STRATEGIC POLITICAL RESOURCE ALLOCATION

by

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Economics is the study of the allocation of resources. Since Arrow's Fundamental Welfare Theorems, we know that competitive-markets achieve Pareto allocations when governments correct market failures. Thus, it has largely been the mission of economists to serve as ‘Market Engineers’: To identify and quantify market failures so the government can implement Pareto-improving policy (make everyone better without making anyone worse).

Do Pareto- improving policies get implemented? How does policy become implemented? Achieving a Pareto efficient allocation of a nation’s resources requires studying the implementation of policy, and therefore studying the allocation of political resources that influence policy.

Policy implementation begins with the electoral process. In this dissertation, I use auction analysis, econometrics, and game theory to study political resource allocations in the electoral process.

This dissertation consists of three research papers:

Finance-Augmented Median-Voter Model

Vote Empirics

Colonel Blotto Strategies

The *Finance-Augmented Median-Voter Model* postulates that candidates' campaign expenditures are bids in a first-price asymmetric all-pay auction in order to explain campaign expenditure behavior.

Vote Empirics empirically analyzes the impacts of campaign expenditures, incumbency status, and district voter registration statistics on observed vote-share results from the 2004 congressional election.

Colonel Blotto Strategies postulates that parties' campaign allocations across congressional districts may be a version of the classic Col Blotto game from Game Theory. While some equilibrium strategies and equilibrium payoffs have been identified, this paper completely characterizes players' optimal strategies.

In total, this dissertation solves candidates' optimal campaign expenditure strategies when campaign expenditures are bids in an all-pay auction. The analysis demonstrates the need for understanding exactly the impacts of various factors, including strategic expenditures, on final vote results. The research uses econometric techniques to identify the effects. Last, the research derives the complete characterization of Col Blotto strategies. Discussed extensions provide testable predictions for cross-district Party contributions.

I present this research not as a final statement to the literature, but in hopes that future research will continue its explanation of political resource allocation. An even greater hope is that in time this literature will be used to identify optimal "policy-influencing policies"; constitutional election policies that provide for the implementation of Pareto-improving government policies.

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Chapter 1. Finance-Augmented Median-Voter-Model

Abstract

I introduce a micro-founded model of campaign finance where candidates compete for elected office. Their expenditures earn shares of a pivotal voting mass via advertising. Given the discrete nature of electoral outcomes, the expenditures are bids in a first-price asymmetric all-pay auction. In equilibrium, candidates employ mixed-Nash expenditure-level strategies. The model is used to analyze the influence of campaign finance on the electoral process while explaining that (1) candidates rarely employ the strategy of expending 100% of their budgets and that (2) there exists variation in expenditure strategies. Augmenting the Median Voter Model with this paper's simple financial game yields intuitive equilibria and, in the case when the pivotal voting mass goes to zero (the importance of the finance game diminishes), the model returns the classical MVM result.

1 Introduction

A recurring debate in American politics concerns the constitutionality and desirability of campaign finance reform. Calls for deregulation argue that limitations are infringements on the 1st amendment liberty "freedom of speech." On the other hand, those in favor of campaign finance regulation argue that expenditures are too significant a factor in electoral outcomes and should therefore be controlled. The Lucas-critique plagues current finance-reform analysis. Reform policies can result in systematic differences in expenditure behavior. This paper seeks a structural explanation of campaign expenditures which can be used to identify the counterfactual expenditure strategies under different reforms.

The standard model of the electoral process is the Median-Voter-Model. While MVM is simple and intuitive, it does not account for the role of campaign finances. Campaign expenditures constitute a sizeable sum, rarely equal 100% of budgets, and exhibit variation across candidates.

Table 1

Year	Total Presidential Campaign Expenditures	Total House Campaign Expenditures	Total House Finances Raised
2004	$845M	$880M	$1.7B

Table 2

Year	Average Candidate Expenditure	Average Candidate Budget	Variance in Candidates' % Funds Expended
2004	$1M	$2M	4%

How do we explain campaign expenditure behavior? More specifically, can we explain with a micro-founded structural model why candidates rarely employ 100% expenditure strategies, and that there exists variation in expenditure strategies?

Although it has been suggested empirically that expenditures may have minor impact on election results, this research shows that campaign finance can have a non-negligible impact; elections exhibit expenditure behavior like bids in an asymmetric all-pay auction.

The organization for the remainder of this paper is as follows:

- **Section 2** provides a brief review of the debate concerning the influence of campaign finance on the electoral process
- **Section 3** briefly outlines the Finance-Augmented Median Voter Model
- **Section 4**, I explain how to compute the model and derive the equilibrium
- **Section 5** states the equilibrium results and provides qualitative analysis
- **Section 6** concludes.

2 Literature Review

Levitt (94) and Gerber (98) both address the influence of campaign finance on electoral results empirically. They show that candidates' expenditures have marginal impact on votes earned, however their models' candidates always exhaust their budgets. To account for this weakness, and conduct counterfactual analysis of campaign finance reform policies not subject to the Lucas critique, it is desirable to seek a micro-founded structural model of expenditures.

Prat (00) develops a structural model of campaign spending and the electoral process. He asserts that campaign spending buys advertising which sways pivotal voters, yet also assumes that candidates exhaust their entire budget (as well as makes restrictive assumptions on special-interest contribution behavior.)

The contribution of this paper is to explain the role of campaign finances on the electoral process similar to Prat, in that spending sways pivotal voters through advertising, while allowing less than 100% expenditure strategies and variation in expenditure strategies.

Given the discrete nature of electoral outcomes, I model candidates' expenditure decisions as bids in a First-Price Asymmetric All-Pay auction. Amann & Leininger (96) solve Nash strategies to the Fpaapa, and suggest applicability in politico-economy. I use their proof in calculating candidates' optimal expenditure strategies and extend their findings by solving Nash strategies under additive asymmetries in the auction environment.

3 The Model

This model extends the Median-Voter-Model by allowing campaign expenditures to influence the amount of votes received by each candidate. The environment consists of a voting district *d* in which a unit mass of voters belong to one of three categories (b_0, b_1, μ), and there are two office-seeking candidates (*P0, P1*). Voters of type b_0 resolutely back the Party-0 candidate *P0*. Similarly b_1 voters vote for candidate *P1*. More interestingly, voters of type μ respond to political advertising bought with campaign expenditures.

Under a proportional advertising technology, if *P0* expends *x* and *P1* expends *y*, then *P0* earns $\mu \cdot \frac{x}{x+y}$ pivotal voters and *P1* earns $\mu \cdot \frac{y}{x+y}$ pivotal voters.[1]

[1] I discuss more general advertising technologies later.

The players of the game are the 2 candidates who must decide how much they will expend on their campaign. The strategy space for a candidate is any positive dollar level. Candidate *P0* chooses expenditure level $x \geq 0$, and candidate *P1* chooses $y \geq 0$.

Table 3

Candidate	Resolute Voting Mass	Expenditure	Total Votes Earned
P0	b_0	x	$b_0 + \mu \cdot \frac{x}{x+y}$
P1	b_1	y	$b_1 + \mu \cdot \frac{y}{x+y}$

For the payoff-structure, candidates are risk-neutral representatives of the party. The implications of relaxing the payoff structure are discussed in Section V. Candidates' payoffs are allowed to be asymmetric (includes case $v_0 = v_1$) .

Table 4

Candidate	Payoff if Win	Payoff if Lose
P0	$v_0 - x$	$-x$
P1	$v_1 - y$	$-y$

Payoff valuations are assumed common knowledge. In the future, this assumption can be relaxed with only minor complications to the equilibrium. Having political contributions as public record, parties' candidates have a very good idea about how much the office is valued by the other candidate's party. I assume that the candidates know each others' valuations.

The mathematical formulation of the game is as follows:

$$\textbf{P0}: \quad max_x \quad v_0 \mathbb{I}_{win\,0} - x \quad st \quad x \geq 0$$

$$\textbf{P1}: \quad max_y \quad v_1 \mathbb{I}_{win\,1} - y \quad st \quad y \geq 0$$

The condition for which candidate wins is standard 'majority wins':

Table 5

Condition	Winning Candidate
$b_0 + \mu \cdot \frac{x}{x+y} > b_1 + \mu \cdot \frac{y}{x+y}$	*P0*
$b_0 + \mu \cdot \frac{x}{x+y} < b_1 + \mu \cdot \frac{y}{x+y}$	*P1*

4 Derivation of Equilibrium & Calculation of Election Probabilities

THM 1:

Either only 1 candidate runs (the other concedes), or both candidates run and both candidates employ non-trivial mixed-Nash equilibrium expenditure strategies.

Proof of *thm1*:

Consider the two possible conditions.

Table 6

Condition	Qualitative Description	Case
$\lvert b_1 - b_0 \rvert > \mu$	The resolute-voter mass-gap is larger than the mass of available swing voters.	Concession. The candidate from the disadvantaged party never enters.
$\lvert b_1 - b_0 \rvert < \mu$	The mass of voters responsive to advertising can influence the election outcome.	**Lemma 1**: Both candidates run and engage in an expenditure game with a Mixed-Nash Equilibrium.

Proof of lemma1:

In order to leverage the similarity among possible sub-states for $|b_1 - b_0| < \mu$, I fully characterize the parameter state space. I divide the regions according to which candidate would win if both expend their full valuation. This allows me to identify the candidate with the overall (Finance-Augmented Median Voter Model, FAMVM) advantage in that region.

$b \equiv b_1 - b_0$ is the horizontal axis.

μ is the vertical axis

<u>Characterization of districts' $\boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{v}$ state space</u>

Light green indicates set of valuations such that, if both candidates were to expend their valuation, the game would result in a tie.

Figure 1

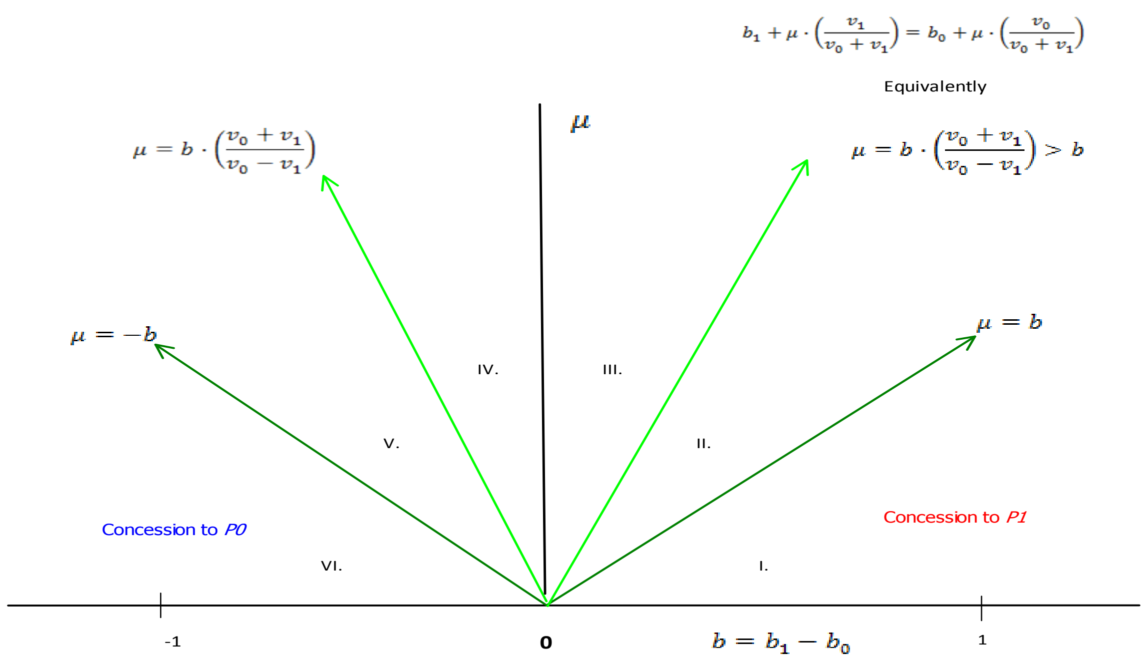


Table 7

Condition	State	Qualitative Description
$0 \leq \mu < (b_1 - b_0) = b$	I.	Concession. Previously Discussed.
$\mu > b > 0$ & $b_1 + \mu \cdot (\frac{v_1}{v_0 + v_1}) > b_0 + \mu \cdot (\frac{v_0}{v_0 + v_1})$	II.	*b>0*, ∴ *P1* has MVM advantage. Valuations s.t. if both fully expend, *P1* would win. *P1* retains an FAMVM advantage.
$\mu > b > 0$ & $b_0 + \mu \cdot (\frac{v_0}{v_0 + v_1}) > b_1 + \mu \cdot (\frac{v_1}{v_0 + v_1})$	III.	*b>0*, ∴ *P1* has MVM advantage. But, valuations s.t. if both fully expend, *P0* would win. *P0* obtains an FAMVM advantage.
$\mu > (-b) > 0$ & $b_1 + \mu \cdot (\frac{v_1}{v_0 + v_1}) > b_0 + \mu \cdot (\frac{v_0}{v_0 + v_1})$	IV.	*b<0*, ∴ *P0* has MVM advantage. But, valuations s.t. if both fully expend, *P1* would win. *P1* obtains an FAMVM advantage.
$\mu > (-b) > 0$ & $b_0 + \mu \cdot (\frac{v_0}{v_0 + v_1}) > b_1 + \mu \cdot (\frac{v_1}{v_0 + v_1})$	V.	*b<0*, ∴ *P0* has MVM advantage. Valuations such that if both fully expend, *P0* would win. *P0* retains an FAMVM advantage.
$0 \leq \mu < (-b)$	VI.	Concession. Previously Discussed.

Consider Case II. (Similar analysis of cases III-V)

$$\textbf{P0}: \quad max_x \quad v_0 \mathbb{I}_{win\,0} - x \quad st \quad x \geq 0$$

$$\textbf{P1}: \quad max_y \quad v_1 \mathbb{I}_{win\,1} - y \quad st \quad y \geq 0$$

Where

$$\{b_0 + \mu \cdot \tfrac{x}{x+y} > b_1 + \mu \cdot \tfrac{y}{x+y}\} \Rightarrow win_0$$

$$\{b_1 + \mu \cdot \tfrac{y}{x+y} > b_0 + \mu \cdot \tfrac{x}{x+y}\} \Rightarrow win_1$$

Equivalently (multiply thru by '*x+y*', use $b = b_1 - b_0$, and collect terms)

$$\{x \cdot (-b + \mu) > y \cdot (b + \mu)\} \Rightarrow \; win_0$$

$$\{y \cdot (b + \mu) > x \cdot (-b + \mu)\} \Rightarrow \; win_1$$

Define G_0 and G_1 to be the expenditure-level strategy CDFs for *P0* and *P1* respectively. Thus, $G_1(x \cdot \frac{\mu-b}{\mu+b}))$ is the probability that *P1* expends less than $x \cdot \frac{\mu-b}{\mu+b})$, yielding that *P0* would win for his expenditure of *x.* Similarly $G_0(y \cdot \frac{\mu+b}{\mu-b}))$ is the probability that *P1* wins for expenditure y. Characterizing these functions is the goal of our analysis. Showing that G_0 and G_1 are non-trivial distributions (not the delta-function) implies that candidates are employing true mixed-Nash strategies in equilibrium.

P0: $\quad max_x \quad v_0 \cdot G_1(x \cdot \frac{\mu-b}{\mu+b})) - x \quad st \quad x \geq 0$

P1: $\quad max_y \quad v_1 \cdot G_0(y \cdot \frac{\mu+b}{\mu-b})) - y \quad st \quad y \geq 0$

To obtain closed-form expressions for G_0 and G_1, differentiate with respect to the endogenous variable (Appendix 1) or, more simply, recognize that a candidate will receive an equal expected payoff for any expenditure-level he would mix among. If he did not, then that expenditure level would no longer be a best-reply, and therefore in the support of his optimal strategy.

Defining k_0 and k_1 to be the optimal expected payoff values for *P0* and *P1* respectively yields:

$$v_0 \cdot G_1\left(x \cdot \frac{\mu-b}{\mu+b} \right. - x = k_0$$

$$v_1 \cdot G_0\left(y \cdot \frac{\mu+b}{\mu-b} \right. - y = k_1$$

Rearranging and changing variables yields the following expressions for G_0 and G_1

$$G_0(x) = \frac{k_1 + x \cdot (\frac{\mu - b}{\mu + b})}{v_1}$$

$$G_1(y) = \frac{k_0 + y \cdot (\frac{\mu + b}{\mu - b})}{v_0}$$

I now impose 2 interpretable economic boundary conditions, Propositions 1 & 2, in order to identify k_0 and k_1.

<u>Proposition1:</u> In case II (and IV) *P0* will never expend more than his valuation v_0.

Expending \$0 earns a 0 payoff while expending >v_0 always earn payoffs<0.

Therefore, there is no chance *P0* expends > v_0. $\therefore$ $G_0(v_0)$=1 ■

By Proposition1:

$$G_0(v_0) = \frac{k_1 + v_0 \cdot (\frac{\mu - b}{\mu + b})}{v_1} = 1 \quad \Rightarrow \quad k_1 = v_1 - v_0 \cdot (\frac{\mu - b}{\mu + b})$$

<u>Proposition2:</u> In case II, *P1* will never expend more than $v_0 \cdot (\frac{\mu - b}{\mu + b})$. $\therefore$ $G_1(v_0 \cdot (\frac{\mu - b}{\mu + b}))$=1.

Considering the MVM advantage and his finances, any extra would be superfluous.

Wtf $\bar{y}$ st: $b_0 + \mu \cdot \frac{v_0}{v_0 + \bar{y}} = b_1 + \mu \cdot (\frac{\bar{y}}{v_0 + \bar{y}}) \quad \Rightarrow \quad \bar{y} = v_0 \cdot (\frac{\mu - b}{\mu + b})$ ■

By Propostion2: $G_1(v_0 \cdot (\frac{\mu - b}{\mu + b})) = \frac{k_0 + V_0 \cdot (\frac{\mu - b}{\mu + b} \cdot (\frac{\mu + b}{\mu - b})}{v_0} = 1 \Rightarrow k_0 = 0$

Corollary to Proposition2: In region II (with risk-neutral preferences), *P1* is able to ensure that *P0* earns 0 expected payoff in equilibrium. $k_0 = 0$

Substituting k_0 and k_1 and simplifying, we characterize G_0 and G_1

$$G_0(x) = \frac{v_1 - v_0 \cdot \dfrac{\mu - b}{\mu + b} + x \cdot \dfrac{\mu - b}{\mu + b}}{v_1}$$

$$G_0(x) = 1 - \frac{v_0}{v_1} \cdot \frac{\mu - b}{\mu + b} + \frac{x}{v_1} \cdot \left(\frac{\mu - b}{\mu + b}\right)$$

$$G_1(y) = \frac{y}{v_0} \cdot \left(\frac{\mu + b}{\mu - b}\right)$$

Regions III-V are analogous. In III, since *P0* has obtained the FAMVM advantage, we now get

$$G_0(x) = \frac{x}{v_1} \cdot \left(\frac{\mu - b}{\mu + b}\right) \quad \& \quad G_1(y) = 1 - \frac{v_1}{v_0} \cdot \frac{\mu + b}{\mu - b} + \frac{y}{v_0} \cdot \left(\frac{\mu + b}{\mu - b}\right)$$

Regions IV (*b*<0), since *P1* has obtained the FAMVM advantage we get the same as II.

$$G_0(x) = 1 - \frac{v_0}{v_1} \cdot \frac{\mu - b}{\mu + b} + \frac{x}{v_1} \cdot \left(\frac{\mu - b}{\mu + b}\right) \quad \& \quad G_1(y) = \frac{y}{v_0} \cdot \left(\frac{\mu + b}{\mu - b}\right)$$

Regions V (*b*<0), since *P0* has retained the FAMVM advantage we get the same as III.

$$G_0(x) = \frac{x}{v_1} \cdot \left(\frac{\mu - b}{\mu + b}\right) \quad \& \quad G_1(y) = 1 - \frac{v_1}{v_0} \cdot \frac{\mu + b}{\mu - b} + \frac{y}{v_0} \cdot \left(\frac{\mu + b}{\mu - b}\right)$$

Thus we have completed the **proof of *lemma1***; both candidates run and employ non-trivial mixed-Nash expenditure strategies for Regions II-V described by above equations.

Further, we have exhausted all cases in our State-Space and have also completed the **proof of *Thm1*** ■

With the characterized mixing strategies, I calculate the probabilities of winning (Region II).

$$\text{Probability } P0 \text{ wins} = \int_0^{v_0} \int_0^{x\cdot\left(\frac{\mu-b}{\mu+b}\right)} dG_1(y)\cdot dG_0(x)$$

$$\text{Probability } P1 \text{ wins} = 1 - \text{Probability } P0 \text{ wins}$$

Evaluation of the integral for the probability of *P0* winning gives

$$\int_0^{v_0} \frac{x}{v_0}\cdot dG_0(x) = \int_0^{v_0} \frac{x}{v_0}\cdot \left(\frac{\mu-b}{\mu+b}\right)\cdot\frac{1}{v_1}dx$$

$$= \frac{x^2}{2\cdot v_0\cdot v_1}\cdot\left(\frac{\mu-b}{\mu+b}\right) \quad \textit{evaluated at } v_0 \textit{ and } 0 \textit{ respectively}$$

$\text{Probability } P0 \text{ wins} =$

$$\frac{v_0}{2\cdot v_1}\cdot\left(\frac{\mu-b}{\mu+b}\right)$$

THM 2: The candidate with the FAMVM advantage has probability of winning the election > ½.

In Region II, we are in the case that

$$b_1 + \mu\cdot\left(\frac{v_1}{v_0+v_1}\right) > b_0 + \mu\cdot\left(\frac{v_0}{v_0+v_1}\right)$$

With $b \equiv b_1 - b_0$, rearranging terms implies

$$v_1\cdot(\mu+b) > v_0\cdot(\mu-b)$$

$$\text{Probability } P0 \text{ wins} = \frac{v_0}{2\cdot v_1}\cdot\left(\frac{\mu-b}{\mu+b}\right) < \frac{1}{2}$$

$$\text{Probability } P1 \text{ wins} = 1 - \frac{v_0}{2 \cdot v_1} \cdot \frac{\mu - b}{\mu + b} > \frac{1}{2}$$

Analogous calculations and analysis hold for Regions III-V. ■

5 Results/Equilibrium

For each possible state of the voting district, the Equilibrium Expenditure Strategy Distributions (CDFs) of the Party Candidates are as follows

Table 8

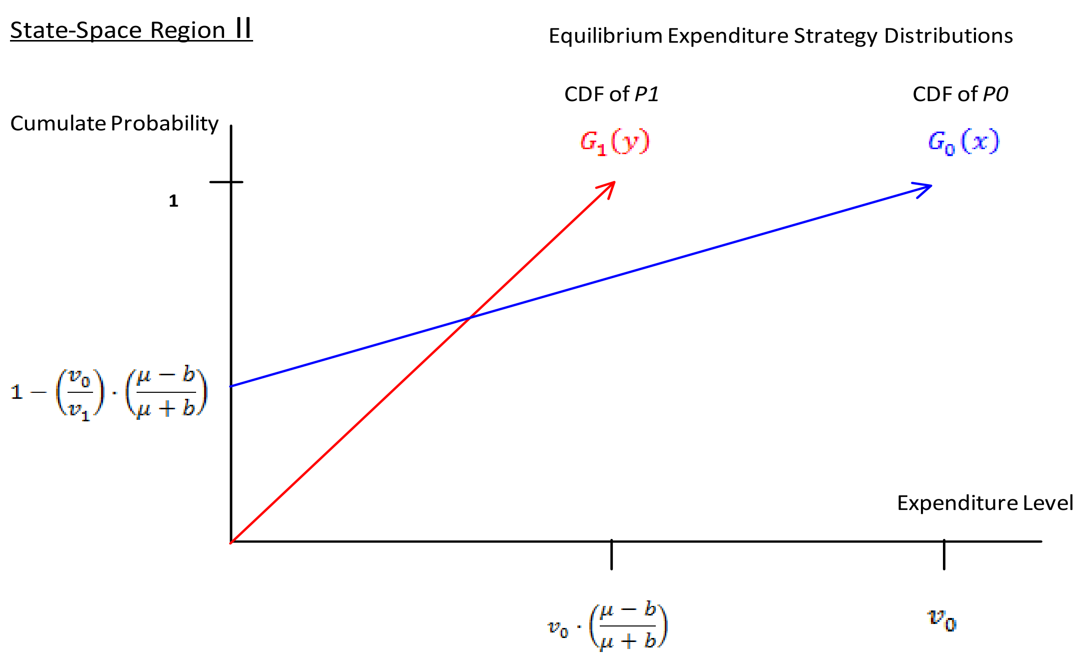
P0 strategy: $G_0(x)$	State	*P1* strategy: $G_1(y)$
$1\mathbb{I}_0$, concede	I	$1\mathbb{I}_0$, win
$1 - \frac{v_0}{v_1} \cdot \frac{\mu - b}{\mu + b} + \frac{x}{v_1} \cdot (\frac{\mu - b}{\mu + b})$	II	$\frac{y}{v_0} \cdot (\frac{\mu + b}{\mu - b})$
$\frac{x}{v_1} \cdot (\frac{\mu - b}{\mu + b})$	III	$1 - \frac{v_1}{v_0} \cdot \frac{\mu + b}{\mu - b} + \frac{y}{v_0} \cdot (\frac{\mu + b}{\mu - b})$
$1 - \frac{v_0}{v_1} \cdot \frac{\mu - b}{\mu + b} + \frac{x}{v_1} \cdot (\frac{\mu - b}{\mu + b})$	IV	$\frac{y}{v_0} \cdot (\frac{\mu + b}{\mu - b})$
$\frac{x}{v_1} \cdot (\frac{\mu - b}{\mu + b})$	V	$1 - \frac{v_1}{v_0} \cdot \frac{\mu + b}{\mu - b} + \frac{y}{v_0} \cdot (\frac{\mu + b}{\mu - b})$
$1\mathbb{I}_0$, win	VI	$1\mathbb{I}_0$, concede

The Probabilities of a Candidate Winning in each Region

Table 9

$P0$	State	$P1$
0	I	1
$\frac{v_0}{2 \cdot v_1} \cdot \frac{\mu - b}{\mu + b} < \frac{1}{2}$	II	$1 - \frac{v_0}{2 \cdot v_1} \cdot \frac{\mu - b}{\mu + b} > \frac{1}{2}$
$1 - \frac{v_1}{2 \cdot v_0} \cdot \frac{\mu + b}{\mu - b} > \frac{1}{2}$	III	$\frac{v_1}{2 \cdot v_0} \cdot \frac{\mu + b}{\mu - b} < \frac{1}{2}$
$\frac{v_0}{2 \cdot v_1} \cdot \frac{\mu + b}{\mu - b} < \frac{1}{2}$	IV	$1 - \frac{v_0}{2 \cdot v_1} \cdot \frac{\mu + b}{\mu - b} > \frac{1}{2}$
$1 - \frac{v_1}{2 \cdot v_0} \cdot \frac{\mu + b}{\mu - b} > \frac{1}{2}$	V	$\frac{v_1}{2 \cdot v_0} \cdot \frac{\mu + b}{\mu - b} < \frac{1}{2}$
1	VI	0

Probability a Candidate Expends Less Than or Equal to an Expenditure Level
Vs. Expenditure Level
Figure 2



The above graph shows the candidates' strategies for a district in State-Space Region II. If the district were still located in Region II, but rotated more counterclockwise, toward III, *P1*'s advantage wanes. His resolute-voter mass-gap gets smaller and his finance advantage decreases while the mass of pivotal voters and importance of the finance game increase.

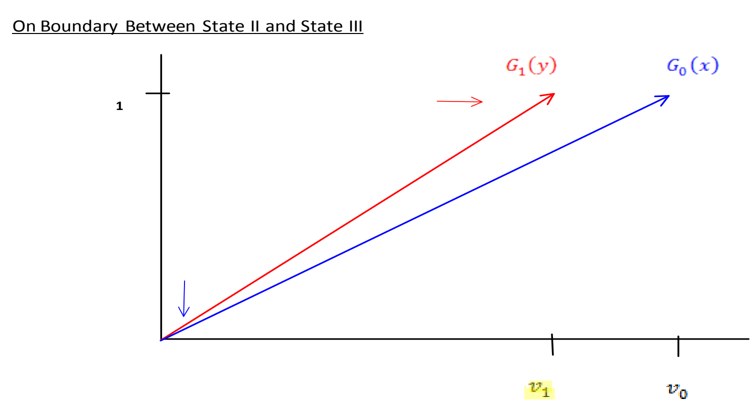
The Equilibrium states that *P1* would now mix over a larger support, with an upper-bound now closer to his valuation. Since, his strategy $G_1(y)$ is still linear, meaning his mixing distribution is still Uniform, his expected expenditure level increases along with variance of expenditure strategy.

Correspondingly, *P0*'s disadvantage wanes, and the size of his mass point at 0 diminishes. In conjunction, the slope of $G_0(x)$ increases meaning *P0* mixes among his non-zero expenditure-levels each with an equally higher probability than before.

Eventually, for a district on the boundary between Regions II and III of the State Space, where the resolute-voter mass-gap advantage and the Finance-Advantage cancel, we are in a symmetric competitive case. The strategies (G distributions) are different since *P1* has more resolute voting mass and *P0* has a larger budget, but both candidates win the office equal probability and neither candidate plays a mass at 0.

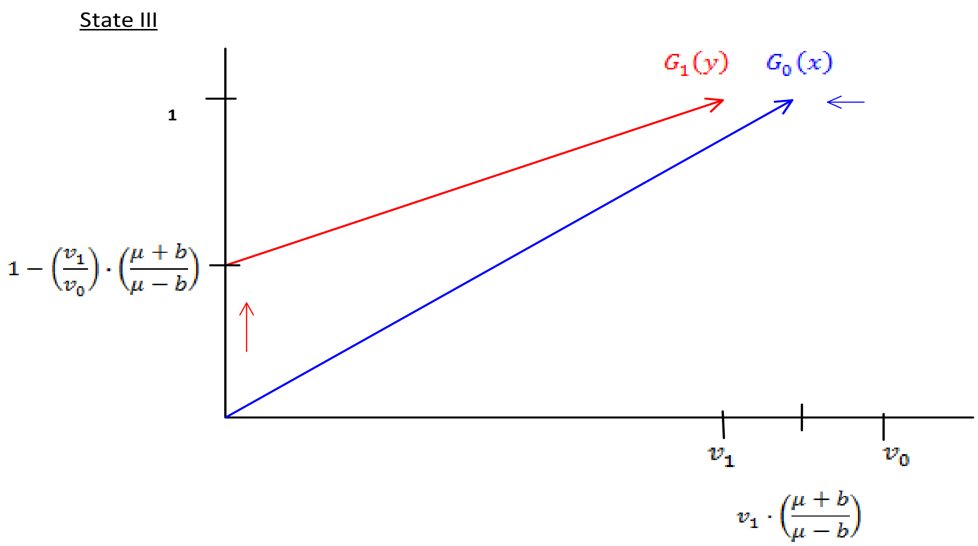
A graphical representation of strategies along the symmetric-competitive case:

Figure 3



Rotating counterclockwise again, into State-Space Region-III, *P1* plays the strategy with the mass point at 0, and *P0* need not mix over his entire support.

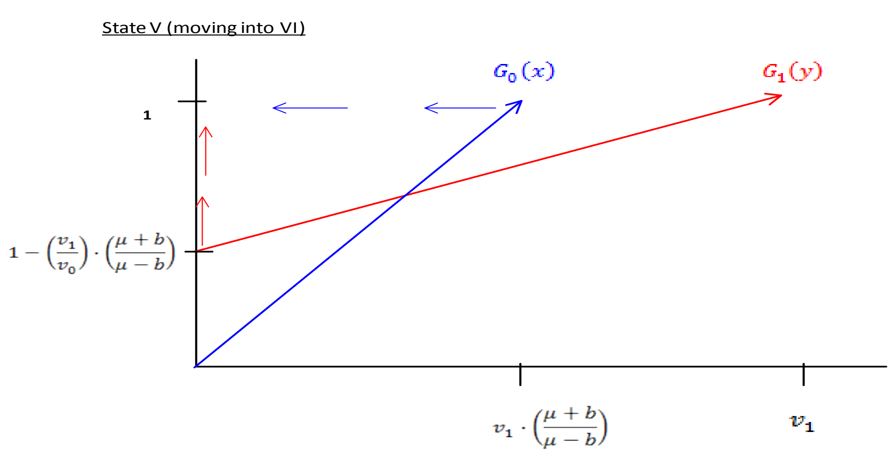
Figure 4



Region IV is trivially similar to region III (the colors, subscripts and *x*'s and *y*'s reverse). The same is true between Regions II and V.

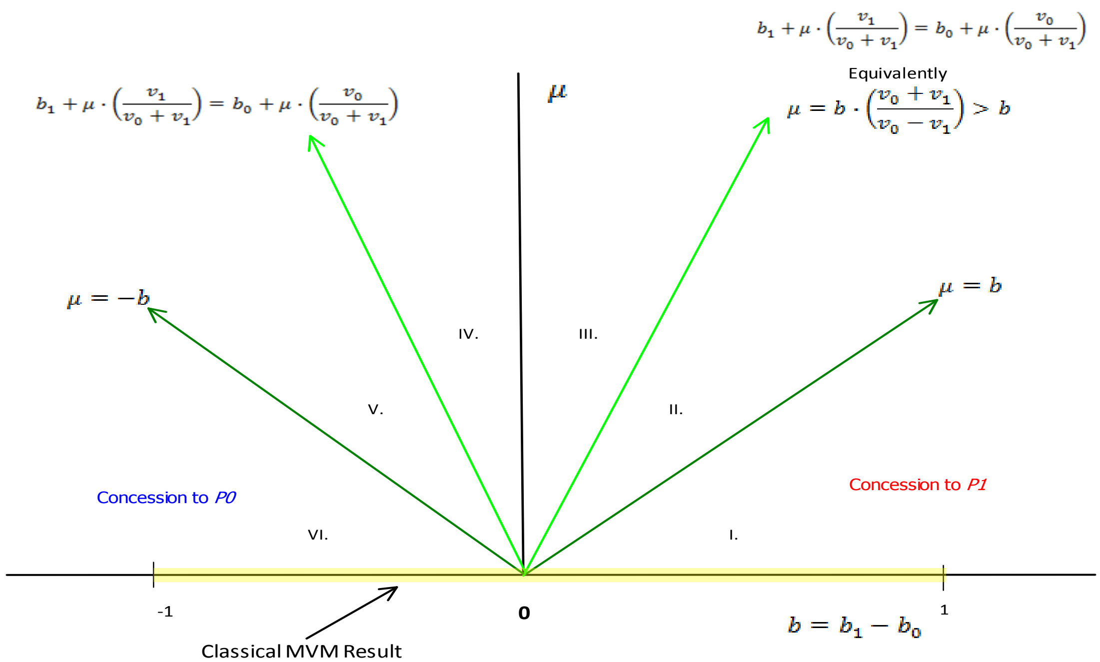
As we rotate from Region V into Region VI (or similarly from Region II into Region I), we converge to the concession strategy. Both players' strategy distributions converge pointwise to the delta function at 0 (remember; in Region IV, V and VI, *b*<0).

Figure 5



Also intuitive, the classical MVM result is embedded in the *b*-axis of the state space, where $\mu = 0$ and finances have no role. Regions I and VI yield MVM.

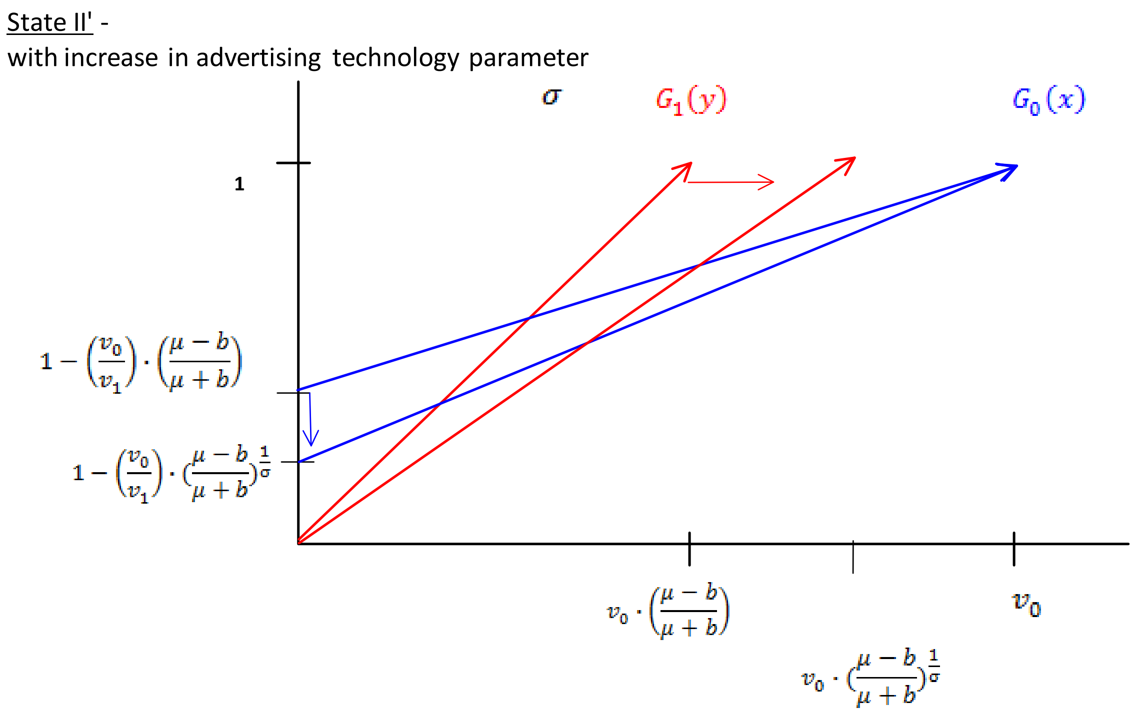
Figure 6



The results are robust for relaxations of the assumptions. I relax the advertising technology from <u>proportional</u> to <u>proportional to the ratio of the expenditures to the power σ.</u> The model still results in uniform candidate mixing strategies among all non-zero expenditure levels (Appendix2). In Region II, increasing the power of the exponents, σ, makes the game more competitively symmetric. I caution the reader however, the state-space is not identical. The symmetric competitive boundary adjusts with σ.

A graphical representation, for an increase in $\sigma > 1$: In Region II' (II'=II after considering new regional boundaries).

Figure 7



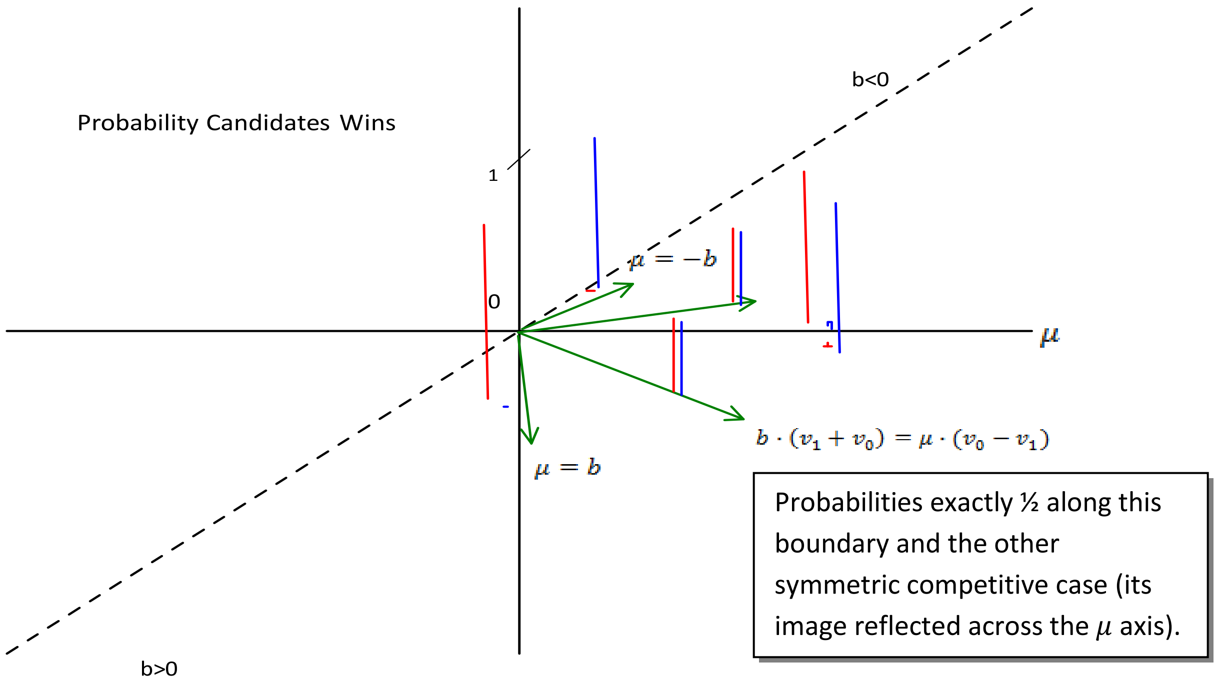
Second, assuming candidate risk-neutrality results in Uniform Distributions of candidate mixing strategies. Introducing risk-aversion will perturb the curvature of the distribution strategies as well as slightly change the supports of the strategy distributions and boundary conditions. Risk-aversion meddles with the simplicity of the all-pay calculations, but is still tractable. It can be shown that, had we made both candidates have the same degree of crra, the candidates would both decrease the probability of very high expenditures and increase the probability of low expenditures; resulting in concave G distribution strategies.

The probabilities each candidate wins can be seen graphically with 3 axes.

On the x-y plane we have the $\underline{\boldsymbol{b}, \boldsymbol{\mu}, \boldsymbol{v}}$ State-Space same as before.

Along the z-axis are the probabilities that the candidates win: blue for $P0$, red for $P1$.

Figure 8



6 Conclusion

This paper has used micro structure to analyze the influence of finances in the electoral process while being mindful that candidates do not always employ 100% expenditure strategies and that there exists variation in expenditure strategies.

It showed that modeling candidates campaign expenditures as bids in an asymmetric all-pay auction yield intuitive mixed-Nash equilibrium strategies. Campaign finances can influence the electoral process: expenditure levels and probabilities that candidates win. All the while, the classical MVM result is preserved in the Finance-Augmented Median Voter Model.

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Appendix 1

Calculation of G's via differential equations

(without making argument that expected payoffs are necessarily constants k_0 and k_1).

From Page 8, in Region II:

$$\textbf{\textit{P0}}: \quad max_x \quad v_0 \cdot G_1\left(x \cdot \frac{\mu-b}{\mu+b} \right. - x \quad st \quad x \geq 0$$

$$\textbf{\textit{P1}}: \quad max_y \quad v_1 \cdot G_0(y \cdot \left(\frac{\mu+b}{\mu-b}\right)) - y \quad st \quad y \geq 0$$

To find the candidates' optimal expenditures, differentiate wrt *x* and *y* respectively.

$$v_0 \cdot g_1\left(x \cdot \frac{\mu-b}{\mu+b} \right. \cdot \frac{\mu-b}{\mu+b} - 1 = 0 \Rightarrow g_1\left(x \cdot \frac{\mu-b}{\mu+b} \right. = \frac{1}{v_0} \cdot \frac{\mu+b}{\mu-b}$$

$$v_1 \cdot g_0\left(y \cdot \frac{\mu+b}{\mu-b} \right. \cdot \frac{\mu+b}{\mu-b} - 1 = 0 \Rightarrow g_0\left(y \cdot \frac{\mu+b}{\mu-b} \right. = \frac{1}{v_1} \cdot \frac{\mu-b}{\mu+b}$$

Here again we can see that the optimal expenditure strategies are to mix uniformly, at locations other than mass points. To account for mass points with this method, recover the Cdfs.

$$G_0(x) = \frac{x}{v_1} \cdot \frac{\mu-b}{\mu+b} + \tilde{k}_1$$

$$G_1(y) = \frac{y}{v_0} \cdot \frac{\mu+b}{\mu-b} + \tilde{k}_0$$

Application of economic boundary conditions, identical to the analysis on page 8, identifies the new, but related constants $\tilde{k}_1$ and $\tilde{k}_0$.

Appendix 2

Generalized Advertising Technology:

Proportion of Pivotal Mass won by *P0* and *P1* for expenditures *x* and *y* respectively

$$P0:\ \mu \cdot \frac{x^\sigma}{x^\sigma + y^\sigma} \quad , \quad P1:\ \mu \cdot \frac{y^\sigma}{x^\sigma + y^\sigma}$$

Region boundaries defining the symmetric cases now change.

New Region II defined by

$$\mu > (b_1 - b_0) \qquad \& \qquad b_1 + \mu \cdot \left(\frac{{v_1}^\sigma}{{v_0}^\sigma + {v_1}^\sigma}\right) > b_0 + \mu \cdot \left(\frac{{v_0}^\sigma}{{v_0}^\sigma + {v_1}^\sigma}\right)$$

P0 will win when

$$y < x \cdot \left(\frac{\mu - b}{\mu + b}\right)^{\frac{1}{\sigma}}$$

And *P1* wins when

$$x < y \cdot \left(\frac{\mu + b}{\mu - b}\right)^{\frac{1}{\sigma}}$$

Yielding the following equations (analogous to those on page 7)

$$v_0 \cdot G_1\left(x \cdot \left(\frac{\mu + b}{\mu - b}\right)^{\frac{1}{\sigma}}\right) - x = k_{0,\sigma}$$

$$v_1 \cdot G_0\left(y \cdot \left(\frac{\mu + b}{\mu - b}\right)^{\frac{1}{\sigma}}\right) - y = k_{1,\sigma}$$

The support for *P0* will still be $[0, v_0]$, however, the new support for *P1* will be $[0, v_0 \cdot (\frac{\mu - b}{\mu + b})^{\frac{1}{\sigma}}]$. Thus we obtain

$$G_0(x) = 1 - \frac{v_0}{v_1} \cdot \left(\frac{\mu - b}{\mu + b}\right)^{\frac{1}{\sigma}} + \frac{x}{v_1} \cdot \left(\frac{\mu - b}{\mu + b}\right)^{\frac{1}{\sigma}}$$

$$G_1(y) = \frac{y}{v_0} \cdot \left(\frac{\mu + b}{\mu - b}\right)^{\frac{1}{\sigma}}$$

Since in Region II *b*>0 and, stipulating $\sigma > 1$

$$(\frac{\mu - b}{\mu + b}) < 1 \Rightarrow (\frac{\mu - b}{\mu + b}) < (\frac{\mu - b}{\mu + b})^{\frac{1}{\sigma}} < 1$$

Thus, the mass point at 0 in *P0*'s strategy decreases for increases in $\sigma = 1$.

Chapter 2: Vote Empirics

Incumbency, Campaign Expenditures, and Partisan Registration: How These Factors Impact Electoral Results?

Abstract

This paper uses aggregate US congressional district level data to identify how incumbency, candidate expenditures, and district voter registration statistics impact final vote shares. We estimate the factor impacts two ways; 1) a reduced-form (semi-parametric) production function specification similar to Levitt (94) and 2) an agent-based discrete-demand estimation set-up as in BLP (95). The first method hypothesizes that the final amount of votes a candidate receives consists of a fraction of the voters registered to his party, his base, plus a non-parametrically estimated fraction of remaining swing voters depending on relative advertising expenditures. By introducing district level partisan voter registration statistics, we mitigate the endogeneity of expenditures and estimate factor impacts consistent with previous literature. The BLP model is more general and includes the first as a special case. We provide and test the restrictions that deliver the classic reduced form regressions. Also, discrete-demand random coefficients provide a natural insertion of voter registration statistics. Last, we discuss proper inclusion of strategic voting and structural models of voter participation into BLP estimates of vote factor impacts.

1 Introduction

While campaign expenditures are a prevalent topic in the media every election cycle, other factors such as incumbency and districting also have real influence on electoral results.

Constitutionalists and policy makers would like to know these factors' impacts. As outlined in the Federalist Papers, Madison organized the Constitution to prevent the culmination of political power by factions. If a group can significantly influence electoral results, they hold real political power; they can influence policy for their gain at the cost of others. The purpose of this paper is positive: to estimate the effects of these three factors on observed aggregated district vote results via two methods: 1) A reduced-form semi-parametric production function, similar to Levitt (94) and 2) An agent-based discrete demand estimate, as in BLP (95). We show that the first can be understood as a special case of the second. Using Congressional election data, we test the viability of the reduced-form literature's restrictions.

The reduced-form vote production function specification hypothesizes that candidates receive a fraction of the voters in their district registered to their party (their base, allowed to depend on incumbency status) and compete for the remaining swing voters thru campaign expenditures. We non-parametrically estimate how candidates' relative campaign expenditures apportion the non-base swing voters. Overall, the model is semi-parametric and turns factors of production (district registration statistics, incumbency, and expenditures) into final vote results. The specification yields results in line with Levitt and Gerber simply by inclusion of voter registration statistics to mitigate the endogeneity, as opposed to instruments or use of repeat races to difference out qualitative regressors. Moreover, the non-parametric estimate of advertising expenditure influences is consistent with IO advertising literature.

BLP's discrete product-demand estimation methodology lends itself naturally to observed aggregate vote share data. The market is a district, the products are the candidates, the characteristics are incumbency status and expenditures, the market share is the vote share, and the consumer-heterogeneity random-coefficient distribution is the partisan voter registration distribution. Rekkas (06) applied the BLP methodology to Canadian elections. She did not have voter registration statistics and had to estimate random-coefficient distribution parameters. This paper uses 2004 US congressional data, includes voter registration statistics, relates the estimation method to seminal reduced-form work, and discusses a critical difference between discrete product-demand and voting for future applications of BLP to politico-economy. In Extensions, we discuss adapting the model to accommodate strategic voting. With product demand, when consumers choose the option not to consume they get no product, yet voters always get a representative. Econometrically, the mass of voters who vote for a candidate are those that either get a very high shock for that candidate or get a very low shock for the other candidate. They may vote to 'block' a candidate. We do not estimate this model in this version of this paper, but discuss steps for doing so.

Last, the paper discusses the possible presence of endogeneity. We argue that our use of exogenous voter registration variation data as a regressor mitigates standard political-campaign endogeneity concerns (larger vote shares imply more support and larger expenditures). Nonetheless, we employ a standard instrument (lagged expenditures) and conduct the Hausman test for endogeneity for both specifications. We find that including partisan voter registration statistics yields consistent factor impact estimates.

The organization for the remainder of this paper is as follows:

- **Section 2** discusses relevant literature and this paper's contribution
- **Section 3** presents the reduced-form semi-parametric model
- **Section 4** presents the discrete-demand model of voting
  - Provides general estimation equation
  - Introduces voter registration distribution as random coefficient distribution
  - Focuses on multi-variate logit estimation with log(expenditures)
- **Section 5** compares the two methodologies and tests the equating restrictions
- **Section 6** explains the data
- **Section 7** provides and interprets empirical results
- **Section 8** concludes
- **Section 9** discusses next-step extensions for structural vote empirics.

2 Literature Review

There have been many studies in Economics and Political Science literature that empirically analyze the relationship between earned vote shares and candidate characteristics. See Gerber (2004) for a thorough summary of the reduced form empirical literature. The contributions of this paper are to perform similar analysis but 1) to include district partisan voter registration statistics as an exogenous source of variation, 2) to estimate a structural discrete-demand model to identify factor influence parameters, and 3) finally to explain the relationship between the two methodologies' estimates.

Gerber (98) and Levitt (94) both estimate electoral factor impacts via reduced-form. They differ in their techniques for dealing with endogeneity; that final vote shares and candidate expenditures may be co-caused by a factor such as candidate quality. An eloquent candidate may garner more supporters, more votes, more financial support, and have larger expenditures. Gerber tries several instruments. Levitt introduces the unobserved qualitative regressor, but then uses repeat races to difference it out and obtain consistent estimates. Alternatively, we introduce a new regressor, partisan voter registration statistics, and argue that this variable explains most indirect causation that may be causing biased estimates.

Berry Levinsohn Pakes (1995) introduce a structural discrete demand model which has been employed across industries to estimate the impacts of product characteristics using observed product market share. In this paper, we apply their methodology to explain the impact of candidate characteristics on observed candidate vote share received. Rekkas (07) applied BLP methodology to Canadian election data. She did not have voter registration statistics and thus had to estimate parameters of the random-coefficient heterogeneity distribution. In addition, the Canadian elections are generally multi-party, requiring more parameters in an already parameter-heavy framework. We apply the discrete-demand analysis to 2004 US House elections, where most districts only had two concerted candidates running. Our district-level partisan voter-registration statistics are naturally suited to serve as the random-coefficient distribution for heterogeneous voters. Last, we discuss extensions to the BLP framework to accommodate strategic voting.

3 The Reduced-Form Semi-Parametric Specification

A candidate's final vote count, denoted b^f, is assumed to be the sum of his base voters, denoted b^o, plus the swing voters he won through his advertising expenditures. The total mass of swing voters, denoted μ, consists of all the non-base voters and are apportioned in a fashion dependent upon the campaign expenditures, x and y. Let x be the candidate's own expenditure, and y is his opponent's. A semi-parametric attempt at the production function at this stage would suggest the following.

$$b^f = b^0 + \mu \cdot f(x,y) + \varepsilon$$

This expression requires modification however, since base voters are not directly observable. We assume that base voters are a fraction, γ_b, of the voters in the district registered to their party, b^r which is observable. In this same fashion, the mass of swing voters must also be refined since some may not participate in the election. Let γ_μ represent the fraction of swing voters that participate in the election. Thus, the expression becomes

$$b^f = \gamma_b \cdot b^r + \gamma_\mu \cdot \mu \cdot f(x,y) + \varepsilon$$

Since swing-voters are all the non-base participating voters, we can express μ in terms of the voter registration statistics. For a candidate from the Democratic Party (Party 0 with expenditure x), with opponent from the Republican Party (Party1 with expenditure y), we write

$$b_0^f = \gamma_b \cdot b_0^r + \gamma_\mu \cdot [1 - \gamma_b \cdot (b_0^r + b_1^r)] \cdot f(x,y) + \varepsilon$$

Last, we control for incumbency. We suggest that being an incumbent garners a candidate a larger fraction of voters registered to his party to be his base. Consequently we reduce the swing mass to account for this enlarged base.

$$b_0^f = (\gamma_b + \gamma_I \cdot \mathbb{I}_{I_0}) \cdot b_0^r + \gamma_\mu \cdot [1 - (\gamma_b + \gamma_I \cdot \mathbb{I}_{I_0}) \cdot b_0^r - (\gamma_b + \gamma_I \cdot \mathbb{I}_{I_1}) \cdot b_1^r] \cdot f(x,y + \varepsilon$$

This estimable expression can be read as follows: Party0's final vote share consists of the fraction of voters registered for his party that were not susceptible to advertising, his base which depends on his incumbency status, plus a fraction of the voting component of advertising susceptible swing voters where the fraction depends on the candidates' relative advertising expenditures.

Ideally, with sufficient data and computational resources we would like to totally non-parametrically identify the factor effects with at least a 3-dimensional Taylor approximation of at least 2nd order; $b_o^f(x, y, i)$. The interaction terms in the multi-dimensional approximation would allow many plausible interpretations, such as that campaign expenditures may affect voter turnout, but come at the cost of introducing more parameters and potentially reducing the significance of the estimate of each. It is for this reason we only use a 2nd order 2-dimensional Taylor polynomial and impose structure to account for the influence of the remaining factors. While we do lose some indirect effects of the factors, we feel that the imposed structural specification capture the major effects of each factor.

Regarding endogeneity, we argue that by including district-level partisan voter registration statistics in the vote production function, the regression does not suffer biased factor impact estimates. We formally test this claim and provide regression results with and without a standard instrument, lagged expenditures (Hausman test of endogeneity; Results, Section 7). We find that we can reject the presence of endogeneity. Moreover, our estimated factor impact parameters from this equation are consistent to those estimated by Levitt and Gerber.

Controlling for district partisan registration statistics, we argue, explains any indirect causation that may be biasing expenditure impact estimates. Moreover, while candidate quality may co-influence vote-share and funds raised, we also argue that candidates spend their desired expenditure level free independent of backer financing. As is supported by the data, candidates can and do go into large amounts of debt spending money they had not raised. Our finding that the marginal returns to advertising expenditures are decreasing justifies why this phenomenon is observed only infrequently. Last, we argue that while Presidential elections are high-visibility and contain many public debates, House elections are generally much lower profile. Voters rarely know more about the candidate than what they observe via advertising, which is already captured in the regression.

4 The Discrete-Demand Model

4.1 Baseline Model

In the nature of BLP (95) we assume a utility-maximizing agent facing a discrete candidate voting decision; whether to vote for the candidate of Party0, Party1, or not vote.

This decision is represented as follows:

$$max_{x_j \epsilon \{x_0, x_1, x_n \emptyset\}} U_i(x_j) = \vec{x}_j \cdot \vec{\beta}' + \delta_j + \varepsilon_{ij}$$

Where the variables have the following interpretations

Table 10

variable	Interpretation
$\vec{x}_j$	Vector of candidate j's observable characteristics: Incumbency status, Campaign expenditure level
$\vec{\beta}$	Vector of coefficients explaining utility impact of candidate characteristics
δ_j	Mean utility from candidate characteristics unobserved by econometrician
ε_{ij}	Idiosyncratic utility error component to agent i from voting for candidate j

For agent i to vote for candidate x_0, the following conditions hold.

Agent i must get more utility from voting for candidate 0 than for candidate 1

$$\vec{x}_0 \cdot \vec{\beta}' + \delta_0 + \varepsilon_{i0} > \vec{x}_1 \cdot \vec{\beta}' + \delta_1 + \varepsilon_{i1}$$

Equivalently, $\varepsilon_{i1} < (\vec{x}_0 - \vec{x}_1) \cdot \vec{\beta}' + (\delta_0 - \delta_1) + \varepsilon_{i0}$

(Utility shock to agent i for candidate1 must be sufficiently less than the one for candidate0)

Agent i must get more utility from voting for candidate 0 than not voting, normalized to 0)

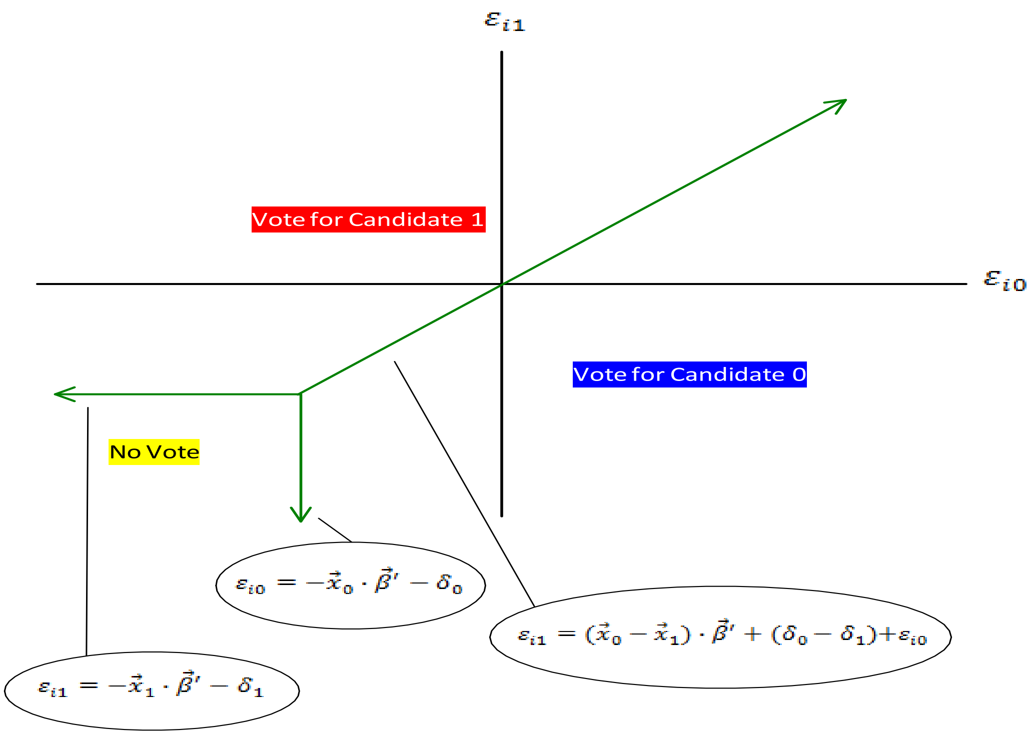
$$\vec{x}_0 \cdot \vec{\beta}' + \delta_0 + \varepsilon_{i0} > 0$$

Equivalently, $\varepsilon_{i0} > -\vec{x}_0 \cdot \vec{\beta}' - \delta_0$

(Utility shock for candidate0 must be sufficiently large to overcome utility from not voting)

These conditions (and the candidate1 analog of the second) can be seen in the following graphical representation. The axes represent the possible utility shocks for the candidates to agent i. The distinct regions indicate the region of the shock space in which agents take particular actions.

Figure 9



The model predicts that the aggregate mass of voters that vote for Candidate 0 (candidate 0's market share) equals the mass of idiosyncratic agents receiving utility shocks that result in the optimal action "vote for candidate 0." In this 2 candidate (2 good) market, that mass is the double integral over the bi-variate density distribution f($\varepsilon_{i0}, \varepsilon_{i1}$).

Predicted Vote Share for Candidate 0 ≡ $b_0(\vec{\beta}, \vec{\delta})$

$$b_0(\vec{\beta}, \vec{\delta}) = \int_{\underline{\varepsilon_{io}}}^{\infty} \int_{-\infty}^{(\vec{x}_0 - \vec{x}_1)\cdot\vec{\beta}' + (\delta_0 - \delta_1) + \varepsilon_{i0}} dF(\varepsilon_{i0}, \varepsilon_{i1})$$

Where $\underline{\varepsilon_{io}} = -\vec{x}_0 \cdot \vec{\beta}' - \delta_0$

We are now able to estimate the effects of each of the factors of vote share $\widehat{\Theta} = (\vec{\beta}, \vec{\delta})$ using observed aggregate district vote shares. The estimation equation using one candidate is:

$$\widehat{\Theta} = argmin_{\Theta=(\vec{\beta},\vec{\delta})} \sum_{i=1}^{N} [b_0^{data} - \int_{-\vec{x}_0 \cdot \vec{\beta}' - \delta_0}^{\infty} \int_{-\infty}^{(\vec{x}_0 - \vec{x}_1) \cdot \vec{\beta}' + (\delta_0 - \delta_1) + \varepsilon_{i0}} dF(\varepsilon_{i0}, \varepsilon_{i1})]^2$$

Assuming the error terms are distributed extreme value simplifies the previous calculations. In this case, the double integral over possible epsilon shock pairs, representing Candidate 0's predicted vote share, reduces to

$$b_0(\vec{\beta}, \vec{\delta}) = \frac{\exp(\vec{x}_0 \cdot \vec{\beta}' + \delta_0)}{1 + \exp(\vec{x}_0 \cdot \vec{\beta}' + \delta_0) + \exp(\vec{x}_1 \cdot \vec{\beta}' + \delta_1)}$$

Using log expenditure, rather than expenditures directly, simplifies the algebra

$$b_0(\vec{\beta}, \vec{\delta}) = \frac{\exp(\vec{x}_{I_o} \cdot \vec{\beta}'_{I_o} + \delta_0) \cdot x^{\rho}}{1 + \exp\left(\vec{x}_{I_o} \cdot \vec{\beta}'_{I_o} + \delta_0\right) \cdot x^{\rho} + \exp(\vec{x}_{I_1} \cdot \vec{\beta}'_{I_1} + \delta_1) \cdot y^{\rho}}$$

Consequently, the estimation equation also simplifies. We follow the argument of Rusk (89) to justify the assumption of extreme-value distribution over idiosyncratic utility shocks.

Last, the model also predicts Republican vote share similarly. Estimating both simultaneously using GMM is feasible given the model's prediction for mass of agents that don't vote. The GMM parameter estimation equation under the bi-variate:
Under extreme-value error distribution the parameter estimation equation:

$$min_{\Theta=(\vec{\beta},\vec{\delta})} \sum_{i=1}^{N} [b_0^{data} - \frac{\exp(\vec{x}_0 \cdot \vec{\beta}' + \delta_0)}{1 + \exp\left(\vec{x}_0 \cdot \vec{\beta}' + \delta_0\right) + \exp(\vec{x}_1 \cdot \vec{\beta}' + \delta_1)}]^2$$

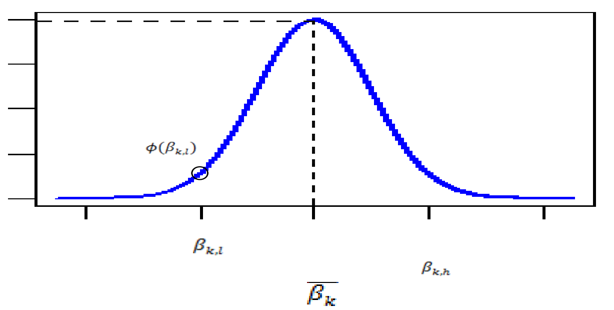
Following Berry (94) this system is invertible and yields a linear estimation equation. We employ the linear regression for the IV and endogeneity tests in section 7.

4.2 Random Coefficients

Prior to estimating this equation, we specify exactly how incumbency, campaign expenditures, and voter registration statistics enter. BLP (95) introduces random coefficients into their regression equation under the argument that different product characteristics can influence heterogeneous consumers differently. In this vote share model we also have heterogeneous agents; those that register Democrat, those that register Republican, and those that register non-partisan.

In BLP, when product characteristic k's coefficient, β_k, is assumed to be random, for example distributed normally as pictured below, the interpretation is as follows:

Figure 10

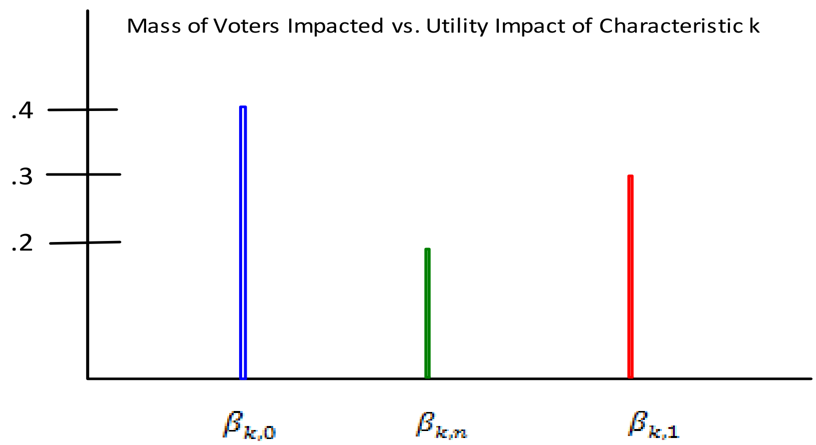


Most consumers' utility are influenced by product characteristic k according to $\overline{\beta_k}$ (a change in characteristic k has a $\overline{\beta_k}$ effect on utility and thus also an observed market share effect). $\phi(\beta_{k,l})$ of consumers are utility influenced differently by characteristic k, by amount $\beta_{k,l}$.

In our vote share model, the random coefficients are slightly simpler. We do not have a continuum of heterogeneous types. Rather, we only have 3 types of agents; those that register Democrat, those that register Republican, and those that do register non-partisan. With only three types of agents, we only need a discrete 3-point distribution of types. Moreover, our data provides the relative populations of each for each district.

In district i where 40% of registered voters register Democrat, 35% Republican, and 25% non-partisan, for candidate characteristic k, the distribution of voters' impacted vs. impact is as follows (where the support values are what is to be estimated).

Figure 11



Agent heterogeneity enters the utility expression as follows.

The utility for an agent from voting for the Democrat candidate, x_0:

$$U(x_0) = .4(\vec{x}_0 \cdot \vec{\beta}'_0 + \delta_{0,0}) + .25(\vec{x}_0 \cdot \vec{\beta}'_n + \delta_{0,n}) + .35(\vec{x}_0 \cdot \vec{\beta}'_1 + \delta_{0,1}) + \varepsilon_0$$

An agent in the district has a .4 chance of being an agent that registered Democrat, in which case his utility is influenced according to the first set of parameters. Similarly, .25 of the voters in the district are registered non-partisan (n), in which case their utility is influenced according to the second set of parameters. Last, the registered Republicans are influenced by the third set. The candidate characteristics consist of the candidate's incumbency status and campaign expenditures. An agent's utility by voting for candidate x_0: $U(x_0) = x_{r,0} \cdot (x_{I_o} \cdot \beta_{I_{0,0}} + x_{e_o} \cdot \beta_{e_{0,0}} + \delta_{0,0}) + x_{r,n} \cdot (x_{I_o} \cdot \beta_{I_{0,n}} + x_{e_o} \cdot$

$e0,n+\delta 0,n \qquad +xr,1\cdot xIo\cdot \beta I0,1+xeo\cdot \beta e0,1+\delta 0,1+\varepsilon 0$

$\beta_{e_{0,n}}$ is the utility impact on non-partisan registered voters from Democratic expenditures. $\beta_{I_{0,1}}$ is the utility impact (and thus observed vote share impact) on Republican-registered voters from the Democratic candidate being the incumbent.

The mean utility to voters registered Democrat from the econometrically unobserved characteristics for the Democratic candidate is $\delta_{0,0}$. The mean utility from the unobserved characteristics to Republican registered voters for the same candidate is $\delta_{0,1}$.

Inserting these random coefficient equations into the general estimation equation becomes unsightly and is only presented in the appendix. Assuming the extreme-value distribution on the error term, we can again provide the closed-form predicted vote share for the Democratic candidate,

$$b_0(\vec{\beta}, \vec{\delta}) = \frac{\exp(x_{r,0} \cdot (x_{I_o} \cdot \beta_{I_{0,0}} + x_{e_o} \cdot \beta_{e_{0,0}} + \delta_{0,0}) + x_{r,n} \cdot (x_{I_o} \cdot \beta_{I_{0,n}} + x_{e_o} \cdot \beta_{e_{0,n}} + \delta_{0,n}) + x_{r,1} \cdot (x_{I_o} \cdot \beta_{I_{0,1}} + x_{e_o} \cdot \beta_{e_{0,1}} + \delta_{0,1}))}{1 + \sum_j \exp[\vec{x}_j \cdot \vec{\beta}_j{}' + \vec{\delta}_j]}$$

4.3 Estimation Equation

The model predicts fractions of registered voters that vote Republican, vote Democrat, and that do not choose to vote. GMM easily allows us to use two of these three predictions to identify model parameters for generic shock distributions. Further, we can easily accommodate instrument specifications via GMM. In particular, our general estimation set up takes the following form:

$$\widehat{\Theta} = argmin_{\Theta=(\vec{\beta},\vec{\delta})} \sum_{i=1}^{N} m_i(\Theta)' \cdot \Omega \cdot m_i(\Theta)$$

Where

$$m_i(\Theta) = \begin{matrix} b_0^{\text{data}} - b_0^{\text{predicted}} \\ b_1^{\text{data}} - b_1^{\text{predicted}} \\ instrument\ restrictions \end{matrix}$$

and Ω= optimal weighting matrix

5 Methodology Comparison

The relationship between the two estimation methodologies is that the reduced-form approach is a special case of the discrete demand approach with several restrictive assumptions in place.

1. The reduced-form approach stipulates that a positive fraction of voters registered to the party necessarily vote for that party. Within the framework of BLP, this stipulation implies that for some agents there does not exist a pair of epsilon shocks large enough to induce the agent to either vote for the other party's candidate, or abstain. Mathematically, the stipulation imposes a restriction on the epsilon shock distributions; that the distributions have zero mass beyond some regional boundary.
2. As currently specified, the reduced-form equation says that both parties get the same fraction of base voters from among their registered voters after controlling for incumbency. Within the BLP framework, this requires $\delta_{0,0} = \delta_{1,1}$ and $\delta_{0,1} = \delta_{1,0}$. Democratic registered voters get the same unobserved base-level utility from Democratic candidates as Republican registered voters get from the Republican Candidate, and that Democratic voters get the same unobserved base-level utility from Republican candidates as Republican voters get from Democratic candidates.
3. The reduced-form equation stipulation that all non-base swing voters are subject to the same f(x,y) apportionment rules is equivalent to the BLP framework with the restriction that $\beta_{e_{0,0}} = \beta_{e_{1,1}} = \beta_{e_{0,1}} = \beta_{e_{1,0}} = \beta_{e_{0,n}} = \beta_{e_{1,n}}$.
4. The reduced-form stipulation that both parties registered voters are influenced equivalently by incumbency implies that $\beta_{I_{0,0}} = \beta_{I_{1,1}}$.
5. The stipulation that non-partisan voters are not influenced by incumbency, nor are Republican voters by Democratic incumbency and Democratic voters by Republican incumbency requires the BLP coefficient restriction that $\beta_{I_{0,n}} = \beta_{I_{0,n}} = \beta_{I_{0,n}} = \beta_{I_{0,n}} = 0$.

6. Assumptions about the joint distribution of candidate shocks imply an exact strategic relationship on relative expenditure levels. For example, assuming the shocks are distributed logit implies that larger expenditures by the Democratic shifts the indifference boundary (fig pg 6) yielding more mass (hence vote share) in favor of the Democratic candidate: how much specifically depends on the logit assumption.

6 Data

The original data set consists of US Congressional district-level voter registration statistics by party, the party of incumbency, and candidate disbursements from the 2004 Congressional House elections. The voter registration statistics come from the states' Secretary of State Offices, and incumbency and disbursements from the Federal Election Commission. Summary Statistics:

Table 11

Variable	Obs	Mean	Std. Dev.	Min	Max
Dem Incumbent	105	0.4571429	0.5005491	0	1
Rep Incumbent	105	0.4857143	0.502193	0	1
% Registered Dem	105	0.4157627	0.1108439	0.24358	0.69135
% Registered Rep	105	0.3464863	0.1059193	0.1006	0.62965
% Registered Other	105	0.2377509	0.0763271	0.08755	0.4882
Dem Expenditure	105	688288.3	636094.2	1994	2752272
Rep Expenditure	105	688288.3	636094.2	1994	2752272
% Dem Votes	105	0.3490476	0.1171663	0.07	0.62
% Rep Votes	105	0.3248571	0.1345871	0.06	0.59
% Not Voting	105	0.3254364	0.088934	0.0012909	0.6044151
Dem Expend 2002	93	692776.9	594569.9	6757	2985329
Rep Expend 2002	90	829494.9	1079567	1532	8150237
Senate Race	105	0.9047619	0.2949514	0	1

We collected data on 154 Congressional districts from 17 states. We do not have data on districts from states that fit at least one of the following categories

- charge for voter registration statistics
- do not keep voter registration statistics by party
- keep voter registration statistics by county, yet have multiple districts per county.

From these states, we only use districts in which

- Two candidates run, one Republican one Democrat
- Both make concerted expenditures, above $1000.

Candidate expenditure level data comes from the Federal Election Commission databases http://www.fec.gov/press/press2005/20050609candidate/house.pdf. We define candidate expenditures as their "net disbursements." The graph displays variation in relative expenditure levels across districts. The variation is sufficient to non-parametrically identify the coefficients of the Taylor approximation of the advertising technology function.

7 Results

7.1 Reduced-Form Estimates

To estimate the non-parametric component of our regression equation, we must specify a known value for our Taylor approximation to be an approximation around. We stipulate that when both candidates expend the mean level of expenditures, they split the advertising response swing mass evenly. This translates into a 2^{nd}-order Taylor approximation around the point ($775K, $775K, 1/2). We tried alternate specified values and they did not greatly change the results and are presented in the appendix.

The regression equation results are

$$b_0^f = (\gamma_b + \gamma_I \cdot \mathbb{I}_{I_0}) \cdot b_0^r + \gamma_\mu \cdot [1 - (\gamma_b + \gamma_I \cdot \mathbb{I}_{I_0}) \cdot b_0^r - (\gamma_b + \gamma_I \cdot \mathbb{I}_{I_1}) \cdot b_1^r] \cdot f(x,y) + \varepsilon$$

Table 12

b_0^f	Robust		R-squared =	0.971
	Coef.	**Std. Err.**	**t**	**P>|t|**
$\boldsymbol{\gamma_b}$	-0.0844223	0.1256908	-0.67	0.503
$\boldsymbol{\gamma_I}$	0.4356608	0.0736011	5.92	0
$\boldsymbol{\gamma_\mu}$	0.6536725	0.047061	13.89	0
$\boldsymbol{f_x}$	2.23E-08	3.05E-08	0.73	0.465
$\boldsymbol{f_y}$	3.92E-09	3.56E-08	0.11	0.913
$\boldsymbol{f_{xx}}$	-2.73E-14	2.17E-14	-1.26	0.21
$\boldsymbol{f_{yy}}$	1.66E-14	1.32E-14	1.26	0.211
$\boldsymbol{f_{xy}}$	9.46E-15	2.43E-14	0.39	0.698

The estimates of γ_b and γ_I says that on average, candidates receive 34% of their registered party voters and another 45% if they are the incumbent. While the estimated advertising technology function is found to be non-linear and thus have non-constant effects, we can still make intuitive inferences from the signs of the coefficients.

Table 13

$f_x > 0$	Larger expenditures earn you more votes
$f_y < 0$	Larger expenditures by your opponent are a detriment to your votes
$f_{xx} < 0$	Decreasing marginal returns to advertising
$f_{yy} > 0$	Increasing marginal returns to opponent advertising

The non-parametric estimates of the coefficients of the Taylor approximation strongly suggest that the advertising technology is consistent with a common advertising technology specification from previous literature

$$f(x, y) = \frac{x^\sigma}{x^\sigma + y^\sigma}$$

The 2nd order Taylor approximation coefficients of $f(x, y) = \frac{x^\sigma}{x^\sigma + y^\sigma}$ at (x, y=x, .5) are

Table 14

f_x	f_y	f_{xx}	f_{yy}	f_{xy}
$\frac{\sigma}{4x}$	$-\frac{\sigma}{4x}$	$-\frac{\sigma}{4x^2}$	$\frac{\sigma}{4x^2}$	0

These coefficients are very similar to our estimated coefficients in signs and in relative magnitudes. In fact, given the point we approximate around, equal x and y at mean the expenditure level (\$775K≅ $1e^6$), a $\sigma \cong .1$ yields all appropriate orders on the coefficients. In addition, the estimated coefficient on f_{xy} is insignificant, consistent with the approximation.

The following table provides all the parameter estimates when the advertising technology is replaced with this one-parameter common advertising technology. Note that the structural parameter estimates remain the same.

Table 15

b_0^f	Robust		R-squared =	0.9698
	Coef.	**Std. Err.**	**t**	**P>|t|**
γ_b	-0.0562745	0.1151149	-0.49	0.626
γ_I	0.3930532	0.0536565	7.33	0
γ_μ	0.6494347	0.0423069	15.35	0
σ	0.0347582	0.0299608	1.16	0.249

Of interest, the decreasing marginal returns to own advertising are consistent with the notion that swing voters actually have a distribution where some are closely ideologically aligned with your party and others are less. The first dollar wins over the swing voters nearest your platform, and it becomes increasingly harder to win over candidates further away from your platform, to compel them to traverse more ideological distance and vote for your party.

Running the same regressions to identify the parameters using the Republican vote shares yields very different estimates.

Table 16

b_1^f	Robust		R-squared =	0.9763
	Coef.	**Std. Err.**	**t**	**P>|t|**
γ_b	0.4994873	0.0731154	6.83	0
γ_I	0.2664751	0.0529905	5.03	0
γ_μ	0.4010537	0.0469246	8.55	0
f_x	1.95E-07	6.80E-08	2.87	0.005
f_y	-1.19E-07	6.70E-08	-1.77	0.08
f_{xx}	-8.45E-14	2.77E-14	-3.05	0.003
f_{yy}	4.06E-14	6.56E-14	0.62	0.538
f_{xy}	6.65E-14	5.75E-14	1.16	0.25

Either the reduced-form specification is the wrong functional form, republican voters respond to factors differently and the specification lacks sufficient parameters, or both. We now present the discrete-demand model which has the capability to handle each of these possibilities.

7.2 <u>Discrete Demand Estimates</u>

A strong attribute of the discrete-demand model with random coefficients is the large number of interpretable parameters. Of course, restrictions can always be imposed in order to reduce the number of parameters and improve standard errors, and the restrictions themselves can be tested. Ideally, as more data becomes available, we would like to re-estimate the model again allowing all factors to effect voters heterogeneously, the whole suite of 24 parameters, as well as distribution parameters on the tri-variate utility shock distribution. In the interim, we provide the specification most similar to the reduced-form specification:

Normalized Multi-variate logistic with log(expenditures)

- $\delta_{0,0} = \delta_{1,1},\ \delta_{0,1} = \delta_{1,0}$
- $\beta_{e_{0,0}} = \beta_{e_{1,1}} = \beta_{e_{0,1}} = \beta_{e_{1,0}} = \beta_{e_{0,n}} = \beta_{e_{1,n}}$
- $\beta_{I_{0,0}} = \beta_{I_{1,1}}$
- $\beta_{I_{0,n}} = \beta_{I_{0,n}} = \beta_{I_{0,n}} = \beta_{I_{0,n}} = 0$

Table 17

W restrictions	Coef.	Std. Err.	z	P>|z|
β_{I_0}	0.3177208	0.1162822	2.73	0.006
β_s	0.1211039	0.1657457	0.73	0.465
$\beta_{\ln(e_0)}$	0.0862229	0.0475662	1.81	0.07
δ_0	-1.292568	0.6175504	-2.09	0.036
β_{I_1}	0.3180796	0.1656882	1.92	0.055
β_s	0.1211039	0.1657457	0.73	0.465
$\beta_{\ln(e_1)}$	0.2072883	0.0646763	3.21	0.001
δ_1	-2.977698	0.7974894	-3.73	0

7.3 Endogeneity Tests

Given the challenges of employing instrumental variables in non-linear regressions, we defer to the discrete-demand model's linear regression (multivariate logistic shocks under inversion) to test the hypothesis that inclusion of voter registration statistics mitigates the endogeneity of candidate expenditures. We provide the factor impact parameter estimates for

1. the model without registration statistics with and without lagged expenditures to instrument for campaign expenditures
2. the model with voter registration statistics with and without lagged expenditures

The first set of estimates is without using voter registration statistics or lagged expenditures. The second set is without voter registration statistics, but with lagged expenditures to instrument for expenditures. The difference between the results indicates the presence of endogeneity. An unaccounted-for regressor, such as candidate candor, is co-causal to both vote-share and expenditures.

Table 18

no IV no vr stat	**Coef.**	**Std. Err.**	**z**	**P>|z|**
β_{I_0}	0.5198979	0.0555395	9.36	0
β_s	0.1752861	0.1335472	1.31	0.189
$\beta_{\ln(e_0)}$	0.0722224	0.0185874	3.89	0
δ_0	-1.284006	0.2522709	-5.09	0
β_{I_1}	0.5198979	0.0555395	9.36	0
β_s	0.1752861	0.1335472	1.31	0.189
$\beta_{\ln(e_1)}$	0.0722224	0.0185874	3.89	0
δ_1	-1.421965	0.2544053	-5.59	0

Table 19

IV no v r stats	**Coef.**	**Std. Err.**	**z**	**P>|z|**
β_{I_0}	0.3604734	0.1064815	3.39	0.001
β_s	0.1118347	0.1779473	0.63	0.53
$\beta_{\ln(e_0)}$	0.127989	0.0475618	2.69	0.007
δ_0	-1.835196	0.6185379	-2.97	0.003
β_{I_1}	0.3604734	0.1064815	3.39	0.001
β_s	0.1118347	0.1779473	0.63	0.53
$\beta_{\ln(e_1)}$	0.127989	0.0475618	2.69	0.007
δ_1	-1.978821	0.6154552	-3.22	0.001

The Hausman test of Endogeneity (test of estimate equality) demonstrates that, with >98% confidence, we fail to reject the presence of endogeneity. For instance, $\hat{\rho}$, the exponent of expenditures under the first specification is .07, but after instrumenting, is .12 .

$$b_0(\vec{\beta},\vec{\delta}) = \frac{\exp(\vec{x}_{I_o} \cdot \vec{\beta}'_{I_o} + \delta_0) \cdot x^{\rho}}{1 + \exp\left(\vec{x}_{I_o} \cdot \vec{\beta}'_{I_o} + \delta_0\right) \cdot x^{\rho} + \exp(\vec{x}_{I_1} \cdot \vec{\beta}'_{I_1} + \delta_1) \cdot y^{\rho}}$$

Endogeneity biases the estimate of the returns of advertising to be too nonlinear; too large an impact of expenditures on vote-share for low dollar expenditures, and too high an impact on vote-share for high dollar expenditures, relative to your opponent.

Now, introducing voter heterogeneity instead of the lagged expenditures, we achieve similar estimates as we got when we instrumented with lagged expenditures.

Table 20

Vote Reg stats no IV	Coef.	Std. Err.	z	P>|z|
β_{I_0}	0.3177208	0.1162822	2.73	0.006
β_s	0.1211039	0.1657457	0.73	0.465
$\beta_{\ln(e_0)}$	0.0862229	0.0475662	1.81	0.07
δ_0	-1.292568	0.6175504	-2.09	0.036
β_{I_1}	0.3180796	0.1656882	1.92	0.055
β_s	0.1211039	0.1657457	0.73	0.465
$\beta_{\ln(e_1)}$	0.2072883	0.0646763	3.21	0.001
δ_1	-2.977698	0.7974894	-3.73	0

For a district with mean voter registration statistics, we now estimate $\hat{\rho}$ to be .12, just as was estimated using lagged expenditures as an estimate. In fact, the Hausman tests rejects the null: the parametric estimates are systematically different.

Thus, we conclude that controlling for co-causal influence on vote share and expenditures using district-level partisan voter registration statistics mitigates as much endogeneity-based estimate bias as does using the instrument lagged expenditures.

8 Conclusion

Methodologically, the generality provided by the discrete-demand model makes it strongly preferable for estimating vote impact effects. Previous reduced-form equations can still be well approximated by imposing restrictions on the discrete-model, but by starting with the discrete demand model the specific restrictions can be tested to see if they are justified. In this paper, even though our reduced form specification is semi-parametric (general for a reduced-form specification), and delivers impact effect estimates consistent with previous literature, we are able to test the restrictions needed to generate the reduced-form specification. The data rejects the imposition of the restrictions.

Moreover, introducing partisan district voter registration statistics through random-coefficients provides a natural insertion of voter heterogeneity. Incorporating registration statistics in this fashion provides interpretable parameters. Also, not having to estimate random coefficient distribution parameters yields more significant estimates for the other parameters.

While we find that impact on overall vote share by candidate expenditures is nonlinear, and at mean expenditure levels the marginal impact is quite small, most of the action of campaign expenditures occurs on the boundary of the non-voting region. Campaign expenditures are more successful at convincing non-voters to vote ('get out the vote' dollars) than at convincing voters with tendencies for the opponent to vote for you. In fact, since we estimate the mean utility for a voter to vote for either candidate to be negative (intuition: without expenditures voters prefer not to vote), expenditures are largely responsible for voter turnout, especially for the challenger. Policy implications of this finding are ambiguous. Expenditures are important but have decreasing marginal impact, and most of their influence is in generating larger voter turnout (itself a controversial topic).

Incumbency is a large significant advantage for a candidate. Normatively, the welfare effect is not well understood. On one hand, with incumbency candidates can commit to longer-term efforts. On the other, decreasing the importance of incumbency results in more competitive electoral races and truer representation.

Less ambiguous, we find that district demographics are an important factor in vote results. Voters with different registration characteristics have different preferences and respond differently to candidate characteristics; incumbency and campaign expenditures. State legislatures are responsible for drawing their state's congressional boundaries (districting). It is well known that these state legislatures can and do manipulate the boundaries to benefit the party in power in the state. This paper demonstrates that voter heterogeneity is large, and thus that the party in power in state legislatures hold real federal political power.

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10 Appendix: Strategic Voting ('Blocking'; the Williams Critique)

Product demand is not perfectly analogous to candidate demand. A consumer must not always obtain a product, yet a voter always gets a representative. The difference manifests itself when considering a voter's incentives to vote for a particular candidate; to vote for a candidate a voter must either receive a lot of utility from voting for that candidate, or be very against being represented by the other candidate. Framing the critique econometrically: interact tastes for voting with the difference between the random utilities of having a particular candidate in office. Modification of the discrete-demand model proceeds as follows:

- The utility from voting Democratic:

$$U_i(x_0) = \vec{x}_0 \cdot \vec{\beta}' + \delta_0 + \varepsilon_{i0}$$

- The utility from voting Republican:

$$U_i(x_1) = \vec{x}_1 \cdot \vec{\beta}' + \delta_1 + \varepsilon_{i1}$$

Conditional on voting, a voter always votes for the candidate he would prefer to have as his representative. We now consider the utility from not voting vs. voting.

- The utility from voting:

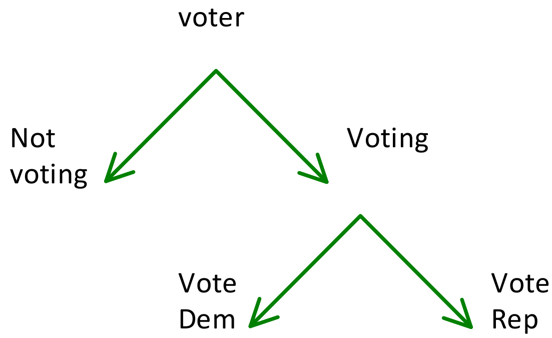
$$U_i(x_v) = prob(pivotal) \cdot [U_i(x_+) - U_i(x_-)] + \delta_v + \varepsilon_{iv}$$
$$U_i(x_+) = \max\{U_i(x_0), U_i(x_1)\}$$
$$U_i(x_-) = \min\{U_i(x_0), U_i(x_1)\}$$

- Normalizing the utility from not-voting to zero.

{We do not explicitly solve or estimate the model, but in doing so, the prob(pivotal) would be a function of the district's voter registration statistics, incumbency, and relative expenditures.}

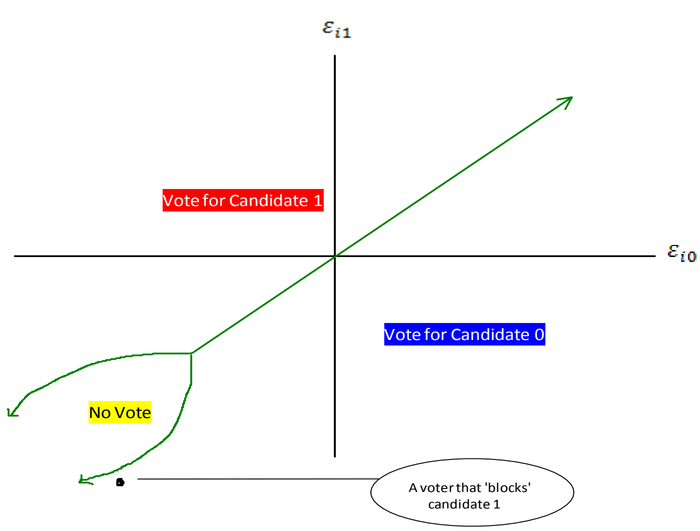
The voter's decision tree:

Figure 12



Solving the voter's participation and incentive indifference conditions yields a similar graphical characterization of what masses of voters choose {vote Dem, vote Rep, no vote} for a given shock distribution. The difference in the model manifests itself in a new non-linear boundary between not-voting and voting Democrat, and not-voting and voting Republican. The intuition is the same as the motivation of the model. If a voter receives a really bad shock for a candidate, he will vote for the other candidate. Since he is going to get a candidate, the voter chooses to strategically block the candidate he has severe distastes for.

Figure 13



Chapter 3: Colonel Blotto Strategies

A Complete Characterization:
Balance Between Balance and Imbalance

Abstract

In Gross and Wagner's (1950) original Colonel Blotto game, two officers must simultaneously decide how to best allocate their finite endowment of resources across multiple battlefields. In their original version of the game (2 battlefields and plurality objectives) the equilibrium payoffs and some equilibrium strategies are known. This paper finishes the characterization by deriving the *complete* set of Nash Equilibrium strategies. As players' resource endowments converge, the set of equilibrium converges and suggests an equilibrium selection. We provide an algorithm for characterization, prove its completeness, and discuss extensions of the analysis.

1 Introduction

The Colonel Blotto game is a constant-sum game of allocation mismatch with application in military campaigns, political campaigns, network defense, and strategic hiring situations such as pro sports and the economics job market. In Gross and Wagner (1950) two officers, Colonel Blotto and Enemy, with potentially asymmetric resources of soldiers B and E, compete in multiple battlefields. The officers must decide how to allocate their limited resources (Air Force assets) across battlefields simultaneously. The officer with more air-assets at a particular field wins that battle. Their objective is to win battles.

Let b_i & e_i denote Blotto's &Enemy's respective allocations to battlefield *i*.

Asymmetric resources, Plurality game:

The probabilities Colonel Blotto wins on each battlefield:

Figure 14

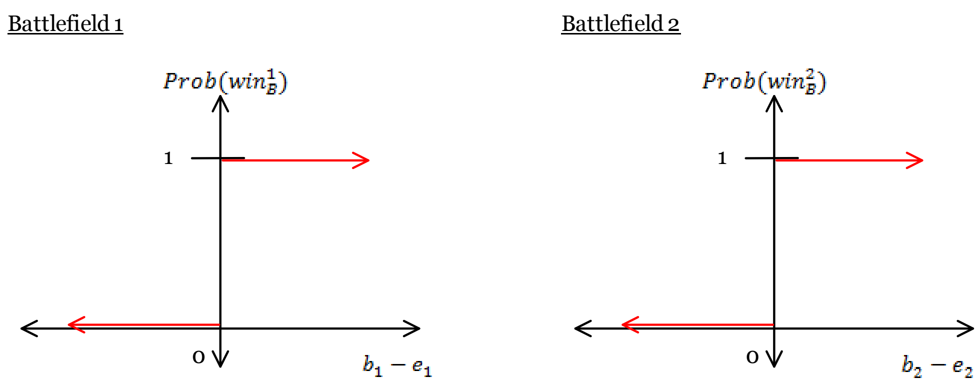


Table 21

	Blotto	Enemy
Objective	$max_{b_1,b_2} \sum_{i=1}^{2} Prob(win_B^i)$	$max_{e_1,e_2} \sum_{i=1}^{2} Prob(win_E^i)$
Constraint	$\sum_{i=1}^{2} b_i \leq B$ & $b_i \geq 0 \,\forall\, i$	$\sum_{i=1}^{2} e_i \leq E$ & $e_i \geq 0 \,\forall\, i$

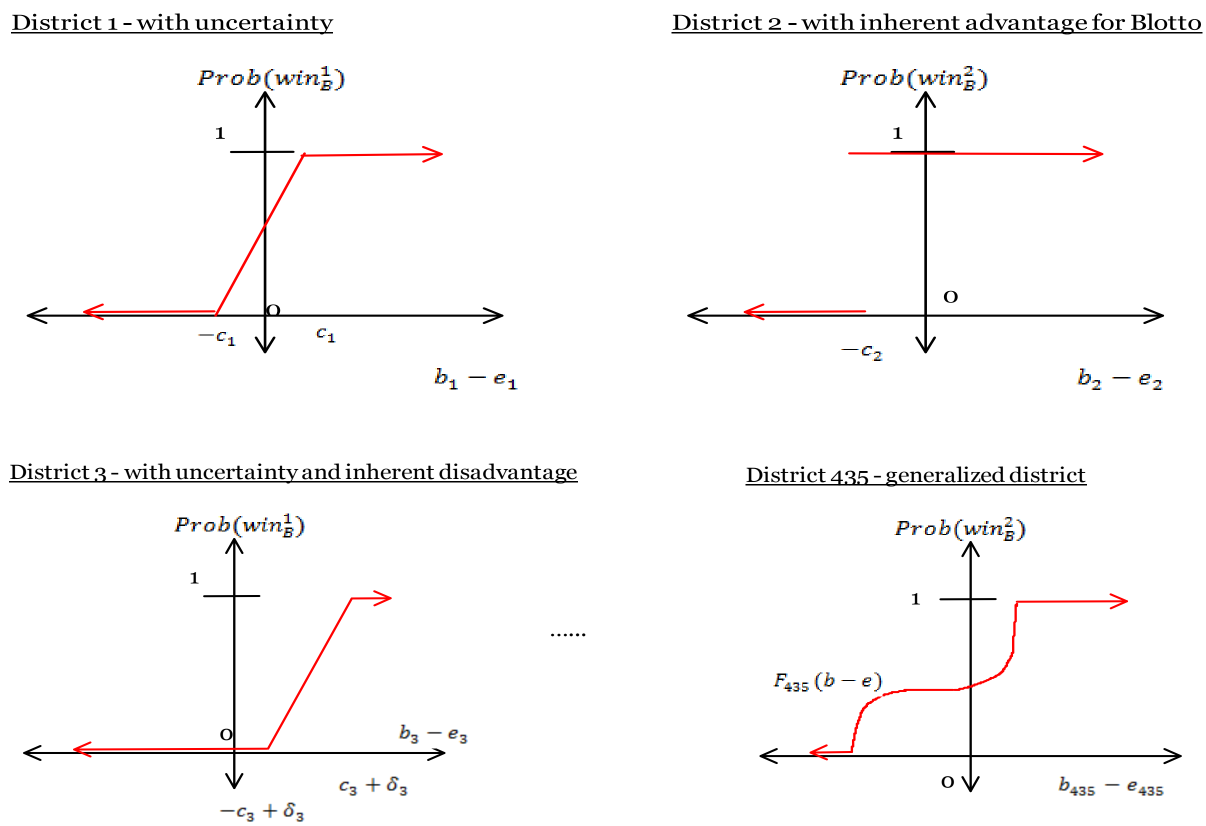
In case of a tie, the battlefield goes to the officer with greater resources. This assumption is standard in the literature (Kvasov) and in equilibrium never occurs.

In the 2008 Congressional elections, the Democratic Party allocated \$748M across 435 districts. The Republican Party allocated \$713M across the same set of districts. Suppose some districts inherently favor the Democrats while others favor the Republicans. Suppose further that that there is uncertainty in the environment; the resource mismatches do not perfectly explain election results in each particular district. Even if Republicans outspend Democrats in a district, the probability Republicans win that district might still be less than one.

Asymmetric Resources, 435 districts, Uncertainty and Battlefield Advantages:

The probabilities Colonel Blotto wins in each district:

Figure 15



Even more confounding, Colonel Blotto's political campaign may have tastes for majority instead of straight plurality. The political party's payoff function may be:

$$max_{\{b_d\}_{d=1}^{D}} (1-\alpha) \sum_{d=1}^{D} \mathbb{I}_{b_d > e_d} + \alpha \mathbb{I}_{majority}$$

Last, it is also possible that different battlefields may have different importance weights. The ultimate goal for Blotto theorists is to characterize all Blotto equilibria (payoffs, strategies) for these generalized battlefields, and also for varying degrees of complementarities between them: from plurality to the straight majority objective. The problem however is that while the game is intuitive and easy to describe, the complete strategy characterization methodology is involved, even for Gross and Wagner (1950) original game.

In this research paper, we provide an algorithm that completely characterizes all Nash equilibrium strategies and prove the characterization's completeness. As players resource endowments converge, the complete set of equilibrium and suggests an equilibrium selection. Last, we discuss applications of the algorithmic analysis: n>3 battlefields, non-constant battlefield weights, inherent battlefield advantages, and natural Nash equilibrium refinements for the game.

The organization for the remainder of this paper is as follows:

- **Section 2** Provides a brief review of preceding literature and our contribution
- **Section 3** Completely Characterizes the Benchmark-Blotto Nash-strategies
    - **3.1** Region 1 Nash Equilibrium construction
    - **3.2** Region 2 Nash Equilibrium construction
    - **3.3** Strategy Characterization algorithm
    - **3.4** Proof of Completeness or Nash Characterization
- **Section 4** Provides a simple re-derivation of the payoff state space
- **Section 5** Concludes

Section2: Literature Review

Borel (1921) posited the first Colonel Blotto game in the early 1900's. Gross and Wagner (1950) found equilibrium strategies and payoffs for 2 battlefields. Similarly, Roberson (2006) has found equilibrium strategies and payoffs when the number of battlefields is greater than 2. … Our contribution in this paper is to completely characterize the set of Nash strategies for the 2 battlefield game, consider some Nash equilibrium refinements, and discuss the application of the strategy characterization algorithm to the n>2 battlefield game and other variants.

Sahuguet and Persico (2006), Kovenock and Roberson (2007), Kvasov (2007), Golman and Page (2009), Blackett (1958), Bellman (1969), Borel (1921), Roberson (2006), Szentes and Rosenthal (2003), and Weinstein (2005) have all done work on Col Blotto games.

Szentes and Rosenthal (2003) equate the Col Blotto game to a multi-unit all-pay auction with a necessarily binding budget constraint. When Col Blotto has tastes for a majority, the game is comparable to a budget-constrained multi-unit all-pay auction with complementarities across the items. In the limiting case that Col Blotto only cares if he wins the majority, the game is comparable to the budget-constrained all-pay version of the chop-sticks auction.

In characterizing the Col Blotto strategies we use two attributes of constant-sum game Nash equilibrium previously identified by Vega-Redondo (2003, 47-50):

1. Players always obtain the same equilibrium payoff in any Nash equilibrium.
2. Every Nash strategy is a best response to any of the opponent's Nash strategies
    a. This property is generally referred to as **"Equilibrium Interchangeability"**
    b. This result implies the product space of Nash strategies is rectangular.

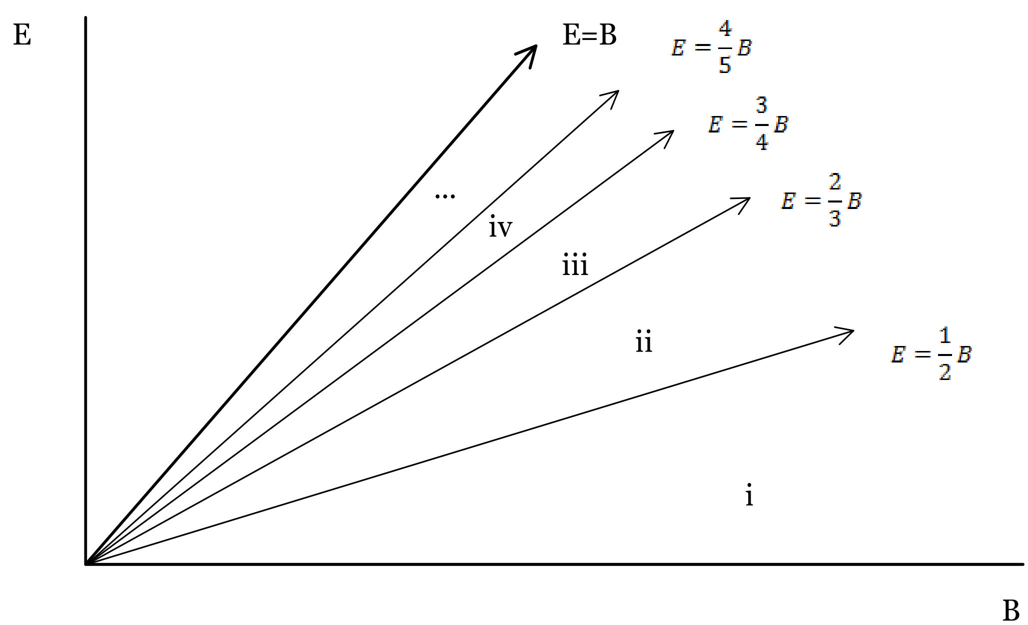
Section3: Complete Nash-Strategy Characterization

In this section we provide the equilibrium payoffs (Roberson 06) for the 2-player 2-battlefied asymmetric-resource Colonel Blotto plurality game. Section 3.1 constructs the necessary conditions for all Nash equilibrium under the case that B>2E (Region 1). Section 3.2 constructs the slightly more complicated necessary conditions for all Nash equilibrium under the case that B<2E, but B>3E/2 (Region 2). Section 3.3 describes the algorithm for identifying all Nash equilibrium for a generic region. Section 3.4 proves that the completeness of the characterization; that the set of conditions imposed by the algorithm is both necessary and sufficient to identify all possible Nash equilibrium strategies to the game.

Figure 15 displays payoffs over the state-space of possible resource-endowments.

Blotto-Enemy Resource State Space

Figure 15



With the following payoffs:

Table 22

Region	Blotto's Expected Payoff	Enemy's Expected Payoff
i	2	0
ii	3/2	1/2
iii	4/3	2/3
iv	5/4	3/4
n	(n+1)/n	(n-1)/n

3.1: Region 1 Characterization

This section completely characterizes the set of Nash equilibrium strategies for when Blotto has more than double the resources of Enemy, B>2E. We first characterize the set of resource-constrained Nash equilibrium strategies, and then relax the assumption to consider all feasible allocations.

The intuition for the region is simple. If Col Blotto has twice the forces of Enemy, he can guarantee himself victory on both battlefields by deploying forces (E+ε, E+ε) and earning payoff of 2. However, there are many other Nash strategies as well. We take the general approach of searching for all resource-constrained mixed-strategies that satisfy first-order necessary conditions.

Colonel Blotto faces the following optimization problem.

$$max_{b_1,b_2} \ \mathbb{I}_{win\, {}^B_1} + \mathbb{I}_{win\, {}^B_2} \quad st \quad b_1 + b_2 \leq B$$

Since left-over resources don't enter the objective function, we temporarily assume the officers' resource constraints binds. We relax this later. Substituting the equality:

$$max_{b_1} \ \mathbb{I}_{b_1 > e_1} + \mathbb{I}_{B - b_1 > E - e_1}$$

Rearranging to isolate the strategic random variable, e_1 (from the perspective of Blotto) yields

$$max_{b_1}\ \mathbb{I}_{b_1>e_1} + \mathbb{I}_{e_1>E-B+b_1}$$

Since we are pursuing a general approach of solving for resource-binding mixed-Nash equilibrium strategies, we define G_B and G_E to represent the two officers' mixing distributions over possible field 1 allocations. Characterizing these CDFs is the objective of our analysis.

For Blotto: $max_{G_B(b_1)} G_E(b_1) + 1 - G_E(E - B + b_1)$

For Enemy: $max_{G_E(e_1)} G_B(e_1) + 1 - G_B(B - E + e_1)$

The intuition for the expressions is sensible. Since CDFs are monotonically increasing, we can see that larger allocations to battlefield 1 increase expected payoff on that battlefield, but also decrease expected payoff on battlefield 2 (since the allocation to battlefield 2 must be smaller) denoted by the negative coefficient on the last term.

Since players earn equal expected payoff (k_B & k_E), for any strategy they mix among:

$$G_E(b_1) + 1 - G_E(E - B + b_1) = k_B$$

$$G_B(e_1) + 1 - G_B(B - E + e_1) = k_E$$

WLOG, assume Blotto has the resource advantage, B>E. Define $\delta = B - E$.

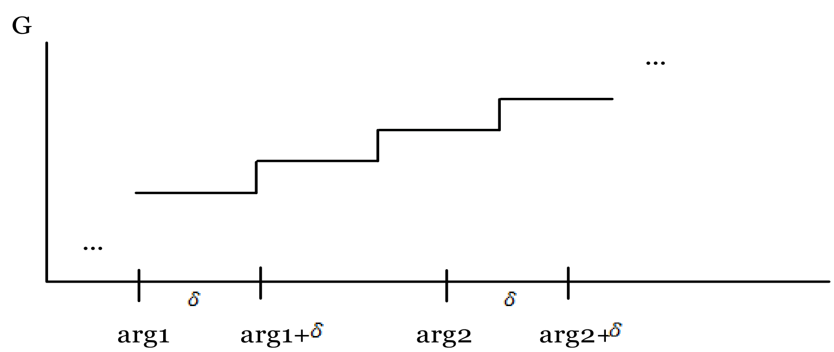
$$G_E(b_1) - G_E(b_1 - \delta) = k_B - 1$$

$$G_B(e_1 + \delta) - G_B(e_1) = 1 - k_E$$

These two equations are necessary conditions for the officers' resource-binding mixed-Nash equilibrium strategies, $G_E$and G_B. As expected, we can see that the advantaged player can always expect payoffs > 1 (disadvantaged<1) since CDFs are weakly increasing. Important for the Nash-strategy set characterization, the officers' optimal resource-binding battlefield-1 allocation mixed strategy must satisfy δ-periodicity.

The necessary conditions state that for any battlefield 1 allocation in Enemy's support, the difference in Blotto's battlefield 1 allocation mixed-strategy cumulative distribution at that allocation and the point δ above is always a constant. A similar statement holds for Enemy. A line (Uniform CDF) satisfies this requirement. In addition, the Heaviside-step-function with standardized step interval length of δ and a constant step-height also satisfies.

Figure 16



In general, any periodic functions with period length of δ or $\frac{\delta}{i}$ $\forall i \epsilon \mathbb{N}$

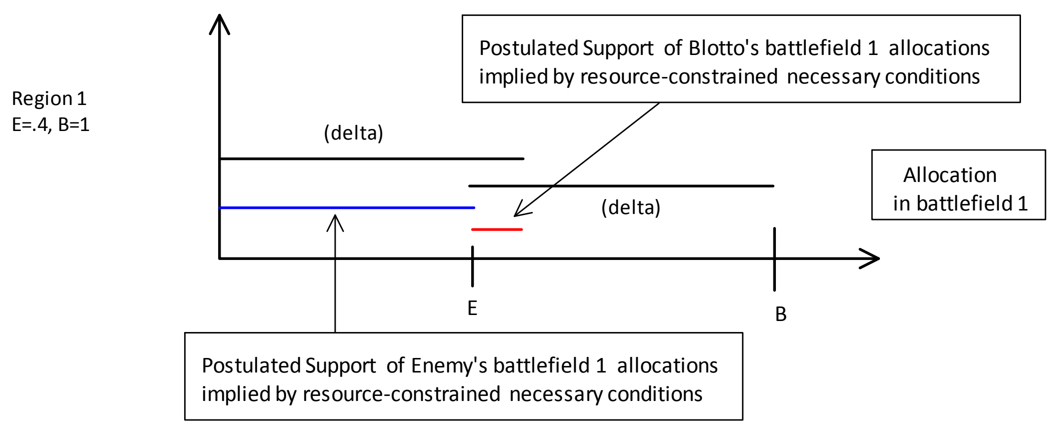
satisfy the FONCs. For example, when $i = 2$ the step-function has period $\frac{\delta}{2}$ and the FONCs are still satisfied. Smooth periodic functions including lines are elements of the class of Heaviside step functions with period length $\frac{\delta}{i}$ when $i \to \infty$.

FONC Property: The essential conclusion is that *every* resource-constrained equilibrium strategy must satisfy δ-periodicity ($i = 1$), and have constant step-height, for all points in the opponents support.

We use the FONC property to conjecture a support, and coarse mixed-Nash mass distribution property, for the players' strategies. The conjectured set of strategies is necessary, proven section 3.4, and is also sufficient in region 1, although subsequent regions require an extra condition for sufficiency.

In the case of region 1, using the fact the equilibrium strategy CDF must be δ-periodic, we postulate the following characterization of the players' supports.

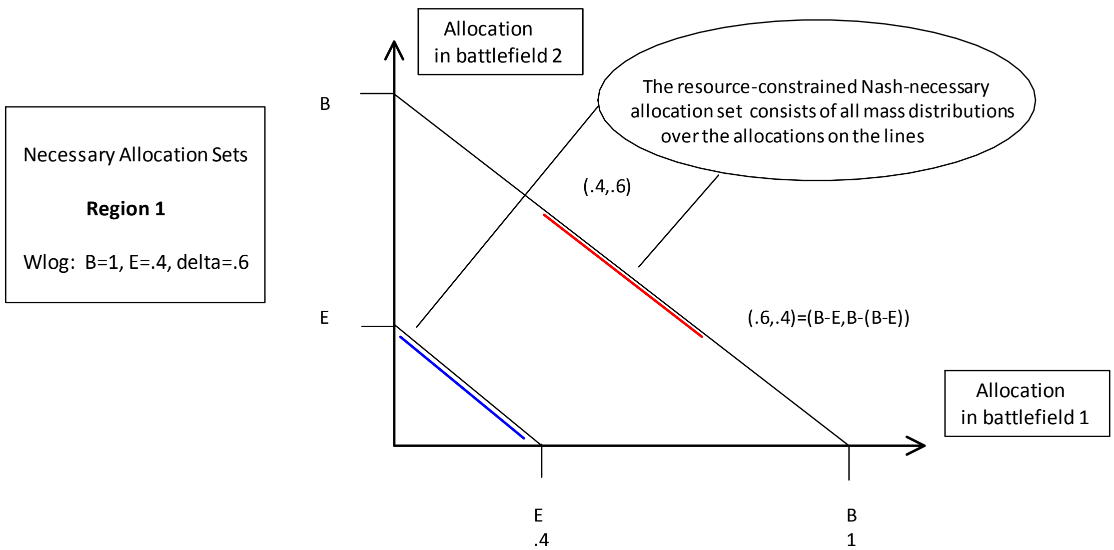
Figure 17



Lemma 1: over the domain [0,E+r] or battlefield 1 allocations, the support of possible equilibrium Blotto allocations and Enemy allocations are disjoint, and their union is full.

Alternatively, we can view this conjectured support of resource-constrained Nash strategies for region 1 on the simplex.

Figure 18



Vega Redondo (03) shows that constant-sum games satisfy **Equilibrium Interchangeability;** any element in a players' set of Nash strategies, when paired with any opponent Nash strategy, are best-responses to each and constitute a Nash equilibrium. Topologically, the product space of the set of Nash strategies is rectangular. Using Equilibrium Interchangeability, we characterize <u>the complete set of resource-constrained Nash equilibrium for region 1</u>:

Table 23

Resource constrained (simplex frontier)	Blotto	Enemy
strategy set	[E,E+r]	[0,E]
strategy payoffs	2	0

The next step of the Region 1 complete strategy characterization construction relaxes the assumption that players only play resource-constrained strategies. Consistent with our intuition that for Blotto to earn expected payoff of 2 he should play at least E in each battlefield, and consistent with our analysis that Blotto should never play any probability mass on allocations that could potentially earn Enemy positive payoff from playing a resource-constrained strategy, we now relax (expand) Blotto's strategy set to include all allocations north-east of (E,E) in Blotto's simplex.

Mathematically, relaxing the assumption that the resource-constraint binds expands our postulated set of Blotto allocations to include all feasible allocations larger than the join of the boundaries of the support of Enemy's strategies. While this now 2-dimensional set is no longer ordered and not representable by a CDF in order to check whether it satisfies FONC property of δ-periodicity over Enemy's support, any of the bivariate-distribution's marginal distributions do. Further, using lemma1, the *join* of Enemy's support boundaries is the *meet* of Blotto.

Last, our construction bounds the region. Again, following the reasoning the Blotto should not play mass on allocations less than the join of Enemy's boundary (in this region (E,E)), we bind the region of Blotto's resource-constraint-relaxed strategies with the lines b1=E and b2=E. Formally, the set of allocations over which our construction allows Blotto to play probability mass is the set of allocations located within the convex-hull of his support boundaries and their *meet*. For Enemy we bound his set of allocations by the same algorithm, which in region 1 is degenerate; it is just the set of all feasible Enemy allocations.

Figure 19

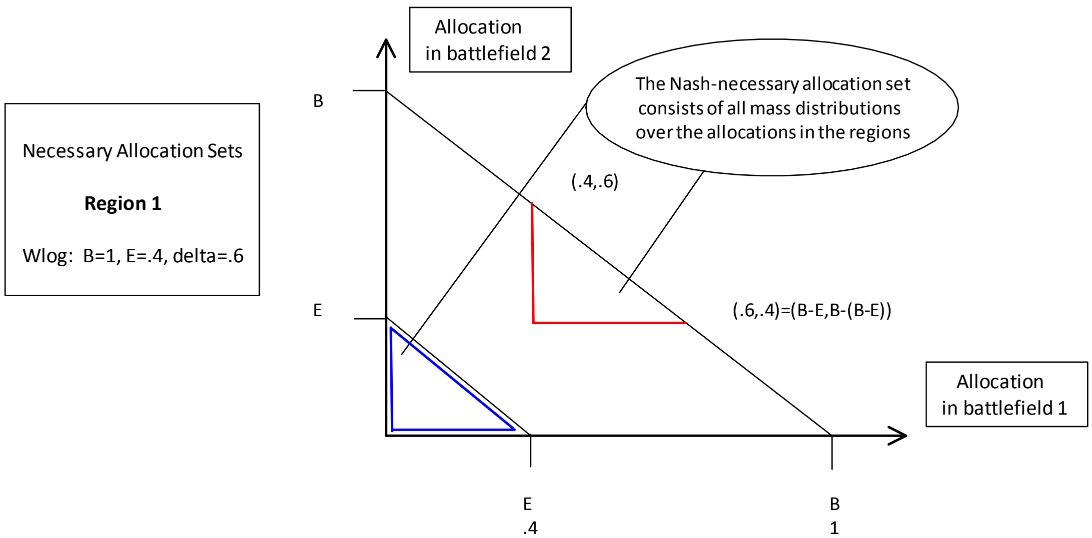


Table 24

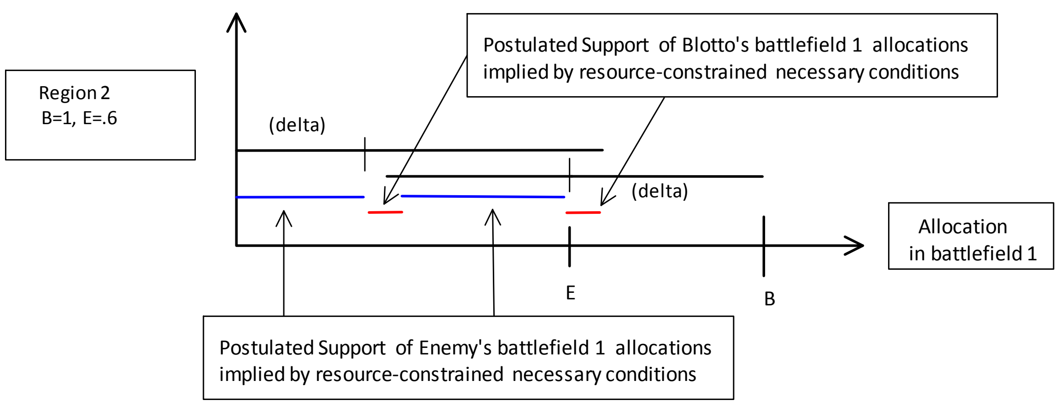
General, all feasible allocations, Region 1	Blotto	Enemy
strategy set	Red Triangle	Blue Triangle
expected payoffs	2	0

3.2 Region 2 Characterization

Region 2 consists of all possible resource endowments where Enemy has more than half of Blotto's resources (E>B/2), but still less than two-thirds (E<2B/3). Blotto's previous (from region 1, section 3.1) resource-minimizing Nash strategy (E,E) is no longer feasible; it now lays north-east of Blotto's feasible allocation simplex frontier. Also, our support construction is slightly different. For resource-endowments in this region, the support-construction component of our strategy characterization algorithm conjectures that Blotto and Enemy each now both have 2 disjoint resource-constrained support cells. In order to satisfy the second characteristic of the FONC Property, that the resource-constrained CDF must have constant step-height, each of these cells must receive equal probability mass (1/2 each).

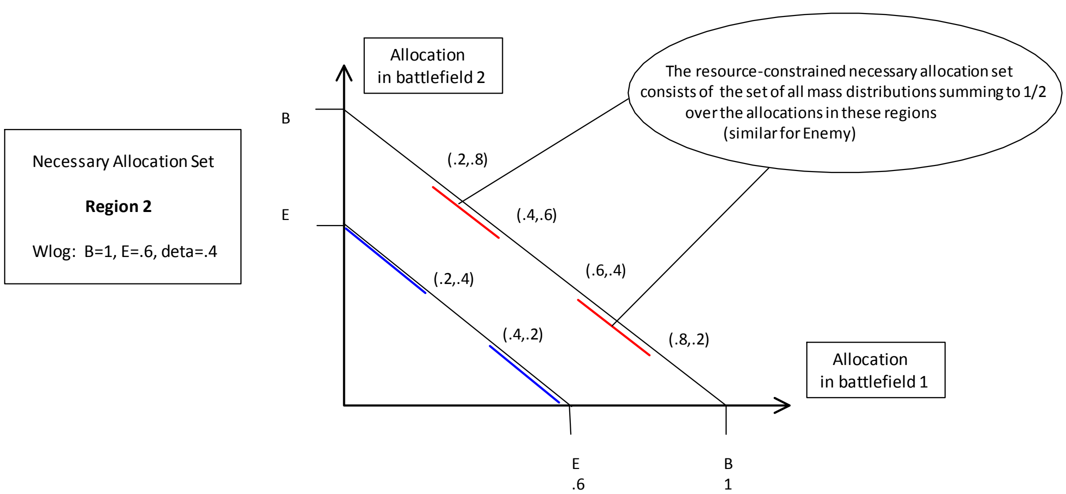
For region 2, the resource-constrained support-construction step of the construction algorithm is demonstrated graphically in Fig 20. The postulated supports again satisfy the properties of lemma 1.

Figure 20



Graphing the predicted resource-constrained supports on the simplex in Region 2 yields:

Figure 21

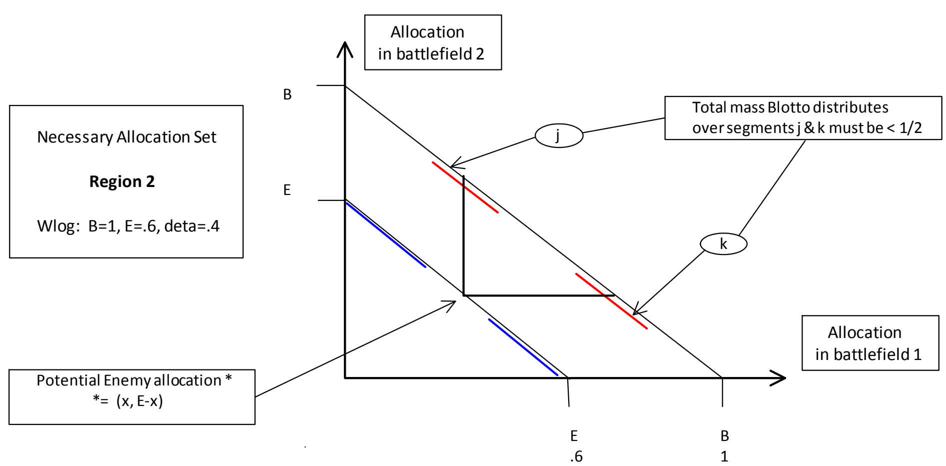


We caution the reader. The analog of region 1 is not as simple; all possible mass distributions over the resource-constrained cells (in this case, ½ on each) do not constitute Nash strategies. We impose a further restriction on the possible mixed-strategy mass distribution for a player; the strategy restriction can be easily interpreted graphically and is depicted in Fig 22. The intuition behind imposing a further restriction on Blotto strategies is that if Blotto plays particular pairs of allocations with a high probability, then Enemy will be incentivized to deviate from his conjectured support to play an alternate allocation and earn higher expected payoffs. A similar story restricts Enemy's potential mixed-strategies. Enemy playing particular pairs of allocations with high probability can incentivize Blotto to deviate from his support to play an alternative allocation and win both battlefields with enough probability to earn larger expected payoff. The conditions must hold for all possible Enemy allocations outside of his support.

Mixed-Strategy Mass Restriction

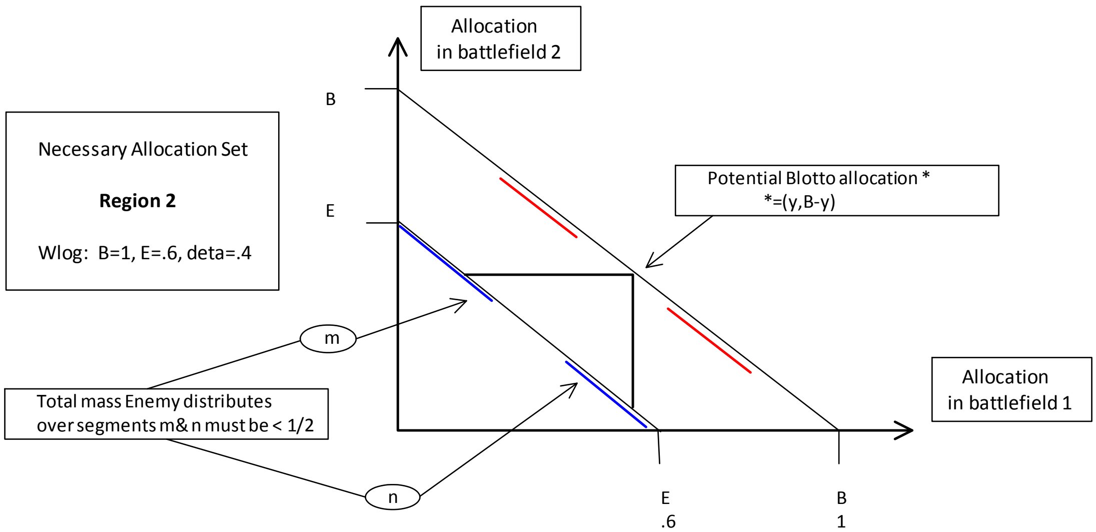
$$G_b(x^*) + \left(1 - G_b(E - x^*)\right) < \frac{1}{2} \qquad \forall x^* \text{not in Enemy}' \text{support}$$

Figure 22



$$G_e(y^*) + \left(1 - G_e(E - y^*)\right) < \frac{1}{2} \qquad \forall y^* \text{not in Blotto's support}$$

Figure 23



The formal characterization at the end of the section proves that such allocations by Enemy are not in his support, therefore are not Nash strategies, and Blotto cannot play strategies in equilibrium to which they would be a best-response imposing the restriction on Blotto's set of mixed-strategies. Similar reasoning is provided for restricting Enemy's mixed-Nash strategies.

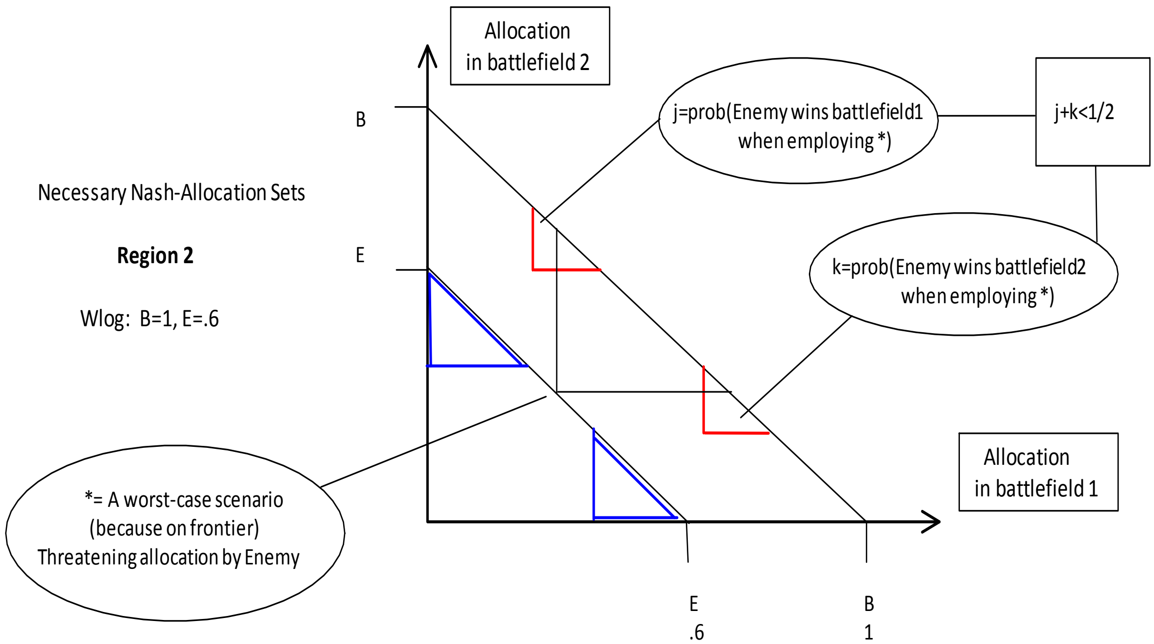
The restricted strategies do provide some intuition however. Consider a pairing of one Enemy support cell and the Blotto support cell directly north-east. All Blotto allocations in his support cell are best-responses to any allocation in the Enemy support cell (or any Enemy allocation strictly south and west of that allocation). If Blotto knew Enemy was playing an allocation in that Enemy support cell, any Blotto allocation in his support cell would guarantee him both battlefields and payoff of 2. Inversely, if Enemy knew Blotto were playing an allocation in his support cell, Enemy's best-response would be to play an allocation in any other support cell to guarantee a payoff of 1. (It is impossible for Enemy to win both).

When Enemy plays an allocation with more resources in battlefield 1, we call this "Enemy attacks heavy in 1" and do so similarly for battlefield 2, and for Blotto. The characterization states that half the time Enemy attacks heavy in battlefield 1, and half the time attacks heavy in battlefield 2. Blotto tries to match the battlefield goes heavy in. Half the time he guesses right and earns payoff 2, and half the time he doesn't for payoff of 1, and overall expected payoff 3/2 leaving Enemy expected payoff ½.

We now relax the assumption that the players play resource-constrained strategies in Region 2. The algorithm relaxing the assumption is analogous to Region 1. For each support cell pair, a best-response to Blotto is any allocation north-east of the *join* of Enemy's support cell boundaries, and by lemma 1, this region can be defined as the set of allocations north-east of the meet of Blotto's support cell boundaries. Enemy's best-response to any allocation in Blotto's support cell is any allocation not south and west of the meet of Blotto's support cell boundaries. Taking the intersection of the boundaries from the neighboring Blotto support cells yields the resource-minimizing Nash allocation for Enemy for that set of strategies. Again, taking the convex hull of the support cell boundaries together with the resource-minimizing allocation binds the set of Nash strategies that the player can play that set's probability mass over.

The final step for the region 2 Nash strategy characterizations is to consider the implication of the mixed-strategy mass restriction under the resource relaxed situation. The mass restriction becomes a mass restriction over the distribution within pairs of trapezoids.

Figure 24



3.3 Strategy Characterization Algorithm

1. Find resource-binding allocation support cell boundaries
2. Take the *meet* of the support cell boundaries
3. Take the *convex-hull* of these three allocations
4. Require each disjoint region to contain equal probability mass
5. Restrict mass distributions considering best-responses

3.4 Formal Characterization

W.L.O.G. assume that Blotto is the advantaged player and normalize his total resources to one (B=1). The complete set of Nash Equilibrium Blotto strategies, Ω^B, is the set of pdfs, $f_B(b_1, b_2)$ allowing the inclusion of atoms, with the following two properties:

Property 1b) The mass of $f_B(b_1, b_2)$ over triangle T_i^b is $\frac{1}{n}$ where T_i^b satisfies: $b_1 \geq E - (n-i)\delta$, $b_2 \geq E - (i-1)\delta$, $b_1 + b_2 \leq 1\ \forall i = 1,2,\dots,n$. Alternatively $\int_{T_i^b} f_B(b_1, b_2) db_1 db_2 = \frac{1}{n}$.

Property 2b) $\forall\ i < n\ \forall\ x_1 \in [E - (n-i)\delta, i\delta]$ the mass of $f_B(b_1, b_2)$ over trapezoid $j_b^{x_1}$ minus the mass over triangle $k_b^{x_1}$ is less than or equal to 0. Here j_b is defined as the portion of T_i^b such that $b_1 < x_1$ and region k_b is defined as the portion of T_{i+1}^b such that $b_2 \geq E - x_1$. Alternatively, $\int_{j_b} f_B(b_1, b_2) db_1 db_2 - \int_{k_b} f_B(b_1, b_2) db_1 db_2 \leq 0$.

The complete set of Nash Equilibrium Enemy strategies, Ω^E, are the set of pdfs, $f_E(e_1, e_2)$, with the following two properties:

Property 1e) Each triangle T_i^e contains mass $\frac{1}{n}$ where T_i^e satisfies: $e_1 > (i-1)\delta$, $e_2 > (n-i)\delta$, $e_1 + e_2 \leq E\ \forall\ i = 1,2,\dots,n$. For $i = 1$ the first inequality is weak. For $i = n$ the second inequality is weak. Alternatively $\int_{T_i^e} f_E(e_1, e_2) de_1 de_2 = \frac{1}{n}$.

Property 2e) $\forall i < n\ \forall\ x_1 \in [i\delta, E - (n-i+1)\delta]$ the mass of $f_E(e_1, e_2)$ over trapezoid $j_e^{x_1}$ minus the mass over triangle $k_e^{x_1}$ is less than or equal 0 where j_e is defined as the portion of T_{i+1}^e such that $e_1 \leq x_1$ and region k_e is defined as the portion of T_i^e such that $e_2 > 1 - x_1$. Alternatively, ${}_{j_e} f_E(e_1, e_2) de_1 de_2 - \int_{k_e} f_E(e_1, e_2) de_1 de_2 \leq 0$.

To show that these two definitions completely characterize the set of Nash Equilibrium we first prove that all strategy pairs satisfying the above conditions constitute a Nash Equilibrium. In a later section we show that no other strategies are a part of any Nash Equilibrium.

Proof: All Characterized Strategies are Part of a Nash Equilibrium.

Proposition 1: Any e_1 that is in[2] $T_{i_e}^e$ is strictly greater than all b_1 that are in $T_{i_b}^b$ where $i_b < i_e$ (should such a $T_{i_b}^b$ exist) and strictly[3] less than all b_1 that are in $T_{i_b}^b$ where $i_b \geq i_e$. Also, any e_2 that is in $T_{i_e}^e$ is strictly greater than all b_2 that are in any $T_{i_b}^b$ where $i_b > i_e$ (should such a $T_{i_b}^b$ exist) and strictly[4] less than all b_2 that are in any $T_{i_b}^b$ where $i_b \leq i_e$.

Consider the bounds for e_1 in $T_{i_e}^e$. It is bounded below by $(i_e - 1)\delta$. Changing the two other constraints to equalities and solving we find that in $T_{i_e}^e$ e_1 is bounded above by $E - (n - i_e)\delta$, the lower bound of b_1 in $T_{i_b}^b$ when $i_b = i_e$. Similar algebra for the other bounds confirms the rest of the proposition.

Given Proposition 1, we know that against any Blotto strategy from our definition, when Enemy plays in T_i^e his probability of winning battlefield 1 is $\frac{i-1}{n}$ and his probability of winning battlefield 2 is $\frac{n-i}{n}$. The total expected payoff is then $\frac{n-1}{n}$ anywhere in *any* T_i^e. Similarly, against any Enemy strategy from above, when Blotto

[2] Here we abuse notation and refer to a coordinate, x_1, as being "in" a two dimensional region so long as there exists another coordinate x_2 such that (x_1, x_2) is in that region.
[3] Weakly in the case where $i_e = n$.
[4] Weakly in the case where $i_e = 1$.

plays in T_i^b his probability of winning battlefield 1 is $\frac{i}{n}$ and his probability of winning battlefield 2 is $\frac{n-i+1}{n}$. The total expected payoff is then $\frac{n+1}{n}$ anywhere in *any* T_i^b.

We now show that there are no allocations for Enemy or Blotto that provide a higher expected payoff than we found in the previous paragraph. Note that if either player were to have a payoff improving deviation from the strategies we defined, they must have a full expenditure payoff improving deviation.[5] Therefore, we only need to show that there are no optimal full expenditure deviations. As all allocations in any T_i^e and any T_i^b have the same expected payoffs for Enemy and Blotto, respectively, we only need to check full expenditure deviations outside of those triangles.

Consider a generic full expenditure deviation (e_1^*, e_2^*). Given that (0,E) is in T_1^e and (E,0) is in T_n^e we know that (e_1^*, e_2^*) must lie between some T_i^e and T_{i+1}^e.[6] Here let (e_1, e_2) be a non-deviating allocation in T_i^e. Examining Property 2b with $x_1 = e_1^*$ we see that that allocation (e_1^*, e_2^*) provides an expected payoff weakly dominated by the expected payoff from playing (e_1, e_2). Given the bounds on such a deviation, we note that the realized payoff to enemy of playing (e_1^*, e_2^*) against the Blotto strategy we describe above will be the same as if he had played (e_1, e_2) unless Blotto plays in T_i^b or T_{i+1}^b. If Blotto plays in T_i^b the deviant strategy *may* do better[7] on Battlefield 1 (and will do the same on Battlefield 2). The cost is that if Blotto plays in T_{i+1}^b the deviant strategy may do worse on Battlefield 2 (and will do the same on Battle field 1). Using the notation of Property 2b, any b_1 in $j_b^{e_1^*}$ will lose to e_1^* (while it would have lost to e_1) and any b_2 in $k_b^{e_1^*}$ will beat e_2^* (while it would have lost to e_2). Property 2b then says that by

---

[5] As the expected payoff functions must be weakly increasing in expenditure in either battlefield.

[6] By "between" we mean that the values of e_1^* and e_2^* lie between any value of e_1 and e_2 in T_i^e and T_{i+1}^e, respectively, respectively.

[7] By "do better" we mean e_1^* would be larger than Blotto's Battlefield 1 allocation, whereas e_1 would be less.

moving from any (e_1, e_2) in T_i^e to (e_1^*, e_2^*) the additional probability of winning on Battlefield 1 must be weakly less than the additional probability of losing on Battlefield 2. Therefore no full expenditure deviation (e_1^*, e_2^*) is payoff improving, and therefore no deviation is payoff improving. The same logic is behind Property 2e) and prevents Blotto from having any payoff improving full expenditure deviations.

Thus, if both Blotto and Enemy were to play strategies as we described, they would both be playing best responses to the other's strategy. Therefore all of the strategies we describe are Nash Equilibrium strategies.

3.3 Sufficiency

In this section we prove that there are no other strategies which could be part of Nash Equilibrium. Because of equilibrium interchangeability, all we need to show in order to prove that a strategy is not a part of *any* Nash Equilibrium is that the strategy does not form *a* Nash Equilibrium when paired with a strategy that we've already shown was a part of a Nash Equilibrium. In particular for Enemy we make use of the strategy $f_E^1(e_1, e_2)$ where he plays uniformly over the full expenditure boundary in each T_i^e. For Blotto we make use of the following two strategies: $f_B^1(b_1, b_2)$ where he plays uniformly over the full expenditure boundary in each T_i^b and $f_B^2(b_1, b_2)$ where he plays with mass $\frac{1}{2n}$ at both $(E-(n-i)\delta, i\delta)$ and $(i\delta, E-(i-1)\delta)$ in each T_i^b.[8]

We now prove the completeness of our definition by contradiction. Suppose there exists a Nash Equilibrium Enemy Strategy that is not described by our definition. Such a strategy must then either violate Property 1e) or satisfy Property 1e) and violate Property 2e). In proving that all our strategies were indeed part of a Nash Equilibrium,

[8] A simple algebraic or graphical analysis shows that these strategies satisfy our conditions.

we've already shown how a violation of Property 2e) provides Blotto with an optimal deviation, so we rule out that possibility.

Consider possible violations of Property 1e): **Deviation 1)** Enemy could play with some mass over a region S where $\exists i$ such that $\forall(e_1^*, e_2^*) \in S, \forall(e_1, e_2) \in T_i^e$, $e_1^* \leq e_1$ and $e_2^* \leq e_2$ with one of those inequalities always holding strictly. We already have a contradiction as this could not be a best response to $f_B^2(b_1, b_2)$ which has blotto playing the lower bounds of $e_1$and e_2 in T_i^e with positive mass. (e_1^*, e_2^*) provides a strictly lower expected payoff than playing in T_i^e.

This only leaves two possible types of deviations by enemy: He could play with mass other than $\frac{1}{n}$ over some T_i^e (**Deviation 2)** and/or he could play with mass over a convex region S where $\forall\ (e_1^*, e_2^*) \in S, \forall(e_1, e_2) \in T_i^e$ either $e_1^* > e_1$ or $e_2^* > e_2$ **(Deviation 3)**. Given the bounds of the T_i^e's any such region S must be a subset of one of the triangles D_j^e, indexed by $j = 1,2, \dots n-1$, where $\forall(e_1, e_2) \in D_j^e, e_1 > E - n - i)\delta, e_2 > E - i\delta, e_1 + e_2 \leq E$.[9]

We simultaneously prove that neither of these deviations is possible. Consider a generic T_i^e and D_i^e and assume that $\forall j = 1,2, \dots, i-1$ the mass over T_j^e is $\frac{1}{n}$ and is 0 over D_j^e.[10] In other words, there has not "yet" been a Deviation 2 or Deviation 3.

Suppose the mass over T_i^e is strictly less than $\frac{1}{n}$. Then, when Blotto plays $(E - (n-i)\delta, i\delta)$,[11] he wins Battlefield 1 with probability less than $\frac{i}{n}$ but still wins Battlefield 2 with probability $\frac{n-i+1}{n}$ for a total expected payoff strictly less than $\frac{n+1}{n}$,

---

[9] Technically the first two inequalities should be weak. However, in strategy $f_B^2(.)$, Blotto plays with positive mass a respective b_1 and b_2 equal to each of the first two boundaries. Therefore, given that ties go to Blotto, playing any mass on either of those boundaries is not a best response for Enemy as he could improve his expected payoff with an ϵ deviation.

[10] If such j's exist.

[11] Which he does with probability $\frac{1}{2n}$ in strategy $f_B^2(.)$

which is Blotto's constant expected payoff in all equilibrium. Similarly, if the mass over T_i^e is strictly more than $\frac{1}{n}$ then when Blotto plays $(E-(n-i)\delta, i\delta)$, he wins Battlefield 1 with probability greater than $\frac{i}{n}$ but still wins Battlefield 2 with probability $\frac{n-i+1}{n}$ for a total expected payoff strictly greater than $\frac{n+1}{n}$ again a contradiction. Therefore, the mass over T_i^e must equal $\frac{1}{n}$.

Now suppose Enemy plays some positive mass over D_i^e. Now, when Blotto plays $(i\delta, E-(i-1)\delta)$[12] he then expects to win in Battlefield 1 with probability greater than $\frac{i}{n}$ and expects to win in Battlefield 2 with probability $\frac{n-i+1}{n}$ therefore his total expected payoff is greater than $\frac{n+1}{n}$ another contradiction. Therefore the mass over D_i^e must equal zero.

As the above analysis holds for all $i = 1,2, \dots, n-1$, simple induction shows that the mass over all such T_i^e and D_i^e must equal $\frac{1}{n}$ and 0, respectively. The remaining mass of $\frac{1}{n}$ must then be distributed over the only region left, T_n^e. We've ruled out any potential Enemy strategies that deviate from our characterization of possible Nash Equilibrium Enemy's strategies. Therefore, the characterization is complete. The proof of the completeness of our characterization of the Blotto strategies follows similarly. Therefore, given equilibrium interchangeability our characterization of the set of Nash Equilibrium is complete.

Q.E.D.

[12]Which he does with probability $\frac{1}{2n}$ in strategy $f_B^2(.)$

Section 4: Re-Derivation of Payoff Space

The preceding analysis is useful for a simple derivation of Fig1, the payoff state-space boundaries. As Enemy's resources increase relative to Blotto, δ consequently decreases (Blotto's triangles shrink and Enemy's grow), eventually an entire extra δ-step cell for the resource-constrained case (triangle on the simplex construction) will fit into both players' allocation support.

Let x_B denote how many δ-steps fit into Blotto's support.

$$B = x_B \cdot \delta + r \qquad \forall \;\; x_B \in \mathbb{N}, \;\; \text{and} \;\; r \in [0, \delta)$$

As Enemy's resources increase (and δ decreases), r will increase until it equals δ. At that point x_B increments and *r* resets. Thus, the boundaries to the state space emerge when r=0. Re-substituting for δ, and solving E in terms of B when r=0 yields exactly the state-space boundaries depicted in Fig1.

$$E = \frac{x_B - 1}{x_B} \cdot B \quad \forall \;\; x_B \in \mathbb{N}$$

Section 5: Conclusion

This paper provides a complete characterization of the Nash equilibrium to the Gross & Wagner's original 2-player 2- battlefield Colonel Blotto game with plurality objectives. We provide an algorithm for constructing the set of all Nash strategies, and prove the completeness of the characterization. A complete characterization of all equilibrium strategies is useful for many reasons. For potential empirical analysis of Blotto applications, it is necessary to know all potential optimal allocation strategies. Second, understanding the complete characterization algorithm lends can aid in solving more complicated and more realistic variants of the game.

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