon(x) - u_y^\epsilon(y)| \leq |x - y|^{1/2}. \tag{5.35}$$

Proof. Let $w = v_y^\epsilon$ be defined as before and consider w_t as in the previous lemma. Since w_t vanishes on more than half the ball of B_r' we have by the Poincare Inequality,

$$\int_{B_r^+} w_t^2 \leq Cr^2 \int_{B_r^+} |\nabla w_t|^2.$$

In particular we produce the following estimate,

$$\frac{1}{r^{n+1}}\int_{B_r^+} w_t^2 \leq \frac{C}{r^{n-1}}\int_{B_r^+} |\nabla w_t|^2 \leq \frac{C}{r^{n-1}}\int_{B_r^+} |\nabla w|^2 \leq C\int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-1}} = C\varphi(r).$$

Moreover, since w_t^2 is subharmonic across $\{y=0\}$, for $s < r - |x|$ and any $|x| \leq r$,

$$w_t^2(x) \leq \frac{n}{\omega_n s^n}\int_{B_s(x)} w_t^2 \leq \frac{n}{\omega_n s^n}\int_{B_r} w_t^2$$

$$\leq C\left(\frac{r}{s}\right)^n \frac{n}{\omega_n r^n}\int_{B_r^+} w_t^2 \leq C\left(\frac{1}{s}\right)^n \varphi(r) r.$$

Now we consider separately the two distinct cases:

Case 1: $2\alpha + \delta_\alpha > 1$.

From the previous lemma this implies that $\varphi(r) \leq C$. Hence in particular,

$$w_t^2 \leq Cr.$$

We observe that,

$$\sup_{B_{r/2}^+} w \leq C[\sup_{B_{r/2}^+} w_t + r^{\alpha+\delta_\alpha}].$$

Thus we obtain,

$$w \leq w_t + r^{\alpha+\delta} \leq Cr^{1/2} + r^{\alpha+\delta_\alpha} \leq \bar{C} r^{1/2}.$$

Case 2: $2\alpha + \delta_\alpha < 1$.

From the previous lemma this implies that $\varphi(r) \leq Cr^{2\alpha+\delta_\alpha-1}$. Hence in particular,

$$w_t^2 \leq Cr^{2\alpha+\delta_\alpha}.$$

This produces for us the estimate,

$$w \leq w_t + r^{\alpha+\delta} \leq Cr^{\alpha+\frac{\delta}{2}} + r^{\alpha+\delta}$$

$$\leq Cr^{\alpha+\frac{\delta_\alpha}{2}}.$$

We observe that we have improved the estimate for w. Set $\alpha_1 = \alpha + \frac{\delta_\alpha}{2}$. If α_1 satisfies the assumption of Case 1, then we are done. If not then using the lemma again we obtain,

$$w \leq Cr^{\alpha+\frac{\delta_\alpha}{2}+\frac{\delta_\alpha}{2}}.$$

We observe that we can iterate this procedure a finite number of times, e.g. k times, until we get $\alpha_k + \frac{\delta_\alpha}{2} > \frac{1}{2}$. Hence after a finite number of iterations we are in Case 1.

Thus in both cases we conclude that,

$$w \leq Cr^{1/2}.$$

Recalling that $w = u_y^\epsilon - (n-1)C_0 y$, we find that,

$$u_y^\epsilon \leq (n-1)C_0 r + Cr^{1/2} \leq \bar{C}r^{1/2}$$

Hence in particular we obtain the uniform estimate,

$$\sup_{B_{r/2}} |u_y^\epsilon| \leq Cr^{1/2}.$$

To conclude we consider the distnct cases as before. □

Finally we remark that as before one can obtain a uniform decay rate in the penalizing paramter ϵ.

Corollary 5. *Let u^ϵ be a solution to the penalized boundary obstacle problem. Then $\forall \alpha < 1$,*

$$\|u^\epsilon\|_{C^{1,\alpha}} \leq C\epsilon^{-\alpha}.$$

Proof. As before we fix a penalizing family,

$$\beta_\epsilon(t) = \begin{cases} \frac{t}{\epsilon} & t < 0. \\ 0 & t \geq 0. \end{cases} \tag{5.36}$$

We consider the scaled function,

$$v^\epsilon(x) = \frac{1}{\epsilon} u^\epsilon(\epsilon x).$$

We note that

$$[v^\epsilon]_y(x) = v^\epsilon(x).$$

Hence we obtain for a constant C independent of ϵ, $\forall \alpha < 1$,

$$\|v^\epsilon\|_{C^{1,\alpha}} \leq C.$$

It follows for a directional derivative τ,

$$
\begin{aligned}
|u^\epsilon_\tau(x) - u^\epsilon_\tau(y)| &= |\epsilon v^\epsilon_\tau(\frac{x}{\epsilon}) - \epsilon v^\epsilon_\tau(\frac{y}{\epsilon})| \\
&= |v^\epsilon_\tau(\frac{x}{\epsilon}) - v^\epsilon_\tau(\frac{y}{\epsilon})| \\
&\leq C|\frac{x}{\epsilon} - \frac{y}{\epsilon}|^\alpha \\
&\leq C\epsilon^{-\alpha}|x-y|^\alpha.
\end{aligned}
$$

□

Chapter 6

Conclusion and Future Directions

This thesis was devoted to studying two classes of problems in the theory of free boundary problems of obstacle-type. In this last chapter we comment on some interesting directions of research left open and some problems the author aims to work on in the future.

The first question is regarding the measure of the set of free boundary points in the implicit constraint obstacle problem, namely the set Γ^d. It is natural to ask if this set is actually finite under the assumption that f is analytic. By projecting points to their closest free boundary point, we see that Γ^d would indeed be finite if one could rule out the scenario that the solution sticks to the obstacle in the negative direction. In a similar direction it will be interesting to understand the free boundary in $2d$ under weaker regularity assumptions on f. Here we have to undertake a finer analysis of the set of free boundary points Γ^d.

Another direction of interest is to study the fully nonlinear stochastic impulse control problem with more general fully nonlinear operators, such as Monge-Ampere type and other degenerate elliptic equations. The goal would be to prove a general modulus of semiconvexity estimate for the solution using properties of the fully nonlinear operator. In this direction it would also be interesting to study measure theoretic estimates for the symmetric difference of free boundaries arising in these problems.

Finally it is of interest to pursue regularity estimates for the level sets of singularly perturbed free boundary problems. In this direction it would be interesting to understand the regularity of level sets for solutions to the penalized fully nonlinear stochastic impulse control problem and the penalized boundary obstacle problem. We would like to show uniform convergence of the level sets to the corresponding free boundary.

Appendices

Appendix A

Elliptic Regularity

In this Appendix we collect some of the main results and tools from second order elliptic equations used in the main body of the work. We naturally consider the division between linear and nonlinear equations. We remark that estimates can be derived as apriori estimates under suitable regularity assumptions on the data, or found after a suitable existence theory for weak solutions. A good reference for a priori estimates is [34], and a nice reference for weak solutions theory is [18].

Linear Theory: We recall some a priori estimates for non-divergence form elliptic equations. We remark that such estimates are applicable to fully nonlinear equations under smoothness assumptions on the solution and operartor.

Suppose Ω is a bounded connected domain in $\mathbb{R}^n$. Consider the operator L in Ω.

$$Lu = a_{ij}(x)D_{ij}u + b(x)D_iu + c(x)u$$

for $u \in C^2 \cap C(\overline{\Omega})$. Assume a_{ij}, b_i, c are continuous and L is uniformly elliptic, i.e. for any $x \in \Omega$, and any $\xi \in \mathbb{R}^n$,

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2.$$

Theorem 24. *(Schauder Interior Estimates) Let $u \in C^{2,\alpha}$ be a solution of $Lu = f$. Assume in addition,*

$$\|a_{ij}\|_{C^{0,\alpha}(\Omega)},\ \|b_i\|_{C^{0,\alpha}(\Omega)},\ \|c\|_{C^{0,\alpha}(\Omega)} \leq \Lambda.$$

Then,

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C(n,\alpha,\lambda,\Lambda)\left(\|u\|_{C^0(\Omega)} + \|f\|_{C^{0,\alpha}(\Omega)}\right).$$

Theorem 25. *(Hopf Lemma) Let B be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B) \cap C(B \cup \{x_0\})$ satisfies $Lu \geq 0$ in B with $c(x) \leq 0$ in B. Assume in addition that $\forall x \in B$ and $u(x_0) \geq 0$,*

$$u(x) < u(x_0).$$

Then for each outward pointing direction ν at x_0, with $\nu \cdot n(x_0) > 0$ there holds,

$$\liminf_{t \to 0^+} \frac{1}{t}[u(x_0) - u(x_0 - t\nu)] > 0.$$

The following estimate does not depend on the smoothness or continuity of the coefficients,

Theorem 26. *(Krylov, Boundary Harnack) Let u solve*

$$\begin{cases} a_{ij}u_{ij} = f & \text{in } B_1^+, \\ u = 0 & \text{in } \partial B_1^+ \cap \{x_n = 0\}. \end{cases}$$

Then for $0 < r \leq 1$,

$$osc_{B_r^+}\left(\frac{u}{x_n}\right) \leq Cr^{\alpha}\left(osc_{B_1^+}\left(\frac{u}{x_n}\right) + \|f\|_{L^\infty}\right)$$

where the constants depend only on ellipticity and dimension.

Fully Nonlinear Theory: We recall some known fact for solutions to fully nonlinear equations of the form,

$$F(D^2u(x), x) = f(x),$$

where $x \in \Omega$, and u, f are functions defined in a bounded domain Ω of $\mathbb{R}^n$. Moreover $F(M, x)$ is a real valued function defined on $\mathbf{S} \times \Omega$, where $\mathbf{S}$ is the space of real $n \times x$ symmetric matrices. We assume that F is a uniformly elliptic operator.

Definition 3. *F is uniformly elliptic if there are two positive constants $\lambda \leq \Lambda$ such that for any $M \in \mathbf{S}$ and $x \in \Omega$,*

$$\lambda\|N\| \leq F(M+N, x) - F(M, x) \leq \Lambda\|N\| \quad \forall N \geq 0,$$

where $N \geq 0$, whenever N is a non-negatve symmetric matrix.

By a property for symmetric matricies,

Lemma 22. *F is uniformly elliptic if and only if $\forall M, N \in \mathbf{S}$, $\forall x \in \Omega$,*

$$F(M+N, x) \leq F(M, x) + \Lambda\|N^+\| - \lambda\|N^-\|.$$

We now define viscosity solutions.

Definition 4. *A continuous function u in Ω is a viscosity subsolution (resp. supersolution) in Ω when the following condition holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, and $u - \varphi$ has a local maximum at x_0, then*

$$F(D^2\varphi(x_0), x_0) \geq f(x_0),$$

with opposite signs for supersolutions.

We now recall some qualitative properties for viscosity solutions to fully nonlinear equations.

Theorem 27. *(ABP Estimate) Let u be a viscosity supersolution in B_d and f a continuous bounded function in B_d. Assume that u is continuous in $\bar{B}_d$ and $u \geq 0$ on ∂B_d. Then*

$$\sup_{B_d} u^- \leq Cd \left(\int_{B_d \cap \{u = \Gamma_u\}} (f^+)^n \right)^{1/n},$$

where Γ_u is the convex envelope of $-u^-$ in B_{2d} and C is a universal constant.

Lemma 23. *(L^ϵ Lemma) Suppose u is a viscosity supersolution in $Q_{4\sqrt{n}}$ and f satisfy:*

$$u \geq 0 \text{ in } Q_{4\sqrt{n}},$$

$$\inf_{Q_3} \leq 1,$$

$$\|f\|_{L^n} \leq \epsilon_0.$$

Then, for positive universal constants d and ϵ,

$$|\{u \geq t\} \cap Q_1| \leq dt^{-\epsilon} \quad \forall t > 0.$$

Theorem 28. *(Weak Harnack Inequality) Let u be a viscosity supersolution in Q_1 that satisfies $u \geq 0$ in Q_1, where f is continuous and bounded in Q_1. Then,*

$$\|u\|_{L^{\frac{\epsilon}{2}}(Q_{1/4})} \leq C(\inf_{Q_{\frac{1}{2}}} u + \|f\|_{L^n(Q_1)}).$$

Theorem 29. *(Harnack Inequality) Let u be a viscosity solution in Q_1 satisfying $u \geq 0$ in Q_1, and f continuous and bounded in Q_1. Then*

$$\sup_{Q_{\frac{1}{2}}} u \leq C(\inf_{Q_{\frac{1}{2}}} u + \|f\|_{L^n(Q_1)}),$$

where C is a universal constant.

Theorem 30. *Let u be a viscosity solution of $F(D^2u) = 0$ in $B_1(0)$. Then $u \in C^{1,\alpha}(\bar{B}_{\frac{1}{2}})$ and*

$$\|u\|_{C^{1,\alpha}(\bar{B}_{\frac{1}{2}})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $0 < \alpha < 1$ and C are universal constants.

Theorem 31. *(Evans-Krylov) Let u be a viscosity solution of a concave equation $F(D^2u) = 0$ in $B_1(0)$. Then $u \in C^{2,\alpha}(\bar{B}_{\frac{1}{2}})$ and*

$$\|u\|_{C^{2,\alpha}(\bar{B}_{\frac{1}{2}})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $0 < \alpha < 1$ and C are universal constants.

Appendix B

Properties of Semiconcave Functions

In this appendix we collect some facts about semiconcave functions that we need in the main body of this work. We start with a definition:

Definition 5. *(Semiconcave Functions) Let $S \subset \mathbb{R}^n$. A function $u : S \to \mathbb{R}^n$ is semiconcave if there exists a nondecreasing upper semi-continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that* $\lim_{\rho \to 0^+} \omega(\rho) = 0$ *and,*

$$\forall \lambda \in (0,1)\ \forall x, y \in S \text{ such that their segment is contained in S},$$

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leq \lambda(1-\lambda)|x-y|\omega(|x-y|).$$

Moreover we say u is semiconcave with linear modulus if $\omega(|x-y|) = k|x-y|$ for some constant k. We now state a general regularity estimate for semiconcave functions with a general modulus of semiconcavity.

Theorem 32. *A semiconcave function $u : S \to \mathbb{R}$ is locally Lipschitz continuous in the interior of S.*

We now introduce a notion of derivative for semiconcave functions:

Definition 6. *(Superdifferential)*

$$D^+u(x) = \{p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0\}.$$

Definition 7. *(Subdifferential)*

$$D^-u(x) = \{p \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0\}.$$

We now come to an important characterization of the superdifferential of a semiconcave function:

Lemma 24. *Let $u : A \to \mathbb{R}^n$ be semiconcave function with modulus ω and let $x \in A$. Then a vector $p \in \mathbb{R}^n$ belongs to $D^+u(x)$ if and only if*

$$u(y) - u(x) \leq \ \langle p, y - x \rangle + \|x - y\| \omega(\|x - y\|) \ \forall y \in A$$

We also state the following result,

Theorem 33. *If $u : A \to \mathbb{R}$ is both semiconcave and semicovex in A, then $u \in C^1(A)$. In addition, on each compact subset of A the modulus of continuity of Du is of the form $c_1\omega(c_2r)$, where ω is a modulus of semiconvexity and of semiconcavity for u, and $c_1, c_2 > 0$ are constants.*

As a corollary we obtain,

Corollary 6. *Let $A \subset \mathbb{R}^n$ be open convex and let $u : A \to \mathbb{R}$ be both semiconvex and semiconcave with a linear modulus and constant C. Then $u \in C^{1,1}(A)$ and the Lipschitz constant of Du is equal to C.*

Appendix C

Variational Inequalities

One can view variational inequalities as a natural generalization of the variational approach to solving Boundary Value Problems in Partial Differential Equations. Many problems in optimization theory and Partial Differential Equations can be solved as a variational inequality. The need to study variational inequalities in PDE arise from solving elliptic and parabolic type equations in a domain with sutiable constraints. For example, variational inequalities are a natural framework in which to study obstacle problems. The conceptual idea is that our solution is constrained to always lie above or below a prescribed obstacle in our domain. A general reference for variational inequalities is [42].

From a functional analytic point of view, variational inequalities can be thought of as a problem characterizing projections onto convex sets. In the theory of Boundary Value Problems one faces a similar characterization and finds that the method of orthogonal projections play an analogous role. In the context of variational inequalities one first proves a Brouwer Fixed Point theorem for a compact, convex set $\mathbb{K} \subseteq \mathbb{R}^n$. Key ingredients in the proof are the unique existence of a projection operator onto any closed convex set, and the Brouwer

Fixed Point Theorem for closed balls $\Sigma \subseteq \mathbb{R}^n$. The interesting point is that one can characterize this projection as a variational inequality:

Lemma 25. *Let $\mathbb{K}$ be a closed convex set of a Hilbert Space $(\ \mathbb{H}, (.)\ )$. Then $y = P_{\mathbb{K}}x$, the projection of x on $\mathbb{K} \iff \langle y, \eta - y\rangle \geq \langle x, \eta - y\rangle\ \forall \eta \in \mathbb{K}$.*

Let us now return to the obstacle problem:

Example 1. *(Obstacle Problem) Let $D \subseteq \mathbb{R}^n$ be a smooth bounded domain with boundary ∂D. Suppose we are also given an Obstacle, a function φ defined on $\bar{D} = D \cup \partial D$ and a smooth function f defined on ∂D. Define:*

$$\mathbb{K} = \{u \in H^1(D)\ :\ u|_{\partial D} = f(x) \text{ and } u \geq \varphi \text{ in } D\}$$

We know that $\mathbb{K}$ is a closed convex set. We look in particular for a unique $u_0 \in \mathbb{K}$ such that u_0 minimizes the Dirichlet Integral:

$$\int_D (\nabla u_0)^2 dx = \min_{v \in \mathbb{K}} \int_D (\nabla v)^2 dx$$

Finding a solution to the obstacle problem is equivalent to finding a unique solution u to following variational problem: $\forall v \in \mathbb{K}$,

$$\langle \nabla u, \nabla (v - u)\rangle \geq \langle f, v - u\rangle$$

A key difference between obstacle type problems and the classical boundary value problems is the existence of the set of points where our solution touches the obstacle. In particular one must also consider the coincidence set:

$$\Lambda = \{x \in D\ :\ u(x) = \varphi(x)\}$$

The boundary of the noncoincidence set is what is called the Free Boundary.

We now consider a general framework in which to understand variational inequalities and in particular variational inequalities admitting implicit constraints. We let E denote a Real Vector Space, and let $C_1 \subseteq C \subseteq E$. We define two functions,

$$\varphi : C_1 \times C \to (-\infty, +\infty] \ \text{ with } \ \varphi(u, .) \neq +\infty \ \ \forall u \in C_1. \tag{C.1}$$

$$f : C_1 \times C \times C \to (-\infty, +\infty) \ \text{ with } \ f(u, v, v) \leq 0 \ \ \forall u \in C_1 \text{ and } \forall v \in C. \tag{C.2}$$

Our problem is to find all vectors $u \in C_1$ such that for some given subset $C_0 \subseteq C_1$:

$$u \in C_0$$

$$\varphi(u, u) + f(u, u, w) \leq \varphi(u, w) \ \ \forall w \in C \tag{C.3}$$

The way one deals with this general problem is to break it up into two consecutive steps. In step one we fix our vector $u \in C_1$. Define:

$$\psi(w) = \varphi(u, w) \ \ \forall w \in C \tag{C.4}$$

$$g(v, w) = f(u, v, w) \ \ \forall v, w \in C \tag{C.5}$$

We look for all vectors $v \in E$ that solve the variational problem,

$$v \in C$$

$$\psi(v) + g(v, w) \leq \psi(w) \ \ \forall w \in C \tag{C.6}$$

We let $S(u)$ denote the set of all solutions to the variational problem. We define the selection map:

$$S : C_1 \to 2^C \tag{C.7}$$

The objective of the first step is to show that under suitable assumptions on our data: E, C, C_1, φ, and f, the selection map has suitably nice properties. In particular we must show that the map is non-empty $\forall u \in C_1$.

In the second step we aim to find all the fixed points of the selection map S belonging to some given set $C_0 \subseteq C_1$. We aim to find all vectors u solving:

$$u \in C_0$$

$$u \in S(u) \tag{C.8}$$

This in principle is the general framework we adopt to help us solve quasivariational inequalities. The idea is to find the correct assumptions on othe functions and sets above that will allow us to show existence and uniqueness for both variational and quasi-variational inequalities. Let us start by stating some results from the classical theory of variational inequalities and then generalize these results to the situation of implicit constraints.

Theorem 34. *Assume that C is a closed convex subset of a Hausdorff Topological Vector Space E, ψ is convex, lower semi-continuous not identically ∞ on E. Furthermore assume that $g(v,v) \leq 0 \ \forall v \in C$, and $g(v,.)$ is concave $\forall v \in C$. Furthermore assume $\forall w \in C \ g(.,w)$ is lower semi-continuous on E. Finally assume that $\exists B \subset E$ and $w_0 \in B \cap C$ such that $\psi(w_0) < \infty$ and $\psi(v) + g(v,w_0) > \psi(w_0) \ \forall v \in C \setminus B$. Then the set of all solutions v to the problem*

$$v \in C,$$

$$\psi(v) + g(v,w) \leq \psi(w) \ \ \forall w \in C,$$

is a non-empty compact subset of $B \cap C$.

As a corollary to this theorem we get our first existence and uniqueness theorem for Variational Inequalities:

Theorem 35. *(Lion-Stampacchia) Let V be a real Hilbert-Space, C a non-empty closed, convex subset of V, $a(.\ ,\ .)$ a coercive continuous bilinear form on V. Then for every continuous linear functional v' on V, there exists a unique vector u that satisfies,*

$$u \in C,$$

$$a(u, u-w) \leq \ \langle v', u-w \rangle \ \ \forall w \in C.$$

When one considers nonlinear variational inequalities the assumption of lower semi-continuity for g is too strong. Instead one can assume that g satisfies a monotonicity requirement as well as being hemicontinuous. Under these modified assumptions one can still prove an existence and uniqueness theorem similar to the theorem stated above. In the second step of our problem we have to show that the selection map has a fixed point in a given subset C_0. For this step of the problem one must also consider topological properties of the mappings and to allow C_0 to have a topology independent of the topology induced from the larger space E. In particular one may assume that C_0 is a locally convex topological space which has a continuous injection into C_1.

Theorem 36. *Let C, C_1, φ, and f be given as above. Define $\psi = \varphi(u, .)$ as before satisfying the assumptions in the previous theorem. Define $g = f(u, ., .)$ satisfying the assumptions in the previous theorem and is monotone and hemicontinuous. Assume furthermore that both satisfy the coercivity condition. Suppose also that C_0 satisfies topological condition mentioned above, $S(C_0) \subseteq C_0$, and the set of all pairs $(u, v) \in C_0 \times C_0$ with $v \in S(u)$ is closed. Then our initial problem admits a solution u.*

The proof for this theorem relies on the following formulation of the Kakutani Fixed Point Theorem:

Theorem 37. *(Kakutani Fixed Point Theorem) Let S be a non-empty, compact, and convex subset of a locally convex topological vector space. Let $\varphi : S \to 2^S$ be a set-valued function which is upper semi-continuous and if the* *image of the map is non-empty, compact and convex. Then φ has a fixed point.*

To be a bit more concrete, let us state how one goes about applying these results in the context of quasi-variational Inequalities. Under suitable monotonicity conditions, coercivity conditions, and semi-continuity conditions, one can use the above results to prove an existence result for quasi-variational inequalities. We obtain the following useful result:

Theorem 38. *(Quasivariational Inequlity Existence) Suppose that:*

$$a(v, w) \text{ is a coercive bilinear form on a Hilbert Space V.}$$

$$v' \text{ is a continuous linear functional on V.}$$

Let Q be a map that associates with each vector u of a convex closed subset C of V a non-empty convex closed subset Q(u) of V. Then there exists a solution to the following QVI:

$$u \in C \;\; u \in Q(u).$$

$$a(u, u - w) \leq \; \langle v', u - w \rangle \;\; \forall w \in Q(u).$$

For a general Banach Space X one can introduce a partial ordering $\leq$ that is induced by the closed positive cone,

$$\mathbf{P} = \{v \in X \mid v \geq 0\}.$$

We assume that X is a vector lattice under this ordering. We can then define the order dual X^*, the closed subspace of the dual space X' generated by the positive cone,

$$\mathbf{P'} = \{x' \in X' \mid \langle x', x\rangle \ \geq 0 \ \forall x \in \mathbf{P}\}.$$

In the context of linear second order elliptic Partial Differential Equations one considers our Banach Space X to be either $H^1(\Omega)$, $H_0^1(\Omega)$, or any closed subset V of $H^1(\Omega)$ such that $H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega)$. In the context of nonlinear second order elliptic Partial Differential Equations we consider the general Sobolev Spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$. On these Banach Spaces one considers the ordering: $u \leq v \iff u(x) \leq v(x) \ a.e. \ \forall x \in \Omega$. Using the order structure of Banach Spaces one can prove comparision theorems that help in the constructive proof of the existence and uniqueness of solutions to specific quasivariational inequalities. A general reference for quasi-variational inequalities is [46].

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