AIP | Chaos
An Interdisciplinary Journal of Nonlinear Science

Instability criteria for steady flows of a perfect fluid

Susan Friedlander and Misha M. Vishik

Citation: Chaos **2**, 455 (1992); doi: 10.1063/1.165888
View online: http://dx.doi.org/10.1063/1.165888
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/2/3?ver=pdfcov
Published by the AIP Publishing

Instability criteria for steady flows of a perfect fluid

Susan Friedlander
Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60680

Misha M. Vishik
Department of Mathematics, University of Texas, Austin, Texas 78712

(Received 17 March 1992; accepted for publication 15 July 1992)

An instability criterion based on the positivity of a Lyapunov-type exponent is used to study the stability of the Euler equations governing the motion of an inviscid incompressible fluid. It is proved that any flow with exponential stretching of the fluid particles is unstable. In the case of an arbitrary axisymmetric steady integrable flow, a sufficient condition for instability is exhibited in terms of the curvature and the geodesic torsion of a stream line and the helicity of the flow.

I. INTRODUCTION

The problem of hydrodynamic stability is a classical problem. In this paper we will discuss conditions under which general flows of an inviscid incompressible fluid in 3D are unstable. We employ an instability criterion derived in a previous paper (see Vishik and Friedlander[1]) based on a Lyapunov-type exponent associated with a system of ODEs. The use of ODEs to obtain instability criteria for PDEs appears to be a powerful technique where it can appropriately be applied. An important paper which introduced a geometric approach to the problem of hydrodynamic stability is that of Arnold.[2] Arnold states that "there appears to be an infinitely great number of unstable configurations." This present paper bears out this observation and illustrates the fact that large classes of flows are unstable.

We briefly mention some of the more recent work whose techniques are related to those that we employ. Eckhoff and Storesletten[3] and Eckhoff[4] showed that local instabilities for hyperbolic systems can essentially be reduced to a local analysis involving ODEs. Bayly[5] studied the stability of quasi-2D steady flows via an analysis of a system of ODEs with the Floquet exponent giving the growth rate of the instability. Lifschitz and Hameiri[6,7] used methods of geometrical optics to examine localized instabilities and to obtain effective stability conditions for general vortex rings via an analysis of the appropriate transport equations.

We now formulate the instability criteria on which the results proved in Secs. II–IV are based. Let $\mathbf{u}(\mathbf{x})$ be a steady solution of Euler's equations governing the motion in 3D of an inviscid, constant density fluid:

$$(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla P, \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=0. \tag{1.2}$$

The 3D vector field $\mathbf{u}(\mathbf{x})$ denotes the velocity and the scalar field $P(\mathbf{x})$ denotes the pressure in the fluid. Using WKB-type asymptotic methods, Vishik and Friedlander[1] prove that the growth rate σ of a small perturbation of an equilibrium solution of Eqs. (1.1) and (1.2) is bounded from below by the following universal quantity of a geometric nature:

$$\overline{\lim_{t\to\infty}}\,(1/t)\log \sup_{\substack{\mathbf{x}_0,\xi_0,\mathbf{b}_0\\ |\mathbf{b}_0|=|\xi_0|=1,\mathbf{b}_0\cdot\xi_0=0}} |\mathbf{b}(\mathbf{x}_0,\xi_0,t)|\leqslant\sigma. \tag{1.3}$$

The vector $\mathbf{b}(\mathbf{x}_0,\xi_0,t)$ is the first term in an expansion of the amplitude of a high frequency wavelet localized at $\mathbf{x}_0$. The vector ξ is the spatial wave number vector for the wavelet. The vectors $\mathbf{b}$ and ξ satisfy the following system of ODE:

$$\dot{\mathbf{x}}=\mathbf{u}(\mathbf{x}), \tag{1.4}$$

$$\dot{\xi}=-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\xi, \tag{1.5}$$

$$\dot{\mathbf{b}}=-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}+2\left[\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}\cdot\xi\right]\xi/|\xi|^2 \tag{1.6}$$

with initial values $\mathbf{x}(0)=\mathbf{x}_0$, $\xi(0)=\xi_0$, $\mathbf{b}(0)=\mathbf{b}_0$, where $\mathbf{b}_0\cdot\xi_0=0$.

The sufficient condition for linear instability (i.e., positivity of the growth rate σ) given by Eq. (1.3) is a precise mathematical formulation of the concept of local instability for fluid motion. We note that the lhs of Eq. (1.3) involves the supremum; hence **any** Lagrangian trajectory of the flow could provide a positive value on the bound from below on σ and hence imply instability. In several papers,[8–10] Friedlander and Vishik have demonstrated the effectiveness of this instability criterion; for example, it is shown that any flow with a hyperbolic stagnation point is unstable.

In the present paper we will complete our study of the types of flows for which the criterion (1.3) is effective. In Sec. II we prove that the existence of a positive Lyapunov exponent at any point $\mathbf{x}_0$ implies that the flow is unstable, i.e., the existence of exponential stretching in the flow implies instability. As was first pointed out by Arnold,[2] all numerical evidence indicates that so-called "*ABC*" flows possess exponential stretching in the volume (although it may be only a small part of the volume when the flow is close to integrable). [See, for example, Arnold[2,11] Henon,[12] and Dombre *et al.*[13]] *ABC* flows are important in dynamo theory and of relevance to the topic of "chaos." A flow with exponential stretching in the volume is an example of a nonintegrable flow. For those values of the parameters

 1054-1500/92/030455-06$001.00

for which there is a stagnation point in the flow (e.g., $A=B=C=1$), instability follows immediately from Friedlander and Vishik.[9] In contrast, in Sec. III we examine integrable flows. We show that $\mathbf{b}(t)$ would grow at most algebraically when $\xi(t)$ is chosen to be an unbounded solution to the cotangent equation (1.5). Hence we seek the possibility of a positive value for the lhs of Eq. (1.3) by examining Eq. (1.6) under the condition that $\xi(t)$ is a bounded solution to Eq. (1.5). A study of the monodromy operator for Eq. (1.6) leads to an explicit condition under which there exists a growing solution $\mathbf{b}(t)$, thus implying instability by criterion (1.3). We remark that an important feature of our results is the fact that the instability criterion involves only calculating averages rather than solving an ODE. In Sec. IV we give a geometric interpretation of a sufficient condition for instability in terms of an average of the curvature and the geodesic torsion of a streamline.

II. EXPONENTIAL STRETCHING

The idea that exponential stretching of fluid particles could imply instability for the Euler equations is originally due to Arnold.[2] In this seminal paper he examined an example of a model flow on a compact manifold M with a Riemannian metric. A vector field $\mathbf{v}$ is defined on M such that $\mathbf{v}$ may be taken to be the velocity field of an irrotational perfect fluid. Every particle of fluid moving in that field stretches in one direction and contracts in another direction. We will now use the inequality (1.3) to prove that, in fact, every flow with exponential stretching somewhere in the flow is linearly unstable.

Proposition 2.1: Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be two linearly independent solutions of Eq. (1.6) with both $\mathbf{b}_1(0)=\mathbf{b}_{1_0}$ and $\mathbf{b}_2(0)=\mathbf{b}_{2_0}$ being perpendicular to ξ_0. Then $\text{Area}(\mathbf{b}_1(t),\mathbf{b}_2(t))=\text{Area}(\mathbf{b}_{1_0},\mathbf{b}_{2_0})|\xi_0|/|\xi(t)|$. (The same observation has been made by B. Bayly, personal communication.)

Proof:

$$\frac{d}{dt}\text{Vol}(\mathbf{b}_1(t),\mathbf{b}_2(t),\xi(t))$$

$$=\det(\dot{\mathbf{b}}_1,\mathbf{b}_2,\xi)+\det(\mathbf{b}_1\dot{\mathbf{b}}_2,\xi)+\det(\mathbf{b}_1\mathbf{b}_2,\dot{\xi}). \tag{2.1}$$

We substitute Eqs. (1.5) and (1.6) into Eq. (2.1) to obtain

$$\frac{d}{dt}\text{Vol}(\mathbf{b}_1,\mathbf{b}_2,\xi)$$

$$=\det\left(-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}_1,\mathbf{b}_2,\xi\right)+\det\left(\mathbf{b}_1,-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}_2,\xi\right)$$

$$+\det\left(\mathbf{b}_1,\mathbf{b}_2,-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\xi\right)$$

$$=-\text{Tr}\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\det(\mathbf{b}_1,\mathbf{b}_2,\xi)+\det(\mathbf{b}_1,\mathbf{b}_2,(\nabla\times\mathbf{u})\times\xi). \tag{2.2}$$

Since $\nabla\cdot\mathbf{u}=0$, it follows that $\text{Tr}(\partial\mathbf{u}/\partial\mathbf{x})=0$. Furthermore $(\nabla\times\mathbf{u})\times\xi$ is perpendicular to ξ and hence this vector is a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$; thus $\det(\mathbf{b}_1,\mathbf{b}_2,(\nabla\times\mathbf{u})\times\xi)=0$. Hence

$$\frac{d}{dt}\text{Vol}(\mathbf{b}_1,\mathbf{b}_2,\xi)=0 \tag{2.3}$$

which implies that

$$\text{Area}(\mathbf{b}_1(t),\mathbf{b}_2(t))|\xi(t)|=\text{Area}(\mathbf{b}_{1_0},\mathbf{b}_{2_0})|\xi_0|. \tag{2.4}$$

■

Theorem 2.1: Let the flow $\dot{\mathbf{x}}=\mathbf{u}(\mathbf{x})$ have a positive Lyapunov exponent at some point $\mathbf{x}_0$. Then the flow $\mathbf{u}(\mathbf{x})$ is linearly unstable as a steady state solution of the Euler equations.

Proof: Let $\eta(t)$ be a solution to the system

$$\dot{\mathbf{x}}=\mathbf{u}(\mathbf{x}),\qquad \mathbf{x}(0)=\mathbf{x}_0,$$
$$\dot{\eta}=\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\eta,\quad \eta(0)=\eta_0 \tag{2.5}$$

such that

$$|\eta(t_i)|/|\eta_0|\geqslant e^{\delta t_i},\quad \delta>0 \tag{2.6}$$

[i.e., $\eta(t_i)$ is exponentially growing for $t_i\to\infty$]. We define a matrix $\Lambda_t = g_*^Tg_*$ where g_* is the Jacobi matrix $g_* =(\partial\mathbf{x}/\partial\mathbf{x}_0)$. From Eq. (2.6) it follows that

$$(\Lambda_{t_i}(\mathbf{x}_0)\eta_0\cdot\eta_0)\geqslant e^{2\delta t_i}|\eta_0|^2, \tag{2.7}$$

and hence the maximal eigenvalue of $\Lambda_{t_i}(\mathbf{x}_0)$ is greater than or equal to $e^{2\delta t_i}$. From the definition of g_*, it follows that

$$|\xi(t_i)|^2=((g_*^{-t_i})^T\xi_0\cdot(g_*^{-t_i})\xi_0)=(\Lambda_{t_i}^{-1}(\mathbf{x}_0)\xi_0\cdot\xi_0). \tag{2.8}$$

The volume preserving property of the flow implies that $|\Lambda_t(\mathbf{x}_0)|=1$; hence there exists an eigenvalue of $\Lambda_{t_i}^{-1}(\mathbf{x}_0)$ which is less than or equal to $e^{-2\delta t_i}$. For ξ_0 being the corresponding eigenvector we have, from Eq. (2.8),

$$|\xi(t_i)|^2\leqslant e^{-2\delta t_i}|\xi_0|^2. \tag{2.9}$$

Let $\mathbf{b}_{1_0}$ and $\mathbf{b}_{2_0}$ be vectors perpendicular to ξ_0 with $|\mathbf{b}_{1_0}|=|\mathbf{b}_{2_0}|=1$. From Proposition 2.1, it follows that

$$|\text{Area}(\mathbf{b}_1(t_i),\mathbf{b}_2(t_i))|\geqslant e^{\delta t_i}. \tag{2.10}$$

Hence at least one of the vectors $\mathbf{b}_1(t_i)$ and $\mathbf{b}_2(t_i)$ has norm greater than or equal to $e^{\delta t_i/2}$. Thus the lhs of Eq. (1.3) is greater than or equal to $\delta/2$, hence implying instability in the flow. ■

III. INTEGRABLE FLOWS

We now turn to the case of integrable flows with smooth velocity fields $\mathbf{u}$. We write the Euler equations (1.1) and (1.2) in the form

$$\mathbf{u}\times(\nabla\times\mathbf{u})=\nabla H, \tag{3.1}$$

$$\nabla\cdot\mathbf{u}=0, \tag{3.2}$$

where the Bernouilli function $H(\mathbf{x}(t))$ is given by

$$H=P+\mathbf{u}^2/2. \tag{3.3}$$

We will use the system of ODEs (1.4)–(1.6) and the condition (1.3) to obtain explicit criteria for instability of general axisymmetric integrable flows with $\nabla H \neq 0$. We consider a compact noncritical level surface $H=H_0$ which, from Eq. (3.1), is necessarily diffeomorphic to the 2D torus (Arnold[14,2]).

Proposition 3.1: Let $\mathbf{x}(t)$ be the solution to Eq. (1.4) with initial condition $\mathbf{x}(0)=\mathbf{x}_0$. Then $\xi(t)=\nabla H(\mathbf{x}(t))$ satisfies Eq. (1.5).

Proof: We use Eq. (3.1) to express $(d/dt)\nabla H$ in terms of $\mathbf{u}$:

$$\frac{d}{dt}\nabla H=\frac{d\mathbf{u}}{dt}\times(\nabla\times u)+\mathbf{u}\times\frac{d}{dt}(\nabla\times\mathbf{u}). \tag{3.4}$$

Now

$$\frac{d\mathbf{u}}{dt}=\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{u}. \tag{3.5}$$

The curl of Eq. (3.1) gives

$$-(\mathbf{u}\cdot\nabla)(\nabla\times\mathbf{u})+(\nabla\times\mathbf{u}\cdot\nabla)\mathbf{u}=0,$$

i.e., we have the steady form of "Kelvins vorticity theorem,"

$$\frac{d}{dt}(\nabla\times\mathbf{u})=\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)(\nabla\times\mathbf{u}). \tag{3.6}$$

Substituting Eqs. (3.5) and (3.6) into Eq. (3.4) gives

$$\begin{aligned}\frac{d}{dt}\nabla H&=\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{u}\times(\nabla\times\mathbf{u})+\mathbf{u}\times\left(\frac{\partial\mathbf{u}}{\partial x}\right)(\nabla\times\mathbf{u})\\&=-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\nabla H,\end{aligned} \tag{3.7}$$

i.e., ∇H is a solution to Eq. (1.5). ■

Proposition 3.2: Let $\mathbf{b}_0$ be perpendicular to $\nabla H(\mathbf{x}_0)$ and let $\xi(t)=\nabla H(\mathbf{x}(t))$, $t\geqslant 0$. Then the solution $\mathbf{b}(t)$ to Eq. (1.6), with initial condition $\mathbf{b}_0$, is bounded.

Proof: Proposition 3.1, together with Eqs. (1.4)–(1.6) implies

$$\begin{aligned}\frac{d}{dt}(\mathbf{b}\cdot\mathbf{u})&=-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}\cdot\mathbf{u}+\mathbf{b}\cdot\left(\frac{\partial\mathbf{u}}{\partial x}\right)\mathbf{u}\\&=\mathbf{b}\cdot\left(\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\right)\mathbf{u}\\&=\mathbf{b}\cdot(\nabla\times\mathbf{u})\times\mathbf{u}=-\mathbf{b}\cdot\xi=0\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\frac{d}{dt}(\mathbf{b}\cdot\nabla\times\mathbf{u})&=-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}\cdot(\nabla\times\mathbf{u})+\mathbf{b}\cdot\left(\frac{\partial\mathbf{u}}{\partial x}\right)(\nabla\times\mathbf{u})\\&=\mathbf{b}\cdot(\nabla\times\mathbf{u})\times(\nabla\times\mathbf{u})=0.\end{aligned} \tag{3.9}$$

Since the trajectory lies on a noncritical compact level surface, Eq. (3.1) together with Eqs. (3.8) and (3.9) imply that $\mathbf{b}$ is bounded. ■

Proposition 3.3: In 2D the vector $\mathbf{b}$ satisfying Eq. (1.6) is bounded along any closed streamline lying on a noncritical level set of H.

Proof: Let ξ_0 be perpendicular to $\mathbf{b}_0$. We use Eqs. (1.5) and (1.6) to write

$$\begin{aligned}\frac{d}{dt}\det(\mathbf{b},\xi)&=\det\left(-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b},\xi\right)+\det\left(\mathbf{b},-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\xi\right)\\&=-\mathrm{Tr}\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\det(\mathbf{b},\xi)-\det((\nabla\times\mathbf{u})\times\mathbf{b},\xi).\end{aligned} \tag{3.10}$$

Now in 2D the vector $(\nabla\times\mathbf{u})\times\mathbf{b}$ is parallel to ξ. Also $\nabla\cdot\mathbf{u}=0$ implies that $\mathrm{Tr}(\partial\mathbf{u}/\partial\mathbf{x})$ is zero, hence Eq. (3.10) gives the result

$$\frac{d}{dt}\det(\mathbf{b},\xi)=0. \tag{3.11}$$

Therefore $|\mathbf{b}|\cdot|\xi|$ is a constant along any streamline and $\mathbf{b}$ is bounded provided $\xi\cdot\mathbf{u}\neq 0$. In the case where $\xi\cdot\mathbf{u}=0$, the vector ξ must be parallel to ∇H and hence by Proposition 3.2, we again conclude that $\mathbf{b}$ is bounded. ■

Remark: Proposition 3.3 shows that in 2D the lhs of Eq. (1.3) is zero on any nondegenerate streamline. However the degenerate streamlines (i.e., the separatrix) could easily provide a positive lower bound for σ. In fact, Friedlander and Vishik,[9] Lifschitz and Hameiri[6] prove the instability of any flow in 2D or 3D with a hyperbolic stagnation point.

Proposition 3.4: Let $\xi(t)$, with $\xi(0)=\xi_0$, be an *unbounded* solution to Eq. (1.5). Let $\mathbf{b}(t)$, with $\mathbf{b}(0)=\mathbf{b}_0\perp\xi_0$, be the solution to Eq. (1.6). Then

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log|\mathbf{b}(t)|=0.$$

Proof: Let $\eta(t)$, with $\eta(0)=\eta_0$, be a solution to the tangent equation $\dot{\eta}=(\partial\mathbf{u}/\partial\mathbf{x})\eta$. According to a theorem of Arnold,[2,11,14] we could introduce a coordinate system (α,β,z) in the neighborhood of the torus $H=H_0$ such that z is a smooth function of H, and

$$\mathbf{u}=\omega_1(z)\frac{\partial}{\partial\alpha}+\omega_2(z)\frac{\partial}{\partial\beta}, \tag{3.12}$$

$$\nabla\times\mathbf{u}=-\omega_2'(z)\frac{\partial}{\partial\alpha}+\omega_1'(z)\frac{\partial}{\partial\beta}, \tag{3.13}$$

α,β mod 2π. In this notation the evolution of the tangent vector η is given by

$$\begin{aligned}\eta&=\eta^\alpha\frac{\partial}{\partial\alpha}+\eta^\beta\frac{\partial}{\partial\beta}+\eta^z\frac{\partial}{\partial z}\\&\mapsto(\eta^\alpha+t\omega_1'\eta^z)\frac{\partial}{\partial\alpha}+(\eta^\beta+t\omega_2'\eta^z)\frac{\partial}{\partial\beta}+\eta^z\frac{\partial}{\partial z}.\end{aligned} \tag{3.14}$$

The evolution of the 1-form ξ is given by

$$\begin{aligned}\xi&=\xi_\alpha\,d\alpha+\xi_\beta\,d\beta+\xi_z\,dz\\&\mapsto\xi_\alpha\,d\alpha+\xi_\beta\,d\beta+(\xi_z-t\omega_1'\xi_\alpha-t\omega_2'\xi_\beta)dz.\end{aligned} \tag{3.15}$$

From Eq. (3.15) it follows that $|\xi|$ is either bounded (when $\omega_1'\xi_\alpha+\omega_2'\xi_\beta=0$) or growing linearly with t. Under the assumption of the proposition, we consider the situation where $|\xi|$ grows linearly with t.

Using Eq. (1.6) we write

$$\frac{d}{dt}(\mathbf{b}\cdot\mathbf{u})=\mathbf{b}\cdot\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{u}-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}\cdot\mathbf{u}+2\left[\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}\cdot\xi\right](\xi\cdot\mathbf{u})/|\xi|^2. \tag{3.16}$$

We use Eq. (3.1) and the fact that

$$\left(\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\right)\mathbf{u}=(\nabla\times\mathbf{u})\times\mathbf{u}$$

to rewrite Eq. (3.16) in the form

$$\frac{d}{dt}(\mathbf{b}\cdot\mathbf{u})=-\mathbf{b}\cdot(\nabla H-\nabla H\cdot\xi)\xi/|\xi|^2)$$
$$+2\left[\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}\cdot\xi\right](\xi\cdot\mathbf{u})/|\xi|^2. \tag{3.17}$$

Similarly,

$$\frac{d}{dt}(\mathbf{b}\cdot\nabla\times\mathbf{u})=2\left[\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{b}\cdot\xi\right](\xi\cdot\nabla\times\mathbf{u})/|\xi|^2, \tag{3.18}$$

and trivially

$$\frac{d}{dt}(\mathbf{b}\cdot\xi/|\xi|)=0. \tag{3.19}$$

It follows from Eqs. (1.5), (3.5), and (3.6) that $(d/dt)(\xi\cdot\mathbf{u})$ and $(d/dt)(\xi\cdot\nabla\times\mathbf{u})$ are zero. Therefore it follows that as $t\to\infty$ the direction of ξ approaches the direction of ∇H. Hence we can apply Gronwall's Lemma to the system (3.17)–(3.19) to obtain the result that

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log|\mathbf{b}(t)|=0. \qquad ■$$

Remarks: The proof given above shows that $\mathbf{b}(t)$ could grow at most algebraically when $\xi(t)$ is unbounded. Furthermore, Eq. (3.15) shows that $\xi(t)$ is bounded if and only if

$$\xi(t)=\psi(\mathbf{x}(t)), \tag{3.20}$$

where $\psi(\mathbf{x})$ is a smooth covector field (i.e., 1-form) on the torus $H=H_0$.

We now restrict our attention to axisymmetric toroidal equilibria. Let $\mathbf{u}$ and H be axisymmetric with respect to rotations about the axis of the nested toroidal surfaces $H=H_0$ constant. From Proposition 3.4 it follows that only a bounded vector field ξ is a possible choice for ξ that will lead to an exponentially growing solution $\mathbf{b}$ to Eq. (1.6). We will consider in more detail the construction of a smooth axisymmetric field $\psi(\mathbf{x})$ such that $\xi(t)=\psi(\mathbf{x}(t))$ is a bounded solution to Eq. (1.5) for any trajectory $\mathbf{x}(t)$ lying on the torus $H=H_0$. We write

$$\psi(\mathbf{x})=c_1(\mathbf{x})\mathbf{u}+c_2(\mathbf{x})\nabla\times\mathbf{u}+c_3(\mathbf{x})\nabla H, \tag{3.21}$$

where c_1, c_2, c_3 are axisymmetric functions of $\mathbf{x}$. As we have previously noted, $\xi\cdot\mathbf{u}$ and $\xi\cdot\nabla\times\mathbf{u}$ are constants. We write

$$\left.\begin{array}{l}\psi(\mathbf{x}(t))\cdot\mathbf{u}(\mathbf{x}(t))=d_1\\ \psi(\mathbf{x}(t))\cdot\nabla\times\mathbf{u}(\mathbf{x}(t))=d_2\end{array}\right\}, \tag{3.22}$$

where d_1 and d_2 are constants. Substituting Eq. (3.21) into Eq. (3.22) gives

$$\left.\begin{array}{l}c_1(\mathbf{x})|\mathbf{u}|^2+c_2(\mathbf{x})\mathbf{u}\cdot\nabla\times\mathbf{u}=d_1\\ c_1(\mathbf{x})\mathbf{u}\cdot\nabla\times\mathbf{u}+c_2(\mathbf{x})|\nabla\times\mathbf{u}|^2=d_2\end{array}\right\}. \tag{3.23}$$

The functions $c_1(\mathbf{x})$ and $c_2(\mathbf{x})$ are determined from Eq. (3.23) and the function $c_3(\mathbf{x})$ satisfies

$$\psi\cdot\nabla H/|\nabla H|^2=c_3(\mathbf{x}(t)). \tag{3.24}$$

Hence,

$$\begin{aligned}\frac{d}{dt}c_3(\mathbf{x}(t))=&-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\psi\cdot\nabla H/|\nabla H|^2\\ &-\psi\cdot\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\nabla H/|\nabla H|^2\\ &+2(\psi\cdot\nabla H)\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\nabla H\cdot\nabla H/|\nabla H|^4\\ =&-\left(\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)+\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\right)(c_1\mathbf{u}+c_2\nabla\times\mathbf{u})\\ &\cdot\nabla H/|\nabla H|^2.\end{aligned} \tag{3.25}$$

Therefore a single-valued function $c_3(\mathbf{x}(t))$ can be uniquely determined by the condition $\langle c_3\rangle=0$, where the mean value is taken over the torus with respect to the normalized invariant measure proportional to $d(\text{area})/|\nabla H|$. Such a value for $c_3(\mathbf{x}(t))$ exists if and only if the mean value of the rhs of Eq. (3.25) is zero, i.e.,

$$\left\langle\left(\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)+\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\right)(\mathbf{c}_1\mathbf{u}+c_2\nabla\times\mathbf{u})\cdot\nabla H/|\nabla H|^2\right\rangle=0. \tag{3.26}$$

According to the Birkhoff–Khinchin ergodic theorem, for any axisymmetric function f on the torus $H=H_0$

$$\langle f\rangle=\frac{1}{T}\int_0^T f(\mathbf{x}(t))dt, \tag{3.27}$$

where T is the period of the helical curve $\mathbf{x}(t)$.

Let $\psi_1(\mathbf{x})$ be constructed by determining $c_1(x)$, $c_2(x)$, and $c_3(x)$ from Eqs. (3.23) and (3.26) with the condition $\langle c_3(\mathbf{x})\rangle=0$. The general $\psi(\mathbf{x})$ is $m_1\psi_1(\mathbf{x})+m_2\nabla H$, where m_1 and m_2 are constants. We consider a choice for $\xi(t)$ of the form

$$\xi(t)=\nabla H+\epsilon\psi_1(\mathbf{x}(t)), \tag{3.28}$$

where ϵ is a small parameter. We seek the solution $\mathbf{b}(t)$ to Eq. (1.6) in the form of an expansion in powers of ϵ:

$$\mathbf{b}(t)=\mathbf{B}(t)+\epsilon\mathbf{B}_1(t)+\epsilon^2\mathbf{B}_2+\cdots. \tag{3.29}$$

We will study the monodromy operator for Eq. (1.6). The condition $(\xi\cdot\mathbf{b})=0$ implies

$$(\mathbf{B}\cdot\nabla H)=0 \tag{3.30}$$

and

$$(\mathbf{B}\cdot\psi_1)+(\mathbf{B}_1\cdot\nabla H)=0. \tag{3.31}$$

Let $\mathbf{B}(0)=\mathbf{b}_0$ and

$$\dot{\mathbf{B}}=-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{B}+2\left(\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\mathbf{B}\cdot\nabla H\right)\nabla H/|\nabla H|^2. \tag{3.32}$$

We choose

$$\mathbf{B}_1(0) = -(\mathbf{b}_0 \cdot \psi_1)\nabla H/|\nabla H|^2|_{\mathbf{x}=\mathbf{x}_0}. \qquad (3.33)$$

We substitute the expansions in powers of ϵ for $\mathbf{b}$ and ξ into Eqs. (3.17) and (3.18) and use Eq. (3.31) to obtain

$$\frac{d}{dt}(\mathbf{B}_1 \cdot \mathbf{u}) = (\mathbf{B} \cdot \psi_1) + 2\left(\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{B} \cdot \nabla H\right) D_1/|\nabla H|^2 \qquad (3.34)$$

and

$$\frac{d}{dt}(\mathbf{B}_1 \cdot \nabla \times \mathbf{u}) = 2\left(\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{B} \cdot \nabla H\right) D_2/|\nabla H|^2, \qquad (3.35)$$

where the constants D_1 and D_2 equal $(\psi_1 \cdot \mathbf{u})$ and $(\psi_1 \cdot \nabla \times \mathbf{u})$, respectively. Hence,

$$\mathbf{B}_1(T) \cdot \mathbf{u}(\mathbf{x}_0) = \int_0^T \left\{\mathbf{B} \cdot \psi_1 + 2\left(\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{B} \cdot \nabla H\right) D_1/|\nabla H|^2\right\} dt \qquad (3.36)$$

and

$$\mathbf{B}_1(T) \cdot \nabla \times \mathbf{u}(\mathbf{x}_0) = \int_0^T \left\{2\left(\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{B} \cdot \nabla H\right) D_2/|\nabla H|^2\right\} dt. \qquad (3.37)$$

Let $(\mathbf{v}, \mathbf{w}, \nabla H/|\nabla H|^2)$ be the dual basis to $(\mathbf{u}, \nabla \times \mathbf{u}, \nabla H)$. The basis for the 2D space perpendicular to $\xi = \nabla H + \epsilon\psi_1$ could be written in the form,

$$\mathbf{v}_\epsilon = \mathbf{v} - \epsilon(\mathbf{v} \cdot \psi_1)\nabla H/|\nabla H|^2 + \epsilon^2(\mathbf{v} \cdot \psi_1)(\psi_1 \cdot \nabla H)\nabla H/|\nabla H|^2 + \cdots,$$

$$\mathbf{w}_\epsilon = \mathbf{w} - \epsilon(\mathbf{w} \cdot \psi_1)\nabla H/|\nabla H|^2 + \epsilon^2(\mathbf{w} \cdot \psi_1)(\psi_1 \cdot \nabla H)\nabla H/|\nabla H|^2 + \cdots.$$

We have the following monodromy operator:

$$\begin{pmatrix}\mathbf{v}_\epsilon\\ \mathbf{w}_\epsilon\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{v}_\epsilon\\ \mathbf{w}_\epsilon\end{pmatrix} + \epsilon J_1\begin{pmatrix}\mathbf{v}_\epsilon\\ \mathbf{w}_\epsilon\end{pmatrix} + \epsilon^2 J_2\begin{pmatrix}\mathbf{v}_\epsilon\\ \mathbf{w}_\epsilon\end{pmatrix} + 0(\epsilon^3), \qquad (3.38)$$

where

$$J_1 = \begin{pmatrix} \int_0^T \left\{\mathbf{v} \cdot \psi_1 + 2D_1\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{v} \cdot \nabla H/|\nabla H|^2\right\} d\tau, & 2D_2\int_0^T \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{v} \cdot \nabla H/|\nabla H|^2\, d\tau \\ \int_0^T \left\{\mathbf{w} \cdot \psi_1 + 2D_1\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{w} \cdot \nabla H/|\nabla H|^2\right\} d\tau, & 2D_2\int_0^T \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{w} \cdot \nabla H/|\nabla H|^2\, d\tau \end{pmatrix}. \qquad (3.39)$$

From Proposition 2.1 and the fact that

$$(\nabla H + \epsilon\psi_1)(\mathbf{x}(T)) = (\nabla H + \epsilon\psi_1)(\mathbf{x}(0)),$$

we observe that the determinant of the monodromy matrix is identically 1. In terms of the expansion in powers of ϵ the monodromy matrix has the form

$$I + \epsilon J_1 + \epsilon^2 J_2 + \cdots.$$

Hence,

$$1 = 1 + \epsilon \operatorname{Tr} J_1 + \epsilon^2(\operatorname{Tr} J_2 + \det J_1) + \cdots. \qquad (3.40)$$

It follows from Eq. (3.40) that

$$\operatorname{Tr} J_1 = 0$$

and

$$\operatorname{Tr} J_2 = -\det J_1. \qquad (3.41)$$

Thus the equation for the Floquet exponents λ associated with the monodromy operator for Eq. (1.6) is

$$\lambda^2 - \lambda(2 + \epsilon^2 \operatorname{Tr} J_2 + \cdots) + 1 = 0. \qquad (3.42)$$

Hence it follows from Eqs. (3.41) and (3.42) that a sufficient condition to have one Floquet exponent with modulus greater than unity is

$$\det J_1 < 0. \qquad (3.43)$$

Thus Eq. (3.43) gives a sufficient condition for the lhs of Eq. (1.3) to be positive, hence implying instability for fluid flow. Using Proposition (3.1) in the expression for J_1 given by Eq. (3.39), we may write the sufficient condition for instability in the form

$$D_2\left[\int_0^T \mathbf{v} \cdot \psi_1\, dt \int_0^T \mathbf{w} \cdot (\nabla H)^{\cdot}\ /|\nabla H|^2 dt - \int_0^T \mathbf{w} \cdot \psi_1\, dt \int_0^T \mathbf{v} \cdot (\nabla H)^{\cdot}\ /|\nabla H|^2 dt\right] > 0. \qquad (3.44)$$

We note that in terms of the "action-angle" coordinates introduced in the proof of Proposition 3.4, the 1-form ψ_1 can be written as

$$\psi_1 = -\omega_2'\, d\alpha + \omega_1' d\beta, \quad \mathrm{mod}\ dz, \qquad (3.45)$$

hence from Eq. (3.13) and the definition of D_2 as $\psi_1 \cdot \nabla \times \mathbf{u}$ we have

$$D_2 = \omega_1'^2 + \omega_2'^2 > = 0. \qquad (3.46)$$

Thus a sufficient condition for instability for a general axisymmetric steady integrable flow is

$$\begin{vmatrix} \int_0^T \mathbf{v} \cdot \psi_1\, dt, & \int_0^T \mathbf{w} \cdot \psi_1\, dt \\ \int_0^T V\, dt, & \int_0^T W\, dt \end{vmatrix} > 0, \qquad (3.47)$$

where

$$V = \mathbf{v} \cdot (\nabla H)^{\cdot}\ /|\nabla H|^2$$

and

$$W=\mathbf{w}\cdot(\nabla H)\ /|\nabla H|^2. \tag{3.48}$$

IV. A GEOMETRICAL INTERPRETATION OF THE INSTABILITY CRITERION

From the definition of $\mathbf{v}$ and $\mathbf{w}$ as the dual vectors to $\mathbf{u}$ and $\nabla\times\mathbf{u}$ it follows that

$$\psi_1=c_1\mathbf{u}+c_2\nabla\times\mathbf{u}+c_3\nabla H=D_1\mathbf{v}+D_2\mathbf{w}+c_3\nabla H. \tag{4.1}$$

The boundedness condition on ξ given by Eq. (3.26) can be written in the form

$$2\int_0^T(c_1\mathbf{u}+c_2\nabla\times\mathbf{u})\cdot(\nabla H)\ /|\nabla H|^2\,dt-\int_0^T c_1\,dt=0. \tag{4.2}$$

We substitute for ψ_1 from Eq. (4.1) and use the definition of V and W given in Eq. (3.48) to write Eq. (4.2) in the form

$$D_1\int_0^T(V-\mathbf{v}^2/2)dt+D_2\int_0^T(W-\mathbf{v}\cdot\mathbf{w}/2)dt=0. \tag{4.3}$$

We substitute for $\psi_1\cdot\mathbf{v}$ and $\psi_1\cdot\mathbf{w}$ using Eq. (4.1) and eliminate D_1 from Eq. (4.3) to write the instability condition (3.44) in the form

$$\left(-D_2^2\Big/\int_0^T(2V-\mathbf{v}^2)dt\right)\left[2\int_0^T\left(\mathbf{w}\int_0^T V\,dt-\mathbf{v}\int_0^T W\,dt\right)^2dt-\left(\int_0^T V\,dt\right)\times\left[\int_0^T\mathbf{v}^2\,dt\int_0^T\mathbf{w}^2\,dt-\left(\int_0^T\mathbf{v}\cdot\mathbf{w}\,dt\right)^2\right]\right]>0. \tag{4.4}$$

A sufficient condition for Eq. (4.4) to hold is

$$\int_0^T V\,dt\leqslant 0. \tag{4.5}$$

In order to give a geometric interpretation of the terms in the above sufficient criterion for instability we introduce notation from differential geometry. It follows from the definition of V and W in Eq. (3.48) that

$$(\nabla H)\ /|\nabla H|^2=V\mathbf{u}+W\nabla\times\mathbf{u}-\left(\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\nabla H\cdot\nabla H\right)\nabla H/|\nabla H|^4 \tag{4.6}$$

and

$$V=(|\nabla\times\mathbf{u}|^2\mathbf{u}\cdot(\nabla H)-\mathbf{u}\cdot\nabla\times\mathbf{u}(\nabla\times\mathbf{u})\cdot(\nabla H)\ )/|\nabla H|^2, \tag{4.7}$$

$$W=(|\mathbf{u}|^2(\nabla\times\mathbf{u})\cdot(\nabla H)-(\mathbf{u}\cdot\nabla\times\mathbf{u})\mathbf{u}\cdot(\nabla H)\ )/|\nabla H|^2. \tag{4.8}$$

We eliminate the $(\nabla\times\mathbf{u})\cdot(\nabla\mathrm{H})$ from Eq. (4.7) using Eq. (4.8) to obtain

$$V=\mathbf{u}\cdot(\nabla H)\ |\mathbf{u}|^2-(\mathbf{u}\cdot\nabla\times\mathbf{u})W/|\mathbf{u}|^2. \tag{4.9}$$

In terms of elementary concepts from differential geometry (see, for example, Hsiung[15]) the quantities W and $\mathbf{u}\cdot(\nabla H)\ /|\mathbf{u}|^2$ may be written as

$$W=-\tau_g|\mathbf{u}|^2/|\nabla H|^2 \tag{4.10}$$

and

$$\mathbf{u}\cdot(\nabla H)\ /|\mathbf{u}|^2=-\kappa\hat{n}\cdot\nabla H, \tag{4.11}$$

where κ is the *curvature* of the streamline, τ_g is the *geodesic torsion* of the streamline and $\hat{n}$ is the principal normal to the streamline. Thus from Eq. (4.9) the instability criterion (4.5) may be written in terms of geometric quantities as the condition

$$\int_0^T\kappa\hat{n}\cdot\nabla H-\tau_g(\mathbf{u}\cdot\nabla\times\mathbf{u})/|\nabla H|^2\,dt\geqslant 0. \tag{4.12}$$

The condition (4.12) can be viewed as a generalized Rayleigh criterion for the instability of an arbitrary steady integrable flow. We note that for particular flows where either the helicity $\mathbf{u}\cdot\nabla\times\mathbf{u}$ is zero or the geodesic torsion τ_g is zero, the criterion (4.12) reduces to the simple condition on the curvature

$$\int_0^T\kappa\hat{n}\cdot\nabla H\,dt\geqslant 0. \tag{4.13}$$

We remark that this condition is in agreement with the condition derived by Lifschitz and Hameiri[7] for the instability of a vortex ring without swirl.

ACKNOWLEDGMENTS

S. Friedlander acknowledges NSF Grant No. DMS 9000137. M. Vishik acknowledges NSF Grant No. DMS 9105688 and is grateful to the University of Chicago for its extremely kind hospitality in 1990–1992. We express our gratitude and appreciation for the excellent typing of the manuscript by S. Roper.

[1] M. M. Vishik and S. Friedlander, J. Math. Pure et Appliques (to be published).
[2] V. I. Arnold, Appl. Math. Mech. 36, 236 (1992).
[3] K. S. Eckhoff and L. Storesletten, J. Fluid Mech. 99, 433 (1980).
[4] K. S. Eckhoff, J. Differential Equations 40, 94 (1981).
[5] B. Bayly, Phys. Fluids 31, 56 (1988).
[6] A. Lifschitz and E. Hameiri, Phys. Fluids A 3, 2644 (1991).
[7] A. Lifschitz and E. Hameiri, Commun. Pure Appl. Math. (to be published).
[8] S. Friedlander and M. M. Vishik, Chaos 1, 198 (1991).
[9] S. Friedlander and M. M. Vishik, Phys. Rev. Lett. 66, 2204 (1991).
[10] S. Friedlander and M. M. Vishik, *Instability criteria in fluid dynamics. Proc. NATO Workshop on Topological Methods in Fluid Dynamics*, edited by H. K. Moffatt (Kluwer Academic, Dordrecht, 1992).
[11] V. I. Arnold, Annales de l'Institut Fourier 16, 347 (1966).
[12] M. Henon, Comptes Rendues de l'Acad. Sci. 262, 312 (1966).
[13] T. Dombre, U. Frisch, J. M. Greene, M. Henon, A. Mehr, and A. M. Soward, J. Fluid Mech. 167, 353 (1986).
[14] V. I. Arnold, Sel. Math. Sov. 5, 327 (1986) (an English translation).
[15] C. C. Hsiung, *A First Course on Differential Geometry* (Wiley, New York, 1981).