The Dissertation Committee for Ernest Eugene Fontes
certifies that this is the approved version of the following dissertation:

A Weighty Theorem of the Heart for the Algebraic K-Theory of Higher Categories

Committee:

Andrew J. Blumberg, Supervisor

Clark Barwick

David Ben-Zvi

Andrew Neitzke

A Weighty Theorem of the Heart for the Algebraic K-Theory of Higher Categories

by

Ernest Eugene Fontes, A.B., M.A.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2017

In loving memory of Ernest and Sylvia Fontes

Acknowledgments

This thesis is the result of many years of toil and suffering during my tenure in graduate school. For my survival to see the successful completion of this project, I owe thanks to a great number of people.

First and foremost, thanks to my advisor Andrew Blumberg, for your overwhelming support, for guiding me when I was blind, for recommendations that enriched my life, and for all the math. I wouldn't have made it without you pushing me. Thanks to Ben Antieau, Clark Barwick, David Ben-Zvi, David Gepner, and Randy McCarthy, for discussions that helped me shape this thesis into its final form. Thanks to Cary Malkiewich, Mona Merling, and Sarah Yeakel, for company at numerous conferences and for many conversations that kept me excited about math. Thanks to Mike Hopkins, for getting me hooked on homotopy theory. Thanks to the Ithaca High School mathematics department for providing me with an outstanding foundation for my later studies.

Thanks to my family, for your love and boundless support, and for making those journeys down to Texas. Thanks for the camaraderie of my many wonderful friends, with a special shout-out to the pizza crew and the wine seminar for permitting me to indulge in my favorite hobbies and share them with you. A few special friends require additional recognition for their support during the last few years. Thanks to Aaron Royer, my mathematical elder brother, for years of good company, for late-night board games, for all our fun times in the Bay, and for the math, the math, the math, and the math. Thanks to Jen Berg, for emotional support and baking in equal measures all throughout graduate school. Thanks to Lisa Piccirillo, for goading me towards indulgence, for baking pies dangerously well, for the aesthetically-pleasing diagrams, and for being the best officemate I could ask for. Thanks to Jonathan Campbell, for our escalating hobbies, for force-feeding me the Mountain Goats discography, for many many trips to Beware Coffee, for beard-grooming tips, and for keeping me optimistic when it was tough.

Research takes a lot of time and I was lucky to be able to do it in some great places, so thanks to MSRI, Small Victory, and Beware Coffee for the hospitality I needed at critical points in my PhD. Thanks Carly, most of this thesis was written while listening to you. These acknowledgments are by no means comprehensive, but in an effort for completion I will end thus: if you made me smile in the last seven years, then thank you, this is for you.

A Weighty Theorem of the Heart for the Algebraic K-Theory of Higher Categories

Publication No. ____________________

Ernest Eugene Fontes, Ph.D.
The University of Texas at Austin, 2017

Supervisor: Andrew J. Blumberg

We introduce the notion of a bounded weight structure on a stable ∞-category and prove a generalization of Waldhausen's sphere theorem for the algebraic K-theory of higher categories. The algebraic K-theory of a stable ∞-category with a bounded non-degenerate weight structure is proven to be equivalent to the algebraic K-theory of the heart of the weight structure. We relate this theorem to previous results as well as new applications.

Table of Contents

Chapter 1

Introduction

A weight structure is a collection of data on a triangulated category which is designed to provide a weak notion of cellular filtrations. In this paper, we prove that if a stable ∞-category $\mathcal{C}$ admits a suitable weight structure on its homotopy category then the algebraic K-theory of $\mathcal{C}$ is equivalent to that of a subcategory, namely the algebraic K-theory of the heart of the weight structure $\mathcal{C}_{\heartsuit w}$. The heart of the weight structure is the collection of building blocks for this cellular theory: the "spheres" in the weight structure which can be used to construct objects of $\mathcal{C}$ through gluing constructions. Morally, our main theorem shows that the algebraic K-theory of a (stable) category of finite cell objects is equivalent to the algebraic K-theory of the spheres.

Theorem 1.0.1 (Weighty theorem of the heart, 5)**.** *If $\mathcal{C}$ is a stable ∞-category equipped with a bounded non-degenerateweight structure w, then the inclusion of the heart of the weight structure $\mathcal{C}_{\heartsuit w} \hookrightarrow \mathcal{C}_{\heartsuit}$ induces an equivalence on algebraic K-theory*

$$K(\mathcal{C}) \simeq K(\mathcal{C}_{\heartsuit w}).$$

Algebraic K-theory is an invariant of rings that arose from considering Euler characteristics. The classical algebraic K-groups K_0, K_1, and K_2 were defined algebraically and were known to fit into exact sequences like those for cohomology theories. Quillen introduced a homotopical construction of an algebraic K-theory spectrum whose homotopy groups extended the classical definitions to define higher algebraic K-theory groups K_i for $i \geq 0$ [19].

The applications for algebraic K-theory span several fields. The algebraic K-theory of rings of integers in number fields contains arithmetic information: the class group, the Brauer group, and so on [23]. On the other hand, the Whitehead torsion of a manifold M is controlled by $K_1(\mathbb{Z}[\pi_1 M])$. In a vast generalization of this, Waldhausen showed that the algebraic K-theory of $\Sigma^\infty(\Omega M)_+$, the "spherical group ring of the loop space of M", contains information about a stabilization of $B\mathrm{Diff}(M)$ [22].

Following Quillen and Waldhausen, the modern view is that algebraic K-theory is a functor of modules categories (or their subcategories) to spectra. Recent work has extended the construction of algebraic K-theory to higher categories that behave like module categories and produced universal characterizations of the algebraic K-theory functor in this setting [2, 3]. One consequence of this work is that most of Quillen and Waldhausen's foundational theorems about the behavior of algebraic K-theory have been established in a very general context. In particular, this framework has permitted new localization [2, 4], devissage [1], and approximation [2, 13] results.

This paper essentially completes the program of lifting fundamental theorems of Quillen and Waldhausen to the higher categorical context. Our weighty theorem of the heart is the higher-categorical analog of Waldhausen's *sphere theorem* [22, 1.7.1]. In Waldhausen's setting, a category $\mathcal{C}$ equipped with cofibrations, weak equivalences, a cylinder functor, and a well-behaved homology theory is shown to have the same K-theory, after stabilizing, as the stable homology spheres. Waldhausen's homology theory is required to satisfy a condition which makes it display all objects of $\mathcal{C}$ as cell objects weakly built out of homology spheres.

For the weighty theorem of the heart, a weight structure replaces Waldhausen's homology functor to provide cellular filtrations. Weight structures were introduced by Bondarko in [7] and studied extensively on triangulated categories [7–9, 9, 11]. Advantageously, weight structures are specified entirely on the homotopy category of our stable ∞-category $\mathcal{C}$. The data of a weight structure consists of choices of subcategories of objects built with cells in degrees $\leq n$ (or $\geq n$) so that every object in $\mathcal{C}$ has at least one associated n-skeleton for all $n \in \mathbb{Z}$. These skeleta are not assumed to be functorial in any way and in fact rarely are. We emphasize that a weight structure should be thought of as providing weak n-skeleta for all objects in $\mathcal{C}$.

It would be appropriate at this point to issue a word of clarification about the various theorems of the heart. Neeman proved a theorem of the heart for the K-theory of triangulated categories [16–18] which was later proven more generally for exact ∞-categories by Barwick [1]. In both cases, the theorem said that a bounded t-structure on the homotopy category induced an equivalence $K(\mathcal{C}) \simeq K(\mathcal{C}_{\heartsuit t})$ between the K-theory of the category and that of the heart of the t-structure. While there are superficial similarities between the theorems—and, as it turns out, the definitions of weight and t-structures—these theorems have little to do with one another. A t-structure provides a Postnikov tower for every object of $\mathcal{C}$ whereas a weight structure provides a cellular filtration. Furthermore, while t-structures (and Postnikov towers) are functorial, weight structures (and cellular filtrations) rarely are.

Weight structures are less structured than t-structures. For example, the hearts of t-structures are abelian categories while the hearts of weight structures are merely additive. This flexibility makes it is easier to place a weight structure on a category than it is to construct a t-structure. On the other hand, it also makes the proof of our weighty theorem of the heart not as conceptually elegant as that for Barwick's theorem of the heart.

Our weighty theorem of the heart is an example of an equivalence between K-theory spectra that is *not* induced by an equivalence of derived categories. Quillen's devissage theorem [19, 4] provides such an equivalence. Blumberg and Mandell's devissage theorem for ring spectra [4] and the closely-related theorem of the heart due to Barwick [1] are the only other such non-trivial equivalences known to the author.

The two familiar examples of weight structures to keep in mind are on finite spectra and bounded chain complexes over a ring R. Finite spectra admit a weight structure where weights record having the homotopy type of a cell complex with cells in certain degrees. The heart of the weight structure on finite spectra consists of all finite wedge sums of the sphere spectrum. For bounded chain complexes, the weight structure identifies chain complexes quasi-isomorphic to complexes of projectives concentrated in specified degrees. The heart of this weight structure comprises the projective modules included as complexes concentrated in degree 0.

In addition to lifting Waldhausen's sphere theorem to quasicategories, the weighty theorem of the heart generalizes several previous results in the literature. Bondarko proves a version of the theorem on K_0 for triangulated categories equipped with bounded weight structures [8, 5.3.1]. We reproduce his result as a corollary of our weighty theorem of the heart.

Corollary 1.0.2 (5.0.1)**.** *If $\mathcal{T}$ is a triangulated category equipped with a non-degenerate bounded weight structure w, then $K_0(\mathcal{T}) \simeq K_0(\mathcal{T}_{\heartsuit w})$.*

The projective weight structure on bounded chain complexes gives an equivalence of $K(R)$ with the K-theory of the category of finitely-generate projective R-modules. Generalizing this weight structure slightly, we produce a formulation of the Gillet–Waldhausen theorem [21, 1.11.7].

Corollary 1.0.3 (Gillet–Waldhausen theorem; 3.2.1)**.** *If $\mathcal{A}$ is an abelian category with enough projectives, the algebraic K-theory of $\mathcal{A}$ (the algebraic K-theory of the derived category of compact objects in $\mathcal{A}$) is equivalent to the algebraic K-theory of the compact projective objects in $\mathcal{A}$.*

$$K(\mathcal{A}^\omega) \simeq K(Proj(\mathcal{A}^\omega))$$

The chief motivation for the weighty theorem of the heart is that $\mathcal{C}_{\heartsuit w}$ is simpler than $\mathcal{C}$ and so its K-theory can be described in an alternate manner.

In particular, all cofiber sequences in the heart split in the homotopy category. Hence, $K(\mathcal{C}_{\heartsuit w})$ permits a description in the spirit of Quillen's plus construction for K-theory:

Theorem 1.0.4 (5.2.1)**.** *If $\mathcal{C}_{\heartsuit w}$ is the heart of a weight structure, then all cofiber sequences split in the homotopy category and*

$$K(\mathcal{C}_{\heartsuit w}) \simeq \coprod_{[X] \in \operatorname{ob} \iota \mathcal{C}_{\heartsuit w}} B \operatorname{Aut}(X)^+$$

where $[X]$ is an equivalence class of objects in $\mathcal{C}_{\heartsuit w}$ and $(-)^+$ denotes the group completion of the topological monoid $B \operatorname{Aut}(X)$.

In chapter 2, we define weight structures and recount some of their properties. We develop a yoga for manipulating weights and recount results of Bondarko on generating weight structures on categories. Finally, we compare the language of weight structures to Waldhausen's formulation of the sphere theorem.

In chapter 3, we enumerate examples of weight structures and explain applications of the weighty theorem of the heart in each case. Several conjectural examples are mentioned that merit exploration in future work.

In chapter 4, we build the technical tools we require for the proof of the main theorem. We introduce cellular filtrations arising from weight structures and prove that they form Waldhausen ∞-categories. We show that localizing these categories at equivalences reflected from $\mathcal{C}$ induces a suitable model for the K-theory of $\mathcal{C}$.

Finally, chapter 5 comprises the proof of the weighty theorem of the heart. The proof relies on tools developed in chapters 2 and 4 as well as results of Barwick [2]. At the end of the chapter, we prove that the K-theory of the heart of a weight structure admits a description analogous to Quillen's plus construction.

Chapter 2

Weight structures on stable ∞-categories

In this chapter, we define weight structures and provide an overview of their basic properties.

2.1 Definitions

Definition 2.1.1 ([9]). Let $\mathcal{T}$ be a triangulated category. A *weight structure* w on $\mathcal{T}$ is a pair of full subcategories $\mathcal{T}_{w\leq 0}$ and $\mathcal{T}_{w\geq 0}$ (closed under retract and finite coproducts) satisfying the following properties. We adopt the notation that $\mathcal{T}_{w\leq n} := \Sigma^n \mathcal{T}_{w\leq 0}$ and $\mathcal{T}_{w\geq n} := \Sigma^n \mathcal{T}_{w\geq 0}$.

1. We have the inclusions $\mathcal{T}_{w\geq 1} \subseteq \mathcal{T}_{w\geq 0}$ and $\mathcal{T}_{w\leq -1} \subseteq \mathcal{T}_{w\leq 0}$.

2. (Orthogonality) For $X \in \mathcal{T}_{w\leq 0}$ we have $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for any $Y \in \mathcal{T}_{w\geq 1}$.

3. (Weight decomposition) For any $X \in \mathcal{T}$ there exists a distinguished triangle
$$X' \longrightarrow X \longrightarrow X''$$
with $X' \in \mathcal{T}_{w\leq 0}$ and $X'' \in \mathcal{T}_{w\geq 1}$.

We will call $\mathcal{T}_{\heartsuit w} := \mathcal{T}_{w\leq 0} \cap \mathcal{T}_{w\geq 0}$ the *heart* of the weight structure.

We have chosen to use the *homological* sign convention for weight structures and all examples and statements will use that convention. Some papers in the literature (*e.g.,* [9]) opt to use cohomological signs, writing $\mathcal{T}^{w\geq 0}$ for what we denote by $\mathcal{T}_{w\leq 0}$. Due to a lack of consensus in the literature, we use the convention that appears more agreeable for homotopy theorists.

As defined, weight structures are overdetermined. Each subcategory of the weight structure $\mathcal{T}_{w\leq 0}$ or $\mathcal{T}_{w\geq 0}$ determines the other by orthogonality. That is, $\mathcal{T}_{w\leq 0}$ is precisely the full subcategory on objects X with $\mathcal{T}(X, Y) = 0$ for all $Y \in \mathcal{T}_{w\geq 1}$. By translating (through suspension), we can provide weight decompositions of an object X at any degree. That is, the degree-zero decomposition for $\Sigma^{-n}X$, $A \to \Sigma^{-n}X \to B$ provides a degree-n-decomposition $\Sigma^n A \to X \to \Sigma^n B$ with $\Sigma^n A \in \mathcal{T}_{w\leq n}$ and $\Sigma^n B \in \mathcal{T}_{w\geq n+1}$.

Note that by suspending and desuspending we can provide weight decompositions of an object X at any degree. That is, the degree-zero decomposition for $\Sigma^{-n}X$, $A \to \Sigma^{-n}X \to B$ provides a degree-n-decomposition $\Sigma^n A \to X \to \Sigma^n B$ where $\Sigma^n A \in \mathcal{D}_{w\leq n}$ and $\Sigma^n B \in \mathcal{C}_{w\geq n+1}$.

The homotopy category of a stable ∞-category is triangulated [15, 1.1.2.15]. Weight structures are defined on stable ∞-categories by way of the homotopy category.

Definition 2.1.2. Let $\mathcal{C}$ be a stable ∞-category. A *weight structure* on $\mathcal{C}$ will be a weight structure on the triangulated category $h\mathcal{C}$.

Full subcategories of ∞-categories are specified by full subcategories of their homotopy categories. A weight structure on a stable ∞-category $\mathcal{C}$ is equivalently defined by two full ∞-subcategories $\mathcal{C}_{w\leq 0}$ and $\mathcal{C}_{w\geq 0}$ (with $\mathcal{C}_{w\geq n}$ and $\mathcal{C}_{w\leq n}$ defined as above) and (co)fiber sequences for each object

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & B & \longrightarrow & \Sigma A \end{array}$$

where both squares are pushouts in $\mathcal{C}$ and $A \in \mathcal{C}_{w\leq n}$ and $B \in \mathcal{C}_{w\geq n+1}$. Furthermore, the orthogonality condition requires that $\pi_0\mathcal{C}(A, B) = 0$ for all $A \in \mathcal{C}_{w\leq n}$ and $B \in \mathcal{C}_{w\geq n+1}$. We will further abuse notation by referring to $\mathcal{C}_{\heartsuit w} := \mathcal{C}_{w\leq 0} \cap \mathcal{C}_{w\geq 0}$ as the heart of the weight structure on $\mathcal{C}$.

Definition 2.1.3. We say that a weight structure w on a triangulated category $\mathcal{T}$ is *non-degenerate* if $\bigcap_{n\to\infty} \mathcal{T}_{w\geq n} = 0$ and $\bigcap_{n\to-\infty} \mathcal{T}_{w\leq n} = 0$.

A weight structure w on a stable ∞-category $\mathcal{C}$ is non-degenerate precisely when $\bigcap_{n\to\infty} \mathcal{C}_{w\geq n}$ and $\bigcap_{n\to-\infty} \mathcal{C}_{w\leq n}$ are equivalent to the subcategory of zero objects $\mathcal{C}_0$ in $\mathcal{C}$,

Throughout the paper, we will view the stable ∞-category $\mathcal{C}$ as a Waldhausen ∞-category equipped with the maximal pair structure in which all edges are ingressive. The heart $\mathcal{C}_{\heartsuit w}$ is given a sub-Waldhausen structure: an edge is ingressive only when its cofiber also lives in $\mathcal{C}_{\heartsuit w}$.

Several examples of weight structures are worked out in chapter 3. We mention the following example for the reader to keep in their mind.

Example 2.1.1. *The* Postnikov weight structure *on finite spectra takes*

$$\mathrm{Sp}_{w\geq n} = \{E : \pi_*(E) = 0, \forall * < n\}$$

that is, the n-connective spectra, and

$$\mathrm{Sp}_{w\leq n} = \{E : H\mathbb{Z}_*(E) = 0, \forall * > n \textit{ and } H\mathbb{Z}_n(E) \textit{ free}\}.$$

The heart $\mathrm{Sp}_{\heartsuit w}$ *consists spectra weakly equivalent to finite wedge sums of copies of S^0.*

Weight structures generalize cellular structures. $\mathcal{C}_{w\leq n}$ is the subcategory of "cell-less-than-n" objects and $\mathcal{C}_{w\geq n}$ is the "cell-greater-than-n" subcategory. The weight decomposition is analogous to the inclusion of an n-skeleton into X.

Weight structures are not formally dual to t-structures. The decompositions arising from t-structures are unique as can be summarized by the existence of a localizing "truncation" functor $\mathcal{C} \to \mathcal{C}_{t\geq n}$ for all n. In contrast, weight structure decompositions have no such unicity: there can be many choices of n-skeleta for each object. Instead for weight decompositions, we have the following result in the homotopy category $h\mathcal{C}$.

Proposition 2.1.2. *Let $A_{w\leq n} \to X \to B_{w\geq n+1}$ denote a weight decomposition of X at degree n. If $n \leq m$ then we get maps*

$$\begin{array}{ccccc} A_{w\leq n} & \longrightarrow & X & \longrightarrow & B_{w\geq n+1} \\ \downarrow{\scriptstyle \exists} & & \| & & \downarrow{\scriptstyle \exists} \\ A'_{w\leq m} & \longrightarrow & X & \longrightarrow & B'_{w\leq m} \end{array}$$

making the diagram commute. If $n < m$ then the induced maps are unique.

As a consequence of this proposition, maps between skeleta are determined up to homotopy provided they map into a strictly "higher" skeleton.

Remark 2.1.1. As a subcategory of C, $\mathcal{C}_{w\leq n}$ is closed under forming fibers. Likewise, $\mathcal{C}_{w\geq n}$ is closed under forming cofibers. If we consider $\mathcal{C}$ as a maximal exact ∞-category in the sense of [1], $\mathcal{C}_{w\geq n}$ is a Waldhausen ∞-subcategory and $\mathcal{C}_{w\leq n}$ is a coWaldhausen ∞-subcategory.

Definition 2.1.4. The weight structure on $\mathcal{C}$ is *bounded* if

$$\bigcup_{n\geq 0} (\mathcal{C}_{w\geq -n} \cap \mathcal{C}_{w\leq n}) = \mathcal{C}.$$

The weight structure defined above on the category of finite spectra is bounded. The same weight structure on the category of (not necessarily finite) spectra is not bounded but its "bounded closure" $\bigcup_{n\geq 0} (\mathcal{C}_{w\geq -n} \cap \mathcal{C}_{w\leq n})$ consists of the finite spectra.

2.2 Properties of weight structures

In this section we establish some basic properties of weight structures and how weights interact with forming fibers and cofibers. The punchline of the section is that a weight structure provides cellular decompositions of objects.

When convenient, we will will use subscripts to denote the weights of given objects. That is, $A_{w\leq n}$ will denote that the object A has weight $w \leq n$ in $\mathcal{C}$.

Proposition 2.2.1. *$\mathcal{C}_{w\leq n}$ determines $\mathcal{C}_{w\geq n+1}$ and vice-versa: $\mathcal{C}_{w\geq n+1}$ is precisely those $X \in \mathcal{C}$ with $\mathrm{Hom}(Y, X) = 0$ for all $Y \in \mathcal{C}_{w\leq n}$.*

Proof. If X is as given in the statement, then it's weight decomposition at n is a fiber sequence

$$X' \longrightarrow X \longrightarrow X''$$

but $X' \in \mathcal{C}_{w\leq n}$ so $X' \to X$ is the zero map in $h\mathcal{C}$. Thus $X \simeq X''$. The proof is identical to show that an object lives in $\mathcal{C}_{w\geq n+1}$. □

Proposition 2.2.2. *$\mathcal{C}_{w\geq n}$ is closed under retracts and cofibers.*

Proof. Say X is a retract of Y in $\mathcal{C}$ and Y lies in $\mathcal{C}_{w\leq n}$. Then it will be as well in $h\mathcal{C}$. In particular, fix $i : X \to Y$ and $r : Y \to X$ with $r \circ i = \mathrm{id}_X$ in $h\mathcal{C}$. For any $Z \in \mathcal{C}_{w\geq n+1}$ we have induced maps

$$h\mathcal{C}(X, Z) \xrightarrow{r^*} h\mathcal{C}(Y, Z) \xrightarrow{i^*} h\mathcal{C}(X, Z)$$

whose composite must be the identity. This demonstrates $h\mathcal{C}(X, Z)$ as a retract of $h\mathcal{C}(Y, Z)$. The latter is trivial so the former must be as well. The previous proposition concludes that $\mathcal{C}_{w\geq n}$ is closed under retracts.

Now suppose X and Y both live in $\mathcal{C}_{w\geq n}$ and

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cofiber}(f)$$

is a cofiber sequence in $\mathcal{C}$. Rotating forward, we have a cofiber sequence $Y \to \mathrm{cofiber}(f) \to \Sigma X$. Let Z be any object of $\mathcal{C}_{w\geq n+1}$. Since $h\mathcal{C}(-, Z)$ carries cofiber sequences to fiber sequences, we have the following fiber sequence.

$$h\mathcal{C}(\Sigma X, Z) \longrightarrow h\mathcal{C}(\text{cofiber}(f), Z) \longrightarrow h\mathcal{C}(Y, Z)$$

The axioms for a weight structure tell us that $\Sigma X \in \mathcal{C}_{w\geq n+1} \subseteq \mathcal{C}_{w\geq n}$ and thus all terms of this sequence must be trivial as the outer two are. The previous proposition again concludes the proof. □

Proposition 2.2.3. *$\mathcal{C}_{w\leq n}$ is closed under retracts and forming fibers.*

Proof. Both proofs are nearly identical to those for the previous proposition when one replaces $h\mathcal{C}(-, Z)$ with $h\mathcal{C}(Z, -)$. $h\mathcal{C}(Z, -)$ carries fiber sequences to fiber sequences, and since $\mathcal{C}$ is stable, fiber and cofiber sequences coincide. Backing up a fiber sequence $\text{fiber}(f) \to X \to Y$ to produce $\Sigma^{-1}Y \to \text{fiber}(f) \to X$ and similar arguments conclude the proof. □

Remark 2.2.1. $\mathcal{C}_{w\geq n} \subset \mathcal{C}$ are Waldhausen subcategories of $\mathcal{C}$. If $\mathcal{C}$ is an exact stable ∞-category in the sense of [1] with a weight structure then $\mathcal{C}_{w\leq n} \subset \mathcal{C}$ are coWaldhausen subcategories.

The following is a lemma about triangulated categories that is surprisingly useful for manipulating weight structures.

Lemma 2.2.4 ([8, 1.4.1])**.** *Let $X \to A \to B \to \Sigma X$ and $X' \to A' \to B' \to \Sigma X$ be two distinguished triangles in a triangulated category $h\mathcal{C}$.*

1. *If $h\mathcal{C}(B, \Sigma A') = 0$ then for any $g : X \to X'$ there exist $h : A \to A'$ and $i : B \to B'$ completing g to a map of distinguished triangles.*

2. If furthermore $h\mathcal{C}(B, A') = 0$ then h and i are unique.

Proof. By the axioms for a triangulated category it suffices to provide one of the two desired maps. Applying $h\mathcal{C}(B, -)$ to the second distinguished triangle yields the following exact sequence.

$$h\mathcal{C}(B, A') \longrightarrow h\mathcal{C}(B, B') \longrightarrow h\mathcal{C}(B, \Sigma X') \longrightarrow h\mathcal{C}(B, \Sigma A')$$

The assumption $h\mathcal{C}(B, \Sigma A') = 0$ lets us lift the composite $B \to \Sigma X \xrightarrow{\Sigma g} \Sigma X'$ to $i : B \to B'$. If the second assumption holds this map is determined uniquely by g. □

Now let $A_{w \leq n} \to X \to B_{w \geq n+1}$ and $A'_{w \leq m} \to X \to B_{w \geq m+1}$ denote two weight decomposition of X at degrees n and m, respectively.

Corollary 2.2.5. *There are maps $a : A_{w \leq n} \to A'_{w \leq m}$ and $b : B_{w \geq n+1} \to B'_{w \geq m+1}$ that assemble into a map of distinguished triangles*

$$\begin{array}{ccccc} A_{w \leq n} & \longrightarrow & X & \longrightarrow & B'_{w \geq n+1} \\ \downarrow{\scriptstyle a} & & \| & & \downarrow{\scriptstyle b} \\ A'_{w \leq m} & \longrightarrow & X & \longrightarrow & B'_{w \geq m+1} \end{array}$$

whenever $m \geq n$. If $m \geq n+1$ then these maps are unique in $h\mathcal{C}$.

Proof. Apply Lemma 2.2.4 to the sequences provided with the map $\mathrm{id} : X \to X$. □

Note that the maps are unique up to choice of the two decompositions (which are not *a priori* unique).

Corollary 2.2.6. *If X has weight $w \geq n$ then for all $k \geq n-1$ any weight decomposition*

$$A_{w \leq k} \longrightarrow X_{w \geq n} \longrightarrow B_{w \geq k+1}$$

is equivalent to the trivial decomposition

$$* \longrightarrow X_{w \geq n} \longrightarrow X_{w \geq n}$$

Proof. Since $k+1 \geq n$, X lies in $\mathcal{C}_{w \geq k+1}$. By the lemma we have maps between the two sequences. The unicity of maps into and out of $*$ makes these unique. Thus $A \simeq *$ and $B \simeq X$. □

Corollary 2.2.7. *If X has weight $w \leq n$ then for all $k \geq n$ any weight decomposition*

$$A_{w \leq k} \longrightarrow X_{w \leq n} \longrightarrow B_{w \geq k+1}$$

is equivalent to the trivial decomposition

$$X_{w \leq n} \longrightarrow X_{w \leq n} \longrightarrow *$$

Proof. The proof is identical to the last corollary. □

Proposition 2.2.8. *If X has weight $w \geq n$ then for all $k \geq n$ any weight decomposition*

$$A_{w \leq k} \longrightarrow X_{w \geq n} \longrightarrow B_{w \geq k+1}$$

has A in $\mathcal{C}_{w \geq n}$.

Proof. Note that $\Sigma^{-1}B$ has weight $w \geq k$ by the axioms. By assumption, this means that $\Sigma^{-1}B \in \mathcal{C}_{w\geq n}$. Thus, rotating back the fiber sequence for the decomposition yields the fiber sequence

$$(\Sigma^{-1}B)_{w\geq n} \longrightarrow A \longrightarrow X_{w\geq n}$$

which demonstrates $A \in \mathcal{C}_{w\geq n}$ by Proposition 2.2.1. □

Proposition 2.2.9. *If X has weight $w \leq n$ then for all $k \leq n-1$ any weight decomposition*

$$A_{w\leq k} \longrightarrow X_{w\leq n} \longrightarrow B_{w\geq k+1}$$

has B in $\mathcal{C}_{w\leq n}$.

Proof. The proof is identical to that for the previous proposition. □

Proposition 2.2.10. *Suppose $\mathcal{C}$ is a stable ∞-category with a non-degenerate weight structure. Maps are detected by maps into or out of the heart in the following sense.*

For any $X \in \mathcal{C}$, $\pi_0 h\mathcal{C}(X,Y) = 0$ for all $Y \in \mathcal{C}_{w\geq 0}$ if and only if $\pi_0 h\mathcal{C}(X,Q) = 0$ for all $Q \in \mathcal{C}_{w=i}$ for $i \geq 0$.

Likewise, for any $Y \in \mathcal{C}$, $\pi_0 h\mathcal{C}(X,Y) = 0$ for all $X \in \mathcal{C}_{w\leq 0}$ if and only if $\pi_0 h\mathcal{C}(Q,Y) = 0$ for all $Q \in \mathcal{C}_{w=i}$ with $i \leq 0$.

Proof. Both proofs are identical so we check the first. The forward direction is trivial. For the reverse implication, fix a map $f : X \to Y$ in $h\mathcal{C}$. Pick a weight decomposition at degree 0 for Y. By proposition 2.2.8, this takes the form

$$A_{w=0} \longrightarrow Y_{w\geq 0} \longrightarrow B_{w\geq 1}$$

and applying $\pi_0 h\mathcal{C}(X,-)$ produces a long exact sequence on mapping groups. By assumption, $\pi_0 h\mathcal{C}(X,A) = \pi_0 h\mathcal{C}(X,\Sigma A) = 0$, so we have that $\pi_0 h\mathcal{C}(X,Y)$ is isomorphic to $\pi_0 h\mathcal{C}(X,B)$ and B has weight $w \geq 1$. We can iterate this argument: replace Y with B in this argument and take a weight decomposition at degree 1. Inductively, we can conclude that $\pi_0 h\mathcal{C}(X,Y)$ is isomorphic to $\pi_o h\mathcal{C}(X,\widetilde{B})$ where $\widetilde{B}$ can be constructed in an arbitrarily high weight $w \geq n$. As $n \to \infty$, we conclude that $\pi_0 h\mathcal{C}(X,Y) \cong 0$ as only zero objects have arbitrarily high weights due to the non-degeneracy of w. □

2.3 Generating weight structures

Suppose $\mathcal{C}$ is a stable ∞-category and $\mathcal{H}$ is a subcategory in $\mathcal{C}$. A natural question is whether there exists a weight structure on $\mathcal{C}$ with $\mathcal{H}$ as its heart. We will require that $\mathcal{H}$ is closed under retracts and finite coproducts.

Definition 2.3.1. We say that $\mathcal{H}$ *weakly generates* $\mathcal{C}$ if $X \in \mathcal{C}$ and

$$\pi_0 h\mathcal{C}(\Sigma^n S, X) = 0$$

for all $n \in \mathbb{Z}$ and for all $S \in \mathcal{H}$, then X is a zero object in $\mathcal{C}$.

Definition 2.3.2. We say that $\mathcal{H}$ is *negative* if for all $n > 0$ we have

$$\pi_0 h\mathcal{C}(S, \Sigma^n S') = 0$$

for all $S, S' \in \mathcal{H}$.

For spectrally-enriched categories, Blumberg and Mandell introduce a very similar notion to a negative subcategory, namely a *connective class.* This definition is essential to their form of the sphere theorem which is discussed in [5, 3.4].

Proposition 2.3.1 ([8, 4.3.2.III(ii) and 4.5.2])**.** *Suppose the objects of $\mathcal{H}$ are compact, $\mathcal{H}$ is negative, and $\mathcal{H}$ weakly generates $\mathcal{C}$. Suppose further that all finite cell complexes constructed from $\mathcal{H}$ exist in $\mathcal{C}$.*

Let $\mathcal{C}_-$ be the full subcategory of $\mathcal{C}$ of objects X so that $\forall S \in \mathcal{H}$ there exists a $N \in \mathbb{Z}$ so that $\pi_0\mathcal{C}(Y, \Sigma^n S) = 0$ for all $n > N$. Then $\mathcal{C}_-$ admits a weight structure with $\mathcal{H}$ contained in its heart.

We introduce two examples of such weight structures now and provide a more detailed discussion in chapter 3.

Example 2.3.2. *Let R be a commutative ring. Let $\mathcal{C} = \mathrm{Ch}_R$ denote the stable ∞-category of bounded-above chain complexes of finitely-generated R-modules. There is a weight structure on Ch_R where $\mathrm{Ch}_{R,w\geq 0}$ contains complexes whose homology is concentrated in non-negative degrees, and $\mathrm{Ch}_{R,w\leq 0}$ contains complexes which are quasi-isomorphic to complexes of projectives whose homology is non-positive degrees (or, equivalently, complexes concentrated in non-positive degrees and projective in degree 0). The heart of this weight structure is the finitely-generated projective R-modules included as complexes concentrated in degree 0.*

Example 2.3.3. *Let* Sp^{fin} *denote the stable* ∞*-category of finite spectra.* Sp^{fin} *admits a weight structure generated by the sphere spectrum* S^0*.* $\mathrm{Sp}^{fin}_{w\geq 0}$ *consists of all connective spectra and* $\mathrm{Sp}^{fin}_{w\leq 0}$ *consists of spectra whose integral homology is concentrated in non-positive degrees. These are precisely spectra which occur as* k*-skeleta for other spectra for* $k \leq 0$*.*

Definition 2.3.3. Let $\mathcal{C}$ be a stable ∞-category equipped with a weight structure and a t-structure. We say that these structures are *left adjacent* (respectively, *right adjacent*) if $\mathcal{C}_{t\geq 0} = \mathcal{C}_{w\geq 0}$ (respectively, $\mathcal{C}_{t\leq 0} = \mathcal{C}_{w\leq 0}$).

When a weight structure is left adjacent to a t-structure, the orthogonality relations of the two structures interact to permit more specific descriptions of the hearts $\mathcal{C}_{\heartsuit w}$ and $\mathcal{C}_{\heartsuit t}$.

If X is in the heart of the weight structure, then X is in $\mathcal{C}_{w\geq 0} = \mathcal{C}_{t\geq 0}$ so for any Y in $\mathcal{C}_{t\leq -1}$ we have $\pi_0\mathcal{C}(X, Y) = 0$. Likewise, X also lies in $\mathcal{C}_{w\leq 0}$ so the orthogonality relations for the weight structure make it admit no maps (on π_0) to $\mathcal{C}_{w\geq 1} = \mathcal{C}_{t\geq 1}$. As a consequence of the overdetermined nature of weight and t-structures (*cf.* Proposition 2.2.1), the heart of the weight structure consists of precisely those objects X with $\pi_0\mathcal{C}(X, Y) = 0$ for $Y \in \mathcal{C}_{t\geq 1} \cup \mathcal{C}_{t\leq -1}$.

A similar analysis can be applied to $\mathcal{C}_{\heartsuit t}$ to deduce that the heart can be detected by $\mathcal{C}_{w\leq -1}$ and $\mathcal{C}_{w\geq 1}$. Together with Proposition 2.2.1, we arrive at the following description.

Proposition 2.3.4. *Suppose* $\mathcal{C}$ *admits left adjacent weight and* t*-structures.* X *is in* $\mathcal{C}_{\heartsuit w}$ *if and only if* $\pi_0\mathcal{C}(X, \Sigma^i B) = 0$ *for all* $B \in \mathcal{C}_{\heartsuit t}$ *for* $i \neq 0$*. Likewise,*

Y is in $\mathcal{C}_{\heartsuit t}$ if and only if $\pi_0\mathcal{C}(\Sigma^i A, Y) = 0$ for all $A \in \mathcal{C}_{\heartsuit w}$ for $i \neq 0$.

Example 2.3.5. *In the category of finite spectra, the cellular weight structure is left adjacent to the Postnikov t-structure. The heart of the weight structure consists of wedge sums of the sphere spectrum and the heart of the t-structure is the Eilenberg–Mac Lane spectra. The proposition notes that the former are (equivalently) the spectra with whose cohomology is concentrated in degree 0 (for all HG), while the latter are precisely those spectra with homotopy groups concentrated in degree 0.*

2.4 On weights and Waldhausen's sphere theorem

The purpose of this section is to place Waldhausen's original sphere theorem within our setting of weight structures on stable ∞-categories. Proposition 2.4.1 proves that our weighty sphere theorem from chapter 5 generalizes Waldhausen's. We take the rest of the section to explore the limits how analogous language can be lifted from Waldhausen's setting to the world of weight structures.

As originally formulated in [22, 1.7], Waldhausen's sphere theorem applies to a Waldhausen category $\mathcal{C}$ equipped with a cylinder functor that satisfies the cylinder axiom. The category must be further equipped with a $\mathbb{Z}$-graded homology functor H_* which carries cofiber sequences to long exact sequences in some abelian target category. Furthermore, weak equivalences in $\mathcal{C}$ are required to be precisely isomorphisms on homology. Finally, the hypothesis for the sphere theorem is that any m-connected map $X \to Y$ (with respect to H_*)

can be factored as

$$X_m \rightarrowtail X_{m+1} \rightarrowtail \cdots \rightarrowtail X_n \xrightarrow{\simeq} Y$$

where the quotients X_{k+1}/X_k are all homology spheres of dimension $k+1$. In this case, the sphere theorem says that the K-theory of the stabilization of $\mathcal{C}$ (under the suspension defined by the cylinder functor) is equivalent to the K-theory of the stabilization (under suspension again) of the homology spheres. In Waldhausen's context, a *homology n-sphere* is an object X whose homology $H_i(X)$ is 0 unless $i = n$ and then lies in some fixed full subcategory $\mathcal{E}$ of the abelian target category of H_* which is closed under extensions and retracts.

Proposition 2.4.1. *If $\mathcal{C}$ is a Waldhausen category satisfying the hypotheses of Waldhausen's sphere theorem, then the stable ∞-category $\mathrm{Stab}(\mathcal{C})$ admits a bounded and non-degenerate weight structure whose heart is equivalent to the stabilized homology spheres in $\mathcal{C}$ if it has a set of compact generators which:*

- *generate the ∞-category under finite colimits,*
- *are homology 0-spheres, and*
- *form a negative class in $\mathcal{C}$.*

Although Waldhausen does not require these additional assumptions, they are true in the cases he studies.

We prove this proposition by defining a weight structure on Stab($\mathcal{C}$) where objects are in weight $w \geq n$ if their homology is concentrated in degrees $* \geq n$ and in weight $w \leq n$ if their homology is concentrated in degrees $* \leq n$ and $H_n(X) \in \mathcal{E}$. Under the hypotheses listed, we can generate a weight structure on Stab($\mathcal{C}$) using proposition 2.3.1. The heart is precisely the homology n-spheres as claimed.

We can transplant Waldhausen's language to the setting of weight structures on stable ∞-categories. Specifically, we can view weight structures as providing a language for discussing connectivity of maps without specifying compact generators whose (co)homology theories measure connectivity.

Definition 2.4.1. A map $f : X \to Y$ in $\mathcal{C}$ with cofiber Cf will be called n-connected if Cf lives in $\mathcal{C}_{w \geq n+1}$.

Proposition 2.4.2. *The composite of two n-connected maps is n-connected.*

Proof. Say $f : X \to Y$ and $g : Y \to Z$ are n-connected. Write D for the given pushout in the following diagram.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & Cf_{w \geq n+1} & \longrightarrow & D \\ & & \downarrow & \ulcorner & \downarrow \\ & & * & \longrightarrow & Cg_{w \geq n+1} \end{array}$$

here Cf and Cg are the respective cofibers and D is evidently the cofiber of the composite $g \circ f$. Since the top and outer squares on the right of the diagram are pushouts, so is the lower square. The lower square induces a distinguished triangle $Cf_{w \geq n+1} \to D \to Cg_{w \geq n+1}$ in $h\mathcal{C}$ and thus D lies in $\mathcal{C}_{w \geq n+1}$. $\square$

This proposition implies that a weight decomposition at degree $k \leq n$ is guaranteed to yield a degree-k decomposition for Y as well after composing with an n-connected map $f : X \to Y$.

Proposition 2.4.3. *If the weight structure on $\mathcal{C}$ is bounded, any n-connected map $f : X \to Y$ factors*

$$X = X_n \longrightarrow X_{n+1} \longrightarrow \cdots X_m \xrightarrow{\simeq} Y$$

with X_k/X_{k-1} in $\mathcal{C}_{w=k}$ for $n+1 \leq k \leq m$.

Proof. We will induct up until the cofiber must be concentrated in weight $w = m$ due to boundedness of the weight structure. The induction essentially proceeds by providing a cellular filtration for Cf (see chapter 4). Fix a diagram in $\mathcal{C}$ for the cofiber sequence for Cf.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0' \\ \downarrow & \ulcorner & \downarrow g & \ulcorner & \downarrow \\ 0 & \longrightarrow & Cf & \xrightarrow{h} & \Sigma X \end{array}$$

We will write $X_n = X$ to start the induction as Cf has weight $w \geq n+1$ by assumption.

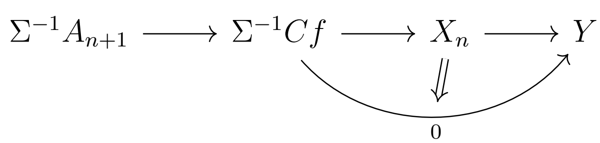
Fix $A_k \to Cf \to B_{k+1}$ weight decompositions at k for all k. Proposition 2.2.8 tells us that A_k lives in weight $n+1 \leq w \leq k$. In particular, A_{n+1} lives in

$\mathcal{C}_{w=n+1}$. Fix a lift of the map $a : A_{n+1} \to Cf$ to $\mathcal{C}$. Form X_{n+1} as the cofiber of the composite

$$\Sigma^{-1} A_{n+1} \xrightarrow{\Sigma^{-1} a} \Sigma^{-1} Cf \xrightarrow{\Sigma^{-1} h} X_n$$

By construction, the cofiber of the map $X_n \to X_{n+1}$ will be equivalent to A_{n+1} which is in $\mathcal{C}_{w=n+1}$ as desired. It remains to show that there is an $(n+1)$-connected map $X_{n+1} \to Y$ to complete the induction.

The composite $\Sigma^{-1} A_{n+1} \to Y$ is homotopic to the zero map because it factors through two consecutive maps in a cofiber sequence.



Thus Y admits a map from X_n. This leads us to consider the following diagram.

$$\begin{array}{ccccc}
X_n & \longrightarrow & X_{n+1} & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_{n+1} & \longrightarrow & Cf \\
 & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & Cf_{n+1}
\end{array}$$

We know that the outer upper square is a pushout along with the upper left. This implies that the upper right one is as well. We form the lower square as the cofiber of the map $A_{n+1} \to Cf$. The outer right square is thus also a pushout and identifies the lower right square as the cofiber of $f_{n+1} : X_{n+1} \to Y$.

The lower right square now tells us that Cf_{n+1} lives in $\mathcal{C}_{w\geq n+2}$ as desired. The relevant cofiber sequence with weights marked is indicated below.

$$(A_{n+1})_{w=n+1} \longrightarrow (Cf)_{w\geq n+1} \longrightarrow Cf_{n+1} \longrightarrow (\Sigma A_{n+1})_{w=n+2}$$

$\square$

Chapter 3

Examples of weight structures and applications

In this chapter, we provide an overview of several examples of weight structures. We produce applications of our weighty theorem of the heart and conjecture about some future directions for research.

3.1 The stable category

Let $\mathrm{Sp}^{\mathrm{fin}}$ denote the category of finite spectra. There is a standard Postnikov t-structure on $h\,\mathrm{Sp}^{\mathrm{fin}}$ where $\mathrm{Sp}^{\mathrm{fin}}_{t\geq 0}$ contains all connective ((-1)-connected) spectra and $\mathrm{Sp}^{\mathrm{fin}}_{t\leq 0}$ consists of all 1-coconnective spectra (*i.e.,* those with homotopy groups concentrated in degrees ≤ 0). The t-structure decompositions are provided by taking n-connected covers and truncating homotopy groups at degree $n-1$. The heart of this t-structure is the abelian category of finitely-generated groups included as the Eilenberg–Maclane spectra.

The weight structure is generated by the sphere spectrum. Let B denote the collection of finite wedges of the sphere spectrum S^0. $\mathrm{Sp}_{w\geq 0}$ is defined to be those spectra E with $h\,\mathrm{Sp}(X, E) = 0$ for any X in $\Sigma^k B$ for $k < 0$. In other words, $\mathrm{Sp}_{w\geq 0}$ is the subcategory of connective spectra. $\mathrm{Sp}_{w\leq 0}$ is the defined via the orthogonality condition: $E \in \mathrm{Sp}_{w\leq 0}$ if $h\,\mathrm{Sp}(E,Y) = 0$ for all $Y \in \mathrm{Sp}_{w\geq 1}$. $\mathrm{Sp}_{w\leq 0}$ turns out to contain those spectra which are 0-skeleta—which can be distinguished by their homology: $E \in \mathrm{Sp}_{w\leq n}$ if and only if $H\mathbb{Z}_*(E) = 0$ for $* > 0$ and $H\mathbb{Z}_0(E)$ is a free abelian group. The heart $\mathrm{Sp}_{\heartsuit_w}$ of this weight structure consists of all spectra weakly equivalent to finite wedges of S^0.

The main theorem implies that $K(S^0)$ can be identified with the algebraic K-theory of the category of spectra weakly equivalent to finite wedges of the sphere spectrum.

3.2 Chain complexes

A standard example in the literature for triangulated categories states that the derived category for an abelian category $\mathcal{A}$ with enough projectives has a t-structure whose heart is $\mathcal{A}$ and a weight structure whose heart is the category of projectives $\mathrm{Proj}(\mathcal{A})$. We consider this carefully when $\mathcal{A} = \mathrm{Mod}(R)$ is the category of finitely-generated R-modules for a commutative ring R.

Let $\mathrm{Proj}(R)$ denote the subcategory of projective modules in $\mathrm{Mod}(R)$. Let $\mathrm{Ch}^{\mathrm{bdd}}(\mod (R))$ be the category of bounded homological chain complexes of finitely-generated R-modules. We put a weight structure on $\mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Mod}(R))$ by setting $\mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Mod}(R))_{w\leq n}$ (likewise $\mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Mod}(R))_{w\geq n}$) to consist of chain complexes quasi-isomorphic to complexes of projectives concentrated in degree $\leq n$ (likewise, $\geq n$). Weight decompositions are provided by projective replacement and the "dumb truncation" as indicated by the following diagram

$$\begin{array}{ccccccccccc}
\cdots & \longleftarrow & M_{n-1} & \longleftarrow & M_n & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & M_{n-1} & \longleftarrow & M_n & \longleftarrow & M_{n+1} & \longleftarrow & M_{n+2} & \longleftarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & M_{n+1} & \longleftarrow & M_{n+2} & \longleftarrow & \cdots
\end{array}$$

where each row is a chain complex and the vertical maps indicate morphisms in $\mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Proj}(R))$. Note that each object in $\mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Mod}(R))$ has a large number of decompositions at any degree that need not be quasi-isomorphic. The heart of $\mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Mod}(R))$ consists of complexes quasi-isomorphic to projective complexes concentrated in degree 0 and can be identified with $\mathrm{Proj}(R)$.

This weight structure is generated by the free module R in the sense of section 2.3. The closure of $\{R\}$ under finite coproducts and retracts are precisely the finitely-generated projective modules, considered as complexes concentrated in degree 0. This is precisely the heart of the weight structure. We also note that the weight and t-structures are left adjacent, in the sense that $\mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Mod}(R))_{w\geq 0} = \mathrm{Ch}^{\mathrm{bdd}}(\mathrm{Mod}(R))_{t\geq 0}$.

The weighty theorem of the heart shows that this weight structure induces an equivalence $K(R) \simeq K(\mathrm{Proj}(R))$, where $K(R)$ is defined to be the K-theory of the derived category of bounded complexes of finitely-generated R-modules. We note that our weight structure above only relied on the fact that $\mathrm{Mod}(R)$ is an abelian category with enough projectives. In general, if $\mathcal{A}$ is any abelian category with enough projectives, then the compact objects in its derived category, $\mathrm{Ch}^{\mathrm{bdd}}(\mathcal{A}^{\omega})$ admit a projective weight structure as well. Weight decompositions are provided by projective replacement and dumb truncation. The heart of this weight structure will be equivalent to $\mathrm{Proj}(\mathcal{A}^\omega)$ the category of projective compact objects in $\mathcal{A}$. The weighty theorem of the heart now implies the following formulation of the Gillet–Waldhausen theorem [21, 1.11.7] for the inclusion $\mathrm{Proj}(\mathcal{A}^\omega) \to \mathrm{Ch}^{\mathrm{bdd}}(\mathcal{A}^\omega)$.

Corollary 3.2.1 (Gillet–Waldhausen theorem)**.** *If $\mathcal{A}$ is an abelian category with enough projectives, the algebraic K-theory of $\mathcal{A}$ (the algebraic K-theory of the derived category of compact objects in $\mathcal{A}$) is equivalent to the algebraic K-theory of the compact projective objects in $\mathcal{A}$.*

$$K(\mathcal{A}^\omega) \simeq K(Proj(\mathcal{A}^\omega))$$

We note that the category of bounded chain complexes admits another weight structure as well. Write $\mathcal{C}_{w'\leq 0}$ for chain complexes with homology concentrated in degrees ≤ 0 and $\mathcal{C}_{w'\geq 0}$ for chain complexes with homology concentrated in degrees ≥ 0 and injective in degree 0. Any complex in $\mathcal{C}_{w'\geq 0}$ is quasi-isomorphic to a complex of injective modules concentrated in degrees ≥ 0 and the orthogonality relation is easily checked on those replacements. For any complex, a weight decomposition can be constructed by replacing with an equivalent complex of injective modules and performing the "dumb truncations". This injective weight structure is *right adjacent* to the standard t-structure.

3.3 Categories of motives

Bondarko establishes an interesting weight structures on Voevodsky's triangulated category of motives [7]. These are constructed from the bounded chain complexes of smooth varieties (with smooth correspondences as morphisms) by localizing and forming the idempotent completion. Within this category, the *Chow motives* are cut out by smooth projective varieties (with morphisms smooth correspondences modulo rational equivalences). Bondarko builds a "Chow" weight structure on $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ the category of effective geometric motives whose heart is the effective Chow motives [8]. His K_0 version of the sphere theorem computes $K_0(\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}) \simeq K_0(\mathrm{Chow}^{\mathrm{eff}})$. The Chow weight structure gives a bounded weight structure on the underlying stable ∞-category of effective geometric motives. Our theorem proves that the algebraic K-theory of the effective geometric motives is equivalent to that of the effective Chow motives.

3.4 Conjectural weight structures

Blumberg, Gepner, and Tabuada introduce a category $\mathcal{M}_{\mathrm{loc}}$ of "localizing noncommutative (spectral) motives" in [3]. This category is constructed from the category of spectrum-valued presheaves on the ∞-category of small stable ∞-categories. This is the category where non-connective K-theory is co-represented. Blumberg has conjectured that $\mathcal{M}_{\mathrm{loc}}$ admits a weight structure whose heart is the dualizable objects—the smooth and proper dg-categories.

Following Hill–Hopkins–Ravenel, the category of genuine G-spectra admits interesting "slice filtrations". These are equivariant analogues for Postnikov towers. The author conjectures that there are adjacent slice weight structures generated by wedge sums of the regular representation spheres. The heart of this weight structure would contain finite wedge sums of all finite-dimensional representation spheres concentrated in virtual degree 0.

Chapter 4

Bounded cell complexes

In this chapter, we define cellular weight filtrations and develop some of their properties. The proof of our main theorem will rely on careful manipulation of these cellular filtrations. Throughout, we will assume $\mathcal{C}$ is a stable ∞-category, viewed as a Waldhausen ∞-category equipped with the maximal pair structure, and w is a bounded weight structure on $\mathcal{C}$.

4.1 Definitions

In preparation for the proof of the weighty sphere theorem, we study an ancillary object: the ∞-category of bounded cell complexes in $\mathcal{C}$.

Definition 4.1.1. Suppose $\mathcal{C}$ is a stable Waldhausen ∞-category. A *relative cell complex* in $\mathcal{C}$ is a functor $A : (N\mathbb{Z})^\sharp \to \mathcal{C}$ of Waldhausen ∞-categories so that any quotient A_i/A_{i-1} is in $\mathcal{C}_{w=i}$. $\lim_\mathbb{Z}$ and $\operatorname{colim}_\mathbb{Z}$ define functors from the category of relative cell complexes to $\mathcal{C}$. A *cell complex* will be a relative cell complex which $\lim_\mathbb{Z}$ takes to a zero object of $\mathcal{C}$. Write $\operatorname{Cell}\mathcal{C} \subset \operatorname{Fun}_{\mathrm{Wald}_\infty}((N\mathbb{Z})^\sharp, \mathcal{C})$ for the full ∞-subcategory of cell complexes in $\mathcal{C}$. We will write A_∞ for $\operatorname{colim}_\mathbb{Z} A$ and $A_{-\infty}$ for $\lim_\mathbb{Z} A$ and will say that A *is a filtration for* A_∞.

By definition, all the morphisms $A_n \to A_m$ in the diagram for a cell complex A are ingressions in $\mathcal{C}$. Furthermore, two cell complexes $A_\bullet$ and $B_\bullet$ in $\operatorname{Cell}\mathcal{C}$ are equivalent if there is a map between them that restricts levelwise to equivalences in $\mathcal{C}$, *i.e.,* levelwise these edges must lie in $i\mathcal{C}$.

Let $i_{\leq n} : \mathbb{Z}_{\leq n} \to \mathbb{Z}$ be the inclusion of the poset of integers $\leq n$. $i_{\leq n}$ induces a functor $i^*_{\leq n} : \operatorname{Fun}_{\mathrm{Wald}_\infty}((N\mathbb{Z})^\sharp, C) \to \operatorname{Fun}_{\mathrm{Wald}_\infty}((N\mathbb{Z}_{\leq n})^\sharp, C)$ which admits a left adjoint. The adjoint is induced by the map $p_{\leq n} : \mathbb{Z} \to \mathbb{Z}_{\leq n}$ which is the identity on $\mathbb{Z}_{\leq n}$ and collapses all larger integers to n. Write tr_n for the composite $i^*_{\leq n} \circ p^*_{\leq n}$, the degree-$n$ truncation of a (relative) cell complex. Likewise $i_{\geq n} : \mathbb{Z}_{\geq n} \to \mathbb{Z}$ induces a functor on relative cell complexes which admits a right adjoint induced by the map $p_{\geq n} : \mathbb{Z} \to \mathbb{Z}_{\geq n}$ which is the identity $\geq n$ and collapses all integers below n to n. Write cotr_n for the composite $p^*_{\geq n} \circ i^*_{\geq n}$, the degree-$n$ cotruncation.

Definition 4.1.2. We call a (relative) cell complex A *bounded* if $A \simeq \operatorname{tr}_n A \simeq \operatorname{cotr}_m A$ for some finite n and m. Write $\operatorname{Cell}^{\mathrm{bdd}}\mathcal{C}$ for the full ∞-subcategory of $\operatorname{Cell}\mathcal{C}$ on bounded cell complexes. If a bounded complex A is equivalent to its n-truncation $\operatorname{tr} A \simeq A$, then we say that A has *degree* $\leq n$. Write $\operatorname{Cell}^{\mathrm{bdd}}_n \mathcal{C}$ for the full ∞-subcategory of $\operatorname{Cell}^{\mathrm{bdd}}\mathcal{C}$ on degree $\leq n$ cell complexes.

A cell complex A with $A \simeq \operatorname{cotr}_n A$ must have $\lim_{\mathbb{Z}} A \simeq \lim_Z \operatorname{cotr}_n A \simeq A_n$ as a zero object. Thus, bounded cell complexes are finite-stage cellular constructions in the weight structure on $\mathcal{C}$ that begin with a zero object. The subcategories $\operatorname{Cell}^{\mathrm{bdd}}_n \mathcal{C}$ filter $\operatorname{Cell}^{\mathrm{bdd}}\mathcal{C}$. That is, under the inclusion maps $\operatorname{colim}_n \operatorname{Cell}^{\mathrm{bdd}}_n \mathcal{C} \simeq \operatorname{Cell}^{\mathrm{bdd}}\mathcal{C}$.

Proposition 4.1.1. *If A is a bounded cell complex in $\mathcal{C}$, then A_i is in $\mathcal{C}_{w\leq i}$ for all i. If A_n is in $\mathcal{C}_{w\geq n}$ then A_{n-1} is in $\mathcal{C}_{w\geq n-1}$ as well. In particular, if $A \in \mathrm{Cell}_n^{bdd}\,\mathcal{C}$ and $A_\infty \in \mathcal{C}_{w=n}$ then $A_i \in \mathcal{C}_{w=i}$ for all i.*

Proof. Induct up the filtration starting with a zero object $A_{\leq -N}$ in $\mathcal{C}_{\leq -N}$ as in §2. For the second part of the proposition, use proposition 2.2.2 for the cofiber sequence $A_n \to A_n/A_{n-1} \to \Sigma A_{n-1}$. The final statement follows by induction down from n. □

Proposition 4.1.2. *If $\mathcal{C}$ is a stable ∞-category equipped with a bounded, non-degenerate weight structure, then every object in $\mathcal{C}$ admits a bounded cellular filtration.*

Proof. Suppose X is an object of $\mathcal{C}$. Then even if the weight structure on $\mathcal{C}$ is not bounded, we can fix weight decompositions $A_{w\leq n} \to X$ for all $n \in \mathbb{Z}$. Corollary 2.2.5 implies the existence of maps $A_n \to A_m$ in the homotopy category for $n \leq m$. These can be lifted to a coherent diagram $N\mathbb{Z} \to \mathcal{C}$ but for a bounded weight structure this is even simpler. In this case, X has weight $-N \leq w \leq N$ for some $N \geq 0$. Set $A_i = X$ for $i \geq N$ and use the weight decomposition starting with $A_N = X$ to inductively find weight decompositions for A_n at weight $n-1$ to get $A_{n-1} \to A_n$. Proposition 2.2.9 implies that the fiber A_n/A_{n-1} is in weight $w = n$ as desired. Each of these maps can be lifted from the homotopy category to $\mathcal{C}$. For $i \leq -N$, A_i is a zero object of $\mathcal{C}$ by the non-degeneracy of the weight structure. The compositions of these maps and the retraction of $\mathbb{Z}$ onto Δ^{2N} as the interval $[-N, N]$ induce the desired functor $N\mathbb{Z} \to \mathcal{C}$. $\square$

4.2 Waldhausen structure on cell complexes

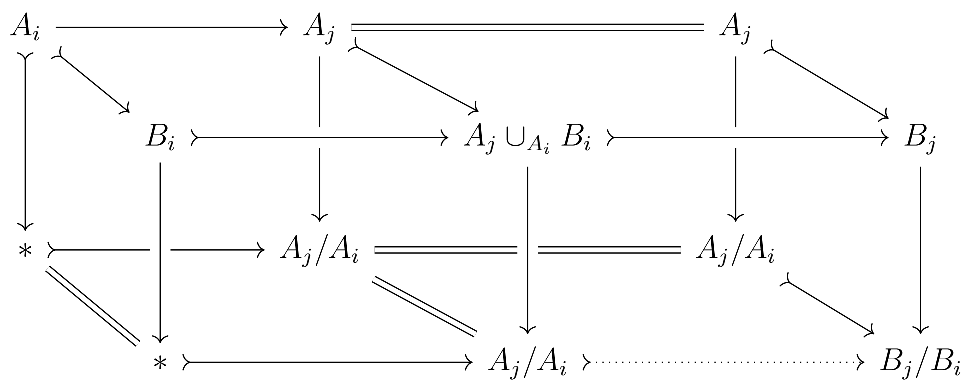
We want to give $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ and $\mathrm{Cell}^{\mathrm{bdd}}_n\,\mathcal{C}$ Waldhausen ∞-category structures. This amounts to selecting ingressive edges. If we think of cell complexes as diagrams in $\mathcal{C}$, an edge in $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ is a diagram

$$\begin{array}{ccccccccc} \cdots & \rightarrowtail & A_{i-1} & \rightarrowtail & A_i & \rightarrowtail & A_{i+1} & \rightarrowtail & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \rightarrowtail & B_{i-1} & \rightarrowtail & B_i & \rightarrowtail & B_{i+1} & \rightarrowtail & \cdots \end{array}$$

and we have a choice of which edges to make ingressive. Just requiring that all vertical maps are ingressions in $\mathcal{C}$ does not imply that the induced maps $A_j/A_i \to B_j/B_i$ are ingressions. As noted in [2, 5.6] and [22, 1.1.2], we need a latching condition on the diagrams: that for any $i < j$, the map from $A_j \cup_{A_i} B_j \to B_i$ is an ingression in $\mathcal{C}$.

$$\begin{array}{ccccc} A_i & \rightarrowtail & A_j & & \\ \downarrow & \ulcorner & \downarrow & \searrow & \\ B_i & \rightarrowtail & A_j \cup_{A_i} B_i & & \\ & \searrow & & \searrow & \\ & & & & B_j \end{array}$$

This result follows by considering the following commuting cube.



If $A_j \cup_{A_i} B_i \rightarrowtail B_j$ is ingressive then so is the dotted edge.

We require further that the cofiber of an ingressive map $A \rightarrowtail B$ in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ is also a degree-n bounded cell complex in $\mathcal{C}$. The cofiber is computed levelwise and we want, in particular, for the cofiber of $B_{i-1}/A_{i-1} \rightarrowtail B_i/A_i$ to have weight $w = i$. This cofiber is identified with the cofiber of the map $A_i \cup_{A_{i-1}} B_{i-1} \rightarrowtail B_i$ which we will require to have weight $w = i$. More generally, an ingression $A \to B$ will be levelwise ingressions $A_i \rightarrowtail B_i$ with the map $A_j \cup_{A_i} B_i \rightarrowtail B_j$ an ingression in $\mathcal{C}$ with cofiber of weight $j + 1 \leq w \leq i$.

Following Barwick (see [2, 5.6] and the following discussion), it is easier to define the Waldhausen structure on $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ as follows.

Definition 4.2.1. For $i \leq j$, write $e_{i,j} : \Delta^1 \to N\mathbb{Z}$ for the map hitting i and j. Let $(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})_\dagger$ be the smallest subcategory spanned by the edges $f : \Delta^1 \to \mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$, which we will write $A \to B$, for which the square $e_{i,j}^* f : \Delta^1 \times \Delta^1 \to \mathcal{C}$

is either of the form

$$\begin{array}{ccc} A_i & \rightarrowtail & A_j \\ \downarrow & \ulcorner & \downarrow \\ B_i & \rightarrowtail & B_j \end{array}$$

where all the edges are ingressive and the square is a pushout square in $\mathcal{C}$, or of the form

$$\begin{array}{ccc} A_i & \rightarrowtail & A_j \\ \downarrow{\scriptstyle\sim} & & \downarrow \\ B_i & \rightarrowtail & B_j \end{array}$$

with the cofiber of the map $A_j \rightarrowtail B_j$ having weight $i+1 \leq w \leq j$, where here the left arrow is an equivalence in $\mathcal{C}$ and the right is an ingression.

We write $(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})_\dagger$ for the subcategory of ingressions in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$. Likewise, write $(\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})_\dagger$ for the ingressions in $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$.

Lemma 4.2.1. *An edge f of $\mathrm{Cell}_n^{bdd}\,\mathcal{C}$ is ingressive if and only if for any $e_{i,j} : \Delta^1 \to N\mathbb{Z}$ and any diagram X from the pair ∞-category*

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d] & 1 \arrow[d] \arrow[ddr, bend left] & \\ 2 \arrow[r, tail] \arrow[drr, tail, bend right] & \infty' \arrow[dr] & \\ & & \infty \end{tikzcd}$$

where $X|_{0,1,2,\infty}$ is a pushout square, the marked edges are ingressions, and $X|_{0,1,2,\infty'} = e_{i,j}^* f : (\Delta^1)^\sharp \times (\Delta^1)^\sharp \to \mathcal{C}$, then $X(\infty) \to X(\infty')$ is an ingression in $\mathcal{C}$ with cofiber in $\mathcal{C}_{i+1 \leq w \leq j}$.

Proof. Ingressions on $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ are defined by their restrictions along the $e_{i,j}$. The resulting types of squares in the definition all admit the desired property:

in the first case we merely note that zero objects are in every weight and the second case the requirement on the vertical map in the square is precisely what is required for the map from the pushout. Hence all ingressive maps satisfy the lemma.

For the converse, we can factor any map satisfying the condition into a composite of maps satisfying the definition. Say $f : A \to B$ satisfies the lemma. There is some k so that $\operatorname{cotr}_k A \simeq A$ and $\operatorname{cotr}_k B \simeq B$. Then A_k and B_k are both zero objects in $\mathcal{C}$, so the map $\operatorname{tr}_k f : \operatorname{tr}_k A \to \operatorname{tr}_k B$ is an equivalence and hence is ingressive in $\operatorname{Cell}_n^{\mathrm{bdd}} \mathcal{C}$. Form the pushout $\operatorname{tr}_k B \cup_{\operatorname{tr}_k A} A$ levelwise. The map from A to this pushout is directly an ingression.

$$\begin{array}{ccccccc}
A_k & \rightarrowtail & A_{k+1} & \rightarrowtail & A_{k+2} & \rightarrowtail & \cdots \\
\downarrow{\scriptstyle\simeq} & \ulcorner & \downarrow & \ulcorner & \downarrow & \ulcorner & \\
B_k & \rightarrowtail & B_k \cup_{A_k} A_{k+1} & \rightarrowtail & B_k \cup_{A_k} A_{k+2} & \rightarrowtail & \cdots
\end{array}$$

All the squares in this diagram are pushouts by [14, 4.4.2.1], and hence the map directly satisfies the definition of ingressive. f induces a map $\operatorname{tr}_k B \cup_{\operatorname{tr}_k A} A \to \operatorname{tr}_{k+1} B \cup_{\operatorname{tr}_{k+1} A} A$ which we write below as the second row of maps.

$$\begin{array}{ccccccc}
A_k & \rightarrowtail & A_{k+1} & \rightarrowtail & A_{k+2} & \rightarrowtail & \cdots \\
\downarrow{\scriptstyle\simeq} & \ulcorner & \downarrow & \ulcorner & \downarrow & \ulcorner & \\
B_k & \rightarrowtail & B_k \cup_{A_k} A_{k+1} & \rightarrowtail & B_k \cup_{A_k} A_{k+2} & \rightarrowtail & \cdots \\
\downarrow{\scriptstyle\simeq} & & \vdots & \ulcorner & \downarrow & \ulcorner & \\
B_k & \rightarrowtail & B_{k+1} & \rightarrowtail & B_{k+1} \cup_{A_{k+1}} A_{k+2} & \rightarrowtail & \cdots
\end{array}$$

Repeated application of [14, 4.4.2.1] demonstrates that the marked squares are pushouts, and application of the hypothesis shows that the dotted arrow induced by f is ingressive in $\mathcal{C}$ and has a cofiber of the appropriate weight. Induction now factors f as a composite of ingressions $A \to \operatorname{tr}_k B \cup_{\operatorname{tr}_k} A \to \cdots \to \operatorname{tr}_n B \cup_{\operatorname{tr}_n A} A \simeq B$ as $\operatorname{tr}_n B \simeq B$ and $\operatorname{tr}_n A \simeq A$. □

Proposition 4.2.2. *The pair ∞-category $(\mathrm{Cell}^{bdd}\,\mathcal{C}, (\mathrm{Cell}^{bdd}\,\mathcal{C})_\dagger)$ of bounded cell complexes and the pair ∞-category $(\mathrm{Cell}_n^{bdd}\,\mathcal{C}, (\mathrm{Cell}_n^{bdd}\,\mathcal{C})_\dagger)$ of bounded and n-truncated cell complexes form Waldhausen ∞-categories.*

Proof. As $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ is the colimit of the subcategories $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ and the same is true for the ingressions, it suffices to check that $(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}, (\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})_\dagger)$ form a Waldhausen ∞-category.

As zero objects in $\mathcal{C}$ are in all weights, the constant diagram at a zero object in $\mathcal{C}$ forms a zero object in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$. For any A in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$, the cofiber of the map $A_i \to A_j$ has weight $i+1 \leq w \leq j$, so the map $0 \to A$ satisfies the lemma above and hence is ingressive.

Now suppose $A \rightarrowtail B$ is an ingression in $\mathrm{Cell}_n^{\mathrm{bdd}}$ and $A \to C$ is an arbitrary map. The pushout in diagrams $B \cup_A C$ is formed levelwise with $(B \cup_A C)_i = B_i \cup_{A_i} C_i$. The maps $B_i \cup_{A_i} C_i \to B_j \cup_{A_j} C_j$ are ingressions in $\mathcal{C}$. As equivalences are checked levelwise, $\operatorname{tr}_n(B \cup_A C) \simeq \operatorname{tr}_n B \cup_{\operatorname{tr}_n A} \operatorname{tr}_n C$ is equivalent to $B \cup_A C$. Since the same holds for cotrunctation, if $B \cup_A C$ is a cellular complex it will lie in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$. It remains to show that $B \cup_A C$ is a cellular complex and the map $C \to B \cup_A C$ is an ingression in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$. Both amount to checking that certain

Write D for the pushout $B \cup_A C$. As pushouts commute, $D_j/D_i \simeq (B_j/B_i) \cup_{A_j/A_i} (C_j/C_i)$.

$$\begin{array}{ccc} A_j/A_i & \rightarrowtail & B_j/B_i \\ \downarrow & \ulcorner & \downarrow \\ C_j/C_i & \rightarrowtail & D_j/D_i \end{array}$$

Hence, the cofiber of the top and bottom map are equivalent in $h\mathcal{C}$, so

$$(D_j/D_i)/(C_j/C_i) \simeq (B_j/B_i)/(A_j/A_i)$$

which is equivalent to $B_j/(A_j \cup_{A_i} B_i)$ by commuting pushouts again. The latter is in weight $i+1 \leq w \leq j$ by assumption on the map $A \to B$. We note that since weights (both $\mathcal{C}_{w \geq i+1}$ and $\mathcal{C}_{w \leq j}$) are closed under extension by definition, the cofiber sequence $C_j/C_i \rightarrowtail D_j/D_i \to (D_j/D_i)/(C_j/C_i)$ now shows that D_j/D_i also has weight $i+1 \leq w \leq j$ as desired. Hence, D lies in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ and the map $C \to D$ is an ingression. We note that this analysis did not require the particular model of D as the levelwise pushout, so we also conclude that any pushout of an ingression in $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ is also an ingression. □

4.3 Localizing cell complexes

By construction, the equivalences in $\mathrm{Cell}\,\mathcal{C}$ are those maps which induce equivalences in $\mathcal{C}$ degreewise. This is too rigid: two cell complexes are only equivalent if all the n-skeleta are equivalent. We would like to make all cell complexes for a single object in $\mathcal{C}$ equivalent to each other.

We regard the functor $\mathrm{colim}_{\mathbb{Z}}$ as taking a cell complex to the object in $\mathcal{C}$ it models. We are primarily interested in bounded complexes, which are filtered by the subcategories $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ of complexes A that precisely carry the data of a cellular filtration for $A_\infty \simeq A_n$.

Denote by $v\,\mathrm{Cell}\,\mathcal{C}$ (or $v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ or $v\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$) the subcategory of $\mathrm{Cell}\,\mathcal{C}$ which $\mathrm{colim}_{\mathbb{Z}}$ takes to equivalences in $\mathcal{C}$. We'd like to localize the bounded cell complexes at $v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ which will require Barwick's labeled Waldhausen ∞-categories [2, 2.9]. The pair $(\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}, v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})$ will be our virtual Waldhausen ∞-category surrogate for $\mathcal{C}$.

In the proof of the main theorem, we compare the K-theory of the localization $(v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ to that of the unlocalized cell complexes $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$. The following result compares this directly to $K(\mathcal{C})$.

Proposition 4.3.1. *The K-theory of the localization $(v\,\mathrm{Cell}^{bdd}\,\mathcal{C})^{-1}\,\mathrm{Cell}^{bdd}\,\mathcal{C}$ is equivalent to $K(\mathcal{C})$.*

Proof. Not every map in $(v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})$ is necessarily ingressive. However, we can apply Fiore's approximation theorem [13] in this situation. Write F for the comparison functor $(v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C} \to \mathcal{C}$ induced by $\mathrm{colim}_{\mathbb{Z}}$. By proposition 4.1.2, F is essentially surjective. By construction of $v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$, F reflects equivalences in $\mathcal{C}$. Finally, any diagram indexed by a finite poset in the maximal Kan subcategory of $(v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ admits a colimit in $(v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ which can be constructed as the colimit of a diagram indexed on a finite poset in $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ where all maps induce equivalences on $\mathrm{colim}_{\mathbb{Z}}$. These colimits are constructed levelwise and since the poset is finite there is some N where all terms achieve $\mathrm{colim}_{\mathbb{Z}}$, so we directly see that

$\text{colim}_{\mathbb{Z}}$ preserves those colimits. Hence F satisfies the hypotheses of [13, 4.5] to show that it induces an equivalence of homotopy categories on the subcategories of ingressions. The approximation theorem [13, 4.10] now implies that F induces an equivalence on K-theory. $\square$

Chapter 5

The weighty theorem of the heart

This chapter provides the proof of theorem 5. Throughout, we let $\mathcal{C}$ denote a fixed stable ∞-category equipped with a bounded weight structure w.

Theorem (Weighty theorem of the heart)**.** *If $\mathcal{C}$ is a stable ∞-category equipped with a bounded non-degenerateweight structure w, then the inclusion of the heart of the weight structure $\mathcal{C}_{\heartsuit_w} \hookrightarrow \mathcal{C}_{\heartsuit}$ induces an equivalence on algebraic K-theory*

$$K(\mathcal{C}) \simeq K(\mathcal{C}_{\heartsuit_w}).$$

Recall that $\mathcal{C}_{\heartsuit_w}$ is given a Waldhausen ∞-category structure where ingressions are precisely those maps admitting cofibers in $\mathcal{C}_{\heartsuit_w}$.

Bondarko proves this theorem on K_0-groups by considering bounded weight structures on triangulated categories [8, 5.3.1]. We reproduce his result by passing to the underlying stable ∞-category, applying the weighty theorem of the heart, and taking π_0.

Corollary 5.0.1 (*cf.* [8, 5.3.1])**.** *If $\mathcal{T}$ is a triangulated category equipped with a non-degenerate bounded weight structure w, then $K_0(\mathcal{T}) \simeq K_0(\mathcal{T}_{\heartsuit_w})$.*

At the end of the chapter, we study the K-theory of the heart of a weight structure. All ingressions in $\mathcal{C}_{\heartsuit_w}$ split in the homotopy category, so the K-theory admits a description in the style of Quillen's plus construction for K-theory.

5.1 Proving the weighty theorem of the heart

Using the technology of cellular filtrations constructed in chapter 4, we study the K-theory of $\mathcal{C}$ through the labeled pair Waldhausen ∞-category $(\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}, v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})$. We use the localization theorem to relate the algebraic K-theory of this pair to the algebraic K-theory of $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$.

Theorem 5.1.1 ([2, 9.24])**.** *Suppose $(\mathcal{A}, w\mathcal{A})$ is a labeled Waldhausen ∞-category that has enough cofibrations. Suppose $\phi : Wald_\infty \to E$ is an additive theory with left derived functor Φ. Then the inclusion $i : \mathcal{A}^w \to \mathcal{A}$ and the morphism of virtual Waldhausen ∞-categories $e : \mathcal{A} \to \mathcal{B}(\mathcal{A}, w\mathcal{A})$ give rise to a fiber sequence*

$$\begin{array}{ccc} \phi(\mathcal{A}^w) & \longrightarrow & \phi(\mathcal{A}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Phi(\mathcal{B}(\mathcal{A}, w\mathcal{A})). \end{array}$$

Here, $\mathcal{B}(\mathcal{A}, w\mathcal{A}))$ is the virtual Waldhausen ∞-category corresponding to the pair $(\mathcal{A}, w\mathcal{A})$. For our result ϕ will be K-theory, and the derived K-theory on the virtual Waldhausen ∞-category of the pair $K(\mathcal{B}(\mathcal{A}, w\mathcal{A}))$ will be written simply as the K-theory of the pair $K(\mathcal{A}, w\mathcal{A})$. In this theorem, $\mathcal{A}^w$ denotes the full subcategory of w-acyclic objects in $\mathcal{A}$. In our setting, the

v-acyclic objects of $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ are classified by the following computation.

Proposition 5.1.2. *If A is a v-acyclic object of* $\mathrm{Cell}^{bdd}\,\mathcal{C}$ *then A_n has weight $w = n$ in $\mathcal{C}$ for all n.*

Lemma 5.1.3. *If*

$$A \longrightarrow B \longrightarrow C$$

is a cofiber sequence in $\mathcal{C}$ with $B \in \mathcal{C}_{w\geq m}$ and $C \in \mathcal{C}_{w\geq n+1}$ and $m \geq n+1$ then $A \in \mathcal{C}_{w\geq n}$.

Proof. For any $X \in \mathcal{C}_{w\leq n-1}$ we obtain the following exact sequence of mapping spaces in $h\mathcal{C}$.

$$hC(X, \Sigma^{-1}C) \longrightarrow hC(X, A) \longrightarrow hC(X, B)$$

Now since $C \in \mathcal{C}_{w\geq n+1}$, $\Sigma^{-1}C$ lives in $\mathcal{C}_{w\geq n}$ so the left mapping space is trivial. The same is true of the right mapping space since $B \in \mathcal{C}_{w\geq m}$ and $m \geq n+1 \geq n$. We conclude that A is in $\mathcal{C}_{w\geq n}$. □

Proof of proposition 5.1.2. The proof follows from induction down from the finite stage where A achieves its colimit, a zero object. Say that $A \in \mathrm{Cell}^{\mathrm{bdd}}_n\,\mathcal{C}$ so that we have an equivalence $* \simeq A_n$, letting us conclude that A_n has weight $w = n$. Induction using the lemma above when $m = k+1$ lets us conclude that A_k has weight $w \geq k$ for all k. A similar induction from below demonstrates that $A_n \in \mathcal{C}_{w\leq n}$ for any sequence in A. This completes the proof. □

Remark 5.1.1. As $\mathcal{C}_{w=n}$ consists of the "pure" objects in the weight structure, $(\operatorname{Cell}^{\mathrm{bdd}} \mathcal{C})^v$ can be thought of as "finite Kozsul resolutions" between zero objects in $\mathcal{C}$. We note that $\mathcal{C}_{\heartsuit w} = \mathcal{C}_{w=0}$ is equivalent (via suspension) to $\mathcal{C}_{w=n}$ for any n.

To use the localization theorem, we must check that the marked cell filtrations satisfy the technical hypothesis of having enough cofibrations.

Proposition 5.1.4. *When $\mathcal{C}$ is a stable ∞-category equipped with the maximal Waldhausen ∞-category structure, the labeled Waldhausen ∞-category $(\operatorname{Cell}^{bdd} \mathcal{C}, v \operatorname{Cell}^{bdd} \mathcal{C})$ has enough cofibrations.*

Proof. By [2, 9.22], it is sufficient to construct a functorial mapping cylinder M on arrows

$$M : \operatorname{Fun}(\Delta^1, \operatorname{Cell}^{\mathrm{bdd}} \mathcal{C}) \to \operatorname{Fun}(\Delta^1, \operatorname{Cell}^{\mathrm{bdd}} \mathcal{C})$$

that produces ingressive arrows, preserves v-equivalences, and comes with a natural transformation $\eta : \mathrm{id} \to M$ which is an objectwise labeled by v-equivalences.

We construct M for arrows in $\operatorname{Fun}(N\mathbb{Z}, \mathcal{C})$ and show that it produces arrows in $\operatorname{Cell}^{\mathrm{bdd}} \mathcal{C}$. Let sh_{-1} denote the functor induced on $\operatorname{Fun}(N\mathbb{Z}, \mathcal{C})$ by the map $z \mapsto z - 1$ on $\mathbb{Z}$. Note that $\mathrm{sh}_{-1}(A)_i = A_{i-1}$. Also note that there is a natural transformation $\mathrm{sh}_{-1} \to \mathrm{id}$ whose levelwise maps are just the structure maps, *i.e.*, $A_{i-1} \to A_i$.

For $f : A \to B$ an arrow, Mf will be defined to be the pushout

$$\begin{array}{ccccc} \mathrm{sh}_{-1}(A) & \xrightarrow{\mathrm{sh}_{-1}(f)} & \mathrm{sh}_{-1}(B) & \longrightarrow & B \\ \downarrow & \ulcorner & & & \downarrow \\ A & & \longrightarrow & & Mf \end{array}$$

and since colimits are computed levelwise on diagram categories, we observe that

$$(Mf)_i \simeq B_i \cup_{A_{i-1}} A_i.$$

We regard $M(f)$ as the arrow $A \to Mf$ and will write Mf only for the target object. We note that the construction is functorial on the arrow category for $\mathrm{Fun}(N\mathbb{Z}, \mathcal{C})$.

First we check that Mf is an endofunctor for arrows in $\mathrm{Cell}^{\mathrm{bdd}}(\mathcal{C})$. By commuting pushouts, we find that $(Mf)_{i+1}/(Mf)_i$ is the (homotopy) pushout

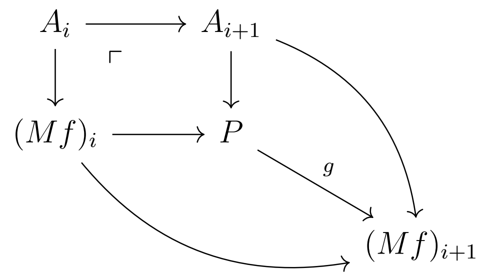
$$\begin{array}{ccc} A_i/A_{i-1} & \longrightarrow & B_{i+1}/B_i \\ \downarrow & \ulcorner & \downarrow \\ A_{i+1}/A_i & \longrightarrow & (Mf)_{i+1}/(Mf)_i \end{array}$$

but due to the weights of the objects, the top and left map are both 0, so $(Mf)_{i+1}/(Mf)_i$ splits up as a wedge sum

$$(Mf)_{i+1}/(Mf)_i \simeq A_{i+1}/A_i \vee B_{i+1}/B_i \vee \Sigma(A_i/A_{i-1})$$

which has weight $w = i + 1$ as desired. Since A and B are bounded, so will Mf. Hence $M(f)$ is an arrow in $\mathrm{Cell}^{\mathrm{bdd}}(\mathcal{C})$ if f is as well.

Next we check that $M(f)$ is a cofibration in $\mathrm{Cell}^{\mathrm{bdd}}(\mathcal{C})$. It suffices to check the latching condition for the map g.



From the construction of Mf, we have the following pushout squares

$$\begin{CD} A_{i-1} @>>> A_i @>>> A_{i+1} \\ @VVV @VVV @VVV \\ B_i @>>> (Mf)_i @>>> P \end{CD}$$

and the outer square is also a pushout square by [14, 4.4.2.1]. We map the outer pushout square to

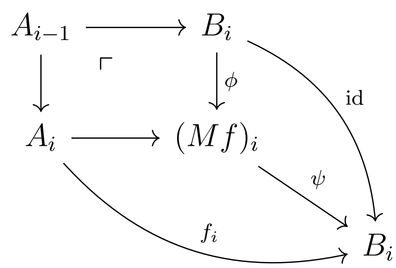
$$\begin{CD} A_i @>>> A_{i+1} \\ @VVV @VVV \\ B_{i+1} @>>> M_{i+1} \end{CD}$$

by the obvious maps. We take the cofiber of this map of pushout squares which, since colimits commute, is also a pushout square.

$$\begin{CD} A_i/A_{i-1} @>>> * \\ @VVV @VVV \\ B_{i+1}/B_i @>>> M_{i+1}/P \end{CD}$$

The weights of A_i/A_{i-1} and B_{i+1}/B_i imply that $M_{i+1} \simeq B_{i+1}/B_i \vee \Sigma(A_i/A_{i-1})$ which lives in weight $w = i + 1$ as desired.

Now we claim that M is a mapping cylinder for f with respect to the v-equivalences. That is, we show that the natural map $B \to Mf$ is a v-equivalence. Denote this map $\phi : B \to Mf$. The maps $f : A \to B$ and $\mathrm{id} : B \to B$ also induce a map $\psi : Mf \to B$. Levelwise, these maps appear as



and $\psi \circ \phi \simeq \mathrm{id}$. ψ is a right inverse for ϕ as well if f_i factors the map from $A_i \to (Mf)_i$ through B_i via ϕ. This will be satisfied once A and B both achieve their limits. Equivalently, the vertical cofibers in the square

$$\begin{array}{ccc} A_{i-1} & \longrightarrow & B_i \\ \downarrow & \ulcorner & \downarrow \\ A_i & \longrightarrow & (Mf)_i \end{array}$$

must be equivalent. The left is A_i/A_{i-1} and the right is $(Mf)_i/B_i$. Once A no longer has any cells, we conclude that $\phi_i : B_i \to (Mf)_i$ is an equivalence. Hence, ϕ is a v-equivalence. Since ϕ induces the desired natural transformation $\mathrm{id} \to M$, we conclude that $(\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}, v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})$ has enough cofibrations.

□

We apply Barwick's localization theorem to the algebraic K-theory of the labeled Waldhausen ∞-category $(\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}, v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})$ to produce the following pushout diagram.

$$\begin{array}{ccc} K((\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C})^{v}) & \longrightarrow & K(\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}) \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & K(\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}, v\,\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}) \end{array}$$

The top map is induced by the inclusion of the v-acyclics into $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$. The proof will proceed by factoring this map on K-theory and analyzing the resulting diagram.

Integral to our argument are two functorial ways of embedding $\mathcal{C}_{\heartsuit w}$ into cell complexes. On one hand, we can include an n-spherical object a_n as the cell filtration concentrated in degree n, where we attach a_n to a zero object and then keep the filtration constant. This functor essentially includes a_n as the filtration $0 \to a_n$ living in $\mathrm{Cell}^{\mathrm{bdd}}_n\,\mathcal{C}$. We will abuse notation and write the resulting complex simply as a_n when its weight is clear. We can also include a_n into the v-acyclic cell filtrations by including it as the filtration $0 \to a_n \to 0'$ where a_n is attached to 0 and then immediately killed at the next level. This will be referred to as the *cone* on a_n, written $\mathrm{cone}(a_n)$, and includes a_n into $(\mathrm{Cell}^{\mathrm{bdd}}_{n+1}\,\mathcal{C})^{v}$. Coherent functoriality of these maps is ensured by the following construction.

By [14, 3.2.2], the source map $s : Fun(\Delta^1, \mathcal{C}_{w=i}) \to \mathcal{C}_{w=i}$ is a cartesian fibration and if we denote the full subcategory of zero objects in $\mathcal{C}$ by $\mathcal{C}_0$, pulling back s over the inclusion of $\mathcal{C}_0 \to \mathcal{C}_{w=i}$ yields a cartesian fibration $\mathcal{C}_0 \times_s \mathrm{Fun}(\Delta^1, \mathcal{C}_{w=i})$ which at a zero object $*$ classifies the ∞-category of arrows $* \to X$ in $\mathcal{C}$ where X has weight $w = i$. Let $p_i : N\mathbb{Z} \to \Delta^1$ be defined by $p_i(j) = 0$ for $j < i$ and $p_i(j) = 1$ for $j \geq i$. Pullback along p_i and inclusion into $\mathcal{C}$ induces a map $\mathcal{C}_0 \times_s \mathrm{Fun}(\Delta^1, \mathcal{C}_{w=i}) \to \mathrm{Fun}(N\mathbb{Z}, \mathcal{C})$. This carries an arrow $* \to X$ with target in weight $w = i$ to a filtration that is evidently cellular and bounded as the level quotients are all zero objects except for at degree i where it is equivalent to X. We will denote the resulting functor from $\mathcal{C}_{w=i} \to \mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ by c_i, or, when clear, with no decoration, as this is the natural way to include $\mathcal{C}_{w=i}$ into $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$.

To produce the cone functor, we use that the maps ev_0 and ev_1 : $\mathrm{Fun}(\Delta^2, \mathcal{C}_{w=i}) \to \mathcal{C}_{w=i}$ are also cartesian fibrations. We pull back the map (ev_0, ev_2) from $\mathrm{Fun}(\Delta^2, \mathcal{C}_{w=i}) \to \mathcal{C}_{w=i} \times \mathcal{C}_{w=i}$ along the inclusion of $\mathcal{C}_0 \times \mathcal{C}_0$. Above a pair of zero objects this classifies composable pairs of arrows $0 \to X \to 0'$ with X in $\mathcal{C}_{w=i}$. We pull back along the map $N\mathbb{Z} \to \Delta^2$ which collapses $\mathbb{Z}$ onto the interval $[i-1, i+1]$ as above. Observe that the resulting cell filtration lives in $\mathrm{Cell}_{i+1}^{\mathrm{bdd}}\,\mathcal{C}$ and $\mathrm{colim}_{\mathbb{Z}}$ takes it to a zero object. Denote the corresponding functor $\mathcal{C}_{w=i} \to (\mathrm{Cell}_{i+1}^{\mathrm{bdd}}\,\mathcal{C})^v$ by cone.

$(n-1)$-truncation tr_{n-1} is an exact functor $\mathrm{Cell}^{\mathrm{bdd}}\,\mathcal{C}$ to $\mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C}$ by lemma 4.2.1. Suppressing the inclusion $\mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C} \to \mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ as well as the restriction to the subcategory, we consider tr_{n-1} as an endofunctor of $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$. The truncation tr_{n-1} comes with a natural transformation to id by construction. We take the (homotopy) cofiber of this map of endofunctors $\mathrm{tr}_{n-1} \to \mathrm{id}$ in the arrow ∞-category of functors. Denote the resulting endofunctor of $\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$ by q_n.

The functor q_n is constructed to be a homotopy-coherent model for the top level quotient $A \mapsto A_n/A_{n-1}$. In particular, since $\operatorname{colim}_{\mathbb{Z}} \operatorname{tr}_{n-1} A \simeq A_{n-1}$ and $\operatorname{colim}_{\mathbb{Z}} A \simeq A_n$, we observe that $\operatorname{colim}_{\mathbb{Z}} q_{n-1}A \simeq A_n/A_{n-1}$ in the homotopy category of C. Alternatively, the cofiber is computed pointwise, and we observe that the image of q_n is equivalent to $C_{w=n}$ included as constant cell complexes concentrated in degree n.

Proposition 5.1.5. *The map*

$$K(\operatorname{Cell}_{n-1}^{bdd}\mathcal{C}) \times K(\mathcal{C}_{w=n}) \xrightarrow{\vee} K(\operatorname{Cell}_n^{bdd}\mathcal{C})$$

is an equivalence, where we include $\mathcal{C}_{w=n}$ as constant cell complexes concentrated in degree n.

Proof. The pushout square of endofunctors

$$\begin{array}{ccc} \operatorname{tr}_{n-1} & \longrightarrow & \operatorname{id} \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & q_n \end{array}$$

implies by [2, 7.4.(5)] that $K(\operatorname{tr}_{n-1} \oplus q_n)$ is equivalent to $K(\operatorname{id})$ on $K(\operatorname{Cell}_n^{\mathrm{bdd}}\mathcal{C})$. Hence the inclusion $(\operatorname{Cell}_n^{\mathrm{bdd}}\mathcal{C})^v \to \operatorname{Cell}_n^{\mathrm{bdd}}\mathcal{C}$ factors on K-theory as

$$\begin{array}{ccc} K((\operatorname{Cell}_n^{\mathrm{bdd}}\mathcal{C})^v) & \xrightarrow{\operatorname{tr}_{n-1} \oplus q_n} & K((\operatorname{Cell}_{n-1}^{\mathrm{bdd}}\mathcal{C}) \times \operatorname{im} q_n) \simeq K((\operatorname{Cell}_{n-1}^{\mathrm{bdd}}\mathcal{C}) \times \mathcal{C}_{w=n}) \\ & \searrow^{\text{incl}} & \downarrow \vee \\ & & K(\operatorname{Cell}_n \mathcal{C}) \end{array}$$

where the middle term is determined by the images of tr_{n-1} and q_n, which on $(\operatorname{Cell}_n^{\mathrm{bdd}}\mathcal{C})^v$ are an $(n-1)$-truncated cell complex and a cell complex concentrated in degree n, which is the essential image of the weight-n-spheres under the constant-cell-complex functor $\mathcal{C}_{w=n} \to \operatorname{Cell}_n^{\mathrm{bdd}} \mathcal{C}$. The vertical map $\vee$ is the wedge of the inclusion and the constant-cell-filtration functor from $_{w=n}$.

Furthermore, the wedge product map is a right inverse to $\operatorname{tr}_{n-1} \oplus q_n$ before taking K-theory, so we conclude that all the maps are equivalences on K-theory and hence the wedging map is an equivalence on K-theory. $\square$

We can induct down on n to arrive at the following result.

Corollary 5.1.6. *$K(\operatorname{Cell}_n^{bdd} \mathcal{C})$ is equivalent to $\prod_{i \le n} K(\mathcal{C}_{w=i})$ under the map induced by $q = q_n \oplus q_{n-1} \operatorname{tr}_{n-1} \oplus q_{n-2} \operatorname{tr}_{n-2} \oplus \cdots$.*

Now we can factor the pushout square from the fibration theorem into two pushout squares

$$\begin{array}{ccccc} K((\operatorname{Cell}_n^{\mathrm{bdd}} \mathcal{C})^v) & \longrightarrow & K(\operatorname{Cell}_{n-1}^{\mathrm{bdd}} \mathcal{C}) \times K(\mathcal{C}_{w=n}) & \xrightarrow{\simeq} & K(\operatorname{Cell}_n^{\mathrm{bdd}} \mathcal{C}) \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & P & \longrightarrow & K(\operatorname{Cell}_n^{\mathrm{bdd}} \mathcal{C}, v \operatorname{Cell}_n^{\mathrm{bdd}} \mathcal{C}) \end{array}$$

and reduce the proof to two outstanding claims.

Claim 5.1.7. *$K(\operatorname{Cell}_n^{bdd} \mathcal{C}, v \operatorname{Cell}_n^{bdd} \mathcal{C}) \simeq K((v \operatorname{Cell}_n^{bdd} \mathcal{C})^{-1} \operatorname{Cell}_n^{bdd} \mathcal{C})$.*

Claim 5.1.8. *The pushout P is equivalent to $K(\mathcal{C}_{w=n}) \simeq K(\mathcal{C}_{\heartsuit w})$.*

As pushouts of equivalences are equivalences, the bottom map in the right square is an equivalence. Hence, these claims and proposition 4.3.1 let us conclude that $K(\mathcal{C}_{\heartsuit w} \simeq K(\mathcal{C})$ after passing to the colimit over n. The equivalence is induced by the inclusion of $\mathcal{C}_{w=n}$ into $\operatorname{Cell}_n^{\mathrm{bdd}} \mathcal{C}$ as constant cellular filtrations at degree n. Under localization at the equivalences $v\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$, this corresponds to the inclusion map $\mathcal{C}_{\heartsuit w} \simeq \mathcal{C}_{w=n} \to \mathcal{C}$ as asserted in the theorem.

Proof of claim 5.1.7. [2, 10.15] implies that the K-theory space of the pair $(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}, v\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})$ is weakly homotopy equivalent to

$$\Omega\,\mathrm{colim}\, vS_\bullet(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})$$

where the nerve direction is taken over maps in $v\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}$. Likewise, the K-theory space of the localization is weakly equivalent to

$$\Omega\,\mathrm{colim}\,\iota S_\bullet((v\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})$$

and localization induces a simplicial functor

$$L_\bullet : S_\bullet(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}) \to S_\bullet((v\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C}).$$

Since localization carries v-labeled edges to equivalences, this maps the v-nerve fully faithfully into the ι-nerve. All we need to check is essential surjectivity. It suffices to check levelwise. This will follow from general behavior of diagrams and localizations.

Any zig-zag

$$A \xleftarrow[\sim_v]{} A' \longrightarrow B \ \text{ or } \ A \longrightarrow B \xleftarrow[\sim_v]{} B'$$

in the simplicial localization receives a map from a zig-zag with only identity arrows reversed

$$\begin{array}{ccccc} A & \xleftarrow[\sim_v]{} & A' & \longrightarrow & B \\ {\scriptstyle\sim_v}\uparrow & & \| & & \| \\ A' & = & A' & \longrightarrow & B \end{array} \quad \text{or} \quad \begin{array}{ccccc} A & \longrightarrow & B & \xleftarrow[\sim_v]{} & B' \\ \| & & {\scriptstyle\sim_v}\uparrow & & \| \\ A & \longrightarrow & B' & = & B' \end{array}$$

and we note that the vertical maps are all v-equivalences, hence equivalences in the localization. The lower zig-zags are hit by the localization functor. Hence, any sequence of maps in any level of $S_\bullet((v\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})^{-1}\,\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})$ receives a map from a sequence of maps in the image of L from $vS_\bullet(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})$. We conclude that L induces a weak equivalence of the nerves as desired. $\square$

Claim 5.1.8 will follow from proving that the truncation functor induces an equivalence on K-theory from $(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})^v$ to $\mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C}$. We will prove this by identifying $K((\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})^v)$ with $\prod_{i\leq n-1} K(\mathcal{C}_{w=i}$ in $K(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})$ which is identified with $\prod_{i\leq n} K(\mathcal{C}_{w=i})$ using corollary 5.1.6.

Lemma 5.1.9. *The truncation functor* $\mathrm{tr}_{n-1} : (\mathrm{Cell}_n^{bdd}\,\mathcal{C})^v \to \mathrm{Cell}_{n-1}^{bdd}\,\mathcal{C}$ *induces an equivalence on K-theory.*

Proof. The truncation functor maps the acyclic n-cell complexes fully and faithfully into the $(n-1)$-cell complexes by forgetting the zero object at degree n. By corollary 5.1.6, $q : \mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C} \to \prod_{i\leq n-1} \mathcal{C}_{w=i}$ induces an equivalence on K-theory whose inverse is induced by the map W which forms the wedge sum of constant cell filtrations. Hence it suffices to check that tr_{n-1} is essentially surjective onto the image of W to induce an equivalence of K-theory. This will follow by inducting down on cells. In particular, we have the following diagram of functors

$$\begin{array}{ccccc}
(\mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C})^v & \xrightarrow{\mathrm{tr}_{n-2}} & \mathrm{Cell}_{n-2}^{\mathrm{bdd}} \times \mathcal{C}_{w=n-1}\mathcal{C} & \underset{\mathrm{id}\times W}{\overset{q}{\rightleftarrows}} & \mathcal{C}_{w=n-1} \times \prod_{i\leq n-2} \mathcal{C}_{w=i} \\
\Big\downarrow{\scriptstyle \mathrm{incl}} & & \Big\downarrow{\scriptstyle \vee} & & \Big\downarrow{\scriptstyle \simeq} \\
(\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})^v & \xrightarrow{\mathrm{tr}_{n-1}} & \mathrm{Cell}_{n-1} & \underset{W}{\overset{q}{\rightleftarrows}} & \prod_{i\leq n-1} \mathcal{C}_{w=i}
\end{array}$$

where all functors in the right square induce equivalences on K-theory. A bounded cell filtration in $\mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C}$ corresponds to a sequence of level quotients $(a_i) \in \prod_{i \le n-1} \mathcal{C}_{w=i}$ where all but finitely many are zero objects. W sends this to the wedge $\bigvee_{i \le n-1} a_i$ where the weight of each a_i indicates in which degree its filtration is concentrated in $\mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C}$.

If we inductively assume that $\bigvee_{i \le n-2} a_i$ is in the essential image of tr_{n-2} in $\mathrm{Cell}_{n-2}^{\mathrm{bdd}}\,\mathcal{C}$, then observe that $\mathrm{cone}(a_{n-1})$ is sent to a_{n-1} under tr_{n-1}, so $a_{n-1} \vee \bigvee_{i \le n-2} a_i$ will also be in the essential image. Boundedness implies that this induction terminates in finite steps once all cells are coned off. □

The homotopy fibers of the vertical maps in the pushout square

$$\begin{array}{ccc} K((\mathrm{Cell}_n^{\mathrm{bdd}}\,\mathcal{C})^v) & \longrightarrow & K(\mathrm{Cell}_{n-1}^{\mathrm{bdd}}\,\mathcal{C}) \\ \big\downarrow & \ulcorner & \big\downarrow \\ * & \longrightarrow & P \end{array}$$

must agree, so in light of the previous lemma, we conclude that $P \simeq K(\mathcal{C}_{w=n})$. This completes the proof of claim 5.1.8 as well as the proof of the weighty sphere theorem.

5.2 On the K-theory of the heart of a weight structure

Let $\mathcal{C}_{\heartsuit_w}$ denote the heart of a weight structure on a stable ∞-category $\mathcal{C}$. In this section, we prove that the K-theory of $\mathcal{C}_{\heartsuit_w}$ can be expressed in terms of the equivalence classes of objects in $\mathcal{C}_{\heartsuit_w}$.

Theorem 5.2.1. *If $\mathcal{C}_{\heartsuit w}$ is the heart of a weight structure, then all cofiber sequences split in the homotopy category and*

$$K(\mathcal{C}_{\heartsuit w}) \simeq \coprod_{[X] \in \mathrm{ob}\, \iota \mathcal{C}_{\heartsuit w}} B\,\mathrm{Aut}(X)^{+}$$

where $[X]$ is an equivalence class of objects in $\mathcal{C}_{\heartsuit w}$ and $(-)^{+}$ denotes the group completion of the topological monoid $B\,\mathrm{Aut}(X)$.

Proposition 5.2.2. *If $f : A \to B$ is an ingression in $\mathcal{C}_{\heartsuit w}$, then f splits in the homotopy category $h\mathcal{C}$.*

Proof. Write $A \to B \to C$ for the cofiber sequence associated to f, considered as an exact triangle in $h\mathcal{C}$. By assumption, C lies in $\mathcal{C}_{\heartsuit w}$ as well. Lemma 2.2.4 implies that there are maps g and h extending $\mathrm{id} : A \to A$ to a map of exact triangles.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}} \\ A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \end{array}$$

Thus $g \circ f \simeq \mathrm{id}_A$ in $h\mathcal{C}$. □

[2, 10.9] implies that the K-theory space of $\mathcal{C}_{\heartsuit w}$ is given by

$$K(\mathcal{C}_{\heartsuit w}) \simeq \Omega \left(\mathrm{colim}\, \iota S_{\bullet}(\mathcal{C}_{\heartsuit w}) \right).$$

Fix choices of representatives X for every homotopy class $[X]$ of objects in $\mathcal{C}_{\heartsuit w}$ and let $\mathcal{W}$ be the subcategory of $\mathcal{C}_{heartw}$ on the objects X where morphisms in $\mathcal{W}$ are precisely those in $\iota \mathcal{C}_{\heartsuit w}$. That is, $\mathcal{W}$ is the full subcategory of $\iota \mathcal{C}_{\heartsuit w}$ on the chosen vertices X.

Proposition 5.2.2 implies that the inclusion of $\mathcal{W}$ into $\mathcal{C}_{\heartsuit_w}$ induces an equivalence on classifying spaces. Note that $\mathcal{W}$ equipped with the maximal pair structure is a sub-Waldhausen ∞-category of $\mathcal{C}_{\heartsuit_w}$. The equivalence on classifying spaces implies the same equivalence after taking $S_\bullet$:

$$\operatorname{colim} \iota S_\bullet(\mathcal{C}_{\heartsuit_w}) \simeq \operatorname{colim} \iota S_\bullet \mathcal{W}.$$

But $\mathcal{W}$ is discrete as all the maps are equivalences. Write $\mathcal{W}_X$ for the full subcategory on the object X. The K-theory space $\Omega(\operatorname{colim} \iota S_\bullet \mathcal{W}_X)$ is the realization of a bisimplicial set, with one coordinate ranging over the nerve ι and the other over the $S_\bullet$-construction. If we realize the ι-direction first, $\iota S_1 \mathcal{W}_X$ forms the classifying space $B\operatorname{Aut}(X)$. As we realize the $S_\bullet$-direction, we form a delooping of a group completion. Hence, $\Omega(\operatorname{colim} \iota S_\bullet(\mathcal{W}_X)) \simeq B\operatorname{Aut}(X)^+$. Because $\mathcal{W}$ is discrete, we can assemble the realizations for each $\mathcal{W}_X$ into one for $\mathcal{W}$. We conclude with the following formula for $K(\mathcal{C}_{\heartsuit_w})$.

$$K(\mathcal{C}_{\heartsuit_w}) \simeq K(\mathcal{W}) \simeq \coprod_{[X] \in \operatorname{ob} \iota \mathcal{C}_{\heartsuit_w}} B\operatorname{Aut}(X)^+$$

Bibliography

[1] Clark Barwick, *On exact ∞-categories and the theorem of the heart*, Compos. Math. **151** (2015), no. 11, 2160–2186.

[2] ———, *On the algebraic K-theory of higher categories*, J. Topol. **9** (2016), no. 1, 245–347.

[3] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838.

[4] Andrew J. Blumberg and Michael A. Mandell, *The localization sequence for the algebraic K-theory of topological K-theory*, Acta Math. **200** (2008), no. 2, 155–179.

[5] ———, *Localization for THH(ku) and the topological Hochschild and cyclic homology of Waldhausen categories*, ArXiv e-prints (November 2011), available at 1111.4003.

[6] ———, *Localization theorems in topological Hochschild homology and topological cyclic homology*, Geom. Topol. **16** (2012), no. 2, 1053–1120.

[7] Mikhail V. Bondarko, *Weight structures and motives; comotives, coniveau and Chow-weight spectral sequences, and mixed complexes of sheaves: a survey*, ArXiv e-prints (March 2009), available at 0903.0091.

[8] ———, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory **6** (2010), no. 3, 387–504.

[9] ———, *Weights and t-structures: in general triangulated categories, for 1-motives, mixed motives, and for mixed Hodge complexes and modules*, ArXiv e-prints (November 2010), available at 1011.3507.

[10] ———, *Weight structures and 'weights' on the hearts of t-structures*, Homology Homotopy Appl. **14** (2012), no. 1, 239–261.

[11] ———, *Gersten weight structures for motivic homotopy categories; direct summands of cohomology of function fields and coniveau spectral sequences*, ArXiv e-prints (December 2013), available at 1312.7493.

[12] ———, *Weights for relative motives: relation with mixed complexes of sheaves*, Int. Math. Res. Not. IMRN **17** (2014), 4715–4767.

[13] Thomas M. Fiore, *Approximation in K-theory for Waldhausen Quasicategories*, ArXiv e-prints (March 2013), available at 1303.4029.

[14] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[15] ———, *Higher algebra*, Preprint from author's website, 2014.

[16] Amnon Neeman, *K-theory for triangulated categories. III(A). The theorem of the heart*, Asian J. Math. **2** (1998), no. 3, 495–589.

[17] ———, *K-theory for triangulated categories. III(B). The theorem of the heart*, Asian J. Math. **3** (1999), no. 3, 557–608.

[18] ———, *K-theory for triangulated categories $3\frac{3}{4}$: a direct proof of the theorem of the heart*, K-Theory **22** (2001), no. 1-2, 1–144.

[19] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[20] Marco Schlichting, *Negative K-theory of derived categories*, Math. Z. **253** (2006), no. 1, 97–134.

[21] R. W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, 1990, pp. 247–435.

[22] Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), 1985, pp. 318–419.

[23] Charles A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.