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Towards a Self-dual Geometric Langlands Program

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Towards a Self-dual Geometric Langlands Program

by

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2018

For Dakota Zipporah,

who abhors public displays of affection.

Acknowledgments

Winners of Academy Awards, after gaining possession of the coveted statue, are given a limited amount of time in which they traditionally attempt to thank as many people as possible before the dramatic orchestral soundtrack kicks in and they are escorted off the stage. I have neither time pressure nor a string section to motivate brevity, and so hope that the following comprises a fairly complete list without being so long as to be, as my people would say, "taking the piss".

I would first like to thank my advisors, David Ben-Zvi and Andrew Neitzke, who originally suggested this project and who have provided me throughout with useful guidance, advice, and criticism. Andy and David have different but complementary personalities and styles, and I have benefited from their separate and combined wisdom (and funding). They are also both good and kind people – I count myself fortunate to have been advised by them.

Next, I wish to thank Andrew Blumberg, Sean Keel, and David Nadler, for serving on my dissertation committee and for providing me with additional academic and career guidance. "Find someone you expect will have a similar experience to you on the job market and drink with them," is the best advice I never took.

The mathematics and physics faculty at the University of Texas at Austin have ensured that my six years at the University of Texas at Austin have been intellectually stimulating, and the mathematics administration and support staff make it possible behind the scenes. I especially wish to single out Dan Freed, with whom I have had many enlightening discussions, and whose clarity in thought and exposition is something I aspire to.

I also acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 “RNMS: Geometric Structures and Representation Varieties” (the GEAR Network).” Our time together was fleeting, but I remember you fondly.

The last of the academic acknowledgements go out to all of the people with whom I have discussed mathematics and physics during my graduate career. There are legitimately too many people in this group to name individually; a partial list consisting of those who I believe will recognise some direct influence on this thesis includes: Rustam Antia-Riedel, Chiara Damiolini, Tudor Dimofte, Ron Donagi, Sam Gunningham, Tom Mainiero, and Sebastian Schulz.

Now for some more personal thanks. My parents Carol and Martin, my older sister Alison, my brother-in-law James and my niece Evie: thank you for the lifetime of support – I may live on the other side of the world, but Australia will never stop being home. My in-laws Annie, Tudey, Karl, Doris, Peter, Brad, Rory, Django, and the extended Derryberry tribe who constitute approximately a quarter of Ohio: thank you for your support, and for accepting me into your family. And my Austin friends and family, past and present: Alice and Tom; Amelia, Brian, and Stephanie; Eleisha, Rustam, Sebastian and Bea; the Bochenkov/a clan (Ani, Katya and Marina); the Bonneu-Alvarez family (Esther, Sebastien, Margaux and Benjamin); the Munguia tribe (headed by the estimable Aunt Michel).

Some papers contain a line acknowledging the hospitality of the prestigious institution at which the manuscript was written. *This* dissertation was completed while attending Passover Seders in Boston and New York. Thanks to Alan, Laura, Maura, Richard and Ruth for the hospitality and the matzos.

Finally and most importantly, I thank my spouse Dakota Zipporah Derryberry. Without her none of this would have been possible or worthwhile.

Towards a Self-dual Geometric Langlands Program

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The University of Texas at Austin, 2018

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This thesis is comprised of two logically separate but conjecturally related parts.

In the first part of the thesis I study theories of class $\mathcal{S}$ [32] via the formalism of relative quantum field theories [30]. From this physical formalism, and by analogy to the physical derivation of usual geometric Langlands [45, 86], I conjecture the existence of a self-dual version of the geometric Langlands program.

In the second part of the thesis I study shifted Cartier duality for the moduli of Higgs bundles. The main results are: (1) a criteria for ramification of L-valued cameral covers, (2) a generalisation of the Langlands duality/mirror symmetry results for the moduli of Higgs bundles of [24,37], and (3) the existence of a self-dual version of the moduli of Higgs bundles. This self-dual space is conjecturally the target space for a theory of class $\mathcal{S}$ compactified on a torus, and provides positive evidence for the self-dual geometric Langlands program.

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Chapter 1

Introduction

In his remarkable article *"Harmonic analysis as the exploitation of symmetry – a historical survey"* [54], George Mackey details how advances in the seemingly disparate fields of number theory, probablility theory, and mathematical physics may all be viewed through the lens of harmonic analysis. The central theme of his article – pithily expressed in the title – is the power and applicability of representation theory ("the exploitation of symmetry") to the aforementioned topics, especially the theory of unitary representations of (commutative and noncommutative) groups.

If one thinks of classical harmonic analysis as the exploitation of *manifest* symmetry – e.g. as the study of functions on a symmetric space – then the theme of this dissertation might reasonably be said to be the exploitation of *hidden* symmetry. The hidden symmetries in question begin life as (a priori) non-geometric "dualities" of quantum field theories, which after careful analysis yield subtle mathematical consequences: an outer automorphism of the $\mathcal{N} = (2, 2)$ supersymmetry algebra, for instance, leads to the mathematical and physical program of "Mirror Symmetry" [42] (a program which has spawned myriad subfields; for a dramatically incomplete list see [17, 35, 47, 59, 74, 80]).

The first part of this thesis (Chapter 2) is an extended review of duality in quantum field theory, both the general principles and specific examples. At the conclusion of this review I describe a self-dual version of the geometric Langlands program, obtained via exploitation of the symmetries of certain two dimensional quantum field theories derived from the theories of class $\mathcal{S}$ of Gaiotto, Moore and Neitzke [32]. When viewed in two dimensions these symmetries are hidden, however by lifting the theory to four dimensions they become geometrically manifest. A further lifting of the theory to a relative quantum field theory in six dimensions reveals that this same symmetry is responsible for the phenomenon of electric-magnetic duality in four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [30, 86]. This electric-magnetic duality was famously used by Kapustin and Witten to derive the (usual) geometric Langlands program [45], justifying the appellation "self-dual geometric Langlands" for Conjectures 1–3.

The second part of this thesis (Chapters 3–5) comprises a mathematical exploration of the physical predictions of Chapter 2. This part contains the main results of the thesis: I prove a generalisation of the results on Langlands duality for Hitchin systems of [24,37] (Theorems 5.5.1 and 5.5.2), and as a corollary derive the existence of a self-dual moduli space predicted by Conjecture 1 (Corollary 5.5.3). I achieve this by constructing a moduli space of "$\widetilde{G}$-Higgs bundles of arbitrary degree" as a slice inside of a larger moduli space of $\widetilde{G}_\tau$-Higgs bundles – this is a generalisation of the procedure of cutting the moduli space of "SL_n-Higgs bundles of degree d" out of the moduli space of GL_n-Higgs bundles (c.f. [7] for the analogous principal bundle construction). The duality results then follow from an analysis of the local structure of these slice moduli spaces, and an application of the Langlands duality results of Donagi and Pantev [24]. I also prove an intermediate result that may be of independent interest regarding when a $\mathbb{G}_m$-equivariant map from the total space of a line bundle to a finite type affine $\mathbb{G}_m$-scheme V will intersect a conical divisor $D \subset V$ (Theorem 4.1.2).

1.1 Outline of dissertation

A chapter-by-chapter summary of this thesis is as follows:

Chapter 1 (as you have just seen) serves as a brief introduction to the thesis. For convenience and reference I have collected the notation and conventions I use in this thesis together in Section 1.2.

Chapter 2 contains the physical content of this thesis. I begin with an extended review of duality in quantum field theories (Section 2.1) and relative quantum field theories (Section 2.2), before discussing the derivation of electric-magnetic duality in $\mathcal{N} = 4$ supersymmetric Yang-Mills and the geometric Langlands program (Section 2.3). I conclude this chapter by outlining a self-dual version of the geometric Langlands program, derived from the theories of class $\mathcal{S}$ of Gaiotto, Moore and Neitzke [32] (Section 2.4; Conjectures 1–3).

In Chapter 3 I review the mathematical background prerequisite for the original work in Chapters 4–5. This background falls into two categories: the modern perspective on shifted Cartier duality for commutative group stacks (Sections 3.1–3.2), and a review of the theory of Higgs bundles with a focus on the connection between the Hitchin fibration, cameral covers, and the group scheme of regular centralisers (Sections 3.3–3.4).

Chapter 4 contains an intermediate result on ramification of cameral covers that is required for the analysis of Chapter 5. I first prove in Theorem 4.1.2 a general statement about when the sections of an bundle associated to a $\mathbb{G}_m$ space V intersect with a divisor induced by a conical divisor $D \subset V$ (Section 4.1), and from this I deduce a criterion for detecting ramification of cameral covers (Section 4.2).

Chapter 5 is the final chapter in the body of the thesis, and contains the main duality results: Theorem 5.5.1, Theorem 5.5.2, and Corollary 5.5.3. As a preliminary result, I compare the Hitchin fibres for isogeneous simple groups (Section 5.1). I then construct and study the local structure of the moduli spaces $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)$ and $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$ (Section 5.2) before comparing the sheaves of regular centralisers for $\widetilde{G}$, G_{ad} and $\widetilde{G}_{\tau}$ (Section 5.3). As an intermediate step I describe the dual of $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)$ (Section 5.4) before presenting and proving the main duality results for $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$ (Section 5.5). I conclude by presenting a variety of examples to contextualise the duality theorems and suggest future applications (Section 5.6).

Finally, there are three Appendices dealing with topics not appropriate to the body of the dissertation: a review of the theory of reductive algebraic groups (Appendix A), a discussion of fixed points of the action of the Weyl group on a chosen maximal torus (Appendix B), and some results on the structure of the reductive group $\widetilde{G}_{\tau}$ and its Langlands dual group (Appendix C).

1.2 Notation and conventions

In this section I will make note of various conventions in notation and terminology that appear throughout this dissertation. First-time readers may wish to skim this section to check for unfamiliar notation, however there is no content (lemmata, theorems, etc.) that is strictly prerequisite for the rest of the dissertation.

1.2.1 Lie theoretic conventions

This section deals only with Lie theoretic notation and conventions: for definitions and properties of reductive algebraic groups, see Appendix A.

In the following, G is most generally a complex reductive algebraic group,[1] however at times I will note further assumptions of simplicity, simple connectivity, etc. Lie algebras will be denoted by lower case fraktur font, so for instance the Lie algebra of G will be denoted by $\mathfrak{g}$. Given a semisimple group G, I will denote by $\widetilde{G}$ the corresponding simply-connected form and by G_{ad} the corresponding adjoint form.

A choice of Borel subgroup of G will usually be denoted B, with Lie algebra $\mathfrak{b}$. The unipotent radical of B will be denoted by U, and a choice of maximal torus will be denoted by H with Lie algebra $\mathfrak{h}$. The notation T is reserved for an algebraic torus that is *not* the maximal torus of a group G, and the (abelian) Lie algebra of such a torus is denoted $\mathfrak{t}$.

[1] In [63, 64] the more general setup of torsors for non-constant group schemes is considered; by contrast I will always think of G as the *constant* group scheme $G \times X \to X$. I work in a topology where G-torsors are locally trivial, i.e. in the étale or analytic topology.

The rank $\dim(H)$ of a reductive algebraic group G will be denoted by $\mathrm{rank}(G)$, or just by r.

When considering the Weyl group of a particular maximal torus $H \subset G$ I will use the notation $W_G(H) = N_G(H)/H$; when I do not need to emphasise the maximal torus H I will just write W.

The set of roots of the group G will be denoted by R, and a choice of positive roots will be denoted R^+. Given a choice of positive roots, the corresponding simple roots will be denoted S.

If M is a set or space with a G-action (e.g. M is a representation of G) I will denote by M^G the fixed points of the G-action (e.g. the G-invariant subspace of the representation).

Finally, there are many notations in the literature for the lattices that appear in the study of reductive algebraic groups. As it can sometimes be difficult to keep straight what each piece of notation means (particularly across different references) I have opted to use a notation that makes manifest the input data and the variance for each lattice without being cumbersome. As above, let T denote an algebraic torus, and let G denote a reductive algebraic group with chosen maximal torus H:

- Denote the character lattice of T by $X^\bullet(T) := \mathrm{Hom}(T, \mathbb{C}^\times)$, and the cocharacter lattice by $X_\bullet(T) := \mathrm{Hom}(\mathbb{C}^\times, T)$. When convenient, these can be identified as subgroups $X^\bullet(T) \subset \mathfrak{t}^*$ and $X_\bullet(T) \subset \mathfrak{t}$.

- Denote by $X^\bullet(G, H) := X^\bullet(H)$ the character lattice corresponding to a choice of maximal torus $H \subset G$; similarly denote the corresponding cocharacter lattice by $X_\bullet(G,H)$. When convenient these can be identified as subgroups $X^\bullet(G,H) \subset \mathfrak{h}^*$ and $X_\bullet(G,H) \subset \mathfrak{h}$.

When G is semisimple and H is a choice of maximal torus I will sometimes use the notation

$$\Lambda_R = X^\bullet(G_{\mathrm{ad}}, H_{\mathrm{ad}}) \quad \text{and} \quad \Lambda_W = X^\bullet(\widetilde{G}, \widetilde{H}) \tag{1.1}$$

for the root and weight lattice, and

$$\Pi_R = X_\bullet(G_{\mathrm{ad}}, H_{\mathrm{ad}}) = \Lambda_R^\wedge = \mathrm{Hom}_{\mathbb{Z}}(\Lambda_R, \mathbb{Z}) \quad \text{and} \quad \Pi_W = X_\bullet(\widetilde{G}, \widetilde{H}) = \Lambda_W^\wedge \tag{1.2}$$

for the coroot and coweight lattice. I have tried to use this notation only in situations where the more precise notation would prove unwieldy, as the condensed notation (1) fails to keep track of the group G and (2) fails to distinguish whether or not I have chosen a maximal torus.

1.2.2 Geometric conventions

A general complex scheme or manifold will be denoted by X, with structure sheaf $\mathcal{O}_X$, and a general test scheme will be denoted S. The constant sheaf on X valued in A is denoted A_X. The notation C will be reserved for the situation where the space in question is a Riemann surface or and algebraic curve (usually, but not always, of genus $g > 1$).

Given a space X and spaces equipped with maps to X, $Y_1 \to X$ and $Y_2 \to X$, I will denote by $\mathrm{Hom}_X(Y_1, Y_2)$ the collection of maps $Y_1 \to Y_2$ in the slice category of spaces with a map to X (i.e. maps which commute with the "structure maps" to X).

Given a group G, I will use the algebro-geometric terminology *G-torsor* to refer to a principal G-bundle. I.e. a G-torsor over a space X is a space $P \to X$ equipped with a (right) G-action, such that (1) the map $(\mathrm{id}_P, \mathrm{act}) : P \times G \to P \times_X P$ is an isomorphism and (2) P admits local sections.[2] Here the terms "space" and "local" are deliberately vague, as this definition is applicable to many different categories and Grothendieck topologies.

As a general rule, stacky moduli spaces are denoted via calligraphic and italic fonts, while coarse moduli spaces are denoted via bold font. Stacky quotients are denoted by square brackets $[\,/\,]$: if X is equipped with a right action of G, then $[X/G]$ represents the stack with presentation given by the groupoid [51, §2.4.3]

$$\begin{array}{c} X \times G \\ {\scriptstyle s}\big\downarrow\big\downarrow{\scriptstyle t} \\ X \end{array} \qquad s(x,g) = x, \quad t(x,g) = x \cdot g. \tag{1.3}$$

Given two stacks $\mathcal{Y}$ and $\mathcal{Z}$, I will denote by $\mathcal{M}ap(\mathcal{Y}, \mathcal{Z})$ the functor whose S points are given by $\mathrm{Map}_S(\mathcal{Y} \times S, \mathcal{Z} \times S)$ for any affine scheme S. Similarly, if $\mathcal{A}$, $\mathcal{B}$ are commutative group stacks, I will denote by $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ the commutative group stack whose S-points are given by $\mathrm{Hom}_S(\mathcal{A} \times S, \mathcal{B} \times S)$ for any affine scheme S [1, XVIII].

Finally and importantly: from Important Remark! 5.2.9 onwards, I will be

[2]Given the assumption of this dissertation that we will only deal with constant group schemes and that the map $P \to X$ is faithfully flat, condition (1) is sufficient to ensure that P has local sections in the étale topology. Furthermore, had G been a non-constant group scheme over X, the map in condition (1) would have had domain $P \times_X G$.

implicitly restricting away from the discriminant locus of the Hitchin base (see Definition 3.3.5 and (3.53)). The duality results of Chapter 5 will hold over this dense open set of $\mathbf{Hitch}_{\mathfrak{g}}(C)$ – the question of whether or not this duality may be extended over the discriminant is still open.[3]

1.2.3 Duality conventions

Much of this thesis has to do with the interplay between various standard dualities. To distinguish between them I use the following notation:

- ${}^L(-)$ denotes an object obtained via Langlands duality, e.g. the Langlands dual group ${}^LG$.
- $(-)^\vee$ denotes the Pontrjagin dual group $\mathrm{Hom}(-, U(1))$ or $\mathrm{Hom}(-, \mathbb{G}_m)$, depending on context.
- $(-)^\wedge$ denotes the dual lattice to an abelian group, $(-)^\wedge := \mathrm{Hom}(-, \mathbb{Z})$.
- $(-)^D$ denotes the Cartier dual $\mathrm{Hom}(-, \mathcal{O}^\times[1])$ or $\mathrm{Hom}(-, B\mathbb{G}_m)$, depending on context. E.g. if A is an abelian variety then A^D is the usual dual abelian variety.

[3]Partial results in this direction have been obtained by Arinkin and Fedorov [4, 5].

Chapter 2

Physics and Duality

In this chapter I will discuss the background and motivation for the self-dual geometric Langlands program. Since this material is largely drawn from physics, readers should be aware that this section will involve some amount on non-rigorous physical reasoning (such as manipulation of path integrals).

In Section 2.1 I will begin by discussing quantum field theories and some of the geometries that may be associated with such a theory. I will describe the notion of a duality of quantum field theories, and explore the ways in which such dualities manifest geometrically.

In Section 2.2 I will describe the notion of a relative quantum field theory after [30]. This formalises the notion of an quantum field theory "with anomaly", and (as we shall see) can be used to engineer dualities of quantum field theories. In this section I will also introduce "Theory $\mathfrak{X}$" (otherwise known as the "6d (2,0) superconformal field theory", see [72, 85]) and theories of class $\mathcal{S}$ [31, 32], two relative quantum field theories of particular importance to this dissertation.

In Section 2.3 I will review how the geometric Langlands program may be derived from Theory $\mathfrak{X}$, as discussed in [45, 76, 86].

Finally, in Section 2.4 I will sketch how a new "self-dual geometric Langlands program" might be obtained from Theory $\mathfrak{X}$, via theories of class $\mathcal{S}$. An overview of the relationship between the theories introduced in Sections 2.2-2.4 is presented in Figure 2.4.

2.1 Quantum field theories, geometry and duality

Let S be a structure that can be placed on a manifold[1] (e.g. smooth structure, Riemannian metric, spin structure, supermanifold structure, G-bundle with connection, etc.). The physical concept at the heart of this dissertation is the following:

Quasi-Definition 2.1.1 (Quantum Field Theory)**.** An *(extended) d-dimensional S-structured quantum field theory (QFT)*, $\mathcal{Z}$, is a procedure for functorially assigning

- a $\mathbb{C}$-number $\mathcal{Z}(M^d)$ to every closed d-manifold with structure S (the *correlation function* or *path integral*),
- a $\mathbb{C}$-vector space $\mathcal{Z}(N^{d-1})$ to every closed $(d-1)$-manifold with structure S (the *space of states*),
- a $\mathbb{C}$-linear category $\mathcal{Z}(P^{d-2})$ to every closed $(d-2)$-manifold with structure S,[2]
- higher (appropriately $\mathbb{C}$-linear) categorical data to higher codimension manifolds with structure S,

[1]More accurately, S should collect together different compatible structures for different dimensional manifolds. For instance, S may specify a symplectic structure for a $2k$-dimensional manifold and a contact structure for a $(2k-1)$-dimensional manifold. See [75] for more discussion on this point.

[2]In the case $d = 2$ this category may be interpreted as the *category of boundary conditions* for the theory.

subject to *unitarity* and *locality* constraints.

Furthermore, for every $k < d$ there is a collection of *k-dimensional submanifold operators* $\{\mathcal{O}^{(k)}\}$ that may be used to decorate a given manifold, e.g. we may evaluate the correlation function of a collection of operators

$$\mathcal{Z}(M^d; \mathcal{O}^{(k_1)}_{a_1}, \ldots, \mathcal{O}^{(k_l)}_{a_l}) \in \mathbb{C}. \tag{2.1}$$

When $k = 0, 1, 2$ these are sometimes called *local*, *line/loop* and *surface* operators respectively; when $k = d - 1$ these are sometimes called *domain walls*.

Remark 2.1.1. The locality constraint of Quasi-Definition 2.1.1 may be mathematically interpreted as saying that the domain of the functor $\mathcal{Z}$ is some kind of *bordism d-category*[3], where e.g. the objects are $(d-k)$-manifolds with some sort of structure (supermanifold, Riemannian structure, spin structure, equipped with a principal bundle, etc.), the morphisms are $(d - k + 1)$-manifolds with boundary (and structure), the 2-morphisms are $(d - k + 2)$-manifolds with corners, etc.

Example 1. It is possible to give a rigorous version of Quasi-Definition 2.1.1 in the case where our QFT is a *topological quantum field theory (TQFT)*. In [52] Lurie defines an fully extended topological field theory valued in a symmetric monoidal (∞, n)-category $\mathcal{C}$ to be a symmetric monoidal functor from a domain bordism (∞, n)-category to $\mathcal{C}$. Moreover, the *cobordism hypothesis* (due to Baez-Dolan [6], Lurie [52], and others) states that such TQFTs satisfy the strongest possible locality constraint: namely, they are determined by what they evaluate to on a connected 0-manifold (i.e. a point).

[3]Or (∞, d)-category [52].

Example 2. One "obvious" example of a QFT which looks boring but will in fact be extremely important later is the *trivial d-dimensional QFT*, triv^d. triv^d assigns the number 1 to every d-manifold, the 1-dimensional vector space $\mathbb{C}$ to every $(d-1)$-manifold, the $\mathcal{C}$-linear category $\mathrm{Vect}_{\mathbb{C}}$ to every $(d-2)$-manifold, and so on, assigning an n-categorical version of a $\mathbb{C}$-linear symmetric monoidal unit to every $(d-n)$-manifold.

2.1.1 Examples of QFTs

While results such as the cobordism hypothesis are elegant and conceptually useful, in practice – i.e. when one wants to study a *particular* QFT – one does not specify only the minimal amount of local data required to give a well-defined theory and attempt to compute non-trivial correlation functions. Instead there are physical techniques one can draw on to construct a quantum field theory, such as the following.

Quasi-Definition 2.1.2 (Path Integral Quantisation)**.** Recall a classical Lagrangian field theory on a d-manifold M^d is specified by the data of

1. a *space of classical fields* $\mathcal{F}$, usually the sections of some fibre bundle over M,
2. a *Lagrangian density* $\mathcal{L}$, which is (roughly) a local functional of the fields $\mathcal{F}$ valued in densities on M, and
3. a *variational 1-form* γ, whose role I will not discuss in the sequel.

(For rigorous definitions of the above and an introduction to classical field theory, one should consult the remarkable set of notes [22].) Given the above data, one can form the *classical action*

$$\begin{aligned} S &: \mathcal{F} \to \mathbb{R} \\ S[\phi] &:= \int_M \mathcal{L}[\phi] \end{aligned} \tag{2.2}$$

and the *classical equations of motion* are derived by finding the critical points of S with respect to the functional derivative $\frac{\delta}{\delta\phi}$.[4]

Given a classical Lagrangian field theory one defines its *path integral quantisation* by considering the fundamental object of study to be the *path integral*

$$\mathcal{Z}(M^d) := \int_{\mathcal{F}} \mathcal{D}\phi e^{iS[\phi]}. \tag{2.3}$$

In this formulation submanifold operators are often functionals $\mathcal{O}$ on $\mathcal{F}$,[5] and one calculates correlation functions of operators by inserting them into the integral:

$$\mathcal{Z}(M^d; \mathcal{O}) := \int_{\mathcal{F}} \mathcal{D}\phi \mathcal{O}(\phi) e^{iS[\phi]}. \tag{2.4}$$

The path integral formalism also allows one to derive the vector spaces of states and (higher) categorical data of Quasi-Definition 2.1.1, although I will not describe how to do so here. For some details and examples see [44] and [18].

Remark 2.1.2. When studying operator insertions it is common to consider the *normalised correlation functions*

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_l \rangle_M := \frac{\mathcal{Z}(M; \mathcal{O}_1, \ldots, \mathcal{O}_l)}{\mathcal{Z}(M)} = \frac{\int_{\mathcal{F}} \mathcal{D}\phi \mathcal{O}_1(\phi) \cdots \mathcal{O}_l(\phi) e^{iS[\phi]}}{\int_{\mathcal{F}} \mathcal{D}\phi e^{iS[\phi]}} \tag{2.5}$$

[4] I am brushing over important details here, e.g. the integral $S[\phi]$ may not always converge. To deal with such issues I again refer the reader to [22].

[5] There are more general submanifold operators that alter the path integral by, for instance, prescribing certain boundary conditions or singularities along a given submanifold.

rather than the path integrals themselves, e.g. compare the perturbative and formal sections of [83].

With the technique of path integral quantisation at our disposal, we can now give examples of QFTs which arise via quantisation of classical theories. For detailed analysis of the classical theories listed below, see [22].

Example 3. The simplest example of a non-trivial QFT is the quantisation of the *free real scalar field of mass* m on a oriented manifold M equipped with a metric η (of Euclidean or Lorentz signature). The classical theory has $\mathcal{F} = \mathrm{Map}(M, \mathbb{R})$ and

$$\mathcal{L}^{(\mathbb{R}\text{-scalar})}[\phi] := \frac{1}{2} d\phi \wedge \star_\eta d\phi - \frac{m^2}{2} \phi^2 d\mathrm{vol}_\eta, \tag{2.6}$$

where $\star_\eta$ is the Hodge star and $d\mathrm{vol}_\eta$ is the volume form associated to η.

Example 4. A natural generalisation of Example 3 is given by the class of theories known as *(non-linear) σ-models*. These are obtained by replacing

(1) the real numbers $\mathbb{R}$ with a Riemannian manifold (X, g) (the *target space*), and

(2) the mass term $\frac{1}{2}m^2\phi^2$ with a potential energy function $V : X \to \mathbb{R}$.

The space of fields is now $\mathcal{F} = \mathrm{Map}(M, X)$ and the Lagrangian density is

$$\mathcal{L}^{(\mathrm{NLSM})}[\Phi] := \left(\frac{1}{2} \|d\Phi\|^2 - \Phi^*(V) \right) d\mathrm{vol}_\eta \tag{2.7}$$

where the norm $\| - \|$ is computed using both metrics η and g.

Example 5. In order to make contact with the real world (e.g. the theory of electromagnetism) one needs to consider the class of theories known as *gauge theories*, which transform "global symmetries" – informally, transformations of the classical fields which are the same at all points of spacetime – with "local" or "gauge symmetries" – where the fields are allowed to transform in a different way at each point of spacetime. Mathematically this is done by expanding the space of fields to include certain principal bundles with connection, and redefining the original fields of the theory to live in associated bundles, where they are acted on by the corresponding covariant derivative.[6]

The basic example of a gauge theory is a *pure gauge theory* or *pure Yang-Mills theory*. Start with the data of:

(1) a Lie group G with Lie algebra $\mathfrak{g}$, and

(2) a bi-invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$.

Pure Yang-Mills theory with gauge group G is constructed by gauging the trivial global action of G on the trivial QFT of Example 2. Explicitly, the space of fields on an oriented (pseudo-)Riemannian manifold (M, η) is $\mathcal{F} = \mathrm{Conn}_G(M)$, the space of connections on principal G-bundles over M, and the Lagrangian density is given

[6] This is really a two-step process: (1) choose a way to extend the theory for each **fixed** principal bundle with connection (i.e. redefine the original fields), and (2) expand the space of fields to include principal bundles with connection. A physicist might say that they had (1) *coupled the theory to a background field* and (2) *gauged the symmetry*.

by

$$\mathcal{L}^{(\mathrm{YM}_G)}[\nabla] := -\frac{1}{2}\langle F_\nabla \wedge \star_\eta F_\nabla \rangle, \tag{2.8}$$

where F_∇ is the curvature 2-form of the connection ∇.

Example 6. As a final example for this section, we consider a synthesis of Examples 4 and 5 called a *gauge theory with matter*. Suppose that G is a compact Lie group, and let R be a finite dimensional complex G-representation equipped with a G-invariant Hermitian inner product $(-,-)$. Given a principal G-bundle $P \to M$, denote by $R_P \to M$ the associated bundle $(P \times R)/G$. Then the space of fields is

$$\mathcal{F}(G,R) := \{((P,\nabla),\Phi) \,|\, (P,\nabla) \in \mathrm{Conn}_G(M),\ \Phi \in \mathrm{Map}_M(M,R_P)\} \tag{2.9}$$

and the Lagrangian density is

$$\mathcal{L}[\nabla,\Phi] := \mathcal{L}^{(R)}[\nabla,\Phi] + \mathcal{L}^{(\mathrm{YM}_G)}[\nabla], \tag{2.10}$$

where

$$\mathcal{L}^{(R)}[\nabla,\Phi] = \frac{1}{2}\|d_\nabla \Phi\|^2 d\mathrm{vol}_\eta \tag{2.11}$$

and d_∇ denotes the induced covariant derivative in the associated bundle.

2.1.2 Duality in QFT

Implicit in the discussion of Example 5 is the idea that quantum field theories are not rigid objects – that is to say that they can have interesting (non-trivial) automorphisms. With that in mind, consider the following:

Quasi-Definition 2.1.3 (Duality of QFTs)**.** A *duality of quantum field theories* is an isomorphism of QFTs $\mathcal{Z}_1$ and $\mathcal{Z}_2$. If $\mathfrak{D} : \mathcal{Z}_1 \to \mathcal{Z}_2$ is duality of QFTs I will say that $\mathfrak{D}$ *exhibits* $\mathcal{Z}_1$ *and* $\mathcal{Z}_2$ *as duals*, or informally just that $\mathcal{Z}_1$ *and* $\mathcal{Z}_2$ *are dual.*

Remark 2.1.3. Quasi-Definition 2.1.3 is a more liberal use of the term "duality" than one usually finds in the literature (either mathematical or physical), as it encompasses (for instance) global symmetries of a QFT. I find it more satisfying, however, to use this as an umbrella definition and then discuss specific interesting examples of duality than to introduce specific examples in isolation and then gesticulate in the direction of some mysterious unifying principle.

I do believe that there ought to be a quasi-definition – or even definition! – of duality that only encompasses "conventionally interesting" examples, but I do not know of any such formulation at present.

Remark 2.1.4. Suppose that $(\mathcal{F}_i, \mathcal{L}_i)$, $i = 1, 2$, are two classical field theories whose path integral quantisations are dual. Observe that:

- This implies that

$$\int_{\phi \in \mathcal{F}_1} D\phi e^{i \int \mathcal{L}_1[\phi]} = \int_{\psi \in \mathcal{F}_2} D\psi e^{i \int \mathcal{L}_2[\psi]}, \tag{2.12}$$

  but it is **not** necessarily the case that $\mathcal{L}_2$ may be obtained from $\mathcal{L}_1$ by a simple change of variables $\mathcal{F}_1 \to \mathcal{F}_2$ (see Section 2.1.4).

- (2.12) by itself does not imply that the quantisations are dual, since a QFT consists of more data than just the partition functions of manifolds. E.g. there must be a one-to-one mapping of between the submanifold operators of the two theories.

Example 7. Arguably the most famous duality in physics is *mirror symmetry*, which is an equivalence of two quite different looking 2d TQFTs known as the "A-model" and the "B-model" [42,75]. The input for these theories includes manifolds with extra structure X and $X^\vee$: when X and $X^\vee$ are both Calabi-Yau manifolds the induced equivalence between the categories of boundary conditions is Kontsevich's celebrated *homological mirror symmetry conjecture* [47], which claims that the bounded derived category of coherent sheaves on $X^\vee$ is equivalent to the Fukaya category of X.

Remark 2.1.5. The concept of duality in physics is not unique to quantum field theory: one can talk about dualities of classical theories, string dualities, and more.

2.1.3 Effective theories and moduli spaces

The approach I have taken so far supposes that quantum field theories are "ideal", in the sense that I implicitly assumed they give well-defined, sensible answers at arbitrarily high energies. In practice, however, many quantum field theories break down above some finite energy scale. One way to understand a theory with such singular behaviour is to interpret it as an *effective field theory (EFT)*.

The motivation for EFT is fairly straightforward: if you want to calculate the trajectory of a cricket ball you don't start with the equations of general relativity.[7] That is to say that in order to understand the low energy (*infrared* or *IR*) behaviour

[7] Despite appearances, the duration of a cricket match is **not** a time-dilation effect.

of a theory $\mathcal{Z}$ one does not need perfect knowledge of the theory at arbitrarily high energies (the *ultraviolet* or *UV* range). Instead, one may study the IR physics by constructing a new *effective* theory $\mathcal{Z}_\Lambda^{(IR)}$ that gives a good approximation to the theory $\mathcal{Z}$ at energy scales less than Λ, but which fails to approximate $\mathcal{Z}$ at energy scales higher than Λ. This failure might involve divergences of observable quantities, such as happens when one interprets a QFT with a UV divergence as an EFT, or it may simply return physically unreasonable answers, as happens if one attempts to apply Newtonian mechanics to arbitrarily fast or massive bodies.

Geometry plays a central role in the study of the IR physics of a QFT for the following reason:

Quasi-Definition 2.1.4 (Target space)**.** The IR physics of a QFT $\mathcal{Z}$ is described by a σ-model of maps to a space $\mathcal{M}(\mathcal{Z})$, or just $\mathcal{M}$. I will refer to the space $\mathcal{M}$ as the *target space* of the theory.

Remark 2.1.6. The target space of a low energy σ-model is often referred to as the *moduli space of vacua* of the theory (see e.g. the glossary of [21]). This is not entirely correct: the moduli space of vacua may be independently defined as the space of ground states of the theory, and while this often agrees with the target space of Quasi-Definition 2.1.4 they are not equivalent. In particular, they will differ for a 2d QFT with a continuous target space.

Now, observe that the target space $\mathcal{M}(\mathcal{Z})$ associated to the theory $\mathcal{Z}$ encodes information about the IR physics[8] of $\mathcal{Z}$ only. The quantum IR physics may be

[8] This terminology is potentially confusing, since defining what is meant by "low energy" itself

obtained as the quantisation of two (or more) different classical σ-models: as such, given a duality $\mathcal{Z}_1 \simeq \mathcal{Z}_2$ one might find two different descriptions of the classical σ-models. While the corresponding target spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ will not be isomorphic, it is natural to expect that the geometry of $\mathcal{M}_1$ and $\mathcal{M}_2$ should be related in *some* way. To explore what sort of relationship one might expect, I need to introduce various methods by which a d-dimensional QFT may be used to produce a $(d-k)$-dimensional QFT.

Quasi-Definition 2.1.5. The following terminology is used inconsistenly across the literature: the convention that I have chosen is adopted from Gregory Moore's *Felix Klein Lectures* [58, §2.2]. Let $\mathcal{Z}$ be a d-dimensional QFT, and let S be a compact k-manifold.

(1) **VerKleinung:** The *VerKleinung of $\mathcal{Z}$ along S* is the $(d-k)$-dimensional QFT which assigns to an l-manifold M the number/vector space/(higher) category $\mathcal{Z}(M \times S)$.[9]

(2) **Compactification:** The *compactification of $\mathcal{Z}$ on S*, denoted $\mathcal{Z}[S]$, is the IR limit of the VerKleinung of $\mathcal{Z}$ along S, where we take all distance scales on S to be small.

(3) **Dimensional Reduction:** Suppose that $\mathcal{Z}$ is a Lagrangian field theory with field space $\mathcal{F}$ and Lagrangian density $\mathcal{L}$, and that G is a Lie group that acts

involves making choices (e.g. one might integrate out all particles with mass above a chosen energy scale Λ) which lead to different IR limits. In this situation I wish to take the "absolute" low energy limit, i.e. integrating out *all* massive particles.

[9] I will not define any notation for VerKleinung, as I will not be making use of this procedure except in so far as it is required for the definition of compactification.

transitively on S. Then the *dimensional reduction of $\mathcal{Z}$ with respect to the G-action* (or dimensional reduction *on S*), denoted $\mathcal{Z}^G$ (or $\mathcal{Z}^S$), is the Lagrangian QFT obtained by taking as field space the G-invariant fields $\mathcal{F}^G$, and by taking as Lagrangian density $\frac{1}{\mathrm{vol}(G)}\mathcal{L}|_{\mathcal{F}^G}$.[10]

Remark 2.1.7. Since by definition a topological quantum field theory has no metric/length/energy dependence, the procedures of VerKleinung and compactification coincide for TQFTs.

We now have the language to describe a particularly famous geometric duality conjecture:

Example 8. In [74] Strominger, Yau and Zaslow study the compactification of type IIA and type IIB string theory on a pair of Calabi-Yau 3-folds X and Y. Under the assumption that the resulting physical theories are mirror dual,[11] an analysis of certain moduli of vacua associated to the compactified theories suggests that given a nonsingular "special Lagrangian" torus X_b in X, the moduli space of flat $U(1)$-connections on small special Lagrangian deformations of X_b provides a coordinate chart for Y. The *SYZ mirror symmetry proposal* then conjectures that the collection of all such coordinate charts cover an open dense subspace of Y.

A geometric consequence of this proposal is that the Calabi-Yau 3-folds X and Y are fibrations over a common base B, and that for a dense set of $b \in B$ the fibres X_b and Y_b are "dual special Lagrangian tori" in the sense that there is

[10] G may have infinite volume, in which case $\mathrm{vol}(G)$ must be treated formally or regularised.

[11] In the sense of string theory, which is different from but may be related to the mirror symmetry of Example 7.

a natural identification of Y_b with the moduli space of flat $U(1)$-connections on X_b (and vice-versa). Such a setup is called an *SYZ fibration*, and X and Y are called *SYZ mirror dual*.

Remark 2.1.8. Experts will notice that for the purposes of exposition I have made some omissions in my description of SYZ mirror symmetry: X and Y are integrable systems over B (in particular they are only generically fibrations), I have not defined the term “special Lagrangian”, and I have omitted any mention of the large complex structure limit. Although important these topics are not necessary for either motivation or understanding of this dissertation, and their inclusion would only serve as a distraction.

Remark 2.1.9. The methods used in this dissertation will be primarily algebro-geometric in nature, and so a priori it seems that the SYZ proposal of Example 8 would be of little relevance (living in the world of symplectic and holomorphic symplectic geometry). Nevertheless, the geometric duality that I will be interested in may be interpreted as an instance of SYZ mirror symmetry via the following trick:

Suppose that instead of just being Calabi-Yau, the manifolds X and Y are *hyperkähler* – recall that X is hyperkähler if it is equipped with a Riemannian metric g and a triple (I, J, K) of complex structures[12] which satisfy the quaternion relations and which all exhibit g as a Kähler metric on X. Write the corresponding Kähler forms as ω_I, ω_J and ω_K. Then $\Omega_I := \omega_J + i\omega_K$ is a holomorphic symplectic form for complex structure I, and it is an exercise in linear algebra to show that if $\Omega_I|_T \equiv 0$ for

[12]In fact by taking linear combinations $aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$ one sees that there is an entire $\mathbb{CP}^1 \cong S^2$ worth of compatible complex structures.

a half-dimensional submanifold T (say that T is *complex Lagrangian* for I) then T is special Lagrangian for complex structure J. Furthermore, there is a correspondence between holomorphic line bundles on (T, I) and flat $U(1)$-bundles on T (thought of as a special Lagrangian submanifold in complex structure J).

Therefore, if X and Y are SYZ mirror dual in the sense of Example 8 with respect to complex structures J_X, J_Y, then with respect to complex structures I_X, I_Y the fibres X_b and Y_b will generically be compact complex tori which are dual in the sense of abelian varieties. This setup, which will be generalised and made precise in Chapter 3, is the *algebraic* version of SYZ mirror symmetry that I will be concerned with in Chapter 5.

2.1.4 Extended example: Reduction of $U(1)$ gauge theory from 4d to 2d

Before delving in to the more complicated QFTs that motivated this thesis, it is instructive to consider a toy example that brings together many of the topics discussed in Sections 2.1.1, 2.1.2, and 2.1.3. Namely, we will compactify (nonsupersymmetric) 4d pure $U(1)$ gauge theory on a torus $S^1_r \times S^1_R$ in two different ways. The two different compactifications yield dual 2d σ-models, and so produce dual moduli of vacua. This duality will be manifest in the complex structure of the moduli spaces (it will be the self-duality of an elliptic curve), but non-manifest in the Riemannian metrics (which will generically be different).

2.1.4.1 Pure 4d $U(1)$ gauge theory

Specialising Example 5 to the case $G = U(1)$, the setup for 4d $U(1)$ gauge theory is as follows: spacetime is a 4-manifold $M^{(4)}$ equipped with a metric η of signature $(+,-,-,-)$, and there is a single $U(1)$-gauge field (i.e. connection on a principal circle bundle L) $A^{(4)}$ with field strength $F^{(4)} = dA^{(4)}$. The Lagrangian for this theory is given by

$$\mathcal{L}^{(4)} = -\frac{\mathrm{Im}(\tau)}{4\pi} F^{(4)} \wedge \star F^{(4)} + \frac{\mathrm{Re}(\tau)}{4\pi} F^{(4)} \wedge F^{(4)} \tag{2.13}$$

where τ is a complex parameter with strictly positive imaginary part, and where the field strength is normalised so that $[F^{(4)}] \in H^2(M^{(4)}; 2\pi\mathbb{Z})$. In the quantum field theory associated to this Lagrangian, we wish to calculate the functional integral

$$\mathcal{Z} = \sum_L \frac{1}{\mathrm{vol}(\mathcal{G}_L)} \int \mathcal{D}A e^{i\int \mathcal{L}^{(4)}}, \tag{2.14}$$

where $\mathcal{G}_L$ is the gauge group of the circle bundle L, and $\mathrm{vol}(\mathcal{G}_L)$ is a normalisation factor formally keeping track of the (infinite) volume of this group.

2.1.4.2 Compactifying to three dimensions

We first reduce to a 3d theory by compactifying $\mathcal{L}^{(4)}$ on a circle of radius R, i.e. set $M^{(4)} = M^{(3)} \times S^1_R$. Taking a Fourier expansion of the connection and plugging in the equations of motion[13] for $\mathcal{L}^{(4)}$, one can derive that the nonzero Fourier modes become, in the 3d theory, massive particles of mass proportional to $\frac{1}{R}$. By letting R be very small, for example of order the Planck length, we can guarantee that no

[13] And also fixing Lorentz gauge $d \star A = 0$.

excitations of the nonzero Fourier modes can occur in the low energy 3d effective theory [73].

Thus we may assume that our connection is constant in the S^1_R direction: precisely, we can always perform a *partial* gauge fixing by imposing the condition $\frac{\partial A_\vartheta}{\partial \vartheta} = 0$, where $\frac{\partial}{\partial \vartheta}$ is the generator of isometric rotations of the circle, and in this choice of gauge we may impose nonexistence of nonzero Fourier modes through the constraint $\partial_\vartheta A_i = 0$. Having imposed this constraint, we integrate out the circle direction to find the 3d effective Lagrangian

$$\mathcal{L}^{(3)}_{kin} = -\frac{R}{2}\mathrm{Im}(\tau) F \wedge \star F + \frac{R}{2}\mathrm{Im}(\tau) d\sigma \wedge \star d\sigma + \frac{R}{2}\mathrm{Re}(\tau) \cdot 2F \wedge d\sigma \tag{2.15}$$

where $A^{(4)} = A + \sigma d\vartheta$. The residual gauge symmetries are given by $U(1)$-gauge transformations of A on $M^{(3)}$, and by certain affine transformations in the variable ϑ (since the second derivative of an affine transformation is zero). Thus, A is a connection on $M^{(3)}$ with field strength $F = dA$, and σ is a scalar field which by invariance of our theory under the residual gauge transformation $e^{\frac{i\vartheta}{R}}$ is valued in $\mathbb{R}/\frac{1}{R}\mathbb{Z}$.

2.1.4.3 Dualising the 3d theory

The classical equations of motion for a $U(1)$ gauge field A with field strength $F = dA$ are given by $d\star F = 0$. In dimension three this implies that (at least locally) $\star F = d\phi$ for some function ϕ, or equivalently $F = \star d\phi$. It is tempting, therefore, to suggest that in three dimensions there should be a dual description of $U(1)$ gauge theory as a (real) scalar field theory. Let us now show that we can indeed reformulate the Lagrangian $\mathcal{L}^{(3)}_{kin}$ in this way, with the caveat that since in a quantum field theory we must integrate over *all* fields, and not just the on-shell fields,[14] the reformulation is somewhat subtle.

Consider adding the Lagrange multiplier

$$\mathcal{L}^{(3)}_{lag} = R d\gamma \wedge F \tag{2.16}$$

to our Lagrangian, where $\gamma : M^{(3)} \to \mathbb{R}/\frac{1}{R}\mathbb{Z}$ is again a circle-valued function. Note that by adding this term to our Lagrangian we have changed our perspective on whether F or A is a fundamental field in our theory. One can show that performing the path integral over γ recovers our original Lagrangian $\mathcal{L}^{(3)}_{kin}$ with the correct quantisation and normalisation conditions for the field strength.[15]

We could also perform the integral over the 2-form field F first. Letting

$$F' = F - \star\left(\frac{1}{\mathrm{Im}(\tau)} d\gamma + \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} d\sigma\right), \tag{2.17}$$

which is a linear shift in F (and so $\mathcal{D}F' = \mathcal{D}F$ in the path integral measure), the

[14]The solutions to the classical equations of motion.

[15]One needs to take care with this integral, since γ is circle valued, not real valued. Specifically, the calculation requires fixing for each homotopy class of maps to the circle $\alpha \in H^1(M^{(3)};\mathbb{Z})$ a 1-form $d\gamma_\alpha$ which represents α in de Rham cohomology, writing our variable as $d\gamma = d\gamma_\alpha + d\gamma_\mathbb{R}$ for $\gamma_\mathbb{R}$ a real valued scalar field, and then performing an integral over $\gamma_\mathbb{R}$ and a sum over $H^1(M^{(3)};\mathbb{Z})$. Recovering the original Lagrangian uses a Poisson resummation trick that requires a Poincaré duality theorem, and fixing 1-form representatives requires Hodge theory. On a non-compact manifold these theorems do not hold for smooth forms: thankfully, we can use L^2-cohomology instead.

Lagrangian becomes

$$\begin{aligned}\mathcal{L}^{(3)}_{kin} + \mathcal{L}^{(3)}_{lag} \\ = -\frac{R}{2}\mathrm{Im}(\tau)\left[F' \wedge \star F' - \left(\frac{1}{\mathrm{Im}(\tau)}d\gamma + \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}d\sigma\right) \wedge \star \left(\frac{1}{\mathrm{Im}(\tau)}d\gamma + \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}d\sigma\right)\right. \\ \left. - d\sigma \wedge \star d\sigma\right]. \end{aligned} \tag{2.18}$$

The integrand of the path integral now factors into a piece which (after Wick rotation) is Gaussian in F', and a piece which is independent of F'. Performing the Gaussian integral and absorbing the result into the normalisation factor of the path integral, we are left with the path integral for the Lagrangian

$$\begin{aligned}\widehat{\mathcal{L}^{(3)}} = -\frac{R}{2}\mathrm{Im}(\tau)\left[- \left(\frac{1}{\mathrm{Im}(\tau)}d\gamma + \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}d\sigma\right) \wedge \star \left(\frac{1}{\mathrm{Im}(\tau)}d\gamma + \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}d\sigma\right)\right. \\ \left. - d\sigma \wedge \star d\sigma\right]. \end{aligned} \tag{2.19}$$

After some straightforward algebraic manipulations, and rescaling

$$\tilde{\gamma} = R\gamma, \tilde{\sigma} = R\sigma : M^{(3)} \to \mathbb{R}/\mathbb{Z}, \tag{2.20}$$

this becomes

$$\widehat{\mathcal{L}^{(3)}} = \frac{1}{2\mathrm{Im}(\tau)R}(d\tilde{\gamma} + \tau d\tilde{\sigma}) \wedge \star(d\tilde{\gamma} + \bar{\tau} d\tilde{\sigma}). \tag{2.21}$$

Now consider the elliptic curve with modulus τ, $E_\tau = \frac{\mathbb{C}}{\mathbb{Z}\oplus\tau\mathbb{Z}}$, with (local) coordinates $z, \bar{z}$ induced from $\mathbb{C}$. Equip E_τ with a rescaling of the standard metric

$$g := \frac{1}{2\mathrm{Im}(\tau)R}dz d\bar{z}. \tag{2.22}$$

Let $\|\cdot\|$ be the norm induced on $T^*M^{(3)} \otimes TE_\tau$ by the metrics η and g. Then setting

$$\Phi(x) = \tilde{\gamma}(x) + \tau\tilde{\sigma}(x), \tag{2.23}$$

which is well-defined since $\tilde{\gamma}$ and $\tilde{\sigma}$ are $\mathbb{R}/\mathbb{Z}$-valued, we have that

$$\frac{1}{2}\|d\Phi\|^2 d\mathrm{vol}_\eta = \frac{1}{2\mathrm{Im}(\tau)R}(d\tilde{\gamma} + \tau\tilde{\sigma}) \wedge \star(d\tilde{\gamma} + \bar{\tau}\tilde{\sigma}). \tag{2.24}$$

Written this way, we recognise $\widehat{\mathcal{L}^{(3)}}$ as the Lagrangian for a σ-model of maps $\Phi : (M^{(3)}, \eta) \to (E_\tau, g)$ (c.f. Example 4). We can calculate the volume of the target space: the volume form is given by

$$d\mathrm{vol}_g = \frac{i}{4\mathrm{Im}(\tau)R} dz \wedge d\bar{z} = \frac{1}{2\mathrm{Im}(\tau)R} d\mathrm{Re}(z) \wedge d\mathrm{Im}(z) \tag{2.25}$$

and so by integrating over the fundamental domain given by the parallelogram defined by the vectors 1 and τ in $\mathbb{C}$, we find that

$$\mathrm{vol}(E_\tau, g) = \frac{1}{2\mathrm{Im}(\tau)R}\int_0^{\mathrm{Im}(\tau)}\int_0^1 d\mathrm{Re}(z) d\mathrm{Im}(z) = \frac{1}{2R}. \tag{2.26}$$

2.1.4.4 Compactification to two dimensions

We now compactify on another circle, this time of radius r, by taking $M^{(3)} = \Sigma \times S^1_r$ for Σ a 2-manifold equipped with a Lorentz signature metric η. An analysis of the Fourier modes similar to the one performed for the reduction to three dimensions shows that the nonzero Fourier modes correspond to massive particles which may be ignored in the low energy effective theory.

Thus we may derive the IR effective Lagrangian for the 2d reduced theory by simply integrating $\widehat{\mathcal{L}^{(3)}}$ on the circle of radius r, with all fields independent of the circle direction. This results in the Lagrangian

$$\mathcal{L}^{(2)} = \frac{1}{2} \cdot 2\pi r \|d\Phi\|^2 d\mathrm{vol}_\eta. \tag{2.27}$$

Rescale the metric on E_τ to

$$h_{r,R} := 2\pi r g = \frac{\pi r}{\mathrm{Im}(\tau) R} dz d\bar{z}. \tag{2.28}$$

Then

$$\mathcal{L}^{(2)} = \frac{1}{2} \|d\Phi\|^2 d\mathrm{vol}_\eta \tag{2.29}$$

is the Lagrangian for a sigma model of maps $\Phi : (\Sigma, \eta) \to (E_\tau, h_{r,R})$, where the target space now has volume

$$\mathrm{vol}(E_\tau, h_{r,R}) = \pi \frac{r}{R}. \tag{2.30}$$

Now, we could have chosen to reduce on the two circles in the *opposite* order, resulting in a 2d sigma model with target space $(E_\tau, h_{R,r})$ of volume $\mathrm{vol}(E_\tau, h_{R,r}) = \pi \frac{R}{r}$. These two theories are dual to each other: concretely, they correspond to different choices of three manifold on which we dualise the 3d gauge field ($\Sigma \times S^1_r$ or $\Sigma \times S^1_R$). So we see that:

Quasi-Theorem 2.1.1. *The Kähler manifold $(E_\tau, h_{R,r})$ is related by a nontrivial duality to the Kähler manifold $(E_\tau, h_{r,R})$.*

Remark 2.1.10. Note that the target space E_τ is an elliptic curve and is therefore self-dual as an abelian variety (this is the manifest duality that was promised at the beginning of Section 2.1.4).

To conclude this section I will make one final observation (which serves as foreshadowing for Section 2.4). Namely, if we calibrate the radii of the two circles so that $r = R$ then the metric on the target space becomes independent of the radii, and we find that:

Quasi-Corollary 2.1.2. *The Kähler manifold* $\left(E_\tau, \frac{\pi}{Im(\tau)} dz d\bar{z}\right)$ *is related to itself by a nontrivial* self-duality *of quantum field theories.*

2.2 Relative quantum field theories

Given a QFT with a global symmetry, Example 5 and Footnote 6 describe how this global symmetry might be promoted to a local symmetry through a "coupling and gauging" procedure. It is possible for this procedure to be obstructed, however, as the next example demonstrates.

Example 9. Let Σ be a Riemann surface with nondegenerate metric η, and let G be a finite dimensional Lie group equipped with a left-invariant Riemannian metric that is induced by a left-invariant metric ρ on $\mathfrak{g}$ (I will suggestively write $\rho(x, y) = \mathrm{tr}(xy)$). The Lagrangian for a σ-model of maps $g : \Sigma \to G$ is given by[16]

$$\mathcal{L}(x) = \frac{1}{2}\|dg(x)\|^2 d\mathrm{vol}_\eta(x). \tag{2.32}$$

[16]This is often written (e.g. [84]) in local coordinates and for a matrix Lie group G as

$$\|dg(x)\|^2 = \eta^{ij}\,\mathrm{tr}\left(g(x)^{-1}\frac{\partial g}{\partial x^i}(x) g(x)^{-1}\frac{\partial g}{\partial x^j}(x)\right). \tag{2.31}$$

Including also an interesting topological term $\Gamma(g)$ [81, 82], the *Wess-Zumino term*, we obtain the *WZW action*

$$S[g] := -\frac{1}{8\pi}\int_{\Sigma} \|dg(x)\|^2 d\mathrm{vol}_{\eta}(x) - i\Gamma(g) \tag{2.33}$$

which is well-defined moduli $2\pi i\mathbb{Z}$ (once the trace-form has been appropriately normalised). Then the *Wess-Zumino-Witten (WZW) model at level $k \in \mathbb{Z}$* is the 2d conformal QFT defined by the path integral

$$\mathcal{Z}_k(\Sigma) := \int_{\mathrm{Map}(\Sigma,G)} \mathcal{D}g e^{-kS[g]}. \tag{2.34}$$

The action $S[g]$ is invariant under the action of $G \times G$ on G given by $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$, so the WZW model has a global $G \times G$ symmetry. Considering only the right multiplication action, the WZW model has a global G-symmetry: we would like to (1) couple to and (2) gauge this G-symmetry.

Unfortunately,[17] there is a well-known obstruction to gauging this G-symmetry [84]. Namely, while it is possible to couple the theory to a G-bundle with connection – i.e. redefine g to be a section of a G-bundle P with connection ∇ on Σ and define an action $S[g, \nabla]$ such that at the trivial connection d_{dR}, $S[g, d_{\mathrm{dR}}] = S[g]$ – it is not possible to do so in a gauge invariant way. The best one can achieve is an action whose behaviour under a gauge transformation $h : \Sigma \to G$ is

$$S[gh^{-1}, (P, h^*(\nabla))] = S[g, (P, \nabla)] + A[h, (P, \nabla)] \tag{2.35}$$

[17] Or fortunately, depending on your perspective.

where the anomalous term A depends only on h and ∇ (not on g or the complex structure on Σ). Consider the path integral of this G-coupled theory

$$\mathcal{Z}_k(\Sigma; (P, \nabla)) := \int_{\mathrm{Map}_\Sigma(\Sigma, P)} \mathcal{D}g e^{-kS[g,(P,\nabla)]} \tag{2.36}$$

which we may think of as a function on the space $\mathcal{A}(\Sigma, G)$ of all G-connections on Σ. The transformation law (2.35) for $S[g, (P, \nabla)]$ implies that the path integral transforms as

$$\mathcal{Z}_k(\Sigma; (P, h^*(\nabla))) = \alpha(h, \nabla)\mathcal{Z}_k(\Sigma; (P, \nabla)) \tag{2.37}$$

where the anomalous multiplicative term α again depends only on h and ∇. α is non-trivial (this is the aforementioned obstruction to gauging the global G-symmetry), however there is a $\mathrm{Map}(\Sigma, G)$-equivariant holomorphic line bundle $\mathcal{L}^{\otimes k}$ on $\mathcal{A}(\Sigma, G)$, the (k^{th} tensor power of the) *prequantum line bundle*, and $\mathcal{Z}_k(\Sigma; (P, \nabla))$ **can** be interpreted as a gauge invariant holomorphic section of $\mathcal{L}^{\otimes k}$ (and therefore as a physical state in 3d Chern-Simons gauge theory at level k [83]).

Remark 2.2.1. In [84, §2.2] Witten describes how the line bundle $\mathcal{L}$ and the path integral $\mathcal{Z}_k$ descend to give a holomorphic section of a line bundle (also denoted $\mathcal{L}$) on $\mathcal{M}(\Sigma, G)$, the moduli space of *flat* connections modulo gauge transformations. I.e. the path integral $\mathcal{Z}_k(\Sigma)$ is an element of the finite dimensional vector space $H^0(\mathcal{M}(\Sigma, G); \mathcal{L}^{\otimes k})$, which is the vector space assigned to the surface Σ in Chern-Simons gauge theory [83, §3.1].

The physical notion of an anomalous field theory (as in Example 9) is captured by the following formalism of Freed and Teleman.

Definition 2.2.1 (Relative Quantum Field Theory [30])**.** Given a $(d+1)$-dimensional QFT α, denote by $\alpha_{\leq d}$ its truncation to manifolds of dimension $\leq d$. Then a *quantum field theory* $\mathcal{Q}$ *relative to* α is either a homomorphism

$$\mathcal{Q} : (\mathrm{triv}^{d+1})_{\leq d} \to \alpha_{\leq d}, \tag{2.38}$$

or a homomorphism

$$\mathcal{Q} : \alpha_{\leq d} \to (\mathrm{triv}^{d+1})_{\leq d}. \tag{2.39}$$

Example 10. To see why Definition 2.2.1 captures the anomaly of the WZW model, observe that a relative QFT $\mathcal{Q} : (\mathrm{triv}^{d+1})_{\leq d} \to \alpha_{\leq d}$ assigns to a d-manifold M^d a linear map

$$\mathcal{Q}(M^d) : (\mathrm{triv}^{d+1})(M^d) = \mathbb{C} \to \alpha(M^d) \tag{2.40}$$

or equivalently (by taking the image of $1 \in \mathbb{C}$) $\mathcal{Q}(M^d) \in \alpha(M^d)$. But now letting $\mathcal{Q} = \mathcal{Z}_k$ be the WZW model and $\alpha = CS_{G,k}$ be Chern-Simons gauge theory at level k, the conclusion of Example 9 translates into

$$\mathcal{Z}_k(\Sigma) \in H^0(\mathcal{M}(\Sigma, G); \mathcal{L}^{\otimes k}) = CS_{G,k}(\Sigma). \tag{2.41}$$

Remark 2.2.2. Following [30], when I wish to emphasise that a QFT is **not** relative (i.e. it is an QFT in the sense of Quasi-Definition 2.1.1) I will say that it is an *absolute* d-dimensional QFT.

Example 11. A QFT relative to triv^{d+1} is an absolute d-dimensional QFT.

The observation of Example 11 leads to the following definition:

Definition 2.2.2. An *absolution* of a relative QFT $\mathcal{Q} : (\mathrm{triv}^{d+1})_{\leq d} \to \alpha_{\leq d}$ (resp. $\mathcal{Q} : \alpha_{\leq d} \to (\mathrm{triv}^{d+1})_{\leq d}$) is another relative QFT $\mathcal{A} : \alpha_{\leq d} \to (\mathrm{triv}^{d+1})_{\leq d}$ (resp. $\mathcal{A} : (\mathrm{triv}^{d+1})_{\leq d} \to \alpha_{\leq d}$). If $\mathcal{A}$ is an absolution of $\mathcal{Q}$, say that $\mathcal{A}$ *absolves* $\mathcal{Q}$.[18]

With this definition, the composition of a relative QFT with an absolution is an absolute d-dimensional QFT $\mathcal{A} \circ \mathcal{Q} : (\mathrm{triv}^{d+1})_{\leq d} \to (\mathrm{triv}^{d+1})_{\leq d}$. I will postpone examples of absolutions to Sections 2.3 and 2.4.

2.2.1 Theory $\mathfrak{X}$

One motivation for introducing the formalism of relative QFTs in [30] was the desire to understand the structure of a mysterious 6-dimensional theory discovered in [72,85], particularly those features predicted in [86, §4] which relate to the geometric Langlands program (I defer discussion of this relation to Section 2.3). This theory, known as *Theory* $\mathfrak{X}$, is a 6d (0,2)-superconformal field theory with no known (or expected[19]) classical description.

As explained in [30, Data 5.1], the data required to specify a Theory $\mathfrak{X}$ is

(1) A real Lie algebra $\mathfrak{g}$ with an invariant inner product $\langle -, - \rangle$ such that all coroots have square length 2, and

(2) A full lattice Γ in a choice of Cartan subalgebra $\mathfrak{h}$, such that Γ contains the coroot lattice of $\mathfrak{g}$ and such that $\langle -, - \rangle$ is integral and even on Γ.

[18] *"Absolution is the process that frees a quantum field theory from the sin of being relative."* [61]

[19] Arguments for why Theory $\mathfrak{X}$ cannot be the quantisation of a classical theory may be found in [86, §4]. One observation that argues against the existence of a Lagrangian is that after a perturbation, the IR physics may be described as a theory of gerbes with a self-dual curvature 3-form H; however, the standard kinetic term for such a field would be $H \wedge \star H = H \wedge H = 0$.

The conditions placed on the inner product imply that the Lie algebra $\mathfrak{g}$ must be reductive with simply-laced semisimple subalgebra. The case of an abelian Lie algebra leads to a theory that is expected to be non-interacting, and I will not discuss this case. Instead, I will focus on the case where $\mathfrak{g}$ is semisimple (or usually just simple) and simply-laced, the lattice is exactly the coroot lattice Π_R. The inner product is then forced to be a specific normalisation of the Killing form of $\mathfrak{g}$.

Remark 2.2.3. Note that if $\widetilde{G}$ is the simply-connected Lie group with Lie algebra $\mathfrak{g}$, the centre of the group may be expressed as $Z(\widetilde{G}) = \Pi_W/\Pi_R$ (notation as in (1.2)), and the inner product $\langle -, - \rangle$ induces a symmetric perfect pairing $Z(\widetilde{G}) \times Z(\widetilde{G}) \to U(1)$.

Given the above data, [30, Expectation 5.3] predicts the existence of a 7d TQFT $\alpha_{\mathfrak{g}}$ and a 6d QFT $\mathfrak{X}_{\mathfrak{g}}$ relative to $\alpha_{\mathfrak{g}}$. Explicitly, at the first two category levels:

- To a 6-manifold X $\alpha_{\mathfrak{g}}$ assigns a (finite dimensional) vector space, and the partition function of $\mathfrak{X}_{\mathfrak{g}}$ is a vector $\mathfrak{X}_{\mathfrak{g}}(X) \in \alpha_{\mathfrak{g}}(X)$.
- To a 5-manifold Y $\alpha_{\mathfrak{g}}$ assigns a linear category,[20] and the space of states of $\mathfrak{X}_{\mathfrak{g}}$ is an object $\mathfrak{X}_{\mathfrak{g}}(Y) \in \alpha_{\mathfrak{g}}(Y)$.

A discussion of the predicted structure of $\alpha_{\mathfrak{g}}$ can be found in [30, §5] – I will restrict my discussion here to a description of the partition vector $\mathfrak{X}_{\mathfrak{g}}(X)$ (following [76,86]).

[20] Modelled on **topological** vector spaces.

Let X be a compact oriented 6-manifold, and consider the middle cohomology group $H^3(X; Z(\widetilde{G}))$. The composition of cup product, the perfect pairing on $Z(\widetilde{G})$, and evaluation against the fundamental class yields a nondegenerate skew pairing

$$\omega : H^3(X; Z(\widetilde{G})) \times H^3(X; Z(\widetilde{G})) \to U(1). \tag{2.42}$$

Such a pairing defines a $U(1)$ central extension known as the *Heisenberg group*,

$$1 \to U(1) \to \mathcal{H}(X, \omega) \to H^3(X; Z(\widetilde{G})) \to 0 \tag{2.43}$$

characterised by the property that any lifts[21] $\Phi(a), \Phi(b) \in \mathcal{H}(\omega)$ of elements $a, b \in H^3(X; Z(\widetilde{G}))$ will satisfy the Heisenberg commutation relation

$$\Phi(b)\Phi(a) = \omega(a, b)\Phi(a)\Phi(b). \tag{2.44}$$

The Stone-von Neumann Theorem [65, Ch.2] states that up to non-canonical isomorphism there is a unique irreducible representation of $\mathcal{H}(X, \omega)$ on which the central $U(1)$ acts via scalar multiplication. Then $\alpha_{\mathfrak{g}}(X)$ is supposed to be the underlying vector space of this representation.

Here we encounter a problem which, to the best of my knowledge, remains unresolved: namely, to define $\alpha_{\mathfrak{g}}(X)$ it is not sufficient to provide an isomorphism class of vector spaces – one must specify a *representative* for this isomorphism class. This requires a choice of Lagrangian (i.e. maximal isotropic) subgroup $L \subset H^3(X; Z(\widetilde{G}))$[22]

[21]Note that Φ cannot be a homomorphism (the extension is non-split).

[22]In the interests of radical transparency, it also requires a choice of splitting $L \to \mathcal{H}(X, \omega)$, which is important but which I will deemphasise in this narrative.

(the representation is constructed by considering a class of L-invariant functions): denote the corresponding representation by $\alpha_{\mathfrak{g}}(X; L)$.

Now, given two choices of Lagrangian subgroup L_1 and L_2[23] there **is** a canonical "Fourier transform" isomorphism $\alpha_{\mathfrak{g}}(X; L_1) \to \alpha_{\mathfrak{g}}(X; L_2)$, providing a glimmer of hope that the vector space might be canonically defined after all! Unfortunately our hope is destined to be dashed upon the rocks of reality: given *three* Lagrangian subgroups L_1, L_2 and L_3, the composition

$$\alpha_{\mathfrak{g}}(X; L_1) \to \alpha_{\mathfrak{g}}(X; L_2) \to \alpha_{\mathfrak{g}}(X; L_3) \to \alpha_{\mathfrak{g}}(X; L_1) \tag{2.45}$$

is not necessarily the identity, but is instead multiplication by some scalar $c(L_1, L_2, L_3)$ [65, Ch.4]. Therefore, absent a choice of Lagrangian subgroup, the canonically defined object is really

$$\mathbb{P}\alpha_{\mathfrak{g}}(X) := \mathbb{P}(\alpha_{\mathfrak{g}}(X; L)) \quad \text{for any Lagrangian subgroup } L. \tag{2.46}$$

Following [76], I will set this problem aside for the moment in favour of choosing a decomposition $H^3(X; Z(\widetilde{G})) \cong A \oplus B$ where A, B are maximal isotropic (and so in duality with each other via the pairing ω), and choosing splittings $\Phi_A : A \to \mathcal{H}(X, \omega)$ and $\Phi_B : B \to \mathcal{H}(X, \omega)$. The action of the elements $\Phi_A(a)$ on $\alpha_{\mathfrak{g}}(X; A)$ may be simultaneously diagonalised by a basis $\{Z_b(X)\}_{b \in B}$ on which the action of $\mathcal{H}(X, \omega)$ is determined by

$$\Phi_A(a) Z_b(X) = \omega(a, b) Z_b(X) \quad \text{and} \quad \Phi_B(b) Z_{b'}(X) = Z_{b+b'}(X). \tag{2.47}$$

[23] Satisfying a compatibility condition which depends on the splittings of Footnote 22.

Then the partition vection of Theory $\mathfrak{X}$ (with respect to all the choices we have been forced to make) is given by

$$\mathfrak{X}_{\mathfrak{g}}(X) = (Z_b(X))_{b\in B} \in \alpha_{\mathfrak{g}}(X;A). \tag{2.48}$$

Remark 2.2.4. Suppose you were to now choose *another* Lagrangian subgroup $L \subset H^3(X; Z(\widetilde{G}))$ (and splitting Φ_L), not necessarily related to A or B. Then the space of L-invariants $\alpha_{\mathfrak{g}}(X;A)^L$ is 1-dimensional, and so the projection of $\mathfrak{X}_{\mathfrak{g}}(X)$ to this subspace gives us an honest partition function $\mathfrak{X}_{\mathfrak{g}}(X;L)$. For $L = A$ this is given by $\mathfrak{X}_{\mathfrak{g}}(X;A) = Z_0$, while for $L = B$ it is given by $\mathfrak{X}_{\mathfrak{g}}(X;B) = \sum_{b\in B} Z_b$.

This suggests that if one could specify a choice of such a subgroup $L(X)$ in a consistent/functorial manner for all X, this might be enough to determine an absolution of Theory $\mathfrak{X}$.

Remark 2.2.5. Following on from Remark 2.2.4, in Sections 2.3 and 2.4 I will explain how upon compactification to a lower dimensional theory we can find ways to consistently choose maximal isotropic splittings $A(X) \oplus B(X)$ and Lagrangian subgroups $L(X)$. The problems of well-definedness which plagued our discussion of Theory $\mathfrak{X}$ are then in some sense resolved once we pass to compactifications – this is one way in which we understand the theories obtained from Theory $\mathfrak{X}$ better than we understand Theory $\mathfrak{X}$ itself.

Example 12 (Theories of class $\mathcal{S}$.)**.** The QFTs that are most relevant to this dissertation are the *theories of class* $\mathcal{S}$ of Gaiotto, Moore and Neitzke [32]. This class of theories is obtained by compactifying (a particular twist[24] of) Theory $\mathfrak{X}_{\mathfrak{g}}$ on a Riemann surface C (potentially with decorated punctures, although I will not analyse the punctured case). The resulting theory, denoted $\mathcal{S}_{\mathfrak{g}}[C]$, is still a relative QFT [31, 76]: in Section 2.4 I will discuss how to absolve $\mathcal{S}_{\mathfrak{g}}[C]$ using the observation of Remark 2.2.4.

2.3 Geometric Langlands from Theory $\mathfrak{X}$

Before discussing absolutions and dualities of $\mathcal{S}_{\mathfrak{g}}[C]$, it is instructive to consider a more familiar duality: S-duality[25] in 4d $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, henceforth SYM or $\mathrm{SYM}_G(\tau)$ when I wish to specify the gauge group G and complexified coupling constant τ (c.f. Section 2.1.4.1), and its relation to the geometric Langlands program [45].

As outlined in [86] and discussed in [76], one way to construct $\mathcal{N} = 4$ SYM is by compactifying Theory $\mathfrak{X}$ on an elliptic curve. Restricting to the case of a simply-laced[26] gauge group G, this construction works as follows:

Consider the theory $\mathfrak{X}_{\mathfrak{g}}$ where $\mathfrak{g}$ is a simply-laced simple Lie algebra, with corresponding simply-connected group $\widetilde{G}$ and adjoint group G_{ad}. Choose a complexified coupling constant $\tau \in \mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$, and consider the elliptic curve with modulus τ, $E_\tau = \frac{\mathbb{C}}{\mathbb{Z} \oplus \tau\mathbb{Z}}$. Note that it is important to remember τ as an element

[24] For a discussion of topological twisting one can consult the book [42]; for references that discuss the twistings relevant to theories of class $\mathcal{S}$ see [58, 75].

[25] A.k.a. electric-magnetic duality [34], a.k.a. Montonen-Olive duality [57].

[26] Construction of SYM for non-simply laced gauge group involves "folding" of Lie algebras [8], the details of which are unclear to me (particularly for theories with no description in terms of classical fields).

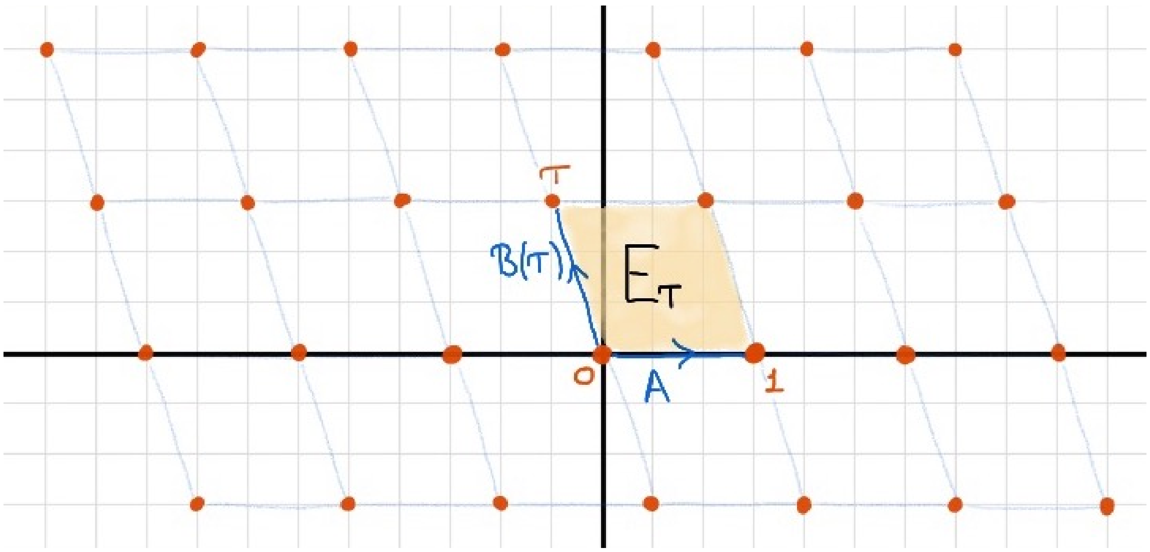

Figure 2.1: Lattice defining E_τ, with fundamental domain shaded and distinguished cycles A and $B(\tau)$ labeled.

of the upper half-plane and **not** just as an element of the parameter space of elliptic curves, $\mathbb{H}/SL_2(\mathbb{Z})$, as this then specifies a consistent choice of basis ("A" and "B" cycle) for the homology of E_τ,

$$H_1(E_\tau) = \mathbb{Z}A \oplus \mathbb{Z}B(\tau), \tag{2.49}$$

by taking the image of the straight-line paths $0 \to 1$ and $0 \to \tau$ in E_τ (see Figure 2.1). Denote the Poincaré duals to A and $B(\tau)$ in $H^1(E_\tau; Z(\widetilde{G}))$ by W and $H(\tau)$ respectively, so that we have a splitting

$$H^1(E_\tau; Z(\widetilde{G})) = Z(\widetilde{G})W \oplus Z(\widetilde{G})H(\tau) \tag{2.50}$$

into electric (**W**ilson) and magnetic ('t **H**ooft) lines.

This data is sufficient to give a well-defined 4d relative QFT $\mathfrak{X}_\mathfrak{g}[E_\tau]$: suppose that M is a 4-manifold with $H^1(M; Z(\widetilde{G})) = H^3(M; Z(\widetilde{G})) = 0$.[27] There is a decomposition

$$H^3(M \times E_\tau; Z(\widetilde{G})) = \underbrace{\Big(H^2(M; Z(\widetilde{G})) \otimes W\Big)}_{\text{electric fluxes through 2-cycles of } M} \oplus \underbrace{\Big(H^2(M; Z(\widetilde{G})) \otimes H(\tau)\Big)}_{\text{magnetic fluxes through 2-cycles of } M} \tag{2.51}$$

and so we can define a basis $\{Z_\nu(\tau; M)\}_{\nu \in H^2(M; Z(\widetilde{G}))} = \{Z_{\nu \otimes H(\tau)}(M \times E_\tau)\}_{\nu \in H^2(M; Z(\widetilde{G}))}$ of $\alpha_{\mathfrak{g}}(\tau; M) = \alpha_{\mathfrak{g}}(M \times E_\tau; H^2(M; Z(\widetilde{G})) \otimes W)$ that satisfies the relations (2.47). Then the partition vector of the 4d theory $\mathfrak{X}_{\mathfrak{g}}[E_\tau]$ is

$$\mathfrak{X}_{\mathfrak{g}}[E_\tau](M) = (Z_\nu(\tau; M))_{\nu \in H^2(M; Z(\widetilde{G}))} \in \alpha_{\mathfrak{g}}(\tau; M). \tag{2.52}$$

The component $Z_\nu(\tau; M)$ may be identified as the partition function of $\mathcal{N} = 4$ SYM with gauge group G_{ad}, restricted to G_{ad}-bundles with characteristic class $\nu \in H^2(M; Z(\widetilde{G}))$ [76, 78, 85]. So, consider the absolutions of $\mathfrak{X}_{\mathfrak{g}}[E_\tau]$ given by the maximal isotropic subgroups $H^2(M; Z(\widetilde{G})) \otimes W$ and $H^2(M; Z(\widetilde{G})) \otimes H(\tau)$ – according to Remark 2.2.4 these have partition functions given by

$$Z_0(\tau; M) = \mathrm{SYM}_{\widetilde{G}}(\tau; M) \tag{2.53}$$

$$\sum_{\nu \in H^2(M; Z(\widetilde{G}))} Z_\nu(\tau; M) = \mathrm{SYM}_{G_{\mathrm{ad}}}(\tau; M) \tag{2.54}$$

i.e. the absolutions are SYM with gauge group the simply-connected group (2.53) and adjoint group (2.54) respectively.

Remark 2.3.1. Other choices of gauge group can be achieved by considering isotropic subgroups formed by taking linear combinations $eW + mH(\tau)$. The situation for type A Lie algebras is spelled out in [76, §4.2].

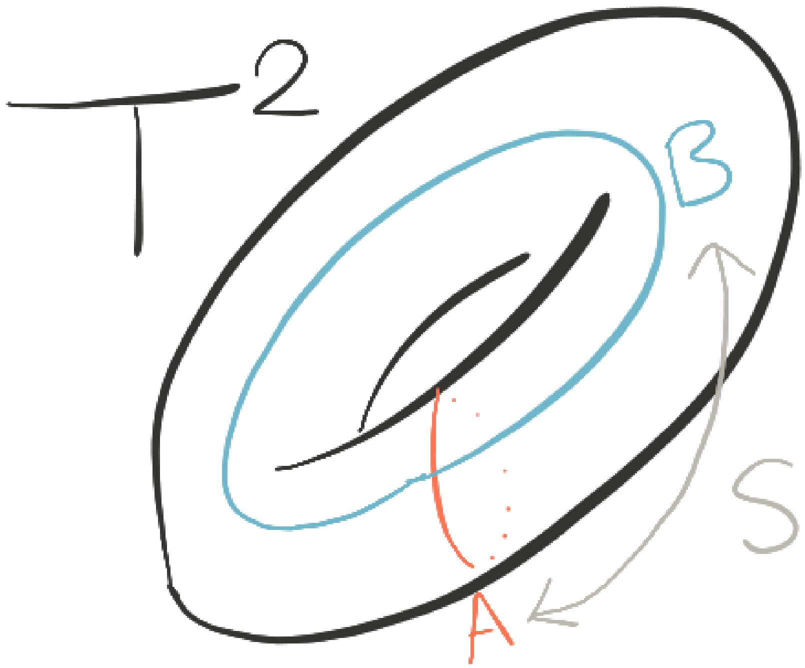

Figure 2.2: $SL_2(\mathbb{Z})$ acts on the data that determines the SYM gauge group.

Now, consider the action of the S-transformation in the mapping class group of the torus, $\mathrm{MCG}(T^2) = SL_2(\mathbb{Z})$ (Figure 2.2). $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts simultaneously on $\mathbb{H}$ and $H^1(T^2)$ via

$$\tau \mapsto -\frac{1}{\tau} \quad \text{and} \quad \begin{matrix} W \\ H \end{matrix} \mapsto \begin{matrix} -H \\ W \end{matrix} . \tag{2.55}$$

This action then induces a duality of QFTs that on the level of partition functions is given by

$$\mathrm{SYM}_{\widetilde{G}}(\tau; M) = Z_0(\tau; M) = \sum_{\nu} Z_\nu \left(-\frac{1}{\tau}; M\right) = \mathrm{SYM}_{G_{\mathrm{ad}}} \left(-\frac{1}{\tau}; M\right) \tag{2.56}$$

Since for simply-laced groups ${}^L(\widetilde{G}) = G_{\mathrm{ad}}$, this provides a geometric realisation of S-duality for SYM; the physical duality of QFTs is the shadow of a geometric symmetry only visible from the point of view of Theory $\mathfrak{X}$.

The relation between S-duality in SYM_G and the geometric Langlands program was analysed in [45] by further compactifying on a Riemann surface C to obtain a 2d QFT, $\sigma_G[C]$, which is a σ-model with target space the moduli of G-Higgs bundles $\mathbf{Higgs}_G(C)$ (see Section 3.3.4 for the definition). There is a continuous family of topological twists for any such σ-model [45, §5], and S-duality thus predicts a family of dualities between the resulting topological field theories, which I will now briefly review.

[27]These assumptions are not necessary, but they simplify the following formulae.

2.3.1 Dolbeault geometric Langlands

The first twist of $\sigma_G[C]$ to consider gives rise to the B-model (Example 7) on the moduli of Higgs bundles. The category of branes[28] for the B-model is given by the bounded derived category of coherent sheaves on the target space, and it is known that S-duality in 4d SYM descends to T-duality[29] of the target spaces for the corresponding 2d σ-models [12, 36].

S-duality of SYM therefore predicts an equivalence of categories

$$D^b_{coh}(\mathbf{Higgs}_G(C)) \simeq D^b_{coh}(\mathbf{Higgs}_{{}^LG}(C)), \tag{2.57}$$

sometimes called the *Dolbeault geometric Langlands conjecture.* The mathematical status of this conjecture is given by the following theorem, proven by Hausel and Thaddeus in the case of $G = SL_n(\mathbb{C})$ [37] and by Donagi and Pantev for arbitrary reductive groups [24]:

Theorem 2.3.1. *Over a dense subset of the base of the Hitchin fibration there is an equivalence of derived categories of coherent sheaves*

$$D^b_{coh}(\mathcal{H}iggs_G(C)) \simeq D^b_{coh}(\mathcal{H}iggs_{{}^LG}(C)), \tag{2.58}$$

implemented by a fibrewise Fourier-Mukai transform.

Remark 2.3.2. The definitions of the moduli *stack* of Higgs bundles $\mathcal{H}iggs_G(C)$ and the Hitchin fibration may be found in Sections 3.3.1 and 3.3.3 respectively.

[28] A.k.a. boundary conditions.
[29] A.k.a. SYZ mirror symmetry (Example 8).

Later in this dissertation, Theorem 5.5.1 will provide a further generalisation of Theorem 2.3.1.

2.3.2 De Rham geometric Langlands

In [45] the authors are most interested not in arbitrary topological twists of $\sigma_G[C]$, but in twists which are induced by topological twists of 4d SYM. The twist of $\sigma_G[C]$ discussed in Section 2.3.1 is *not* induced by a twist of SYM – however it admits a 2-parameter family of deformations which are, and which are the primary focus of [45].

The parameters, call them ϵ_1 and ϵ_2, both deform $\mathbf{Higgs}_G(C)$, but they do so in nonsymmetric ways:

(1) The first deforms the *complex structure* of $\mathbf{Higgs}_G(C)$, and via the non-abelian Hodge theorem the resulting space may be identified as the space $\mathbf{LocSys}_G(C)$ of flat G-connections on C [70]. The resulting theory is the B-model on $\mathbf{LocSys}_G(C)$.

(2) The second quantizes the symplectic structure, which via the identification $\mathcal{O}(\mathbf{Higgs}_G(C)) = \mathcal{O}(T^*\mathbf{Bun}_G(C))$[30] results in the "non-commutative" space of differential operators (or "$\mathcal{D}$-modules") on the moduli of G-bundles on C, $\mathbf{Bun}_G(C)$. The resulting theory (whose category of branes is $\mathcal{D}$-mod$(\mathbf{Bun}_G(C))$) may be identified with the A-model on $\mathbf{LocSys}_G(C)$.[31]

[30] On the level of stacks, $\mathcal{H}iggs_G(C) = T^*\mathcal{B}un_G(C)$; on the level of coarse moduli spaces there is an open dense inclusion $T^*\mathbf{Bun}_G(C) \subset \mathbf{Higgs}_G(C)$, whose complement has high codimension.

[31] The complex structure on $\mathbf{LocSys}_G(C)$ is a priori different here to the complex structure for the B-model. See [45, §5.3] for details.

Moreover, under S-duality the two parameters (ϵ_1, ϵ_2) transform in the fundamental representation of $SL_2(\mathbb{Z})$,[32] and so S-duality between categories of branes becomes:

Conjecture (De Rham Geometric Langlands)**.** There is an equivalence of derived categories

$$D^b_{coh}(\mathbf{LocSys}_{^L G}(C)) \simeq \mathcal{D}\text{-mod}(\mathbf{Bun}_G(C)).$$

Remark 2.3.3. The above conjecture is often just called "**The** *geometric Langlands conjecture*", and is due to Laumon, Beilinson and Drinfeld [9, 10, 28, 50]. There are known counterexamples to Conjecture 2.3.2 exactly as it is stated here; the state of the art formulation (which requires the use of derived algebraic geometry) may be found in the paper [3] of Arinkin and Gaitsgory.

Remark 2.3.4. The nomenclature "de Rham" (and "Dolbeault" in Section 2.3.1) is by analogy with the terminology used by Simpson for the moduli spaces involved in the non-abelian Hodge theorem [70] (which are the targets of the corresponding 2d σ-models we are discussing). To complete the analogy, there is also a *Betti geometric Langlands* due to Ben-Zvi and Nadler [11].

Remark 2.3.5. Strictly speaking, de Rham geometric Langlands is obtained by turning on a *single* deformation parameter at a time. Turning on both at once results in a "quantum geometric Langlands correspondence" – e.g. see Teschner's exploration of the AGT correspondence [2, 77].

[32] This is because the parameters ϵ_1 and ϵ_2 come from turning on a *Nekrasov Ω-background* [62]. Mathematically this means that we begin to work $E_\tau \cong S^1 \times S^1$-equivariantly, resulting in a family of theories over the graded $H^\bullet_{S^1 \times S^1}(*) = \mathbb{C}[\epsilon_1, \epsilon_2]$-plane.

2.3.3 Geometric Satake

There are two basic types of line operator in SYM: electric *Wilson lines* and magnetic *'t Hooft lines*. Both are derived from *surface* operators in $\mathfrak{X}_{\mathfrak{g}}$ in which one surface direction is "wrapped" around a cycle of E_τ (notation as in (2.49)):

(1) The Wilson lines correspond to surface operators in $\mathfrak{X}_{\mathfrak{g}}$ in which one surface direction is wrapped around the cycle A in E_τ.

(2) The 't Hooft lines correspond to surface operators in $\mathfrak{X}_{\mathfrak{g}}$ in which one surface direction is wrapped around the cycle $B(\tau)$ in E_τ.

(3) There are also mixed line operators corresponding to surface operators with one direction wrapped around a linear combination of the A and $B(\tau)$ cycles.

Not all of these line operators will be compatible with any given topological twisting of $\sigma_G[C]$. In the two topological twists of 4d SYM described in Section 2.3.2 the allowed line operators are:

(1) For the twist that leads to the B-model with target $\mathbf{LocSys}_G(C)$, the only topological line operators are the Wilson lines. The category of Wilson lines is given by the category of representations of G, $\mathrm{Rep}(G)$.

(2) For the twist that leads to A-model with target $\mathbf{LocSys}_G(C)$, the only topological line operators are the 't Hooft lines. The category of 't Hooft lines is given by the category of (equivariant, perverse) sheaves on the affine Grassmannian of G, $\mathcal{P}_{L^+G}(\mathrm{Gr}_G)$.

Since the operator S exchanges the cycles A and $B(\tau)$ on E_τ (see Figure 2.2 again). S-duality exchanges the Wilson and 't Hooft lines, which manifests as the following mathematical theorem:

Theorem 2.3.2 (Geometric Satake Theorem, [33, 53, 56])**.** *There is an equivalence of tensor categories* $Rep({}^L G) \simeq \mathcal{P}_{L^+G}(Gr_G)$.

2.4 Self-dual Geometric Langlands

Now, recall the relative theories of class $\mathcal{S}$ of Example 12. Taking a cue from the requirement in the SYM case that we ought to remember $\tau \in \mathbb{H}$ and not just in $\mathbb{H}/SL_2(\mathbb{Z})$, here I claim that we ought to also remember a pants decomposition of C. The pants decomposition yields a canonical splitting of $H_1(C)$ into A and B cycles [15], and so provides a well-defined partition vector for the relative class $\mathcal{S}$ theory.

As was the case in Section 2.3, in order to absolve this relative theory the extra data required is a choice of Lagrangian subgroup $\Gamma \subset H^1(C; Z(\widetilde{G}))$ (physically: a collection of mutually local line operators [31, 76]). Denote the absolved theory by $\mathcal{S}_{\mathfrak{g}}[C; \Gamma]$.

Now, reduce further on an torus $T^2 = S^1 \times S^1$ where the circles have the same radius and T^2 is equipped with the product metric. The resulting 2d theory, which I will denote by $\Sigma_{\mathfrak{g}}[C; \Gamma]$, is now equipped with an $\mathrm{MCG}(T^2) = SL_2(\mathbb{Z})$ of *self*-dualities (c.f. Quasi-Corollary 2.1.2), as the S^1-factors in the T^2 are indistinguishable and (more importantly) the $SL_2(\mathbb{Z})$-action leaves invariant the data Γ which was

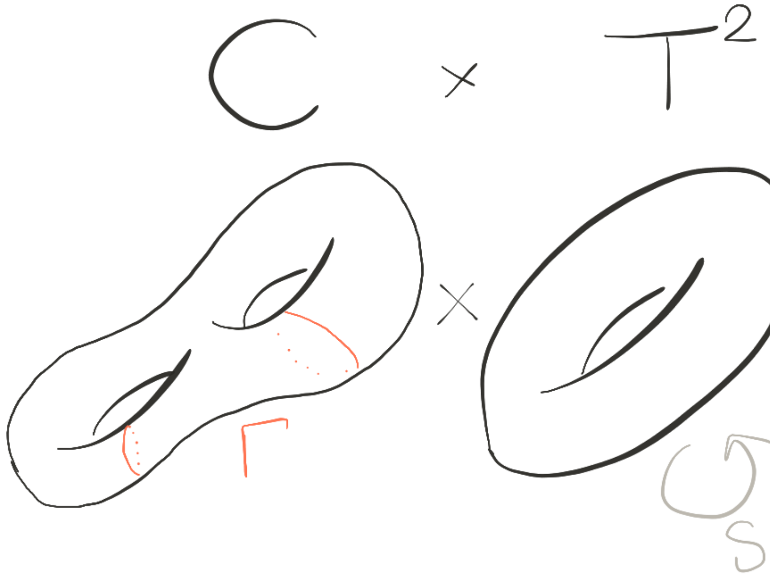

Figure 2.3: $SL_2(\mathbb{Z})$ preserves the data Γ that determines the theory of class $\mathcal{S}$.

required to absolve the theory (Figure 2.3).

By analogy with the work of [45] relating $\mathrm{SYM}_G(\tau)$ and geometric Langlands, we may think of the theories $\Sigma_{\mathfrak{g}}[C;\Gamma]$ as encoding the structure of the following *self-dual geometric Langlands program.*

2.4.1 Self-dual Dolbeault Langlands

There is a canonical torus fibration $\mathcal{M}_{\mathfrak{g}}(C;\Gamma)$ associated to $\mathcal{S}_{\mathfrak{g}}[C;\Gamma]$, called the *Seiberg-Witten integrable system* of $\mathcal{S}_{\mathfrak{g}}[C;\Gamma]$ (in fact such a torus fibration exists for *any* 4d $\mathcal{N}=2$ theory [27,68,69]) – in particular, this integrable system is the moduli space of vacua for the 2d theory $\Sigma_{\mathfrak{g}}[C;\Gamma]$.[33] In [76] (and [31] for type A_1), $\mathcal{M}_{\mathfrak{g}}(C;\Gamma)$ is predicted to be $\frac{\mathbf{Higgs}_{\widetilde{G}}(C)}{\Gamma}$, where the action of $H^1(C; Z(\widetilde{G}))$ on $\mathbf{Higgs}_{\widetilde{G}}(C)$ is that of tensoring by a principal $Z(\widetilde{G})$-bundle.

The self-duality induced by the S-transformation of $\mathrm{MCG}(T^2)$ then suggests the following conjecture:

Conjecture 1 (Self-dual Dolbeault Langlands)**.** The target space of the σ-model $\Sigma_{\mathfrak{g}}[C;\Gamma]$ is self SYZ mirror dual. Moreover, there is a Fourier-Mukai transform implementing a self-equivalence of the derived category $D^b_{coh}(\mathcal{M}_{\mathfrak{g}}(C;\Gamma))$.

In Theorem 5.5.2 and Corollary 5.5.3 I will demonstrate that over a dense open subset of the Hitchin base a "stacky" version of Conjecture 1 holds. Moreover, the neutral component of the coarse moduli space studied in Section 5.5 is exactly the self-dual abelian scheme $\frac{\mathbf{Higgs}_{\widetilde{G}}(C)}{\Gamma}$ predicted by [31, 76].

2.4.2 Self-dual de Rham and quantum Langlands

Just as was the case in Section 2.3.2, the action of $SL_2(\mathbb{Z})$ on $\Sigma_{\mathfrak{g}}[C;\Gamma]$ can be viewed as a discrete shadow of a more refined homotopical $S^1 \times S^1$ symmetry. Working equivariantly with respect to this $S^1 \times S^1$-action (i.e. turning on a Nekrasov Ω-background, c.f. Footnote 32) we again spread our theory out as a family over the $H^\bullet_{S^1\times S^1}(*) \simeq \mathbb{C}[\epsilon_1, \epsilon_2]$-plane, yielding a natural 2-parameter deformation of self-dual Dolbeault Langlands.

[33] In fact, up to a scaling factor in the metric, this is also the target space for the low energy effective theory associated to the 3d QFT obtained by compactifying $\mathcal{S}_{\mathfrak{g}}[C;\Gamma]$ on a circle.

The parameters ϵ_1 and ϵ_2 here have the same origin as the ones that appeared in the deformations of Dolbeault Langlands, and so (ϵ_1, ϵ_2) transforms in the fundamental representation of $SL_2(\mathbb{Z})$. I therefore propose the following:

Conjecture 2 (Self-dual de Rham and quantum Langlands)**.** There exists a two parameter family of theories $\Sigma_{\mathfrak{g}}^{(\epsilon_1,\epsilon_2)}[C;\Gamma]$ that quantises self-dual Dolbeault Langlands, in the sense that $\Sigma_{\mathfrak{g}}^{(0,0)}[C;\Gamma] = \Sigma_{\mathfrak{g}}[C;\Gamma]$. Moreover, S-duality induces an equivalence between the categories of branes $\mathfrak{B}_{\mathfrak{g}}^{(\epsilon_1,\epsilon_2)}[C;\Gamma]$ in both the *self-dual de Rham* (1-parameter, i.e. $\mathfrak{B}_{\mathfrak{g}}^{(\epsilon,0)}[C;\Gamma] \simeq \mathfrak{B}_{\mathfrak{g}}^{(0,\epsilon)}[C;\Gamma]$) and *self-dual quantum* (full 2-parameter) Langlands theories.

Remark 2.4.1. In particular, a satisfactory resolution to Conjecture 2 would require explicitly identifying both the category of branes $\mathfrak{B}_{\mathfrak{g}}^{(\epsilon_1,\epsilon_2)}[C;\Gamma]$ and the corresponding action of S-duality.

Remark 2.4.2. Note that since for $\Sigma_{\mathfrak{g}}[C;\Gamma]$ there do not appear to be any preferred cycles on T^2 (as opposed to the cycles determined by τ for SYM), it is possible that the quantised theories may be completely determined up to equivalence by how many deformation parameters are turned on.

Regardless of this, I do not expect that the *derivation* of the ϵ_1 and ϵ_2 deformations will be symmetric. By analogy with usual geometric Langlands I suspect that one will manifest as a complex structure deformation and one will be a quantisation of the symplectic structure, and that this self-duality will appear as a highly non-trivial identification between a priori different categories.

2.4.3 Self-dual geometric Satake

Finally, recall that the line operators in $\mathrm{SYM}_G(\tau)$ all arose from surface operators in $\mathfrak{X}_{\mathfrak{g}}$ with one direction wrapped around a cycle of the torus E_τ. Since these 6d surface operators are constant on C, upon reducing to $\mathcal{S}_{\mathfrak{g}}[C;\Gamma]$ we obtain a collection of 4d surface operators, labelled by points of C (together with some extra discrete data).

One interesting problem would be to explicitly identify the 2-category of surface operators $\mathfrak{S}_{\mathfrak{g}}[C;\Gamma]$ in the 4d theory $\mathcal{S}_{\mathfrak{g}}[C;\Gamma]$. This, however, is an extremely hard problem. To reduce to a related but potentially more tractable problem, consider wrapping one direction of a surface operator from $\mathfrak{S}_{\mathfrak{g}}[C;\Gamma]$ on a cycle in T^2 to obtain a line operator in $\Sigma_{\mathfrak{g}}[C;\Gamma]$ – I will call such line operators *special*.[34]

In a 2-dimensional theory the collection of line operators form a monoidal category which acts on the category of branes by modifying boundary conditions. Denote the tensor category of special line operators by $\mathfrak{L}_{\mathfrak{g}}[C;\Gamma]$. Since objects of $\mathfrak{L}_{\mathfrak{g}}[C;\Gamma]$ are labelled by 1-cycles on T^2 I expect that $SL_2(\mathbb{Z})$ will act by non-trivial autoequivalences on $\mathfrak{L}_{\mathfrak{g}}[C;\Gamma]$, in a manner compatible with the $SL_2(\mathbb{Z})$-action on $\mathfrak{B}_{\mathfrak{g}}[C;\Gamma]$.

Conjecture 3 (Self-dual geometric Satake conjecture)**.** *S-duality is a non-trivial autoequivalence of $\mathfrak{L}_{\mathfrak{g}}[C;\Gamma]$, intertwining the action of $\mathfrak{L}_{\mathfrak{g}}[C;\Gamma]$ on $\mathfrak{B}_{\mathfrak{g}}[C;\Gamma]$.*

[34]Observe that these operators are the analogs for $\Sigma_{\mathfrak{g}}[C;\Gamma]$ of those line operators in $\sigma_G[C]$ induced by Wilson and 't Hooft lines in SYM.

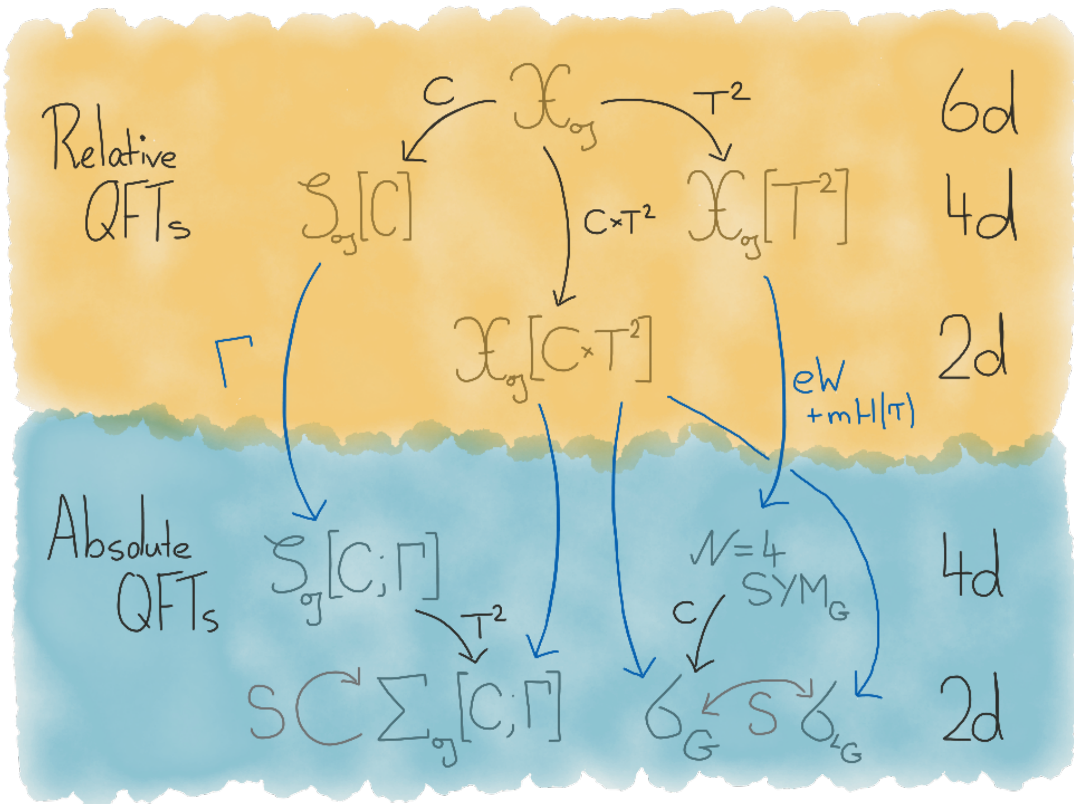

Figure 2.4: QFTs obtained from Theory $\mathfrak{X}$. Black lines are compactifications, blue lines are absolutions, grey lines are S-dualities.

Remark 2.4.3. As in Remark 2.4.1, a satisfactory resolution to Conjecture 3 would require explicitly identifying the tensor category $\mathfrak{L}_{\mathfrak{g}}[C;\Gamma]$ and the autoequivalence determined by S-duality.

Chapter 3

Cartier duality and Higgs bundles

In this chapter I will review the mathematical background prerequisite for the original work of Chapters 4 and 5. There are two broad themes to this material: *Cartier duality of commutative group stacks* (Sections 3.1 and 3.2), and *Higgs bundles and the Hitchin fibration* (Sections 3.3 and 3.4).

In Section 3.1 I recall the definition of a commutative group stack and give examples. I discuss the concept of an action of a commutative group stack, and give a procedure for constructing examples of such actions.

In Section 3.2 I describe a generalisation of duality for abelian varieties known as shifted Cartier duality. I discuss how Cartier duality acts on various classes of commutative group stacks which are important for this dissertation. As an extended example, in Section 3.2.1 I consider the Cartier dual for the moduli stack of torus bundles – the first mathematic instance of Langlands duality in this thesis.

In Section 3.3 I introduce the moduli stack of Higgs bundles and explore how many geometric structures – chiefly the Hitchin fibration and the Hitchin section – arise naturally out of representation theory via a mapping stack construction. I recall the parametrisation of the Hitchin base in terms of cameral covers, and also briefly discuss the coarse moduli space of semistable Higgs bundles.

Finally, in Section 3.4 I explain how over a dense subset of the Hitchin base the Hitchin fibration may be studied via an abelianisation procedure involving cameral covers. In contrast to abelianisation of Higgs bundles via spectral covers [38, 40] the abelian bundles we consider are not line bundles but are instead bundles for the group scheme of regular centralisers [25, 63, 64] – roughly, bundles with structure group a maximal torus and with nice (abelian) degenerations at the branch points of the cameral cover.

3.1 Commutative group stacks

Categorical background, e.g. material on symmetric monoidal categories, may be found in [46,55]. Background on stacks and descent theory may be found in [51,79]. As always, k denotes an algebraically closed field.

Definition 3.1.1. A *Picard groupoid* is a symmetric monoidal category in which every object is invertible (with respect to the monoidal structure) and every morphism is invertible (in the usual sense).

Remark 3.1.1. Given a Picard groupoid $(\mathcal{C}, \otimes)$ the set[1] of equivalence classes of objects $\pi_0\mathcal{C}$ is a commutative group in a canonical way.

The canonical example of a Picard groupoid, which in particular explains the nomenclature, is as follows:

[1] I am implicitly assuming that $\mathcal{C}$ is essentially small. From now on I will not mention such set-theoretic caveats, as they are not essential to the content of this thesis.

Example 13. Let X be a complex manifold, and consider the category whose objects are holomorphic line bundles on X and whose morphisms are given by isomorphisms of holomorphic line bundles. Tensor product of line bundles endows this category with the structure of a Picard groupoid, and the commutative group obtained by taking π_0 is exactly the *Picard group* of holomorphic line bundles on X.

Definition 3.1.2. Let X be a space endowed with a Grothendieck topology. A *commutative group stack on* X is a sheaf of Picard groupoids on X.

Remark 3.1.2. I have left the meaning of "space" in Definition 3.1.2 deliberately ambiguous. In this thesis I will consider the following two cases:

(1) X is an algebraic stack over a base scheme S, with the fppf topology (c.f. [1, XVIII 1.4] and [14, 16]).

(2) X is a complex variety with the analytic topology (c.f. [24, 26]).

Example 14. Given two commutative group stacks $\mathcal{A}$ and $\mathcal{B}$ over X there is a commutative group stack $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ whose U-points are given by the category $\mathrm{Hom}_U(\mathcal{A} \times_X U, \mathcal{B} \times_X U)$ [14, Definition 2.4 and Example 2.8].

The following three examples are central to the spaces I will study in Chapter 5:

Example 15. An *abelian variety* A is a group scheme which is a complete variety over k. Given a scheme X, an *abelian scheme* over X is a smooth group scheme over X whose fibres are abelian varieties – this provides a class of examples of commutative group stacks.

Note that when $k = \mathbb{C}$ abelian varieties correspond analytically to projective compact complex tori – this is the situation of most relevance to this thesis. More information on abelian varieties and schemes may be found in [60, 65].

Example 16. More generally, any sheaf of abelian groups K over X may be regarded as a commutative group stack with discrete objects (and trivial automorphisms).

Example 17. Given a sheaf of abelian groups K over X, the classifying stack BK whose U-points are $BK(U) =$ (groupoid of $K|_U$-torsors on U) is a commutative group stack. BK may be presented as the stack quotient $[*/K]$ where $*$ denotes the trivial sheaf of abelian groups on X, c.f. (1.3).

Remark 3.1.3. In light of the presentation $BK = [*/K]$ we may interpret the classifying stack of K over X to be the quotient of X by the trivial action of K. This perspective is especially useful when X is defined to be the parameter space of equivalence classes of objects which admit non-trivial automorphisms: then, even though the group of automorphisms K acts trivially on the space X, the quotient stack $[X/K]$ remembers the fact that the objects parametrised by X admit non-identity automorphisms.

Remark 3.1.4. There is a convenient reformulation of the theory of commutative group stacks in terms of complexes of sheaves, due to Deligne [1, XVIII, 1.4]. Let $\mathrm{Ch}^{[-1,0]}(X)$ denote the 2-category given by:

- Objects are complexes of abelian sheaves on X concentrated in degrees -1 and 0, $A^\bullet = [A^{-1} \to A^0]$, such that A^{-1} is injective.[2]

[2] The injectivity assumption implies that the quotient prestack $[A^0/A^{-1}]$ is already a stack.

- Morphisms are chain maps of complexes.

- 2-morphisms are homotopies of chain maps.

Given a complex of abelian sheaves of the form $A^{-1} \to A^0$ the quotient stack $[A^0/A^{-1}]$ is a commutative group stack on X. This construction gives an equivalence between $\mathrm{Ch}^{[-1,0]}(X)$ and the 2-category of commutative group stacks on X [1, XVIII, 1.4.17]. This may be interpreted as a (length 1) form of the Dold-Kan correspondence between simplicial objects and chain complexes.

3.1.1 Gerbes and "stacky" actions

Following the terminology of [25, 26], consider the following notion of a principal bundle with structure "group" BK:

Definition 3.1.3. 1. Let $\mathcal{P}$ be a Picard groupoid. A category $\mathcal{C}$ is a *gerbe over* $\mathcal{P}$ if

(a) $\mathcal{P}$ acts on $\mathcal{C}$ as a tensor category, and

(b) for any object $C \in \mathcal{C}$ the functor

$$\begin{aligned} \mathcal{P} &\longrightarrow \mathcal{C} \\ P &\longmapsto P \cdot C \end{aligned} \tag{3.1}$$

is an equivalence of categories.

2. Let $\mathcal{A}$ be a commutative group stack over X. A stack $\mathcal{E}$ over X is a *gerbe over* $\mathcal{A}$ if

(a) for every open set[3] U of X, $\mathcal{E}(U)$ is a gerbe over $\mathcal{A}(U)$, compatible with pullbacks [25, §3.6], and

(b) there exists a covering $\mathcal{U} \to X$ such that $\mathcal{E}(\mathcal{U})$ is non-empty (i.e. local sections exist).

If $\mathcal{A} = BK$ for K a sheaf of abelian groups on X I will refer to $\mathcal{E}$ as a *K-gerbe.*

Remark 3.1.5. The definition of a K-gerbe I have given in Definition 3.1.3 is more accurately called a *K-banded gerbe.* Since I am interested only in gerbes for sheaves of **abelian** groups, this abuse of terminology is both quite minor and extremely common in the literature (e.g. [24–26,41]). For the more general definition of a gerbe see [23, II.Appendix].

The actions by commutative group stacks that I will make use of in this thesis are mostly of the following, quite concrete form:

Proposition 3.1.1. *Given the data of*

(1) two groups T and G,

(2) a homomorphism $\zeta : T \to Z(G)$,

(3) a T-module V,

(4) a $G/\zeta(T)$-module M, and

(5) a T-invariant linear map $l : V \to M^G$,

[3] Or étale/fppf map, depending on which Grothendieck topology we are using.

there exists a functor

$$F(\zeta, l) : V/T \times M/G \to M/G \tag{3.2}$$

defined on objects by $l + id_M$ *and on morphisms by* $(t, g) \mapsto \zeta(t)g$. *This descends to an action morphism on the quotient stacks* $[V/T] \times [M/G] \to [M/G]$.

Furthermore, if T *and* G *are equipped with extra geometric structure (e.g. group schemes, Lie groups) and* X *is a geometric space of the same type, there is an induced action morphism on the mapping stacks*

$$\mathcal{M}ap\,(X, [V/T]) \times \mathcal{M}ap\,(X, [M/G]) \to \mathcal{M}ap\,(X, [M/G])\,. \tag{3.3}$$

Proof. Provided the functor $F(\zeta, l)$ is well-defined the induced action on stack quotients follow automatically from the functor of points perspective, and the induced action on mapping stacks follows from this together with the universal property of the product. Well-definedness of $F(\zeta, l)$ follows from an easy calculation. □

Example 18. Suppose $V = M = 0$ and T is a central subgroup of G. Then the induced action (3.3) is given by twisting a G-bundle by a T-bundle:

$$\begin{aligned} \mathcal{B}un_T(X) \times \mathcal{B}un_G(X) &\to \mathcal{B}un_G(X) \\ (L, P) &\mapsto L \otimes P \end{aligned} \tag{3.4}$$

If L and P are defined by Čech 1-cocycles λ_{ij} and g_{ij} respectively, then $L \otimes P$ is defined by the 1-cocycle $\lambda_{ij} \cdot g_{ij}$.

Example 19. In Example 18 suppose that $T = \mathbb{C}^\times$ acts via diagonal matrices on $G = GL_n(\mathbb{C})$. Then the induced action

$$\begin{aligned} \mathcal{P}ic(X) \times \mathcal{B}un_{GL_n}(X) &\to \mathcal{B}un_{GL_n}(X) \\ (L, E) &\mapsto L \otimes E \end{aligned} \tag{3.5}$$

is *literally* given by taking the tensor product of a line bundle L with a rank n vector bundle E.

The following action will be important in Chapter 5.

Example 20. Let $\widetilde{G}$ be a connected and simply-connected simple group, T be an algebraic torus, and $\tau : Z(\widetilde{G}) \to T$ be an embedding. Let $\widetilde{G}_\tau := \frac{\widetilde{G}\times T}{Z(\widetilde{G})}$, identify $T = Z(\widetilde{G}_\tau)$, and define $\zeta : T \to \widetilde{G}_\tau \times \mathbb{G}_m$ by $\zeta(t) = ([1_{\widetilde{G}}, t], 1_{\mathbb{G}_m})$.

Then letting V be the trivial T-module (and so necessarily "l" is the zero map) and M be the $\widetilde{G}_\tau \times \mathbb{G}_m$-module $\mathfrak{g}_\tau = \mathrm{Lie}(\widetilde{G}_\tau)$ (via the adjoint action and scalar multiplication), Proposition 3.1.1 produces a functor

$$F(\zeta, 0) : BT \times \mathfrak{g}_\tau/\widetilde{G}_\tau \times \mathbb{G}_m \to \mathfrak{g}_\tau/\widetilde{G}_\tau \times \mathbb{G}_m. \tag{3.6}$$

Passing to mapping stacks yields an action

$$\zeta_* : \mathcal{B}un_T(X) \times \mathrm{Map}\left(X, \left[\mathfrak{g}_\tau/\widetilde{G}_\tau \times \mathbb{G}_m\right]\right) \to \mathrm{Map}\left(X, \left[\mathfrak{g}_\tau/\widetilde{G}_\tau \times \mathbb{G}_m\right]\right) \tag{3.7}$$

and so by restricting to maps classifying a fixed line bundle L (see Section 3.3.1) we obtain an action map

$$\zeta_* : \mathcal{B}un_T(X) \times \mathcal{H}iggs_{\widetilde{G}_\tau}(X; L) \to \mathcal{H}iggs_{\widetilde{G}_\tau}(X; L). \tag{3.8}$$

3.2 Shifted Cartier duality

Given an abelian variety A the *dual abelian variety* A^D is the moduli space of multiplicative line bundles on A [65]. Recalling that $B\mathbb{G}_m$ is the classifying stack for $\mathbb{G}_m$-torsors, i.e. algebraic line bundles, the "multiplicative" condition may be translated into the statement that

$$A^D := \mathcal{H}om(A, B\mathbb{G}_m) \tag{3.9}$$

This example may be generalised as follows:

Definition 3.2.1. Let $\mathcal{A}$ be a commutative group stack over X. The *shifted Cartier dual* or *1-Cartier dual of* $\mathcal{A}$ is the commutative group stack $\mathcal{A}^D := \mathcal{H}om(\mathcal{A}, B\mathbb{G}_m)$.

Remark 3.2.1. When working over $\mathbb{C}$ in the analytic topology, the definition/notation $\mathcal{A}^D = \mathcal{H}om(\mathcal{A}, B\mathcal{O}^\times)$ is sometimes used [24, 26].

I adopt the following definition after [16, Def. 1.2.1]:

Definition 3.2.2. A commutative group stack $\mathcal{A}$ is *reflexive* if the canonical morphism $\mathcal{A} \to (\mathcal{A}^D)^D$ is an isomorphism.

Example 21 (Dualising sheaves and classifying stacks [14, Cor. 3.5-6])**.** Let K be a sheaf of abelian groups on X. Then:

1. There is a canonical isomorphism $(BK)^D \simeq K^\vee = \mathcal{H}om(K, \mathbb{G}_m)$.

2. There is a canonical homomorphism $B(K^\vee) \to K^D$, which is an isomorphism if the sheaf $\mathcal{E}xt^1(K, \mathbb{G}_m) = 0$.

In particular, if K is locally finitely generated (i.e. locally a finite direct sum of $\mathbb{Z}^s$ and a finite group) then K is a reflexive commutative group stack.

Definition 3.2.3. Given a commutative group stack $\mathcal{A}$ over X there are two associated sheaves of abelian groups [14, Def. 2.9]:

- the coarse moduli sheaf $\pi_0(\mathcal{A})$, and
- the automorphism group of a neutral section $\pi_1(\mathcal{A})$.

A sequence of commutative group stacks $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is *exact* if both sequences of sheaves of abelian groups

$$\pi_0(\mathcal{A}) \to \pi_0(\mathcal{B}) \to \pi_0(\mathcal{C}) \tag{3.10}$$

$$\pi_1(\mathcal{A}) \to \pi_1(\mathcal{B}) \to \pi_1(\mathcal{C}) \tag{3.11}$$

are exact.

The following proposition is immediate:

Proposition 3.2.1. *Shifted Cartier duality is an exact, contravariant, involutive autoequivalence on the 2-category of reflexive commutative group stacks.*

One might fear that the hypotheses of Proposition 3.2.1 are too restrictive to apply to any interesting examples. The following proposition, together with reflexivity of abelian varieties [65, Cor. 10.2] and Example 21 proves that we need not worry:

Proposition 3.2.2. *Suppose that a commutative group stack $\mathcal{A}$ over X is locally isomorphic to a product of reflexive commutative group stacks. Then $\mathcal{A}$ is a reflexive commutative group stack.*

Proof. The canonical map $\mathcal{A} \to (\mathcal{A}^D)^D$ is an isomorphism if and only if it is an isomorphism locally on X – but this is exactly our hypothesis (c.f. [26, Prop. A.6] and [19, Appendix A]). $\square$

3.2.1 Dualising $\mathcal{B}un_T(X)$

Let T be an algebraic torus, and let X be a smooth, projective, connected curve over k. Recall that the moduli stack of T-bundles on X is the commutative group stack $\mathcal{B}un_T(X) = \mathcal{M}ap(X, BT)$. Denote by $\mathbf{Bun}_T(X)$ the corresponding coarse moduli space, and by $\mathcal{B}un_T^0(X)$ and $\mathbf{Bun}_T^0(X)$ the corresponding neutral components.

Proposition 3.2.3. *Let T be an complex algebraic torus with Langlands dual torus ${}^LT$ (see Theorem A.4.2). Then the moduli stacks and coarse moduli spaces of T-bundles have the following Cartier duals:*

$$\mathcal{B}un_T(X)^D = \mathcal{B}un_{{}^LT}(X) \tag{3.12}$$

$$\mathcal{B}un_T^0(X)^D = \boldsymbol{Bun}_{{}^LT}(X) \tag{3.13}$$

$$\boldsymbol{Bun}_T^0(X)^D = \boldsymbol{Bun}_{{}^LT}^0(X) \tag{3.14}$$

Remark 3.2.2. Proposition 3.2.3 is well-known, and follows from autoduality of the Jacobian (as I will outline below). I include the proof here as, although it is not difficult, I am unaware of anywhere in the literature where it has been written out in full.

The hard part in the proof of Proposition 3.2.3 – which I am outsourcing! – is the following formulation of geometric class field theory (due to Justin Campbell):

Theorem 3.2.4 ([16, Thm 1.2.2]). *If $\mathcal{A}$ is a reflexive commutative group stack then restriction along the Abel-Jacobi map*

$$\mathcal{H}om(\mathcal{P}ic(X), \mathcal{A}) \to \mathcal{M}ap(X, \mathcal{A}) \tag{3.15}$$

is an isomorphism.

Proof of Proposition 3.2.3. Theorem 3.2.4 and Proposition 3.2.1 give the following chain of natural isomorphisms:

$$\begin{aligned}\mathcal{B}un_T(X) &= \mathcal{M}ap(X, BT)\\ &= \mathcal{H}om(\mathcal{P}ic(X), BT)\\ &= \mathcal{H}om((BT)^D, \mathcal{P}ic(X)^D)\\ &= \mathcal{H}om(X^\bullet(T), \mathcal{P}ic(X)^D).\end{aligned} \tag{3.16}$$

Given a lattice L, define $L^\wedge = \mathrm{Hom}(L, \mathbb{Z})$. The natural map $L^\wedge \otimes_{\mathbb{Z}} \mathcal{A} \to \mathrm{Hom}(L, \mathcal{A})$ is an isomorphism, so using that $X^\bullet(T)^\wedge = X_\bullet(T) = X^\bullet({}^L T)$ gives

$$\begin{aligned}\mathcal{B}un_T(X) &= \mathcal{H}om(X^\bullet(T), \mathcal{P}ic(X)^D)\\ &= X^\bullet(T)^\wedge \otimes_{\mathbb{Z}} \mathcal{P}ic(X)^D\\ &= X_\bullet(T) \otimes_{\mathbb{Z}} \mathcal{P}ic(X)^D\\ &= X^\bullet({}^L T) \otimes_{\mathbb{Z}} \mathcal{P}ic(X)^D.\end{aligned} \tag{3.17}$$

Dualising this expression gives

$$\begin{aligned}
\mathcal{B}un_T(X)^D &= \mathcal{H}om(\mathcal{B}un_T(X), B\mathbb{G}_m) \\
&= \mathcal{H}om(X^\bullet(^LT) \otimes_{\mathbb{Z}} \mathcal{P}ic(X)^D, B\mathbb{G}_m) \\
&= \mathcal{H}om(X^\bullet(^LT), \mathcal{H}om(\mathcal{P}ic(X)^D, B\mathbb{G}_m)) \\
&= \mathcal{H}om(X^\bullet(^LT), \mathcal{P}ic(X)) \\
&= \mathcal{H}om(\mathcal{P}ic(X)^D, B(^LT)) \\
&= \mathcal{H}om(\mathcal{P}ic(X), B(^LT)) \\
&= \mathcal{M}ap(X, B(^LT)) \\
&= \mathcal{B}un_{^LT}(X).
\end{aligned} \tag{3.18}$$

Now,

$$0 \to \mathcal{B}un_T^0(X) \to \mathcal{B}un_T(X) \to X_\bullet(T) \to 0 \tag{3.19}$$

dualises to

$$0 \to B(^LT) \to \mathcal{B}un_{^LT}(X) \to \mathcal{B}un_T^0(X)^D \to 0 \tag{3.20}$$

i.e. $\mathcal{B}un_T^0(X)^D = \mathbf{Bun}_{^LT}(X)$, and

$$0 \to \mathbf{Bun}_T^0(X) \to \mathbf{Bun}_T(X) \to X_\bullet(T) \to 0 \tag{3.21}$$

dualises to

$$0 \to B(^LT) \to \mathcal{B}un_{^LT}^0(X) \to \mathbf{Bun}_T^0(X)^D \to 0 \tag{3.22}$$

i.e. $\mathbf{Bun}_T^0(X)^D = \mathbf{Bun}_{^LT}^0(X)$. □

3.3 Higgs bundles and cameral covers

For the rest of this chapter I consider a fixed Riemann surface (or complex smooth projective algebraic curve) C, often assumed to have genus > 1. Denote by $K_C \to C$ the canonical bundle of C, and let G be a complex reductive algebraic group. The following (standard) notion of a Higgs bundle is attributable to Hitchin [38, 40]:

Definition 3.3.1. A *K_C-valued G-Higgs bundle on C* is a pair (E, φ), where

- $E \to C$ is a holomorphic G-bundle, and
- $\varphi \in H^0(C; \mathrm{ad}(E) \otimes K_C)$, i.e. φ is a global section of the bundle $\mathrm{ad}(E) \otimes K_C$.

Here, $\mathrm{ad}(E)$ is the vector bundle associated to E via the adjoint representation of G on $\mathfrak{g} = \mathrm{Lie}(G)$. This is sometimes also denoted $\mathrm{ad}(E) = \mathfrak{g}_E$.

Although this is the most common definition of a Higgs bundle in the literature, it turns out that the analysis of this and other related moduli spaces can be profitably approached through a more abstract notion of a Higgs bundle. As per [25], we may think of the above definition of a Higgs bundle as specifying *complete spectral data* – a decomposition into eigenspaces and the corresponding eigenvalues – for a Higgs field; the following more abstract definition corresponds to specifying only a decomposition into eigenspaces for a Higgs field:

Definition 3.3.2. Recall that the *locus of regular elements* in $\mathfrak{g}$ is the locus of elements whose centralisers have minimal possible dimension,

$$\mathfrak{g}^{reg} := \{x \in \mathfrak{g} \mid \dim(Z_G(x)) = \mathrm{rank}(G) = r\}. \tag{3.23}$$

A *regular G-Higgs bundle on C* is a pair $(E, \mathfrak{c}_C)$, where

(1) E is a principal G-bundle over C, and

(2) $\mathfrak{c}_C$ is a vector subbundle of $\mathfrak{g}_E$ of rank r such that $[\mathfrak{c}_C, \mathfrak{c}_C] = 0$, and such that locally $\mathfrak{c}_C$ is the sheaf of centralisers of a section of $E \times^G \mathfrak{g}^{reg}$.

Example 22. To understand the idea that a Higgs bundle specifies some complete/partial spectral data, consider the situation of a $GL_n\mathbb{C}$-Higgs bundle on a one point space $*$. A regular Higgs bundle is then given by an n-dimensional complex vector space V, and a commutative subalgebra $\mathfrak{c}_* \subset \mathrm{End}(V)$ which is n-dimensional as a complex vector space and admits a single regular generator.

Any basis of $\mathfrak{c}_*$ as a $\mathbb{C}$-vector space defines a maximally commuting set of linear operators on V, and hence a decomposition of V into one-dimensional subspaces $V = \bigoplus_{i=1}^n L_i$ which are simultaneous eigenlines for the n operators; the L_i are independent of the choice of basis of operators. A regular Higgs bundle with values[4] (c.f. Definition 3.3.1) in this situation then picks out a particular element of $\mathfrak{c}_*$ to be the Higgs field, and this is the same data as specifying the eigenvalue of the Higgs field on each of the eigenlines L_i.

Remark 3.3.1. In [25] it is proved that there is a smooth irreducible complex scheme $\overline{G/N}$, a partial compactification of the quotient varient G/N which parametrises

[4]More accurately we are specifying here the data of a *regularised* Higgs bundle with values in the trivial bundle: see [25] for this notion. The distinction is in whether the subbundle $\mathfrak{c}_C$ is given explicitly as data; if the Higgs field φ is regular, however, there is no choice to speak of, as $\mathfrak{c}_C$ is forced to be the centraliser of φ.

Cartan subalgebras of $\mathfrak{g}$, which parametrises the regular centralisers in $\mathfrak{g}$. Thus, one could equivalently define a G-Higgs bundle to be a pair (E, σ) where

(1) E is a principal G-bundle over C, and

(2) σ is a G-equivariant map $\sigma : E \to \overline{G/N}$.

A morphism of G-Higgs bundles $\Phi : (E, \sigma) \to (F, \tau)$ is a morphism Φ of principal G-bundles such that $\tau \circ \Phi = \sigma$,

$$\begin{tikzcd} E \arrow[dd, "\Phi"] \arrow[dr, "\sigma"] & \\ & \overline{G/N} \\ F \arrow[ur, "\tau"] & \end{tikzcd} \tag{3.24}$$

With this notion of morphism we obtain a category of Higgs bundles, and these fit together to define the *stack of abstract Higgs bundles on C*, the S-points of which are the category of G-Higgs bundles on $C \times S$.

3.3.1 General analysis of Higgs bundles with values

As my primary object of study will be not abstract Higgs bundles but Higgs bundles with values in K_C, I must also describe how the collection of all K_C-valued Higgs bundles may be given the structure of a stack. For this, consider more generally the notion of an *L-valued G-Higgs bundle on X*,[5] where X is some complex scheme.

[5] The definition of which is identical to Definition 3.3.1 with every occurance of "K_C" replaced by "L".

Consider the stack quotient $[\mathfrak{g}/G]$, where G acts on its Lie algebra $\mathfrak{g}$ via the adjoint action. A map from X to $[\mathfrak{g}/G]$ is given by

(1) a principal G-bundle E on X, together with

(2) a section of the adjoint bundle $\mathfrak{g}_E$.

Comparing this to Definition 3.3.1 of a Higgs bundle with values, we see that such a map is equivalent to an $\mathcal{O}_X$-valued G-Higgs bundle on X.

We may extend this description to Higgs bundles with values in other line bundles as follows: consider the multiplicative group $\mathbb{G}_m$, i.e. the group scheme whose R valued points are given by $R^\times = \mathrm{Spec}(R[t, t^{-1}])$. There is an action of $\mathbb{G}_m$ on $\mathfrak{g}$ which commutes with the adjoint action of G, given by $\lambda \cdot x = \lambda x$ for $\lambda \in \mathbb{G}_m$ and $x \in \mathfrak{g}$. We may therefore consider the stack quotient $[\mathfrak{g}/G \times \mathbb{G}_m]$. A map from X to this quotient is given by

(1) a principal G-bundle E on X, and

(2) a line bundle (i.e. principal $\mathbb{G}_m$-bundle) L on X, together with

(3) a section of the vector bundle $\mathfrak{g}_E \otimes L \to X$.

Hence the stack $\mathcal{M}ap(X, [\mathfrak{g}/G \times \mathbb{G}_m])$ is exactly the moduli stack of G-Higgs bundles on X with values in *some* line bundle. Composition of such a map with the natural projection map $[\mathfrak{g}/G \times \mathbb{G}_m] \to B\mathbb{G}_m$ classifies the line bundle of values for the corresponding Higgs bundle.

Definition 3.3.3. Denote by $\mathcal{H}iggs_G(X, L)$ the *moduli stack of G-Higgs bundles on X with values in L*, i.e. the substack of $\mathcal{M}ap(X, [\mathfrak{g}/G \times \mathbb{G}_m])$ whose projection to $\mathcal{M}ap(X, B\mathbb{G}_m) = \mathcal{B}un_{\mathbb{G}_m}(X)$ classifies the line bundle L.

3.3.2 The Chevalley morphism and the Kostant section

By using the description of the moduli of Higgs bundles as a mapping stack into $[\mathfrak{g}/G \times \mathbb{G}_m]$ we are able to identify certain geometric features which are induced from the representation theory of G, are so are insensitive to the geometry of the space X and the line bundle L. In particular, we are interested in a canonical projection that exists for each moduli space, and a section of this projection whose only geometric dependence is on a choice of square root of of L. Before describing these features, let us review the relevant representation theoretic facts.

Fix the data of a maximal torus and a Borel subgroup $H \hookrightarrow B \subset G$. This determines a set of simple roots S in the root system R of the Lie algebra $\mathfrak{g}$. Denote the root space of $\mathfrak{g}$ corresponding to $\alpha \in R$ by $\mathfrak{g}_\alpha$, and choose a nonzero vector $x_\alpha \in \mathfrak{g}_\alpha$ for each simple $\alpha \in S$. For each simple root α there is then a unique element $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = \alpha^\vee$, the coroot corresponding to α (determined by the normalisation condition $\alpha(\alpha^\vee) = 2$). Background and a proof of the following result can be found in [20, 48]:

Theorem 3.3.1. *The elements $x_+ = \sum_{\alpha \in S} x_\alpha$ and $x_- = \sum_{\alpha \in S} x_{-\alpha}$ are regular nilpotent elements of $\mathfrak{g}$.*

Consider the adjoint action of G on $\mathfrak{g}$, and the induced action of the Weyl group $W := W_G(H) = N_G(H)/H$ on the Lie algebra of the maximal torus $\mathfrak{h}$, where

$N_G(H)$ denotes the normaliser in G of the torus H. These induce actions of G and W on the algebras $\mathbb{C}[\mathfrak{g}]$ and $\mathbb{C}[\mathfrak{h}]$ respectively, and we define $\mathfrak{c} := \mathrm{Spec}(\mathbb{C}[\mathfrak{h}]^W) = \mathfrak{h}/W$. We then have the following theorem of Kostant [49]:

Theorem 3.3.2. *1. The restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ induces an isomorphism on the subalgebras of invariants $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$. Moreover, $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial algebra generated by homogeneous elements $P_1, \ldots, P_r$ of degrees $m_1 + 1, \ldots, m_r + 1$.*

2. The Chevalley *or* characteristic polynomial map *$\chi : \mathfrak{g} \to \mathfrak{c}$, induced by the above isomorphism, is $\mathbb{G}_m$-equivariant with respect to the weight one action of $\mathbb{G}_m$ on $\mathfrak{g}$, and the action on $\mathfrak{c}$ defined by*

$$\lambda \cdot (P_1, \ldots, P_r) = (\lambda^{m_1+1} P_1, \ldots, \lambda^{m_r+1} P_r).$$

3. The restriction of χ to the regular locus $\mathfrak{g}^{reg} \subset \mathfrak{g}$ is smooth, and each fibre is a single G-orbit.

4. Let $\mathfrak{g}^{x_+} \subset \mathfrak{g}$ denote the Lie algebra centraliser of x_+ (i.e. the kernel of $\mathrm{ad}(x_+)$ *acting on $\mathfrak{g}$). Then the affine subspace $x_- + \mathfrak{g}^{x_+}$ is contained in the regular locus $\mathfrak{g}^{reg}$, and the Chevalley map restricts to an isomorphism $x_- + \mathfrak{g}^{x_+} \cong \mathfrak{c}$.*

Remark 3.3.2. The inverse to the Chevalley map on $x_- + \mathfrak{g}^{x_+}$ is called the *Kostant section*, and is often denoted by κ:

$$\begin{tikzcd} \mathfrak{g}^{reg} \arrow[r, hook] & \mathfrak{g} \arrow[d, "\chi"] \\ & \mathfrak{c} \arrow[ul, "\kappa"] \end{tikzcd}$$

Observe that the subspace $x_- + \mathfrak{g}^{x_+}$ is not stable under the weight one action of $\mathbb{G}_m$, hence the Kostant section has no chance of being $\mathbb{G}_m$-equivariant. We fix this as in [63]: recall that, if one writes a root $\alpha \in R$ as $\alpha = \sum n_i \alpha_i$, where the sum is over the set of simple roots and the coefficients are entirely contained in either $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{\leq 0}$, then the *height* of α is defined to be

$$\mathrm{ht}(\alpha) := \sum n_i. \tag{3.25}$$

Define an action $\rho : \mathbb{G}_m \to \mathrm{Aut}(\mathfrak{g})$ by taking the trivial action on $\mathfrak{h}$ and acting on the root space $\mathfrak{g}_\alpha$ as

$$\rho(\lambda) \cdot \mathfrak{g}_\alpha = \lambda^{\mathrm{ht}(\alpha)} \mathfrak{g}_\alpha, \tag{3.26}$$

and then further define the action $\rho_+ : \mathbb{G}_m \to \mathrm{Aut}(\mathfrak{g})$ by

$$\rho_+(\lambda) = \lambda \rho(\lambda). \tag{3.27}$$

ρ_+ acts by scaling x_+ and so preserves $\mathfrak{g}^{x_+}$, and

$$\rho_+(x_-) = \lambda \rho(\lambda) x_- = \lambda \lambda^{-1} x_- = x_-; \tag{3.28}$$

hence the $\mathbb{G}_m$-action ρ_+ preserves the Kostant section $x_- + \mathfrak{g}^{x_+}$. Furthermore, we have (see [63]):

Proposition 3.3.3. *The map κ is $\mathbb{G}_m$-equivariant with respect to the action of $\mathbb{G}_m$ on $\mathfrak{c}$ given in the previous theorem, and the action ρ_+ on the Kostant section defined above.*

3.3.3 The Hitchin fibration

I will now describe the canonical geometric data induced by the representation theory of G.

3.3.3.1 The Hitchin base

For $L \to X$ a line bundle, corresponding to the $\mathbb{G}_m$-torsor $(L - 0) \to X$, consider the associated $\mathfrak{c}$-bundle on X

$$\mathfrak{c}_L := \mathfrak{c} \times^{\mathbb{G}_m} (L - 0). \tag{3.29}$$

Concretely, one can write $\mathfrak{c}_L = (L \otimes \mathfrak{h})/W$.

Definition 3.3.4. $\mathcal{H}itch_{\mathfrak{g}}(X, L)$ is the functor whose S-points are given by

$$\mathrm{Hom}(S, \mathcal{H}itch_{\mathfrak{g}}(X, L)) = \mathrm{Hom}_X(S \times X, \mathfrak{c}_L). \tag{3.30}$$

Now, since by definition $\mathfrak{c} = \mathfrak{h}/W = \mathfrak{g}//G$, the Chevalley morphism factors through the stack $[\mathfrak{g}/G]$. Since it is $\mathbb{G}_m$-equivariant, it further descends to give a map

$$\chi : [\mathfrak{g}/G \times \mathbb{G}_m] \to [\mathfrak{c}/\mathbb{G}_m] \tag{3.31}$$

which by abuse of notation I will also denote by χ. For X a complex scheme, consider the mapping stack $\mathcal{M}ap(X, [\mathfrak{c}/\mathbb{G}_m])$. χ further induces a map

$$\mathcal{M}ap(X, [\mathfrak{g}/G \times \mathbb{G}_m]) \to \mathcal{M}ap(X, [\mathfrak{c}/\mathbb{G}_m]), \tag{3.32}$$

which on the subfunctors determined by the projection to $\mathcal{M}ap(X, B\mathbb{G}_m)$ give maps

$$h_L : \mathcal{H}iggs_G(X, L) \to \mathcal{H}itch_{\mathfrak{g}}(X, L). \tag{3.33}$$

Definition 3.3.5. The maps h_L are called *Hitchin maps*, and we define the *Hitchin base* to be the $\mathbb{C}$-points $\mathbf{Hitch}_{\mathfrak{g}}(X, L) = \mathrm{Hom}_X(X, \mathrm{tot}(\mathfrak{c}_L)) = H^0(X; (L \otimes \mathfrak{h})/W)$.

The Hitchin base $\mathbf{Hitch}_{\mathfrak{g}}(X, L)$ is an affine space, and it represents the functor $\mathcal{H}itch_{\mathfrak{g}}(X, L)$. Moreover, it parametrises the *L-valued cameral covers of X*, which we define after [24] as follows:

Definition 3.3.6. A *cameral cover* of X is a scheme $\tilde{X}$ together with a map $p : \tilde{X} \to X$ and a W-action along the fibres of p satisfying:

1. p is finite and flat over X.
2. As an $\mathcal{O}_X$-module with W-action $p_*\mathcal{O}_{\tilde{X}}$ is locally isomorphic to $\mathcal{O}_X \otimes \mathbb{C}[W]$.
3. Locally with respect to the étale (or analytic) topology on X, $\tilde{X}$ is a pullback of the W-cover $\mathfrak{h} \to \mathfrak{h}/W$.

An *L-valued cameral cover* of X is a cameral cover $p : \tilde{X} \to X$ together with a W-equivariant embedding $\tilde{\sigma} : \tilde{X} \to \mathrm{tot}(L \otimes \mathfrak{h})$.

Remark 3.3.3. The embedding $\tilde{\sigma}$ realises the cameral cover $\tilde{X}$ as a pullback

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\sigma}"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near end] & \mathrm{tot}(L \otimes \mathfrak{h}) \arrow[d] \\ X \arrow[r, "\sigma"] & \mathrm{tot}\,(\mathfrak{c}_L) \end{tikzcd} \tag{3.34}$$

for a unique $\sigma \in \mathbf{Hitch}_{\mathfrak{g}}(X, L)$.

Remark 3.3.4. If $p : \tilde{X} \to X$ satisfies only conditions (1) and (2), we say that it is a *W-cover* of X. We say that condition (3) defines a "cameral" cover because it locally implies that we can label the sheets (and ramification pattern) of p using the Weyl chambers (and root hyperplanes) of $\mathfrak{g}$.

3.3.3.2 The Hitchin sections

Our next goal is to use the Kostant section $\kappa : \mathfrak{c} \to \mathfrak{g}$ to obtain sections of the various Hitchin maps h_L defined above. The most straightforward way to do so would be to show that the Kostant section itself descends to the stack quotient to provide a section $[\mathfrak{c}/\mathbb{G}_m] \to [\mathfrak{g}/G \times \mathbb{G}_m]$: using the diagonal morphism $\mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ we have a map

$$[(x_- + \mathfrak{g}^{x_+})/\rho_+(\mathbb{G}_m)] \to [\mathfrak{g}/\rho(\mathbb{G}_m) \times \mathbb{G}_m] \tag{3.35}$$

where the second factor is the action by homotheties. Here however we encounter a complication: the action ρ does not necessarily factor through $\mathbb{G}_m \to H \to G$ as a cocharacter, and thus obstructs the easy existence of a section.[6]

In fact in ρ does not factor through a cocharacter then the desired section does not always exist. Assume, however, that ρ does not factor through a cocharacter and that the line bundle L admits a square root $L^{1/2}$. In this situation a section does exist, and we construct it as follows. Consider the homomorphism $\phi : SL_2\mathbb{C} \to G$

[6] E.g. ρ factors through a cocharacter for $G = SL_3(\mathbb{C})$, but **not** for $G = SL_2(\mathbb{C})$.

determined by the principal[7] $\mathfrak{sl}_2\mathbb{C}$ (c.f. Theorem 3.3.1)

$$(x_+, h, x_-) \quad \text{where } h = [x_+, x_-]. \tag{3.36}$$

We can then define an action on $\mathfrak{g}$ using the diagonal $\mathbb{G}_m \subset SL_2\mathbb{C}$ by taking

$$\tilde{\rho}(\lambda) \cdot x = \mathrm{Ad}_{\phi(\lambda)}(x), \quad \text{and} \quad \tilde{\rho}_+(\lambda) \cdot x = \lambda^2 \, \mathrm{Ad}_{\phi(\lambda)}(x). \tag{3.37}$$

The action $\mathrm{Ad}_{\phi(\lambda)}$ is the square of the action ρ defined previously, and the shift by λ^2 ensures that $\tilde{\rho}_+$ preserves the Kostant section $x_- + \mathfrak{g}^{x_+}$. The Chevalley map is equivariant with respect to this action of $\mathbb{G}_m$ on $\mathfrak{g}$, and the square of the action previously defined for $\mathfrak{c}$.

Denote by $\mathbb{G}_m^{[2]} \to \mathbb{G}_m$ the squaring morphism of the multiplicative group, and let $\mathbb{G}_m^{[2]}$ act on $\mathfrak{c}$ by the square of the usual action (i.e. through the squaring homomorphism to $\mathbb{G}_m$ acting via the usual action). Then we have the sequence of maps

$$\begin{tikzcd}
{[\mathfrak{c}/\mathbb{G}_m^{[2]}]} \arrow[r, "\kappa", "\sim"'] \arrow[drrr, dashed, bend right] & {[(x_- + \mathfrak{g}^{x_+})/\rho_+(\mathbb{G}_m^{[2]})]} \arrow[r] & {[\mathfrak{g}/\rho(\mathbb{G}_m^{[2]}) \times \mathbb{G}_m^{[2]}]} \arrow[r, equal] & {[\mathfrak{g}/\tilde{\rho}(\mathbb{G}_m) \times \mathbb{G}_m^{[2]}]} \arrow[d] \\
& & & {[\mathfrak{g}/G \times \mathbb{G}_m^{[2]}]}
\end{tikzcd} \tag{3.38}$$

[7] An $\mathfrak{sl}_2$-triple is called *principal* if its elements are regular.

We therefore have the following induced diagram on mapping stacks:

$$\begin{tikzcd} \mathrm{Map}(X, [\mathfrak{g}/G \times \mathbb{G}_m^{[2]}]) \arrow[r] \arrow[d, shift left] & \mathrm{Map}(X, [\mathfrak{g}/G \times \mathbb{G}_m]) \arrow[d] \\ \mathrm{Map}(X, [\mathfrak{c}/\mathbb{G}_m^{[2]}]) \arrow[r] \arrow[u, bend left, "\kappa_*"] & \mathrm{Map}(X, [\mathfrak{c}/\mathbb{G}_m]) \end{tikzcd} \tag{3.39}$$

Now, consider an element in the Hitchin base $\mathcal{H}itch_{\mathfrak{g}}(X, L)$ where L admits a square root $L^{1/2}$. Recall that an S-point in the Hitchin base is a map in $\mathrm{Hom}_X(S \times X, \mathfrak{c}_L)$. Locally, we can find square-roots of this section, i.e. maps to $\mathfrak{c}_{L^{1/2}}$, and to these we may apply the Kostant section κ_* and pushfoward again via the squaring map. Pushing forward along the squaring map eliminates the possible $\{\pm 1\}$ discrepancies among our choices of lifts, and so these local maps patch together to give us a well-defined S-point of $\mathcal{H}iggs_{\mathfrak{g}}(X, L)$. I.e., if L admits a square-root, we may use this to construct a *Hitchin section*

$$\begin{tikzcd} \mathcal{H}iggs_G(X, L) \arrow[d, "h_L"] \\ \mathcal{H}itch_{\mathfrak{g}}(X, L) \arrow[u, bend left, "s_L"] \end{tikzcd} \tag{3.40}$$

Remark 3.3.5. If the action of ρ does factor through a cocharacter (as for $SL_3(\mathbb{C})$), the section (3.40) exists for any line bundle L.

3.3.4 Coarse moduli spaces of semistable Higgs bundles

Although most of my analysis will be done on the moduli stack of Higgs bundles, I will also be able to draw some conclusions about the moduli space of semistable Higgs bundles. The stability conditions arise naturally both from the point of view of physics [13] and nonabelian Hodge theory [70], and while the moduli space contains strictly less information than the moduli stack of Higgs bundles it has a much richer geometric structure.

Recall that a Higgs bundle (E, ϕ) on a Riemann surface C is *semistable* if for any parabolic subgroup $P \subset G$ and P-Higgs subbundle $(F, \psi) \subset (E, \varphi)$, the degree of F is less than or equal to zero (where degree refers to the degree of the vector bundle $\mathfrak{p}_F$ associated to F by the adjoint action of P). If the inequality is strict, we say that (E, φ) is *stable*.

Definition 3.3.7. Denote by $\mathbf{Higgs}_G(C, K_C)$ the moduli space of semistable K_C-valued G-Higgs bundles on C.

Remark 3.3.6. There is a natural open substack of semistable K_C-valued G-Higgs bundles on C, $\mathcal{H}iggs_G^{ss}(C, K_C) \subset \mathcal{H}iggs_G(C, K_C)$ which maps to this coarse moduli space, a fact which I will exploit to transfer results proved using the moduli stack onto the moduli space.

It is known that the moduli spaces $\mathbf{Higgs}_G(C, K_C)$ can be equipped with a natural hyperkähler structure [38, 45]; indeed a standard method of constructing these moduli spaces involves taking the hyperkähler quotient – a technique which allows one to take the quotient of a group action on a hyperkähler space in such a way as to induce a hyperkähler structure on the quotient [39] – of all solutions to Hitchin's equations by the action of gauge transformations. One of the results of Chapter 5 will be an SYZ mirror symmetry statement for certain quotients of $\mathbf{Higgs}_G(C, K_C)$ which themselves inherit a hyperkähler structure (c.f. Remark 2.1.9).

3.4 The group scheme of regular centralisers

The Hitchin fibration described in section 3.3.3 is a powerful tool that we may use in order to study the geometry of the moduli of Higgs bundles. Following the ideas of Donagi, Gaitsgory and Ngô [25, 64], I will now review a uniform approach to understanding the fibres of the Hitchin map via the group scheme of regular centralisers.

3.4.1 The schemes of centralisers

Consider the *group scheme of centralisers* $I \to \mathfrak{g}$ defined by

$$I = \{(x, g) \in \mathfrak{g} \times G \mid \mathrm{Ad}_g(x) = x\} \subset \mathfrak{g} \times G. \tag{3.41}$$

This map is very poorly behaved: observe for instance that it interpolates between the fibre of a regular semisimple element, which is an algebraic torus of dimension $r = \mathrm{rank}(G)$, and the fibre over 0, which is a copy of G. When restricted to the regular locus, however, I^{reg} becomes a smooth commutative group scheme of relative dimension r, whose generic fibre (over a semisimple element) is an algebraic torus.

Example 23. The only rank one complex simple Lie algebra is $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, for which G might equal $SL_2\mathbb{C}$ or $PGL_2\mathbb{C}$. In both situations the centraliser of a regular semisimple element (i.e. nonzero diagonalisable matrix) is a rank one algebraic torus, conjugate to the standard diagonal torus

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}_{\lambda \in \mathbb{C}^\times} \subset G. \tag{3.42}$$

The centraliser of any regular nilpotent element is related by conjugation to the centraliser of the standard regular nilpotent element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which one can calculate to be $\mathbb{C}$ for $G = PGL_2\mathbb{C}$, and $\mathbb{C} \amalg \mathbb{C} = \mathbb{C} \times (\mathbb{Z}/2\mathbb{Z})$ for $G = SL_2\mathbb{C}$.

Definition 3.4.1. Recalling that the Kostant section κ is valued in the regular locus of $\mathfrak{g}$, we define the *group scheme of regular centralisers* J by

$$J = \kappa^* I^{reg} \to \mathfrak{c}. \tag{3.43}$$

Since I^{reg} is a smooth commutative group scheme, so is J. Consider the pullback by the Chevalley map $\chi^* J \to \mathfrak{g}$: by construction this is equipped with an isomorphism over the regular locus $(\chi^* J)|_{\mathfrak{g}^{reg}} \xrightarrow{\sim} I|_{\mathfrak{g}^{reg}}$, and this extends uniquely to a homomorphism of group schemes $\chi^* J \to I$ since J is smooth, I is affine, and $\chi^* J \setminus \chi^* J|_{\mathfrak{g}^{reg}}$ is closed of high codimension [64].

Remark 3.4.1. Note that although the comparison map $(\chi^* J)|_{\mathfrak{g}^{reg}} \xrightarrow{\sim} I|_{\mathfrak{g}^{reg}}$ extends over all of $\mathfrak{g}$, it is very far from being an isomorphism – indeed, over $0 \in \mathfrak{g}$ it is the zero map.

3.4.2 Schemes of centralisers and automorphisms of Higgs bundles

I will now recall how to relate the group schemes I and J to the geometry of the moduli stack of Higgs bundles by understanding them as inducing automorphisms of Higgs bundles. I is equipped with a natural action of $G \times \mathbb{G}_m$, given by

$$(h, t) \cdot (x, g) = (t\,\mathrm{Ad}_h(x), hgh^{-1}) \tag{3.44}$$

which covers the $G \times \mathbb{G}_m$-action on $\mathfrak{g}$. So, I descends to a group scheme

$$[I/G \times \mathbb{G}_m] \to [\mathfrak{g}/G \times \mathbb{G}_m], \tag{3.45}$$

which by abuse of notation I will also denote by $I \equiv [I/G \times \mathbb{G}_m]$. Observe that the fibre over a closed point x is

$$I_x = \{(x, g) \mid \mathrm{Ad}_g(x) = x\} = \{x\} \times C_G(x) \tag{3.46}$$

which are the automorphisms of the Higgs bundle with values in the trivial line bundle

$$\begin{array}{c} G \times * \\ \big\downarrow \\ * \end{array}, \quad \phi(*) = x \in \mathfrak{g}. \tag{3.47}$$

In general, given a map $h_{E,\phi} : X \times S \to [\mathfrak{g}/G \times \mathbb{G}_m]$ classifying a family of Higgs bundles (E, ϕ) on X, we have that $h^*_{E,\phi}(I) = \mathrm{Aut}(E, \phi)$, essentially by definition [63]. Similarly, J descends to a group scheme

$$[J/\mathbb{G}_m] \to [\mathfrak{c}/\mathbb{G}_m], \tag{3.48}$$

and the morphism $\chi^* J \to I$ descends to a map

$$\begin{tikzcd} \chi^*([J/\mathbb{G}_m]) \arrow[rr] \arrow[dr] & & {[I/G \times \mathbb{G}_m]} \arrow[dl] \\ & {[\mathfrak{g}/G \times \mathbb{G}_m]} & \end{tikzcd} \tag{3.49}$$

which is an isomorphism over the locus $[\mathfrak{g}^{reg}/G \times \mathbb{G}_m]$. The following is due to [63]:

Proposition 3.4.1. $[\mathfrak{g}^{reg}/G] \to \mathfrak{c}$ *is a J-gerbe, and is in fact the trivial J-gerbe trivialised by the Kostant section.*

Proof. The smooth surjective morphism over $\mathfrak{c}$

$$G \times \mathfrak{c} \to \mathfrak{g}^{reg} \quad \text{given by} \quad (g, a) \mapsto \mathrm{Ad}_g(\kappa(a)) \tag{3.50}$$

has fibres given by J, and is G-equivariant where the G-action on the left hand side is given by left multiplication on G. Taking the quotient by the G-action then gives the desired isomorphism

$$[\mathfrak{c}/J] \xrightarrow{\sim} [\mathfrak{g}^{reg}/G]. \tag{3.51}$$

□

I will now consider a Picard stack on the affine Hitchin base $\mathbf{Hitch}_{\mathfrak{g}}(X, L)$, defined as follows [63]: recall that a point $\sigma : S \to \mathbf{Hitch}_{\mathfrak{g}}(X, L)$ is equivalent to a map

$$h_\sigma : X \times S \to [\mathfrak{c}/\mathbb{G}_m] \tag{3.52}$$

which lies over the map $X \to B\mathbb{G}_m$ that classifies the line bundle $L \to X$. By pulling back the smooth commutative group scheme $J \to [\mathfrak{c}/\mathbb{G}_m]$ along h_σ, we obtain a smooth family of commutative group schemes $J_\sigma = h_\sigma^* J \to X \times S$.

Consider the category of J_σ-torsors on $X \times S$, $\mathrm{Tors}_{J_\sigma}(X \times S)$. The assignment $\sigma \mapsto \mathrm{Tors}_{J_\sigma}(X \times S)$ defines a Picard stack on $\mathbf{Hitch}_{\mathfrak{g}}(X, L)$ which we denote $\mathcal{T}ors_J$.

Proposition 3.4.2. *There is an action of $Tors_{J_\sigma}(X \times S)$ on the fibre $\mathcal{H}iggs_G(X, L)_\sigma :=$ $h_L(S)^{-1}(\sigma)$ of the (S-points of the) Hitchin map.*

Proof. This arises from the following construction: given a sheaf $\mathscr{G} \to B$ of groupoids (in sets) on B, a sheaf of abelian groups $\mathcal{A} \to B$, and a homomorphism $\mathcal{A} \to \mathrm{End}(\mathrm{id}_{\mathscr{G}})$, one can twist any global section (i.e. object of $\mathscr{G}(B)$) by an $\mathcal{A}$-torsor. In our situation, $\mathscr{G} = \mathcal{H}iggs_G(X, L)_\sigma$, $\mathcal{A} = J_\sigma$, and the homomorphism is induced by the map $\chi^* J \to I$. □

Remark 3.4.2. Since I can be described as the sheaf of automorphisms on the stack $[\mathfrak{g}/G \times \mathbb{G}_m]$, Proposition 3.4.2 can also be viewed as a relative-over-$\mathfrak{g}$ version of Proposition 3.1.1.

Through this action we may interpret the moduli of Higgs bundles $\mathcal{H}iggs_G(X, L)$ as a kind of partial compactification of $\mathcal{T}ors_J$ as follows. Write $\mathcal{H}iggs_G(X, L)^{reg}$ for the subfunctor classifying those maps $h_{E,\phi} : X \times S \to [\mathfrak{g} \otimes L/G]$ which factor through the open substack $[\mathfrak{g}^{reg} \otimes L/G]$. Assume that L admits a square root, so that the Hitchin fibration admits a Kostant section. Then since by construction the Kostant section takes values in the regular locus, we have the following [64, Prop 4.3.3]:

Proposition 3.4.3. *$\mathcal{H}iggs_G(X, L)^{reg}$ is open in $\mathcal{H}iggs_G(X, L)$ with non-empty fibres over $\boldsymbol{Hitch}_{\mathfrak{g}}(X, L)$. Moreover, $\mathcal{T}ors_J$ acts on this locus simply-transitively.*

3.4.3 The Hitchin fibration away from the discriminant locus

Consider the branch locus of the generically étale Galois W-cover $\mathfrak{h} \to \mathfrak{c}$, which we denote by $\mathfrak{D}_{\mathfrak{g}}$ since it may be identified with the divisor given by vanishing of the discriminant

$$\prod_{\alpha \in R} d\alpha, \tag{3.53}$$

where the product is over the roots of G.[8] I adopt the follow definiton after [63]:

Definition 3.4.2. Call $\sigma \in \mathbf{Hitch}_{\mathfrak{g}}(X, L)$ *very regular* if the image of the associated map $h_\sigma : X \to \mathfrak{c}_L$ is transverse to the divisor $\mathfrak{D}_L = \mathfrak{D}_{\mathfrak{g}} \times^{\mathbb{G}_m} L$.

[8] Since $\alpha : H \to \mathbb{G}_m$, $d\alpha : \mathfrak{h} \to \mathbb{G}_a$.

Remark 3.4.3. Geometrically, Definition 3.4.2 means that the associated cameral cover $p_\sigma : \tilde{X}_\sigma \to X$ has simple Galois ramification, i.e. all of the ramification points of p_a have ramification index one [24]; moreover in this situation $\tilde{X}$ is smooth [64].

If L is very ample, then the very regular locus is open and dense in $\mathbf{Hitch}_{\mathfrak{g}}(X, L)$ [63]. Denote the complement of this locus by $\Delta_{\mathfrak{g}}$ (or just Δ if $\mathfrak{g}$ is clear from context), so that the very regular locus is $\mathbf{Hitch}_{\mathfrak{g}}(X, L) \setminus \Delta$.

Proposition 3.4.4. *[63, Prop 4.3] For $\sigma \in \boldsymbol{Hitch}_{\mathfrak{g}}(X, L)\backslash\Delta$, the groupoid $Tors_{J_\sigma}(X \times S)$ acts simply-transitively on $\mathcal{H}iggs_G(X, L)_\sigma$; i.e. $\mathcal{H}iggs_G(X, L)_\sigma$ is a J_σ-gerbe. Moreover, if the Hitchin section exists, it trivialises this gerbe.*

Proof. This follows from [63, Prop 4.2], which says that for σ very regular, $\mathcal{H}iggs_G(X, L)_\sigma \subset \mathcal{H}iggs_G(X, L)^{reg}$. $\square$

Let $(X, L) = (C, K_C)$, and recall the coarse moduli space of semistable K_C-valued Higgs bundles $\mathbf{Higgs}_{\mathfrak{g}}(C, K_C)$. The above groupoid level analysis, together with the fact that Higgs bundles with very regular characteristics are stable, yields the following corollary upon passage to equivalence classes:

Corollary 3.4.5. *The Hitchin fibre $\boldsymbol{Higgs}_G(C, K_C)_\sigma$ lying over a very regular characteristic $\sigma \in \boldsymbol{Hitch}_{\mathfrak{g}}(C, K_C)$ is a torsor for $H^1(C; J_\sigma)$, the group of equivalence classes of J_σ-torsors on C.*

Chapter 4

Ramification of cameral covers

In this chapter I discuss the conditions under which a cameral cover will be ramified. The method I use do this is analogous to how one detects ramification of spectral covers in the case of $GL_n(\mathbb{C})$-Higgs bundles valued in the canonical bundle of a curve: there, since the spectral cover is locally cut out by the characteristic polynomial of a matrix of holomorphic 1-forms, ramification is determined by the discriminant of the characteristic polynomial (which detects repeated roots). The key observation to make is that branch points can be identified as the zeros of a section of some power of the canonical bundle.

Since I am working with principal Higgs bundles, and hence with cameral covers rather than spectral covers, it is not possible to take the characteristic polynomial of a Higgs field. Instead one locally applies the Chevalley map to the Higgs field, and detects ramification using the collection of root hyperplanes in $\mathfrak{c} = \mathfrak{h}/W$. In order to deal with this more general situation, I will first prove a more general statement about when the sections of an bundle associated to a $\mathbb{G}_m$ space V intersect with a divisor induced by a conical divisor $D \subset V$. The application to cameral covers is then straightforward and immediate.

4.1 Twisting of $\mathbb{G}_m$-spaces and conical divisors

Let k be an algebraically closed field, and let V be an affine k-scheme of finite type equipped with a $\mathbb{G}_m$-action; i.e.

$$V = \mathrm{Spec}(A) \quad \text{for} \quad A = \bigoplus_{n \in \mathbb{Z}} A_n \text{ a graded ring.} \tag{4.1}$$

I wish to understand what it means to "twist" V by another scheme with a $\mathbb{G}_m$-action.

4.1.1 Calculations over Spec(k)

To begin with, let us work over the point $\mathrm{Spec}(k)$, and let $W = \mathrm{Spec}(B)$ be a scheme with the same properties as V. To construct the twist described approximately by

$$W \times^{\mathbb{G}_m} V \text{``=''} (W \times V)/(w, v) \sim (\lambda^{-1}w, \lambda v) \text{ for } \lambda \in \mathbb{G}_m \tag{4.2}$$

we proceed as follows. Let $\bar{B}$ be the graded k-algebra whose underlying algebra is B and with grading given by $\bar{B}_n = B_{-n}$. Then define

$$W \times^{\mathbb{G}_m} V := \mathrm{Spec}\left((\bar{B} \otimes_k A)^{\mathbb{G}_m}\right), \tag{4.3}$$

i.e. the GIT quotient $W \times^{\mathbb{G}_m} V = (\bar{W} \times V)//\mathbb{G}_m$, where $\bar{W}$ signifies that we have taken the inverse to the usual $\mathbb{G}_m$-action on W.

Remark 4.1.1. It is reasonable to ask whether or not this GIT quotient in fact defines a *geometric* quotient, as was suggested by our approximate description of the twist (4.2). This will not always be the case: for instance, if $V = \mathbb{A}_k^N$ and $W = \mathbb{A}_k^M$ then it is a short exercise to show that the described GIT quotient will yield $\mathbb{A}_k^{NM}$, which for $N, M > 2$ cannot be a geometric quotient for dimension reasons.

I am interested in a situation where this does happen to give a geometric quotient, however: namely, when W is the total space of a line bundle on a smooth (affine) k-variety with its zero section removed. Much of the analysis that follows does not rely on whether or not the GIT quotient is a geometric quotient, and we will use the notation $W \times^{\mathbb{G}_m} V$ to refer to the GIT quotient without worrying too much about whether or not the quotient is geometric.

Unwrapping the definition of $W \times^{\mathbb{G}_m} V$, we find that

$$(\bar{B} \otimes_k A)^{\mathbb{G}_m} = (\bar{B} \otimes_k A)_0 = \bigoplus_n (\bar{B})_{-n} \otimes A_n = \bigoplus_n B_n \otimes A_n =: C. \tag{4.4}$$

Note that C has a natural grading given by

$$C_n = B_n \otimes A_n, \tag{4.5}$$

i.e. there is a residual action of $\mathbb{G}_m$ on $W \times^{\mathbb{G}_m} V$, which we can interpret as acting on either the V or the W factor (with the action being "balanced out" by the quotient, i.e. independent of whether one chooses to act on V or on W and hence well-defined).

Example 24. Suppose that $W = \mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$. Inverting the $\mathbb{G}_m$-action on W places t in degree -1 and t^{-1} in degree +1, so that

$$\mathbb{G}_m \times^{\mathbb{G}_m} V = \mathrm{Spec}\left(\bigoplus_n k \cdot t^n \otimes_k A_n\right) = \mathrm{Spec}\left(\bigoplus_n t^n A_n\right) \cong \mathrm{Spec}(A)$$

since $\bigoplus_n t^n A_n \cong A$ naturally as graded k-algebras. I.e. there is a natural isomorphism

$$\mathbb{G}_m \times^{\mathbb{G}_m} V \cong V$$

as k-schemes equipped with a $\mathbb{G}_m$-action. Since this is supposed to be an associated bundle construction, this is as we would expect.

Now, suppose that we are given $f \in A_d$, $d > 0$, assumed to be neither a unit nor nilpotent. The vanishing locus of f defines a codimension 1 subscheme of V,

$$D = \mathrm{Spec}(A/(f)) \subset \mathrm{Spec}(A) = V \tag{4.6}$$

which, since f is homogeneous, is also equipped with an $\mathbb{G}_m$-action. Consider the graded subring of B defined by[1]

$$B^+_{(d)} := \bigoplus_{n \geq 0} B_{dn}, \quad (B^+_{(d)})_i = B_{di}. \tag{4.7}$$

Then the element f induces a graded k-algebra homomorphism

$$\begin{array}{ccc} B^+_{(d)} & \xrightarrow{\varphi} & \bigoplus_{n\in\mathbb{Z}} B_n \otimes A_n \\ b_{dn} & \longmapsto & b_{dn} \otimes f^n \end{array} \tag{4.8}$$

where $b_{dn} \in B_{dn}$. Note that this is a well-defined homomorphism since no strictly positive degree element of $B^+_{(d)}$ can be a unit – indeed this was the reason for the truncation, and we will soon see in the motivating example of twisting a $\mathbb{G}_m$-space by a line bundle that such a construction appears organically.

Thus, we have an $\mathbb{G}_m$-equivariant map

$$W \times^{\mathbb{G}_m} V \to \mathrm{Spec}(B^+_{(d)}). \tag{4.9}$$

I will now examine in more detail the case where W is the total space of a line bundle with its zero section removed.

[1]This is essentially the d^{th} Veronese ring of B but with a twisted grading and truncated to non-negative degrees.

Example 25. In the simplest situation, let $W = \mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$. Then $k[t, t^{-1}]^+_{(d)} = k[t^d]$, and the map φ is

$$\begin{array}{ccccc} k[t^d] & \longrightarrow & \bigoplus t^n A_n & \xrightarrow{\sim} & A \\ t^d & \longmapsto & t^d f & \longmapsto & f \end{array}$$

i.e. this corresponds to the map f itself, which we may think of a $\mathbb{G}_m$-equivariant map $V \to \mathbb{A}^1_k$ (where the $\mathbb{G}_m$ action on $\mathbb{A}^1_k$ has weight d).

4.1.2 Twisting construction in families

I now want to perform a version of the above construction in families, with the goal of answering the question: *If f is a homogeneous function on a space V equipped with $\mathbb{G}_m$-action, what sort of function does it induce on the twist of V by a line bundle?*

So, suppose that $X = \mathrm{Spec}(R)$ is a smooth affine k-variety and L is a locally free R-module of rank 1 with dual $L^\vee = \mathrm{Hom}_R(L, R)$. The corresponding geometric line bundle is given by

$$\mathcal{L} = \underline{Spec}_X(\mathrm{Sym}_R(L^\vee)) \to X. \tag{4.10}$$

As before, $V = \mathrm{Spec}(A)$ is an affine k-scheme with a $\mathbb{G}_m$-action, however for ease of exposition I will additionally assume that A is non-negatively graded. We have

$$\begin{tikzcd} \mathcal{L} = \underline{Spec}_X(\mathrm{Sym}_R(L^\vee)) \arrow[dr] & & X \times V = \underline{V} \arrow[dl] \\ & X & \end{tikzcd} \tag{4.11}$$

and so we form the scheme over X

$$V_{\mathcal{L}} := (\mathcal{L} \setminus 0) \times_X^{\mathbb{G}_m} \underline{V}. \tag{4.12}$$

Because of our assumption that A is non-negatively graded, the process of taking $\mathbb{G}_m$-invariants kills off any negatively graded functions on $\mathcal{L} \setminus 0$, i.e. functions with a pole along the zero section. Hence, we may equivalently work with

$$V_{\mathcal{L}} = \mathcal{L} \times_X^{\mathbb{G}_m} \underline{V}. \tag{4.13}$$

Remark 4.1.2. Note that although the GIT quotients above yield the same space, $(\mathcal{L} \setminus 0) \times^{\mathbb{G}_m} \underline{V}$ is a geometric quotient of $(\mathcal{L} \setminus 0) \times_X \underline{V}$, while $\mathcal{L} \times_X^{\mathbb{G}_m} \underline{V}$ is *not* a geometric quotient of $\mathcal{L} \times_X \underline{V}$. It is a distinction probably worth remembering: the former description is geometrically accurate, while the latter is algebraically convenient in our situation.

Now, we explicitly have that

$$\begin{aligned} V_{\mathcal{L}} = \mathcal{L} \times_X^{\mathbb{G}_m} \underline{V} &= \underline{Spec}_X \left((\overline{\mathrm{Sym}_R(L^\vee)} \otimes_R (R \otimes_k A))^{\mathbb{G}_m} \right) \\ &= \underline{Spec}_X \left((\overline{\mathrm{Sym}_R(L^\vee)} \otimes_k A)^{\mathbb{G}_m} \right) \\ &= \underline{Spec}_X \left((\overline{\mathrm{Sym}_R(L^\vee)} \otimes_k A)_0 \right) \\ &= \underline{Spec}_X \left(\bigoplus_{n \geq 0} \mathrm{Sym}_R(L^\vee)_n \otimes_k A_n \right). \end{aligned} \tag{4.14}$$

Given $f \in A_d$ as before, let $B := A/(f)$ and

$$D = \mathrm{Spec}(B) \subset \mathrm{Spec}(A) = V. \tag{4.15}$$

Since f is homogeneous, B is (non-negatively) graded, and so we may form

$$D_{\mathcal{L}} = \underline{Spec}_X \left(\bigoplus_{n \geq 0} \mathrm{Sym}_R(L^\vee)_n \otimes_k B_n \right). \tag{4.16}$$

Since L is assumed locally free, the homogeneous components of the symmetric algebra are flat R-algebras, and so the quotient map $A \twoheadrightarrow B$ induces an R-linear quotient map

$$\bigoplus_{n \geq 0} \mathrm{Sym}_R(L^\vee)_n \otimes_k A_n \twoheadrightarrow \bigoplus_{n \geq 0} \mathrm{Sym}_R(L^\vee)_n \otimes_k B_n, \tag{4.17}$$

and so a closed embedding over X

$$\begin{tikzcd} D_{\mathcal{L}} \arrow[rr] \arrow[dr] & & V_{\mathcal{L}} \arrow[dl] \\ & X & \end{tikzcd} \tag{4.18}$$

Now, L is a locally free (graded) R-module of rank 1, hence so is $L^{\otimes d}$, and we have $(L^{\otimes d})^\vee = (L^\vee)^{\otimes d}$. The following isomorphism may be checked locally, where it becomes Example 25:

$$\mathrm{Sym}_R((L^\vee)^{\otimes d}) \cong \mathrm{Sym}_R(L^\vee)^+_{(d)}. \tag{4.19}$$

Relative Spec of this ring is exactly the d^{th} line bundle tensor power $\mathcal{L}^{\otimes d}$, and so the graded R-linear map induced by f

$$\begin{aligned} \mathrm{Sym}_R(L^\vee)^+_{(d)} &\xrightarrow{\varphi} \textstyle\bigoplus_{n \geq 0} \mathrm{Sym}_R(L^\vee)_n \otimes_k A_n \\ p_{dn} &\longmapsto p_{dn} \otimes f^n \end{aligned} \tag{4.20}$$

induces a $\mathbb{G}_m$-equivariant map over X

$$\begin{tikzcd} V_{\mathcal{L}} \arrow[rr, "\Phi"] \arrow[dr] & & \mathcal{L}^{\otimes d} \arrow[dl] \\ & X & \end{tikzcd} \tag{4.21}$$

Proposition 4.1.1. $\Phi^{-1}(0) = D_{\mathcal{L}}$

Proof. The zero section is defined by the map

$$z : \mathrm{Sym}_R(L^\vee)^+_{(d)} \to R$$

which is projection onto the 0-graded piece, i.e. projection onto $\mathbb{G}_m$-invariants. Thus the fibre over the zero section is defined by the pushout of R-algebras

$$\begin{tikzcd} \mathrm{Sym}_R(L^\vee)^+_{(d)} \arrow[r, "\varphi"] \arrow[d, "z"'] \arrow[dr, phantom, "\ulcorner", very near end] & \bigoplus_{n\geq 0} \mathrm{Sym}_R(L^\vee)_n \otimes_k A_n \arrow[d] \\ R \arrow[r] & R \bigotimes_{\mathrm{Sym}_R(L^\vee)^+_{(d)}} \left(\bigoplus_n \mathrm{Sym}_R(L^\vee)_n \otimes_k A_n \right) \end{tikzcd} \tag{4.22}$$

Due to our assumption that f is not nilpotent φ is injective, and so we may identify this pushout with

$$\bigoplus_n \mathrm{Sym}_R(L^\vee)_n \otimes_k (A/(f))_n = \bigoplus_n \mathrm{Sym}_R(L^\vee)_n \otimes_k B_n; \tag{4.23}$$

taking relative Spec over X then completes the proof. $\square$

Remark 4.1.3. By working in affine patches, the same analysis applies more generally to any smooth k-variety X (not necessarily affine).

We now have the following commutative diagram:

$$\begin{tikzcd} D_{\mathcal{L}} \arrow[r] \arrow[d, hook] & 0 \arrow[d, hook] \\ V_{\mathcal{L}} \arrow[r, "\Phi"] \arrow[dr] & \mathcal{L}^{\otimes d} \arrow[dl] \\ & X \end{tikzcd} \tag{4.24}$$

Now, suppose that $\sigma : X \to V_{\mathcal{L}}$ is a section. Then

$$\sigma^{-1}(D_{\mathcal{L}}) = \sigma^{-1}(\Phi^{-1}(0)) = (\Phi \circ \sigma)^{-1}(0) \tag{4.25}$$

and $\Phi \circ \sigma : X \to \mathcal{L}^{\otimes d}$ is a section of a line bundle. Therefore $\Phi \circ \sigma$ is nowhere vanishing if and only if it trivialises $\mathcal{L}^{\otimes d}$, and so we conclude that:

Theorem 4.1.2. *Let V, f be as above, and suppose that $\mathcal{L} \to X$ is a line bundle on a smooth k-variety whose d^{th} power is non-trivial. Then any section of the associated V-bundle $V_{\mathcal{L}} \to X$ has nonempty intersection with the divisor $D_{\mathcal{L}} \subset V_{\mathcal{L}}$.*

4.2 Application to cameral covers

We may immediately apply Theorem 4.1.2 to get the following result:

Theorem 4.2.1. *Suppose that $\mathfrak{g}$ is a simple Lie algebra and choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let W and R denote the corresponding Weyl group and root system, respectively. Let X be a smooth complex algebraic variety, and let L be a complex line bundle on X such that $L^{\otimes |R|}$ is non-trivial. Then every L-valued $\mathfrak{g}$ cameral cover of X is ramified.*

Proof. This is Theorem 4.1.2 applied to the situation where $V = \mathfrak{c} = \mathfrak{g}//G = \mathfrak{h}/W$ is the adjoint quotient, and

$$f = \prod_{\alpha \in R} d\alpha. \tag{4.26}$$

The $d\alpha$ are all linear functions on $\mathfrak{h}$, so f is homogenous of degree $|R|$, and since the Weyl group permutes the roots α this function descends to the quotient $\mathfrak{c}$. Finally, as noted in section 3.4.3, the vanishing locus of f is exactly the branch locus for the universal cameral cover $\mathfrak{h} \to \mathfrak{c}$. $\square$

Remark 4.2.1. In fact the above theorem can be sharpened slightly: it holds exactly for simply-laced root systems, however in the non simply-laced case we have a factorisation into a product over the short roots R_s and long roots R_l,

$$\prod_{\alpha \in R} d\alpha = \left(\prod_{\alpha_s \in R_s} d\alpha_s \right) \left(\prod_{\alpha_l \in R_l} d\alpha_l \right) \tag{4.27}$$

where each factor is individually Weyl group invariant. Therefore we only need to require that $L^{\otimes |R_l|}$ and $L^{\otimes |R_s|}$ are both nontrivial, and we may conclude that we have ramification points in our cameral cover corresponding to both long and short roots.

Chapter 5

Duality for quotients of the moduli of Higgs bundles

Having dealt with the neccessary background and preliminary results in Chapters 3 and 4, in this chapter I present the main results of this thesis: a generalisation of the Langlands duality and mirror symmetry results of [24,37] (Theorems 5.5.1 and 5.5.2), and the existence of self-dual moduli stacks conjecturally related to theories of class $\mathcal{S}$ (Corollary 5.5.3 and Conjecture 1).

I begin in Section 5.1 by comparing the Hitchin fibres for isogenous simple groups, in particular observing that isogenous simple groups have isogenous Hitchin fibres (Theorem 5.1.3). This leads to a particularly nice comparison theorem relating the Hitchin Pryms of Langlands dual groups (Theorem 5.1.4).

In Section 5.2 I introduce the main object of interest in this thesis, the moduli space $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$ of "$\widetilde{G}$-Higgs bundles of arbitrary degree, modulo $Z(\widetilde{G})$" (this quasi-definition will be elucidated over the course of the chapter, for instance in Example 29). I will also introduce as an intermediary object of study a moduli stack $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)$ which has a clearer modular interpretation, but which behaves poorly under Cartier duality. The end of this section (5.2.2) is dedicated to studying the geometry of $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)$ and $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$ locally over the Hitchin base.

In Section 5.3 I make use of the sheaves of regular centralisers to identify the Hitchin Prym for G_{ad} as a quotient of the Hitchin Prym for $\widetilde{G}_\tau$.

In Section 5.4 I study the behaviour of $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)$ under Cartier duality. Although there is not a clear modular description of this dual in terms of the Langlands dual group, this analysis is required for the main theorems.

Section 5.5 contains the main results of this thesis: (1) the Langlands duality interpretation of $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)^D$ (Theorem 5.5.1), (2) the identification of the Cartier dual to certain finite group quotients of $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$, and (3) the existence of self-dual moduli stacks conjecturally related to $\Sigma_{\mathfrak{g}}[C;\Gamma]$ (Corollary 5.5.3).

Finally, in Section 5.6 I conclude the thesis with a collection of examples illustrating the duality results of Section 5.5. I discuss how these results relate to theories of class $\mathcal{S}$ (Examples 30 and 31), compare with the results of [24, 37] (Examples 32 and 33), and discuss Theorem 5.5.1 for Higgs bundles of type B_n and C_n (Example 34).

5.1 Comparison of Hitchin fibres for isogenous simple groups

In what follows I will make heavy use of comparisons between Hitchin Pryms (Definition 5.1.1) for different reductive groups belonging to the same isogeny class. Let G be a reductive algebraic group and denote by $J^G_\sigma \to C$ the pullback of the group scheme of regular centralisers for G by the map $\sigma : C \to [\mathfrak{c}/\mathbb{G}_m]$ classifying a point in the Hitchin base; i.e. $J^G_\sigma = \sigma^{-1}J^G$, where J^G is the group scheme of regular centralisers for G.

Restrict to the situation where G a simple group. The following claim may be checked quickly via a computation on stalks:

Lemma 5.1.1. *Let $G \to G/Z$ denote an isogeny of simple groups, so that Z is a discrete subgroup of the centre $Z(G)$. There is a short exact sequence of commutative group schemes over $[\mathfrak{c}/\mathbb{G}_m]$,*

$$0 \to Z_{\mathfrak{c}} \to J^G \to J^{G/Z} \to 0. \tag{5.1}$$

Since pullback of sheaves is exact there is an analogous exact sequence over any other $[\mathfrak{c}/\mathbb{G}_m]$-scheme. In particular, corresponding to a point σ in the Hitchin base we have a short exact sequence of sheaves over C

$$0 \to Z_C \to J_\sigma^G \to J_\sigma^{G/Z} \to 0. \tag{5.2}$$

Suppose now that $\widetilde{G}$ is a connected and simply-connected simple group. Taking the long exact sequence of the sequence (5.2) yields

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] & \Gamma(C; J_\sigma^{\widetilde{G}}) \arrow[r] & \Gamma(C; J_\sigma^{\widetilde{G}/Z}) \arrow[dll, rounded corners] \\
& H^1(C; Z) \arrow[r] & H^1(C; J_\sigma^{\widetilde{G}}) \arrow[r] & H^1(C; J_\sigma^{\widetilde{G}/Z}) \arrow[dll, rounded corners] \\
& H^2(C; Z) & &
\end{tikzcd} \tag{5.3}$$

Definition 5.1.1. For simply connected $\widetilde{G}$, the *Hitchin Prym for $\widetilde{G}$ associated to σ* is

$$H^1(C; J_\sigma^{\widetilde{G}}) \quad (\cong \mathbf{Higgs}_{\widetilde{G}}(C)_\sigma). \tag{5.4}$$

For a general reductive group G define the *Hitchin Prym* to be the identity component of $H^1(C; J^G_\sigma)$.

Remark 5.1.1. Note that for a non simply-connected semisimple group $\widetilde{G}/Z$ the Hitchin Prym is given by

$$\ker[H^1(C; J_\sigma^{\widetilde{G}/Z}) \to H^2(C; Z)] \quad (\cong \mathbf{Higgs}^0_{\widetilde{G}/Z}(C)_\sigma). \tag{5.5}$$

Remark 5.1.2. In order to identify the cohomology group $H^1(C; J^G_\sigma)$ with the Hitchin fibre $\mathbf{Higgs}_G(C)_\sigma$ I have implicitly trivialised the gerbe of Higgs bundles [25] using a Hitchin section (3.40).

The Hitchin Pryms are known to be abelian varieties [24], and a rephrasing of Corollary 3.4.5 yields that the fibres of the Hitchin fibration for $\widetilde{G}/Z$ which lie over very regular characteristics are torsors for the $\mathbf{Higgs}^0_{\widetilde{G}/Z}(C)_\sigma$.

Rewrite the exact sequence associated to (5.2) as

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r] & \Gamma(C; J_\sigma^{\widetilde{G}}) \arrow[r] & \Gamma(C; J_\sigma^{\widetilde{G}/Z}) \arrow[dlll, rounded corners] \\ H^1(C; Z) \arrow[r] & \mathbf{Higgs}_{\widetilde{G}}(C)_\sigma \arrow[r] & \mathbf{Higgs}^0_{\widetilde{G}/Z}(C)_\sigma \arrow[r] & 0. \end{tikzcd} \tag{5.6}$$

Lemma 5.1.2. *Suppose that C is a smooth, proper, irreducible curve over $\mathbb{C}$, that the line bundle classified by σ has nontrivial $|R|^{th}$ power, and that σ is a very regular characteristic. Then the map on global sections $\Gamma(C; J_\sigma^{\widetilde{G}}) \to \Gamma(C; J_\sigma^{\widetilde{G}/Z})$ is surjective.*

Remark 5.1.3. In what follows I will make use of an alternative and more explicit description of the sheaf of regular centralisers, which is due to [25]. Denote by

$\pi_\sigma : \tilde{C}_\sigma \to C$ the cameral cover of C classified by $\sigma : C \to [\mathfrak{c}/\mathbb{G}_m]$, and consider the sheaf on $\tilde{C}_\sigma$ of holomorphic maps to a choice of maximal torus $H \subset G$, $H(\mathcal{O}_{\tilde{C}_\sigma})$. Push this sheaf down to C and take $W \equiv W_G(H)$-invariants, calling the result $\overline{H}_{\tilde{C}_\sigma}$,

$$\overline{H}_{\tilde{C}_\sigma}(U) = \left((\pi_\sigma)_* H(\mathcal{O}_{\tilde{C}_\sigma})^W\right)(U) = \mathrm{Hom}_W(\tilde{U}_\sigma, H), \tag{5.7}$$

i.e. W-equivariant maps from the induced cameral cover $\tilde{U}_\sigma$ to the maximal torus H. Denote by $\mathfrak{D}_\sigma^\alpha$ the fixed point scheme of the root reflection $s_\alpha \in W$ acting on $\tilde{C}_\sigma$, and define a subsheaf $H_{\tilde{C}_\sigma} \subset \overline{H}_{\tilde{C}_\sigma}$ by

$$H_{\tilde{C}_\sigma}(U) = \{t \in \overline{H}_{\tilde{C}_\sigma}(U) \,|\, (\alpha \circ t)|_{\mathfrak{D}_\sigma^\alpha} = +1 \text{ for each } \alpha \in R\}. \tag{5.8}$$

Then according to [25, Theorem 11.6] there is an isomorphism between J_σ^G and $T_{\tilde{C}_\sigma}$. I will use the description given by the latter in the proof of Lemma 5.1.2.

Proof. First observe that since C is proper so is $\tilde{C}_\sigma$, and since σ is assumed to be very regular $\tilde{C}_\sigma$ is non-singular. Thus, since H is affine, any map from $\tilde{C}_\sigma$ to H will be locally constant; i.e.

$$\overline{H}_{\tilde{C}_\sigma}(C) = \mathrm{Hom}_W(\tilde{C}_\sigma, H) \subset \mathrm{Hom}_W(\pi_0(\tilde{C}_\sigma), H). \tag{5.9}$$

Write $\tilde{C}_\sigma$ in terms of components as

$$\tilde{C}_\sigma = \coprod_{i=1}^{|\pi_0(\tilde{C}_\sigma)|} \tilde{C}^{(i)}. \tag{5.10}$$

Since the W-action is transitive on sheets, by choosing a component $\tilde{C}^{(i)} \in \pi_0(\tilde{C}_\sigma)$ we can identify

$$\overline{H}_{\tilde{C}_\sigma}(C) = \mathrm{Hom}_W(\tilde{C}_\sigma, H) = \mathrm{Hom}_{\mathrm{Stab}_W(\tilde{C}^{(i)})}(\tilde{C}^{(i)}, H) \subset H^{\mathrm{Stab}_W(\tilde{C}^{(i)})}, \tag{5.11}$$

where $\mathrm{Stab}_W(\tilde{C}^{(i)}) \subset W$ is the stabiliser of $\tilde{C}^{(i)}$ under the induced action of W on $\pi_0(\tilde{C}_\sigma)$. If the cover $\tilde{C}_\sigma$ were unramified this inclusion would be an equality; in order to take into account possible ramification, let $S(x) \subset \mathrm{Stab}_W(\tilde{C}^{(i)})$ denote the stabiliser of the closed point $x \in \tilde{C}^{(i)}$. Then

$$\overline{H}_{\tilde{C}_\sigma}(C) = \mathrm{Hom}_{\mathrm{Stab}_W(\tilde{C}^{(i)})}(\tilde{C}^{(i)}, H) = \bigcap_{x \in \tilde{C}^{(i)}} H^{S(x)} \subset H^{\mathrm{Stab}_W(\tilde{C}^{(i)})}. \tag{5.12}$$

The stabiliser of any point in $\tilde{C}_\sigma$ may be identified with the stabiliser of its image (under any local trivialisation where we identify $\tilde{C}_\sigma \to C$ as pulled back from $\mathfrak{h} \to \mathfrak{c}$), and all such stabilisers are Weyl subgroups of W.[1]

From the assumption that the line bundle classified by σ has nontrivial $|R|^{th}$ power, we know from Theorem 4.2.1 and the subsequent remark that there exists a root[2] α such that $s_\alpha \in \mathrm{Stab}_W(\tilde{C}^{(i)})$ and $\mathfrak{D}^\alpha_\sigma \cap \tilde{C}^{(i)} \neq \emptyset$. Thus there is some finite subset of roots $R' \subset R$ – which must contain both a long and a short root, in the non-simply laced case, and which is closed under the action of $\mathrm{Stab}_W(\tilde{C}^{(i)})$ on R – such that

$$\overline{H}_{\tilde{C}_\sigma}(C) = \bigcap_{x \in \tilde{C}^{(i)}} H^{S(x)} = H^{\langle s_\alpha | \alpha \in R' \rangle} = \bigcap_{\alpha \in R'} T^{s_\alpha}. \tag{5.13}$$

Now, according to Theorem B.1.2,

$$\begin{aligned} H^{s_\alpha, +1} &:= \{ tZ \in (H/Z)^{s_\alpha} \mid \alpha(tZ) = +1 \} \\ &= \{ tZ \in (H/Z)^{s_\alpha} \mid s_\alpha(tz) = tz \text{ for all } tz \in tZ \}. \end{aligned} \tag{5.14}$$

[1] Specifically, for $\tilde{x} \in \tilde{C}_\sigma$ they are the subgroups generated by the root reflections corresponding to root hyperplanes containing the image of $\tilde{x}$ under $\tilde{C}_\sigma \to \mathfrak{h}$.

[2] In the simply-laced case, and in the non-simply laced case that there exists both a long and a short root.

The top condition, $\alpha(tZ) = +1$, is exactly the extra condition that distinguishes $\overline{(H/Z)}_{\tilde{C}_\sigma}$ from $(H/Z)_{\tilde{C}_\sigma} \cong J_\sigma^{\widetilde{G}/Z}$; the bottom condition cuts out exactly the subset of $(H/Z)^{s_\alpha}$ whose preimages under $H \to H/Z$ are also fixed points of s_α. Hence we have that

$$H_{\tilde{C}_\sigma}(C) = \bigcap_{\alpha \in R'} H^{s_\alpha,+1} \to \bigcap_{\alpha \in R'} (H/Z)^{s_\alpha,+1} = (H/Z)_{\tilde{C}_\sigma}(C) \tag{5.15}$$

is a surjection, which is exactly the statement we wished to prove. □

Example 26. How could this have failed? Suppose that C is an irreducible complex projective variety that admits a connected étale double cover: all double covers are $\mathfrak{sl}_2\mathbb{C}$ cameral covers, so we are implicitly assuming that our double cover is cameral and valued in some line bundle which has trivial square. As observed in the proof of Lemma 5.1.2, global sections of J in this case are given by

$$J^{SL_2}(C) = T^W_{SL_2} \cong \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad J^{PGL_2}(C) = T^W_{PGL_2} = \mathbb{Z}/2\mathbb{Z}, \tag{5.16}$$

where the Weyl group invariants in this case were calculated in Example 38. Thus, in this example $J^{PGL_2}(C) \ncong J^{SL_2}(C)/Z(SL_2\mathbb{C})$.

From Lemma 5.1.2 we obtain a comparison theorem relating any Hitchin Prym to the Hitchin Prym for the connected simply-connected group:

Theorem 5.1.3. *Let $\widetilde{G}$ be a simple, connected, simply-connected group and $\widetilde{G} \to \widetilde{G}/Z$ an isogeny. Then for $\sigma \in \boldsymbol{Hitch}_{\mathfrak{g}}(C, K_C) \setminus \Delta_{\mathfrak{g}}$ there is an isomorphism of abelian varieties*

$$\boldsymbol{Higgs}^0_{\widetilde{G}/Z}(C)_\sigma = \frac{\boldsymbol{Higgs}_{\widetilde{G}}(C)_\sigma}{H^1(C, Z)}. \tag{5.17}$$

Proof. By Lemma 5.1.2, $H^1(C;Z) \to \mathbf{Higgs}_{\widetilde{G}}(C)_\sigma$ is injective, thus the long exact sequence of (5.2) involving the Hitchin Pryms breaks up into two short exact sequences; the isomorphism of the theorem is the content of the bottom sequence. □

Remark 5.1.4. To really get value out of Theorem 5.1.3 one should assume that the genus of C is at least 2, so that the very regular locus is open and dense in the Hitchin base.

Theorem 5.1.4. *Let $\widetilde{G}$ be a simple, connected, simply-connected group, and let $\widetilde{{}^LG}$ denote the simply-connected cover of its Langlands dual group. Then*

$$(\boldsymbol{Higgs}_{\widetilde{{}^LG}}(C)_\sigma)^D = \frac{\boldsymbol{Higgs}_{\widetilde{G}}(C)_\sigma}{H^1(C;Z(\widetilde{{}^LG}))^\vee}. \tag{5.18}$$

Proof. By [24, Theorem A] we have that $\mathbf{Higgs}_{\widetilde{G}}(C)_\sigma = (\mathbf{Higgs}^0_{{}^LG_{\mathrm{ad}}}(C)_\sigma)^D$. Dualising the isogeny of abelian varieties from Theorem 5.1.3

$$0 \to H^1(C;Z(\widetilde{{}^LG})) \to \mathbf{Higgs}_{\widetilde{{}^LG}}(C)_\sigma \to \mathbf{Higgs}^0_{{}^LG_{\mathrm{ad}}}(C)_\sigma \to 0 \tag{5.19}$$

we obtain the dual isogeny

$$0 \to H^1(C;Z(\widetilde{{}^LG}))^\vee \to (\mathbf{Higgs}^0_{{}^LG_{\mathrm{ad}}}(C)_\sigma)^D \to (\mathbf{Higgs}_{\widetilde{{}^LG}}(C)_\sigma)^D \to 0. \tag{5.20}$$

□

5.2 Construction and local structure of $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$ and $\mathcal{M}^\bullet_{\widetilde{G}}(C)$

In their proof of Langlands duality for SL/PGL-Hitchin systems [37], Hausel and Thaddeus make use not just of the moduli of SL_n-Higgs bundles but of the moduli space of "degree d" SL_n-Higgs bundles. This does not literally make sense as written (as an SL_n-bundle has trivial determinant and is thus degree zero) – what is really meant by this is "GL_n-Higgs bundles with determinant a *fixed line bundle of degree d* and *trace-free Higgs field*".

To generalise the results of [24, 37], and to prove the existence of a self-dual space, I will now construct a generalisation of this space for $\widetilde{G}$-Higgs bundles, where $\widetilde{G}$ may be any connected simply-connected semisimple group (c.f. [7] for an analogous construction for the moduli stack of bundles).

5.2.1 Construction of $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)$

Let μ_N denote the group of N^{th} roots of unity with generator $\omega := e^{\frac{2\pi i}{N}}$. Observe that a homomorphism $\tau : (\mu_N)^s \to (\mathbb{C}^\times)^s$ is determined by an $s \times s$-matrix $A = (A_{ji}) \in \mathrm{Mat}_{s\times s}(\mathbb{Z}/N\mathbb{Z})$ by setting

$$\begin{aligned} (\mu_N)^s &\xrightarrow{\tau} (\mathbb{C}^\times)^s \\ (\omega^{\vec{a}}) &\longmapsto (\omega^{A\vec{a}}) \end{aligned} \tag{5.21}$$

where $\vec{a} \in (\mathbb{Z}/N\mathbb{Z})^s$ and $(\omega^{\vec{a}}) = (\omega^{a_1}, \ldots, \omega^{a_s}) \in (\mu_N)^s$.

Definition 5.2.1. Call an homomorphism $\tau : (\mu_N)^s \to (\mathbb{C}^\times)^s$ a *special embedding* if it can be represented by a matrix in the image of the map $SL_s(\mathbb{Z}) \to SL_s(\mathbb{Z}/N\mathbb{Z})$.

More generally, let K be a finite abelian group equipped with an isomorphism $k : K \simeq \mu_{N_1} \times \cdots \times \mu_{N_s}$, and let T is an complex algebraic torus of rank s. I will call a homomorphism $\tau : K \to T$ a *special embedding* if the map it induces $\tau \circ k^{-1} : (\mu_{lcm(N_1,\ldots,N_s)})^s \to (\mathbb{C}^\times)^s$ is a special embedding for some isomorphism $T \simeq (\mathbb{C}^\times)^s$.

Remark 5.2.1. If τ is a special embedding with respect to *some* isomorphism $T \simeq (\mathbb{C}^\times)^s$, it is in fact an isomorphism with respect to *all* such isomorphisms.

Remark 5.2.2. It is not difficult to show that $\tau : (\mu_N)^s \to (\mathbb{C}^\times)^s$ is an embedding if and only if any matrix A which represents it is in $GL_s(\mathbb{Z}/N\mathbb{Z})$. In particular, special embeddings are embeddings.

Now, let $\widetilde{G}$ be a connected simply-connected simple group with centre $Z(\widetilde{G})$, fix a trivialisation $k : Z(\widetilde{G}) \to \mu_{N_1} \times \cdots \times \mu_{N_s}$, and let $\tau : Z(\widetilde{G}) \to T$ be a special embedding of $Z(\widetilde{G})$ into a complex algebraic torus (whose rank s is necessarily equal to the number of cyclic factors in $Z(\widetilde{G})$, by the definition of a special embedding).[3]

Definition 5.2.2. Define a group $\widetilde{G}_\tau$ by the equation

$$\widetilde{G}_\tau := \frac{\widetilde{G} \times T}{Z(\widetilde{G})}, \tag{5.22}$$

where $Z(\widetilde{G}) \subset \widetilde{G}$ is the inclusion homomorphism.

Proposition 5.2.1. *The group $\widetilde{G}_\tau$ is independent of the choice of special embedding, up to non-canonical isomorphism.*

Proof. Suppose that τ_1, τ_2 are two special embeddings, and consider them as maps from $(\mu_N)^s \to (\mathbb{C}^\times)^s$ where N is the lowest common multiple of the orders of the cyclic factors of $Z(\widetilde{G})$. Let A_1, A_2 be representative matrices for the special embeddings. We wish to find an automorphism $\beta : (\mathbb{C}^\times)^s \to (\mathbb{C}^\times)^s$ such that $\beta \circ \tau_1 = \tau_2$.

[3]This value of s is moreover the minimal possible rank for a torus admitting an embedding of $Z(\widetilde{G})$.

As observed above, β will be represented by some matrix $B \in \mathrm{Mat}_{s\times s}(\mathbb{Z})$. For $\beta \circ \tau_1 = \tau_2$ to hold, we need that for all $\vec{a} \in (\mathbb{Z}/N\mathbb{Z})^s$, $\beta \circ \tau_1(\omega^{\vec{a}}) = (\omega^{BA_1\vec{a}}) = (\omega^{A_2\vec{a}}) = \tau_2(\omega^{\vec{a}})$, which occurs if and only if $BA_1 \equiv A_2$ modulo N. But by the definition of a special embedding the matrices representing τ_1 and τ_2 may be lifted to matrices in $SL_s(\mathbb{Z})$, which I will also denote by A_1 and A_2, and so it suffices to take $B = A_2A_1^{-1}$.

To complete the proof of the proposition, it suffices to observe that $[\mathrm{id}_{\widetilde{G}} \times \beta]$ is a well-defined isomorphism $\widetilde{G}_{\tau_1} \simeq \widetilde{G}_{\tau_2}$. $\square$

Remark 5.2.3. It is reasonable to ask whether we really needed to consider special embeddings, or whether any matrix $A \in GL_s(\mathbb{Z}/N\mathbb{Z})$ would suffice. In fact, we need to require at least that A is in the image of the map $GL_s(\mathbb{Z}) \to GL_s(\mathbb{Z}/N\mathbb{Z})$. Suppose that $\det(A) \neq \pm 1$ modulo N, so that A cannot be lifted to $GL_s(\mathbb{Z})$. It is possible to find an automorphism γ of $(\mu_N)^s$, represented by a matrix C, such that $\det(AC) = 1$. In order for this to induce an isomorphism as in Proposition 5.2.1, α would need to extend to an automorphism of the group $\widetilde{G}$, necessarily not an inner automorphism. But, for example, $\mathrm{Out}(SL_8\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$ while $\mathrm{Aut}(Z(SL_8\mathbb{C})) = \mathrm{Aut}(\mathbb{Z}/8\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ – so there are necessarily automorphisms of the centre which do not extend to automorphisms of the entire group.

Remark 5.2.4. Note that the isomorphism β in Proposition 5.2.1 is **not** unique. For example, if $s = 2$ we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$, and both are in $SL_2(\mathbb{Z})$.

The group $\widetilde{G}_\tau$ comes equipped with two projections

$$\begin{tikzcd} & \widetilde{G}_\tau \arrow[dl, "p"'] \arrow[dr, "\partial"] & \\ G_{\mathrm{ad}} & & T/Z(\widetilde{G}) \end{tikzcd} \tag{5.23}$$

Note that $T/Z(\widetilde{G}) \simeq T$ non-canonically: for the moment I will not choose such an isomorphism, preferring to work with $T/Z(\widetilde{G})$.

Example 27. Let $\widetilde{G} = SL_n$ and $\tau : Z(SL_n) = \mu_n \subset \mathbb{G}_m$.[4] Then $\widetilde{G}_\tau = GL_n$ and the maps p and ∂ are

$$\begin{tikzcd} & GL_n \arrow[dl, "p"'] \arrow[dr, "\det"] & \\ PGL_n & & \mathbb{G}_m \end{tikzcd} \tag{5.24}$$

The Lie algebra of $\widetilde{G}_\tau$ is

$$\mathfrak{g}_\tau = \mathfrak{g} \oplus \mathfrak{t} \tag{5.25}$$

where $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$ and $\mathfrak{t} = \mathrm{Lie}(T)$. Let $H \subset \widetilde{G}$ be a maximal torus with Lie algebra $\mathfrak{h}$ so that

$$H_\tau = \frac{H \times T}{Z(\widetilde{G})} \tag{5.26}$$

is a maximal torus of $\widetilde{G}_\tau$ with Lie algebra $\mathfrak{h}_\tau = \mathfrak{h} \times \mathfrak{t}$. Since $\mathfrak{t}$ is abelian, the quotient $\mathfrak{c}_\tau = \mathfrak{h}_\tau / W$ (where $W \equiv W_{\widetilde{G}_\tau}(H_\tau) = W_{\widetilde{G}}(H)$ is the Weyl group, see Appendix B) is

$$\mathfrak{c}_\tau = (\mathfrak{h}/W) \times \mathfrak{t} = \mathfrak{c} \times \mathfrak{t} \tag{5.27}$$

[4]Although some results will require that we work over $\mathbb{C}$, many of the constructions – such as this one – are independent of the ground ring.

where $\mathfrak{c} = \mathfrak{h}/W$ is the adjoint quotient for the group $\widetilde{G}$. Thus there is a "Hitchin map" between stacks

$$\chi_\tau = \chi \times \mathrm{id}_{\mathfrak{t}} : \left[\mathfrak{g}_\tau/\widetilde{G}_\tau \times \mathbb{G}_m\right] = \left[(\mathfrak{g} \times \mathfrak{t})/\widetilde{G}_\tau \times \mathbb{G}_m\right] \to [\mathfrak{c}/\mathbb{G}_m] \times [\mathfrak{t}/\mathbb{G}_m] = [\mathfrak{c}_\tau/\mathbb{G}_m]\,. \tag{5.28}$$

The maps p and ∂ induce maps

$$\begin{tikzcd}[column sep=small] & \left[(\mathfrak{g} \times \mathfrak{t})/\widetilde{G}_\tau \times \mathbb{G}_m\right] \arrow[dl, "p_*"'] \arrow[dr, "\partial_*"] & \\ {[\mathfrak{g}/G_{\mathrm{ad}} \times \mathbb{G}_m]} & & \left[\mathfrak{t}/(T/Z(\widetilde{G})) \times \mathbb{G}_m\right] \simeq B(T/Z(\widetilde{G})) \times [\mathfrak{t}/\mathbb{G}_m] \end{tikzcd} \tag{5.29}$$

and so for a space[5] X there are maps

$$\begin{tikzcd}[column sep=small] & \mathrm{Map}\left(X, \left[(\mathfrak{g} \times \mathfrak{t})/\widetilde{G}_\tau \times \mathbb{G}_m\right]\right) \arrow[dl, "p_*"'] \arrow[dr, "\partial_*"] & \\ \mathrm{Map}\left(X, [\mathfrak{g}/G_{\mathrm{ad}} \times \mathbb{G}_m]\right) & & \mathcal{B}un_{T/Z(\widetilde{G})}(X) \times \mathrm{Map}\left(X, [\mathfrak{t}/\mathbb{G}_m]\right) \end{tikzcd} \tag{5.30}$$

Supposing now that the pushforwards to $B\mathbb{G}_m$ all classify the line bundle $L \to X$, we obtain maps

$$\begin{tikzcd}[column sep=small] & \mathcal{H}iggs_{\widetilde{G}_\tau}(X, L) \arrow[dl, "p_*"'] \arrow[dr, "\partial_*"] & \\ \mathcal{H}iggs_{G_{\mathrm{ad}}}(X, L) & & \mathcal{B}un_{T/Z(\widetilde{G})}(X) \times H^0(X; \mathfrak{t} \otimes L) \end{tikzcd} \tag{5.31}$$

[5]Depending on the category: Complex scheme, algebraic variety, projective algebraic curve/Riemann surface, etc.

Example 28. In the running SL_n/GL_n example (5.24), these maps are

$$\begin{tikzcd} & \mathcal{H}iggs_{GL_n}(X,L) \arrow[dl, "p_*"'] \arrow[dr, "\det\times\mathrm{tr}"] & \\ \mathcal{H}iggs_{PGL_n}(X,L) & & \mathcal{P}ic(X)\times H^0(X;L) \end{tikzcd} \tag{5.32}$$

Now, choose an isomorphism $t : T \cong \mathbb{G}_m^s$. Under this isomorphism $Z(\widetilde{G})$ is sent to a product of groups of roots of unity, so t induces an isomorphism

$$T/Z(\widetilde{G}) \cong \frac{\mathbb{G}_m}{\mu_{i_1}} \times \cdots \times \frac{\mathbb{G}_m}{\mu_{i_s}} \tag{5.33}$$

and by taking i_j^{th} powers componentwise we obtain an isomorphism $T/Z(\widetilde{G}) \cong \mathbb{G}_m^s$. This isomorphism of groups allows us to further identify

$$\mathcal{B}un_{T/Z(\widetilde{G})}(X) \cong \mathcal{P}ic(X) \times \cdots \times \mathcal{P}ic(X). \tag{5.34}$$

Remark 5.2.5. Choosing a different trivialisation $T \cong \mathbb{G}_m^s$ induces a unique automorphism of $\frac{\mathbb{G}_m}{\mu_{i_1}} \times \cdots \times \frac{\mathbb{G}_m}{\mu_{i_s}}$ and so ultimately a unique automorphism of the stack $\mathcal{P}ic(X) \times \cdots \times \mathcal{P}ic(X)$.

Now, suppose that $X = C$ is a Riemann surface, or a smooth complex projective algebraic curve. Choose a point $x \in C$ and for $\vec{p} = (p_1, \ldots, p_s)$ denote

$$\mathcal{O}(\vec{p}x) = (\mathcal{O}(p_1x), \ldots, \mathcal{O}(p_sx)) \in \mathcal{B}un_{T/Z(\widetilde{G})}(C) \tag{5.35}$$

where we have used the isomorphism (5.34). Define a lattice by $\Lambda(x) = \{\mathcal{O}(\vec{p}x) \,|\, \vec{p} \in \mathbb{Z}^s\} \subset \mathcal{B}un_{T/Z(\widetilde{G})}(C)$. Passing to the group of connected components of $\mathcal{B}un_{T/Z(\widetilde{G})}(C)$ exhibits an isomorphism

$$\begin{tikzcd} \Lambda(x) \arrow[rr, "\cong"] \arrow[dr, "\iota"'] & & X_\bullet(T/Z(\widetilde{G})) \\ & \mathcal{B}un_{T/Z(\widetilde{G})}(C) \arrow[ur] & \end{tikzcd} \tag{5.36}$$

and so yields a splitting $\iota_x : X_\bullet(T/Z(\widetilde{G})) \hookrightarrow \mathcal{B}un_{T/Z(\widetilde{G})}(C)$.

Definition 5.2.3. Define $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C, L)$ to be the pullback of stacks over the *trace-free locus* of the Hitchin base $\{0\} \subset H^0(C; \mathfrak{t} \otimes L)$

$$\begin{tikzcd}
\mathcal{H}iggs^\bullet_{\widetilde{G}}(C, L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{H}iggs_{\widetilde{G}_\tau}(C, L) \arrow[d, "\partial_*"] & \\
X_\bullet(T/Z(\widetilde{G})) \arrow[r, "\iota_x"'] & \mathcal{B}un_{T/Z(\widetilde{G})}(C) \arrow[r, hook, "\mathrm{id}\times 0"] & \mathcal{B}un_{T/Z(\widetilde{G})}(C) \times H^0(C; \mathfrak{t} \otimes L)
\end{tikzcd} \tag{5.37}$$

Remark 5.2.6. Note that given another point $y \in C$, the embeddings ι_x and ι_y differ by the automorphism of $\mathcal{B}un_{T/Z(\widetilde{G})}(C)$ given by tensoring with the $T/Z(\widetilde{G})$-bundles $\mathcal{O}(\vec{p}(y - x))$. Since $\mathcal{O}(\vec{p}(y - x)) \in \mathbf{Bun}^0_{T/Z(\widetilde{G})}(C)$ and $\mathbf{Bun}^0_{T/Z(\widetilde{G})}(C)$ is a divisible abelian group, this also yields an automorphism of $\mathcal{H}iggs_{\widetilde{G}_\tau}(C, L)$. Hence, by uniqueness of pullbacks the stacks $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C, L)$ for various choices of $x \in C$ are all isomorphic.

Remark 5.2.7. When $L = K_C$, I will often omit the line bundle from the notation, e.g.

$$\mathcal{H}iggs_G(C, K_C) \equiv \mathcal{H}iggs_G(C), \quad \mathbf{Hitch}_\mathfrak{g}(C, K_C) = \mathbf{Hitch}_\mathfrak{g}(C), \quad \text{etc.} \tag{5.38}$$

5.2.2 Local description over $\mathbf{Hitch}_\mathfrak{g}(C)$ and definition of $\mathcal{M}^\bullet_{\widetilde{G}}(C)$

Recall that the Hitchin base is defined by $\mathbf{Hitch}_\mathfrak{g}(C, L) = H^0(C; \mathfrak{c}_L)$, so that for $\mathfrak{g}_\tau$

$$\begin{aligned}
\mathbf{Hitch}_{\mathfrak{g}_\tau}(C, L) &= H^0(C; \mathfrak{c}_L \times (\mathfrak{t} \otimes L)) \\
&= H^0(C; \mathfrak{c}_L) \times H^0(C; \mathfrak{t} \otimes L) \\
&= \mathbf{Hitch}_\mathfrak{g}(C, L) \times H^0(C; \mathfrak{t} \otimes L).
\end{aligned} \tag{5.39}$$

Restricting to the case of $L = K_C$, the following square commutes (though is **not** cartesian):

$$\begin{tikzcd}
\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C) \arrow[r] \arrow[d] & \mathcal{H}iggs_{\widetilde{G}_\tau}(C) \arrow[d] \\
\mathbf{Hitch}_{\mathfrak{g}}(C) \arrow[r, "\mathrm{id}\times 0"'] & \mathbf{Hitch}_{\mathfrak{g}}(C) \times H^0(C; \mathfrak{t}\otimes K_C)
\end{tikzcd} \tag{5.40}$$

Remark 5.2.8. From now on I will implicitly restrict $\mathcal{H}iggs_{\widetilde{G}_\tau}(C)$ to the trace-free locus $\mathbf{Hitch}_{\mathfrak{g}}(C) \times \{0\} \subset \mathbf{Hitch}_{\mathfrak{g}} \times H^0(C; \mathfrak{t}\otimes K_C) = \mathbf{Hitch}_{\mathfrak{g}_\tau}(C)$, as this is the appropriate place to compare $\mathcal{H}iggs_{\widetilde{G}_\tau}(C)$ with $\mathcal{H}iggs_{\widetilde{G}}(C)$.

Note that

$$\begin{aligned}
Z(\widetilde{G}_\tau) \cong T &\hookrightarrow \widetilde{G}_\tau = \tfrac{\widetilde{G}\times T}{Z(\widetilde{G})} \\
t &\longmapsto [(1_{\widetilde{G}}, t)]
\end{aligned} \tag{5.41}$$

hence $\mathcal{H}iggs_{\widetilde{G}_\tau}(X)|_{\mathbf{Hitch}_{\mathfrak{g}}(C)\setminus\Delta} \to \mathbf{Higgs}_{\widetilde{G}_\tau}(X)|_{\mathbf{Hitch}_{\mathfrak{g}}(C)\setminus\Delta}$ is a (locally trivial) $Z(\widetilde{G}_\tau) = T$-gerbe [24].

Remark 5.2.9 (Important Remark!). From now on I will assume that we are working away from the discriminant locus (3.53), and **I will omit the explicit restriction symbol** "$|_{\mathbf{Hitch}_{\mathfrak{g}}(C)\setminus\Delta}$".

In other words, locally the stack $\mathcal{H}iggs_{\widetilde{G}_\tau}(C)$ decomposes as the product

$$\mathcal{H}iggs_{\widetilde{G}_\tau}(C) \cong \mathbf{Higgs}_{\widetilde{G}_\tau}(C) \times BT \tag{5.42}$$

and moreover, the coarse moduli space $\mathbf{Higgs}_{\widetilde{G}_\tau}(C)$ splits locally into the product of its neutral component and its group of connected components (since its group of connected components is the free group $\pi_0(\mathbf{Higgs}_{\widetilde{G}_\tau}(C)) = \pi_0(\mathbf{Bun}_{T/Z(\widetilde{G})}(C)) =$

$X_\bullet(T/Z(\widetilde{G})))$, i.e. locally

$$\mathcal{H}iggs_{\widetilde{G}_\tau}(C) \simeq \mathbf{Higgs}^0_{\widetilde{G}_\tau}(C) \times X_\bullet(T/Z(\widetilde{G})) \times BT. \tag{5.43}$$

Next I wish to understand the local structure of $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$. A (closed) point of $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$ is given by

- a $\widetilde{G}_\tau$-bundle $P \to C$
- a Higgs field $\phi \in H^0(C; \mathfrak{c}_{K_C})$ (i.e. "tracefree"), and
- an isomorphism $\psi : \partial_*(P) \simeq \mathcal{O}(\vec{p}x)$ (for some $\vec{p} \in \mathbb{Z}^s$).

More generally, an S-point of $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$ is given by

- a $\widetilde{G}_\tau$-bundle $P_S \to C \times S$
- a Higgs field $\phi_S \in H^0(C \times S; \mathrm{pr}_C^*(\mathfrak{c}_{K_C}))$, and
- an isomorphism $\psi_S : \partial_*(P_S) \simeq \mathrm{pr}_C^*(\mathcal{O}(\vec{p}x))$ (for some $\vec{p} \in \mathbb{Z}^s$).

Note that the action of BT which was previously given by tensoring with the pullback of a T-bundle on S must be restricted: now only T-bundles $\mathcal{T}_S \to S$ satisfying

$$\partial_*(\mathcal{T}_S) \simeq \mathcal{O}_S \tag{5.44}$$

may act on the moduli space. These are exactly those T-bundles which are induced from $Z(\widetilde{G})$-bundles via τ,

$$0 \longrightarrow BZ(\widetilde{G}) \xrightarrow{B\tau} BT \xrightarrow{B\partial} B(T/Z(\widetilde{G})) \longrightarrow 0, \tag{5.45}$$

so we see that one effect of pulling back is a "reduction of structure group" from BT to $BZ(\widetilde{G})$.

To see what happens to the abelian variety component in the local decomposition (5.43), note that the component defined by the cartesian diagram

$$\begin{tikzcd} \mathcal{H}iggs^0_{\widetilde{G}}(C,L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \mathcal{H}iggs^\bullet_{\widetilde{G}}(C,L) \arrow[d] \\ * \arrow[r, "0"'] & X_\bullet(T/Z(\widetilde{G})) \end{tikzcd} \tag{5.46}$$

may be identified as $\mathcal{H}iggs^0_{\widetilde{G}}(C,L) \simeq \mathcal{H}iggs_{\widetilde{G}}(C,L)$, the usual moduli of Higgs bundles for the simply-connected simple group $\widetilde{G}$. So locally $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$ decomposes as

$$\mathcal{H}iggs^\bullet_{\widetilde{G}}(C) \cong \mathbf{Higgs}_{\widetilde{G}}(C) \times X_\bullet(T/Z(\widetilde{G})) \times BZ(\widetilde{G}). \tag{5.47}$$

The natural map $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C) \to \mathcal{H}iggs_{\widetilde{G}_\tau}(C)$ is locally

$$\begin{tikzcd} \mathbf{Higgs}_{\widetilde{G}}(C) \arrow[d, hook] & \times & X_\bullet(T/Z(\widetilde{G})) \arrow[d, hook, "\mathrm{id}"] & \times & BZ(\widetilde{G}) \arrow[d, hook, "B\tau"] \\ \mathbf{Higgs}^0_{\widetilde{G}_\tau}(C) & \times & X_\bullet(T/Z(\widetilde{G})) & \times & BT \end{tikzcd} \tag{5.48}$$

and the projection $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C) \to \mathcal{H}iggs_{G_{\mathrm{ad}}}(C)$ is locally

$$\begin{tikzcd} \mathbf{Higgs}_{\widetilde{G}}(C) \arrow[d, "\text{isogeny}"'] & \times & X_\bullet(T/Z(\widetilde{G})) \arrow[d] & \times & BZ(\widetilde{G}) \arrow[d] \\ \mathbf{Higgs}^0_{G_{\mathrm{ad}}}(C) & \times & Z(\widetilde{G}) & \times & * \end{tikzcd} \tag{5.49}$$

There is another important stack which admits a map from $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$, constructed as follows. Recall from (3.8) that $\mathcal{B}un_T(C)$ acts on $\mathcal{H}iggs_{\widetilde{G}_\tau}(C)$ (see Section 3.1.1 for a description of this action). Via the splitting $\iota_x : X_\bullet(T) \to \mathcal{B}un_T(C)$ we may restrict this to an action of $X_\bullet(T)$, which may be thought of concretely as tensoring Higgs bundles with $\mathcal{O}(\vec{p}x)$ as $\vec{p}$ ranges over $\mathbb{Z}^s$, and this action restricts to give an action on $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$.

Definition 5.2.4. Denote by $\mathcal{M}^\bullet_{\widetilde{G}}(C)$ the stack $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)/X_\bullet(T)$.

Proposition 5.2.2. *Locally the quotient map $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C) \to \mathcal{M}^\bullet_{\widetilde{G}}(C)$ is given by*

$$\begin{array}{ccccccc} \mathcal{H}iggs^\bullet_{\widetilde{G}}(C) & \simeq & \boldsymbol{Higgs}_{\widetilde{G}}(C) & \times & X_\bullet(T/Z(\widetilde{G})) & \times & BZ(\widetilde{G}) \\ \downarrow & & \| & & \downarrow & & \| \\ \mathcal{M}^\bullet_{\widetilde{G}}(C) & \simeq & \boldsymbol{Higgs}_{\widetilde{G}}(C) & \times & Z(\widetilde{G}) & \times & BZ(\widetilde{G}) \end{array} \tag{5.50}$$

Proof. The action of $X_\bullet(T)$ on $\mathbf{Higgs}_{\widetilde{G}}(C)$ and $BZ(\widetilde{G})$ is trivial, so it suffices to check this claim for the group of connected components. For this, is suffices to check the corresponding claim for the moduli space of bundles (not Higgs bundles). Consider the generalisation of the Kümmer sequence[6]

$$1 \to Z(\widetilde{G}) \to T(\mathcal{O}_C) \to (T/Z(\widetilde{G}))(\mathcal{O}_C) \to 1. \tag{5.51}$$

The H^0 row of the corresponding long exact sequence in cohomology is exact (since C is compact/projective); starting at H^1 the long exact sequence is

$$\begin{tikzcd} 0 \arrow[r] & H^1(C; Z(\widetilde{G})) \arrow[r] & H^1(C; T(\mathcal{O}_C)) \arrow[r] & H^1(C; (T/Z(\widetilde{G}))(\mathcal{O}_C)) \arrow[dll, rounded corners] \\ & H^2(C; T(\mathcal{O}_C)) & & \end{tikzcd} \tag{5.52}$$

[6]This becomes the Kümmer sequence for $T = \mathbb{G}_m$ and $Z(\widetilde{G}) = \mu_n$.

Now, $H^2(C; T(\mathcal{O}_C)) = 0$ – this follows analytically by taking the long exact sequence of the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 1$ and observing that there are no (2,0)-forms on C, and it follows algebraically from the existence of an injective comparison map $H^2_{et}(C; \mathbb{G}_m) \to H^2(C^{an}; \mathcal{O}_C^\times)$ [26].

Identifying $H^2(C; Z(\widetilde{G})) = Z(\widetilde{G})$ canonically and using the identification $H^1(C; T(\mathcal{O}_C)) = \mathbf{Bun}_T(C)$, (5.52) becomes

$$0 \to H^1(C; Z(\widetilde{G})) \to \mathbf{Bun}_T(C) \to \mathbf{Bun}_{T/Z(\widetilde{G})}(C) \to Z(\widetilde{G}) \to 0 \tag{5.53}$$

The map out of $H^1(C; Z(\widetilde{G}))$ factors through the identity component of $\mathbf{Bun}_T(C)$, and so the content of (5.53) may be split into the two identifications: $\mathbf{Bun}^0_{T/Z(\widetilde{G})}(C) \cong \frac{\mathbf{Bun}^0_T(C)}{H^1(C; Z(\widetilde{G}))}$ and

$$Z(\widetilde{G}) \cong \frac{\pi_0(\mathbf{Bun}_{T/Z(\widetilde{G})}(C))}{\pi_0(\mathbf{Bun}_T(C))} = \frac{X_\bullet(T/Z(\widetilde{G}))}{X_\bullet(T)} \tag{5.54}$$

which is what we wished to show. $\square$

Example 29. In the running example with $\widetilde{G} = SL_n$, $\mathcal{M}^\bullet_{SL_n}(C)$ may be thought of as encoding the observation that the moduli spaces $\mathbf{Higgs}^d_{SL_n}$ depend only on d mod n, and that tensoring with the line bundle $\mathcal{O}(x)$ is an isomorphism $\mathbf{Higgs}^d_{SL_n} \cong \mathbf{Higgs}^{d+n}_{SL_n}$.

5.3 Comparing sheaves of regular centralisers

In (5.48) we observed that $\mathbf{Higgs}_{\widetilde{G}}(C)$ appears as an abelian subvariety of $\mathbf{Higgs}^0_{\widetilde{G}_\tau}(C)$. Since upon Cartier duality subobjects become quotient objects, and under Langlands duality simply-connected groups are sent to adjoint groups, it should be the case that $\mathbf{Higgs}^0_{G_{\mathrm{ad}}}(C)$ may be realised as a quotient of $\mathbf{Higgs}^0_{\widetilde{G}_\tau}(C)$. To see that this is indeed possible, I will compare the sheaves of regular centralisers $J^{\widetilde{G}_\tau}$ and $J^{\widetilde{G}}$.

Proposition 5.3.1. *(1)* $J^{G_1 \times G_2} = J^{G_1} \times J^{G_2}$

(2) If $T \cong (\mathbb{C}^\times)^n$ *then* $J^T = T$.

(3) There is a short exact sequence of sheaves

$$0 \to Z(\widetilde{G}) \to J^{\widetilde{G}} \times T \to J^{\widetilde{G}_\tau} \to 0. \tag{5.55}$$

Proof. (1) Follows from the fact that the Lie algebra of $G_1 \times G_2$ is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, and the adjoint action factors as $G_1 \times G_2 \to \mathrm{End}(\mathfrak{g}_1) \oplus \mathrm{End}(\mathfrak{g}_2) \subset \mathrm{End}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$.

(2) Since T is abelian the adjoint action is trivial, so $Z_T(x) = T$ for every $x \in \mathfrak{t}$.

(3) This can be checked locally, as per Lemma 5.1.1.

□

Proposition 5.3.2. *There are isomorphisms of abelian schemes (over the complement of the discriminant locus in the Hitchin base)* $\boldsymbol{Higgs}^0_{G_{\mathrm{ad}}}(C) \cong \frac{\boldsymbol{Higgs}^0_{\widetilde{G}_\tau}(C)}{\boldsymbol{Bun}^0_T(C)}$ *and* $\frac{\boldsymbol{Bun}^0_T(C)}{H^1(X;Z(\widetilde{G}))} \cong \frac{\boldsymbol{Higgs}^0_{\widetilde{G}_\tau}(C)}{\boldsymbol{Higgs}_{\widetilde{G}}(C)}$.

Proof. Pulling the short exact sequence (5.55) back via some cameral cover of C

yields

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(C; Z(\widetilde{G})) \arrow[r] & \Gamma(C; J_{\widetilde{G}} \times T) \arrow[r] & \Gamma(C; J_{\widetilde{G}_\tau}) \arrow[dll, rounded corners, to path={}] \\
& H^1(C; Z(\widetilde{G})) \arrow[r] & H^1(C; J_{\widetilde{G}} \times T) \arrow[r] & H^1(C; J_{\widetilde{G}_\tau}) \\
& H^2(C; Z(\widetilde{G})) = Z(\widetilde{G}) \arrow[r] & 0
\end{tikzcd}
\tag{5.56}
$$

where the vanishing of $H^2(C; J_{\widetilde{G}} \times T)$ is observed in [24, §5].

Let $K_\tau = \ker(H^1(C; J_{\widetilde{G}_\tau}) \to H^2(C; Z(\widetilde{G})))$.[7] Then (5.56) becomes

$$
\begin{tikzcd}
0 \arrow[r] & Z(\widetilde{G}) \arrow[r] & \Gamma(C; J_{\widetilde{G}}) \times T \arrow[r] & \Gamma(C; J_{\widetilde{G}_\tau}) \\
H^1(C; Z(\widetilde{G})) \arrow[r] & H^1(C; J_{\widetilde{G}}) \times H^1(C; T) \arrow[r] & K_\tau \arrow[r] & 0.
\end{tikzcd}
\tag{5.57}
$$

Since the map $H^1(C; Z(\widetilde{G})) \to H^1(C; T) = \mathbf{Bun}_T(C)$ is itself an embedding[8] (and in fact it factors through $H^1(C; T)^0 = \mathbf{Bun}^0_T(C)$), the above sequence splits into two short exact sequences, yielding

$$K_\tau = \frac{\mathbf{Higgs}_{\widetilde{G}}(C) \times \mathbf{Bun}_T(C)}{H^1(C; Z(\widetilde{G}))} \tag{5.58}$$

and (restricting the the neutral component)

$$\mathbf{Higgs}^0_{\widetilde{G}_\tau}(C) = \frac{\mathbf{Higgs}_{\widetilde{G}}(C) \times \mathbf{Bun}^0_T(C)}{H^1(C; Z(\widetilde{G}))}. \tag{5.59}$$

[7] Note that this is not necessarily connected, i.e. is not necessarily the neutral component.
[8] Reduce to the case $H^1(C; \mu_n) \subset H^1(C; \mathbb{G}_m)$.

The isomorphism $\frac{\mathbf{Bun}^0_T(C)}{H^1(X;Z(\widetilde{G}))} \cong \frac{\mathbf{Higgs}^0_{\widetilde{G}_\tau}(C)}{\mathbf{Higgs}_{\widetilde{G}}(C)}$ follows immediately from (5.59), and the isomorphism $\mathbf{Higgs}^0_{G_{\mathrm{ad}}}(C) \cong \frac{\mathbf{Higgs}^0_{\widetilde{G}_\tau}(C)}{\mathbf{Bun}^0_T(C)}$ follows from (5.59) and the identification $\mathbf{Higgs}^0_{G_{\mathrm{ad}}}(C) \cong \frac{\mathbf{Higgs}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))}$ of Theorem 5.1.3. □

5.4 Dualising $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$

At a first glance one might expect that the stacks $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$ will provide the correct generalisation of the Langlands duality results of [24, 37]. In this section we will see that this is not quite correct, since by remembering all of the connected components of $\mathbf{Higgs}_{\widetilde{G}_\tau}(C)$ this stack is keeping track of too much information (or perhaps better, it is keeping track of components and automorphisms in a non-symmetric manner). Regardless, I will describe the structure of the Cartier dual $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$ so that in Section 5.5 I can show that the moduli space $\mathcal{M}^\bullet_{\widetilde{G}}(C)$ *is* well-behaved under Cartier duality.

As a first step I will "measure the difference" between the stacks $\mathcal{H}iggs^\bullet_{\widetilde{G}}(C)$ and $\mathcal{H}iggs_{\widetilde{G}_\tau}(C)$, i.e.

Proposition 5.4.1. *There are isomorphisms of commutative group stacks*

$$\mathcal{H}iggs_{\widetilde{G}_\tau}(C)/\mathcal{H}iggs^\bullet_{\widetilde{G}}(C) \cong \frac{\mathcal{B}un_{T/Z(\widetilde{G})}(C)}{X_\bullet(T/Z(\widetilde{G}))} \cong \mathcal{B}un^0_{T/Z(\widetilde{G})}(C). \tag{5.60}$$

Proof. The second isomorphism is immediate – we have already seen that a choice of point $x \in C$ gives a splitting of the map $\mathcal{B}un_{T/Z(\widetilde{G})}(C) \to \pi_0(\mathcal{B}un_{T/Z(\widetilde{G})}(C)) = X_\bullet(T/Z(\widetilde{G}))$. Hence it suffices to prove the first isomorphism, which follows by composing the defining pullback square (5.37) with the pullback square

$$\begin{tikzcd} X_\bullet(T/Z(\widetilde{G})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{B}un_{T/Z(\widetilde{G})}(C) \arrow[d] \\ 0 \arrow[r] & \frac{\mathcal{B}un_{T/Z(\widetilde{G})}(C)}{X_\bullet(T/Z(\widetilde{G}))} \end{tikzcd} \tag{5.61}$$

to obtain the pullback square

$$\begin{tikzcd} \mathcal{H}iggs^\bullet_{\widetilde{G}}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{H}iggs_{\widetilde{G}_\tau}(C) \arrow[d] \\ 0 \arrow[r] & \frac{\mathcal{B}un_{T/Z(\widetilde{G})}(C)}{X_\bullet(T/Z(\widetilde{G}))} \end{tikzcd} \tag{5.62}$$

This can be seen to yield a short exact sequence of commutative group stacks via the local description of the maps given in Section 5.2.2. $\square$

Now, consider the following short exact sequences of commutative group stacks and their coarse moduli spaces:

$$\begin{gathered} 0 \longrightarrow \mathcal{H}iggs^\bullet_{\widetilde{G}}(C) \longrightarrow \mathcal{H}iggs_{\widetilde{G}_\tau}(C) \longrightarrow \mathcal{B}un^0_{T/Z(\widetilde{G})}(C) \longrightarrow 0 \\ 0 \longrightarrow \mathbf{Higgs}^\bullet_{\widetilde{G}}(C) \longrightarrow \mathbf{Higgs}_{\widetilde{G}_\tau}(C) \longrightarrow \mathbf{Bun}^0_{T/Z(\widetilde{G})}(C) \longrightarrow 0 \end{gathered} \tag{5.63}$$

Using Proposition 3.2.3 and the identifications given in Appendix C (as well as another dualisation result from [24], namely ${}^L\mathcal{H}iggs^0 = \mathbf{Higgs}^D$) these dualise to the short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathbf{Bun}_{{}^L(T/Z(\widetilde{G}))}(C) \longrightarrow \mathcal{H}iggs_{(\widetilde{{}^LG})_{L_\tau}}(C) \longrightarrow \mathcal{H}iggs^\bullet_{\widetilde{G}}(C)^D \longrightarrow 0 \\ 0 \longrightarrow \mathbf{Bun}^0_{{}^L(T/Z(\widetilde{G}))}(C) \longrightarrow \mathcal{H}iggs^0_{(\widetilde{{}^LG})_{L_\tau}}(C) \longrightarrow \mathbf{Higgs}^\bullet_{\widetilde{G}}(C)^D \longrightarrow 0 \end{gathered} \tag{5.64}$$

From the exact sequences (5.64), we are led to study the quotient stacks $\frac{\mathcal{H}iggs_{\widetilde{G}_\tau}(C)}{\mathbf{Bun}_T(C)}$ and $\frac{\mathcal{H}iggs^0_{\widetilde{G}_\tau}(C)}{\mathbf{Bun}^0_T(C)}$. By Proposition 5.3.2, $\frac{\mathbf{Higgs}^0_{\widetilde{G}_\tau}(C)}{\mathbf{Bun}^0_T(C)} \cong \mathbf{Higgs}^0_{G_{\mathrm{ad}}}(C)$, so that $\frac{\mathcal{H}iggs^0_{\widetilde{G}_\tau}(C)}{\mathbf{Bun}^0_T(C)}$ is a T-gerbe over $\mathbf{Higgs}^0_{G_{\mathrm{ad}}}(C)$. This result in fact extends to the non-neutral connected components as well:

Proposition 5.4.2. *The stack* $\frac{\mathcal{H}iggs_{\widetilde{G}_\tau}(C)}{\boldsymbol{Bun}_T(C)}$ *is a* T*-gerbe over* $\boldsymbol{Higgs}_{G_{\mathrm{ad}}}(C)$.

Proof. The exact sequence of groups

$$1 \to T \to \widetilde{G}_\tau \to G_{\mathrm{ad}} \to 1 \tag{5.65}$$

yields the short exact sequence of sheaves of regular centralisers

$$1 \to T(\mathcal{O}_C) \to J_{\widetilde{G}_\tau} \to J_{G_{\mathrm{ad}}} \to 1. \tag{5.66}$$

Global sections of (5.66) remain exact, so starting at H^1 the associated long exact sequence of cohomology gives

$$0 \to H^1(C; T(\mathcal{O}_C)) \to H^1(C; J_{\widetilde{G}_\tau}) \to H^1(C; J_{G_{\mathrm{ad}}}) \to H^2(C; T(\mathcal{O}_C)). \tag{5.67}$$

We have already seen that $H^2(C; T(\mathcal{O}_C)) = 0$ during the course of the proof of Proposition 5.2.2, and so this becomes the short exact sequence of coarse moduli spaces

$$0 \to \mathbf{Bun}_T(C) \to \mathbf{Higgs}_{\widetilde{G}_\tau}(C) \to \mathbf{Higgs}_{G_{\mathrm{ad}}}(C) \to 0. \tag{5.68}$$

Since $\mathcal{H}iggs_{\widetilde{G}_\tau}(C)$ is locally isomorphic to $\mathbf{Higgs}_{\widetilde{G}_\tau}(C) \times BT$ the result follows. $\square$

Combining this result with the short exact sequences (5.64) gives the following corollary:

Corollary 5.4.3. *(a)* $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)^D$ *is an* ${}^L(T/Z(\widetilde{G}))$*-gerbe over* $\boldsymbol{Higgs}_{{}^LG_{\mathrm{ad}}}(C)$*.*

(b) $\boldsymbol{Higgs}^{\bullet}_{\widetilde{G}}(C)^D$ *is an* ${}^L(T/Z(\widetilde{G}))$*-gerbe over* $\boldsymbol{Higgs}^0_{{}^LG_{\mathrm{ad}}}(C)$*.*

Notation 5.4.1. To declutter the notation, from now on I will denote $\frac{\mathcal{H}iggs_{\widetilde{G}_\tau}(C)}{\mathbf{Bun}_T(C)}$ by $\mathcal{Q}^{\bullet}_{\widetilde{G}}(C)$.

5.5 Dualising $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$

As per Example 29, the moduli space $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$ may be interpreted as the "moduli of $\widetilde{G}$-Higgs bundles on C of arbitrary degree, modulo uninteresting isomorphisms". The main results of this thesis – namely the generalisation of [24, 37] to incorporate "non-zero degrees" for all semisimple groups (Theorems 5.5.1 and 5.5.2) and the existence of self-dual moduli stacks associated to simply-laced Lie algebras (Corollary 5.5.3) – boil down to the fact that the moduli space $\mathcal{M}^{\bullet}_{\widetilde{G}}(C)$ behaves nicely under Cartier duality.

There is an action of $H^1(C; Z(\widetilde{G}))$ on $\mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C)$, induced by the $\mathcal{B}un_T(C)$ action on $\mathcal{H}iggs_{\widetilde{G}_\tau}(C)$ and the trivialisation of the gerbe $\mathcal{B}un_T(C)$ over $\mathbf{Bun}_T(C)$ given by the choice of point $x \in C$.[9] This action is free away from the discriminant locus of $\mathbf{Hitch}_{\mathfrak{g}}(C)$, a fact which may be checked locally.

Theorem 5.5.1. *Away from the discriminant locus in the Hitchin base there is an*

[9]The existence of such a trivialisation may be easier to see from the Cartier dual perspective, where it becomes the splitting of the map $\mathcal{B}un_{{}^LT}(C) \to \pi_0(\mathcal{B}un_{{}^LT}(C)) = X_{\bullet}({}^LT)$.

isomorphism of commutative group stacks

$$\left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))}\right)^D \cong \mathcal{M}^{\bullet}_{\widetilde{^LG}}(C). \tag{5.69}$$

Proof. Consider the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_{\bullet}(T) \times H^1(C;Z(\widetilde{G})) \arrow[r] \arrow[d] & \mathbf{Bun}_T(C) \arrow[r] \arrow[d] & \mathbf{Bun}^0_{T/Z(\widetilde{G})}(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H}iggs^{\bullet}_{\widetilde{G}}(C) \arrow[r] \arrow[d] & \mathcal{H}iggs_{\widetilde{G}_\tau}(C) \arrow[r] \arrow[d] & \mathcal{B}un^0_{T/Z(\widetilde{G})}(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))} \arrow[r] \arrow[d] & \mathfrak{Q}^{\bullet}_{\widetilde{G}}(C) \arrow[r] \arrow[d] & B(T/Z(\widetilde{G})) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{5.70}$$

Dualising this diagram gives

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & X_\bullet({}^L(T/Z(\widetilde{G}))) & \longrightarrow & \mathcal{Q}^\bullet_{\widetilde{G}}(C)^D & \longrightarrow & \left(\frac{\mathcal{M}^\bullet_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))}\right)^D & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathbf{Bun}_{{}^L(T/Z(\widetilde{G}))}(C) & \to & \mathcal{H}iggs_{(\widetilde{{}^LG})_{L_\tau}}(C) & \longrightarrow & \mathcal{H}iggs^\bullet_{\widetilde{G}}(C)^D & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathbf{Bun}^0_{{}^L(T/Z(\widetilde{G}))}(C) & \longrightarrow & \mathcal{B}un^0_{{}^LT}(C) & \longrightarrow & B({}^LT) \times B(H^1(C;Z(\widetilde{{}^LG}))) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array} \tag{5.71}$$

But by the definition of $\mathcal{Q}^\bullet_{\widetilde{G}}(C)$ and Proposition 5.4.2, $\mathcal{Q}^\bullet_{\widetilde{G}}(C)^D \cong \mathcal{H}iggs^\bullet_{\widetilde{{}^LG}}(C)$, and so the first row of the diagram can be rewritten as

$$\left(\frac{\mathcal{M}^\bullet_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))}\right)^D \cong \frac{\mathcal{H}iggs^\bullet_{\widetilde{{}^LG}}(C)}{X_\bullet({}^L(T/Z(\widetilde{G})))} =: \mathcal{M}^\bullet_{\widetilde{{}^LG}}(C). \tag{5.72}$$

□

Now, take a subgroup $\Gamma \subset H^1(C;Z(\widetilde{G}))$ and consider the "intermediate quotient" stack $\frac{\mathcal{M}^\bullet_{\widetilde{G}}(C)}{\Gamma}$. Recall that $H^1(C;Z(\widetilde{G}))$ is equipped with a non-degenerate skew pairing (c.f. (2.42)), and denote by $\mathrm{ann}(\Gamma)$ the annihilator of Γ with respect to this pairing.

Theorem 5.5.2. *Away from the discriminant locus in the Hitchin base there is an* *isomorphism of commutative group stacks*

$$\left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma}\right)^{D} \cong \frac{\mathcal{M}^{\bullet}_{\widetilde{^{L}G}}(C)}{\mathrm{ann}(\Gamma)}. \tag{5.73}$$

Proof. Consider the quotient map

$$\gamma : \frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma} \to \frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{H^1(C; Z(\widetilde{G}))} \tag{5.74}$$

with kernel $H^1(C; Z(\widetilde{G}))/\Gamma$. Locally the map (5.74) is

$$\begin{array}{ccccccc} \frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma} & \simeq & \frac{\mathbf{Higgs}_{\widetilde{G}}(C)}{\Gamma} & \times & Z(\widetilde{G}) & \times & BZ(\widetilde{G}) \\ {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\text{isogeny}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{id}_{Z(\widetilde{G})}} & & \big\downarrow{\scriptstyle\mathrm{id}_{BZ(\widetilde{G})}} \\ \frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))} & \simeq & \frac{\mathbf{Higgs}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))} & \times & Z(\widetilde{G}) & \times & BZ(\widetilde{G}) \end{array} \tag{5.75}$$

Under Cartier duality $(-)^D$, the map γ dualises locally to

$$\begin{array}{ccccccc} \left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))}\right)^{D} & \simeq & \left(\frac{\mathbf{Higgs}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))}\right)^{D} & \times & BZ(\widetilde{^{L}G}) & \times & Z(\widetilde{^{L}G}) \\ {\scriptstyle\gamma^D}\big\downarrow & & {\scriptstyle\text{dual isogeny}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{id}_{BZ(\widetilde{^{L}G})}} & & \big\downarrow{\scriptstyle\mathrm{id}_{Z(\widetilde{^{L}G})}} \\ \left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma}\right)^{D} & \simeq & \left(\frac{\mathbf{Higgs}_{\widetilde{G}}(C)}{\Gamma}\right)^{D} & \times & BZ(\widetilde{^{L}G}) & \times & Z(\widetilde{^{L}G}) \end{array} \tag{5.76}$$

The kernel of the dual isogeny is $(H^1(C; Z(\widetilde{G}))/\Gamma)^{\vee}$, so we have a short exact sequence

$$0 \to (H^1(C; Z(\widetilde{G}))/\Gamma)^{\vee} \to \left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{H^1(C; Z(\widetilde{G}))}\right)^{D} \to \left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma}\right)^{D} \to 0. \tag{5.77}$$

The theorem now follows from the identification $\left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{H^1(C;Z(\widetilde{G}))}\right)^{D} \cong \mathcal{M}^{\bullet}_{\widetilde{^{L}G}}(C)$ of Theorem 5.5.1, and the identification $(H^1(C; Z(\widetilde{G}))/\Gamma)^{\vee} \cong \mathrm{ann}(\Gamma)$ due to non-degeneracy of the skew-pairing on $H^1(C; Z(\widetilde{G}))$. ∎

In particular, we immediately deduce from Theorem 5.5.2 the existence of a collection of self-dual commutative group stacks.

Corollary 5.5.3. *In the setup of Theorem 5.5.2, suppose that* $\widetilde{G} = \widetilde{{}^L G}$ *(e.g.* $\widetilde{G}$ *is ADE type), and that* Γ *is a Lagrangian subgroup of* $H^1(C; Z(\widetilde{G}))$ *(i.e.* $\Gamma = \operatorname{ann}(\Gamma)$*). Then away from the discriminant locus of the Hitchin base*

$$\left(\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma}\right)^D \cong \frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma}, \tag{5.78}$$

i.e. $\frac{\mathcal{M}^{\bullet}_{\widetilde{G}}(C)}{\Gamma}$ *is a self-dual commutative group stack.*

Remark 5.5.1. As per Remark 2.1.9 and Conjecture 1, Corollary 5.5.3 may be interpreted as the statement that the target space of the 2d QFT $\Sigma_{\mathfrak{g}}[C;\Gamma]$ is self SYZ mirror dual.

Finally, we may deduce from the above results the following (non-stacky) corollary:

Corollary 5.5.4. *With notation as above and away from the discriminant locus in the Hitchin base,* $\frac{\boldsymbol{Higgs}_{\tilde{G}}(C)}{\Gamma}$ *and* $\frac{\boldsymbol{Higgs}_{\widetilde{{}^L G}}(C)}{\operatorname{ann}(\Gamma)}$ *are torsors for dual abelian schemes. In particular, if* $\tilde{G} = \widetilde{{}^L G}$ *and* $\operatorname{ann}(\Gamma) = \Gamma$ *then* $\frac{\boldsymbol{Higgs}_{\tilde{G}}(C)}{\Gamma}$ *is a self-dual abelian scheme.*

Proof. This follows from the previous results by restricting to the neutral component of the coarse moduli space. $\square$

5.6 Examples of dual spaces

To conclude I will describe how the results of Section 5.5 apply to various examples.

Example 30. An analysis of A_1 theories of class $\mathcal{S}$ was performed in [31]. There Gaiotto, Moore and Neitzke explain that a line operator in the A_1 theory corresponds to a simple closed path on C, and that a collection of line operators may be simultaneously included in the theory only if a "mutual locality condition" is satisfied. Geometrically, the mutual locality condition on a collection of line operators $\mathscr{L}$ becomes the requirement that that the number of intersection points of any two paths in $\mathscr{L}$ be even – by passing to Poincaré dual cocycles, this induces an isotropic subgroup of $H^1(C;\mu_2)$ with respect to the skew-pairing (2.42).

This example generalises to any group of ADE type by choosing a symplectic basis for $H_1(C)$ with respect to the intersection pairing (which may further be divided into a pair of bases for maximal isotropic subgroups, called A and B cycles) and taking as generators for Γ the Poincaré duals in $H^1(C;Z(\widetilde{G}))$ of an isotropic subset of this basis – for instance taking as a basis for Γ the Poincaré duals of all the A-cycles yields a maximal isotropic subgroup $\Gamma \subset H^1(C;Z(\widetilde{G}))$, and so by Corollary 5.5.3 and Remark 5.5.1 a self-dual target space for the 2d theory $\Sigma_{\mathfrak{g}}[C;\Gamma]$.

Example 31. As in Example 30 consider the A_1 theory, and suppose that we have chosen a collection of mutually local line operators $\Gamma \subset H^1(C;\mu_2)$ for the theory $\mathcal{S}_{\mathfrak{sl}_2}[C;\Gamma]$. If the collection of line operators is non-maximal, so that Γ is isotropic but not Lagrangian, then we may identify the Cartier/SYZ dual of the corresponding space/stack of Higgs bundles as per Theorem 5.5.2 and Corollary 5.5.4. However, this space does not have an obvious physical interpretation as the target space of a 2d σ-model, since the corresponding collection of line operators $\mathrm{ann}(\Gamma)$ no longer satisfies the mutual locality condition and cannot be used to absolve the relative theory $\mathcal{S}_{\mathfrak{sl}_2}[C]$.

Example 32. Theorem 5.5.1 in fact gives another derivation of the SYZ mirror symmetry results of Hausel and Thaddeus for SL/PGL-Higgs bundles [37]. To see this, observe that for type A_{n-1} (5.69) becomes

$$\left(\frac{\mathcal{M}^\bullet_{SL_n\mathbb{C}}(C)}{H^1(C;\mathbb{Z}/n\mathbb{Z})}\right)^D \cong \mathcal{M}^\bullet_{SL_n\mathbb{C}}(C). \tag{5.79}$$

The right hand side of this equation is the moduli stack of $GL_n\mathbb{C}$-Higgs bundles (E,ϕ) equipped with an isomorphism $\det(E) \simeq \mathcal{O}_C(dx)$ for some degree $d \in \mathbb{Z}/n\mathbb{Z}$[10] and such that $\operatorname{tr}\phi = 0$, and the object we are dualising on the left hand side is the moduli space of $PGL_n\mathbb{C}$-Higgs bundles equipped with the gerbe of liftings of the universal projective Higgs bundle to a universal GL_n-Higgs bundle (again, tracefree and equipped with an isomorphism $\det(E) \simeq \mathcal{O}_C(dx)$). The exact form of [37, Thm. 3.7] then ought to follow from an argument similar to the proof of [24, Cor. 5.5] (the details of which will appear in a future paper).

Example 33. Consider the group $G = SO(2n)$. This is a self Langlands dual group, and so by the results of Donagi and Pantev [24] gives rise to a self-dual moduli space of Higgs bundles. It is natural to ask whether or not this space fits into the story of this dissertation.

In fact it does: for simplicity I will discuss this duality on the level of coarse moduli spaces. The centre of the universal cover $\tilde{G} = Spin(2n)$ is either $\mu_2 \times \mu_2$ (if $2n = 4k$) or μ_4 (if $2n = 4k+2$). The central subgroup corresponding to $SO(2n)$ is

[10] The dependence on d mod n rather than $d \in \mathbb{Z}$ is observed in [37].

either the diagonal copy of $\mu_2 \subset \mu_2 \times \mu_2$ or the unique μ_2 subgroup of μ_4 – in either case this subgroup is isotropic with respect to the natural pairing on $Z(\widetilde{G})$, and so induces an isotropic subgroup $H^1(C;\mu_2) \subset H^1(C;Z(\widetilde{G}))$. By nondegeneracy of the skew-pairing on $H^1(C;Z(\widetilde{G}))$ this subgroup is maximal isotropic, and the resulting abelian scheme $\frac{\mathbf{Higgs}_{Spin(2n)}(C)}{H^1(C;\mu_2)}$ is isomorphic to $\mathbf{Higgs}^0_{SO(2n)}(C)$, the moduli space of $SO(2n)$-Higgs bundles with vanishing second Stiefel-Whitney class.

To make this example extremely concrete, consider the first non-trivial case $G = SO(4)$. The universal cover is $\tilde{G} = Spin(4) = SU(2) \times SU(2)$ with centre $\mu_2 \times \mu_2$, corresponding to the μ_2 centres of each of the $SU(2)$ factors. $Spin(4)$ double covers the spaces $SO(3) \times SU(2)$, $SU(2) \times SO(3)$, and $SO(4)$, corresponding respectively to the subgroups $\mu_2 \times 1$, $1 \times \mu_2$, and the diagonal subgroup Δ. Denote the unique nondegenerate pairing on μ_2 by Υ_2; then the pairing on the central $\mu_2 \times \mu_2$ is

$$\Upsilon((a,b),(c,d)) = \Upsilon_2(a,c)\Upsilon_2(b,d). \tag{5.80}$$

On the diagonal subgroup corresponding to $SO(4)$, this pairing is identically 1, since $\Upsilon((a,a),(b,b)) = \Upsilon_2(a,b)^2 = 1$. Hence the subgroup $H^1(C;\Delta) \subset H^1(C;\mu_2 \times \mu_2)$ is isotropic, and by nondegeneracy of the cup product pairing and of Υ on $\mu_2 \times \mu_2$ it is maximal isotropic and the results of the previous paragraph apply.

Example 34. Finally, it is interesting to consider what the duality of Theorem 5.5.1 looks like for the simply-connected groups $Sp(2n)$ and $Spin(2n+1)$, whose Lie algebras are exchanged by Langlands duality.

First, consider the isomorphism

$$\left(\frac{\mathcal{M}^{\bullet}_{Sp(2n)}(C)}{H^1(C;\mu_2)}\right)^D \cong \mathcal{M}^{\bullet}_{Spin(2n+1)}(C). \tag{5.81}$$

The stack we are dualising on the left hand side of (5.81) is the moduli space of $PSp(2n) = Sp(2n)/\mu_2$-Higgs bundles equipped with the gerbe of liftings of the universal $PSp(2n)$-Higgs bundle to a universal symplectic Higgs bundle. To interpret the right hand side, use the standard embedding $\mu_2 = Z(Spin(2n+1)) \subset \mathbb{C}^\times$ to construct

$$\frac{Spin(2n+1)\times\mathbb{C}^\times}{\mu_2} = Spin^c(2n+1)_{\mathbb{C}} \tag{5.82}$$

the complexification of the compact group $Spin^c(2n+1)$. Fix a point $x \in C$. Then the moduli stack $\mathcal{M}^{\bullet}_{Spin(2n+1)}(C)$ may be identified as the stack of $Spin^c(2n+1)_{\mathbb{C}}$-Higgs bundles (E,ϕ) equipped with an isomorphism

$$\partial_*(E) \simeq \begin{cases} \mathcal{O}_C & \text{or} \\ \mathcal{O}_C(x) & \end{cases} \tag{5.83}$$

and with ϕ "tracefree" (c.f. (5.31)). Specifically, the neutral component $\mathcal{M}^{0}_{Spin(2n+1)}(C)$ may be identified with the usual moduli stack $\mathcal{H}iggs_{Spin(2n+1)}(C)$, and the non-neutral component $\mathcal{M}^{1}_{Spin(2n+1)}(C)$ may be identified as the moduli stack of $Spin^c(2n+1)_{\mathbb{C}}$-Higgs bundles (E,ϕ) equipped with an isomorphism $\partial_*(E) \simeq \mathcal{O}_C(x)$. Since $H^2(C;\mu_2) = \mu_2$ we have that $\mathcal{M}^{\bullet}_{Spin(2n+1)}(C) = \mathcal{M}^{0}_{Spin(2n+1)}(C) \coprod \mathcal{M}^{1}_{Spin(2n+1)}(C)$.

Next consider the isomorphism

$$\left(\frac{\mathcal{M}^{\bullet}_{Spin(2n+1)}(C)}{H^1(C;\mu_2)}\right)^D \cong \mathcal{M}^{\bullet}_{Sp(2n)}(C). \tag{5.84}$$

We have already seen one interpretation of the left hand side in terms of $Spin^c(2n+1)_{\mathbb{C}}$-Higgs bundles – another interpretation is that on the left hand side we are dualising the moduli space of $SO(2n+1)$-Higgs bundles equipped with the gerbe of liftings of the universal $SO(2n+1)$-Higgs bundle to a universal $Spin(2n+1)$-Higgs bundle.

To interpret the right hand side we again construct the corresponding group $\widetilde{G}_\tau$ – this time the group is

$$Sp^c(2n)_{\mathbb{C}} := \frac{Sp(2n, \mathbb{C}) \times \mathbb{C}^\times}{\mu_2}, \tag{5.85}$$

the complexification of the compact group $Sp^c(2n) = \frac{Sp(2n) \times U(1)}{\mu_2}$.[11] Then $\mathcal{M}^\bullet_{Sp(2n)}(C)$ is – imprecisely – the stack of $Sp^c(2n)_{\mathbb{C}}$-Higgs bundles "with fixed second Stiefel-Whitney class, again considered up to parity". The precise interpretation of the two connected components is analogous to the interpretation for $Spin(2n+1)$: $\mathcal{M}^0_{Sp(2n)}(C)$ is isomorphic to the moduli stack $\mathcal{H}iggs_{Sp(2n)}(C)$, and $\mathcal{M}^1_{Sp(2n)}(C)$ may be identified as the moduli stack of $Sp^c(2n)_{\mathbb{C}}$-Higgs bundles (E, ϕ) equipped with an isomorphism $\partial_*(E) \simeq \mathcal{O}_C(x)$, and satisfying $\mathrm{tr}(\phi) = 0$.

[11] Ideally there ought to be a relation between this group and the noncompact group $Mp^c(2n)$ of automorphisms of the unitary Heisenberg group associated to the standard real symplectic vector space $(\mathbb{R}^{2n}, \omega_{std})$ [29, 66, 67] – $Mp^c(2n)$ is to the metaplectic group $Mp(2n)$ as $Spin^c(n)$ is to the ordinary spin group $Spin(n)$.

Appendices

Appendix A

Review of reductive algebraic groups

In the following appendix we work over an algebraically closed field k. References for this material are [43, 71].

A.1 Linear algebraic groups

Definition A.1.1. An *algebraic group* is a group object in the category of algebraic varieties. Explicitly, it is an algebraic variety G equipped with an identity element $1_G \in G$, multiplication map $\mu : G \times G \to G$ and an inversion map $\iota : G \to G$ satisfying the usual group axioms, and such that the maps μ and ι are morphisms of algebraic varieties.

Definition A.1.2. A *closed subgroup* H of an algebraic group G is a subgroup which is closed in the Zariski topology.

Remark A.1.1. A closed subgroup H of an algebraic group G can be given the structure of an algebraic group, such that the inclusion map $H \subset G$ is a homomorphism of algebraic groups.

Definition A.1.3. A *linear algebraic group* is an algebraic group G whose underlying variety is affine.

Example 35. The following are all examples of linear algebraic groups. The meaning of "standard" group structure will be explicated below in Example 36:

(a) The *additive group* $\mathbb{G}_a$, whose underlying variety is $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$, with the standard additive structure.

(b) The *multiplicative group* $\mathbb{G}_m$, whose underlying variety is $\mathrm{Spec}(k[t, t^{-1}])$, with the standard multiplicative structure.

(c) The *general linear group* $GL(V)$ of a (finite dimensional) vector space V, with group structure given by matrix multiplication. To see that $GL(V)$ can be given the structure of an affine variety, observe that it can be obtained as an open subvariety of $\mathrm{End}(V) \cong \mathbb{A}_k^{\dim(V)^2}$ by localising away from the determinant function $\det \in \mathcal{O}(\mathrm{End}(V))$.

(d) Any Zariski closed subgroup of $GL(V)$ inherits the structure of a linear algebraic group. This includes the familiar examples of $SL(V)$, $SO(V)$, $Sp(V)$, diagonal matrices, upper triangular matrices, strictly upper triangular matrices, and finite groups.

Example 36. It is instructive to consider (linear) algebraic groups from the functor-of-points perspective. Recall that a k-scheme S defines a functor (which I will also denote by S):

$$\begin{aligned} S : \mathrm{CAlg}_k &\longrightarrow \mathrm{Set} \\ R &\longmapsto S(R) := \mathrm{Hom}_{k-\mathrm{Sch}}(\mathrm{Spec}(R), S) \end{aligned} \tag{A.1}$$

If G is an algebraic group then its functor of points factors through the category of groups:

$$\begin{aligned} G : \mathrm{CAlg}_k &\longrightarrow \mathrm{Grp} \\ R &\longmapsto G(R) := \mathrm{Hom}_{k-\mathrm{Sch}}(\mathrm{Spec}(R), G) \end{aligned} \tag{A.2}$$

We call the set/group $G(R)$ the *R-points of G*. From this point of view we can describe the linear algebraic groups of Example 35 as follows:

(a) The R-points of the additive group are given by $\mathbb{G}_a(R) = R$ with addition as the group operation.

(b) The R-points of the multiplicative group are given by $\mathbb{G}_m(R) = R^\times$, the group of multiplicative units in R.

(c) The R-points of $GL_n(k)$ are given by $GL_n(R)$, the $n \times n$ matrices with entries in R whose determinant is in $R^\times$.

Example 37. A class of algebraic groups which are *not* linear algebraic groups but which are essential to this dissertation are given by *abelian varieties*. An abelian variety over k is an algebraic group A over k whose underlying variety is complete. As this implies that any map from A to an affine scheme is constant, these cannot be linear algebraic groups.

A.2 Lie algebras

Recall the abstract definition of a Lie algebra:

Definition A.2.1. A *Lie algebra over k* is a pair $(\mathfrak{v}, [-,-])$ where

- $\mathfrak{v}$ is a k-vector space, and
- $[-,-] : \mathfrak{v} \otimes_k \mathfrak{v} \to \mathfrak{v}$ is a k-bilinear pairing which is skew-symmetric and satisfies the *Jacobi identity*:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \text{for all } x,y,z \in \mathfrak{v}. \tag{A.3}$$

Given a linear algebraic group G over k with coordinate ring $k[G]$, consider the *Lie algebra of k-linear derivations*

$$\mathcal{D}_G = \mathrm{Der}_k(k[G], k[G]) := \{D \in \mathrm{Hom}_{\mathrm{Vect}_k}(k[G], k[G]) \mid D(fg) = f \cdot Dg + g \cdot Df\} \tag{A.4}$$

i.e. the Lie algebra of vector fields on G. The actions of G on itself by left and right translation induce actions of G on $k[G]$, denoted L and R respectively, and consequently induce G-actions (by conjugation) on $\mathcal{D}_G$.

Definition A.2.2. The *Lie algebra of* G, denoted $\mathrm{Lie}(G)$ or $\mathfrak{g}$, is the Lie algebra of left-invariant vector fields on G,

$$\mathrm{Lie}(G) = \mathfrak{g} := \{D \in \mathcal{D}_G \mid D = L_g \circ D \circ L_g^{-1} \text{ for all } g \in G\}. \tag{A.5}$$

Remark A.2.1. The actions of left and right translation commute, and so R induces an action of G on $\mathfrak{g}$.

Next, consider the homomorphism

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}(G) \\ g &\longmapsto (h \mapsto ghg^{-1}) \end{aligned} \tag{A.6}$$

This fixes the identity $1 \in G$ and induces automorphisms of the tangent space T_1G.

Definition A.2.3. The representation of G on T_1G induced by conjugation is called the *adjoint action of* G as is denoted $\mathrm{Ad} : G \to GL(T_1G)$.

Proposition A.2.1. *[71, §4.4] There is a vector space isomorphism $\mathfrak{g} \cong T_1G$ under which the action of G induced by right translation is identified with* Ad.

Definition A.2.4. The differential of Ad at the identity is denoted $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(T_1G)$, and is called the *adjoint action* (for reasons made clear by the following Proposition A.2.2).

Proposition A.2.2. *[71, §4.4] Under the identification $\mathfrak{g} \cong T_1G$, the adjoint action satisfies* $\mathrm{ad}(x)(y) = [x, y]$. *I.e.* ad *is a Lie algebra homomorphism, and so defines an action of the Lie algebra $\mathfrak{g}$ on itself.*

Remark A.2.2. Dual to the adjoint actions, there are induced *coadjoint actions* on the dual of the Lie algebra,

$$\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*) \tag{A.7}$$

$$\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*) \tag{A.8}$$

A.3 Types of linear algebraic group and Lie algebra

Definition A.3.1. Let V be a finite dimensional vector space over k.

(1) An endomorphism $A \in \mathrm{End}_k(V)$ is *semisimple* if there is a basis of V in which A is a diagonal matrix.

(2) An endomorphism $A \in \mathrm{End}_k(V)$ is *nilpotent* if there is $N \in \mathbb{Z}_{>0}$ such that $A^N = 0$.

(3) An endomorphism $A \in \mathrm{End}_k(V)$ is *unipotent* if $A - 1$ is nilpotent.

Let G be a linear algebraic group over k.

(1) An element $g \in G$ is *semisimple* if and only if its image in any finite dimensional faithful representation is semisimple.

(2) An element $g \in G$ is *unipotent* if and only if its image in any finite dimensional faithful representation is unipotent.

Theorem A.3.1 (Jordan Decompositions, [71, §2.4])**.** *Let V be a finite dimensional vector space over k, and let G be a linear algebraic group over k.*

(i) **Additive Jordan Decomposition:** *Let $A \in \mathrm{End}(V)$. There exist unique elements $A_s, A_n \in \mathrm{End}(V)$ such that*

- *A_s is semisimple,*
- *A_n is nilpotent,*
- *A_s and A_n commute, and*
- *$A = A_s + A_n$.*

Furthermore, there are polynomials $P, Q \in k[t]$ without constant term such that $A_s = P(A)$ and $A_n = Q(A)$.

(ii) **Multiplicative Jordan Decomposition:** *Let $A \in GL(V)$. There exist unique elements $A_s, A_u \in GL(V)$ such that*

- *A_s is semisimple,*

- *A_u is unipotent,*
- *A_s and A_u commute, and*
- *$A = A_s A_u$.*

(iii) **Jordan Decomposition in a Linear Algebraic Group:** *Let $g \in G$. There exist unique elements $g_s, g_u \in G$ such that*

- *g_s is semisimple,*
- *g_u is unipotent,*
- *g_s and g_u commute, and*
- *$g = g_s g_u$.*

Definition A.3.2. (1) An linear algebraic group T is an *(algebraic) torus* if it is isomorphic to $\mathbb{G}_m^N$ for some $N \in \mathbb{Z}_{>0}$.[1]

(2) A linear algebraic group U is *unipotent* if all of its elements are unipotent.

(3) A group B is called *solvable* if its *derived series*

$$B^{(0)} := B, \quad B^{(n)} := [B^{(n-1)}, B^{(n-1)}], \tag{A.9}$$

terminates at the identity after finitely many steps, i.e. $B^{(n)} = \{1_B\}$ for some $n \in \mathbb{Z}_{>0}$.

[1]Recall that we are working over an algebraically closed field. Over a non-algebraically closed field $\mathbb{F}$, T is an algebraic torus if its base-change to an algebraic closure $\bar{\mathbb{F}}$ is an algebraic torus; a torus which is already isomorphic to $\mathbb{G}_m^N$ over $\mathbb{F}$ is called $\mathbb{F}$-*split*.

(4) A group N is called *nilpotent* if its *lower central series*

$$N_1 := N, \quad N_n := [N_{n-1}, N], \tag{A.10}$$

terminates at the identity after finitely many steps, i.e. $N_n = \{1_N\}$ for some $n \in \mathbb{Z}_{>0}$.

Definition A.3.3. Let G be a linear algebraic group over k.

(1) A closed subgroup P of G is *parabolic* if G/P is a complete variety.

(2) A *Borel subgroup* B of G is a (closed, connected) maximal solvable subgroup of G. Equivalently, it is a minimal parabolic subgroup of G.

(3) A *maximal torus* H of G is a subtorus of G not strictly contained in any other subtorus.

(4) A *Cartan subgroup* of G is the identity component of the centraliser of a maximal torus.

(5) The *radical of G, $R(G)$,* is the maximal closed, connected, normal, solvable subgroup of G.

(6) The *unipotent radical of G, $R_u(G)$,* is the maximal closed, connected, normal, unipotent subgroup of G. Equivalently, it is the group of unipotent elements in the radical $R(G)$.

Definition A.3.4. Let G be a linear algebraic group over k.

(1) G is *simple* if has no proper, connected, closed, normal subgroup.

(2) G is *semi-simple* if $R(G) = \{1_G\}$.

(3) G is *reductive* if $R_u(G) = \{1_G\}$.

Remark A.3.1. Simple linear algebraic groups (as defined in A.3.4) are occasionally referred to as *quasi-simple*, e.g. [71, §8.1.12]. This is because they are **not** simple in the purely group theoretic sense. This use of the term "simple" is standard practice, and in context does not usually cause confusion.

The above concepts for linear algebraic groups have analogues in the theory of Lie algebras.

Definition A.3.5. Let $\mathfrak{g}$ be a Lie algebra over k.

1. An element $x \in \mathfrak{g}$ is *semisimple* if its image in any finite-dimensional representation is semisimple.

2. An element $x \in \mathfrak{g}$ is *nilpotent* if its image in any finite-dimensional representation is nilpotent.

Definition A.3.6. 1. A Lie algebra $\mathfrak{b}$ is called *solvable* if its *derived series*

$$\mathfrak{b}^{(0)} := \mathfrak{b}, \quad \mathfrak{b}^{(n)} := [\mathfrak{b}^{(n-1)}, \mathfrak{b}^{(n-1)}], \tag{A.11}$$

terminates at zero after finitely many steps, i.e. $\mathfrak{b}^{(n)} = \{0\}$ for some $n \in \mathbb{Z}_{>0}$.

2. A Lie algebra $\mathfrak{n}$ is called *nilpotent* if its *lower central series*

$$\mathfrak{n}_1 := \mathfrak{n}, \quad \mathfrak{n}_n := [\mathfrak{n}_{n-1}, \mathfrak{n}], \tag{A.12}$$

terminates at zero after finitely many steps, i.e. $\mathfrak{n}_n = \{0\}$ for some $n \in \mathbb{Z}_{>0}$.

Definition A.3.7. Let $\mathfrak{g}$ be a Lie algebra over k.

(1) $\mathfrak{g}$ is *simple* if it is non-abelian and has no non-trivial proper ideals.

(2) $\mathfrak{g}$ is *semi-simple* if it is a direct sum of simple Lie algebras.

(3) $\mathfrak{g}$ is *reductive* if it is the direct sum of a semi-simple Lie algebra and an abelian Lie algebra.

Definition A.3.8. Let $\mathfrak{g}$ be a semi-simple Lie algebra over k.

(1) A *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$.

(2) A *parabolic subalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ is a subalgebra containing a Borel subalgebra of $\mathfrak{g}$.

(3) A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a self-normalising nilpotent subalgebra.

Remark A.3.2. If we restrict our focus to reductive algebraic groups and Lie algebras, the Cartan subgroups are exactly the maximal tori, and similarly the Cartan subalgebras are exactly the maximally commuting subalgebras of semisimple elements (which occur as the Lie algebras of maximal tori).

Remark A.3.3. The Additive Jordan Decomposition of Theorem A.3.1 carries over to give a Jordan decomposition for elements of semisimple Lie algebras over k.

A.4 Classification of reductive algebraic groups

For this section, let G denote a *reductive* algebraic group over an algebraically closed field k, let $B \subset G$ be a choice of Borel subgroup, and let $H \subset B$ be a choice of maximal torus.

A.4.1 Abstract root datum

Definition A.4.1. A *root datum* is the data of a quadruple $\Psi = (X, R, X^\vee, R^\vee)$ consisting of

(1) free abelian groups X and $X^\vee$ of finite rank, equipped with a perfect pairing $\langle -, - \rangle : X \times X^\vee \to \mathbb{Z}$,[2] and

(2) finite subsets R and $R^\vee$ of X and $X^\vee$ respectively, together with a bijection

$$\begin{aligned} R &\longrightarrow R^\vee \\ \alpha &\longmapsto \alpha^\vee \end{aligned}$$

We call R the set of *roots* and $R^\vee$ the set of *coroots*.

This data is subject to the following conditions:

(a) If $\alpha \in R$ then $\langle \alpha, \alpha^\vee \rangle = 2$.

(b) If $\alpha \in R$ then $s_\alpha(R) = R$ and $s_\alpha^\vee(R^\vee) = R^\vee$, where for $\lambda \in X$, $x \in X^\vee$,

$$\begin{aligned} s_\alpha(\lambda) &:= \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \\ s_\alpha^\vee(x) &:= x - \langle \alpha, x \rangle \alpha^\vee \end{aligned} \tag{A.13}$$

The map s_α is called the *root reflection* corresponding to α.

Definition A.4.2. Given a root datum $\Psi = (X, R, X^\vee, R^\vee)$, the *Weyl group of* Ψ, denoted $W(\Psi)$, is the subgroup of $\mathrm{Aut}(X)$ generated by the root reflections $\{s_\alpha\}_{\alpha \in R}$.

[2] I.e. $X^\vee \cong \mathrm{Hom}(X, \mathbb{Z}) = X^\wedge$. This is an annoying notational inconsistency, which should not cause confusion in context.

Definition A.4.3. Given a root datum $\Psi = (X, R, X^\vee, R^\vee)$, the *root lattice* is $\Lambda_R := \mathbb{Z} \cdot R \subset X$ and the *coroot lattice* is $\Pi_R := \mathbb{Z} \cdot R^\vee \subset X^\vee$.

Remark A.4.1. Suppose $R \neq \emptyset$ and consider the real vector vector space $V := \mathbb{R} \otimes \Lambda_R$. Then R is a *root system* in V [71, §7.4.1], i.e. R satisfies:

(a) R is finite, generates V, and $0 \notin R$.

(b) If $\alpha \in R$ there exists $\alpha^\vee \in V^*$ such that $\langle \alpha, \alpha^\vee \rangle = 2$, and $s_\alpha(R) = R$.

(c) If $\alpha \in R$ then $\alpha^\vee(R) \subset \mathbb{Z}$.

Furthermore, a root system is called *reduced* if $c\alpha \in R$ for some $\alpha \in R$ and $c \in \mathbb{Q}$ implies that $c = \pm 1$.

Definition A.4.4. Let $\Psi = (X, R, X^\vee, R^\vee)$ be a root datum. A subset $R^+ \subset R$ is a *system of positive roots* if there exists $x \in X^\vee$ such that $\langle \alpha, x \rangle \neq 0$ for all $\alpha \in R$, and

$$R^+ = \{\alpha \in R \mid \langle \alpha, x \rangle > 0\}. \tag{A.14}$$

Definition A.4.5. Let Ψ be a root datum with a choice of positive roots R^+. A root $\alpha \in R$ is called *simple* if it cannot be written as the sum of two positive roots.

A.4.2 Root datum from reductive algebraic groups

In this section I describe how a connected reductive algebraic group gives rise to a root datum. Recall that we have a choice of maximal torus and Borel, $H \subset B \subset G$, and that these have Lie algebras $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$.

Definition A.4.6. The *Weyl group associated to* $G \supset H$ is $W_G(H) := N_G(H)/H$.

Recall that there is a canonical representation associated to G called the *adjoint action*, $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Restricting this action to H, we may decompose $\mathfrak{g}$ into isotypic components labelled by the character lattice $X^\bullet(G, H)$:

$$\mathfrak{g} \cong \bigoplus_{\lambda \in X^\bullet(G,H)} \mathfrak{g}_\lambda. \tag{A.15}$$

Definition A.4.7. The elements of the set

$$R(G, H) := \{\alpha \in X^\bullet(G, H) \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\} \tag{A.16}$$

are called the *roots of* G *relative to* H. This will sometimes be denoted by R when context makes G, H clear.

Recall that there is a natural integer valued pairing $\langle -, - \rangle$ between characters (maps *to* $\mathbb{G}_m$) and cocharacters (maps *from* $\mathbb{G}_m$) defined by

$$\begin{aligned} X^\bullet(G, H) \times X_\bullet(G, H) &\longrightarrow \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z} \\ (\lambda, x) &\longmapsto (\lambda \circ x)(z) = z^{\langle \lambda, x \rangle} \end{aligned} \tag{A.17}$$

Definition A.4.8. Given a root $\alpha \in R(G, H)$ there is a unique element $\alpha^\vee \in X_\bullet(G, H)$ satisfying the conditions of Definition A.4.1 with respect to the pairing (A.17). $\alpha^\vee$ is called the *coroot* corresponding to α, and the set of coroots is denoted $R(G, H)^\vee$ (or $R^\vee$ if G, H are clear from context).

Definition A.4.9. The isotypic components of $\mathfrak{g}$ in equation (A.15) corresponding to roots are called the *root spaces* of $\mathfrak{g}$.

Definition A.4.10. Let $R^+(B)$ (or R^+ is context is clear) denote the subset of $R(G,H)$ corresponding to the Borel subalgebra $\mathfrak{b}$, i.e.

$$\mathfrak{b} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R^+(B)} \mathfrak{g}_\alpha \right). \tag{A.18}$$

Theorem A.4.1 (Classification of reductive algebraic groups.)**.** *Let $H \subset B \subset G$ be as above.*

1. *The quadruple $\Psi(G,H) = (X^\bullet(G,H), R(G,H), X_\bullet(G,H), R(G,H)^\vee)$ is a root datum with reduced root system.*
2. *There is an isomorphism $W(\Psi(G,H)) \cong W_G(H)$.*
3. *There is a one-to-one correspondence between root datum up to isomorphism[3] and connected reductive algebraic groups up to isomorphism.*
4. *The subset $R^+(B)$ of Definition A.4.10 is a system of positive roots (Definition A.4.4). Conversely, any system of positive roots for $R(G,H)$ arises as $R^+(B')$ for some Borel subgroup $B' \supset H$.*

Proof. See [71, Ch.7–10]. □

Proposition A.4.2 (Langlands Duality)**.** *For simplicity, let the ground field be $k = \mathbb{C}$.*

1. *If $\Psi = (X, R, X^\vee, R^\vee)$ is a root datum then so is ${}^L\Psi = (X^\vee, R^\vee, X, R)$.*

[3]There is a natural way to define the notion of a morphism of root systems.

2. *There is an involution* ${}^L(-)$ *on the set of connected complex reductive algebraic groups up to isomorphism, called* Langlands duality. *If* G *is a connected reductive algebraic group and* $H \subset G$ *is a maximal torus, then* ${}^LG$ *is determined up to isomorphism by the root datum* $(X_\bullet(G,H), R(G,H)^\vee, X^\bullet(G,H), R(G,H))$.

3. *If* T *is an algebraic torus, then* ${}^LT \cong X^\bullet(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$.

Appendix B

Fixed points of Weyl group actions

In this appendix I study the action of the Weyl group of a reductive algebraic group G on a choice of fixed maximal torus H, with a view to understanding the fixed loci of root reflections. As we have seen, this is related to understanding the global sections of the scheme of regular centralisers on a smooth proper scheme (c.f. Lemma 5.1.2).

B.1 Fixed points: The (semi)simple case

Assume that G is a simple and connected complex algebraic group, with a choice of maximal torus $H \subset G$. Via the exponential map we have an (analytic and W-equivariant) identification

$$H \cong \frac{X_\bullet(H) \otimes \mathbb{C}}{X_\bullet(H)} = X_\bullet(H) \otimes \mathbb{C}^\times \tag{B.1}$$

where $X_\bullet(H) = \mathrm{Hom}(\mathbb{C}^\times, H) \subset \mathfrak{h}$ is the cocharacter lattice of H.

Recall that the Weyl reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ corresponding to the root α is defined by[1]

$$s_\alpha(\lambda) = \lambda - \lambda(H_\alpha)d\alpha, \tag{B.2}$$

where H_α is the coroot associated to α, i.e. the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying $d\alpha(H_\alpha) = 2$. Dualising this, we have that $s_\alpha \in W_G(H)$ acts on $\mathfrak{h}$ via

$$s_\alpha(x) = x - d\alpha(x)H_\alpha. \tag{B.3}$$

Translating this via the exponential map into a question about fixed points on the maximal torus H, we say that a point $x \in \mathfrak{h}$ is a fixed point of s_α if and only if $s_\alpha(x) \in x + X_\bullet(H)$, which, using our explicit description of s_α, occurs if and only if $d\alpha(x)H_\alpha \in X_\bullet(H)$.

As one application of this lattice theoretic description, we make the following observation:

Proposition B.1.1. *If $h \in H$ is fixed by the action of s_α, then $\alpha(h) = \pm 1$.*

Proof. Let Λ_R denote the root lattice and $X^\bullet(G, H) = X^\bullet(H)$ the character lattice of G, both thought of as embedded in $\mathfrak{h}^*$. We have

$$X_\bullet(H) = \{y \in \mathfrak{h} \mid \lambda(y) \in \mathbb{Z} \text{ for all } \lambda \in X^\bullet(H)\}. \tag{B.4}$$

Represent the fixed $h \in H$ by $x \in \mathfrak{h}$. Since $\Lambda_R \subset X^\bullet(H)$ we have that $s_\alpha(x) \in x + X_\bullet(H)$ implies $d\alpha(d\alpha(x)H_\alpha) \in \mathbb{Z}$, equivalently $2d\alpha(x) \in \mathbb{Z}$, and so $d\alpha(x) \in \frac{1}{2}\mathbb{Z}$.

[1] Recall that our convention is that α defines a character of H, hence its derivative $d\alpha$ defines a linear functional on $\mathfrak{h}$.

But then for some $n \in \mathbb{Z}$

$$\alpha(t) = e^{2\pi i d\alpha(x)} = e^{\pi i n} \in \{\pm 1\}. \tag{B.5}$$

$\square$

Recall that if $G_1 \to G_2$ is an isogeny of simple groups inducing an isogeny on maximal tori $H_1 \to H_2$, then $X_\bullet(G_1, H_1) \subset X_\bullet(G_2, H_2)$. This reflects the fact that if $x \in \mathfrak{h}$ represents a fixed point of s_α acting on $H_1 \subset G_1$, then it also represents a fixed point of s_α acting on $H_2 \subset G_2$. This is not a deep fact: the isogeny is W-equivariant, where $W \equiv W_{G_1}(H_1) = W_{G_2}(H_2)$, since it corresponds to the quotient by a central subgroup and the Weyl group action is induced by conjugation. More interesting is the question of when a fixed element $h_2 \in H_2^{s_\alpha}$ can be *lifted* to a fixed element $h_1 \in H_1^{s_\alpha}$. It turns out that we can give an exact answer to this question when the group we wish to lift to is the simply-connected form of the group.

Theorem B.1.2. *Let $\tilde{G}$ be a simple, connected, simply-connected complex algebraic group, and let $\tilde{G} \to G$ be an isogeny of simple groups. Choose a maximal torus $\widetilde{H} \subset \widetilde{G}$ an denote by H the corresponding maximal tori in G. Suppose that $h \in H$ is fixed by the root reflection $s_\alpha \in W$. Then a preimage $\tilde{h} \in \widetilde{h}$ of h is fixed by s_α if and only if $\alpha(h) = 1$.*

Proof. We first translate this into a statement about lattices and integrality: specifically the claim of the theorem is equivalent to the claim that for any element $x \in \mathfrak{h}$ representing h, $d\alpha(x) \in \mathbb{Z}$ if and only if $d\alpha(x)H_\alpha \in X_\bullet(\widetilde{G}, \widetilde{H})$. In this form, the theorem follows from the fact that the cocharacter lattice for the simply connected form of the group is exactly the coroot lattice (i.e. the integral span of the coroots). $\square$

Remark B.1.1. By considering products of simple groups and their Weyl groups, Theorem B.1.2 immediately extends to all semi-simple complex algebraic groups.

Example 38. Consider the groups $SL_2\mathbb{C}$ and $PGL_2\mathbb{C}$, with a simultaneous choice of Cartan subalgebra $\mathfrak{h} = \{2 \times 2 \text{ traceless complex matrices}\}$. Let $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and consider the character

$$\begin{aligned} d\alpha &: \mathfrak{h} \to \mathbb{C} \\ d\alpha\,(a \cdot h) &= 2a \end{aligned} \tag{B.6}$$

Then the root, weight, and character lattices are given by

$$\begin{aligned} \Lambda_R &= \mathbb{Z} \cdot d\alpha = X^\bullet(PGL_2, H_{\mathrm{ad}}) \\ \Lambda_W &= \frac{1}{2}\mathbb{Z} \cdot d\alpha = X^\bullet(SL_2, H) \end{aligned} \tag{B.7}$$

and the coroot, coweight, and cocharacter lattices are

$$\begin{aligned} \Pi_R &= \mathbb{Z} \cdot h = X_\bullet(SL_2, H) \\ \Pi_W &= \frac{1}{2}\mathbb{Z} \cdot h = X_\bullet(PGL_2, H) \end{aligned} \tag{B.8}$$

The Weyl group in this case is of order 2, with non-trivial element acting on $\mathfrak{h}$ by $s_\alpha(x) = -x$, so that x exponentiates to a fixed point in G if and only if $2x \in X_\bullet(G, H)$. For $G = SL_2\mathbb{C}$ this translates to $d\alpha(x) \in \mathbb{Z}$, which upon exponentiating gives

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

For $G = PGL_2\mathbb{C}$ this translates to $d\alpha(x) \in \frac{1}{2}\mathbb{Z}$, which upon exponentiating gives a new non-trivial fixed element given by the equivalence class of

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Remark B.1.2. This example, and a comparison of $SL_r\mathbb{C}$ and $PGL_r\mathbb{C}$ for $r > 2$, could have been done more directly by using that the Weyl group of type A_{r-1} is the symmetric group on r letters acting on the standard torus of diagonal matrices by permuting the elements on the diagonal. A generalisation of this direct analysis even to other classical groups, however, is difficult and not particularly illuminating.

B.2 Fixed points: The reductive case

B.2.1 The Weyl group of a reductive algebraic group

Let G be (connected) semisimple group, T an algebraic torus, and K a finite central subgroup of $G \times T$, with projection to G denoted $K_G \subset Z(G) \subset G$ and inversion and projection to T denoted by $K_T \subset T$.[2] We want to understand the Weyl group of the quotient

$$G(K;T) = \frac{G \times T}{K}. \tag{B.9}$$

Let $H \subset G$ be a choice of maximal torus. Since $K_G \subset Z(G) \subset H$, we have that $K \subset H \times T$ and the maximal torus of $G(K;T)$ is

$$H(K;T) = \frac{H \times T}{K}. \tag{B.10}$$

First, we consider the Weyl group of the product $G \times T$. T is central, so $N_{G\times T}(H \times T) = N_G(H) \times T$, and thus the Weyl group with respect to this choice of maximal torus is

$$W_{G\times T}(H \times T) = \frac{N_{G\times T}(H \times T)}{H \times T} = N_G(H)/H = W_G(H), \tag{B.11}$$

[2] In other words, if we think of K as a subgroup of both G and T, the equivalence relation we quotient out by is $(g, t) \sim (gk, k^{-1}t)$.

i.e. the Weyl group of the product is the Weyl group of the semisimple factor.

Now, let's consider the Weyl group of the quotient $G(K;T)$. Since this is a covering space of $G \times T$, and there is an alternate characterisation of the Weyl group as generated by reflections in hyperplanes in the Cartan subalgebra, we expect that we should arrive at the same answer again. Still, let us check this directly.

Since elements of the form $[1_G, t] \in G(K;T)$ are central, it suffices to determine when an element of the form $[g, 1_T] \in G(K;T)$ is in the normaliser. It suffices to consider elements of the form $[h, 1_T] \in H(K;T)$ (again, since the image of the subgroup T is central in the quotient). We calculate

$$[g, 1_T][h, 1_T][g^{-1}, 1_T] = [ghg^{-1}, 1_T]. \tag{B.12}$$

This lies in $H(K;T)$ if and only if $ghg^{-1} = h'k$ for some $k \in K$ whose image in K_T is trivial, and $h' \in H$. But $K_G \subset Z(G) \subset H$, so this occurs if and only if $ghg^{-1} \in H$, i.e. $g \in N_G(H)$. The normaliser is therefore

$$N_{G(K;T)}(H(K;T)) = \frac{N_G(H) \times T}{K} \tag{B.13}$$

and so taking the quotient by $H(K;T) = \frac{H \times T}{K}$ we find that

$$W_{G(K;T)}(H(K;T)) = W_G(H). \tag{B.14}$$

B.2.2 Calculation of fixed points

Now, assume that the map $K \to K_T$ is an embedding, and denote by W the canonically isomorphic Weyl groups

$$W \equiv W_{G(K;T)}(H(K;T)) = W_G(H). \tag{B.15}$$

We wish to identify the fixed locus $H(K;T)^W$.

Proposition B.2.1.
$$H(K;T)^W = \frac{H^W \times T}{K}$$

Proof. Restoring the trivial action of the centraliser, this is the same as $H(K;T)^{(N_G(H)\times T)/K}$. Let $[n,t] \in \frac{N_G(H)\times T}{K}$ and $[h,s] \in H(K;T)^W$, so that

$$[h,s] = [n,t][h,s][n^{-1},t^{-1}] = [nhn^{-1},s]. \tag{B.16}$$

This occurs if and only if there is $k \in K$ with images k_G and k_T in K_G and K_T such that $nhn^{-1} = hk_G$ and $s = k_T^{-1}s$. This requires $k_T = 1_T$, and since we assumed that $K \to K_T$ was an embedding this implies $k = 1$ and so $k_G = 1_G$. Therefore,

$$nhn^{-1} = h, \quad \text{i.e. } h \in H^{N_G(H)} = H^W \tag{B.17}$$

and the proposition follows. □

Remark B.2.1. Given that we have spent time in Section B.1 comparing Weyl group fixed points for isogenous simple groups, it is important to note what this does **not** prove: namely, it does not contradict Theorem B.1.2, which gave conditions for when a fixed point in the maximal torus of a simple group may be lifted to a fixed point in the maximal torus of the corresponding simply-connected form of the group.

The assumption that saves us from any inconsistency is the assumption that the map $K \to K_T$ is an embedding: in the setting where we are studying an isogeny of (semi)simple groups we have that T is the trivial group, and so K must also be the trivial group.

Appendix C

Structure results for $\widetilde{G}_\tau$

In this appendix I study the structure of the reductive algebraic group $\widetilde{G}_\tau$, which I used in Chapter 5 to construct the moduli stack $\mathcal{M}^\bullet_{\widetilde{G}}(X)$.

C.1 The Langlands dual of the map τ

Consider the exact sequence of complex algebraic groups

$$1 \to Z(\widetilde{G}) \to \widetilde{G} \times T \to \widetilde{G}_\tau \to 1. \tag{C.1}$$

I claim that there is a dual exact sequence

$$1 \to Z(\widetilde{{}^LG}) \to {}^L(\widetilde{G}_\tau) \to \widetilde{{}^LG} \times {}^LT \to 1. \tag{C.2}$$

Where does this come from? Consider the exact sequence of *abelian* groups

$$1 \longrightarrow Z(\widetilde{G}) \xrightarrow{\ \tau\ } T \longrightarrow T/Z(\widetilde{G}) \longrightarrow 1. \tag{C.3}$$

Taking characters $\mathrm{Hom}(-, \mathbb{C}^\times)$ is a contravariant functor and yields the exact sequence

$$0 \to X^\bullet(T/Z(\widetilde{G})) \to X^\bullet(T) \to Z(\widetilde{G})^\vee \to 0, \tag{C.4}$$

i.e.

$$0 \to X_\bullet({}^L(T/Z(\widetilde{G}))) \to X_\bullet({}^LT) \to Z(\widetilde{{}^LG}) \to 0. \tag{C.5}$$

Apply $- \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{C}^\times$ and take homology to get the exact sequence

$$1 \to \mathrm{Tor}_1^{\mathbb{Z}}(Z(\widetilde{{}^LG}), \mathbb{C}^\times) \to {}^L(T/Z(\widetilde{G})) \to {}^LT \to 1. \tag{C.6}$$

As an abelian group $\mathbb{C}^\times \cong \mathbb{R}^\times_{>0} \times U(1) \cong \mathbb{R} \times U(1)$, and so $\mathrm{Tor}_1^{\mathbb{Z}}(Z(\widetilde{{}^LG}), \mathbb{C}^\times)$ is canonically isomorphic to the torsion subgroup of $Z(\widetilde{{}^LG})$ (which is the entire group, since $Z(\widetilde{G})$ is torsion). I.e. we have an exact sequence

$$1 \longrightarrow Z(\widetilde{{}^LG}) \xrightarrow{{}^L\tau} {}^L(T/Z(\widetilde{G})) \longrightarrow {}^LT \longrightarrow 1. \tag{C.7}$$

So, let $\widetilde{H} \subset \widetilde{G}$ be a maximal torus. Then

- $H_{\mathrm{ad}} = \widetilde{H}/Z(\widetilde{G})$ is a maximal torus for G_{ad}.
- $\widetilde{H} \times T$ is a maximal torus for $\widetilde{G} \times T$.
- $\frac{\widetilde{H}\times T}{Z(\widetilde{G})}$ is a maximal torus for $\widetilde{G}_\tau$.
- ${}^L\left(\frac{\widetilde{H}\times T}{Z(\widetilde{G})}\right)$ is a maximal torus for ${}^L(\widetilde{G}_\tau)$.

So an exact sequence

$$1 \to Z(\widetilde{G}) \to \widetilde{H} \times T \to \frac{\widetilde{H} \times T}{Z(\widetilde{G})} \to 1 \tag{C.8}$$

yields an exact sequence

$$1 \to Z(\widetilde{{}^LG}) \to {}^L\left(\frac{\widetilde{H} \times T}{Z(\widetilde{G})}\right) \to {}^L\widetilde{H} \times {}^LT \to 1 \tag{C.9}$$

and so via the inclusions $\widetilde{H} \subset \widetilde{G}$, ${}^L\left(\frac{\widetilde{H}\times T}{Z(\widetilde{G})}\right) \subset {}^L(\widetilde{G}_\tau)$, the exact sequence (C.1) yields a dual exact sequence (C.2).

C.2 Structure of the Langlands dual group

There is another inclusion

$$\begin{aligned} Z(\widetilde{G}) &\xrightarrow{1\times\tau} \widetilde{G}\times T \\ z &\longmapsto (1_{\widetilde{G}}, \tau(z)) \end{aligned} \tag{C.10}$$

which induces an exact sequence

$$1 \longrightarrow Z(\widetilde{G}) \xrightarrow{\overline{1\times\tau}} \widetilde{G}_\tau \longrightarrow G_{\mathrm{ad}} \times (T/Z(\widetilde{G})) \longrightarrow 1. \tag{C.11}$$

Proposition C.2.1. *The Langlands dual exact sequence is given by*

$$1 \longrightarrow Z(\widetilde{{}^LG}) \xrightarrow{{}^L\iota\times{}^L\tau} \widetilde{{}^LG} \times {}^L(T/Z(\widetilde{G})) \longrightarrow {}^L(\widetilde{G}_\tau) \longrightarrow 1 \tag{C.12}$$

where $\iota : Z(\widetilde{G}) \subset \widetilde{G}$ and ${}^L\iota : Z(\widetilde{{}^LG}) \subset \widetilde{{}^LG}$ are the subgroup inclusions, and ${}^L\tau$ is the map described in section C.1. I.e. we can realise the Langlands dual of $\widetilde{G}_\tau$ as

$${}^L(\widetilde{G}_\tau) \cong \frac{\widetilde{{}^LG} \times {}^L(T/Z(\widetilde{G}))}{Z(\widetilde{{}^LG})} = (\widetilde{{}^LG})_{{}^L\tau}. \tag{C.13}$$

Proof. It suffices to prove the result after replacing the group $\widetilde{G}$ with a choice of maximal torus $\widetilde{H}$. Consider the following commutative diagram, where all rows and columns are exact:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & Z(\widetilde{G}) \arrow[r, "\tau"] \arrow[d, equal] & T \arrow[r] \arrow[d, "\overline{1\times\mathrm{id}}"] & T/Z(\widetilde{G}) \arrow[r] \arrow[d, "1\times\mathrm{id}"] & 1 \\
1 \arrow[r] & Z(\widetilde{G}) \arrow[r, "\overline{1\times\tau}"] \arrow[d] & \frac{\widetilde{H}\times T}{Z(\widetilde{G})} \arrow[r] \arrow[d] & H_{\mathrm{ad}} \times T/Z(\widetilde{G}) \arrow[r] \arrow[d] & 1 \\
 & 0 \arrow[r] & H_{\mathrm{ad}} \arrow[r, equal] \arrow[d] & H_{\mathrm{ad}} \arrow[r] \arrow[d] & 0 \\
 & & 1 & 1 &
\end{tikzcd}
\tag{C.14}
$$

Applying $(-)^\vee := \mathrm{Hom}(-, \mathbb{C}^\times)$ yields another commutative diagram, again with all rows and columns exact:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X^\bullet(H_{\mathrm{ad}}) \arrow[r, equal] \arrow[d] & X^\bullet(H_{\mathrm{ad}}) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X^\bullet(H_{\mathrm{ad}}) \times X^\bullet(T/Z(\widetilde{G})) \arrow[r] \arrow[d] & X^\bullet\left(\frac{\widetilde{H}\times T}{Z(\widetilde{G})}\right) \arrow[r, "\overline{1\times\tau}^\vee"] \arrow[d] & Z(\widetilde{G})^\vee \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X^\bullet(T/Z(\widetilde{G})) \arrow[r] \arrow[d] & X^\bullet(T) \arrow[r, "\tau^\vee"] \arrow[d] & Z(\widetilde{G})^\vee \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{C.15}
$$

Applying $- \otimes^{\mathbf{L}}_{\mathbb{Z}} \mathbb{C}^\times$ and taking homology yields a third commutative diagram with all rows and columns exact:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] \arrow[d] & {}^L\widetilde{H} \arrow[r, equal] \arrow[d] & {}^L\widetilde{H} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & Z({}^L\widetilde{G}) \arrow[r, "{}^L\overline{1\times\tau}"] \arrow[d, equal] & {}^L\widetilde{H} \times {}^L(T/Z(\widetilde{G})) \arrow[r] \arrow[d] & {}^L\left(\frac{\widetilde{H}\times T}{Z(\widetilde{G})}\right) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z({}^L\widetilde{G}) \arrow[r, "{}^L\tau"] \arrow[d] & {}^L(T/Z(\widetilde{G})) \arrow[r] \arrow[d] & {}^LT \arrow[r] \arrow[d] & 1 \\
 & 0 & 1 & 1 &
\end{tikzcd} \tag{C.16}$$

Therefore, composing ${}^L\overline{1 \times \tau}$ with projection to the second factor gives

$$\begin{tikzcd}
Z({}^L\widetilde{G}) \arrow[r, "{}^L\overline{1\times\tau}"] \arrow[dr, "{}^L\tau"'] & {}^L\widetilde{H} \times {}^L(T/Z(\widetilde{G})) \arrow[d] \\
 & {}^L(T/Z(\widetilde{G}))
\end{tikzcd} \tag{C.17}$$

Repeating this argument but with the central column in the first diagram given by

$$1 \longrightarrow \widetilde{H} \longrightarrow \frac{\widetilde{H}\times T}{Z(\widetilde{G})} \longrightarrow T/Z(\widetilde{G}) \longrightarrow 1 \tag{C.18}$$

shows that composition with the first projection is

$$\begin{tikzcd}
Z({}^L\widetilde{G}) \arrow[r, "{}^L\overline{1\times\tau}"] \arrow[dr, "{}^L\iota"'] & {}^L\widetilde{H} \times {}^L(T/Z(\widetilde{G})) \arrow[d] \\
 & {}^L\widetilde{H}
\end{tikzcd} \tag{C.19}$$

Therefore, ${}^L\overline{1 \times \tau} = {}^L\iota \times {}^L\tau$. $\square$

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This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.