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Twisted and Exceptional Tinkertoys for Gaiotto Duality

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Twisted and Exceptional Tinkertoys for Gaiotto Duality

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Twisted and Exceptional Tinkertoys for Gaiotto Duality

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A large class of $4d$ $\mathcal{N}=2$ superconformal field theories arise as compactifications of a $6d$ $(2,0)$ theory of type $\mathfrak{j}=A,D,E$ on a punctured Riemann surface, C. These theories can be classified by listing the allowed fixtures and cylinders which can occur in a pants decomposition of C, and giving the rules for gluing them together. Different pants decompositions of the same surface give different weakly-coupled presentations of the same underlying SCFT, related by S-duality. An even larger class of theories can be constructed in this way by including "twisted" punctures, which carry a non-trivial action of the outer-automorphism group of $\mathfrak{j}$. In this dissertation, we discuss the classification procedure for twisted theories of type D_N, as well as for twisted and untwisted theories of type E_6. Using these results, we write the Seiberg-Witten solutions for all $Spin(n)$ gauge theories with matter in spinor representations which can be realized by compactifying the $(2,0)$ theory. We also study a family of SCFTs arising from the twisted A_{2N} series, whose twisted punctures are still not fully-understood.

Contents

Chapter 1

Introduction

In [1], Gaiotto showed that the marginal deformations of a large class of four-dimensional $\mathcal{N} = 2$ SCFTs can be identified with $\mathcal{M}_{g,n}$, the moduli space of a genus g Riemann surface, C, with n punctures. These theories can be seen to arise from the compactification of a six-dimensional $(2,0)$ theory on C. Different degeneration limits of C correspond to different weakly-coupled descriptions of the same SCFT, related by S-duality. This construction greatly generalizes the original examples of $\mathcal{N} = 2$ S-dualities, first found in [2].

In a degeneration limit, C can be decomposed into a collection of 3-punctured spheres ("fixtures") and cylinders, corresponding to $4d$ "matter" theories and $\mathcal{N} = 2$ vector multiplets, respectively. One can classify the $4d$ theories which arise in this way by listing the allowed fixtures and cylinders which can arise in a pants-decomposition of C, and giving the rules for gluing them together. For a given $(2,0)$ theory, this is a finite list. The classification procedure has been discussed for twisted and untwisted theories of type A in [3, 4] and untwisted theories of type D in [5]. In this dissertation, we will continue this classification procedure for the twisted D-series, as well as the twisted and untwisted E_6 theory. We will also discuss a family of interacting

SCFTs which arise from compactifying the twisted A_{2N} $(2,0)$ theory, whose general twisted punctures are not yet fully-understood.

We begin this chapter by discussing general aspects of the compacification of six-dimensional $(2,0)$ theories on C, following [6]. We then discuss in detail the "local" properties of the codimension-2 defects of the $(2,0)$ theory which live at the punctures, following [7].

1.1 Compactifying the $(2,0)$ theories

The six-dimensional $(2,0)$ superconformal theories enjoy $osp(6,2|4)$ superconformal invariance, and can be constructed by taking a low-energy decoupling limit of type IIB string theory on $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma$, where Γ is a discrete subgroup of $SU(2)$ of type $\mathfrak{j} = A, D$, or E. There is a basis of operators transforming in short representations of $osp(6,2|4)$, labeled by the Casimir operators of $\mathfrak{j}$. Within each short multiplet, there is a subspace V_k of operators with lowest conformal weight, given by twice the exponent d_k of $\mathfrak{j}$, which is an irreducible representation of the $\mathfrak{so}(5)$ R-symmetry. The theory has a Coulomb branch parametrized by expectation values of these chiral operators, which is isomorphic to

$$\mathcal{M} = \frac{(\mathbb{R}^5)^{\text{rank } \mathfrak{j}}}{\mathcal{W}_{\mathfrak{j}}} \tag{1.1}$$

where $\mathcal{W}_{\mathfrak{j}}$ is the Weyl group of $\mathfrak{j}$.

To compactify the $(2,0)$ theory on a Riemann surface C while preserving $4d$ $\mathcal{N} = 2$ supersymmetry, we must perform a partial twisting. The super

Poincaré subalgebra of $\mathfrak{osp}(6,2|4)$ has bosonic part $\mathfrak{so}(5,1) \oplus \mathfrak{so}(5)$, where the spinor representations of $Spin(1,5)$ and $Spin(5)$ are given by $\mathbb{H}^2$, where $\mathbb{H}$ is the quaternions. The Poincaré supercharges therefore transform in the $(\mathbb{C}^4 \otimes \mathbb{C}^4)_+$ of $\mathfrak{so}(5,1) \oplus \mathfrak{so}(5)$, where the subscript $+$ denotes a symplectic Majorana reality constraint. Compactifying on C breaks $\mathfrak{so}(5,1) \oplus \mathfrak{so}(5)$ to $\mathfrak{so}(3,1) \oplus \mathfrak{so}(2)_C \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)_R$, under which the supercharges transform as

$$\left(\left((2,1)_{\frac{1}{2}} \oplus (1,2)_{-\frac{1}{2}}\right) \otimes \left(2_{\frac{1}{2}} \oplus 2_{-\frac{1}{2}}\right)\right)_+ \tag{1.2}$$

The twisting consists of identifying the diagonal $\mathfrak{so}(2)$ of $\mathfrak{so}(2)_C \oplus \mathfrak{so}(2)_R$ with the holonomy algebra of C, leaving us with supercharges transforming under $\mathfrak{so}(3,1) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)'_C$ as

$$(2,1;2)_1 \oplus (2,1;2)_0 \oplus (1,2;2)_0 \oplus (1,2;2)_{-1} \tag{1.3}$$

The middle two summands of (1.3) are uncharged under $\mathfrak{so}(2)'_C$, and so are well-defined four dimensional supercharges, which we denote by $Q^{\alpha A}, \overline{Q}^{\dot{\alpha} A}$.

The moduli space of the four-dimensional theory is obtained essentially by dimensional reduction from the the Coulomb branch of the six-dimensional theory. More precisely, we choose a Cartan subalgebra $\mathfrak{so}(2)_R \oplus \mathfrak{so}(2)$ of the $\mathfrak{so}(5)$ R-symmetry, and let $\mathcal{O}_k$ be the operator in V_k of weight $(d_k, 0)$. This operator has the largest $\mathfrak{so}(2)_R$ charge in the multiplet and hence it must be annihilated by any supercharge, such as $\overline{Q}^{\dot{\alpha} A}$, with positive $\mathfrak{so}(2)_R$ charge. The $4d$ Coulomb branch is parametrized by the vacuum expectation values of these chiral operators.

After the twisting, $\mathcal{O}_k$ is a section of the bundle $K^{\otimes d_k}$ over C, which is also true of its vacuum expectation value $\langle \mathcal{O}_k \rangle$. Since $\mathcal{O}_k$ is annihilated by $\overline{Q}^{\dot{\alpha}A}$, and since $\overline{Q}^{\dot{\alpha}A}$-exact operators have vanishing vev's, $\langle \mathcal{O}_k \rangle$ must be annihilated by $\overline{\partial}$. This is the only condition on $\langle \mathcal{O}_k \rangle$, so the $4d$ Coulomb branch is simply

$$\mathcal{C}_{4d} = \bigoplus_{k=1}^{r} H^0(C, K^{\otimes d_k}). \tag{1.4}$$

The $(2,0)$ theory of type A_{N-1} can also be realized as the low-energy worldvolume theory on N coincident M5-branes. In this case, $\mathcal{C}_{4d}$ has a nice geometric interpretation. The M5-branes are wrapped on a holomorphic cycle C inside a hyperkähler four-manifold Q. We go onto the Coulomb branch by separating the branes so that they wrap some other cycle Σ inside Q, where we take Σ to be a connected divisor inside Q. By viewing Q as a holomorphic symplectic manifold, we can identify a neighborhood of C with the holomorphic cotangent bundle T^*C by picking holomorphic Darboux coordinates (x, z) for Q. A point of $\mathcal{C}_{4d}$ corresponds to picking the coefficients $u_k \in H^0(C, K^{\otimes k})$, of the equation

$$x^N + \sum_{k=2}^{N} u_k(z) x^{N-k} = 0 \tag{1.5}$$

defining $\Sigma \subset T^*C$.

Normalizing the coordinates so that the holomorphic symplectic form is

$$\Omega = \frac{\ell^3}{2\pi^2} dx \wedge dz, \tag{1.6}$$

the projection map $T^*C \to C$ identifies Σ as an N-sheeted cover of C. The distance between the i-th and j-th sheets is a one-form on C, which we denote λ_{ij}.

In [6], it was shown that Σ can be identified with the Seiberg-Witten curve of the $4d$ theory, and the canonical one-form

$$\lambda = x dz \tag{1.7}$$

restricted to Σ can be identified with the Seiberg-Witten differential.

For $(2,0)$ theories of type $\mathfrak{j} \neq A_{N-1}$, the anaylsis is similar, except that the coefficients in (1.5) are no longer just linear functions on the Coulomb branch, but, in general, are polynomial expressions when expressed in terms of the natural linear coordinates at the origin of the Coulomb branch. Additionally, the Coulomb branch can have graded components of degrees other than the expected d_k. In general, the Coulomb branch takes the form [5]

$$E \subset V \tag{1.8}$$

where

$$V = \bigoplus_k H^0(C, K^{\otimes d_k}) \oplus \bigoplus_k W_k \tag{1.9}$$

where the W_k are vector spaces of degree k and E is the subvariety satisfying the collection of polynomial constraints (linear in at least one variable, and of homogeneous degree).

1.2 Relation to Hitchin systems

It is well-known that the Seiberg-Witten solutions of many $\mathcal{N} = 2$ theories can be understood in terms of complex integrable systems. We now show that the relevant integrable system for this class of theories is a *Hitchin system*.

To do so, we consider the theory obtained by further dimensional reduction from $d = 4$ to $d = 3$ on S^1. On general field theory grounds, at low energies the $3d$ effective theory is an $\mathcal{N} = 4$ sigma model into a hyperkähler target space $\mathcal{M}$, where $\mathcal{M}$ is a fibration over $\mathcal{C}_{4d}$ with generic fiber a compact torus [8].

Now consider reversing the order of compactification. By first compactifying on S^1 of radius R, we obtain at low energies $5d$ $\mathcal{N} = 2$ super Yang-Mills theory. We are interested in this five-dimensional theory further compactified on C, with an appropraite topological twist. The moduli space of the resulting $3d$ theory is the space of BPS configurations of the five-dimensional theory which are Poincaré invariant in $3d$. Denote the adjoint scalars of the super Yang-Mills theory by Φ^I, $I = 1, \ldots, 5$, so that

$$\Phi \equiv \frac{1}{2}(\Phi^1 + i\Phi^2) \tag{1.10}$$

has $\mathfrak{so}(2)_R$ charge $+1$, and $\Phi^{3,4,5}$ have charge zero. In the twisted theory, $\Phi = \Phi_z dz$ is a $(1,0)$ form on C. Then the BPS equations are the Hitchin equations for the gauge field $A = A_z dz + A_{\overline{z}} d\overline{z}$ cotangent to C and the adjoint scalar field Φ:

$$\begin{aligned} F + R^2[\Phi, \overline{\Phi}] &= 0, \\ \overline{\partial}_A \Phi \equiv d\overline{z}(\partial_z \Phi + [A_{\overline{z}}, \Phi]) &= 0, \\ \partial_A \overline{\Phi} \equiv dz(\partial_z \overline{\Phi} + [A_z, \overline{\Phi}]) &= 0. \end{aligned} \tag{1.11}$$

Due to the partial topological twist, the BPS protected quantities we study do not depend on the conformal scale of the metric on C, so we do not expect any phase transition when we exchange the relative length scales of S^1 and C. We therefore identify $\mathcal{M}$ with the moduli space of solutions of Hitchin's equations on C. The map $\mathcal{M} \to \mathcal{C}_{4d}$ given by

$$(A, \Phi) \mapsto \{\text{Casimirs of } \Phi\} \tag{1.12}$$

is the well-known *Hitchin fibration*; its fiber over a generic $u \in \mathcal{C}_{4d}$ is indeed an abelian variety, the Prym variety of the projection $\overline{\Sigma}_u \to C$, defined as the kernel of a corresponding map of Jacobians $J(\overline{\Sigma}_u) \to J(C)$.

As discussed above, the $\langle \mathcal{O}_k \rangle$ determine the Seiberg-Witten curve $\Sigma \subset T^*C$. Since we have identified these with the Casimirs Tr Φ^k, Σ given by (1.5) is the *spectral curve* determined by Φ,

$$\det\,(x dz - \Phi) = 0. \tag{1.13}$$

We see that the positions x_i of the sheets of Σ in the cotangent directions can be interpreted as the eigenvalues of the matrix-valued one-form Φ, thus the coefficients $u_k(z)$ are elementary symmetric functions of the eigenvalues, and can be written as polynomials in the $\langle \mathcal{O}_k \rangle$.

In the next subsection, we will discuss a class of codimension-2 defects of the $6d$ $(2,0)$ theories, which wrap four-dimensional spacetime and live at a point on C. The presence of these defects will give rise to singular boundary conditions for Φ, which introduce extra parameters contributing to the dimension of the moduli space of the $4d$ theory. We will discuss the classification of the defects in detail, and show how to compute their contritutions to the $4d$ Higgs and Coulomb branch dimensions, as well as to the central charges of the $4d$ SCFT.

1.3 Local properties of codimension-2 defects of the $6d$ $(2,0)$ theories

We now review the local properties of the half-BPS codimension-2 defects of the $6d$ $(2,0)$ theories which live at the punctures on C, following [7]. For a $(2,0)$ theory of type $J = A, D, E$, these defects are labeled by a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{j}$. For $J = A_{N-1}, D_N$, or E_6, we can further introduce a class of "twisted" defects, around which there is an action of a non-trivial outer-automorphism o of J. Twisted defects are labeled by homomorphisms $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, where $\mathfrak{g}^\vee$ is the subalgebra of $\mathfrak{j}$ invariant under o. The introduction of a defect of type ρ at a point on C will [1]:

- add a flavor symmetry group factor $F(\rho)$,
- increase the dimensions of the Higgs and Coulomb branches by, respectively, $\dim_{\mathbb{H}}\mathcal{H}(\rho)$ and $\dim_{\mathbb{C}}\mathcal{C}_{4d}(\rho)$, and
- increase the effective number of vector and hypermultiplets n_v and n_h by $n_v(\rho)$ and $n_h(\rho)$. Accordingly, the central charges a and c, which are linear combinations of n_h and n_v, also get a contribution.

[1] We assume the number of punctures on C is sufficient to get a $4d$ SCFT in the zero area limit.

All of these quantities are local in the sense that they do not depend of the genus of C or on the properties of the other punctures.

1.3.1 Classification of punctures

As mentioned above, each puncture is labeled by a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$. The adjoint orbit of an element $e \in \mathfrak{g}$, denoted O_e, is its $G_{\mathbb{C}}$ conjugacy class in $\mathfrak{g}$,

$$O_e = \{ad(g) \cdot e \in \mathfrak{g} | g \in G_{\mathbb{C}}\}.$$

The Jacobson-Morozov theorem states that the classification of such homomorphisms ρ, up to conjugacy, is equivalent to the classification of nilpotent elements in $\mathfrak{g}$, also up to conjugacy, through the correspondence $e = \rho(\sigma^+)$. Since e is nilpotent, the orbit O_e is called a *nilpotent orbit*. When $\mathfrak{g}$ is of classical type, nilpotent orbits have a convenient classification in terms of partitions. When $\mathfrak{g} = \mathfrak{su}(N)$, a nilpotent orbit O_e is specified by the decomposition of the N-dimensional fundamental representation into irreducible representations of $\mathfrak{su}(2)$, $N = N_1 + \cdots + N_k$, or, equivalently, by a partition $p = [N_i]$ of N. The N_i are called the *parts* of the partition p.

When $\mathfrak{g} = \mathfrak{so}(N)$, a nilpotent orbit O_e is specified by the decomposition of the N-dimensional vector representation into irreducible representations of $\mathfrak{su}(2)$. This can again be specified by a partition $p = [N_i]$ of N, but with the requirement that any even part must appear an even number of times. Such partitions are caleed B- or *D-partitions*, when N is odd or even, respectively.

Given a partition $p = [N_i]$ satisfying this condition, there is a unique nilpotent orbit, except for the case when all the parts N_i are even and each even integer appears an even number of times. Such a partition is called a *very even* partition, and there are two distinct nilpotent orbits associated to it, exchanged by the outer-automorphism of $\mathfrak{so}(N)$.

Similarly, for a nilpotent element e in $\mathfrak{g} = \mathfrak{sp}(N)$, the corresponding homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{sp}(N)$ determines a partition $p = [N_i]$ of $2N$, with the condition that any odd part in $p = [N_i]$ appears an even number of times. Such a partition is called a *C-partition*, and each such partition corresponds to a unique orbit.

Nilpotent orbits in exceptional $\mathfrak{g}$ also have a conventient classification, though not in terms of partitions. The classification for $\mathfrak{g} = \mathfrak{e}_6$ and $\mathfrak{f}_4$ will be discussed in chapters 4 and 5, respectively.

1.3.2 Global symmetry and central charge k_{4d}

Each puncture carries a flavor symmetry group, $F(\rho)$, given by the subgroup of G commuting with the image of ρ. For classical $\mathfrak{g}$, the Lie algebra $\mathfrak{f}(\rho)$ of $F(\rho)$ is given by

$$\begin{aligned}
&\mathfrak{s}[\oplus_i \mathfrak{u}(r_i)] && \text{when } \mathfrak{g} = \mathfrak{su}(N),\\
&\oplus_{i \text{ odd}}\, \mathfrak{so}(r_i) \oplus \oplus_{i \text{ even}} \mathfrak{sp}(r_i/2) && \text{when } \mathfrak{g} = \mathfrak{so}(N),\\
&\oplus_{i \text{ odd}}\, \mathfrak{sp}(r_i/2) \oplus \oplus_{i \text{ even}} \mathfrak{so}(r_i) && \text{when } \mathfrak{g} = \mathfrak{sp}(N),
\end{aligned}$$

where r_i is the number of parts N_k in the partition $p = [N_k]$ equal to i.

The central charge k_{4d} of each non-abelian factor in $F(\rho)$, defined via the current algebra [2]

$$J^a_\mu(x)J^b_\nu(0) = \frac{3k_{4d}}{4\pi^4}\delta^{ab}\frac{g_{\mu\nu}x^2 - 2x_\mu x_\nu}{(x^2)^4} + \frac{2}{\pi^2}f^{ab}{}_c\frac{x_\mu x_\nu x\cdot J^c}{(x^2)^3},$$

can be determined as follows. Consider putting the $6d$ theory and the defect on a general Riemann surface C, and perform Nekrasov's deformation on the $4d$ side, with parameters $\epsilon_{1,2}$. This will lead to a $2d$ theory on C, which is believed to have the W-symmetry $W(\mathfrak{g},\rho)$ at the parameter $b^2 = \frac{\epsilon_2}{\epsilon_1}$, obtained by quantum Drinfeld-Sokolov reduction of the affine $\mathfrak{g}$ Lie algebra. Since we will make use of it shortly, we note that the $2d$ central charge of $W(\mathfrak{g},\rho)$ is given by

$$c_{2d} = \dim\,\mathfrak{g}_0 - \frac{1}{2}\dim\,\mathfrak{g}_{1/2} + \frac{24}{b^2}\rho_\mathfrak{g}\cdot\rho_\mathfrak{g} + 12\rho_\mathfrak{g}\cdot\frac{h}{2} + 24b^2\frac{h}{2}\cdot\frac{h}{2} \tag{1.14}$$

where $\rho_\mathfrak{g}$ is the Weyl vector of $\mathfrak{g}$, $h = \rho(\sigma^3)$, and we have decomposed $\mathfrak{g}$ into

$$\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_j$$

where j is the eigenvalue of the action of $h/2$.

For a defect of type ρ with flavor symmetry $F(\rho)$, the corresponding W-algebra $W(\mathfrak{g},\rho)$ has the affine Lie subalgebra $\hat{f}(\rho)$. The current algebra level of $\hat{f}(\rho)$ was computed in [9]. Since $F(\rho)$ commutes with $\rho(\mathfrak{su}(2))$, the adjoint representation $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} R_j\otimes V_j$$

where V_j is the irreducible representation of $\mathfrak{su}(2)$ of spin j, and R_j is the corresponding (reducible) representation of $f(\rho)$. Choose generators $T^{a,b}$ of a simple subalgebra $f' \subset f(\rho)$ so that $tr_{f'}T^aT^b = h^\vee(f')\delta^{ab}$, where $h^\vee$ is the dual Coxeter number. Denote by f the natural embedding $f : f(\rho) \to \mathfrak{g}$. The level of $\hat{f}'$ is then given by

$$k_{2d}(f')\delta^{ab} = k_{2d}\frac{1}{h^\vee(\mathfrak{g})}tr_{\mathfrak{g}}f(T^a)f(T^b) + \sum_j 2jtr_{R_j}T^aT^b \tag{1.15}$$

where $k_{2d} = -h^\vee(\mathfrak{g}) + 1/b^2$ is the level of the affine $\mathfrak{g}$ algebra before the Drinfeld-Sokolov reduction.

The relation between the level of the $2d$ current subalgebra $\hat{f}(\rho)$ of $W(\mathfrak{g}, \rho)$ and the level of the $4d$ flavor symmetry $f(\rho)$ was shown in [9] to be given by $k_{4d}(f') = 2k_{2d}(f')|_{1/b^2=0}$. Using (1.15), we find $k_{4d}(f')$ is given by

$$k_{4d}(f')\delta^{ab} = -2tr_{\mathfrak{g}}f(T^a)f(T^b) + 2\sum_j 2jtr_{R_j}T^aT^b = 2\sum_j tr_{R_j}T^aT^b.$$

1.3.3 a and c central charges

Four dimensional conformal field theories have two Weyl anomaly coefficients, a and c, defined via

$$T_\mu{}^\mu = \frac{c}{16\pi^2}(\text{Weyl})^2 - \frac{a}{16\pi^2}(\text{Euler})$$

where

$$(\text{Weyl})^2 = R^2_{\mu\nu\lambda\rho} - 2R^2_{\mu\nu} + \frac{1}{3}R^2,$$
$$(\text{Euler}) = R^2_{\mu\nu\lambda\rho} - 4R^2_{\mu\nu} + R^2.$$

For $4d\,\mathcal{N}=2$ theories, it is convenient to parametrize a and c by the "effective" numbers of vector and hypermultiplets

$$n_v = 4(2a-c), \qquad n_h = 4(5c-4a),$$

which are normalized so that, in a free theory, n_v counts the number of free vector multiplets and n_h counts the number of free hypermultiplets. Adding a defect of type ρ increases these central charges by $n_v(\rho)$ and $n_h(\rho)$. The total n_v and n_h of a $4d\,\mathcal{N}=2$ SCFT take the following form

$$\begin{aligned} n_v &= \sum_i n_v(\rho_i) + (g-1)(\frac{4}{3}h^\vee(J)\ \dim\ J + \text{rank}\ J), \\ n_h &= \sum_i n_h(\rho_i) + (g-1)(\frac{4}{3}h^\vee(J)\ \dim\ J). \end{aligned} \tag{1.16}$$

The global terms, which are proportional to $(g-1)$, were calculated in [10] using the anomaly polynomials of the $6d$ $(2,0)$ theories. The $n_{v,h}(\rho)$, as well as the $2d$ central charge c_{2d} should come from the anomaly polynomial of the defect of type ρ. On flat space, a codimension-2 defect has $a(\rho)$ and $c(\rho)$, and the $SO(2)$ rotation of the transverse space to the defect is a flavor symmetry, with central charge $k_T(\rho)$. The $n_{v,h}(\rho)$ are a certain linear combination of these three fundamental quantities, determined by the R-symmetry twist needed to preserve $4d\,\mathcal{N}=2$ supersymmetry discussed above.

The central charge of the W-algebra $W(\mathfrak{g},\rho)$ should have the same origin. Since the standard W-algebra $W(\mathfrak{j}) = W(\mathfrak{j},\rho_{\text{prin}})$ corresponds to the absence of the defect when $\mathfrak{g}$ is simply-laced, the contribution from the presence of the defect of type ρ is

$$\delta c_{2d}(\mathfrak{g},\rho) = c_{2d}(\mathfrak{g},\rho) - c_{2d}(\mathfrak{j},\rho_{\text{prin}})$$
$$= (\dim\ \mathfrak{g}_0 - \text{rank}\ \mathfrak{j}) + \frac{1}{2}\dim\ \mathfrak{g}_{1/2} + 12(\rho_{\mathfrak{g}}\cdot\frac{h}{2} - \rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}}) + 24b^2(\frac{h}{2}\cdot\frac{h}{2} - \rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}}),$$

where the roots are normalized so that $\rho_{\mathfrak{g}}\cdot\rho_{\mathfrak{g}} = \rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}}$. This suggests that $a(\rho), c(\rho)$ and $k_T(\rho)$, and therefore also $n_{v,h}(\rho)$, can be expressed as linear combinations of

$$\dim\ \mathfrak{g}_0 - \ \text{rank}\ \mathfrak{j},\quad \dim\ \mathfrak{g}_{1/2},\quad \rho_{\mathfrak{g}}\cdot\frac{h}{2} - \rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}},\quad \frac{h}{2}\cdot\frac{h}{2} - \rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}}.$$

From the known results for $n_h(\rho)$ and $n_v(\rho)$ for type $J = A, D$ from the analysis of quiver gauge theories [1, 11], one finds that

$$n_h(\rho) = 8(\rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}} - \rho_{\mathfrak{g}}\cdot\frac{h}{2}) + \frac{1}{2}\dim\ \mathfrak{g}_{1/2}, \tag{1.17}$$
$$n_v(\rho) = 8(\rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}} - \rho_{\mathfrak{g}}\cdot\frac{h}{2}) + \frac{1}{2}(\text{rank}\ \mathfrak{j} - \ \dim\ \mathfrak{g}_0), \tag{1.18}$$

where we note that $\rho_{\mathfrak{j}}\cdot\rho_{\mathfrak{j}} = \frac{1}{12}h^\vee(J)\dim\ J$.

1.3.4 Contributions to Higgs and Coulomb branch dimensions

To determine the contributions of a defect of type ρ to the Higgs and Coulomb branch dimensions, we put the $6d$ $(2,0)$ theory of type J on $\mathbb{R}^{2,1}\times(\text{cigar})\times\tilde{S}^1$, where "cigar" denotes a semi-infinite cigar geometry with the defect placed at the tip. We allow the fields on the cigar to undergo a monodromy by an outer-automorphism o of J upon circling the defect.

Upon reducing along the $U(1)$ isometry of the cigar, we get $5d$ $\mathcal{N}=2$ super Yang-Mills with gauge group G on $\mathbb{R}^{2,1}\times(\text{a half-line})\times\tilde{S}^1$. The defect becomes a boundary condition at the end of the half-line, producing a pole in three of the adjoint scalars. Letting s be the distance to the boundary, this is given by

$$\Phi_{1,2,3}(s) \sim \frac{\rho(\tau_{1,2,3})}{s},$$

where ρ is a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, as discussed above. Further reducing on $\tilde{S}^1$ gives $4d$ $\mathcal{N} = 4$ super Yang-Mills with gauge group G on a half-space with essentially the same boundary condition, which is exactly the Nahm-type boundary conditions studied by Gaiotto and Witten in [12, 13]. The homomorphism ρ is therefore called the *Nahm pole*.

Gaiotto and Witten also considered the S-dual of this boundary condition, which corresponds to inverting the order of reductions on the S^1 of the cigar and $\tilde{S}^1$. They found that S-duality gives $4d$ $\mathcal{N} = 4$ super Yang-Mills with gauge group $G^\vee$, with boundary condition given by coupling the bulk fields to a $3d$ $\mathcal{N} = 4$ superconformal field theory $T^\rho[\mathfrak{g}]$ with $G^\vee$ flavor symmetry, living at the $3d$ boundary of the $4d$ half-space.

If we now undo the reduction on S^1, the codimension-2 defect of $5d$ super Yang-Mills on the cigar is given by coupling the fields to the $3d$ theory $T^\rho[\mathfrak{g}]$, producing a pole in the adjoint scalar field

$$\Phi(z) \equiv \Phi_4(z) + i\Phi_5(z) \sim \tilde{\rho}(\sigma^+)/z,$$

where z is a complex coordinate on the cirgar so that the tip is at $z = 0$, and $\tilde{\rho}$ is a new homomorphism, $\tilde{\rho} : \mathfrak{su}(2) \to \mathfrak{g}^\vee$, determined by the properties of $T^\rho[\mathfrak{g}]$. Since $\Phi(z)$ will become the Higgs field of the Hitchin system which controls the $4d$ Coulomb branch, so $\tilde{\rho}$ is called the *Hitchin pole.*

Unfortunately, we do not understand the $6d$ $(2,0)$ theory well enough to also undo the reduction on $\tilde{S}^1$ and give a precise description of the defect in the $6d$ theory. However, we will still be able to study how the worldvolume fields $\phi_k(z)$ of dimension k behave at the defect, by studying the behavior of $p_k(\Phi(z))$ where p_k is a degree-k invariant polynomial of $\mathfrak{g}$.

1.3.4.1 Contribution to $4d$ Higgs branch

We now determine the local contributions of a defect of type ρ to the dimensions of the $4d$ Higgs branch and Coulomb branch. Consider $5d$ $\mathcal{N}=2$ super Yang-Mills with gauge group J on $\mathbb{R}^{2,1}\times C$, coupled to the $3d$ theory $T^{\rho_i}[\mathfrak{g}]$, which wraps $\mathbb{R}^{2,1}$ and lives at a point on C. In the setup above, this corresponds to reducing the $(2,0)$ theory of type J on $\tilde{S}^1$. This system was studied in [14] for $J=A,D$, but the arguments also hold for J of type E. After reducing on $\tilde{S}^1$, the dimension of the Coulomb branch doubles, while the dimension of the Higgs branch is preserved. The contribution of the defect to the Higgs branch quaternionic dimension is

$$\dim_{\mathbb{H}}\mathcal{H}(\rho)=\dim_{\mathbb{H}}\mathcal{H}(T^\rho[\mathfrak{g}])$$

and the total quaternionic dimension of the Higgs branch is then given by

$$\dim_{\mathbb{H}}\mathcal{H}=\sum_i \dim_{\mathbb{H}}\mathcal{H}(\rho_i)+\text{rank }\mathfrak{g}^\vee.$$

We now determine $\dim_{\mathbb{H}}\mathcal{H}(T^\rho[\mathfrak{g}])$ for arbitrary ρ. When $\rho = 0$ (i.e., the trivial embedding), $T^\rho[\mathfrak{g}]$ is often denoted by $T[\mathfrak{g}]$. Its Coulomb branch is $\mathcal{N}_{\mathfrak{g}^\vee}$ and its Higgs branch is $\mathcal{N}_{\mathfrak{g}}$, where $\mathcal{N}_{\mathfrak{g}}$ denotes the *nilpotent cone* of $\mathfrak{g}$ - the subset of $\mathfrak{g}$ consisting of its nilpotent elements. The dimension of $\mathcal{N}_{\mathfrak{g}}$ is given by

$$\dim_{\mathbb{C}} \mathcal{N}_{\mathfrak{g}} = \dim G - \text{ rank } G.$$

This complex dimension is always even, as $\mathcal{N}_{\mathfrak{g}}$ is a hyperkähler cone, which is expected since $T[\mathfrak{g}]$ has $3d$ $\mathcal{N} = 4$ superconformal symmetry.

For a homomorphism $\rho : \mathfrak{su}(2) \to \mathfrak{g}$, we can give a Higgs vev $e = \rho(\sigma^+)$ to the theory $T[\mathfrak{g}]$. The moduli directions inside $\mathcal{N}_{\mathfrak{g}}$ that are transverse to O_e are in general singular, while the directions along O_e are smooth. Thus, at low energies, the theory $T[\mathfrak{g}]$ becomes $(\dim_{\mathbb{C}} O_e)/2$ free hypermultiplets plus the $3d$ interacting SCFT $T^\rho[\mathfrak{g}]$. The quaternionic Higgs branch dimension of $T^\rho[\mathfrak{g}]$ is therefore

$$\dim_{\mathbb{H}} \mathcal{H}(T^\rho[\mathfrak{g}]) = \frac{1}{2}(\dim G - \text{rank } G - \dim_{\mathbb{C}} O_e).$$

For a nilpotent orbit O_e in a classical Lie algebra, labeled by a partition $p = [N_i]$, its dimension is given by

$$\dim_{\mathbb{C}} O_{[N_i]} = \begin{cases} N^2 - \sum_i s_i^2 & \mathfrak{g} = \mathfrak{su}(N), \\ N(2N+1) - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i \text{ odd}} r_i & \mathfrak{g} = \mathfrak{so}(2N+1), \\ N(2N+1) - \frac{1}{2}\sum_i s_i^2 - \frac{1}{2}\sum_{i \text{ odd}} r_i & \mathfrak{g} = \mathfrak{sp}(N), \\ N(2N-1) - \frac{1}{2}\sum_i s_i^2 + \frac{1}{2}\sum_{i \text{ odd}} r_i & \mathfrak{g} = \mathfrak{so}(2N), \end{cases}$$

where $[s_i]$ is the transpose partition to $[N_i]$, and r_k is the number of times the part k appears in the partition $[N_i]$. The contributions of a defect of type

$\mathfrak{e}_6$ and $\mathfrak{f}_4$ to the $4d$ Higgs and Coulomb branches will be given explicitly in chapters 4 and 5, respectively.

1.3.4.2 Contribution to $4d$ Coulomb branch

The Coulomb branch of the $3d$ theory is given by the moduli space of the Hitchin system of gauge group J, with an outer-automorphism twist o_i around the i-th puncture p_i, coupled to the Coulomb branch of $T^{\rho_i}[\mathfrak{g}]$. To compute the local contribution of a defect to the Coulomb branch, we need to understand the local boundary condition for the Hitchin system near the puncture on C.

The Coulomb branch of the $T^{\rho}[\mathfrak{g}]$ theory is a subset of the Coulomb branch of $T[\mathfrak{g}]$, which is the nilpotent cone $\mathcal{N}_{\mathfrak{g}^\vee}$. When $\mathfrak{g}$ is of classical type, the theories $T^{\rho}[\mathfrak{g}]$ can be constructed via an arrangement of branes. Let p be the partition corresponding to the nilpotent element $e = \rho(\sigma^+)$. When $\mathfrak{g}$ is of type A, C, or D, the Coulomb branch is the closure of a single nilpotent orbit $O_{\tilde{e}}$, where $\tilde{e}$ is a nilpotent element of $\mathfrak{g}^\vee$ [14, 12], whose partition type is given by p^t when $\mathfrak{g}$ is of type A, $(p^{+t})_B$ when $\mathfrak{g}$ is of type C, and $(p^t)_D$ when $\mathfrak{g}$ is of type D. Here, the notation is that, for a partition $p = [N_1, \ldots, N_k]$, p^t is the transpose partition to p, p^+ is the partition $[N_1, \ldots, N_k, 1]$, and $p_{B,D}$ stand for the B- and D-collapses, respectively, of p, and are defined to be the unique maximal B-, D-partition q satisfying $p \geq q$ (in the standard partial order on the partitions). This combinational operation agrees with the map $d : \mathcal{N}_{\mathfrak{g}}/G \to \mathcal{N}_{\mathfrak{g}^\vee}/G^\vee$, defined for any simple Lie algebra $\mathfrak{g}$, known as the

Spaltenstein map.

This implies that the Coulomb branch of $T^{\rho}[\mathfrak{g}]$ is given by the closure of the Spaltenstein dual orbit $d(O_e)$ to the orbit O_e, where $e = \rho(\sigma^+)$.

We are now in a position to compute the contribution to the $4d$ Coulomb branch dimension for a defect of type ρ. Before doing so, we note that the outer-automorphism o introduces a grading for the Lie algebra $\mathfrak{j}$. The outer-automorphism can be trivial, of order 2 (for $\mathfrak{j} = A_{N-1}, D_N, E_6$), or of order 3 (for $\mathfrak{j} = D_4$). The Lie algebra $\mathfrak{j}$ splits into a direct sum of eigenspaces under the action of o:

$$\begin{aligned} \mathfrak{j} &= \mathfrak{j}_1 + \mathfrak{j}_{-1} && \text{for } o \text{ of order 2,} \\ \mathfrak{j} &= \mathfrak{j}_1 + \mathfrak{j}_{\omega} + \mathfrak{j}_{\omega^2} && \text{for } o \text{ of order 3,} \end{aligned} \tag{1.19}$$

where the lower indices denote the eigenvalues under the action of o, e.g., $o(j_{\omega^2}) = \omega^2 j_{\omega^2}$. By definition, $\mathfrak{j}_1 = \mathfrak{g}^{\vee}$. The grading means that, e.g., $[\mathfrak{j}_{\omega}, \mathfrak{j}_{\omega^2}] \subset \mathfrak{j}_1$.

Let us now pick a defect of type ρ, and let z be a local coordinate on C such that the defect is located at the origin. If the outer-automorphism o associated to the defect is trivial, the defect is called *untwisted*, and is labeled by a nilpotent orbit O_{ρ} in $\mathfrak{j}$ (since, for o trivial, $\mathfrak{j} = \mathfrak{g} = \mathfrak{g}^{\vee}$). In this case, the Spaltenstein dual orbit $d(O_{\rho})$ is also in $\mathfrak{j}$. The Higgs field Φ of the Hitchin system behaves as

$$\Phi(z) = \left[\frac{\Phi_{-1}}{z} + \Phi_0 + \dots\right] dz, \tag{1.20}$$

where Φ_{-1} is an element in $d(O_{\rho})$, and Φ_0 is a generic element in $\mathfrak{j}$. The introduction of an untwisted defect of type ρ increases the dimension of the

Coulomb branch by

$$\dim_{\mathbb{C}} \mathcal{C}_{4d}(\rho) = \dim_{\mathbb{H}} \mathcal{C}_{3d}(\rho) = \frac{1}{2}\dim_{\mathbb{C}} d(O_\rho). \tag{1.21}$$

When o is nontrivial, the defect is called *twisted*, and we impose

$$\Phi(e^{2\pi i}z) = g[o(\Phi(z))]g^{-1} \tag{1.22}$$

where g parametrizes the coset $J/G^\vee$. In particular, when o is of order 2, the twisted defect is labeled by a nilpotent orbit in $\mathfrak{g}$, while the Spaltenstein dual orbit lives in $\mathfrak{g}^\vee = \mathfrak{j}_1$. The boundary condition for the Higgs field in this case is

$$\Phi(z) \sim \left[\frac{\Phi_{-1}}{z} + \frac{\Phi_{-1/2}}{z^{1/2}} + \Phi_0 + \dots\right] dz \tag{1.23}$$

where Φ_{-1} is an element of $d(O_\rho)$, $\Phi_{-1/2}$ is a generic element in $\mathfrak{j}_{-1}$, and Φ_0 is a generic element in $\mathfrak{j}_1$.

When o is of order 3, the defect is again labeled by a nilpotent orbit in $\mathfrak{g}$, and the Spaltenstein dual orbit is in $\mathfrak{g}^\vee = \mathfrak{j}_1$, but now the boundary condition for the Higgs field is

$$\Phi(z) \sim \left[\frac{\Phi_{-1}}{z} + \frac{\Phi_{-2/3}}{z^{2/3}} + \frac{\Phi_{-1/3}}{z^{1/3}} + \Phi_0 + \dots\right] dz, \tag{1.24}$$

where Φ_{-1} is an element of $d(O_\rho)$, $\Phi_{-1/3}$ is a generic element in $\mathfrak{j}_\omega$, and $\Phi_{-2/3}$ is a generic element in $\mathfrak{j}_{\omega^2}$.

Altogether, the introduction of a twisted defect of type ρ increases the dimension of the Coulomb branch by

$$\dim_{\mathbb{C}} \mathcal{C}_{4d}(\rho) = \frac{1}{2}\dim_{\mathbb{C}} d(O_\rho) + \frac{1}{2}\dim J/G^\vee. \tag{1.25}$$

The second term in (1.25) may be a half-integer when o is of order 2, but this is not a problem because the twisted punctures always come in pairs.

So, for a theory on a surface of genus g, the total Coulomb branch dimension is

$$\dim_{\mathbb{C}} \mathcal{C}_{4d} = \sum_i \dim_{\mathbb{C}} \mathcal{C}_{4d}(\rho_i) + (g-1) \dim G. \tag{1.26}$$

1.3.4.3 Coulomb branch and Sommers-Achar group

We have computed in (1.26) the local contribution of a defect of type ρ to the complex dimension of the $4d$ Coulomb branch. The $4d$ $\mathcal{N}=2$ theory is superconformal, so its Coulomb branch actually has a finer structure. The scaling symmetry sends $\Phi(z) \to t\Phi(z)$, which preserves the form of the singularities because the nilpotent orbits are cones. The scaling symmetry also makes $\mathcal{C}$ into a cone, and for all known cases $\mathcal{C}$ is a graded vector space. We can therefore choose generators u_i of the chiral ring of the $4d$ Coulomb branch, which form a basis for $\mathcal{C}$ unambiguously. Letting n_k be the total number of u_i whose scaling dimension is k, we should have

$$\dim_{\mathbb{C}}(\mathcal{C}) = \sum_k n_k. \tag{1.27}$$

The n_k receive a local contribution $n_k(\rho)$ from a defect of type ρ. The local contribution $n_k(\rho)$ and the local contribution $n_v(\rho)$ to the effective number of vector multiplets n_v are related by [15, 16]

$$n_v(\rho) = \sum_k (2k-1) n_k(\rho). \tag{1.28}$$

Let $P^{(d_a)}(\Phi)$ $(a = 1, \ldots, \text{rank } J)$ be the degree-d_a symmetric invariant polynomial of $\mathfrak{j}$, so that $P^{(d_a)}$ generates all the invariant polynomials. Let $\phi^{(d_a)}(z)$ be the invariant polynomials constructed from the Higgs field $\Phi(z)$, i.e. $\phi^{(d_a)}(z) \equiv P^{(d_a)}(\Phi(z))$. Then, $\phi^{(d_a)}(z_1)$ and $\phi^{(d_b)}(z_2)$, for any d_a, d_b, z_1, z_2, Poisson-commute by construction, and they are expected to provide a complete set of integrals of motion. The $\phi^{(d_a)}(z)$ are assigned a scaling dimension d_a.

Introducing punctures of type ρ_i at $z = z_i$, the singularities (2.2.1.2), (1.23), (1.24) give rise to a pole of order at most $p_{d_a}(\rho)$ in the $\phi^{d_a}(z)$ at $z = z_i$, where p_{d_a} may be fractional when ρ is a twisted puncture. The number of degrees of freedom in the meromorphic d_a-differential $\phi^{(d_a)}(z)$ is

$$\sum_i p_{d_a}(\rho_i) + (1 - g)(2d_a - 1). \tag{1.29}$$

Considering the second term above to be the contribution from the bulk of the Riemann surfacef, we see that a puncture of type ρ, inserted at $z = 0$, effectively adds $p_{d_a}(\rho)$ Coulomb branch operators of scaling dimension d_a. More concretely, these operators can be identified with the coefficients $\phi_k^{(d_a)}$ of the poles of order z^{-k} in $\phi^{(d_a)}(z)$, where $0 < k \leq p_{d_a}$. However, these coefficients $\phi_k^{(d_a)}$ are not always the most elementary Coulomb branch operators. Rather, they are polynomials in the true generators of the Coulomb branch operators introduced by ρ. Indeed, the coefficients $\phi_k^{(d_a)}$ usually satisfy rather intricate constraints.

We now explain how to obtain the local Coulomb branch operators, which was determined in [7]. Consider the untwisted boundary condition

(2.2.1.2) (the twisted cases can be treated similarly)

$$\Phi(z) = e\frac{dz}{z} + \Phi_0 dz + \dots \tag{1.30}$$

where e is a fixed element in the Spaltenstein dual orbit $d(O_\rho)$. The allowed continuous gauge transformations are of the form

$$g(z) = g_0 + g_1 z + \dots, \quad g_0 \in \mathfrak{g}_e \equiv \{x|[e,x]=0\}, g_{i>0} \in \mathfrak{g}. \tag{1.31}$$

To find the local Coulomb branch operators, one first finds all functions of $\Phi(z)$ (where $\Phi(z)$ takes the form (1.30)) invariant under the connected part of the gauge group, generated by (1.31). One then further imposes invariance under a certain discrete group $\mathcal{C}(O_\rho)$, to be described below. The resulting invariant functions are the local Coulomb branch operators.

The discrete group $\mathcal{C}(O_\rho)$ is known as the *Sommers-Achar group.* Before we identify $\mathcal{C}(O_\rho)$, let us first return to an important property of the Spaltenstein map

$$d : \{\text{nilpotent orbits in } \mathfrak{g}\} \to \{\text{nilpotent orbits in } \mathfrak{g}^\vee\}.$$

When $\mathfrak{g}$ is of type A, d takes a nilpotent orbit defined by a partition p to the nilpotent orbit defined by p^t, the transpose partition to p. We see then that d is an involution on nilpotent orbits in $\mathfrak{g} = \mathfrak{su}(N)$. However, in general, for $\mathfrak{g} \neq A_{N-1}$, d is not an involution, but satisfies $d^3 = d$. It is possible then, that two distinct nilpotent orbits $O_\rho, O_{\rho'}$ in $\mathfrak{g}$ map to the same nilpotent orbit in $\mathfrak{g}^\vee$, $d(O_\rho) = d(O_{\rho'})$.

If a nilpotent orbit O_e is an image of d, i.e. $O_e = d(O_{e'})$, the orbit O_e is called *special*. The Spaltenstein map is an involution $d^2 = 1$ on the set of special orbits. For a special O_e, the set of orbits O such that $d^2(O) = O_e$ is called the *special piece* of O_e; for such non-special O, O_e is the unique smallest special orbit larger than O. We note that, among nilpotent orbits in a fixed $\mathfrak{g}$, there are usually more special orbits than non-special orbits.

We now describe the Sommers-Achar group $\mathcal{C}(O)$. Pick $e \in O$, and let $F(O)$ be the subgroup of G commuting with e, and $F(O)^\circ$ the connected component of $F(O)$ that contains the identity. Let $A(O) = F(O)/F(O)^\circ$ denote the group of components of $F(O)$. $A(O)$ is trivial when $\mathfrak{g} = A_{N-1}$, is $(\mathbb{Z}_2)^k$ for some k when $\mathfrak{g} = B_N, C_N$, or D_N , and is S_k for some k when $\mathfrak{g}$ is exceptional. In particular, $A(O)$ is a Coxeter group.

Sommers and Achar constructed a map $f : O \mapsto (d(O), C(O))$ assigning to a nilpotent orbit in $\mathfrak{g}$ a pair, consisting of the Spaltenstein dual orbit $d(O)$ in $\mathfrak{g}^\vee$ and a conjugacy class $C(O)$ in $\overline{A}(d(O))$, which is a certain quotient of $A(O)$ introduced by Lusztig. These maps have the properties

- $f(O_1) = f(O_2)$ if and only if $O_1 = O_2$,
- $f(O) = (d(O), 1)$ if and only if O is special.

Just as $A(d(O))$, $\overline{A}(d(O))$ is also a Coxeter group. Using this fact, Sommers and Achar assigned a subset of simple reflections $\overline{r}_1, \ldots, \overline{r}_\ell \in \overline{A}(d(O))$ whose product lies in $C(O)$. If we let $r_1, \ldots, r_\ell$ be the corresponding simple reflections in $A(d(O))$, then $\mathcal{C}(O)$ is the subgroup of $A(d(O))$ generated by them.

For nilpotent orbits in classical $\mathfrak{g}$, there is a combinatorial procedure to determine $C(O)$ from the partition labeling O, which is described in [7]. For $\mathfrak{g}$ exceptional, the $C(O)$ have also been determined, and can be found in [7]. The $\mathcal{C}(O)$ for nilpotent orbits in $\mathfrak{e}_6$ and $\mathfrak{f}_4$ will be discussed in chapters 4 and 5.

We note that, so far, $\mathcal{C}(O)$ is purely a Coulomb branch concept. In chapter 4, we will discover an explicit action on $\mathcal{C}(O)$ on the generators of the Higgs branch chiral ring for certain $4d$ $\mathcal{N}=2$ SCFTs.

Chapter 2

The $\mathbb{Z}_2$-Twisted D_N Series

In this chapter, we consider the classification program of twisted theories of type D_N [1]. Preliminary studies of the twisted D_N series were made in [14, 11, 18].

The D_N Dynkin diagram is invariant under a $\mathbb{Z}_2$ outer automorphism group. Correspondingly, the possible twists are classified by giving an element $\gamma \in H^1(C - \{p_i\}, \mathbb{Z}_2)$. The forgetful map, which "forgets" the puncture, p, gives an inclusion

$$H^1(C - \{p_1, \dots \widehat{p}, \dots\}, \mathbb{Z}_2) \hookrightarrow H^1(C - \{p_1, \dots p, \dots\}, \mathbb{Z}_2).$$

If γ descends to a nontrivial element of the quotient, $\frac{H^1(C - \{p_1, \dots p, \dots\}, \mathbb{Z}_2)}{H^1(C - \{p_1, \dots \widehat{p}, \dots\}, \mathbb{Z}_2)}$, then we say that the puncture at p is twisted (otherwise, untwisted). (For the D_4 theory, the $\mathbb{Z}_2$ enhances to a non-abelian S_3 group. The study of the 4D $\mathcal{N} = 2$ SCFTs that arise from such enhancement is work in progress.)

For a given puncture, we explain how to compute all the local properties that contribute to determining the 4D $\mathcal{N} = 2$ SCFT. Among these, are the contribution to the graded Coulomb branch dimensions, the global symmetry

[1] This chapter is based on [17].

group, flavour-current central charges, the conformal-anomaly central charges (a, c), and the "pole structure" and "constraints", which determine the contribution to the Seiberg-Witten curve. From this information, it is possible to determine gauge groups, hypermultiplet matter representations, and other properties.

As an application of our results, we are able to find realizations of $Spin(8)$ gauge theory with matter in the $6(8_v)$, or with matter in the $5(8_v) + 1(8_s)$. These two cases, of vanishing β-function for $Spin(8)$, were the ones that were not captured by the untwisted sector of the D_N series. Similarly, for $Spin(7)$ gauge theory, we find the theory with matter in the $5(7)$, and in the $1(8)+4(7)$; the other combinations with vanishing β-function were already found in the untwisted sector of the D_N series. We also study various realizations of $Sp(N)$ gauge theory, including $Sp(3)$ with matter in the $\frac{11}{2}(6)+\frac{1}{2}(14')$ and in the $3(6) + 1(14')$, where the $14'$ is the 3-index traceless antisymmetric tensor representation.

2.1 The $\mathbb{Z}_2$-twisted D_N Theory

The Coulomb branch geometry of the 4D $\mathcal{N} = 2$ compactification [1, 6] of the 6D $\mathcal{N} = (2, 0)$ theories of type D_N is governed by the Hitchin equations on C with gauge algebra $\mathfrak{so}(2N)$. In particular, the Seiberg-Witten curve Σ is a branched cover of C described by the spectral curve [11],

$$\Sigma : \det(\Phi - \lambda I) = \lambda^{2N} + \sum_{j=1}^{N-1} \phi_{2j}\lambda^{2N-2j} + \tilde{\phi}^2 = 0, \tag{2.1}$$

where Φ is the $\mathfrak{so}(2N)$-valued Higgs field, while the k-differentials ϕ_k ($k = 2, 4, 6, \ldots, 2N-2$) and the Pfaffian N-differential $\tilde{\phi}$ are associated with the Casimirs of the D_N Lie algebra. In the rest of the paper, N will always stand for the rank of D_N.

Introducing punctures on C corresponds to imposing local boundary conditions on the Hitchin fields. We consider untwisted and twisted punctures under the action of the $\mathbb{Z}_2$ outer-automorphism group of the $\mathfrak{so}(2N)$ Lie algebra. Untwisted punctures are labeled by $\mathfrak{sl}(2)$ embeddings in $\mathfrak{so}(2N)$, or, equivalently, by nilpotent orbits in $\mathfrak{so}(2N)$, or by D-partitions[2] of $2N$. Instead of a compact curve, C, consider a semi-infinite cigar, with the puncture at the tip. Reducing along the circle action, we get 5D SYM on a half-space, with a Nahm-type boundary condition of the sort studied by Gaiotto and Witten in [12]. For that reason, we call the D-partition that labels the untwisted puncture the *Nahm pole*.

To describe the local Hitchin boundary condition for an untwisted puncture with Nahm-pole D-partition p, one must recall the Spaltenstein map[3] ,

[2] A D-partition of $2N$ is a partition of $2N$ where each even part appears with even multiplicity. However, "very even" D-partitions — those where all of the parts are even — correspond to not one, but *two*, nilpotent orbits. To distinguish between the two orbits, we assign a red or blue colour to the very-even Young diagrams.

[3] This Spaltenstein map consists in taking the "D-collapse" of the transpose of the D-partition. The D-collapse operation is explained in the untwisted D-series paper [5], as well as in the book [19].

which takes p into a new D-partition $d(p)$, called the *Hitchin pole* of the puncture[4] . Then, the local boundary condition corresponding to p is

$$\Phi(z) = \frac{X}{z} + \mathfrak{so}(2N)$$

where X is an element of the nilpotent orbit[5] associated to $d(p)$, and $\mathfrak{so}(2N)$ above denotes a generic regular function in z valued in $\mathfrak{so}(2N)$.

On the other hand, we have a sector of twisted punctures, with monodromy given by the action of the nontrivial element o of the $\mathbb{Z}_2$ outer automorphism group of D_N. The action of o splits $\mathfrak{so}(2N)$ as

$$\mathfrak{so}(2N) = \mathfrak{so}(2N-1) \oplus o_{-1},$$

where $\mathfrak{so}(2N-1)$ and o_{-1} are the eigenspaces with eigenvalues +1 and -1, respectively. The action of o on the k-differentials is also quite simple:

[4]When p is *non-special* (i.e., when it does not lie in the image of the Spaltenstein map), the information encoded in $d(p)$ must be supplemented by a nontrivial "Sommers-Achar" finite group, C, whose definition can be found in [7]. This additional discrete information encodes the *disconnected* part of the group of gauge transformations which we mod out by in constructing the solutions to the Hitchin system. In particular, it determines the presence (or absence) of the "a-type" constraints, on the gauge-invariant k-differentials. This, in turn affects the local contributions to the graded Coulomb branch dimensions. In the Tables, we denote the Hitchin pole for non-special punctures as a pair $(d(p), C)$.

[5]Using a nilpotent element X in this equation amounts to writing the local boundary condition in the *absence* of mass deformations. The mass-deformed boundary condition involves semisimple (diagonalizable) elements of $\mathfrak{so}(2N)$, whose eigenvalues take values in the Cartan subalgebra of the flavour Lie algebra for the puncture. For the untwisted A series, a recipe for mass-deformed local boundary conditions was given in [20]. A general prescription is given in Sec. 2.4 of [7].

$$
\begin{aligned}
o : \phi_{2k} &\mapsto \phi_{2k} \qquad (k = 1, 2, \ldots, N-1) \\
\tilde{\phi} &\mapsto -\tilde{\phi}
\end{aligned}
\tag{2.2}
$$

Following [7], the twisted punctures of the D_N series are labeled by embeddings of $\mathfrak{sl}(2)$ in $\mathfrak{sp}(N-1)$ (the Langlands dual of $\mathfrak{so}(2N-1)$), or, equivalently, by nilpotent orbits in $\mathfrak{sp}(N-1)$, or by C-partitions[6] of $2N-2$.

To describe the local boundary condition for a twisted puncture, we need to recall the relevant Spaltenstein map[7] . This is a map d that takes a C-partition p of $2N-2$ into a B-partition $d(p)$ of $2N-1$. A B-partition of $2N-1$ labels an $\mathfrak{sl}(2)$ embedding in $\mathfrak{so}(2N-1)$, or equivalently a nilpotent orbit in $\mathfrak{so}(2N-1)$. So, in our nomenclature, the Nahm pole p of a twisted puncture is a C-partition of $2N-2$, and its Hitchin pole[8] is a B-partition $d(p)$ of $2N-1$. The local boundary condition for the Higgs field is then:

$$
\Phi(z) = \frac{X}{z} + \frac{o_{-1}}{z^{1/2}} + \mathfrak{so}(2N-1)
$$

Here X is an element of the $\mathfrak{so}(2N-1)$ nilpotent orbit $d(p)$, while o_{-1} and $\mathfrak{so}(2N-1)$ in the equation above denote generic regular functions in z valued in these linear spaces, respectively.

[6] A C-partition of $2N$ is a partition of $2N$ where each odd part appears with even multiplicity. A B-partition of $2N-1$ is a partition of $2N-1$ where each even part appears with even multiplicity.

[7] This Spaltenstein map consists in adding a part "1" to a C-partition p, taking the transpose, and then doing a B-collapse. The result is always a B-partition. The "B-collapse" is discussed in [14, 7] and in [19].

[8] Again, when the Nahm pole p is non-special, the complete Hitchin pole information is not just $d(p)$, but a pair $(d(p), C)$, with C the Sommers-Achar group [7].

2.1.1 Local Properties of Punctures

2.1.1.1 Global Symmetry Group and Central Charges

The local properties of a puncture that we list in our tables are the pole structure (with constraints), the flavour group (with flavour-current central charges for each simple factor) and the contributions $(\delta n_h, \delta n_v)$ to, respectively, the effective number of hypermultiplets and vector multiplets (or, equivalently, to the conformal-anomaly central charges (a, c)). We will discuss how to compute pole structures and constraints in §4.1.4.3 and §4.1.4.4. Here we want to focus briefly on the other properties.

Given the Nahm partition, for every part l, let its multiplicity be n_l. Then, the flavour group of untwisted and twisted punctures are, respectively,

$$
\begin{aligned}
G_{\text{flavour}} &= \prod_{l\ \text{even}} Sp\left(\tfrac{n_l}{2}\right) \times \prod_{l\ \text{odd}} SO(n_l) \qquad \text{(untwisted)} \\
G_{\text{flavour}} &= \prod_{l\ \text{even}} SO(n_l) \times \prod_{l\ \text{odd}} Sp\left(\tfrac{n_l}{2}\right) \qquad \text{(twisted)}
\end{aligned}
$$

The flavour-current central charges for each simple factor above can be computed using the formulas in Section 3 of [7]. In that reference, one can also see how to compute δn_h and δn_v. Instead of reviewing the general formulas, we find it more useful to discuss an example.

Consider the D_6 twisted puncture with Nahm pole C-partition $[3^2, 1^4]$. The flavour group is $G_{\text{flavour}} = Sp(2) \times SU(2)$. To compute the central charges, we need to know how the adjoint representation of $Sp(5)$ decomposes under the subgroup $SU(2) \times G_{\text{flavour}}$ (the first factor being the embedding of $SU(2)$, corresponding to this partition). The C-partition itself tells us that the fundamental of $Sp(5)$ decomposes as $10 = (1;4,1) + (3;1,2)$. The embedding indices of each factor of $SU(2) \times G_{\text{flavour}} = SU(2) \times Sp(2) \times SU(2)$ in $Sp(5)$ are 8,1 and 3, respectively. With this information, it is not hard to see that the adjoint representation of $Sp(5)$ decomposes as

$$55 = (1;10,1) + (1;1,3) + (3;1,1) + (3;4,2) + (5;1,3). \tag{2.3}$$

Now, to find δn_h and δn_v, we use eq. (3.19) of [7]. In the notation of that paper, we have $\mathfrak{j} = \mathfrak{so}(12)$, $\mathfrak{g} = \mathfrak{sp}(5)$, and, in their respective usual root bases, the Weyl vectors $\rho_{Spin(12)} = (5,4,3,2,1,0,0,0,0,0,0,0)$, $\rho_{Sp(5)} = (5,4,3,2,1,0,0,0,0,0)$. We also find $h/2 = (1,1,0,0,0,-1,-1,0,0,0)$ using, say, the formulas of Section 5.3 of [19]. Since the adjoint representation of $Sp(5)$ decomposes under the Nahm-pole $SU(2)$ as $55 = 13(1) + 9(3) + 3(5)$, we have dim $\mathfrak{g}_0 = 13 + 9 + 3 = 25$ and dim $\mathfrak{g}_{1/2} = 0$. Thus, eq. (3.19) of [7] yields $\delta n_h = 368$ and $\delta n_v = \frac{717}{2}$.

Finally, from (2.3) above as well as eq. (3.20) of [7], we compute the flavour-current central charges for each simple factor of G_{flavour},

$$\begin{aligned} k_{Sp(2)} &= 1 \times l_{Sp(2)}(10) + 2 \times l_{Sp(2)}(4) = 8 \\ k_{SU(2)} &= 1 \times l_{SU(2)}(3) + 1 \times l_{SU(2)}(3) + 4 \times l_{SU(2)}(2) = 12 \end{aligned}$$

where $l_{\mathfrak{h}}(R)$ denotes the index of the representation R of $\mathfrak{h}$.

2.1.1.2 Pole Structures

The pole structure of a puncture is the set of leading pole orders $\{p_2, p_4, p_6, \ldots, p_{2N-2}; \tilde{p}\}$ in the expansion of the k-differentials $\phi_k(z)$ $(k = 2, 4, 6, \ldots, 2N-2)$ and the Pfaffian $\tilde{\phi}(z)$ around the position of the puncture on C. Knowing the pole structures of the various punctures allows us to write down the Seiberg-Witten curve (2.1) of a theory. The pole orders are all integers, except for $\tilde{p}$ in a twisted puncture, which must be a half-integer because of the monodromy (2.2).

We already saw in [5] how to read off the pole structure of an untwisted puncture from its Hitchin-pole D-partition p. Basically, regard p as a partition in the untwisted A-series, use the procedure to write down the pole structure [3], and discard the pole orders that would correspond to ϕ_k with odd k. Finally, divide the pole order p_{2N} of ϕ_{2N} by two, to obtain the pole order $\tilde{p}$ of the Pfaffian $\tilde{\phi}$. p_{2N} will always be even, so that $\tilde{p}$ will come out to be an integer, as expected for an untwisted puncture.

To compute the pole structure of a twisted puncture, we use its Hitchin B-partition p. Simply, add 1 to the first (i.e., the largest) part in p, and use the same procedure to compute the pole structure as for an untwisted D-series puncture. Notice that upon adding 1 to the largest part, the B-partition becomes a partition of $2N$, and one can show that the pole order p_{2N} of ϕ_{2N} is always odd, so that the pole order $\tilde{p}$ of the Pfaffian is a half-integer, as it should be.

For instance, consider the D_6 twisted puncture with Nahm-pole C-partition $[4^2, 1^2]$. The Hitchin B-partition is $[5, 2^2, 1^2]$. Following our prescription, we add 1 to the largest part, so we get $[6, 2^2, 1^2]$, and read off the pole structure as in the untwisted A-series. We thus get $\{1, 2, 3, 4, 5, 5, 6, 6, 7, 7, 7\}$ (corresponding to scaling dimensions $2, 3, 4, \ldots, 11, 12$). We discard the pole orders at odd dimensions, and divide the pole order of $\phi_{12} = \tilde{\phi}^2$ by two, and we are left with the correct pole structure, $\{1, 3, 5, 6, 7; \frac{7}{2}\}$.

2.1.1.3 Constraints

In the untwisted D-series, punctures featured "constraints", which are either: 1) relations among leading coefficients in the k-differentials ("c-constraints"); or 2) expressions defining *new* parameters $a^{(k)}$ of scaling dimension k as, roughly, the square roots of a leading coefficient $c^{(2k)}$ of dimension $2k$ ("a-constraints"). Both kinds of constraints affect the counting of graded Coulomb branch dimensions of the theory, as well as the Seiberg-Witten curve. As expected, we find a-constraints and c-constraints also in the twisted sector. The pole structure and the constraints provide a "fingerprint" [21] that allows us to identify the puncture uniquely.

Let us briefly review our nomenclature. For a puncture at $z = 0$, we consider the coefficients $c_l^{(2k)}$ and $\tilde{c}_l$ of the leading singularities in the expansion in z of the $2k$-differentials ($2k = 2, 4, \ldots, 2N - 2$) and the Pfaffian $\tilde{\phi}$, respectively,

$$\phi_{2k}(z) = \frac{c_l^{(2k)}}{z^l} + \dots$$
$$\tilde{\phi}(z) = \frac{\tilde{c}_l}{z^l} + \dots$$

where $\dots$ denotes less singular terms. (The pole orders l above are, of course, the same as those in the pole structure, so we have $l = p_{2k}$ or $l = \tilde{p}$, respectively; in this subsection we just write l to keep expressions simple.)

An *a-constraint* of scaling dimension $2k$ is an expression linear in $c_l^{(2k)}$ that defines (up to sign) a new parameter $a_{l/2}^{(k)}$ of dimension k,

$$c_l^{(2k)} = \left(a_{l/2}^{(k)}\right)^2 + \dots,$$

where $\dots$ stands for a polynomial in leading coefficients (of dimension less than $2k$) as well as new coefficients $a_{l'}^{(j')}$ (which would themselves be defined by other a-constraints). This polynomial is homogeneous in dimension and pole order, i.e., in every term in the polynomial, the sum of the scaling dimensions of every factor must be $2k$, and the sum of pole orders must be l. The existence of an a-constraint implies that, in counting graded Coulomb branch dimensions, a parameter of scaling dimension $2k$ is to be replaced by one of dimension k.

A *c-constraint* of dimension $2k$ is an expression linear in $c_l^{(2k)}$, which relates it to other leading coefficients, and perhaps also to new parameters a_l^j defined by a-constraints,

$$c_l^{(2k)} = \dots$$

where, again, the ellipsis denotes a homogeneous polynomial in leading coefficients and new parameters. For even N, if the puncture is very-even, a "very-even" c-constraint, which is linear in the leading coefficients of both ϕ_N and the Pfaffian, may appear,

$$c_l^{(N)} \pm 2\tilde{c}_l = \dots$$

Unlike an a-constraint, a c-constraint does not define any new parameters; it simply tells us that $c_l^{(2k)}$ (or, say, $c^{(N)}$ for a very-even c-constraint) is not independent, and so it should not be considered when counting Coulomb branch dimensions.

Finally, at every scaling dimension $2k$, we find at most one constraint, which can be either an a-constraint or a c-constraint.

Below, we present algorithms to compute the scaling dimensions $2k$ at which a-constraints and c-constraints appear for a given puncture. This information is enough to compute the local contribution to the graded Coulomb branch dimensions.

Untwisted punctures Let p be the Nahm pole D-partition of an untwisted puncture. Also, let $q = \{q_1, q_2, \dots\}$ be the transpose partition, and $s = \{s_1, s_2, \dots\}$ the sequence of partial sums of q ($s_i = q_1 + q_2 + \cdots + q_i$). Below, s_1 denotes the first element of s, and p_{last}, the last element of the D-partition p. (By the conditions that define a D-partition, s_1 is always an even number.)

Then, an a-constraint of dimension $2k$ exists if the following conditions are met:

1. $2k$ belongs to s, say, $s_j = 2k$.

2. j is even.

3. If s_j is a multiple of s_1, say, $s_j = rs_1$, one has $r \geq 2\left\lfloor \frac{p_{\text{last}}}{2} \right\rfloor + 1$.

4. s_j is not the last element of s.

On the other hand, a c-constraint of scaling dimension $2k$ exists if the following conditions are met:

1. $2k$ belongs to s, say, $s_j = 2k$.

2. If j is even, one has that: a) s_j is a multiple of s_1, say, $s_j = rs_1$; b) $\left\lfloor \frac{p_{\text{last}}}{2} \right\rfloor + 1 \leq r \leq 2\left\lfloor \frac{p_{\text{last}}}{2} \right\rfloor$; c) s_j is not the last element of s.

3. If j is odd, one has that: a) s_j is neither the first nor the last element of s; b) both s_{j-1} and s_{j+1} are even; c) $s_j = \frac{s_{j-1}+s_{j+1}}{2}$; d) if s_j is divisible by s_1, say, $s_j = rs_1$, one has $r \geq \left\lfloor \frac{p_{\text{last}}}{2} \right\rfloor + 1$.

Finally, if p is very even, an additional, "very-even", c-constraint exists at $2k = N$ if N belongs to s and $N = \frac{s_1 p_{\text{last}}}{2}$. As already mentioned, this very-even c-constraint is linear in *both* leading coefficients $c_l^{(N)}$ and $\tilde{c}_l$. (The pole orders of ϕ_N and $\tilde{\phi}$ are the same if the conditions just mentioned hold, so such a linear constraint is possible.) A generic very-even puncture may or may not have this very-even c-constraint. In particular, a very-even puncture could have a c-constraint of dimension N which is not very even (in the sense that it is not linear in both $c_l^{(N)}$ and $\tilde{c}_l$).

Twisted punctures Suppose we have a twisted puncture labeled by the Nahm-pole C-partition p. Let q be the transpose partition, and s the sequence of partial sums of q. It is convenient to define another sequence s', obtained by adding 2 to every element in s. (As a check, the last element of s' must be $2N$.) Let $s' = \{s'_1, s'_2, \ldots\}$.

Then, an a-constraint of scaling dimension $2k$ exists if the following conditions are met:

1. $2k$ belongs to s', say, $s'_j = 2k$.
2. j is odd.
3. s'_j is not the last element of s'.

On the other hand, a c-constraint of scaling dimension $2k$ exists if the following conditions are met:

1. $2k$ belongs to s', say, $s'_j = 2k$.
2. j is even.
3. s'_j is not the last element of s'

4. Both s'_{j-1} and s'_{j+1} are even, and $s'_j = \frac{s'_{j-1}+s'_{j+1}}{2}$.

Constraint structure The constraints of twisted punctures are very simple. c-constraints are always "cross-terms" between a-constraints, or between an a-constraint and the Pfaffian (where $\phi_{2N} = \tilde{\phi}^2$ is seen as another "a-constraint"). As a schematic example, $c^{(k+m)}$ below is a cross-term for the "squares" at dimensions $2k$ and $2m$:

$$c^{(2k)} = \left(a^{(k)}\right)^2, \qquad c^{(k+m)} = 2a^{(k)}a^{(m)}, \qquad c^{(2m)} = \left(a^{(k)}\right)^2 \tag{2.4}$$

(In an actual example, $k+m$ would always turn out to be even). a-constraints also generically contain cross-terms, in addition to the quadratic term in the new parameter. Many examples can be found in the Tables.

The constraints of untwisted punctures are slightly more complicated, but they resemble very much the constraints of twisted punctures in the A_{2N-1} series [4], so we refrain from repeating the details. To be brief, there is a sequence of c-constraints (illustrated below in an example), all related to each other, and which are associated to the first terms in the set of partial sums s. c-constraints outside this sequence are simply cross-terms between a-constraints and/or the Pfaffian, as in (2.4). For a very-even puncture, the very-even c-constraint, if it exists, becomes part of the sequence just mentioned. As usual, a-constraints can include cross-terms in addition to the quadratic term that defines the new parameter.

Let us discuss the constraints of a D_6 very-even puncture, $[6^2]$. In this case, $q = [2^6]$ and $s = [2, 4, 6, 8, 10, 12]$. Also, $p_{\text{last}} = 6$ and $s_1 = 2$. So, there are c-constraints at $2k = rs_1$ with $4 \le r \le 6$, that is, at $2k = 8, 10$. There is also a very-even c-constraint (at $2k = 6$). All c-constraints in this case constitute the sequence mentioned in the previous paragraph. There are no a-constraints. We can also compute the pole structure to be $\{1, 2, 3, 4, 5; 6\}$. Let us see the structure of these c-constraints by writing:

$$
\begin{aligned}
c_0^{(0)} &= 1, & c_4^{(8)} &= \tfrac{1}{4}\left(t_2^{(4)}\right)^2 + \tfrac{1}{2}t_3^{(6)}t_1^{(2)}, \\
c_1^{(2)} &\equiv t_1^{(2)}, & c_5^{(10)} &= t_3^{(6)}t_2^{(4)}, \\
c_2^{(4)} &\equiv \tfrac{1}{4}\left(t_1^{(2)}\right)^2 + t_2^{(4)}, & c_6^{(12)} &\equiv \left(\tilde{c}_3^{(6)}\right)^2 = \tfrac{1}{4}\left(t_3^{(6)}\right)^2. \\
c_3^{(6)} &\equiv \tfrac{1}{2}t_1^{(2)}t_2^{(4)} + t_3^{(6)}, & &
\end{aligned}
$$

The first line above is trivial, but it facilitates the construction of the other expressions. Disregarding the very-even c-constraint at $2k = 6$ for a moment, the expressions at $2k = 2, 4, 6$ provide definitions for the quantities $t_1^{(2)}$, $t_2^{(4)}$ and $t_3^{(6)}$. Besides, each term in the equations above can be interpreted as either a cross-term or a square of 1, $t_1^{(2)}$, $t_2^{(4)}$ and $t_3^{(6)}$. For example, the term $t_1^{(2)}$ is not a square, so it has to be a cross-term (for 1 and $\frac{1}{4}\left(t_1^{(2)}\right)^2$), which is why we include the term $\frac{1}{4}\left(t_1^{(2)}\right)^2$ in $c_2^{(4)}$. Since $c_2^{(4)}$ cannot be equal to $\frac{1}{4}\left(t_1^{(2)}\right)^2$ (since that would be a c-constraint at $k = 4$), we introduce the new quantity $t_2^{(4)}$. Notice that we have also written $c_6^{(12)}$ as a square of $t_3^{(6)}$. Since ϕ_{12} is the square of the Pfaffian, we must have $t_3^{(6)} = \pm 2\tilde{c}_3$, and we recover the very-even constraint at $2k = 6$. Solving for $t_1^{(2)}$ and $t_2^{(4)}$, we find our actual c-constraints:

$$
\begin{aligned}
c_3^{(6)} \mp 2\tilde{c}_3 &= \tfrac{1}{2} c_1^{(2)} \left(c_2^{(4)} - \tfrac{1}{4} \left(c_1^{(2)} \right)^2 \right), \\
c_4^{(8)} &= \tfrac{1}{4} \left(c_2^{(4)} - \tfrac{1}{4} \left(c_1^{(2)} \right)^2 \right)^2 \pm \tilde{c}_4 c_1^{(2)}, \\
c_5^{(10)} &= \pm \tilde{c}_3 \left(c_2^{(4)} - \tfrac{1}{4} \left(c_1^{(2)} \right)^2 \right).
\end{aligned}
$$

Flipping the sign of $\tilde{c}_3$ switches between the constraints for the red and the blue versions of this puncture.

2.1.2 Collisions

When two punctures collide, a new puncture appears. This process can be described at the level of the Higgs field, using the local boundary conditions discussed in §5.1, or at the level of the k-differentials, using the pole structures and the constraints of §4.1.4.3 and §4.1.4.4. Of course, both mechanisms are quite related, because the k-differentials are, essentially, the trace invariants of the Higgs field. These procedures are analogous to those for the twisted A_{2N-1} series described in [4].

Let us start by discussing collisions using the Higgs field. Consider two untwisted punctures at $z = 0$ and $z = x$ on a plane. The respective local boundary conditions are:

$$
\begin{aligned}
\Phi(z) &= \frac{X_1}{z} + \mathfrak{so}(2N), \\
\Phi(z) &= \frac{X_2}{z - x} + \mathfrak{so}(2N),
\end{aligned}
$$

where X_1 and X_2 are representatives of the respective Hitchin-pole orbits for the punctures. Then, in the collision limit, $x \to 0$, a new untwisted puncture appears at $z = 0$,

$$\Phi(z) = \frac{X_1 + X_2}{z} + \mathfrak{so}(2N).$$

Here, $X_1 + X_2$ is an element of the mass-deformed Hitchin-pole orbit for the new puncture, and the mass deformations correspond to the VEVs of the decoupled gauge group. Taking the mass deformations to vanish, $X_1 + X_2$ becomes the Hitchin-pole nilpotent orbit for the new puncture. The fact that the new residue is $X_1 + X_2$ also follows from the residue theorem applied to the three-punctured sphere that appears in the degeneration limit; another derivation ensues from an explicit ansatz for the Higgs field on the plane with two punctures [4], where the limit $x \to 0$ can be taken.

Now consider an untwisted and a twisted puncture, at $z = 0$ and $z = x$, respectively. The respective local boundary conditions are:

$$\begin{aligned}\Phi(z) &= \frac{X}{z} + \mathfrak{so}(2N),\\ \Phi(z) &= \frac{Y}{z-x} + \frac{o_{-1}}{(z-x)^{1/2}} + \mathfrak{so}(2N-1).\end{aligned}$$

Then, the local boundary condition for the new twisted puncture is:

$$\Phi(z) = \frac{X|_{\mathfrak{so}(2N-1)} + Y}{z} + \frac{o_{-1}}{z^{1/2}} + \mathfrak{so}(2N-1),$$

where $X|_{\mathfrak{so}(2N-1)}$ is the restriction of $X \in \mathfrak{so}(2N)$ to the subalgebra $\mathfrak{so}(2N-1)$.

Finally, consider two twisted punctures at $z = 0$ and $z = x$,

$$\Phi(z) = \frac{Y_1}{z} + \frac{o_{-1}}{z^{1/2}} + \mathfrak{so}(2N-1),$$
$$\Phi(z) = \frac{Y_2}{z-x} + \frac{o_{-1}}{(z-x)^{1/2}} + \mathfrak{so}(2N-1).$$

Then, the local boundary condition for the new untwisted puncture is:

$$\Phi(z) = \frac{Y_1 + Y_2 + o_{-1}}{z} + \mathfrak{so}(2N),$$

where o_{-1} denotes a generic element in such space.

The procedure to collide punctures using k-differentials is explained in [4] for the case of the twisted A_{2N-1} series. The discussion is entirely analogous, so we leave the details to that paper. Here we will just give an example of how to use it.

Consider the collision of three punctures,

$$[2(N-r)-1, 2r+1] \times [2(N-r)-1, 2r+1] \times [2(N-1)],$$

which yields the $[2(N-2r-1), 1^{4r}]$ puncture with an $Sp(r) \times Sp(r)$ gauge group. We will use this result in §2.1.5.4. Let us show how to derive it for the particular case $r = 3$.

The puncture $[2N-7, 7]$ has pole structure $\{1, 2, 3, 4, 5, 6, 6, 6, \ldots, 6; 3\}$, no a-constraints, and three c-constraints at $2k = 8, 10, 12$:

$$
\begin{aligned}
c_4^{(8)} =& \tfrac{1}{4}\left(c_2^{(4)} - \tfrac{1}{4}\left(c_1^{(2)}\right)^2\right)^2 + \tfrac{1}{2}c_1^{(2)}\left(c_3^{(6)} - \tfrac{1}{2}c_1^{(2)}\left(c_2^{(4)} - \tfrac{1}{4}\left(c_1^{(2)}\right)^2\right)\right), \\
c_5^{(10)} =& \tfrac{1}{2}\left(c_2^{(4)} - \tfrac{1}{4}\left(c_1^{(2)}\right)^2\right)\left(c_3^{(6)} - \tfrac{1}{2}c_1^{(2)}\left(c_2^{(4)} - \tfrac{1}{4}\left(c_1^{(2)}\right)^2\right)\right), \\
c_6^{(12)} =& \tfrac{1}{4}\left(c_3^{(6)} - \tfrac{1}{2}c_1^{(2)}\left(c_2^{(4)} - \tfrac{1}{4}\left(c_1^{(2)}\right)^2\right)\right)^2.
\end{aligned}
\tag{2.5}
$$

On the other hand, the puncture $[2(N-1)]$, which is the "minimal" twisted puncture, has pole structure $\{1, 1, 1, \ldots, 1; \frac{1}{2}\}$, and no constraints.

First, consider two $[2N-7, 7]$ punctures on the plane, at positions $z = 0$ and $z = x$, and write down the k-differentials:

$$
\begin{aligned}
\phi_{2k}(z) &= \frac{u_{2k} + v_{2k}z + \ldots}{z^k(z-x)^k} \qquad (2k = 2, 4, 6, 8, 10, 12) \\
\phi_{2k}(z) &= \frac{u_{2k} + \ldots}{z^6(z-x)^6} \qquad (2k = 14, 16, \ldots, 2N-2) \\
\tilde{\phi}(z) &= \frac{\tilde{u} + \ldots}{z^3(z-x)^3}
\end{aligned}
$$

Then, in the $x \to 0$ limit, which corresponds to the collision, we find the pole orders $\{2, 4, 6, 8, 10, 12, 12, 12, \ldots, 12; 6\}$. So, at first sight, we would have gauge-group Casimirs at $2k = 2, 4, 6, 8, 10, 12$. However, the c-constraints (2.5) from the two $[2N-7, 7]$ punctures imply that the leading and subleading coefficients u_{2k} and v_{2k} for $2k = 8, 10, 12$ are dependent on the coefficients u_2, u_4, u_6, and furthermore vanish when we take $u_2, u_4, u_6 \to 0$. Thus, the only independent gauge-group Casimirs are u_2, u_4, u_6, and the massless puncture has pole structure $\{1, 3, 5, 6, 8, 10, 12, 12, \ldots, 12; 6\}$, with no constraints. These properties single out the puncture $[2N-13, 2^6, 1]$, which has $Sp(3)$ flavour symmetry. Thus, the gauge group must be $Sp(3)$.

Colliding the new puncture $[2N-13, 2^6, 1]$ with the minimal twisted puncture is much easier, because none is constrained. So all we need to do is add up pole orders, and identify gauge-group Casimirs. The sum of the pole structures is $\{2, 4, 6, 7, 9, 11, 13, 13, \dots, 13; \frac{13}{2}\}$. Hence, we have again a gauge group with Casimirs $2, 4, 6$, and a new puncture with pole structure $\{1, 3, 5, 7, 9, 11, 13, 13, \dots, 13; \frac{13}{2}\}$, with no constraints. These properties correspond to the puncture $[2N-14, 1^{12}]$, which has flavour symmetry $Sp(6)$. Thus, we are gauging an $Sp(3)$ gauge group out of the $Sp(6)$. Actually, since the two new punctures we find in the subsequent collisions are not maximal, it must be that an $Sp(3) \times Sp(3)$ subgroup (each factor from each of the two cylinders) of $Sp(6)$ is being gauged. (We saw multiple examples of this phenomenon in [5, 4].)

Let us derive the same result by doing the collisions in a different order: first, we collide a $[2N-7, 7]$ puncture (at $z=0$) with the minimal twisted puncture (at $z=x$). We use the k-differentials[9]

[9] In this subsection, we use generic names for Coulomb branch parameters such as u_{2k}, v_{2k}, r_k, etc. They are understood to be different variables in different collisions.

$$\phi_{2k}(z) = \frac{u_{2k} + \dots}{z^k(z-x)} \qquad (2k = 2,4,6,8,10,12),$$
$$\phi_{2k}(z) = \frac{u_{2k} + \dots}{z^6(z-x)} \qquad (2k = 14,16,\dots,2N-2),$$
$$\tilde{\phi}(z) = \frac{\tilde{u} + \dots}{z^3(z-x)^{1/2}}$$

This time, solving the c-constraints is less simple. The constraints are not solvable unless one introduces parameters r_2, r_4, r_6 of dimension 2,4,6 such that:

$$u_2 = r_2 x^{1/2}, \qquad u_4 = -\tfrac{(r_2)^2}{4} + r_4 x^{1/2}, \qquad u_6 = -r_2 r_4 + r_6 x^{1/2}$$

(See Sec. 4.1.3 of [4] for a similar example in more detail.) Then, the constraints imply:

$$u_8 = -\tfrac{1}{4}((r_4)^2 + 2r_2 r_6), \qquad u_{10} = -\tfrac{1}{2} r_6 r_4, \qquad u_{12} = -\tfrac{1}{4}(r_6)^2$$

and in the limit $x \to 0$, we get a pole structure $\{1,3,4,5,6,7,7,7,\dots,\frac{7}{2}\}$, with constraints

$$\begin{aligned} c_3^{(4)} &= -\tfrac{(r_2)^2}{4}, & c_6^{(10)} &= -\tfrac{r_4 r_6}{2} \\ c_4^{(6)} &= -r_2 r_4, & c_7^{(12)} &= -\tfrac{(r_6)^2}{4} \\ c_5^{(8)} &= -(r_4)^2 - \tfrac{r_2 r_6}{2} \end{aligned}$$

that is, we have a-constraints at $2k = 4,8,12$ and c-constraints at $2k = 6,10$. These properties uniquely identify the twisted puncture $[2N-8, 6]$. Notice that there are no gauge-group Casimirs, so our interpretation is that the cylinder is "empty". This is an example of an "atypical degeneration", as we will recall in §2.1.5.4.

Let us now collide the new puncture $[2N-8,6]$ (at $z=0$) with the remaining untwisted puncture $[2N-7,7]$ (at $z=x$). We have the k-differentials

$$\begin{aligned}
\phi_2(z) &= \frac{u_2+\dots}{z(z-x)} \\
\phi_{2k}(z) &= \frac{xu_{2k}+v_{2k}z+\dots}{z^{k+1}(z-x)^k} && (2k=4,6,8,10,12) \\
\phi_{2k}(z) &= \frac{u_{2k}+\dots}{z^7(z-x)^6} && (2k=14,16,18,\dots,2N-2) \\
\tilde{\phi}(z) &= \frac{\tilde{u}+\dots}{z^{7/2}(z-x)^3}
\end{aligned}$$

Taking the collision limit $x\to 0$, we get the pole orders $\{2,4,6,8,10,12,13,13,\dots,13;\frac{13}{2}\}$. So, in principle, the gauge-group VEVs are $u_2,v_4,v_6,v_8,v_{10},v_{12}$. However, v_8,v_{10},v_{12} are polynomials in u_2,v_4,v_6 and in three new parameters r_2,r_4,r_6, of respective dimensions 2,4,6, which arise from combining the a-/c-constraints of $[2N-8,6]$ with the c-constraints of $[2N-7,7]$. So the actual gauge-group VEVs are u_2,v_4,v_6,r_2,r_4,r_6. These VEV dimensions are consistent with an $Sp(3)\times Sp(3)$ gauge group, as before, except that now both $Sp(3)$ factors are supported on a single cylinder. Setting to zero the gauge-group VEVs, we get the massless pole orders $\{1,3,5,7,9,11,13,13,\dots,13;\frac{13}{2}\}$, with no constraints, which, as before, correspond to the $[2N-14,1^{12}]$ puncture.

2.1.3 Gauge Couplings

Consider an $\mathcal{N} = 2$ supersymmetric gauge theory, with simple gauge group, G, and matter content chosen so that the β-function vanishes. This gives rise to a family of SCFTs, parametrized by

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$$

A rich class of (though not all) such theories can be realized as compactifications of the $(2,0)$ theory on a sphere with four *untwisted* punctures. If the four puncture are distinct, then the S-duality group, $\Gamma(2) \subset PSL(2,\mathbb{Z})$, is generated by

$$T^2 : \tau \mapsto \tau + 2, \quad ST^2S : \tau \mapsto \frac{\tau}{1 - 2\tau}$$

The fundamental domain for $\Gamma(2)$ is isomorphic to $\mathcal{M}_{0,4} \simeq \mathbb{C}\mathrm{P}^1$. In particular, the coordinate on the complex plane, f, is given by[10]

[10]Our θ-function conventions are

$$\begin{aligned}\theta_2(0,\tau) &= \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2}\\ \theta_3(0,\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2/2}\\ \theta_4(0,\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}\end{aligned}$$

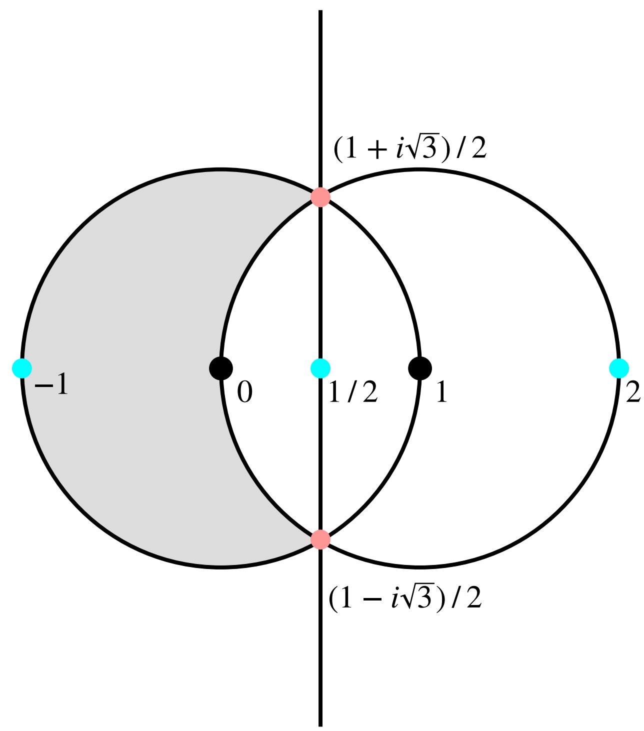
where $q = e^{2\pi i\tau}$.

$$
\begin{aligned}
f(\tau) &= -\frac{\theta_2^4(0,\tau)}{\theta_4^4(0,\tau)} \\
&= -\left(16q^{1/2} + 128q + 704q^{3/2} + \dots\right)
\end{aligned}
$$

Since $\Gamma(2)$ is index-6 in $PSL(2,\mathbb{Z})$, the generators of the latter group act on $\mathcal{M}_{0,4}$ as

$$
T:\, f \mapsto \frac{f}{f-1}, \quad S:\, f \mapsto \frac{1}{f}
$$

These generate an S_3 action on $\mathcal{M}_{0,4}$, as depicted in the figure



The points, $\{0,1,\infty\}$, of the compactification divisor, are fixed points with stabilizer group $\mathbb{Z}_2$. The points $\{-1,1/2,2\}$ are also fixed points with stabilizer group $\mathbb{Z}_2$. Finally, the points $(1 \pm i\sqrt{3})/2$ are fixed points with stabilizer group $\mathbb{Z}_3$. The j-invariant (invariant under the action of $PSL(2,\mathbb{Z})$) is

$$\begin{aligned} j(\tau) &= 256 \frac{\left(1-f+f^2\right)^3}{f^2\left(1-f\right)^2} \\ &= \frac{1}{q} + 744 + 196884q + \dots \end{aligned}$$

Of course, while the j-invariant is invariant under the full $PSL(2,\mathbb{Z})$, the physics generically is not

If two of the punctures are identical, then $\tau \mapsto -1/\tau$ leaves the physics unchanged. The S-duality group is $\Gamma_0(2) \subset PSL(2,\mathbb{Z})$, generated by $T^2 : \tau \mapsto \tau + 2$ and $S : \tau \mapsto -1/\tau$, whose fundamental domain is the $\mathbb{Z}_2$ quotient of $\mathcal{M}_{0,4}$ by $f \mapsto 1/f$. The physics at $f = 0$ and at $f = \infty$ are both that of a weakly-coupled G gauge theory. The other boundary point, $f = 1$, and the interior point, $f = -1$ are fixed-points of the $\mathbb{Z}_2$ action.

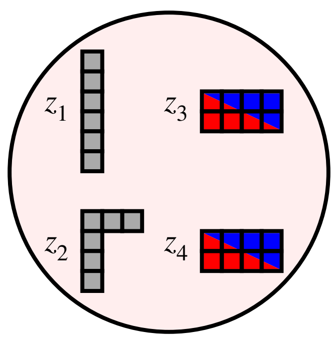
If *three* of the punctures (or all four) are identical, then the S-duality group is the full $PSL(2,\mathbb{Z})$, the physics at all three boundary points is that of a weakly-coupled G-gauge theory and the fundamental domain is just the shaded region in the figure.

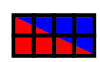
How this picture gets modified, in the presence of twisted punctures, will be one of our main themes in this paper.

2.1.4 Very-even Punctures

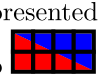
In the A_{2N-1} series, the outer automorphism twists acted trivially on the *set* of nilpotent orbits. So the identities of the untwisted punctures were unaffected by the introduction of twisted punctures. By contrast, in the D_N series (for N even), the outer automorphism twists act by exchanging the "red" and "blue" very-even punctures. Dragging an untwisted very-even puncture around a twisted puncture turns it from red to blue, or vice-versa.

To illustrate the phenomenon, let us look at an example in the twisted D_4 theory.



Here, it is useful to recall [5] that the very-even puncture[11]  has only one constraint, which is a very-even c-constraint,

$$c_3^{(4)} \pm 2\tilde{c}_3 = 0,$$

[11] As in[3, 5, 4], a Nahm-pole partition p is represented by a Young diagram such that the column heights are equal to the parts of p. (So  is the puncture with Nahm pole D-partition $[2^4]$.) In this paper we do not use Young diagrams to represent Hitchin-pole partitions.

where the top (bottom) sign corresponds to a red (blue) puncture.

The Higgs field (with Coulomb branch parameters $u_2, u_4, \tilde{u}, u_6$) yields the differentials

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2 z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{z_{24} z_{34} (dz)^4}{(z-z_1)(z-z_2)^3(z-z_3)^3(z-z_4)^3} \\
&\quad \times \left[u_4 (z-z_3)(z-z_4) z_{12} z_{23} \right. \\
&\quad \left. + 2\tilde{u}(z_2 - z)\left((z-z_3)(z_{13}z_{23}z_{14}z_{24})^{1/2} + (z-z_4) z_{13} z_{23}\right)\right] \\
\phi_6(z) &= \frac{u_6 z_{12} z_{23} z_{24} z_{34}^3 (dz)^6}{(z-z_1)(z-z_2)^3(z-z_3)^4(z-z_4)^4} \\
\tilde{\phi}(z) &= \frac{\tilde{u} z_{24} z_{34}^2 (z_{13}z_{23})^{1/2} (dz)^4}{(z-z_1)^{1/2}(z-z_2)^{3/2}(z-z_3)^3(z-z_4)^3}
\end{aligned}
$$

The powers of $z_{ij} \equiv z_i - z_j$ have been introduced to make the above expressions Möbius-invariant[12] , and hence well-defined on the moduli space. However, the (unavoidable) square-roots mean that moduli space is, itself, a double-cover (in fact, a 4-fold cover, but the SW geometry factors through a $\mathbb{Z}_2$ quotient) of the moduli space of the 4-punctured sphere.

Whether a very-even puncture is red or blue depends on the *relative sign* of the residues of the cubic poles of $\phi_4(z)$ and $\tilde{\phi}(z)$ at the location of

[12]To minimize the number of ensuing branch cuts, we have chosen not to preserve the obvious $z_3 \leftrightarrow z_4$ symmetry. We can restore it by redefining the Coulomb branch parameter

$$\hat{\tilde{u}} = \tilde{u}\left(\frac{z_{13}z_{24}}{z_{12}z_{34}}\right)^{1/2}$$

The resulting theory lives naturally on the 4-fold branched cover of $\mathcal{M}_{0,4}$.

the puncture. But the square-roots are such that if we drag the very-even puncture (say, the one located at z_3) around one of the twisted punctures (say, the one located at z_1), the relative sign changes, indicating that the puncture has changed from red to blue, or vice versa.

Since the formulae are a little bit formidable-looking in their fully Möbius-invariant form, it helps to fix the Möbius invariance by setting

$$(z_1, z_2, z_3, z_4) \to (0, \infty, w^2, 1)$$

The expressions for $\phi_4(z), \tilde{\phi}(z)$ (which are all we need for the present discussion) simplify to

$$\begin{aligned}
\phi_4(z) &= \frac{(w^2-1)\left[u_4(z-w^2)(z-1) + 2\tilde{u}\left(w(z-w^2) + w^2(z-1)\right)\right](dz)^4}{z(z-w^2)^3(z-1)^3} \\
\tilde{\phi}(z) &= \frac{\tilde{u}w(w^2-1)^2(dz)^4}{z^{1/2}(z-w^2)^3(z-1)^3}
\end{aligned}$$

Dragging the point $z_3 = w^2$ around the origin changes the sign of w in the above expressions. This changes the *relative* sign of the residues of ϕ_4 and $\tilde{\phi}$ at $z = w^2$, whilst preserving the relative sign of the residues at $z = 1$.

Of course, the Seiberg-Witten geometry is invariant under the operation of simultaneously flipping all of the colours of all of the very-even punctures. This gives a $\mathbb{Z}_2$ which acts freely on the gauge theory moduli space. We will often find it useful to work on the quotient, fixing the colour of *one* of the very-even punctures.

Having seen the phenomenon is global example, let us recover the same result, working locally on the plane, with the Higgs field itself (rather than the gauge-invariant k-differentials). Consider a very-even Higgs-field residue $B \in \mathfrak{so}(2N)$, which belongs to a, say, red nilpotent orbit. We can write $B = B|_{\mathfrak{so}(2N-1)} + B|_{o_{-1}}$, corresponding to the splitting $\mathfrak{so}(2N) = \mathfrak{so}(2N-1) \oplus o_{-1}$. Then, one can check that the map $B|_{o_{-1}} \mapsto -B|_{o_{-1}}$ puts the residue B in the other (blue) nilpotent orbit. This map defines an isomorphism between the elements of the red and the blue nilpotent orbits.

Now suppose that the twisted puncture (with residue $A \in \mathfrak{so}(2N-1)$ is at $z = 0$ and the very-even puncture (with residue $B \in \mathfrak{so}(2N)$) is at $z = x$. Then, the Higgs field for this system is:

$$\Phi(z) = \tfrac{(z-x)A + zB|_{\mathfrak{so}(2N-1)}}{z(z-x)} + \tfrac{x^{1/2}B|_{o_{-1}} + (z-x)D + \dots}{z^{1/2}(z-x)} + \dots$$

where D is a generic element in o_{-1}, and the $\dots$ denote regular terms. The factor of $x^{1/2}$ is necessary to make Φ well-defined as a one-form. Then, x parametrizes the distance between the very-even puncture and the twisted puncture, and if x circles the origin, $x^{1/2} \to -x^{1/2}$, it enforces $B|_{o_{-1}} \to -B|_{o_{-1}}$, so our red puncture becomes blue, or vice versa.

2.1.5 Atypical Degenerations

2.1.5.1 Atypical Punctures

As an application of the formulas in §2.1.1, let us find the series of punctures with contribution $n_2 = 2$. We will call these “atypical punctures”, as they give rise to theories where the number of simple factors in the gauge group is not equal to the dimension of the moduli space of the punctured Riemann surface, C. We have seen this phenomenon already in the twisted A_{2N-1} series [4].

From our rules for a-constraints, it is easy to see that there are *no* untwisted atypical punctures, and that for a twisted puncture to be atypical, its Nahm pole C-partition must consist of exactly *two* parts. Hence, the atypical punctures are

$$\overset{[2(N-r-1),2r]}{\bullet}, \qquad \text{for } r = 1, 2, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor$$

with the addition of

$$\overset{[N-1,N-1]}{\bullet} \quad \text{if } N \text{ is even.}$$

These arise, respectively, as the coincident limit of

a) $\overset{[2(N-1)]}{\bullet}$ and $\overset{[2(N-r)-1,2r+1]}{\circ}$

b) $\overset{[2(N-1)]}{\bullet}$ and $\overset{[N,N]}{\bullet}$ (for N even)

Normally, the OPE of two (regular) punctures, p and p', yields a third (regular) puncture, p'', *coupled* to a gauge theory, (X, H), where

- The gauge group, H, is a subgroup of the global symmetry group of p''.

- In the coincident limit, the gauge coupling of H goes to zero.

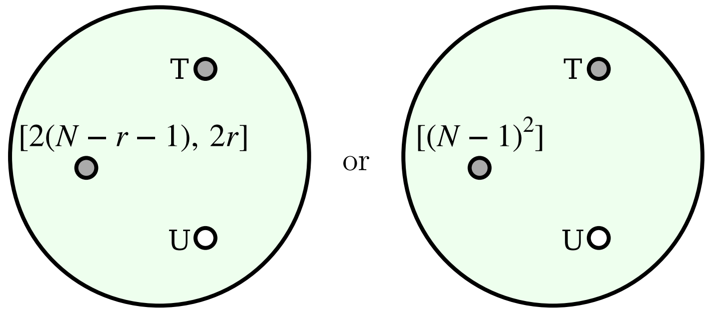
Here, when p'' is atypical, the would-be gauge theory is *empty*: $(X, H) = (\emptyset, \emptyset)$. Instead, the theory with an insertion of p'' has one more simple factor in the gauge group than the "expected" $3g - 3 + n$.

For a surface, C, with n punctures, m of which are atypical, the number of simple factors in the gauge group is $3g-3+n+m$. "Resolving" each atypical puncture by the pair of punctures, above, yields a surface with $n+m$ punctures and the moduli space of the gauge theory is a branched cover of $\mathcal{M}_{g,n+m}$. In contrast to the usual case, where each component of the boundary of the moduli space corresponds to one simple factor in the gauge group becoming weakly-coupled, the boundaries of $\mathcal{M}_{g,n+m}$, where an atypical puncture arises in the OPE, do not typically correspond to any gauge coupling becoming weak (that is, under the branched covering, they are the image of loci in the interior of the gauge theory moduli space).

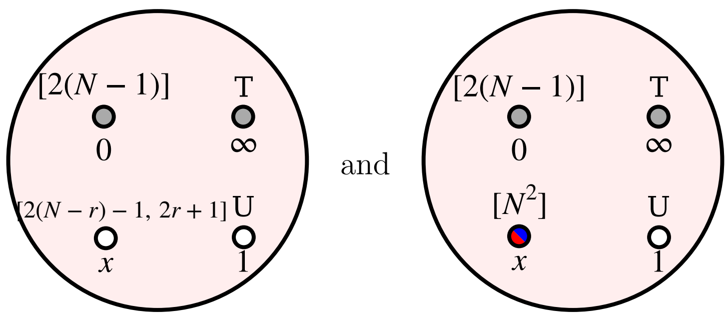
2.1.5.2 Gauge Theory Fixtures

In particular, for $n = 3$, $m = 1$ (or 2), we have a "gauge theory fixture." Resolving the atypical puncture yields a gauge theory moduli space which is branched cover of $\mathcal{M}_{0,4}$. We may well ask, "Where, in the gauge theory moduli space, have we landed, in the coincident limit which yields the atypical puncture?" The answer is that we are at the interior point, "$f(\tau) = -1$", though the mechanics of how this happens varies between the cases.

Let us resolve



to



respectively. We have parametrized $\mathcal{M}_{0,4}$ by x, but the gauge theory moduli space is a branched cover, parametrized by w, with

$$w^2 = x$$

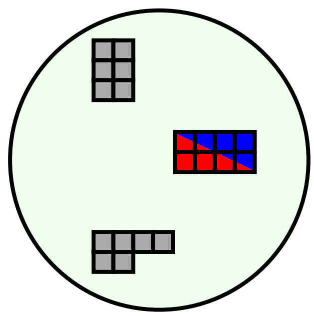
The gauge coupling

$$f(\tau) = \frac{w-1}{w+1} \tag{2.6}$$

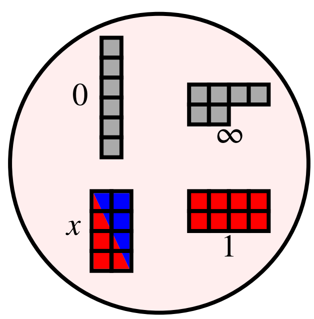
so that $f = 0$ and $f = \infty$ both map to $x = 1$, while $f = 1$ maps to $x = \infty$.

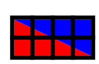
Our gauge-theory fixture is whatever lies over the point $x = 0$. From (2.6), this is the interior point, $f(\tau) = -1$, of the gauge theory moduli space.

As an example, let us consider the D_4 gauge theory fixture



whose resolution is

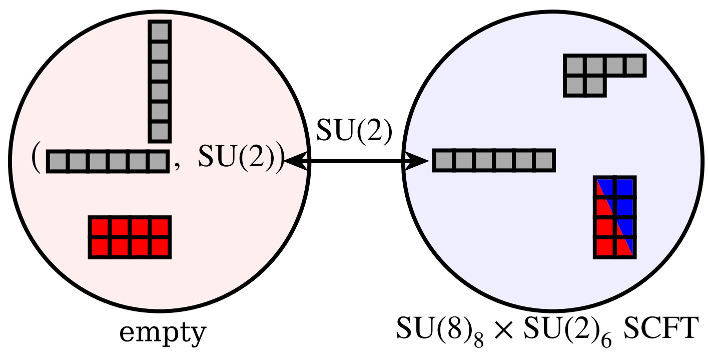


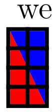
Actually, since we have two very-even punctures, the full moduli space is a 4-sheeted cover of $\mathcal{M}_{0,4}$. The SW geometry is invariant under simultaneously flipping the colours of both punctures, so we can consistently work on the quotient by that $\mathbb{Z}_2$, and take the colour of the  puncture to be red.

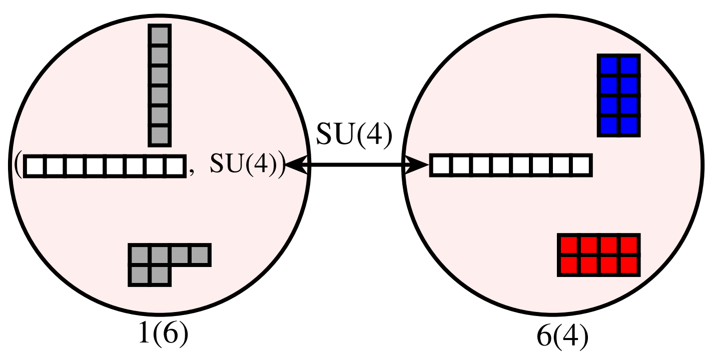
$SU(4)$ gauge theory, with matter in the $1(6) + 4(4)$ was studied in [3]. Near $f(\tau) = 0$, the weakly-coupled description is the Lagrangian field theory.

Near $f(\tau) = 1$, the weakly-coupled description is an $SU(2)$ gauging of the $SU(8)_8 \times SU(2)_6$ SCFT, $R_{0,4}$. Near $f(\tau) = \infty$, the weakly-coupled description is $SU(3)$, with two hypermultiplets in the fundamental, coupled to the $(E_7)_8$ SCFT.

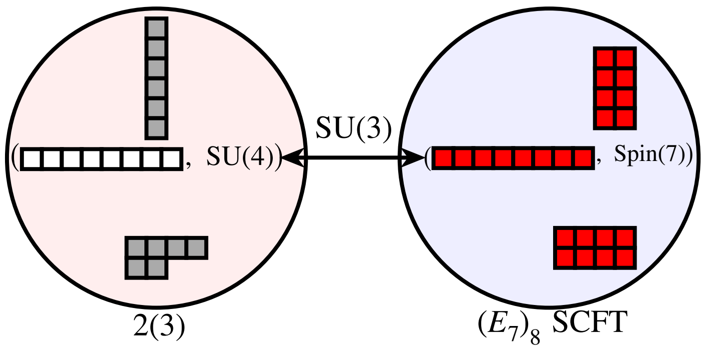
In the present case, the $f \to 1$ theory arises as $x \to \infty$



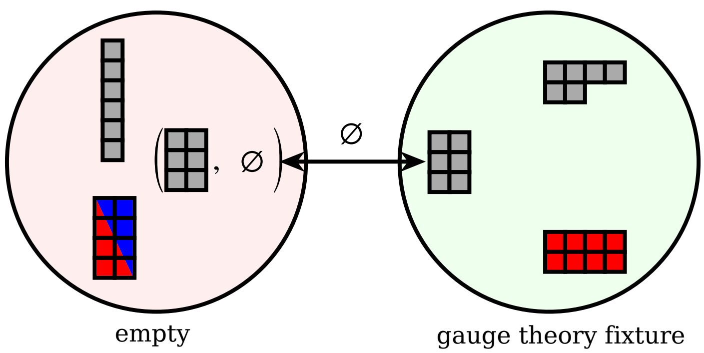
Over $x = 1$, we have two distinct degenerations, which are exchanged by dragging the  puncture around the origin and returning it to its original position: the Lagrangian field theory ($f = 0$)



and the theory at $f = \infty$



Having fixed the behaviour of f over this two-sheeted cover of $\mathcal{M}_{0,4}$, by reproducing the correct asymptotics as $x \to 1$ and $x \to \infty$, we can now take $x \to 0$



and recover that the gauge theory fixture is the aforementioned $SU(4)$ gauge theory at $f(\tau) = -1$.

2.1.5.3 Gauge Theory Fixtures with Two Atypical Punctures

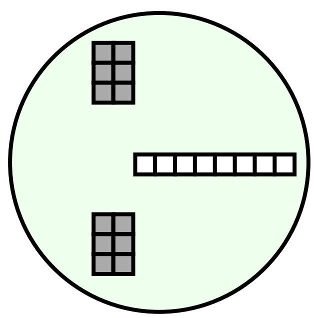
When we resolve the gauge theory fixtures with *two* atypical punctures, we obtain a branched covering of $\mathcal{M}_{0,5}$.

The geometry of $\mathcal{M}_{0,5}$, and the relevant branched covering thereof, were discussed in detail in section 5.1.2 of [4]. Here, we will simply borrow the relevant results.

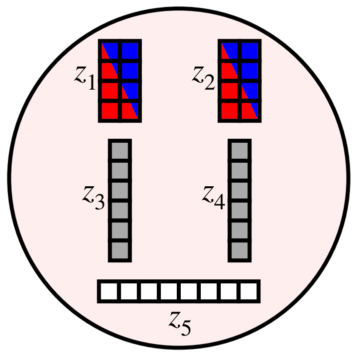
The (compactified) $\mathcal{M}_{0,5}$ is a rational surface. The boundary divisor consists of ten (-1)-curves ($\mathbb{C}\mathrm{P}^1$s with normal bundle $\mathcal{O}(-1)$). We label these curves as D_{ij}, corresponding to the locus where the punctures p_i and p_j collide. The D_{ij}, in turn, intersect in 15 points.

The moduli space of the $(2,0)$ compactification is a branched covering, $\tilde{\mathcal{M}} \to \mathcal{M}_{0,5}$, which is branched over the boundary divisor.

The D_4 gauge theory fixture



is an $Sp(2) \times SU(2)$ gauge theory, with matter in the $6(4,1) + 4(1,2)$, with gauge couplings $(f_{Sp(2)}, f_{SU(2)}) = (-1,-1)$. Resolving the atypical punctures, we have a 5-punctured sphere,



Since the resolution has two very-even punctures, $\tilde{\mathcal{M}}$ is an 8-sheeted branched cover of $\mathcal{M}_{0,5}$. However since the gauge couplings (and the rest of the physics) are invariant under simultaneously flipping the colours of both very-even punctures, we can pass to the quotient, $X = \tilde{\mathcal{M}}/\mathbb{Z}_2$, and it is the geometry of 4-sheeted branched cover, $X \to \mathcal{M}_{0,5}$, that was studied in detail in [4].

Meromorphic functions on $\mathcal{M}_{0,5}$ are rational functions of the cross-ratios

$$s_1 = \frac{z_{13}z_{25}}{z_{15}z_{23}}, \quad s_2 = \frac{z_{14}z_{25}}{z_{15}z_{24}}$$

X is a branched 4-fold cover of $\mathcal{M}_{0,5}$, whose ring of meromorphic functions is generated by rational functions of w_1, w_2

$$w_1^2 = s_1, \quad w_2^2 = s_2$$

The gauge couplings are meromorphic functions on X, given by

$$f_{Sp(2)} = \frac{w_1 - 1}{w_1 + 1}\frac{w_2 + 1}{w_2 - 1}, \quad f_{SU(2)} = \frac{w_1 - 1}{w_1 + 1}\frac{w_2 - 1}{w_2 + 1} \tag{2.7}$$

There is a natural action of the dihedral group, D_4, on X. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup is generated by the deck transformations,

$$\alpha: \; w_1 \to -w_1,\, w_2 \to w_2$$

$$\beta: \; w_1 \to w_1,\, w_2 \to -w_2$$

which act on the gauge couplings as

$$\alpha: \; f_{Sp(2)} \to 1/f_{SU(2)},\, f_{SU(2)} \to 1/f_{Sp(2)}$$

$$\beta: \; f_{Sp(2)} \leftrightarrow f_{SU(2)}$$

Both α and β change the relative colour of the two very-even punctures. The additional generator of D_4,

$$\gamma :, w_1 \leftrightarrow w_2$$

acts as S-duality for the $Sp(2)$,

$$\gamma: \; f_{Sp(2)} \to 1/f_{Sp(2)},\, f_{SU(2)} \to f_{SU(2)}$$

At the boundary, various sheets come together, and the behaviour of the gauge couplings is

- Over D_{15} and D_{25}, both couplings go to $f = 1$, but the ratio $\frac{f_{Sp(2)}-1}{f_{SU(2)}-1}$ is arbitrary.

- Over D_{35}, both couplings are weak ($f = 0$ or $f = \infty$), but the ratio $\frac{f_{Sp(2)}}{f_{SU(2)}}$ is arbitrary.

- Over D_{45}, both couplings are weak ($f_{Sp(2)} = 0$, $f_{SU(2)} = \infty$ or vice-versa), but the product $f_{Sp(2)} \cdot f_{SU(2)}$ is arbitrary.

- Over D_{12}, one coupling is weak ($f = 0$ or ∞), while the other is arbitrary.

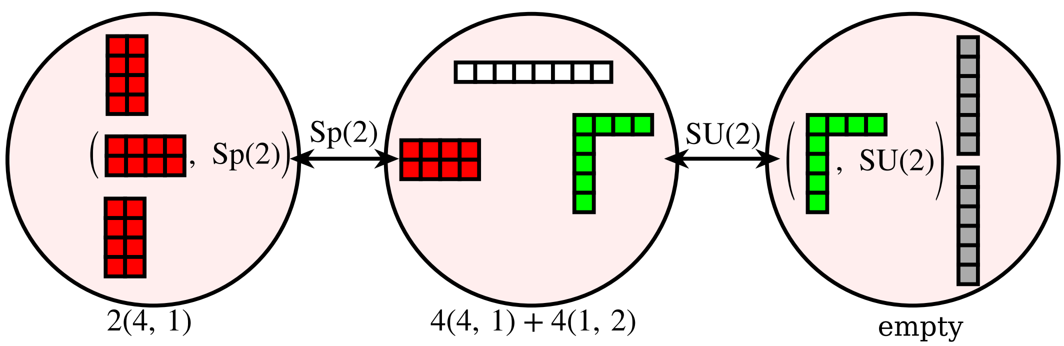
- Over D_{34}, one coupling is $f = 1$, while the other is arbitrary.

- Over D_{13} and D_{23}, $f_{Sp(2)} = 1/f_{SU(2)}$.

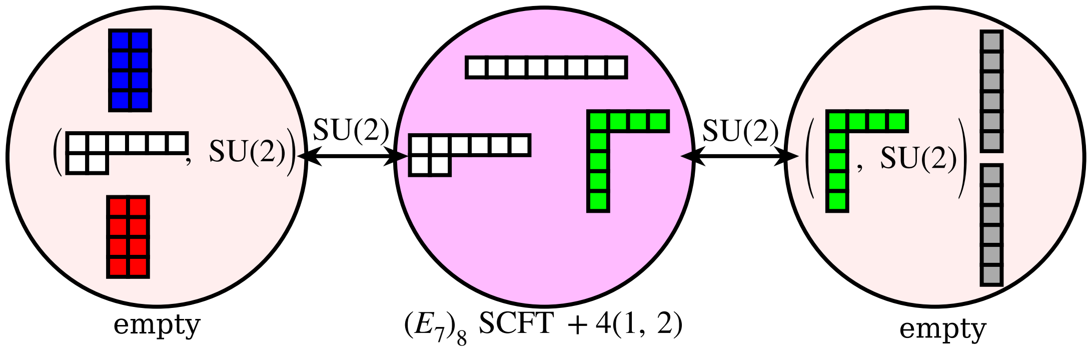
- Over D_{14} and D_{24}, $f_{Sp(2)} = f_{SU(2)}$.

Over the *intersections* of these divisors, we see the various S-duality frames of the gauge theory.

Over $D_{12} \cap D_{34}$, we have

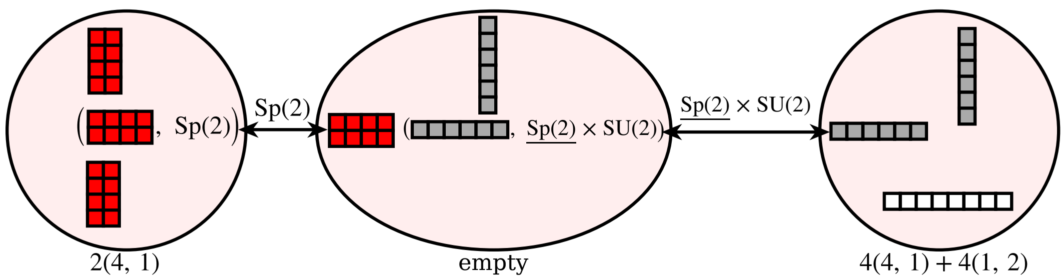


and



In the first case, $f_{Sp(2)} = 0$ or ∞ and $f_{SU(2)} = 1$; in the latter, $f_{Sp(2)} = 1$ and $f_{SU(2)} = 0$ or ∞.

Over $D_{12} \cap D_{35}$ and $D_{12} \cap D_{45}$, we have



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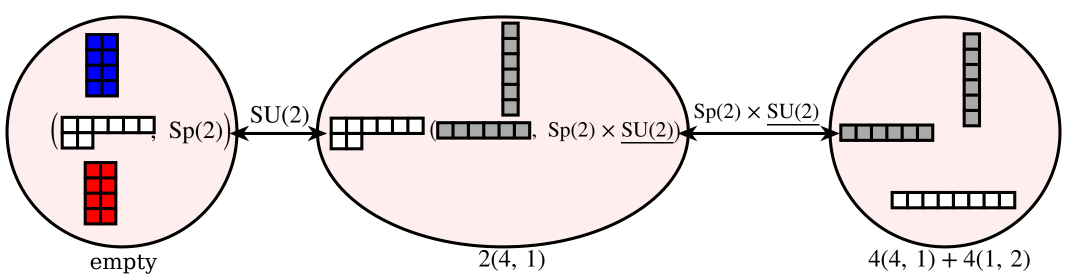
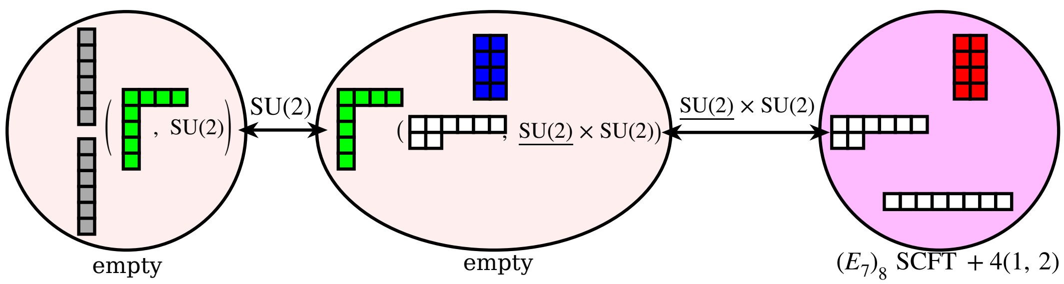
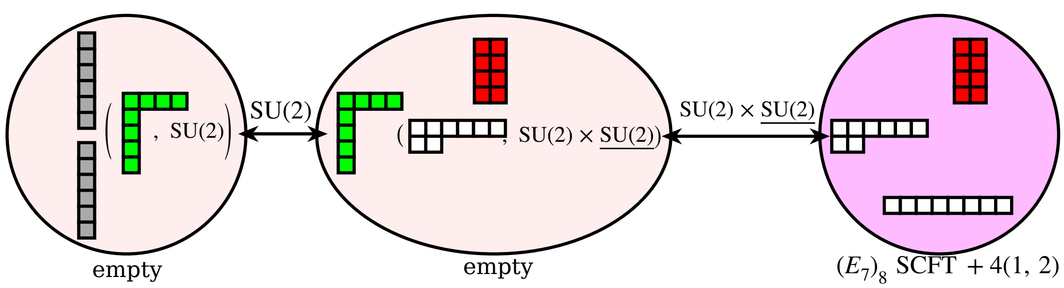


In both cases, the underlined gauge group on the right-hand cylinder is *identified* with the gauge group on the left-hand cylinder. The notation, which we introduced in [4], indicated that when the cylinder on the right pinches off, *both* factors in the gauge group become weakly-coupled ($f \to 0$ or ∞). When the cylinder on the left pinches off, only one of the gauge group factors becomes weakly-coupled.

Over $D_{34} \cap D_{15}$ and $D_{34} \cap D_{25}$, $f_{Sp(2)} = f_{SU(2)} = 1$. So we have



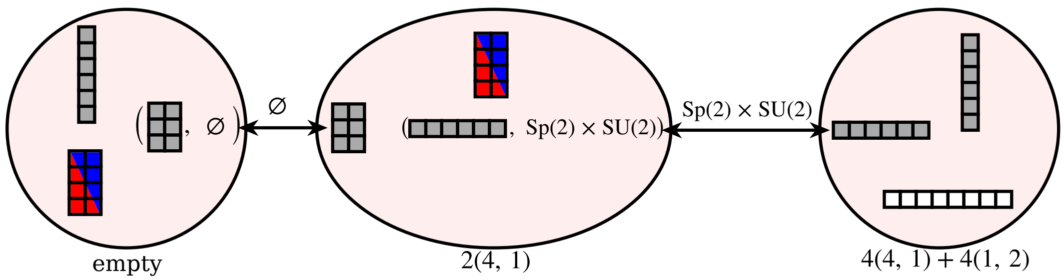
and



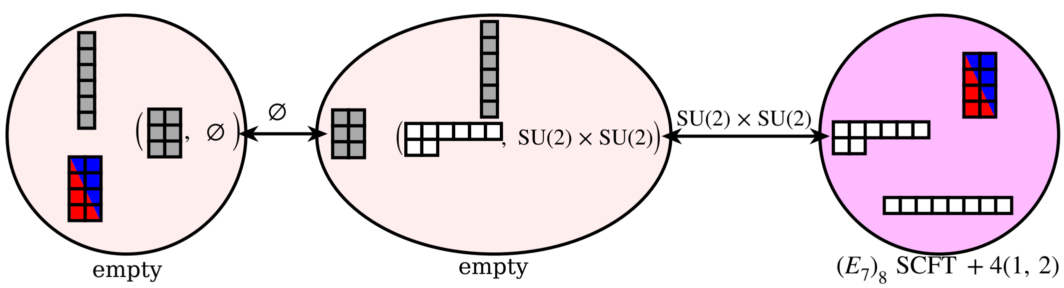
These differ only very subtly, as to "which" $SU(2)$ gauge coupling is controlled by the cylinder on the left. In the first case, it is the $SU(2)$ which couples to the

$(E_7)_8$ (i.e., the one which becomes weakly-coupled at $f_{Sp(2)} = 1$); in the second case, it is the $SU(2)$ which couples to the 4 fundamental hypermultiplets.

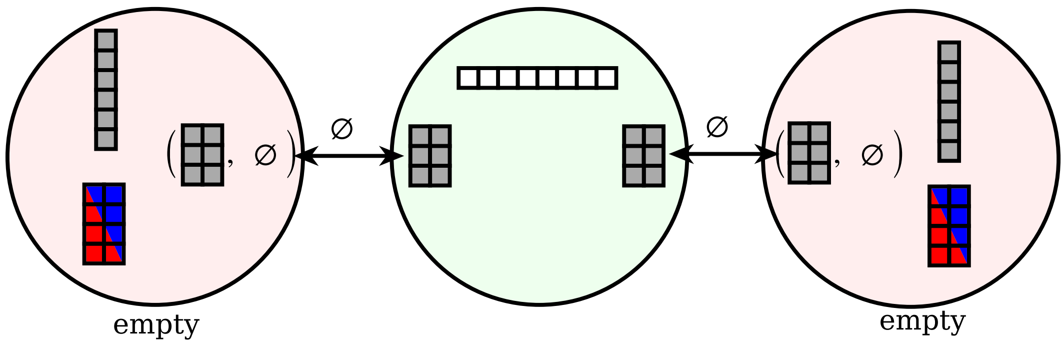
Over $D_{13} \cap D_{45}$, $D_{23} \cap D_{45}$, $D_{14} \cap D_{35}$ and $D_{24} \cap D_{35}$, we have



Over $D_{13} \cap D_{25}$, $D_{14} \cap D_{25}$, $D_{23} \cap D_{15}$ and $D_{24} \cap D_{15}$, we have $f_{Sp(2)} = 1$, $f_{SU(2)} = 1$:



Finally, over $D_{13} \cap D_{24}$ and $D_{14} \cap D_{23}$, we recover our gauge theory fixture, and read off that its gauge theory couplings are $f_{Sp(2)} = f_{SU(2)} = -1$



2.1.5.4 Atypical Degenerations and Ramification

Once we introduce outer-automorphism twists, the moduli space of the gauge theory no longer coincides with $\mathcal{M}_{g,n}$, the moduli space of punctured curves. As we saw, in §2.1.5.1, even the dimensions don't agree, until we "resolve" each atypical puncture, replacing $\mathcal{M}_{g,n}$ by $\mathcal{M}_{g,n+m}$ (for m atypical punctures). Even then, the moduli space of the gauge theory is a branched covering of $\mathcal{M}_{g,n+m}$, branched over various components of the boundary.

Over a generic point on "most" of the components of the boundary, the covering is unramified, and the gauge couplings behave "normally": one (and only one) gauge coupling becomes weak at that irreducible component of the boundary. Here, we would like to catalogue the exceptions: those components of the boundary where

- the covering is ramified
- an "unexpected" (either 0 or 2, in the cases at hand) number of gauge couplings become weak

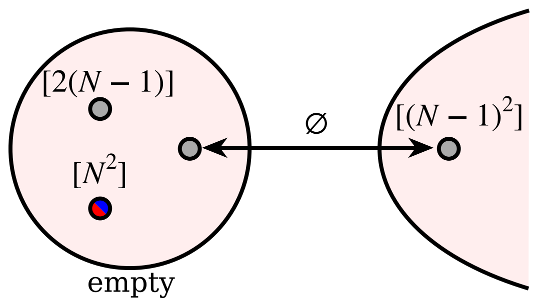
- both

Let us denote, by $D_{p_1,p_2,\ldots p_l}$, the component of the boundary of $\mathcal{M}_{g,n+m}$ where the punctures $p_1, p_2, \ldots p_l$ collide, bubbling off an $(l+1)$-punctured sphere. All of our exceptional cases will involve either D_{p_1,p_2} or D_{p_1,p_2,p_3}.

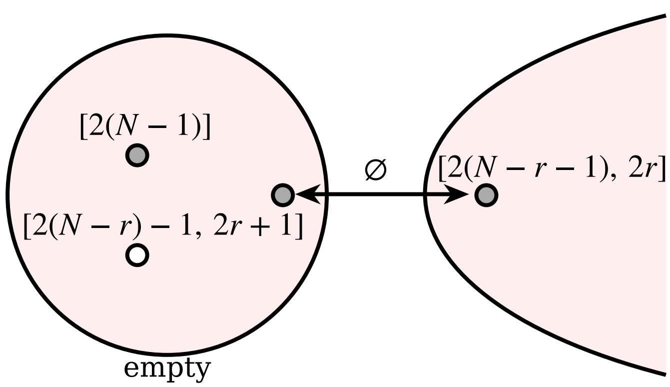
$\underline{D_{T,V}}$ The first source of ramification, as we saw in §2.1.4, is that the outer automorphism changes the colour of a very even puncture from red to blue and vice versa. In general, this changes the physics of the gauge theory. So, for a theory with v very-even punctures, we get a 2^v sheeted cover of the moduli space of curves, ramified (with ramification index 2) over $D_{T,V}$ where "T" denotes any twisted-sector puncture and "V" represents any very-even. As already noted, simultaneously changing the colour of all of the very-even punctures leads to isomorphic physics so we can (and usually will) pass to the $\mathbb{Z}_2$ quotient.

Generically, the gauge couplings behave "normally," with one gauge coupling becoming weak at $D_{T,V}$.

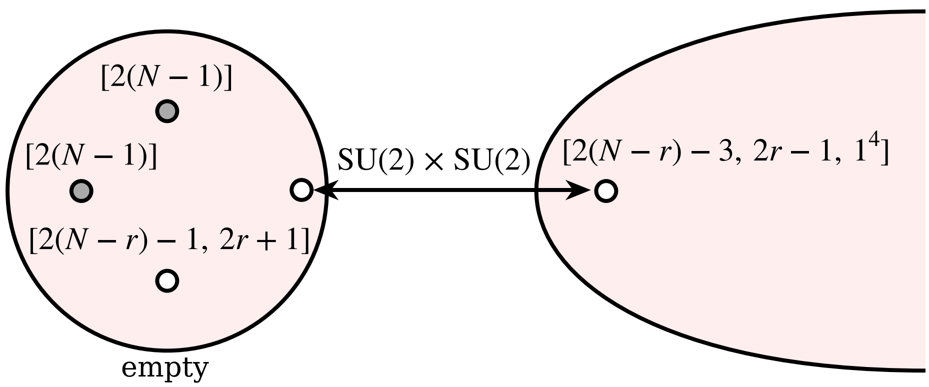
$\underline{D_{[2(N-1)],[N^2]}}$ When N is even, there is one such collision where, in addition to ramification, no gauge coupling becomes weak. Instead, the two punctures fuse (in non-singular fashion) into an atypical puncture.



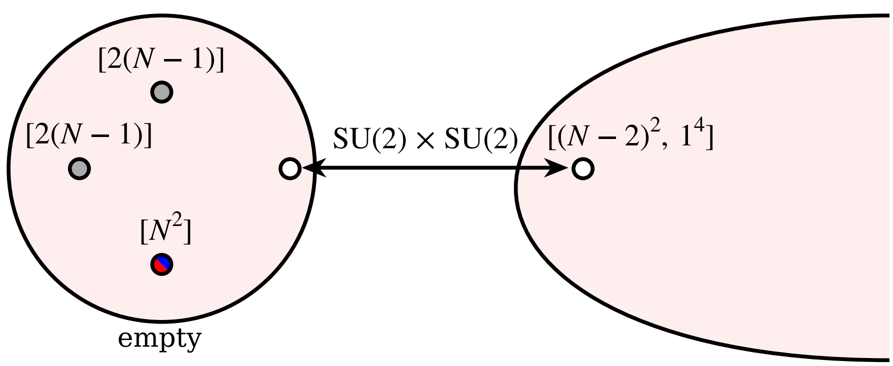
$\underline{D_{[2(N-1)],[2(N-r)-1,2r+1]}}$ For $r = 1, 2, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor$, we again obtain an atypical puncture as the OPE. No gauge coupling become weak, but the moduli space is ramified (with ramification index 2).



$\underline{D_{[2(N-1)],[2(N-1)],[2(N-r)-1,2r+1]}}$ The moduli space is *unramified* over this component of the boundary. Nonetheless, *two* gauge couplings become weak.

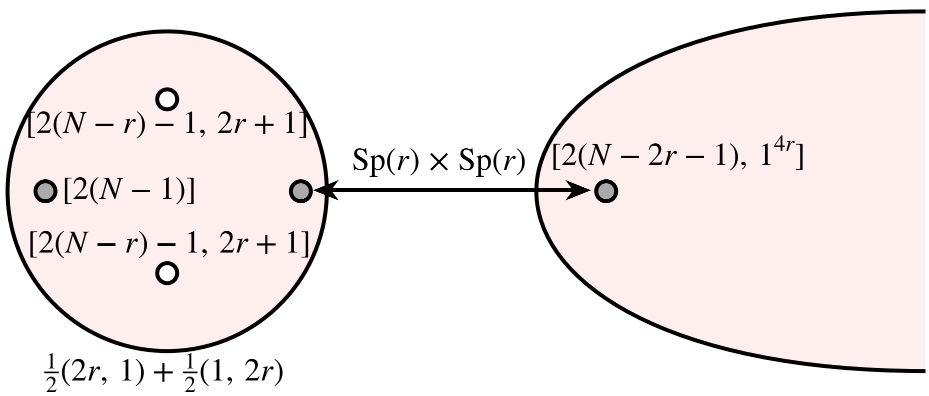


$\underline{D_{[2(N-1)],[2(N-1)],[N^2]}}$ Here, again, an $SU(2) \times SU(2)$ gauge group becomes weak, but now the moduli space is also ramified (with ramification index 2)

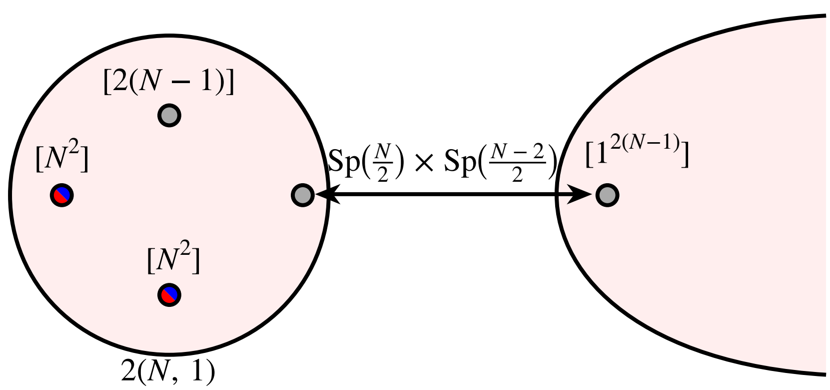


$\underline{D_{t,u,u'}}$ In all of the remaining cases, the moduli space is ramified (with ramification index 2) and two gauge couplings become weak.

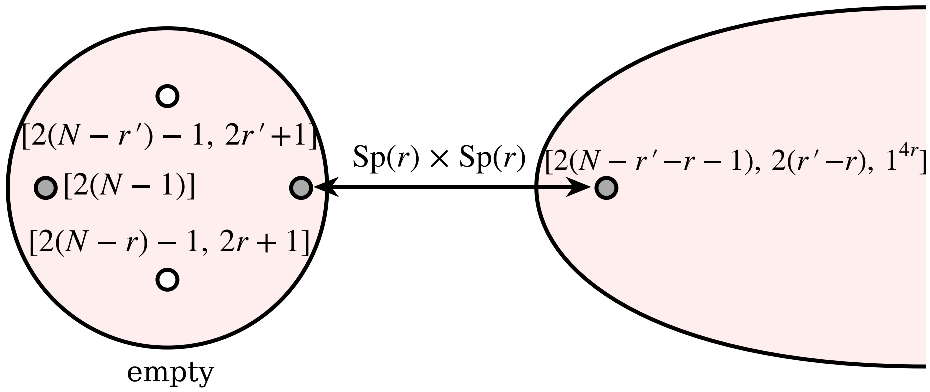
Over $D_{[2(N-1)],[2(N-r)-1,2r+1],[2(N-r)-1,2r+1]}$ (with the same untwisted puncture), we have an $Sp(r) \times Sp(r)$ gauge group becoming weak



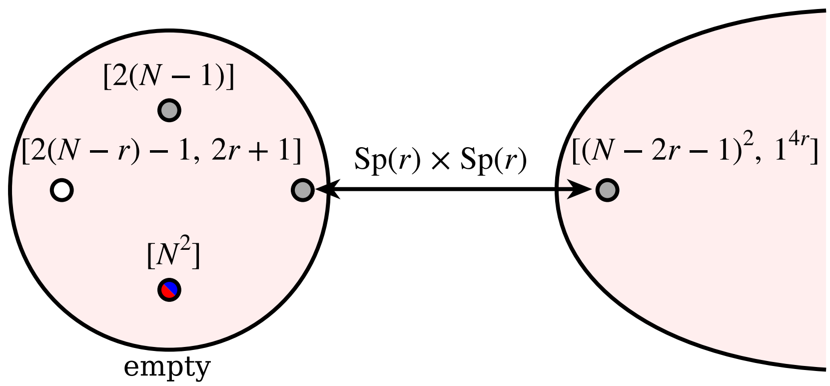
and, for N even, the gauge group which becomes weak is $Sp\left(\frac{N}{2}\right) \times Sp\left(\frac{N-2}{2}\right)$



Over $D_{t,u,u'}$, with $r', r = 1, 2, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor$ (and, without loss of generality, $r' > r$)



and, for N even,



2.1.6 Global Symmetries and the Superconformal Index

2.1.6.1 Computing the Index in the Hall-Littlewood Limit

Each puncture has a "manifest" global symmetry associated to it. The global symmetry group of the SCFT associated to a fixture contains the product of the "manifest" global symmetry groups, associated to each of the punctures, as a subgroup. But, in general, it is larger. Here, we will outline how to use the superconformal index [22, 23, 24, 25] to determine the global symmetry group of the fixture *and* (in the case of a mixed fixture) the number of free hypermultiplets that it contains.

The prescription to compute the superconformal index of an interacting SCFT defined by a D_N-series fixture was given in [26]. For a D_N $\mathbb{Z}_2$-twisted sector fixture with punctures $(\tilde{\Lambda}_1, \tilde{\Lambda}_2, \Lambda_3)$, where $\tilde{\Lambda}$ denotes a twisted puncture and Λ an untwisted puncture, the index is given by [13]

$$
\begin{aligned}
\mathcal{I}(\mathbf{a},\mathbf{b},\mathbf{c}) &= \mathcal{A}(\tau)\mathcal{K}(\mathbf{a}(\tilde{\Lambda}_1))\mathcal{K}(\mathbf{b}(\tilde{\Lambda}_2))\mathcal{K}(\mathbf{c}(\Lambda_3)) \\
&\quad \times \sum_{\lambda'} \frac{P^{\lambda'}_{Sp(N-1)}(\mathbf{a}(\tilde{\Lambda}_1)|\tau) P^{\lambda'}_{Sp(N-1)}(\mathbf{b}(\tilde{\Lambda}_2)|\tau) P^{\lambda=\lambda'}_{SO(2N)}(\mathbf{c}(\Lambda_3)|\tau)}{P^{\lambda=\lambda'}_{SO(2N)}(1,\tau,\tau^2,\dots,\tau^{N-1}|\tau)}.
\end{aligned}
\tag{2.8}
$$

The various elements of this formula are summarized below. Detailed explanations can be found in [26]:

- $\mathcal{A}(\tau)$ is the overall (fugacity-independent) normalization, given by

$$
\mathcal{A}(\tau) = \frac{(1-\tau^{2N})}{(1-\tau^2)^{\frac{N}{2}}} \prod_{j=1}^{N-1} (1-\tau^{4j}).
$$

- P^{λ} are the Hall-Littlewood polynomials of type $SO(2N)$ and $Sp(N)$,

[13] In the following, we need only consider the "Hall-Littlewood" limit of the index, where we restrict to the one-parameter slice in the space of superconformal fugacities given by ($p=0$, $q=0$, $t^{1/2} \equiv \tau$) [24].

given by

$$
\begin{aligned}
&P^{\lambda}_{SO(2N)}(x_1,\ldots,x_N) = \\
&\qquad W_\lambda(\tau)^{-1}\sum_{\sigma\in S_N}\sum_{\substack{s_1,\ldots,s_N=\pm1\\ \prod s_i=+1}} x^{s_1\lambda_1}_{\sigma(1)}\cdots x^{s_N\lambda_N}_{\sigma(N)}\prod_{i<j}\frac{1-\tau^2x_i^{-s_i}x_j^{\pm s_j}}{1-x_i^{-s_i}x_j^{\pm s_j}}, \\
&P^{\lambda}_{Sp(N)}(x_1,\ldots,x_N) = \\
&\qquad W_\lambda(\tau)^{-1}\sum_{\sigma\in S_N}\sum_{s_1,\ldots,s_N=\pm1} x^{s_1\lambda_1}_{\sigma(1)}\cdots x^{s_N\lambda_N}_{\sigma(N)}\prod_{i<j}\frac{1-\tau^2x_i^{-s_i}x_j^{\pm s_j}}{1-x_i^{-s_i}x_j^{\pm s_j}} \\
&\qquad \times\prod_{i=1}^{N}\frac{1-\tau^2x_i^{-2s_i}}{1-x_i^{-2s_i}},
\end{aligned}
\tag{2.9}
$$

where

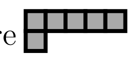
$$
W_\lambda(\tau) = \left(\sum_{\substack{w\in W\\ w\lambda=\lambda}}\tau^{2\ell(w)}\right)^{\frac{1}{2}}
$$

with $\ell(w)$ denoting the length of the Weyl group element w.

- The prescription for writing the $\mathcal{K}$-factors can be found in [26]. Their precise form will not be important here.

- The sum runs over all partitions $\lambda' = (\lambda'_1,\ldots,\lambda'_{N-1})$ corresponding to the highest weight of a finite-dimensional irreducible representation of $Sp(N-1)$ (in the standard orthonormal basis); "$\lambda = \lambda'$" means that we only sum over representations of $SO(2N)$ of the form $\lambda = (\lambda'_1,\ldots,\lambda'_{N-1},0)$.

- The fugacities $\mathbf{a}_I$ dual to the Cartan subalgebra of the flavor symmetry group of the puncture Λ_I ($\tilde{\Lambda}_I$) are assigned by setting the character of the fundamental representation of $SO(2N)$ ($Sp(N-1)$) equal to the sum of $SU(2)$ characters corresponding to the decomposition determined by the puncture, with $SU(2)$ fugacity equal to τ. The multiplicity of each $SU(2)$ representation is then replaced by the character of the fundamental representation of the flavor symmetry determined by that multiplicity. From this equation, one can simply read off the fugacities. [14]

For example, the D_4 twisted puncture  corresponds to the $SU(2)$ embedding under which the 6 of $Sp(3)$ decomposes as $2 + 4(1)$. So setting

$$\chi^{\mathbf{6}}_{Sp(3)}(x_1, x_2, x_3) = 1 \cdot \chi^{\mathbf{2}}_{SU(2)}(\tau) + \chi^{\mathbf{4}}_{Sp(2)}(a_1, a_2) \cdot 1$$
$$\sum_{i=1}^{3}(x_i + x_i^{-1}) = \tau + \tau^{-1} + \sum_{i=1}^{2}(a_i + a_i^{-1})$$

we can take fugacities $x_1 = \tau, x_2 = a_1, x_3 = a_2$.

To determine the global symmetry, as well as any decoupled sector, of an interacting SCFT fixture from its superconformal index, we need only compute (2.8) to order τ^2: as explained in [27], the contribution at order τ is due to free hypermultiplets while the contribution at order τ^2 is due to moment map operators of flavor symmetries.

Computing the index to order τ^2 while keeping only the term $\lambda' = 0$ in

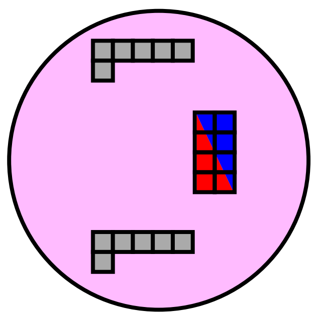
[14] If the puncture is not "very even", different choices of fugacities are related by a Weyl transformation, under which the Hall-Littlewood polynomials are invariant. For "very even" punctures there are two inequivalent choices, which are permuted by the $\mathbb{Z}_2$ outer-automorphism, corresponding to the red and blue coloring. For examples, see [26].

the sum over representations gives the contribution

$$1 + (\chi^{\mathbf{adj}}_{G_1} + \chi^{\mathbf{adj}}_{G_2} + \chi^{\mathbf{adj}}_{G_3})\tau^2,$$

encoding the manifest global symmetry. The global symmetry of the SCFT is enhanced if there are additional terms contributing at order τ^2 coming from the sum over $\lambda' > 0$.

As an example, consider the fixture



.

Letting $(a_1, a_2), (b_1, b_2)$ be $Sp(2)$ fugacities and c an $SU(2)$ fugacity, from (2.8) we find

$$\begin{aligned}
\mathcal{I} = 1 &+ \chi^{\mathbf{2}}_{SU(2)}(c)\tau + [2\chi^{\mathbf{3}}_{SU(2)}(c) + \chi^{\mathbf{10}}_{Sp(2)}(a_1, a_2) + \chi^{\mathbf{10}}_{Sp(2)}(b_1, b_2) + \\
&\chi^{\mathbf{4}}_{Sp(2)}(a_1, a_2)\chi^{\mathbf{4}}_{Sp(2)}(b_1, b_2) \\
&+ \chi^{\mathbf{2}}_{SU(2)}(c)(\chi^{\mathbf{4}}_{Sp(2)}(a_1, a_2) + \chi^{\mathbf{4}}_{Sp(2)}(b_1, b_2))]\tau^2 + \dots \\
&= 1 + \chi^{\mathbf{2}}_{SU(2)}(c)\tau + [2\chi^{\mathbf{3}}_{SU(2)}(c) + \chi^{\mathbf{36}}_{Sp(4)}(a_1, a_2, b_1, b_2) + \\
&\chi^{\mathbf{2}}_{SU(2)}(c)\chi^{\mathbf{8}}_{Sp(4)}(a_1, a_2, b_1, b_2)]\tau^2 + \dots
\end{aligned}$$

The order τ term signals the contribution of a free hypermultiplet in the $\frac{1}{2}(1, 1, 2)$ of $Sp(2) \times Sp(2) \times SU(2)$, the index of which is given by

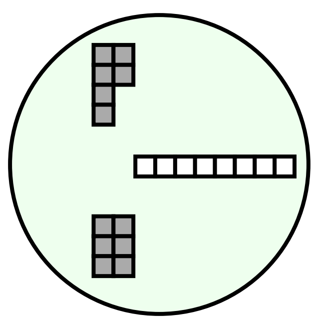
$$\mathcal{I}_{\text{free}} = PE[\tau \chi^{\mathbf{2}}_{SU(2)}(c)] = 1 + \chi^{\mathbf{2}}_{SU(2)}(c)\tau + \chi^{\mathbf{3}}_{SU(2)}(c)\tau^2 + \ldots,$$

where PE denotes the plethystic exponential [26]. Removing the contribution of the free hypermultiplet, the index of the interacting SCFT is given by

$$\begin{aligned}\mathcal{I}_{SCFT} &= \mathcal{I}/\mathcal{I}_{\text{free}} \\ &= 1 + [\chi^{\mathbf{3}}_{SU(2)}(c) + \chi^{\mathbf{36}}_{Sp(4)}(a_1,a_2,b_1,b_2) + \chi^{\mathbf{2}}_{SU(2)}(c)\chi^{\mathbf{8}}_{Sp(4)}(a_1,a_2,b_1,b_2)]\tau^2 \\ &\quad + \ldots \\ &= 1 + \chi^{\mathbf{55}}_{Sp(5)}(a_1,a_2,b_1,b_2,c)\tau^2 + \ldots\end{aligned}$$

and hence this SCFT has an enhanced $Sp(5)$ global symmetry.

We can also use the second order expansion of (2.8) as a check on our identifications for the gauge theory fixtures. For example, the fixture



is an $SU(2)\times SU(2)$ gauge theory with 4 hypermultiplets in the $(2,1)$, 4 hypermultiplets in the $(1,2)$, and 8 free hypermultiplets transforming in the $\frac{1}{2}(2,8_v)$ of the manifest $SU(2)_8 \times SO(8)_{12}$ global symmetry. Thus the manifest global symmetry of this fixture should be enhanced to $SO(8)^2 \times Sp(8)$. Choosing

$(b; c_1, c_2, c_3, c_4)$ as fugacities for the manifest global symmetry, indeed we find the expansion of the index is given by

$$\begin{aligned}\mathcal{I} = 1 + \chi^{\mathbf{2}}_{SU(2)}(b)\chi^{\mathbf{8_v}}_{SO(8)}(c_1, c_2, c_3, c_4)\tau + \\ (2\chi^{\mathbf{28}}_{SO(8)}(c_1, c_2, c_3, c_4) + \chi^{\mathbf{136}}_{Sp(8)}(b, c_1, c_2, c_3, c_4))\tau^2 + \dots\end{aligned}$$

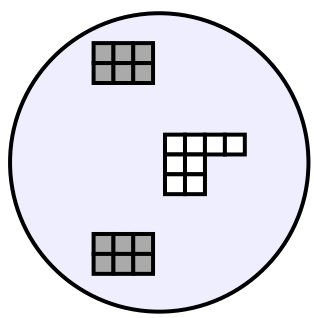
where

$$\begin{aligned}\chi^{\mathbf{136}}_{Sp(8)}(b, c_1, c_2, c_3, c_4) = \chi^{\mathbf{3}}_{SU(2)}(b) + \chi^{\mathbf{28}}_{SO(8)}(c_1, c_2, c_3, c_4) + \\ \chi^{\mathbf{3}}_{SU(2)}(b)\chi^{\mathbf{35_v}}_{SO(8)}(c_1, c_2, c_3, c_4).\end{aligned}$$

We have used this technique to check the global symmetries and the number of free hypermultiplets in our tables of fixtures for the $\mathbb{Z}_2$-twisted D_4 theory.

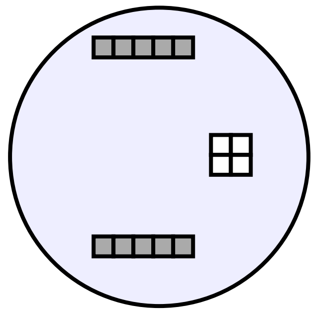
2.1.6.2 The $Sp(4)_6 \times SU(2)_8$ SCFT

Here we use the superconformal index to argue that the D_4 interacting fixture

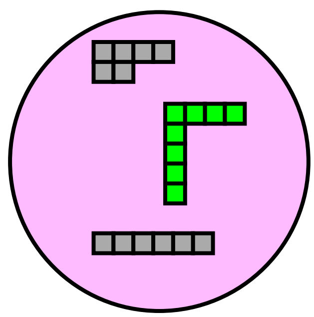


gives rise to the $Sp(4)_6 \times SU(2)_8$ SCFT. For this fixture, we cannot use any S-dualities to study its properties as none of the flavor symmetries carried by the punctures can be gauged.

The $Sp(4)_6 \times SU(2)_8$ SCFT first appeared in [4] as the twisted-sector fixture



in the A_3 theory. It also appears, accompanied by six free hypermultiplets, as



(2.10)

in our list of twisted-sector mixed fixtures in the D_4 theory. In those cases, we are able to use various S-dualities to study it.

Letting a and b be $SU(2)$ fugacities and c_1^2, c_2^2 $U(1)$ fugacities, the expansion of the index of this fixture is given by

$$
\begin{aligned}
\mathcal{I} = 1 + (&\chi^{\mathbf{3}}_{SU(2)}(a) + \chi^{\mathbf{3}}_{SU(2)}(b) + (1 + c_1^2 + c_1^{-2}) + \\
&\chi^{\mathbf{3}}_{SU(2)}(a)\chi^{\mathbf{3}}_{SU(2)}(b)(1 + c_1^2 + c_1^{-2}) + (1 + c_2^2 + c_2^{-2}))\tau^2 + \ldots \\
&= 1 + (\chi^{\mathbf{3}}_{SU(2)}(a) + \chi^{\mathbf{3}}_{SU(2)}(b) + \chi^{\mathbf{3}}_{SU(2)}(c_1) + \chi^{\mathbf{3}}_{SU(2)}(a)\chi^{\mathbf{3}}_{SU(2)}(b)\chi^{\mathbf{3}}_{SU(2)}(c_1) \\
&+ \chi^{\mathbf{3}}_{SU(2)}(c_2))\tau^2 + \ldots \\
&= 1 + (\chi^{\mathbf{36}}_{Sp(4)}(a, b, c_1) + \chi^{\mathbf{3}}_{SU(2)}(c_2))\tau^2 + \ldots ,
\end{aligned}
\tag{2.11}
$$

indicating that the manifest $SU(2)^2_{24} \times U(1)^2$ global symmetry is enhanced to $Sp(4) \times SU(2)$. This, along with the other numerical invariants of this fixture agree with our previous results for the $Sp(4)_6 \times SU(2)_8$ SCFT.

Since $A_3 \cong D_3$, we can use (2.8) to compute the index of the twisted A_3 fixture by appropriately identifying fugacities and replacing $P^{\lambda}_{SO(6)}(P^{\lambda'}_{Sp(2)}) \to P^{\mu}_{SU(4)}(P^{\mu'}_{SO(5)})$ where μ (μ') is the highest weight of the $SU(4)$ ($SO(5)$) representation corresponding to λ (λ'). Letting a be an $SU(2)$ fugacity and $(b_1, b_2), (c_1, c_2)$ $SO(5)$ fugacities, the expansion of the index of the twisted A_3 fixture is

$$
\begin{aligned}
\mathcal{I} = 1 + (&\chi^{\mathbf{3}}_{SU(2)}(a) + \chi^{\mathbf{10}}_{Sp(2)}(\sqrt{b_1 b_2}, \sqrt{\frac{b_1}{b_2}}) + \chi^{\mathbf{10}}_{Sp(2)}(\sqrt{c_1 c_2}, \sqrt{\frac{c_1}{c_2}}) \\
&+ \chi^{\mathbf{4}}_{Sp(2)}(\sqrt{b_1 b_2}, \sqrt{\frac{b_1}{b_2}}))\chi^{\mathbf{4}}_{Sp(2)}(\sqrt{c_1 c_2}, \sqrt{\frac{c_1}{c_2}})\tau^2 + \ldots \\
&= 1 + (\chi^{\mathbf{3}}_{SU(2)}(a) + \chi^{\mathbf{36}}_{Sp(4)}(\sqrt{b_1 b_2}, \sqrt{\frac{b_1}{b_2}}, \sqrt{c_1 c_2}, \sqrt{\frac{c_1}{c_2}}))\tau^2 + \ldots ,
\end{aligned}
$$

in agreement with (2.11). We have checked further that the unrefined indices (obtained by setting all flavor fugacities to "1") of these two fixtures agree to tenth order in τ. The unrefined index of each fixture is given by

$$\mathcal{I} = 1 + 39\tau^2 + 878\tau^4 + 13396\tau^6 + 152412\tau^8 + 1370975\tau^{10} + \dots .$$

We can also compare with the mixed fixture (2.10). After removing the contribution to the index of a free hypermultiplet in the 6 of $Sp(3)$, the index of this fixture is given by

$$\begin{aligned}\mathcal{I} &= 1 + (\chi^{\mathbf{3}}_{SU(2)}(a_2) + \chi^{\mathbf{21}}_{Sp(3)}(b_1, b_2, b_3) + \chi^{\mathbf{2}}_{SU(2)}(a_2)\chi^{\mathbf{6}}_{Sp(3)}(b_1, b_2, b_3) \\ &\quad + \chi^{\mathbf{3}}_{SU(2)}(c))\tau^2 + \dots \\ &= 1 + (\chi^{\mathbf{36}}_{Sp(4)}(a_2, b_1, b_2, b_3) + \chi^{\mathbf{3}}_{SU(2)}(c))\tau^2 + \dots .\end{aligned}$$

Again, the numerical invariants of this fixture imply the SCFT is the $Sp(4)_6 \times SU(2)_8$ theory. We have computed the unrefined index of this fixture to fourth order in τ; removing the contribution of the free hypermultiplet, we find agreement with the fixtures above.

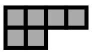
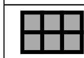
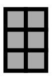
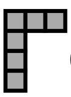
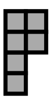
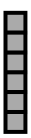
2.2 The $\mathbb{Z}_2$-twisted D_4 Theory

2.2.1 Punctures and Cylinders

2.2.1.1 Regular Punctures

The untwisted sector of regular punctures was discussed in [5]. The $\mathbb{Z}_2$-twisted regular punctures are shown in the Table below.

Table 2.1: $\mathbb{Z}_2$-twisted regular punctures

Flavour C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
	$[7]$	$\{1,3,5;\frac{7}{2}\}$	$-$	$Sp(3)_8$	$(112, \frac{207}{2})$
(ns)	$([5,1^2],\mathbb{Z}_2)$	$\{1,3,5;\frac{5}{2}\}$	$-$	$Sp(2)_7$	$(102, \frac{193}{2})$
	$[5,1^2]$	$\{1,3,5;\frac{5}{2}\}$	$c_5^{(6)} = (a^{(3)})^2$	$SU(2)_6\times U(1)$	$(94, \frac{181}{2})$
	$[3^2,1]$	$\{1,3,4;\frac{5}{2}\}$	$-$	$SU(2)_{24}$	$(88, \frac{171}{2})$
	$[3,2^2]$	$\{1,3,4;\frac{5}{2}\}$	$c_3^{(4)} = (a^{(2)})^2$ $c_4^{(6)} = 2a^{(2)}\tilde{c}_{5/2}$	$SU(2)_8$	$(72, \frac{141}{2})$
(ns)	$([3,1^4],\mathbb{Z}_2)$	$\{1,3,3;\frac{3}{2}\}$	$-$	$SU(2)_5$	$(69, \frac{135}{2})$
	$[3,1^4]$	$\{1,3,3;\frac{3}{2}\}$	$c_3^{(4)} = (a^{(2)})^2$	none	$(64, \frac{127}{2})$
	$[1^7]$	$\{1,1,1;\frac{1}{2}\}$	$-$	none	$(24, \frac{49}{2})$

2.2.1.2 Irregular Punctures

A fairly lengthy list of irregular untwisted punctures, arising from the OPE of untwisted punctures, was discussed in [5]. Additional ones arise from considering the OPE of two $\mathbb{Z}_2$-twisted punctures. Moreover, twisted-sector irregular twisted punctures arise from the OPE of an untwisted puncture and a $\mathbb{Z}_2$-twisted puncture. These two sets of new irregular punctures are listed in the Tables below.

Untwisted

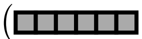
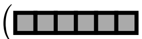
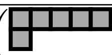
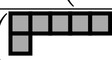
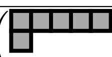
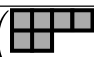
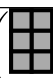
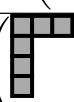
Table 2.2: Untwisted irregular punctures

Irregular puncture	(n_h, n_v)	Flavour Symmetry
$\left(\ydiagram{8}, Sp(2)\right)$	$(112, 118)$	$Sp(2)_0$
$\left(\ydiagram{6,2}, SU(2)\times SU(2)\right)$	$(128, 133)$	$SU(2)_0 \times SU(2)_0$
$\left(\ydiagram{6,1,1}, SU(2)\times SU(2)\right)$	$(136, 140)$	$SU(2)_0 \times SU(2)_0$
$\left(\ydiagram{4,1,1,1,1}, SU(2)\right)$	$(176, 179)$	$SU(2)_0$

As was the case in [5], there are three inequivalent embeddings of $Sp(2) \hookrightarrow Spin(8)$, exchanged by triality, under which one of the 8-dimensional representations decomposes as $5+3(1)$ while the other two decompose as $2(4)$. To indicate which we mean, we assign a green/red/blue colour to $\ydiagram{8}$. The same remark applies to the three index-1 embeddings of $SU(2)\times SU(2)$ in the $SU(2)^3$ of $\ydiagram{6,2}$ which are exchanged by triality.

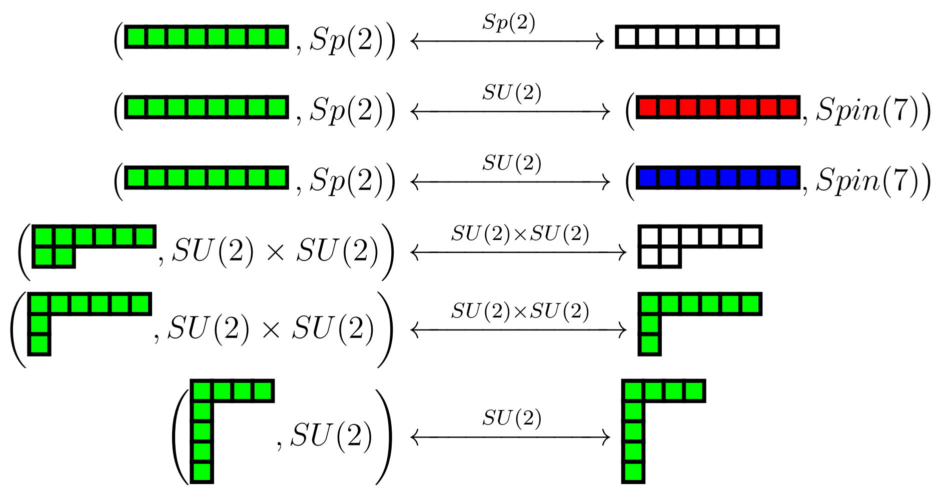
Twisted

Table 2.3: $\mathbb{Z}_2$-twisted irregular punctures

Irregular puncture	(n_h, n_v)	Flavour Symmetry
$\big($$, Sp(2) \times SU(2)\big)$	$\left(112, \frac{223}{2}\right)$	$Sp(2)_4 \times SU(2)_0$
$\big($$, Sp(2)\big)$	$\left(112, \frac{229}{2}\right)$	$Sp(2)_4$
$\big($$, SU(2) \times SU(2)\big)$	$\left(112, \frac{237}{2}\right)$	$SU(2)_0 \times SU(2)_0$
$\big($$, SU(2)\big)$	$\left(112, \frac{243}{2}\right)$	$SU(2)_0$
$\big($$, Sp(2)\big)$	$\left(122, \frac{243}{2}\right)$	$Sp(2)_5$
$\big($$, SU(2) \times SU(2)\big)$	$\left(122, \frac{251}{2}\right)$	$SU(2)_1 \times SU(2)_1$
$\big($$, SU(2)\big)$	$\left(122, \frac{257}{2}\right)$	$SU(2)_1$
$\big($$, SU(2)\big)$	$\left(130, \frac{269}{2}\right)$	$SU(2)_2$
$\big($$, \emptyset\big)$	$\left(152, \frac{315}{2}\right)$	none
$\big($$, SU(2)\big)$	$\left(155, \frac{315}{2}\right)$	$SU(2)_3$
$\big($$, \emptyset\big)$	$\left(155, \frac{315}{2}\right)$	none

2.2.1.3 Cylinders

In addition to the untwisted cylinders of [5], we have



and the twisted sector adds the cylinders

$$\ydiagram{6} \xleftrightarrow{\;Sp(3)\;} \ydiagram{6}$$

$$\left(\ydiagram{6}\,, Sp(2)\times SU(2)\right) \xleftrightarrow{\;Sp(2)\times SU(2)\;} \ydiagram{6}$$

$$\left(\ydiagram{6}\,, Sp(2)\right) \xleftrightarrow{\;Sp(2)\;} \ydiagram{6}$$

$$\left(\ydiagram{6}\,, SU(2)\times SU(2)\right) \xleftrightarrow{\;SU(2)\times SU(2)\;} \ydiagram{6}$$

$$\left(\ydiagram{6}\,, SU(2)\right) \xleftrightarrow{\;SU(2)\;} \ydiagram{6}$$

$$\left(\ydiagram{5,1}\,, Sp(2)\right) \xleftrightarrow{\;Sp(2)\;} \ydiagram{5,1}$$

$$\left(\ydiagram{5,1}\,, SU(2)\times SU(2)\right) \xleftrightarrow{\;SU(2)\times SU(2)\;} \ydiagram{5,1}$$

$$\left(\ydiagram{5,1}\,, SU(2)\right) \xleftrightarrow{\;SU(2)\;} \ydiagram{5,1}$$

$$\left(\ydiagram{4,2}\,, SU(2)\right) \xleftrightarrow{\;SU(2)\;} \ydiagram{4,2}$$

$$\left(\ydiagram{2,2,2}\,, \emptyset\right) \xleftrightarrow{\;\emptyset\;} \ydiagram{2,2,2}$$

$$\left(\ydiagram{3,1,1,1}\,, SU(2)\right) \xleftrightarrow{\;SU(2)\;} \ydiagram{3,1,1,1}$$

$$\left(\ydiagram{2,2,1,1}\,, \emptyset\right) \xleftrightarrow{\;\emptyset\;} \ydiagram{2,2,1,1}$$

2.2.2 Fixtures

2.2.2.1 Free-field Fixtures

Table 2.4: $\mathbb{Z}_2$-twisted free field fixtures

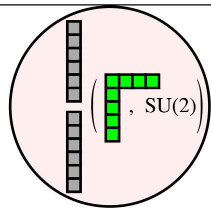
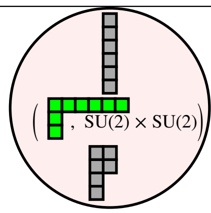
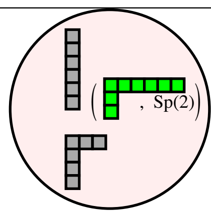
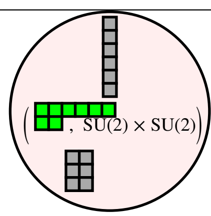
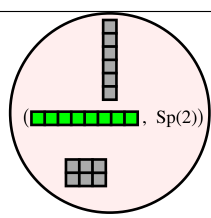
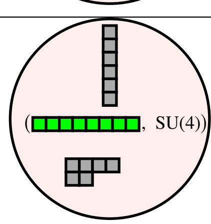
#	Fixture	Number of hypers	Representation
1		0	empty
2		0	empty
3		5	$\frac{1}{2}(2,5)$
4		0	empty
5		0	empty
6		6	$\frac{1}{2}(2,6)$

Table 2.4: $\mathbb{Z}_2$-twisted free field fixtures

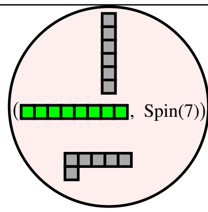
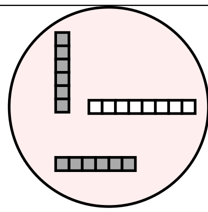
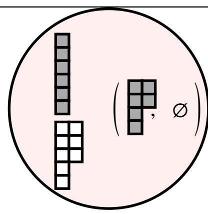
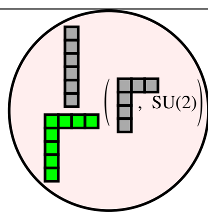
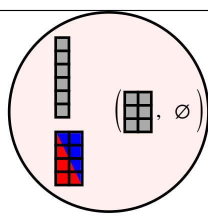
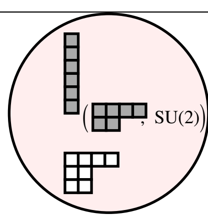
#	Fixture	Number of hypers	Representation
7		14	$\frac{1}{2}(4,7)$
8		24	$\frac{1}{2}(6,8_v)$
9		0	empty
10		3	$\frac{1}{2}(3,2)$
11		0	empty
12		2	$1(2)$

Table 2.4: $\mathbb{Z}_2$-twisted free field fixtures

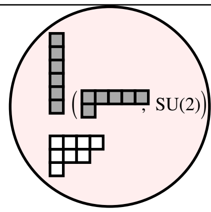
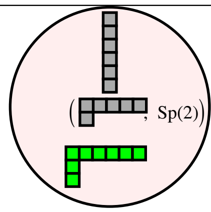
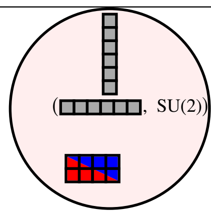
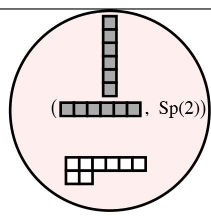
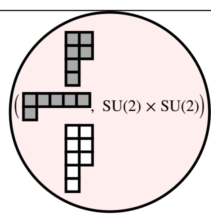
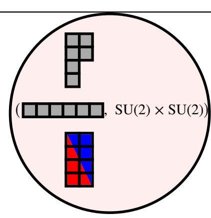
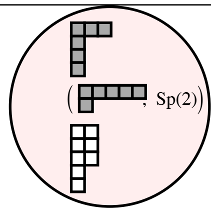
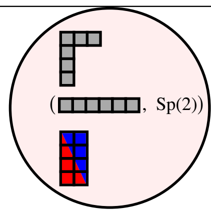
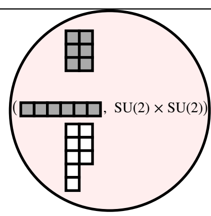
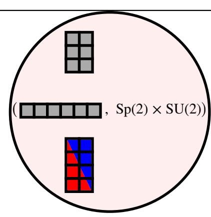
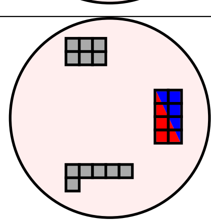
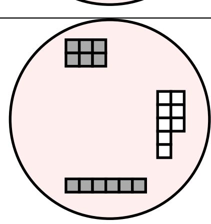
#	Fixture	Number of hypers	Representation
13		1	$\frac{1}{2}(1,2)$
14		10	$\frac{1}{2}(5,4)$
15		0	empty
16		8	$\frac{1}{2}(1,2,2;4)$
17		2	$\frac{1}{2}(2,1)+\frac{1}{2}(1,2)$
18		0	empty

Table 2.4: $\mathbb{Z}_2$-twisted free field fixtures

#	Fixture	Number of hypers	Representation
19		7	$\frac{1}{2}(2,5)+\frac{1}{2}(1,4)$
20		5	$\frac{1}{2}(2,1,5)$
21		0	empty
22		8	$\frac{1}{2}(2,2;4,1)$
23		14	$\frac{1}{2}(3,4,1)+\frac{1}{2}(1,5,2)+\frac{1}{2}(3,1,2)$
24		16	$\frac{1}{2}(1,14')+\frac{1}{2}(3,6)$

2.2.2.2 Interacting Fixtures

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

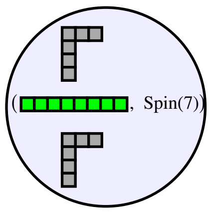
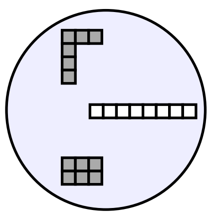
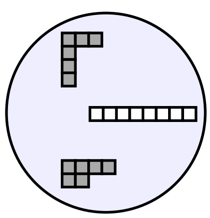
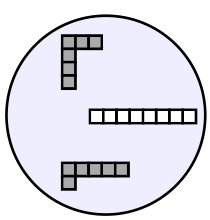
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
1		$(0, 0, 2, 0, 0)$	$(26, 14)$	$Spin(7)_8 \times SU(2)_5^2$
2		$(0, 0, 2, 0, 1)$	$(45, 25)$	$Spin(11)_{12} \times SU(2)_5$
3		$(0, 1, 2, 0, 1)$	$(51, 30)$	$Spin(10)_{12} \times SU(2)_6 \times SU(2)_5$
4		$(0, 0, 2, 0, 2)$	$(59, 36)$	$Spin(9)_{12} \times Sp(2)_7 \times SU(2)_5$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

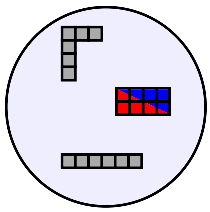
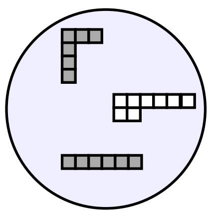
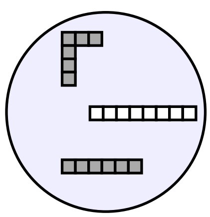
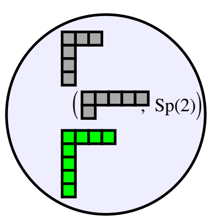
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
5		$(0, 0, 2, 0, 1)$	$(45, 25)$	$Sp(5)_8 \times SU(2)_5$
6		$(0, 0, 3, 0, 1)$	$(53, 32)$	$Sp(4)_8 \times SU(2)_8^2 \times SU(2)_5$
7		$(0, 0, 3, 0, 2)$	$(69, 43)$	$Spin(8)_{12} \times Sp(3)_8 \times SU(2)_5$
8		$(0, 0, 1, 0, 0)$	$(15,7)$	$Sp(3)_5 \times SU(2)_8$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

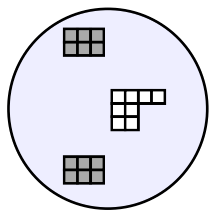
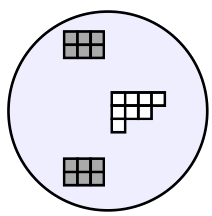
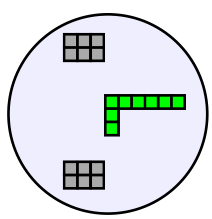
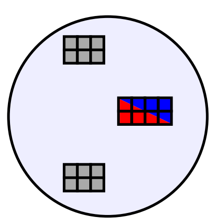
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
9		$(0, 1, 1, 0, 0)$	$(24, 12)$	$Sp(4)_6 \times SU(2)_8$
10		$(0, 0, 1, 0, 1)$	$(31, 18)$	$SU(4)_{12} \times SU(2)_7 \times U(1)$
11		$(0, 0, 2, 0, 1)$	$(40, 25)$	$SU(4)_{12} \times Sp(2)_8$
12		$(0, 0, 2, 0, 1)$	$(40, 25)$	$SU(2)^2_{24} \times Sp(2)_8$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

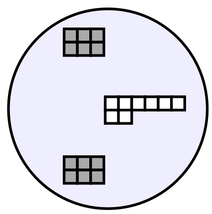
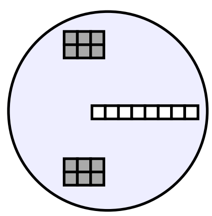
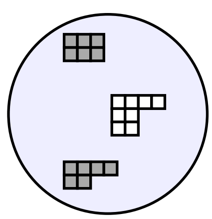
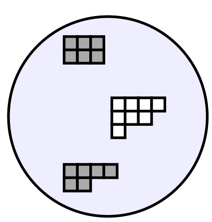
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
13		$(0, 0, 3, 0, 1)$	$(48, 32)$	$SU(2)^2_{24} \times$ $SU(2)^3_8$
14		$(0, 0, 3, 0, 2)$	$(64, 43)$	$Spin(8)_{12} \times$ $(SU(2)_{24})^2$
15		$(0, 2, 1, 0, 0)$	$(30, 17)$	$Sp(2)^2_6 \times$ $SU(2)_6 \times U(1)$
16		$(0, 1, 1, 0, 1)$	$(37, 23)$	$Sp(2)_{12} \times$ $SU(2)_7 \times$ $SU(2)_6$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

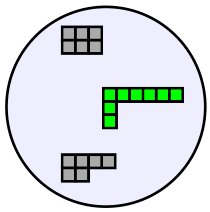
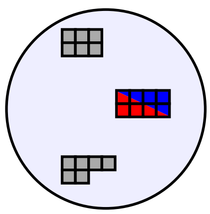
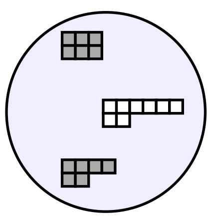
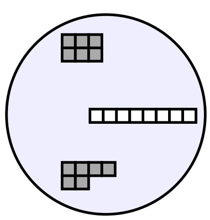
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
17		$(0, 1, 2, 0, 1)$	$(46, 30)$	$Sp(2)_{12} \times Sp(2)_8 \times SU(2)_6$
18		$(0, 1, 2, 0, 1)$	$(46, 30)$	$Sp(2)_8 \times SU(2)_{24} \times SU(2)_6 \times U(1)$
19		$(0, 1, 3, 0, 1)$	$(54, 37)$	$SU(2)_{24} \times SU(2)_8^3 \times SU(2)_6 \times U(1)$
20		$(0, 1, 3, 0, 2)$	$(70, 48)$	$Spin(8)_{12} \times SU(2)_{24} \times SU(2)_6 \times U(1)$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

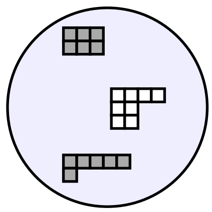
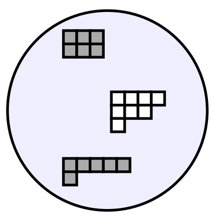
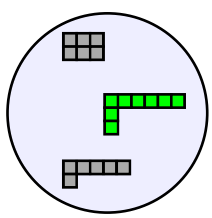
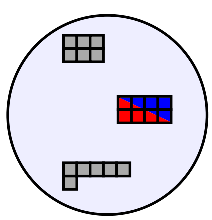
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
21		$(0, 1, 1, 0, 1)$	$(38, 23)$	$Sp(2)_{12} \times Sp(2)_7 \times U(1)$
22		$(0, 0, 1, 0, 2)$	$(45, 29)$	$Sp(2)_7 \times SU(2)_7 \times SU(2)_{12}^2$
23		$(0, 0, 2, 0, 2)$	$(54, 36)$	$Sp(2)_8 \times Sp(2)_7 \times (SU(2)_{12})^2$
24		$(0, 0, 2, 0, 2)$	$(54, 36)$	$Sp(2)_8 \times Sp(2)_7 \times SU(2)_{24}$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

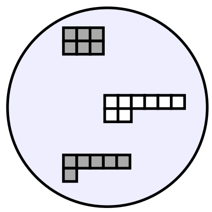
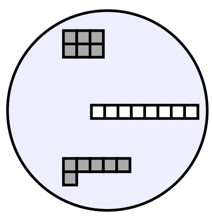
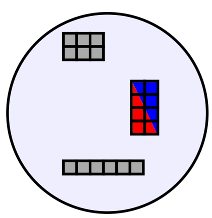
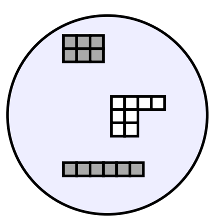
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
25		$(0, 0, 3, 0, 2)$	$(62, 43)$	$Sp(2)_7 \times SU(2)_{24} \times SU(2)_8^3$
26		$(0, 0, 3, 0, 3)$	$(78, 54)$	$Spin(8)_{12} \times Sp(2)_7 \times SU(2)_{24}$
27		$(0, 0, 1, 0, 0)$	$(24, 7)$	$(E_7)_8$
28		$(0, 1, 2, 0, 1)$	$(48, 30)$	$Sp(3)_8 \times SU(2)_{24} \times U(1)^2$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

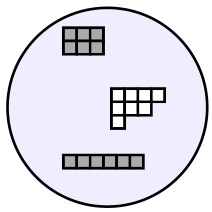
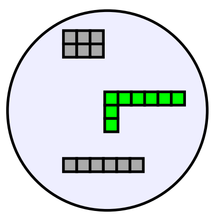
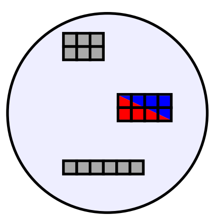
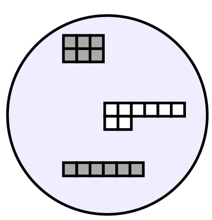
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
29		$(0, 0, 2, 0, 2)$	$(55, 36)$	$Sp(3)_8 \times$ $SU(2)_{24} \times$ $SU(2)_7$
30		$(0, 0, 3, 0, 2)$	$(64, 43)$	$Sp(3)_8 \times$ $Sp(2)_8 \times$ $SU(2)_{24}$
31		$(0, 0, 3, 0, 2)$	$(64, 43)$	$Sp(3)_8 \times$ $Sp(2)_8 \times$ $SU(2)_{24}$
32		$(0, 0, 4, 0, 2)$	$(72, 50)$	$Sp(3)_8 \times$ $SU(2)_{24} \times$ $SU(2)_8^3$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

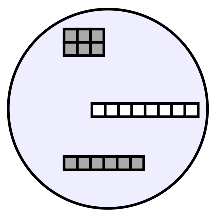
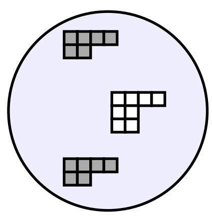
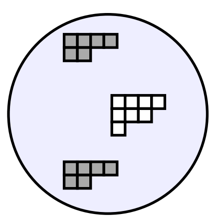
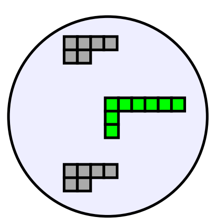
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
33		$(0, 0, 4, 0, 3)$	$(88, 61)$	$Spin(8)_{12} \times Sp(3)_8 \times SU(2)_{24}$
34		$(0, 3, 1, 0, 0)$	$(36, 22)$	$SU(2)_6^6 \times U(1)$
35		$(0, 2, 1, 0, 1)$	$(43, 28)$	$SU(2)_{12}^2 \times SU(2)_6^2 \times SU(2)_7$
36		$(0, 2, 2, 0, 1)$	$(52, 35)$	$Sp(2)_8 \times SU(2)_{12}^2 \times SU(2)_6^2$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

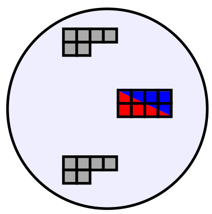
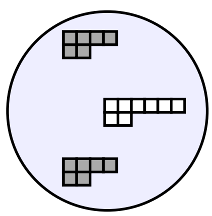
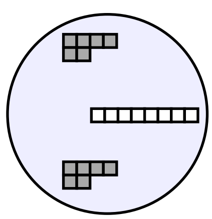
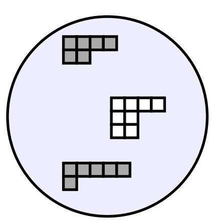
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
37		$(0, 2, 2, 0, 1)$	$(52, 35)$	$Sp(2)_8 \times SU(2)_6^2 \times U(1)^2$
38		$(0, 2, 3, 0, 1)$	$(60, 42)$	$SU(2)_8^3 \times SU(2)_6^2 \times U(1)^2$
39		$(0, 2, 3, 0, 2)$	$(76, 53)$	$Spin(8)_{12} \times SU(2)_6^2 \times U(1)^2$
40		$(0, 2, 1, 0, 1)$	$(44, 28)$	$Sp(2)_7 \times SU(2)_{12}^2 \times SU(2)_6 \times U(1)$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

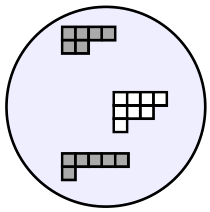
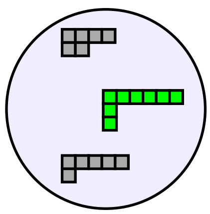
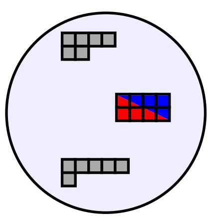
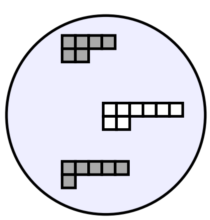
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
41		$(0, 1, 1, 0, 2)$	$(51, 34)$	$Sp(2)_7 \times$ $SU(2)_{24} \times$ $SU(2)_7 \times$ $SU(2)_6$
42		$(0, 1, 2, 0, 2)$	$(60, 41)$	$Sp(2)_8 \times$ $Sp(2)_7 \times$ $SU(2)_{24} \times$ $SU(2)_6$
43		$(0, 1, 2, 0, 2)$	$(60, 41)$	$Sp(2)_8 \times$ $Sp(2)_7 \times$ $SU(2)_6 \times U(1)$
44		$(0, 1, 3, 0, 2)$	$(68, 48)$	$Sp(2)_7 \times$ $SU(2)_8^3 \times$ $SU(2)_6 \times U(1)$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

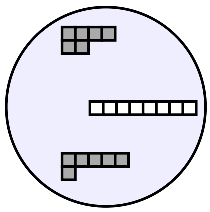
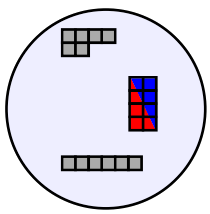
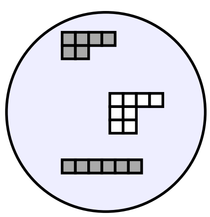
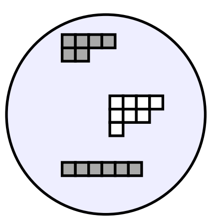
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
45		$(0, 1, 3, 0, 3)$	$(84, 59)$	$Spin(8)_{12} \times$ $Sp(2)_7 \times$ $SU(2)_6 \times U(1)$
46		$(0, 1, 1, 0, 0)$	$(30, 12)$	$SU(2)_6 \times$ $SU(8)_8$
47		$(0, 2, 2, 0, 1)$	$(54, 35)$	$Sp(3)_8 \times$ $SU(2)_6 \times U(1)^3$
48		$(0, 1, 2, 0, 2)$	$(61, 41)$	$Sp(3)_8 \times$ $SU(2)_7 \times$ $SU(2)_6 \times U(1)$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

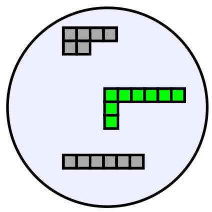
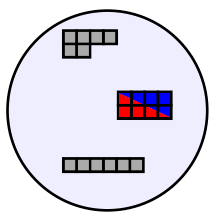
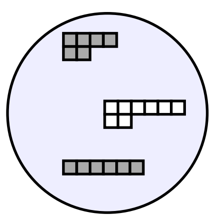
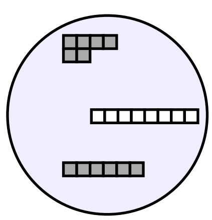
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
49		$(0, 1, 3, 0, 2)$	$(70, 48)$	$Sp(3)_8 \times$ $Sp(2)_8 \times$ $SU(2)_6 \times U(1)$
50		$(0, 1, 3, 0, 2)$	$(70, 48)$	$Sp(3)_8 \times$ $Sp(2)_8 \times$ $SU(2)_6 \times U(1)$
51		$(0, 1, 4, 0, 2)$	$(78, 55)$	$Sp(3)_8 \times$ $SU(2)_8^3 \times$ $SU(2)_6 \times U(1)$
52		$(0, 1, 4, 0, 3)$	$(94, 66)$	$Spin(8)_{12} \times$ $Sp(3)_8 \times$ $SU(2)_6 \times U(1)$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

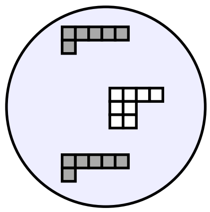
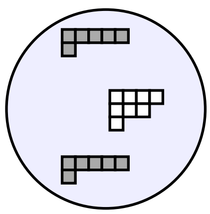
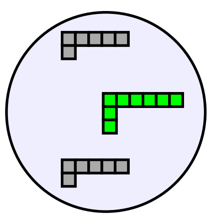
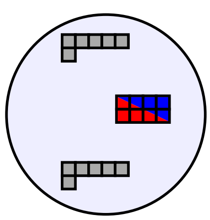
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
53		$(0, 1, 1, 0, 2)$	$(52, 34)$	$Sp(2)^2_7 \times SU(2)_{24} \times U(1)$
54		$(0, 0, 1, 0, 3)$	$(59, 40)$	$Sp(2)^2_7 \times SU(2)_7 \times U(1)$
55		$(0, 0, 2, 0, 3)$	$(68, 47)$	$Sp(2)_8 \times Sp(2)^2_7 \times U(1)$
56		$(0, 0, 2, 0, 3)$	$(68, 47)$	$Sp(2)_8 \times Sp(2)^2_7$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

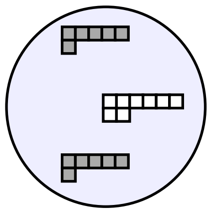
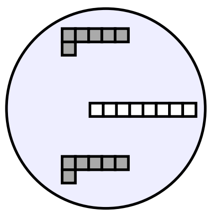
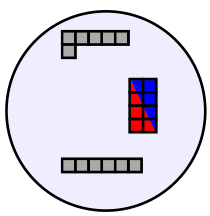
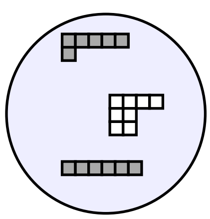
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
57		$(0,0,3,0,3)$	$(76,54)$	$Sp(2)_7^2 \times SU(2)_8^3$
58		$(0,0,3,0,4)$	$(92,65)$	$Spin(8)_{12} \times Sp(2)_7^2$
59		$(0,0,1,0,1)$	$(38,18)$	$Sp(4)_8 \times Sp(2)_7$
60		$(0,1,2,0,2)$	$(62,41)$	$Sp(3)_8 \times Sp(2)_7 \times U(1)^2$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

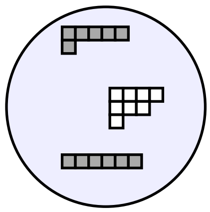
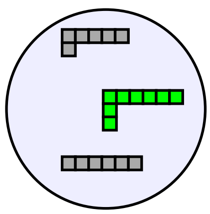
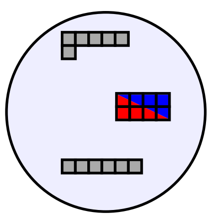
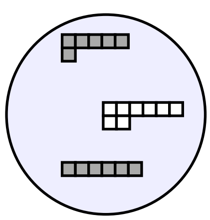
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
61		$(0, 0, 2, 0, 3)$	$(69, 47)$	$Sp(3)_8 \times Sp(2)_7 \times SU(2)_7$
62		$(0, 0, 3, 0, 3)$	$(78, 54)$	$Sp(3)_8 \times Sp(2)_8 \times Sp(2)_7$
63		$(0, 0, 3, 0, 3)$	$(78, 54)$	$Sp(3)_8 \times Sp(2)_8 \times Sp(2)_7$
64		$(0, 0, 4, 0, 3)$	$(86, 61)$	$Sp(3)_8 \times Sp(2)_7 \times SU(2)_8^3$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

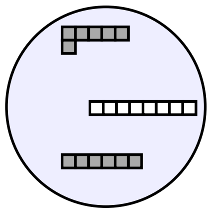
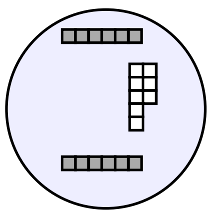
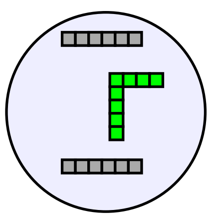
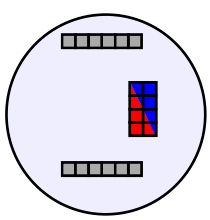
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
65		$(0, 0, 4, 0, 4)$	$(102, 72)$	$Spin(8)_{12} \times Sp(3)_8 \times Sp(2)_7$
66		$(0, 0, 1, 0, 1)$	$(40, 18)$	$Sp(6)_8$
67		$(0, 0, 2, 0, 1)$	$(48, 25)$	$Sp(6)_8 \times SU(2)_8$
68		$(0, 0, 2, 0, 1)$	$(48, 25)$	$Sp(3)_8^2 \times SU(2)_8$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

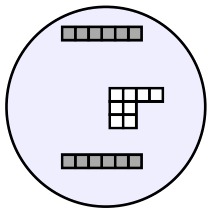
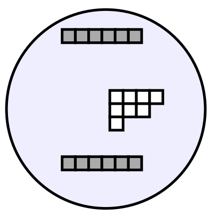
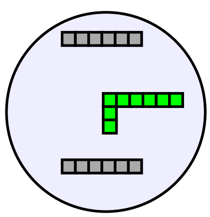
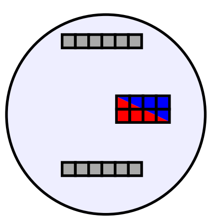
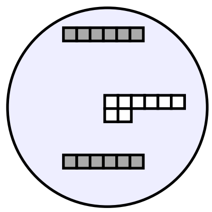
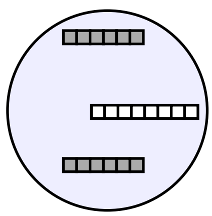
#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
69		$(0,1,3,0,2)$	$(72,48)$	$Sp(3)_8^2 \times U(1)^2$
70		$(0,0,3,0,3)$	$(79,54)$	$Sp(3)_8^2 \times SU(2)_7$
71		$(0,0,4,0,3)$	$(88,61)$	$Sp(3)_8^2 \times Sp(2)_8$
72		$(0,0,4,0,3)$	$(88,61)$	$Sp(3)_8^2 \times Sp(2)_8$

Table 2.5: $\mathbb{Z}_2$-twisted interacting fixtures

#	Fixture	$(d_2, d_3, d_4, d_5, d_6)$	(n_h, n_v)	G_{global}
73		$(0, 0, 5, 0, 3)$	$(96, 68)$	$Sp(3)_8^2 \times SU(2)_8^3$
74		$(0, 0, 5, 0, 4)$	$(112, 79)$	$Spin(8)_{12} \times Sp(3)_8^2$

2.2.2.3 Mixed Fixtures

Three new SCFTs make their appearance in the list of "mixed" fixtures (accompanied by some number of free hypermultiplets).

- The $Sp(4)_7 \times SU(2)_5$ SCFT has Coulomb branch dimensions $(d_2, ..., d_6) = (0, 0, 1, 0, 1)$ and $(n_h, n_v) = (33, 18)$.
- The $Sp(5)_7 \times SU(2)_8$ SCFT has Coulomb branch dimensions $(d_2, ..., d_6) = (0, 0, 1, 0, 1)$ and $(n_h, n_v) = (35, 18)$.
- The $Sp(3)_7 \times Sp(2)_8 \times SU(2)_5$ SCFT has Coulomb branch dimensions $(d_2, ..., d_6) = (0, 0, 2, 0, 1)$ and $(n_h, n_v) = (42, 25)$.

The remaining SCFTs in our list of mixed fixtures include the venerable $(E_6)_6$ theory, the $Sp(5)_7$ theory (which appeared in the untwisted D_4 theory [5]), two theories ($Sp(3)_5 \times SU(2)_8$ and $Spin(7)_8 \times SU(2)_5^2$) which appear above (see also [4]) and three more which appeared in the twisted A_3 theory [4].

Table 2.6: $\mathbb{Z}_2$-twisted mixed fixtures

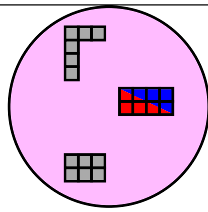
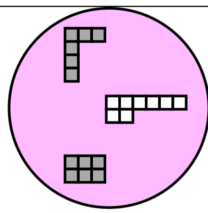
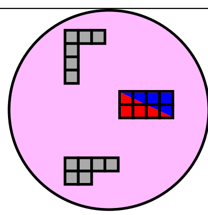
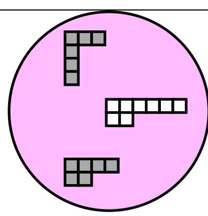
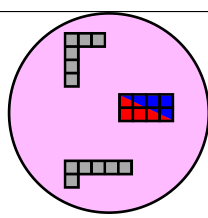
#	Fixture	Theory
1		$\frac{1}{2}(1,3,4) + Sp(3)_5 \times SU(2)_8$
2		$\frac{1}{2}(1,3;2,1,1) + SU(2)_5^2 \times Spin(7)_8$
3		$(1,1,4) + SU(2)_5 \times Sp(3)_6 \times U(1)$
4		$(1,1;2,1,1) + SU(2)_5 \times SU(4)_8 \times Sp(2)_6$
5		$\frac{1}{2}(1,1,4) + Sp(4)_7 \times SU(2)_5$

Table 2.6: $\mathbb{Z}_2$-twisted mixed fixtures

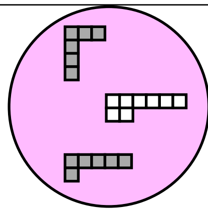
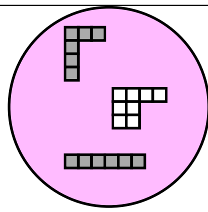
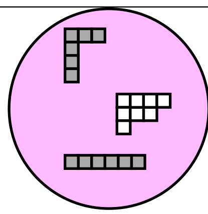
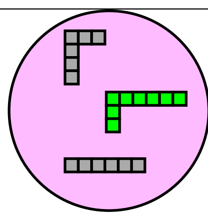
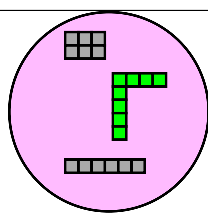
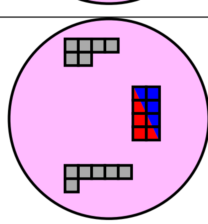
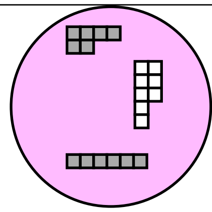
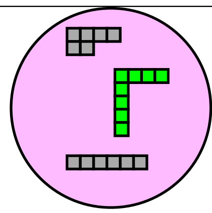
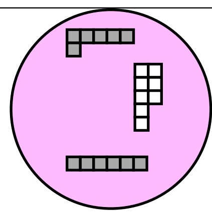
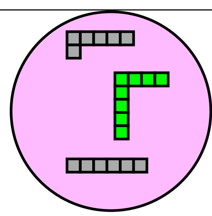
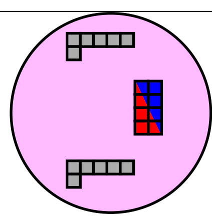
#	Fixture	Theory
6		$\frac{1}{2}(1,1;2,1,1) + Sp(3)_7 \times Sp(2)_8 \times SU(2)_5$
7		$(1,6) + SU(2)_5 \times Sp(3)_6 \times U(1)$
8		$\frac{1}{2}(1,6,1) + Sp(4)_7 \times SU(2)_5$
9		$\frac{1}{2}(1,6,1) + Sp(3)_7 \times Sp(2)_8 \times SU(2)_5$
10		$\frac{1}{2}(3,6,1) + Sp(3)_5 \times SU(2)_8$
11		$(1,1,2) + \frac{1}{2}(1,4,1) + (E_6)_6$

Table 2.6: $\mathbb{Z}_2$-twisted mixed fixtures

#	Fixture	Theory
12		$(1,6)+(E_6)_6$
13		$(1,6,1)+Sp(4)_6\times SU(2)_8$
14		$\frac{1}{2}(1,6)+Sp(5)_7$
15		$\frac{1}{2}(1,6,1)+Sp(5)_7\times SU(2)_8$
16		$\frac{1}{2}(1,1,2)+Sp(5)_7$

2.2.2.4 Gauge Theory Fixtures

For each gauge theory fixture, we list the gauge group, G, and the representation content of the hypermultiplets, $(R_{F_1}, R_{F_2}, R_{F_3}; R_G)$. Here, R_G

is the representation of the gauge group and R_{F_i} is the representation of the semisimple part of the flavour symmetry of the i^{th} puncture (where we work counterclockwise from the upper-left, and omit F_i if it is abelian or empty).

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

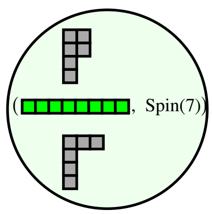
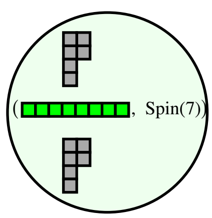
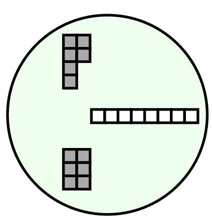
#	Fixture	$(d_2,\ldots,d_6)$	G	# Hypers	Representation
1		$(1,0,1,0,0)$	$Sp(2)$	21	$\frac{1}{2}(1,8;4)$ + $\frac{1}{2}(2,1;5)$
2		$(2,0,0,0,0)$	$SU(2)\times$ $SU(2)$	16	$\frac{1}{2}(R_1;2,1)$ $+\frac{1}{2}(R_2;1,2)$ where $R_i=8$ or $1+7$
3		$(2,0,0,0,0)$	$SU(2)\times$ $SU(2)$	24	$\frac{1}{2}(2,8_v;1,1)$ $+\frac{1}{2}(1,8;2,1)$ $+\frac{1}{2}(1,8;1,2)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

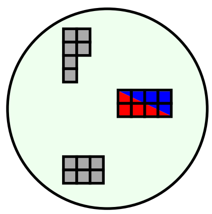
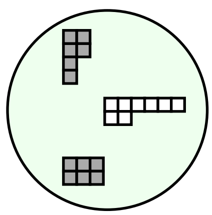
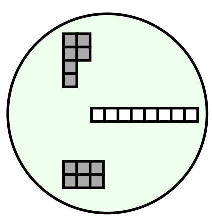
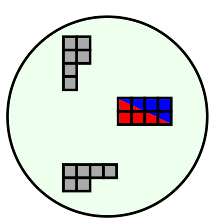
#	Fixture	$(d_2,\ldots,d_6)$	G	# Hypers	Representation
4		$(1,0,0,0,0)$	$SU(2)$	16	$\frac{1}{2}(1,5;2)$ $+\frac{1}{2}(1,4;1)$ $+\frac{1}{2}(3,4;1)$ $+\frac{1}{2}(3,1;2)$
5		$(1,0,1,0,0)$	$Sp(2)$	24	$\frac{1}{2}(2;1,2,1;4)$ $+\frac{1}{2}(2;1,1,2;4)$ $+\frac{1}{2}(1;2,1,1;5)$ $+\frac{1}{2}(3;2,1,1;1)$
6		$(1,0,1,0,1)$	$Sp(3)$	40	$\frac{1}{2}(1,8;6)$ $+\frac{1}{2}(3,1;6)$ $+\frac{1}{2}(1,1;14')$
7		$(1,1,0,0,0)$	$SU(3)$	22	$(2,1;3)$ $+(1,4;3)$ $+(1,4;1)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

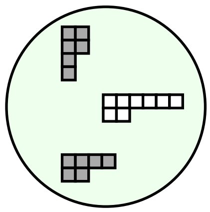
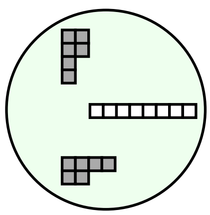
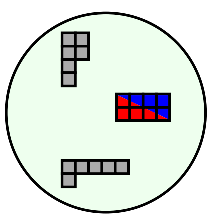
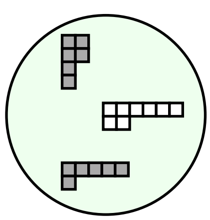
#	Fixture	$(d_2, \ldots, d_6)$	G	# Hypers	Representation
8		$(1,1,1,0,0)$	$SU(4)$	30	$\frac{1}{2}(2;1,1,1;6)$ $+\frac{1}{2}(1;2,1,1;6)$ $+(1;2,1,1;1)$ $+(1;1,2,1;4)$ $+(1;1,1,2;4)$
9		$(1,1,1,0,1)$	$Sp(3)$	46	$\frac{1}{2}(1,8;6)$ $+(1,1;6)$ $+(E_6)_6$
10		$(1,0,0,0,1)$	G_2	30	$\frac{1}{2}(4,1;7)$ $+\frac{1}{2}(1,4;7)$ $+\frac{1}{2}(1,4;1)$
11		$(1,0,1,0,1)$	$Spin(7)$	38	$\frac{1}{2}(4;1,1,1;7)$ $+\frac{1}{2}(1;2,1,1;7)$ $+\frac{1}{2}(1;1,2,1;8)$ $+\frac{1}{2}(1;1,1,2;8)$ $+\frac{1}{2}(1;2,1,1;1)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

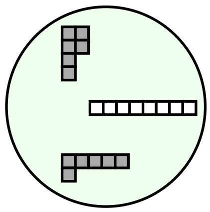
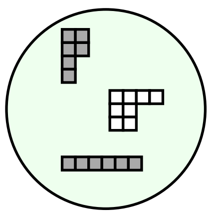
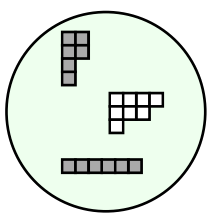
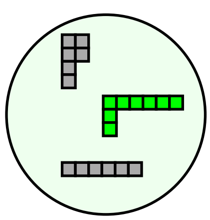
#	Fixture	$(d_2,\dots,d_6)$	G	# Hypers	Representation
12		$(1,0,1,0,2)$	$Spin(7)$	54	$\frac{1}{2}(4,1;7)$ $+ (E_8)_{12}$
13		$(1,1,0,0,0)$	$SU(3)$	24	$(6;3)$ $+ (6;1)$
14		$(1,0,0,0,1)$	G_2	31	$\frac{1}{2}(1,2;7)$ $+ \frac{1}{2}(6,1;7)$ $+ \frac{1}{2}(6,1;1)$
15		$(1,0,1,0,1)$	$Spin(7)$	40	$\frac{1}{2}(6,1;7)$ $+ \frac{1}{2}(1,4;8)$ $+ \frac{1}{2}(6,1;1)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

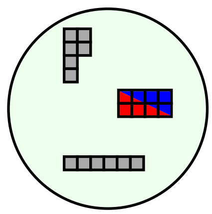
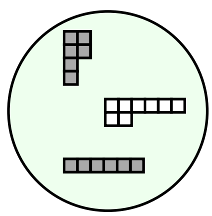
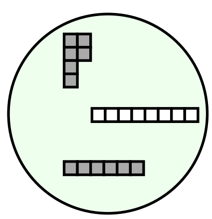
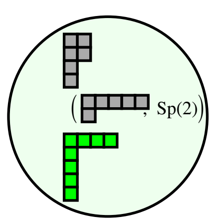
#	Fixture	$(d_2, \ldots, d_6)$	G	# Hypers	Representation
16		$(1,0,1,0,1)$	$Spin(7)$	40	$\frac{1}{2}(6,1;8)$ $+\frac{1}{2}(1,4;8)$
17		$(1,0,2,0,1)$	$Spin(8)$	48	$\frac{1}{2}(6;1,1,1;8_v)$ $+\frac{1}{2}(1;2,1,1;8_v)$ $+\frac{1}{2}(1;1,2,1;8_s)$ $+\frac{1}{2}(1;1,1,2;8_c)$
18		$(1,0,2,0,2)$	$Spin(8)$	64	$\frac{1}{2}(6,1;8)$ $+(E_8)_{12}$
19		$(1,0,0,0,0)$	$SU(2)$	10	$\frac{1}{2}(2,4;2)$ $+\frac{1}{2}(1,4;1)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

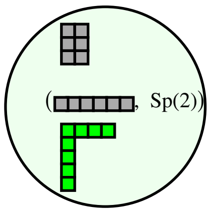
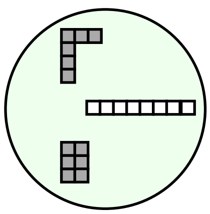
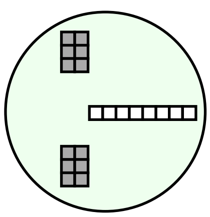
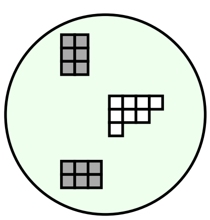
#	Fixture	$(d_2, \ldots, d_6)$	G	# Hypers	Representation
20		$(1,0,0,0,0)$	$SU(2)$	8	$\frac{1}{2}(1,2,4;2)$ or $\frac{1}{2}(1,1,5;2)$ $+\frac{1}{2}(1,3,1;2)$
21		$(1,0,1,0,0)$	$Sp(2)$	29	$\frac{1}{2}(2,1,1;5)$ $+\frac{1}{2}(1,1,8;4)$ $+\frac{1}{2}(1,2,8_v;1)$
22		$(2,0,1,0,0)$	$Sp(2) \times SU(2)$	32	$\frac{1}{2}(2,2,1;4,1)$ $+\frac{1}{2}(1,1,8_v;4,1)$ $+\frac{1}{2}(1,1,8_v;1,2)$
23		$(1,0,0,0,0)$	$SU(2)$	15	$\frac{1}{2}(2,1,2;2)$ $+\frac{1}{2}(1,3,1;2)$ $+\frac{1}{2}(1,1,1;2)$ $+\frac{1}{2}(2,3,1;1)$ $+\frac{1}{2}(1,3,2;1)$ $+\frac{1}{2}(2,1,1;1)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

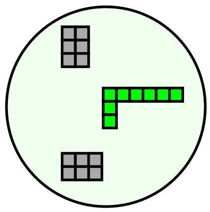
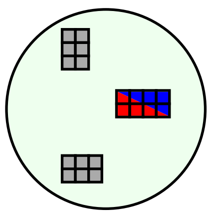
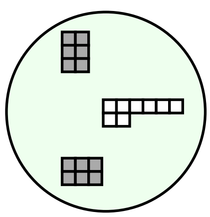
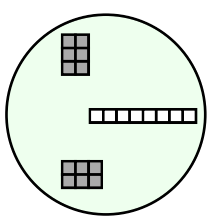
#	Fixture	$(d_2, \dots, d_6)$	G	# Hypers	Representation
24		$(1,0,1,0,0)$	$Sp(2)$	24	$\frac{1}{2}(1,2,4;4)$ $+\frac{1}{2}(2,1,1;5)$ $+\frac{1}{2}(2,3,1;1)$
25		$(1,0,1,0,0)$	$Sp(2)$	24	$\frac{1}{2}(2,2,1;4)$ $+\frac{1}{2}(1,2,4;4)$
26		$(1,0,2,0,0)$	$Sp(2)$	32	$\frac{1}{2}(1,1;1,2,2;4)$ $+(E_7)_8$
27		$(1,0,2,0,1)$	$Sp(3)$	48	$\frac{1}{2}(1,1,8;6)$ $+(E_7)_8$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

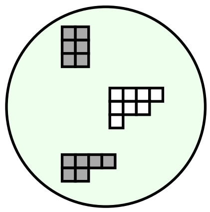
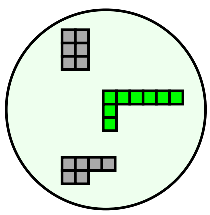
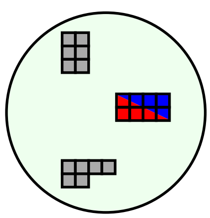
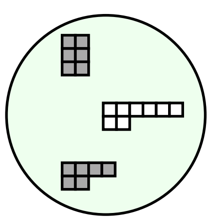
#	Fixture	$(d_2,\dots,d_6)$	G	# Hypers	Representation
28		$(1,1,0,0,0)$	$SU(3)$	21	$(2,1,1;3)$ $+(1,2,1;3)$ $+(1,1,2;3)$ $+(2,1,1;1)$ $+\frac{1}{2}(1,1,2;1)$
29		$(1,1,1,0,0)$	$SU(4)$	30	$\frac{1}{2}(1,2,1;6)$ $+\frac{1}{2}(2,1,1;6)$ $+(2,1,1;1)$ $+(1,1,4;4)$
30		$(1,1,1,0,0)$	$SU(4)$	30	$\frac{1}{2}(1,2,1;6)$ $+(2,1,1;4)$ $+(1,1,4;4)$
31		$(1,1,2,0,0)$	$SU(4)$	38	$(2,1;1,1,1;4)+$ $\frac{1}{2}(1,2;1,1,1;6)$ $+(E_7)_8$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

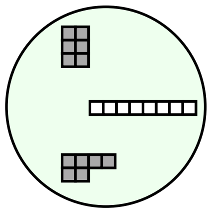
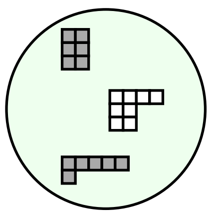
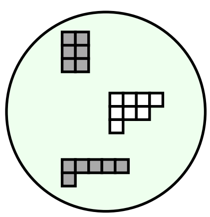
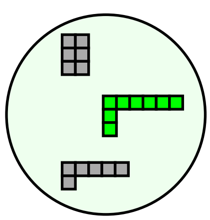
#	Fixture	$(d_2, \ldots, d_6)$	G	# Hypers	Representation
32		$(1,1,2,0,1)$	$Sp(3)$	54	$\frac{1}{2}(1,1,8;6)+$ $SU(2)_6 \times SU(8)_8$
33		$(1,1,0,0,0)$	$SU(3)$	22	$(2,1;3)$ $+\,(1,4;3)$ $+\,(2,1;1)$ $+\,\frac{1}{2}(1,4;1)$
34		$(1,0,0,0,1)$	G_2	29	$\frac{1}{2}(1,1,2;7)$ $+\,\frac{1}{2}(1,4,1;7)$ $+\,\frac{1}{2}(2,1,1;7)$ $+\,\frac{1}{2}(2,1,1;1)$
35		$(1,0,1,0,1)$	$Spin(7)$	38	$\frac{1}{2}(2,1,1;7)$ $+\,\frac{1}{2}(1,4,1;7)$ $+\,\frac{1}{2}(1,1,4;8)$ $+\,\frac{1}{2}(2,1,1;1)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

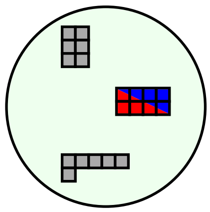
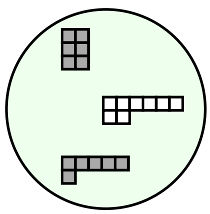
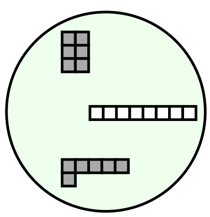
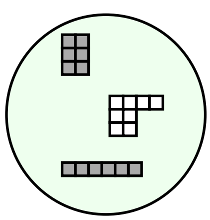
#	Fixture	$(d_2,\ldots,d_6)$	G	# Hypers	Representation
36		$(1,0,1,0,1)$	$Spin(7)$	38	$\frac{1}{2}(2,1,1;8)$ $+\frac{1}{2}(1,1,4;8)$ $+\frac{1}{2}(1,4,1;7)$
37		$(1,0,2,0,1)$	$Spin(7)$	46	$\frac{1}{2}(2,1;1,1,1;8)+$ $\frac{1}{2}(1,4;1,1,1;7)$ $+(E_7)_8$
38		$(1,0,2,0,2)$	$Spin(7)$	62	$\frac{1}{2}(1,4,1;7)+$ $Spin(16)_{12}$ $\times SU(2)_8$
39		$(1,1,1,0,0)$	$SU(4)$	32	$(2,1;4)$ $+(1,6;4)$

Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

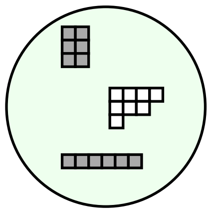
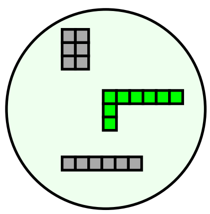
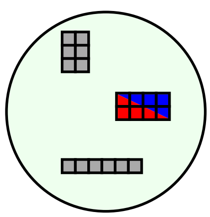
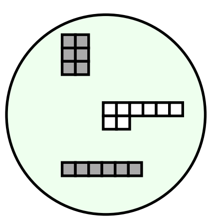
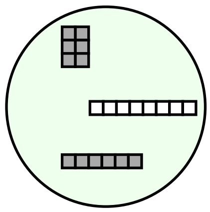
#	Fixture	$(d_2,\ldots,d_6)$	G	# Hypers	Representation
40		$(1,0,1,0,1)$	$Spin(7)$	39	$\frac{1}{2}(2,1,1;8)$ $+\frac{1}{2}(1,6,1;8)$ $+\frac{1}{2}(1,1,2;7)$
41		$(1,0,2,0,1)$	$Spin(8)$	48	$\frac{1}{2}(2,1,1;8_v)$ $+\frac{1}{2}(1,6,1;8_v)$ $+\frac{1}{2}(1,1,4;8_{s/c})$
42		$(1,0,2,0,1)$	$Spin(8)$	48	$\frac{1}{2}(2,1,1;8_{c/s})$ $+\frac{1}{2}(1,6,1;8_v)$ $+\frac{1}{2}(1,1,4;8_{s/c})$
43		$(1,0,3,0,1)$	$Spin(8)$	56	$\frac{1}{2}(2,1;1,1,1;8_{s/c})$ $+$ $\frac{1}{2}(1,6;1,1,1;8_v)$ $+(E_7)_8$

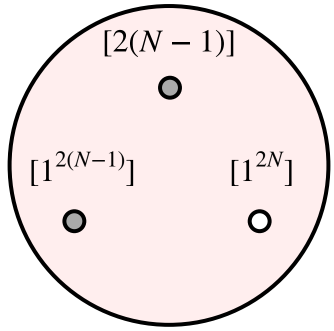
Table 2.7: $\mathbb{Z}_2$-twisted gauge theory fixtures

#	Fixture	$(d_2,\ldots,d_6)$	G	# Hypers	Representation
44		$(1,0,3,0,2)$	$Spin(8)$	72	$\frac{1}{2}(1,6,1;8_v)$ $+ Spin(16)_{12}$ $\times SU(2)_8$

2.3 Applications

2.3.1 $Spin(2N)$ and $Sp(N-1)$ Gauge Theory

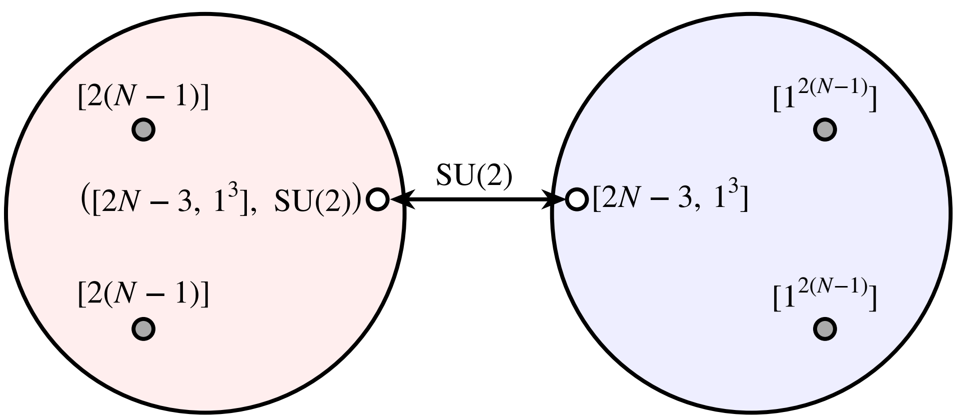
For general N, $SO(2N)$ gauge theory with $2(N-1)$ fundamental hypermultiplets, and $Sp(N-1)$ gauge theory with $2N$ fundamentals, are superconformal. Their construction is well-understood from the orientifold perspective [28, 29, 30, 31, 32]. In particular, the (2,0) theory of type D_N is the theory on $2N$ coincident M5-branes at an orientifold singularity and, in that realization of these theories [11], the key building block is the fixture consisting of a twisted-sector minimal puncture, a twisted-sector full puncture and an untwisted-sector full puncture,



$\frac{1}{2}(2(N-1),\, 2N)$

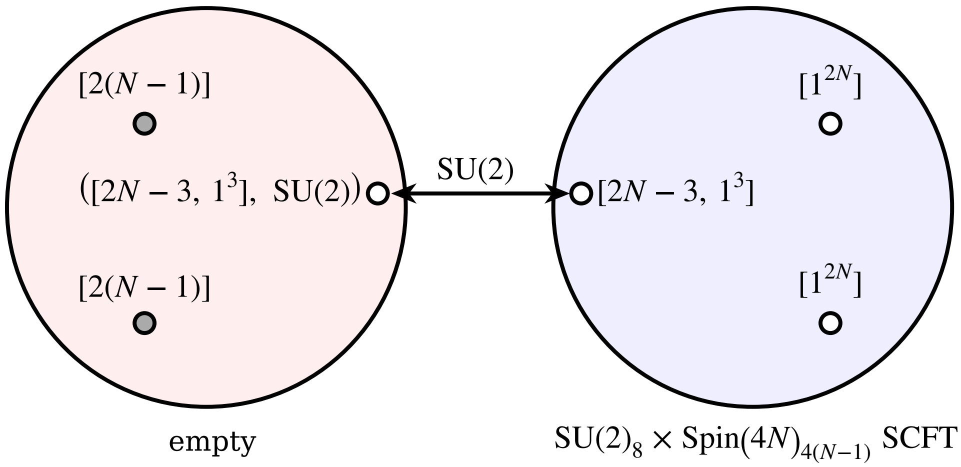
which is a free-field fixture transforming as a bifundamental half-hypermultiplet of $Sp(N-1) \times SO(2N)$. Taking two of these fixtures and connecting them with a $\overset{[1^{2N}]}{\circ}\xleftrightarrow{SO(2N)}\overset{[1^{2N}]}{\circ}$ cylinder yields the aforementioned $SO(2N)$ gauge theory. Connecting them, instead, with a $\overset{[1^{2(N-1)}]}{\bullet}\xleftrightarrow{Sp(N-1)}\overset{[1^{2(N-1)}]}{\bullet}$ cylinder yields the $Sp(N-1)$ gauge theory.

Here, we read off the S-dual strong-coupling descriptions. In the $SO(2N)$ case,



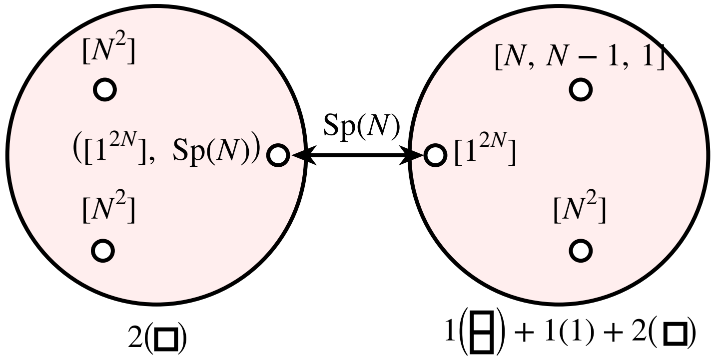
empty

$\mathrm{SU}(2)_8 \times \mathrm{Sp}(2(N-1))_{2N}$ SCFT we have an $SU(2)$ gauging of the $SU(2)_8 \times Sp\big(2(N-1)\big)_{2N}$ SCFT. In the $Sp\big(2(N-1)\big)$ case,



we have an $SU(2)$ gauging of the $SU(2)_8 \times Spin(4N)_{4(N-1)}$ SCFT.

For completeness, let us note that the other $Sp(N)$ gauge theory which is superconformal for *arbitrary* $N > 1$, namely the one with one hypermultiplet in the traceless antisymmetric tensor and four hypermultiplets in the fundamental representation, was already realized (with the addition of a single free hypermultiplet) in the untwisted sector of the A_{2N-1} theory [3]



For this theory, by contrast, all the degeneration limits are (isomorphic) weakly-coupled Lagrangian field theories. The flavour symmetry group for this family of field theories is $F = SU(2)_{2N^2-N-1} \times Spin(8)_{2N}$. As is the case for $SU(2)$, $N_f = 4$, the S-duality, which acts as an S_3 symmetry on $\mathcal{M}_{0,4}$, acts as outer automorphisms of the $Spin(8)$ flavour symmetry. Moreover, the Seiberg-Witten curve takes the absurdly simple form

$$0 = \lambda^{2N} + \sum_{k=1}^{N} u_{2k}\, \eta^k \lambda^{2(N-k)}$$

where the quadratic differential

$$\eta(z) = \frac{z_{13} z_{24} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

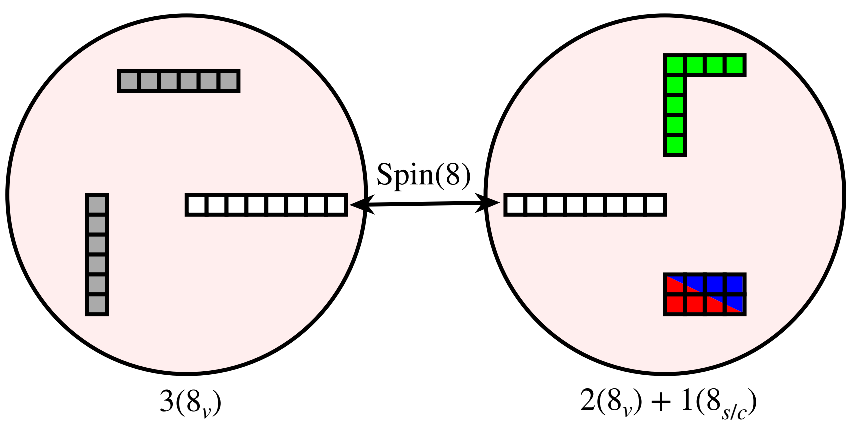
2.3.2 $Spin(8)$, $Spin(7)$ and $Sp(3)$ Gauge Theory

2.3.2.1 $Spin(8)$ Gauge Theory

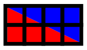
$Spin(8)$ gauge theory, with matter in the $n_v(8_v) + n_s(8_s) + n_c(8_c)$, is superconformal for $n_v + n_s + n_c = 6$. Up to permutations, related to triality, the list of possible values for n_v, n_s, n_c is quite short and we discussed most of them in [5]. There were, however, two cases which were not realizable with only untwisted sector punctures.

One is $n_v = 6$, which is a special case of the construction in §4.4.1. The other case is $n_v = 5$, $n_s = 1$ (which, as we shall presently see, lies in the same moduli space as $n_v = 5$, $n_c = 1$).

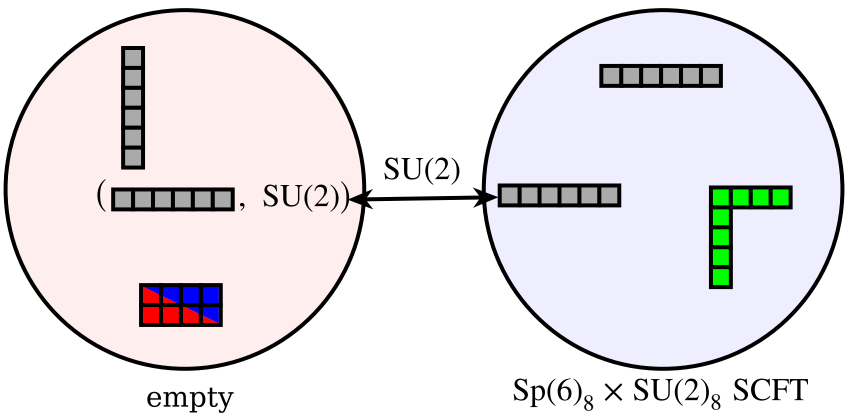
Consider the 4-punctured sphere



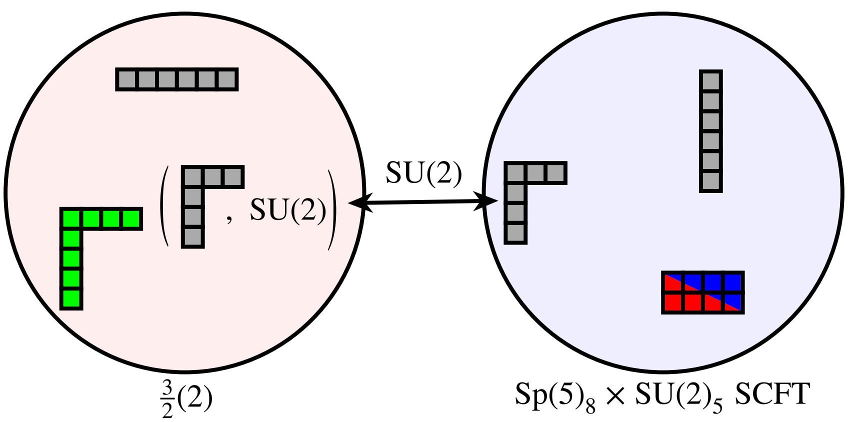
This is a weakly-coupled $Spin(8)$ gauge theory with matter in either the $5(8_v)+1(8_s)$ or the $5(8_v)+1(8_c)$. The two realizations are exchanged by dragging the

 puncture around one of the twisted-sector punctures and returning it to its original location.

The strong coupling limits are $SU(2)$ gauge theories



(where we gauge an $SU(2)$ subgroup of $Sp(6)_8$) and



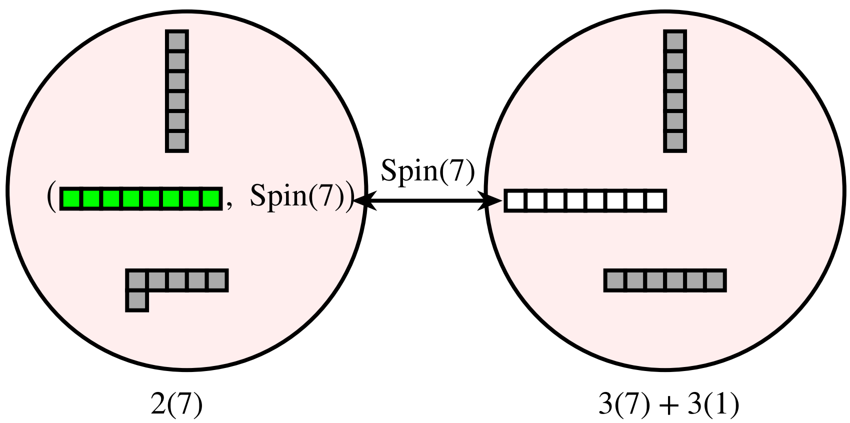
where the $SU(2)_5$ is gauged.

2.3.2.2 $Spin(7)$ Gauge Theory

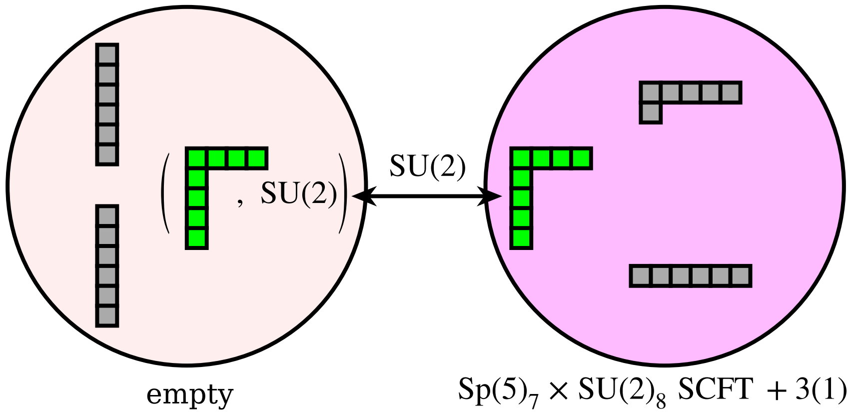
Similar to the case of $Spin(8)$ gauge theory, realizations of most cases of conformally-invariant $Spin(7)$ gauge theory were already discussed in [5]. Here we show realizations of the missing two cases.

5(7)

With the addition of three free hypermultiplets, we have a realization of the theory with 5 hypermultiplets in the vector representation as

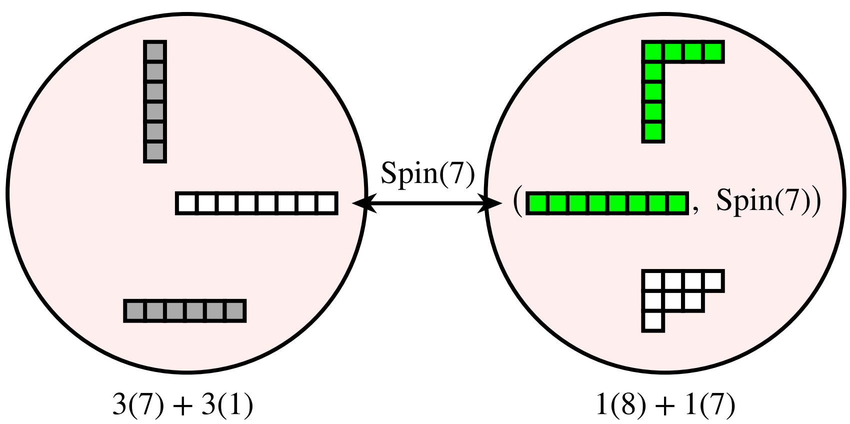


The S-dual theory is an $SU(2)$ gauging of the $Sp(5)_7 \times SU(2)_8$ SCFT, plus 3 free hypermultiplets.

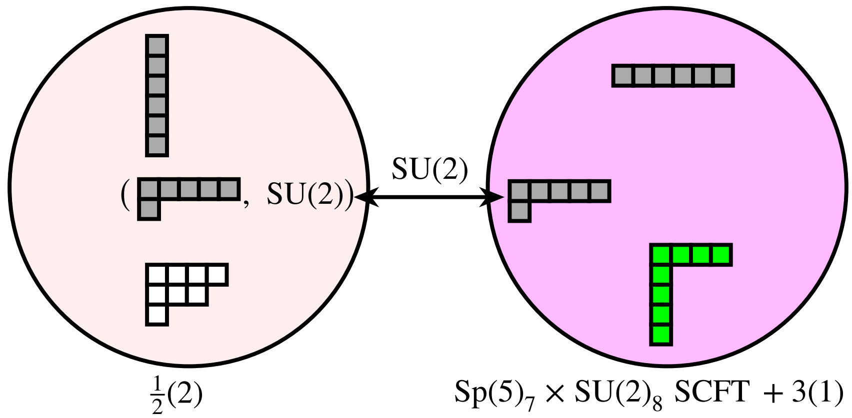


$1(8) + 4(7)$

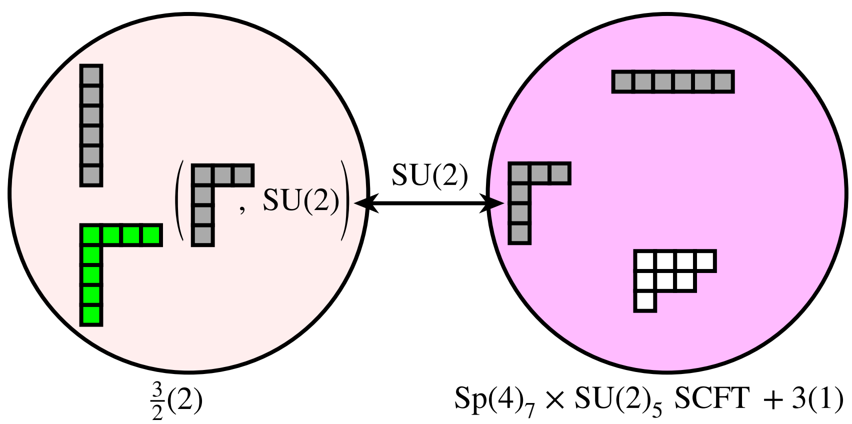
The $Spin(7)$ gauge theory, with one spinor and four vectors, can be realized in a couple of different ways. With the addition of three free hypermultiplets, we have



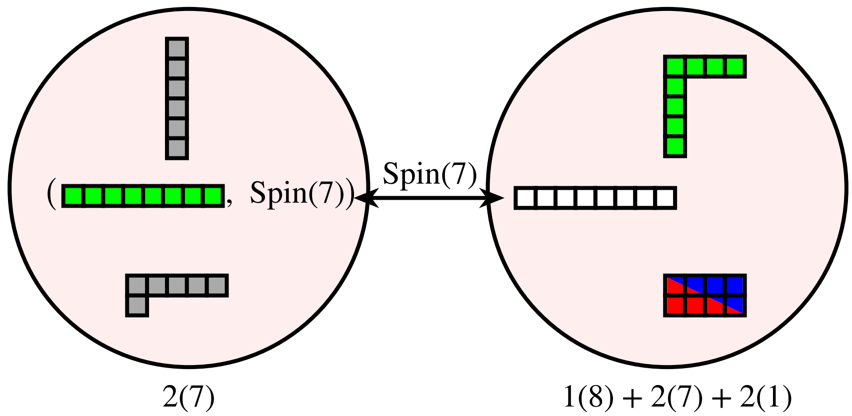
There are two S-dual descriptions. Both are $SU(2)$ gauge theories; one with a half-hypermultiplet in the fundamental, gauging an $SU(2)$ subgroup of the $Sp(5)$ symmetry of the $Sp(5)_7 \times SU(2)_8$ SCFT,



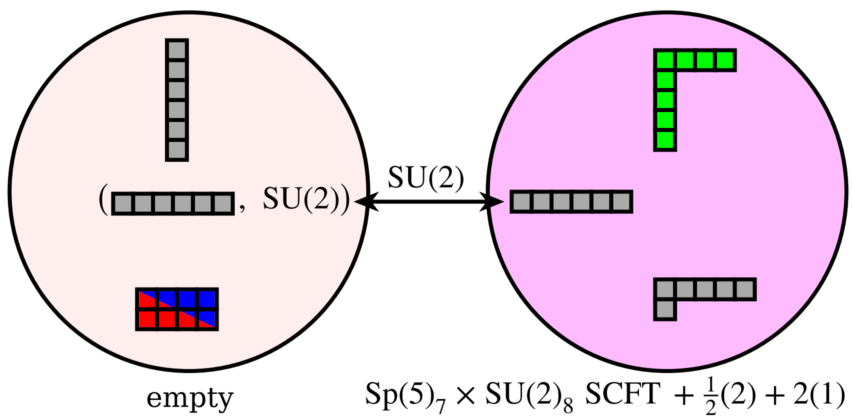
the other with three half-hypermultiplets in the fundamental, gauging the $SU(2)_5$ of the $Sp(4)_7 \times SU(2)_5$ SCFT



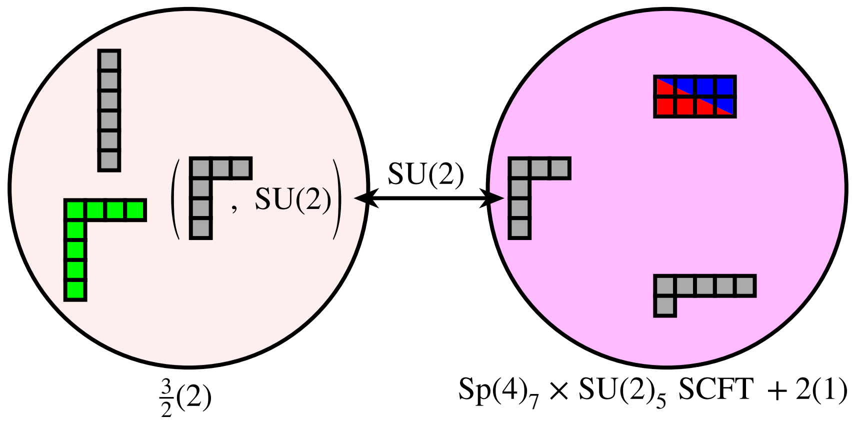
Another realization, with the addition of only two free hypermultiplets, is



where the S-dual theories are



and

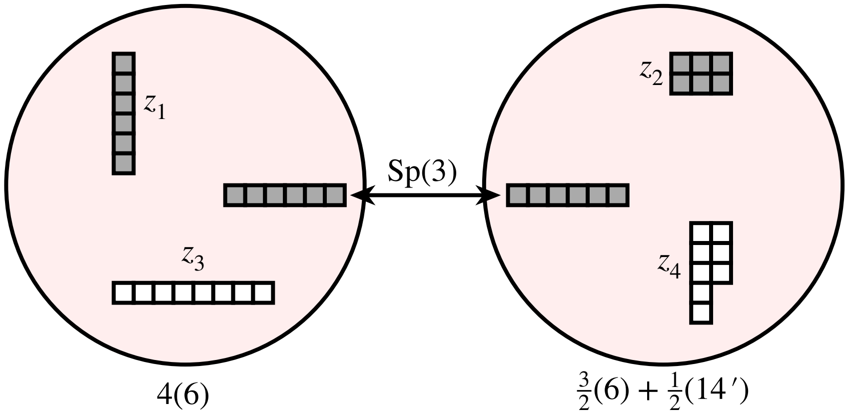


2.3.2.3 $Sp(3)$ Gauge Theory

In this section, we will consider various cases of $Sp(3)$ gauge theory, with vanished β-function. We have already discussed the theory with 8(6) and the theory with $1(14) + 4(6)$ (special cases of the discussion of §4.4.1).

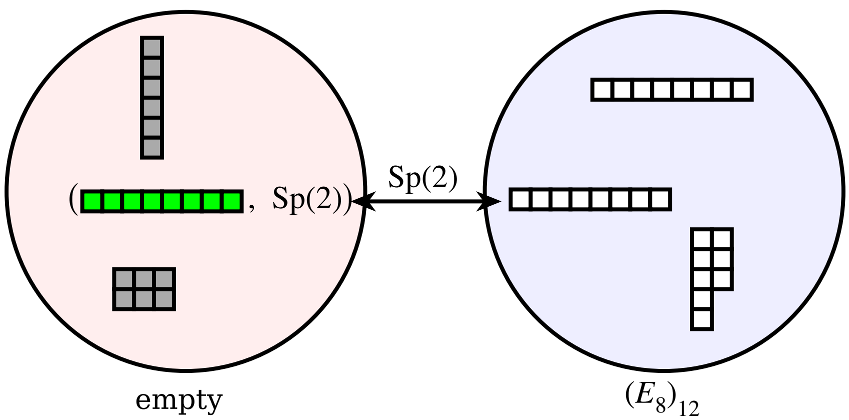
The $14'$, the traceless 3-index antisymmetric tensor representation, is pseudoreal and has index $\ell = 5$. So we can replace five fundamental (half-)hypermultiplets with a $14'$ (half-)hypermultiplet.

<u>$\frac{11}{2}(6) + \frac{1}{2}(14')$</u> With one half-hypermultiplet in the $14'$, we have

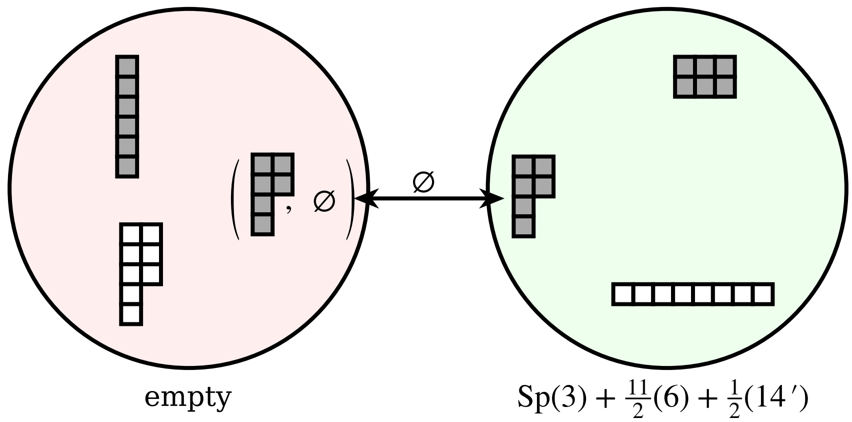


(2.12)

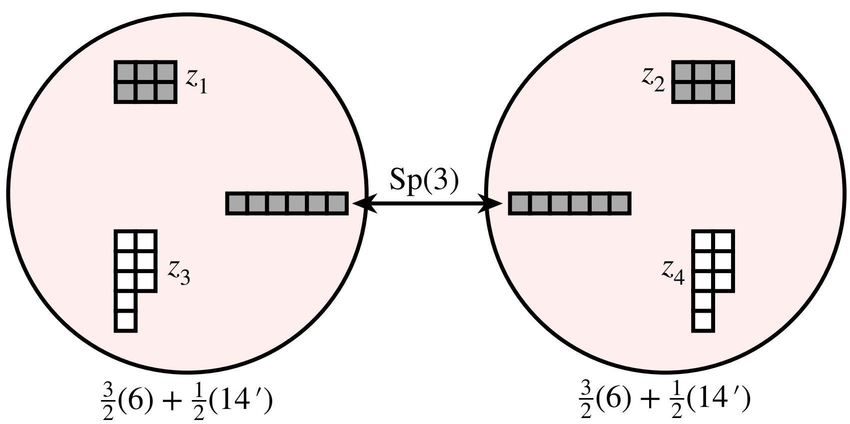
At strong coupling, we have an $Sp(2)$ gauging of the $(E_8)_{12}$ SCFT



The third boundary point involves a gauge-theory fixture

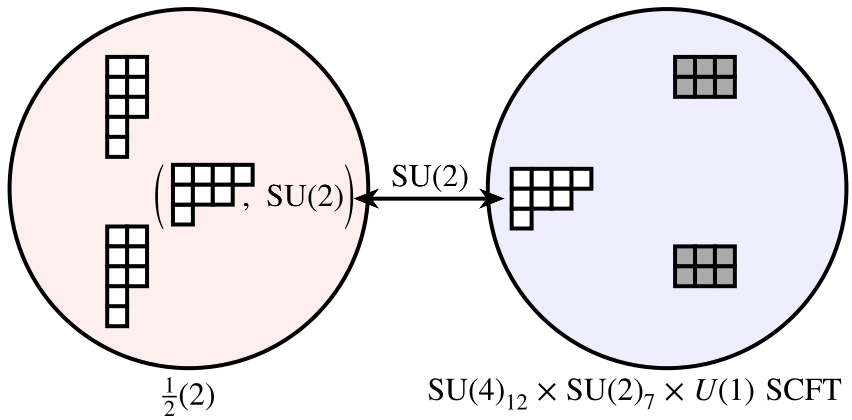


$\underline{3(6)+1(14')}$ With two half- or one full-hypermultiplet in the $14'$, we have



(2.13)

whose S-dual is an $SU(2)$ gauging of the $SU(4)_{12} \times SU(2)_7 \times U(1)$ SCFT, with an additional half-hypermultiplet in the fundamental:



Because, to our knowledge, the Seiberg-Witten solution to this theory has not been studied in the literature, let us present some of the details, here. Setting the locations of the punctures on $C = \mathbb{CP}^1$ as in (2.13), the Seiberg-Witten curve is the locus in T^*C given by the equation

$$0 = \lambda^8 + \sum_{k=1}^{3} \lambda^{8-2k} \phi_{2k}(z) + \left(\tilde{\phi}(z)\right)^2 \tag{2.14}$$

where $\lambda = ydz$ is the Seiberg-Witten differential. In the case at hand,

$$\begin{aligned}
\phi_2(z) &= \frac{u_2 z_{14} z_{23} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{z_{14} z_{23} \left[\frac{1}{4} u_2^2 (z-z_1)(z-z_2) z_{14} z_{23} + u_4 (z-z_3)(z-z_4) z_{12}^2\right] (dz)^4}{(z-z_1)^3 (z-z_2)^3 (z-z_3)^2 (z-z_4)^2} \\
\phi_6(z) &= \frac{u_6 z_{14}^2 z_{23}^2 z_{12}^2 (dz)^6}{(z-z_1)^4 (z-z_2)^4 (z-z_3)^2 (z-z_4)^2} \\
\tilde{\phi}(z) &= 0
\end{aligned}$$

Setting $(z_1, z_2, z_3, z_4) \to (0, \infty, x, 1)$, (2.14) simplifies to

$$0 = y^2 \left[y^6 + y^4 \frac{u_2}{z(z-1)(z-x)} \right.$$
$$\left. + y^2 \frac{1}{z(z-1)(z-x)} \left(\frac{\frac{1}{4}u_2^2}{(z-1)(z-x)} + \frac{u_4}{z^2} \right) + \frac{u_6}{z^4 (z-1)^2 (z-x)^2} \right] \quad (2.15)$$

The S-duality group of this theory is $\Gamma(2)$, and we have $f(\tau) = x$.

Repeating the analysis for (2.12), we find the Seiberg-Witten curve for $Sp(3)$ with $\frac{11}{2}(6) + \frac{1}{2}(14')$ to be

$$0 = y^2 \left[y^6 + y^4 \frac{u_2}{z(z-1)(z-x)} \right.$$
$$\left. + y^2 \frac{1}{z(z-1)(z-x)^3} \left(\frac{1}{4}u_2^2 \frac{(x-1)}{(z-1)} + u_4 \right) + \frac{u_6 (x-1)}{z (z-1)^2 (z-x)^5} \right] \quad (2.16)$$

In this case, the moduli space is the branched double-cover of $\mathcal{M}_{0,4}$, parametrized by $w^2 = x$. The gauge coupling is

$$f(\tau) = \frac{2w}{1+w}$$

In particular, the S-duality group is the $\Gamma_0(2)$, generated by

$$T : \tau \mapsto \tau + 1, \quad ST^2S : \tau \mapsto \frac{\tau}{1 - 2\tau} \quad .$$

Here, T acts as the deck transformation, $w \mapsto -w$, and ST^2S acts trivially on the w-plane. The theory at $f(\tau) = 0$ is the Lagrangian field theory; at $f(\tau) = 1, \infty$ (which project to $x = 1$) we have the $Sp(2)$ gauging of the $(E_8)_{12}$ SCFT. The gauge theory fixture, at $x = \infty$, is the theory at the $\mathbb{Z}_2$-invariant interior point of the moduli space, $f(\tau) = 2$.

Other cases The remaining cases of $Sp(3)$ with vanishing β-function have matter in the

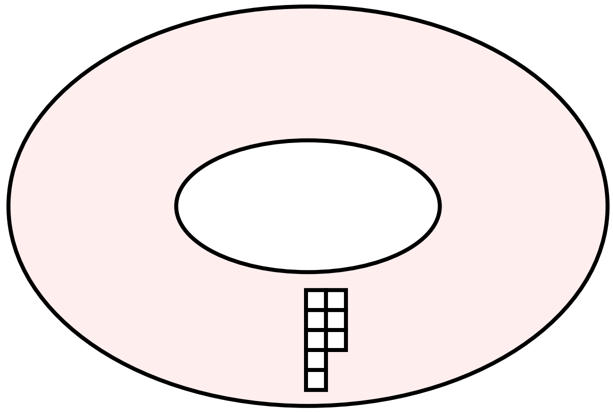
- $2(14)$
- $\frac{3}{2}(6) + 1(14) + \frac{1}{2}(14')$
- $\frac{1}{2}(6) + \frac{3}{2}(14')$

Unfortunately, we don't know how to realize these theories as compactifications from 6 dimensions. Presumably, the methods of [33] can be applied, to recover these cases as well.

2.3.3 Higher Genus

In almost all of the discussion in this paper, we have taken C to be genus-zero. We should close with at least one example of higher-genus, so that we can see the effect of twists around handles of C.

Consider a genus-one curve, with one minimal puncture, in the D_4 theory.



$H^1(T^2 - p, \mathbb{Z}_2) = (\mathbb{Z}_2)^2$. Under the action of the modular group, $H^1(T^2 - p, \mathbb{Z}_2)$ breaks up into two orbits: the zero orbit (the "untwisted theory") and the nonzero orbit ("the twisted theory").

The untwisted theory is a $Spin(8)$ gauging of the $(E_8)_{12}$ SCFT. There are three inequivalent index-2 embeddings of $Spin(8)$ in E_8. They can be characterized by how the 248 decomposes (up to outer automorphisms of $Spin(8)$). Either

$$248 = 3(1) + 5(28) + 35_v + 35_s + 35_c \tag{2.17a}$$

or

$$248 = 1 + 2(8_v) + 3(28) + 35_v + 2(56_v) \tag{2.17b}$$

or

$$248 = 8_v + 8_s + 8_c + 2(28) + 56_v + 56_s + 56_c \tag{2.17c}$$

The untwisted theory corresponds to (2.17a). The twisted theory, depending on the S-duality frame chosen, corresponds either to a $Spin(8)$ gauging of the $(E_8)_{12}$ SCFT using the embedding (2.17b), or to an $Sp(3)$ gauging of the $Sp(6)_8$ SCFT.

For the untwisted theory, the gauge theory moduli space is the fundamental domain for $PSL(2,\mathbb{Z})$ in the UHP, and τ is the modular parameter of the torus. For the twisted theory, the moduli space of the gauge theory is the moduli space of pairs (C,γ), where γ is a nonzero element of $H^1(C,\mathbb{Z}_2)$. This is the fundamental domain of $\Gamma_0(2)$, as discussed in §2.1.3.

Chapter 3

$Spin(n)$ Gauge Theories with Spinors

3.1 Introduction

$\mathcal{N} = 2$ supersymmetric $Spin(n)$ gauge theory, with $n-2$ hypermultiplets in the vector representation, is superconformal for any $n > 2$, and the Seiberg-Witten solutions are known from the mid 1990's [34, 35]. Replacing some number of vectors by hypermultiplets in spinor representations is only possible for sufficiently low n. The corresponding Seiberg-Witten solutions do not seem to be known[1]. For $Spin(5) \simeq Sp(2)$ and Spin(6) $\simeq SU(4)$, the solutions were presented in [3, 4]. The solutions to $Spin(7), Spin(8)$ appeared in our previous papers [5, 17] (see [33] for an alternative formulation). As a further application of [5, 17], we will discuss $Spin(n)$ gauge theories for $n = 9, 10, \ldots, 14$, with matter content such that $\beta = 0$. These are all of the remaining cases where one can have matter in the spinor representation. For $n > 14$, only matter in the vector representation is compatible with $\beta \leq 0$ [2].

These 4D gauge theories can be obtained by compactifying [1, 6] a 6D

[1] The solutions (with arbitrary masses for the vector and spinor hypermultiplets) of the asymptotically-free theories for $n = 8, 10, 12$ were constructed in [36]. The status of Seiberg-Witten solutions, to *various* $\mathcal{N} = 2$ supersymmetric gauge theories, was recently reviewed in [37].

[2] This chapter is based on [38].

(2,0) theory of type D_N on a 4-punctured sphere, where the punctures are labeled by nilpotent orbits in $\mathfrak{d}_N$ (or in $\mathfrak{c}_{N-1}$ for twisted-sector punctures) [5, 7, 17, 11, 18]. When the 4-punctured sphere degenerates into a pair of 3-punctured spheres ("fixtures"), connected by a long thin cylinder, the gauge theory description is weakly-coupled. Fixtures with only hypermultiplets in the vector representation are, necessarily, twisted. With at least one (half-)hypermultiplet in the spinor representation, we can find an untwisted fixture and — wherever possible — we prefer to work in the untwisted theory.

From these realizations as 4-punctured spheres, we construct the corresponding Seiberg-Witten geometries, and discuss the strong-coupling S-dual realizations [2] of the gauge theories.

3.2 Seiberg-Witten Geometry

3.2.1 Seiberg-Witten curve

In the D_N theory, the Seiberg-Witten curve, $\Sigma \subset \mathrm{tot}(K_C)$, is the spectral curve (in the vector representation) for D_N. In other words, it can be written as the locus

$$0 = \lambda^{2N} + \phi_2(z)\lambda^{2N-2} + \phi_4(z)\lambda^{2N-4} + \cdots + \phi_{2N-2}(z)\lambda^2 + \tilde{\phi}(z)^2 \qquad (3.1)$$

where the Seiberg-Witten differential, $\lambda = y dz$, is the tautological 1-form on K_C. Σ is a branched cover of C, of rather high genus. But it admits an obvious involution $\iota\colon \lambda \to -\lambda$. The quotient by this involution is a curve $\tilde{C}$, also a branched cover of C. One finds[3] that $g(\Sigma) - g(\tilde{C}) = N$. The SW solution is obtained by computing the periods of λ over the cycles which are anti-invariant under ι. Said differently, the fibers of the Hitchin integrable system are the Prym variety for $\Sigma \to \tilde{C}$.

For the $Spin(2N)$ gauge theories considered below, the above description is completely adequate, as $\tilde{\phi}(z)$ is nowhere-vanishing on C. For the $Spin(2N-1)$ gauge theories, $\tilde{\phi}(z)$ vanishes identically. So Σ is reducible

$$0 = \lambda^2(\lambda^{2N-2} + \phi_2(z)\lambda^{2N-4} + \phi_4(z)\lambda^{2N-6} + \cdots + \phi_{2N-2}(z)) \quad .$$

Let Σ_0 be the component

$$0 = \lambda^{2N-2} + \phi_2(z)\lambda^{2N-4} + \phi_4(z)\lambda^{2N-6} + \cdots + \phi_{2N-2}(z) \quad .$$

As before, Σ_0 admits an involution $\iota\colon \lambda \to -\lambda$, with quotient $\tilde{C}_0 = \Sigma_0/\iota$, and the SW solution, for the $Spin(2N-1)$ gauge theory, is given by the periods of λ on the anti-invariant cycles. There is one subtlety which did not occur in the previous case: $\phi_{2N-2}(z)$ typically does have zeroes on C, which means that Σ_0 is slightly singular. It has ordinary double-points over the zeroes of $\phi_{2N-2}(z)$. As in Hitchin's original paper [42], we actually work over

[3]For many purposes, it's convenient to replace Σ by the compact curve

$$0 = \lambda^{2N} + \phi_2(z)\lambda^{2N-2}\mu^2 + \phi_4(z)\lambda^{2N-4}\mu^4 + \cdots + \phi_{2N-2}(z)\lambda^2\mu^{2N-2} + \tilde{\phi}(z)^2\mu^{2N}$$

in $\mathrm{tot}(P(K_C \oplus \mathcal{O}))$. Away from the punctures, $\mu \neq 0$ and we can scale it to 1. At the punctures, $\mu = 0$, and the SW curve has interesting ramification over the punctures. The A_{N-1} case [39, 40] is explained in detail in [41]. The generalization to D_N has a few subtleties, which we won't attempt to explicate here.

the resolutions[4], $\hat{\Sigma}_0 \to \tilde{C}_0$, whose Prym variety has the desired dimension, $g(\hat{\Sigma}_0) - g(\tilde{C}_0) = N - 1$.

3.2.2 Calabi-Yau geometry

An alternative formulation [43, 44], more directly related to the Type-IIB description of these 4D theories is as follows. Consider a family of non-compact Calabi-Yau 3-folds, $X_{\vec{u}}$, realized as the hypersurface

$$0 = w^2 + yx^2 - y^{N-1} - \phi_2(z)y^{N-2} - \phi_4(z)y^{N-3} - \cdots - \phi_{2N-2}(z) - 2\tilde{\phi}(z)x$$

in the total space of the bundle $V = \left(K_C^{(N-1)} \oplus K_C^{(N-2)} \oplus K_C^2\right) \to C$. Here, $\vec{u}$ are the Coulomb branch parameters, on which the $\phi_k(z)$ depend, and

$$w = \tilde{w}(dz)^{N-1}, \quad x = \tilde{x}(dz)^{N-2}, \quad y = \tilde{y}(dz)^2$$

are the tautological differentials on V. The $g_s \to 0$ limit of Type IIB on $\mathbb{R}^{3,1} \times X_{\vec{u}}$ is the $4D\ \mathcal{N} = 2$ field theory (decoupled from the bulk gravity).

$X_{\vec{u}}$ has a collection of 3-cycles of the form of an S^2 in the fiber over a curve on C. The Seiberg-Witten solutions to the $Spin(2N)$ theories below are constructed from the periods of the holomorphic 3-form,

$$\Omega = \frac{d\tilde{x} \wedge d\tilde{y} \wedge dz}{\tilde{w}}$$

over a (rational) symplectic basis of these 3-cycles. For the $Spin(2N-1)$ theories, $\tilde{\phi}(z) \equiv 0$, and $X_{\vec{u}}$ has an involution $\iota\colon (w, x) \to (-w, -x)$, under

[4] In the D_4 theory, there are examples of $Spin(8)$ gauge theory, with matter in the $n_s(8_s) + n_c(8_c) + (6 - n_s - n_c)(8_v)$, where $\tilde{\phi}(z)$ has isolated zeroes on C. Over those points, Σ has ordinary double points and, similarly, we work on the resolution, $\hat{\Sigma}$.

which Ω is invariant. ι acts by exchanging two of the S^2s in the fiber (fixing the rest). Integrating Ω over the invariant cycles yields the $2(N-1)$ periods which comprise the solution for the $Spin(2N-1)$ theories.

3.2.3 Dependence on the gauge coupling

The Seiberg-Witten solutions to the $\beta = 0$ gauge theories, which are our focus, have elaborate (but holomorphic) dependence [45] on the complexified gauge coupling

$$\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2} \quad .$$

In particular, *any* such theory, which can be realized by compactifying the (2,0) theory on a 4-punctured sphere, *automatically* has a symmetry under $\Gamma(2) \subset PSL(2,\mathbb{Z})$, generated by

$$T^2 : \tau \mapsto \tau + 2, \quad ST^2S : \tau \mapsto \frac{\tau}{1-2\tau} \quad .$$

That is, the dependence on the gauge coupling is through the function

$$\begin{aligned} f(\tau) &\equiv -\frac{\theta_2^4(0,\tau)}{\theta_4^4(0,\tau)} \\ &= -\left(16q^{1/2} + 128q + 704q^{3/2} + \dots\right) \end{aligned}$$

where $q = e^{2\pi i \tau}$.

In the untwisted theory, $f(\tau)$ is simply identified with the cross-ratio of the 4-punctured sphere:

$$f(\tau) = x \equiv \frac{z_{13}z_{24}}{z_{14}z_{23}} \quad . \tag{3.2}$$

The limit $x \to 0$ is the usual weak-coupling limit. $x \to 1$ and $x \to \infty$ are limits which admit an alternative (physically-distinct) S-dual description as a weakly coupled gauge theory.

When the punctures at z_1 and z_2 are identical, then the theory has a larger symmetry under $\Gamma_0(2) \supset \Gamma(2)$, where the extra generator acts on the x-plane as

$$S : x \mapsto \frac{1}{x} \quad .$$

The theories, below, with two (one full and one minimal) twisted punctures and two untwisted punctures, have a similar story, except that the relation between $f(\tau)$ (which parametrizes the gauge theory moduli space) and the cross-ratio is more complicated. The gauge theory moduli space is a branched double-cover [17] of the moduli space of the 4-punctured sphere, $\mathcal{M}_{0,4}$. Instead of (3.2),

$$w^2 = x \equiv \frac{z_{13} z_{24}}{z_{14} z_{23}} \tag{3.3}$$

and the gauge coupling

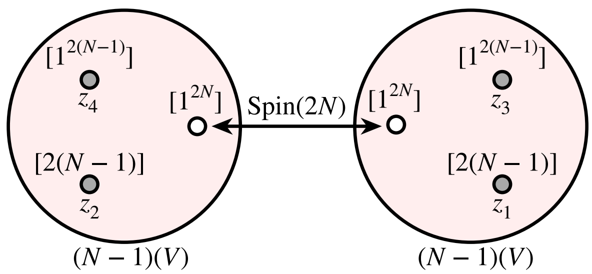
$$f(\tau) = \frac{w-1}{w+1} \quad . \tag{3.4}$$

In particular, this means that $x \to 0$ corresponds to $f(\tau) \to -1$ (i.e. $\tau \to i$), which is an *interior* point of the gauge theory moduli space and intrinsically strongly coupled. As in our previous works on the twisted sector [4, 17], we denote these peculiar degenerations as involving a “gauge theory fixture.” The other degeneration limits have more prosaic interpretations. The limit $f(\tau) \to 1$ projects to $x \to \infty$ and the limits $f(\tau) \to 0$ and $f(\tau) \to \infty$ (which have isomorphic physics) both project to $x \to 1$.

In presenting the solutions, below, we write the dependence on the positions of the four punctures in a manifestly $PSL(2,\mathbb{C})$-invariant form. For calculational purposes, it is invariably easier to fix the $PSL(2,\mathbb{C})$ symmetry by setting $(z_1, z_2, z_3, z_4) = (0, \infty, x, 1)$.

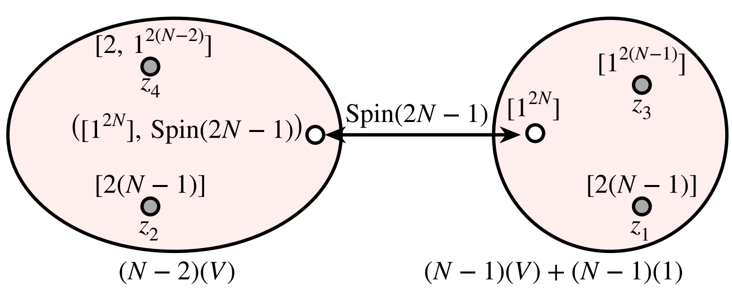
3.3 $Spin(2N) + (2N-2)(V)$ and $Spin(2N-1) + (2N-3)(V)$

Just as $Spin(2N)$ gauge theory with $2(N-1)$ fundamentals is realized as the compactification of the D_N theory with four $\mathbb{Z}_2$-twisted punctures



, (3.5)

there is a universal realization of $Spin(2N-1)$ with $2N-3$ fundamentals plus $(N-1)$ free hypermultiplets as a four-punctured sphere in the (twisted) D_N theory



. (3.6)

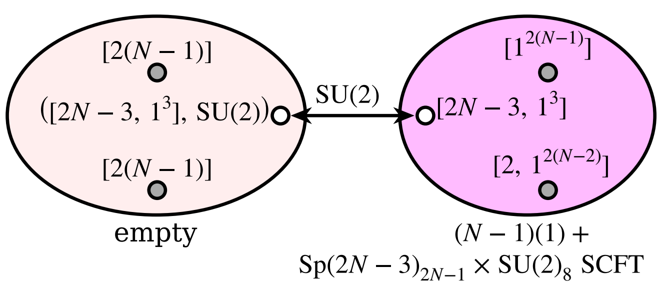
The Seiberg-Witten curve corresponding to (3.5) takes the form of (3.1) where the invariant k-differentials are

$$\begin{aligned}\phi_{2k}(z) &= \frac{u_{2k}\, z_{14} z_{23} z_{34}^{2(k-1)} (dz)^{2k}}{(z-z_1)(z-z_2)(z-z_3)^{2k-1}(z-z_4)^{2k-1}}\\ \tilde{\phi}(z) &= \frac{\tilde{u}\, z_{14}^{1/2} z_{23}^{1/2} z_{34}^{N-1} (dz)^N}{(z-z_1)^{1/2}(z-z_2)^{1/2}(z-z_3)^{(2N-1)/2}(z-z_4)^{(2N-1)/2}}\quad .\end{aligned}$$

The Seiberg-Witten curve for (3.6) takes the same form, but with $\tilde{\phi} \equiv 0$.

This pattern will repeat, in many of the examples below. The $Spin(2N-1)$ theory, with the same number of hypermultiplets in the spinor, but one fewer in the vector representation, is obtained by replacing the puncture at z_4, with one where the last box in the Young diagram is shifted to a new row. Physically, this corresponds to using one of the vector hypermultiplets to Higgs $Spin(2N) \to Spin(2N-1)$. The "surprise" is that integrating out the massive modes has such a simple effect on the Coulomb branch geometry.

The strong-coupling dual of (3.6) is an $SU(2)$ gauging of the $Sp(2N-3)_{2N-1} \times SU(2)_8$ SCFT, with $N-1$ additional free hypermultiplets



.

These theories have vanishing β-function for any N.

Including hypermultiplets in spinor representations will follow a similar pattern, where we will realize $Spin(2N-1)$ and $Spin(2N)$ gauge theories as 4-punctured spheres in the D_N theory. The Seiberg-Witten curve for each of these theories takes the form (3.1). We list the invariant k-differentials for each theory below.

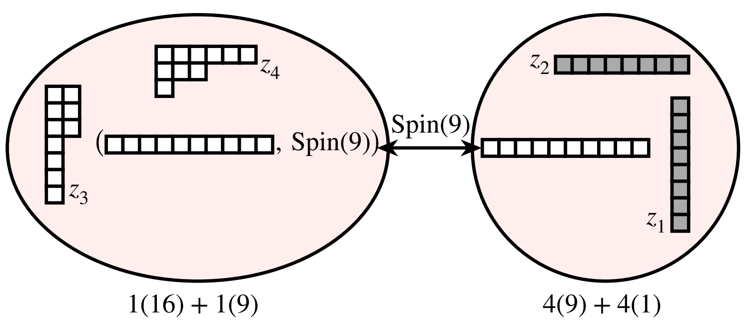
As we saw above, the solutions for $Spin(2N-1)$ is obtained from the corresponding $Spin(2N)$ theory (i.e, the theory with the same number of spinors (ignoring their chirality, for N even) and one more vector) by setting $\tilde{u} = 0$.

3.4 $Spin(9)$ and $Spin(10)$ Gauge Theories

All of the following arise in the D_5 theory, possibly with $\mathbb{Z}_2$-twisted punctures.

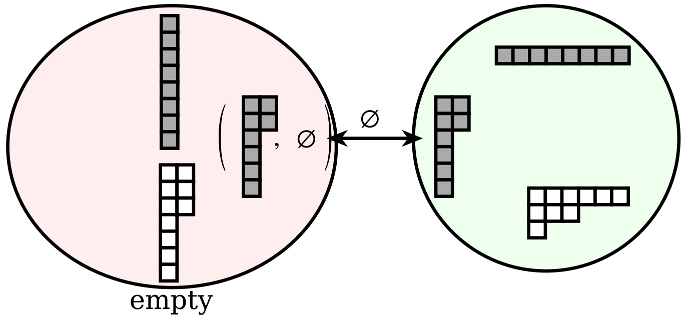
3.4.1 $Spin(9)$

3.4.1.1 $Spin(9) + 1(16) + 5(9)$

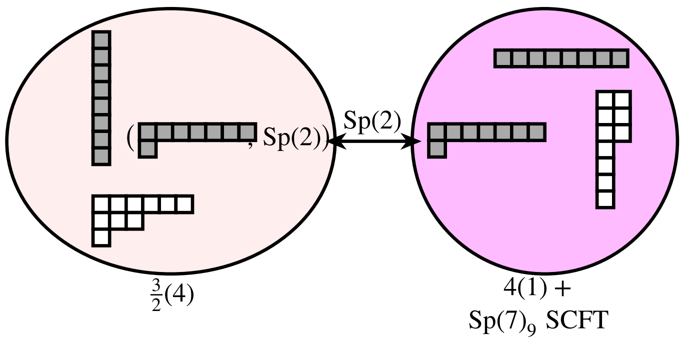


. (3.7)

The other degeneration limits yield a gauge theory fixture

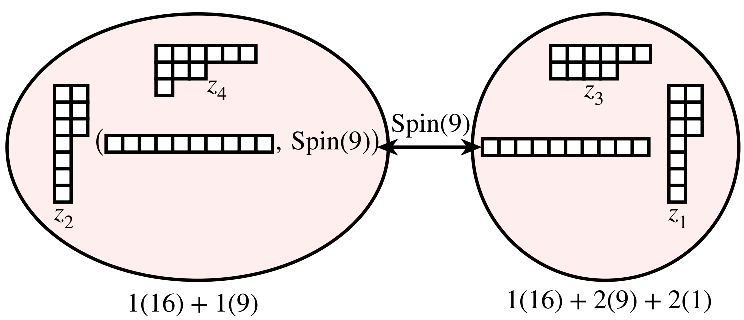


and an $Sp(2)$ gauging of the $Sp(7)_9\,\mathrm{SCFT} + \frac{3}{2}(4) + 4(1)$



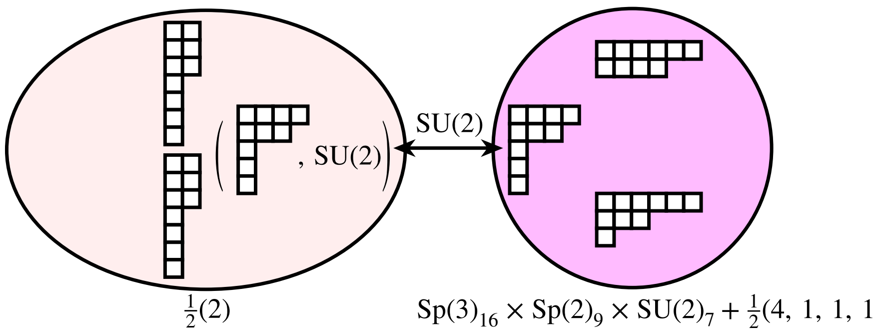
.

3.4.1.2 $Spin(9) + 2(16) + 3(9)$



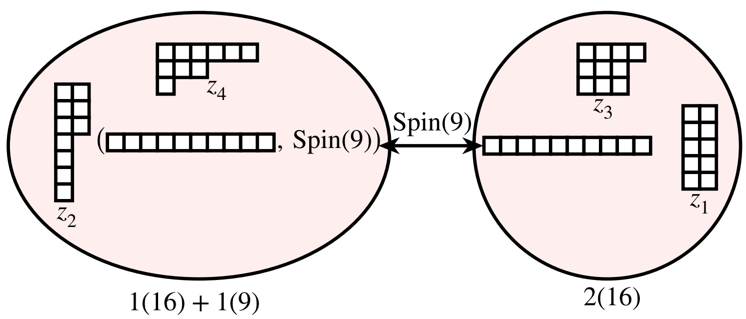
. (3.8)

The S-dual theory is an $SU(2)$ gauging of the $Sp(3)_{16} \times Sp(2)_9 \times SU(2)_7$SCFT+ $\frac{1}{2}(2) + 2(1)$



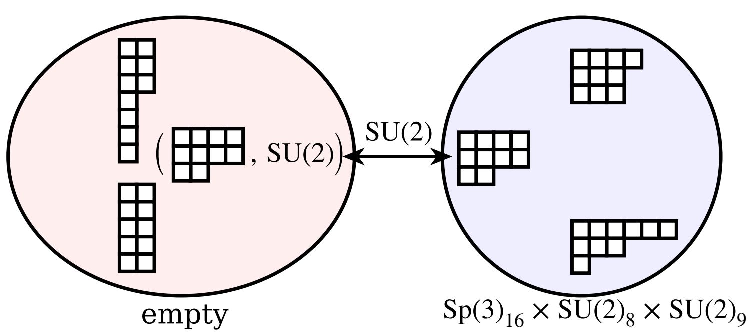
.

3.4.1.3 $Spin(9) + 3(16) + 1(9)$

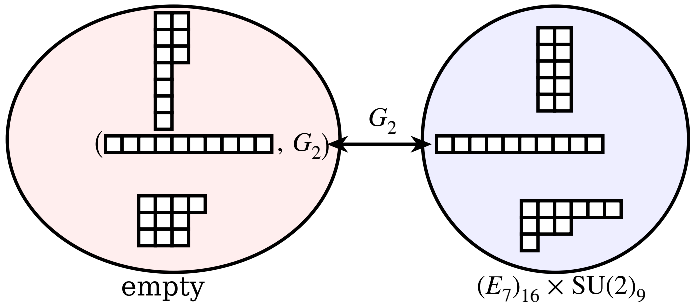


. (3.9)

The S-dual theories are an $SU(2)$ gauging of the $Sp(3)_{16} \times SU(2)_8 \times SU(2)_9$ SCFT



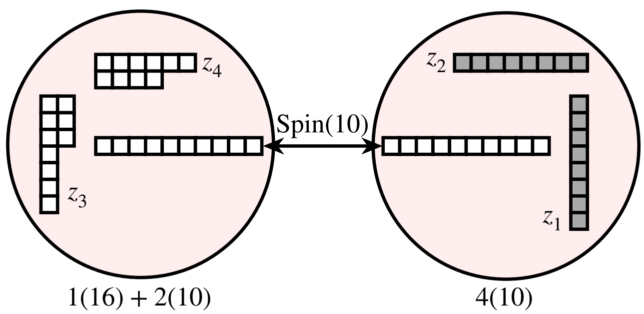
and a G_2 gauging of the $(E_7)_{16} \times SU(2)_9$ SCFT[5]



.

3.4.2 $Spin(10)$

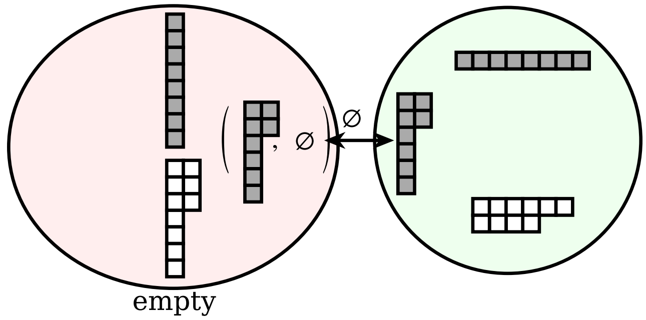
3.4.2.1 $Spin(10) + 1(16) + 6(10)$



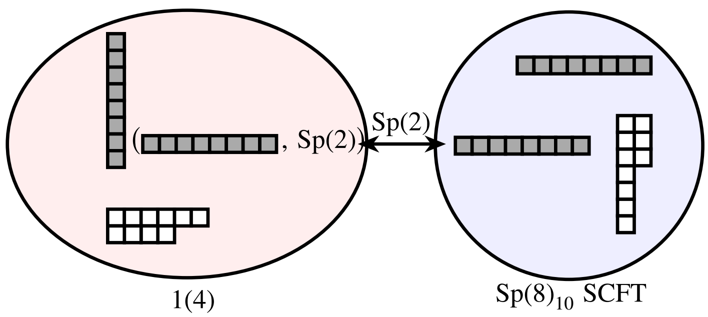
. (3.10)

[5]This interacting fixture is another realization of the $(E_7)_{8n} \times SU(2)_{(n-1)(4n+1)}$ SCFT, which arises on the world volume of n D3-branes probing a III* singularity in F-theory (see [46, 47, 48] and §5.3 of [49]).

The other degenerations involve a gauge theory fixture



and an $Sp(2)$ gauging of the $Sp(8)_{10}\,\mathrm{SCFT} + 1(4)$



.

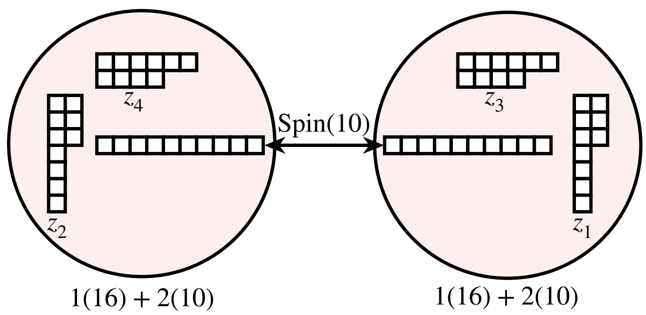
The invariant k-differentials for (3.10) are given by

$$\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{13} z_{24} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\\
\phi_4(z) &= \frac{\left[u_4\,(z-z_3) z_{24} - \frac{1}{4} u_2^2\,(z-z_2) z_{34}\right] z_{13} z_{24}^2 (dz)^4}{(z-z_1)(z-z_2)^3(z-z_3)^2(z-z_4)^3}\\
\phi_6(z) &= \frac{u_6\, z_{13} z_{23} z_{24}^4 (dz)^6}{(z-z_1)(z-z_2)^5(z-z_3)^2(z-z_4)^4}\\
\phi_8(z) &= \frac{u_8\, z_{13} z_{23} z_{24}^6 (dz)^8}{(z-z_1)(z-z_2)^7(z-z_3)^2(z-z_4)^6}\\
\tilde{\phi}(z) &= \frac{\tilde{u}\, z_{13}^{1/2} z_{23}^{1/2} z_{24}^4 (dz)^5}{(z-z_1)^{1/2}(z-z_2)^{9/2}(z-z_3)(z-z_4)^4} \quad .
\end{aligned} \tag{3.11}$$

The gauge theory moduli space is a branched double-cover of $\mathcal{M}_{0,4}$ and the gauge couplings are given by (3.4).

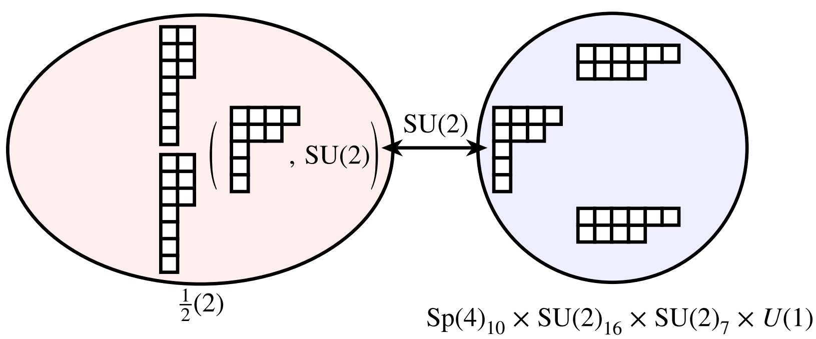
The invariant k-differentials for (3.7) are as above, but with $\tilde{\phi} \equiv 0$.

3.4.2.2 $Spin(10) + 2(16) + 4(10)$



(3.12)

The S-dual is an $SU(2)$ gauging of the $Sp(4)_{10} \times SU(2)_{16} \times SU(2)_7 \times U(1)$ SCFT+ $\frac{1}{2}(2)$



.

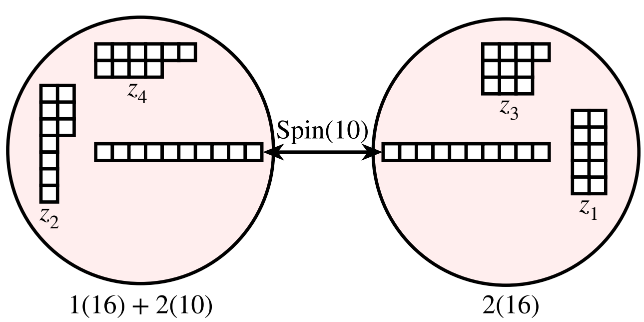
The invariant k-differentials for (3.12) are given by

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\\
\phi_4(z) &= \frac{[u_4\,(z-z_1)(z-z_2)z_{34} + \frac{1}{4}u_2^2\,((z-z_2)(z-z_3)z_{14}}{(z-z_1)^2(z-z_2)^2(z-z_3)^3(z-z_4)^3}\\
&\quad \frac{-(z-z_1)(z-z_4)z_{23})]z_{12}z_{34}^2(dz)^4}{}\\
\phi_6(z) &= \frac{u_6\, z_{12}^2 z_{34}^4 (dz)^6}{(z-z_1)^2(z-z_2)^2(z-z_3)^4(z-z_4)^4}\\
\phi_8(z) &= \frac{u_8\, z_{12}^2 z_{34}^6 (dz)^8}{(z-z_1)^2(z-z_2)^2(z-z_3)^6(z-z_4)^6}\\
\tilde{\phi}(z) &= \frac{\tilde{u}\, z_{12} z_{34}^4 (dz)^5}{(z-z_1)(z-z_2)(z-z_3)^4(z-z_4)^4} \quad .
\end{aligned}
\tag{3.13}
$$

The k-differentials for (3.8) are as above, but with $\tilde{\phi} \equiv 0$.

Since we are in the untwisted theory, the gauge theory moduli space is $\mathcal{M}_{0,4}$ (or more precisely, in this case, its $\mathbb{Z}_2$ quotient), and the gauge coupling is given by (3.2).

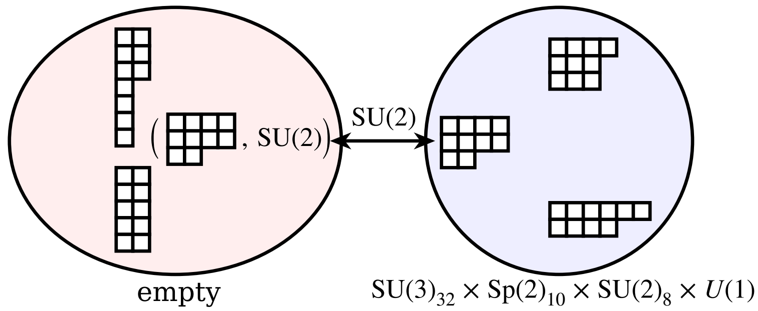
3.4.2.3 $Spin(10) + 3(16) + 2(10)$



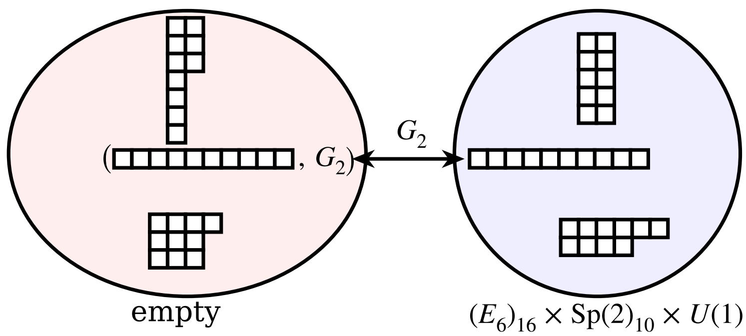
. (3.14)

The S-dual theories are an $SU(2)$ gauging of the $SU(3)_{32} \times Sp(2)_{10} \times SU(2)_8 \times$

$U(1)$ SCFT



and a G_2 gauging of the $(E_6)_{16} \times Sp(2)_{10} \times U(1)$ SCFT



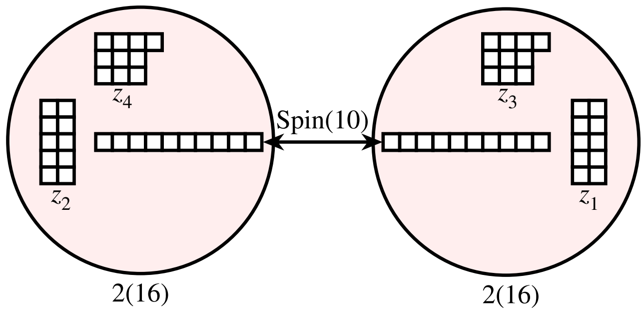
.

The invariant k-differentials for (3.14) are given by

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{\left[u_4\,(z-z_2) z_{14} + \frac{1}{4} u_2^2\,(z-z_4) z_{12}\right] z_{12} z_{34}^2 (dz)^4}{(z-z_1)^2 (z-z_2)^2 (z-z_3)^2 (z-z_4)^3} \\
\phi_6(z) &= \frac{\left[u_6\,(z-z_1) z_{34} + \frac{1}{2} u_2 u_4\,(z-z_4) z_{13}\right] z_{12}^2 z_{34}^3 (dz)^6}{(z-z_1)^3 (z-z_2)^2 (z-z_3)^4 (z-z_4)^4} \\
\phi_8(z) &= \frac{\left[u_8\,(z-z_1) z_{34} + \frac{1}{4} u_4^2\,(z-z_4) z_{13}\right] z_{14} z_{12}^2 z_{34}^4 (dz)^8}{(z-z_1)^4 (z-z_2)^2 (z-z_3)^5 (z-z_4)^6} \\
\tilde{\phi}(z) &= \frac{\tilde{u}\, z_{14} z_{12} z_{34}^3 (dz)^5}{(z-z_1)^2 (z-z_2)(z-z_3)^3 (z-z_4)^4} \quad .
\end{aligned}
\tag{3.15}
$$

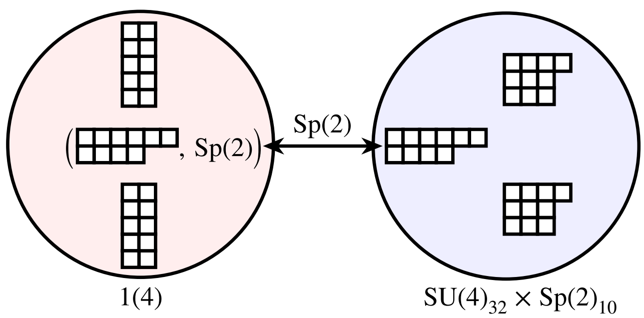
The k-differentials for (3.9) are as above, but with $\tilde{\phi} \equiv 0$.

3.4.2.4 $Spin(10) + 4(16)$



. (3.16)

The S-dual theory is an $Sp(2)$ gauging of the $SU(4)_{32} \times Sp(2)_{10}$ SCFT $+ 1(4)$



.

For this theory, the k-differentials characterizing the Seiberg-Witten curve are

$$\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{u_4\, z_{12}^2 z_{34}^2 (dz)^4}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^2} \\
\phi_6(z) &= \frac{[u_6\,(z-z_1)(z-z_2)z_{34} - \frac{1}{2}u_2(u_4-\frac{1}{4}u_2^2)((z-z_1)(z-z_3)z_{24}}{(z-z_1)^3(z-z_2)^3(z-z_3)^4(z-z_4)^4} \\
&\quad \underline{-(z-z_2)(z-z_4)z_{13})]z_{12}^2 z_{34}^3 (dz)^6} \\
\phi_8(z) &= \frac{[u_8\,(z-z_1)(z-z_2)z_{34} - \frac{1}{4}(u_4-\frac{1}{4}u_2^2)^2((z-z_1)(z-z_3)z_{24}}{(z-z_1)^4(z-z_2)^4(z-z_3)^5(z-z_4)^5} \\
&\quad \underline{-(z-z_2)(z-z_4)z_{13})]z_{12}^3 z_{34}^4 (dz)^8} \\
\tilde{\phi}(z) &= \frac{\tilde{u}\, z_{12}^2 z_{34}^3 (dz)^5}{(z-z_1)^2(z-z_2)^2(z-z_3)^3(z-z_4)^3} \quad .
\end{aligned} \tag{3.17}$$

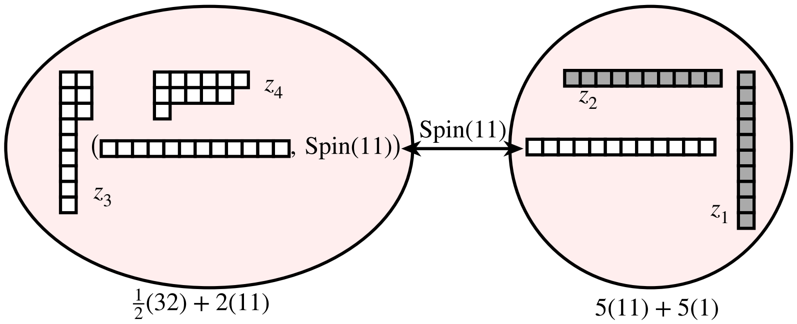
In this case, there are no hypermultiplets in the vector, which one could use to Higgs $Spin(10) \to Spin(9)$. Equivalently, it's not possible to move the last box, in the Young diagram at z_4, to a new row while keeping it a D-partition. So there is no corresponding $Spin(9)$ gauge theory.

3.5 $Spin(11)$ and $Spin(12)$ Gauge Theories

These arise in the compactification of the D_6 theory, possibly with $\mathbb{Z}_2$-twisted punctures.

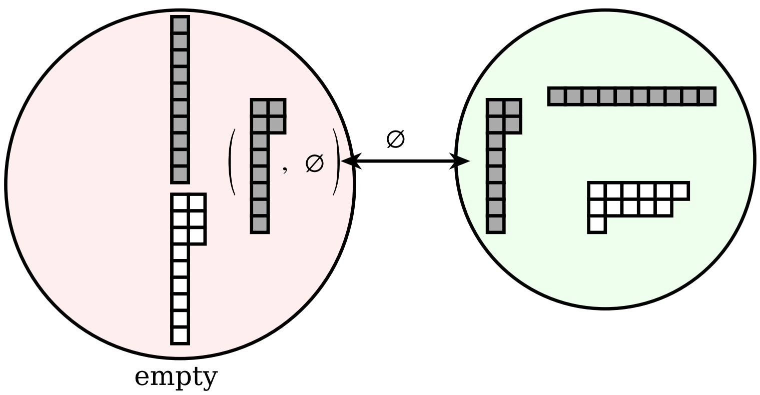
3.5.1 $Spin(11)$

3.5.1.1 $Spin(11) + \frac{1}{2}(32) + 7(11)$

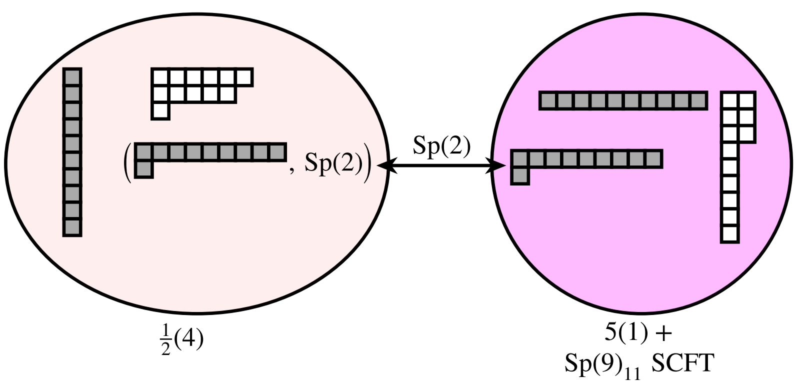


. (3.18)

The other degenerations involve a gauge theory fixture

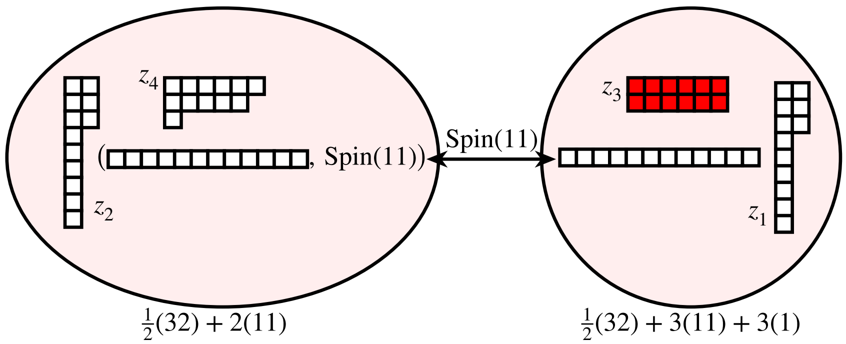


and an $Sp(2)$ gauging of the $Sp(9)_{11}$ SCFT $+ \frac{1}{2}(4) + 5(1)$



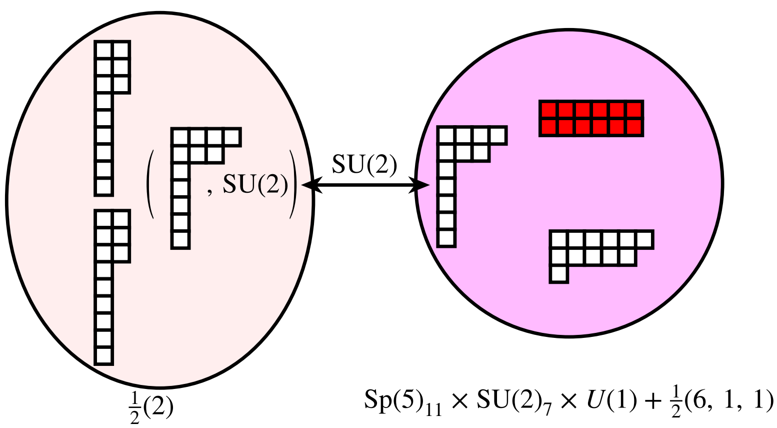
.

3.5.1.2 $Spin(11) + 1(32) + 5(11)$



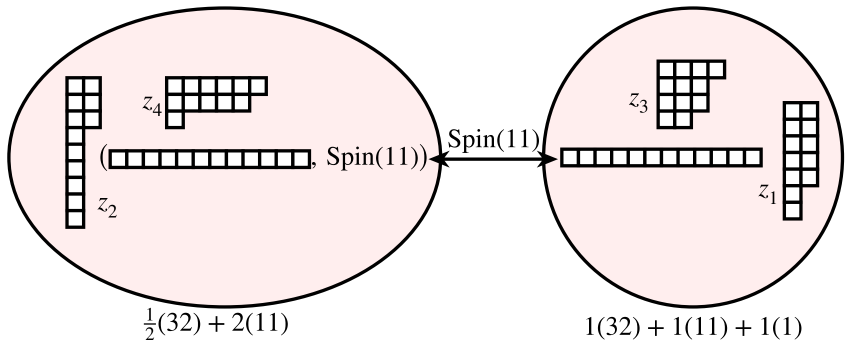
$$. \tag{3.19}$$

The S-dual theory is an $SU(2)$ gauging of the $Sp(5)_{11} \times SU(2)_7 \times U(1)$ SCFT + $\frac{1}{2}(2) + 3(1)$,



.

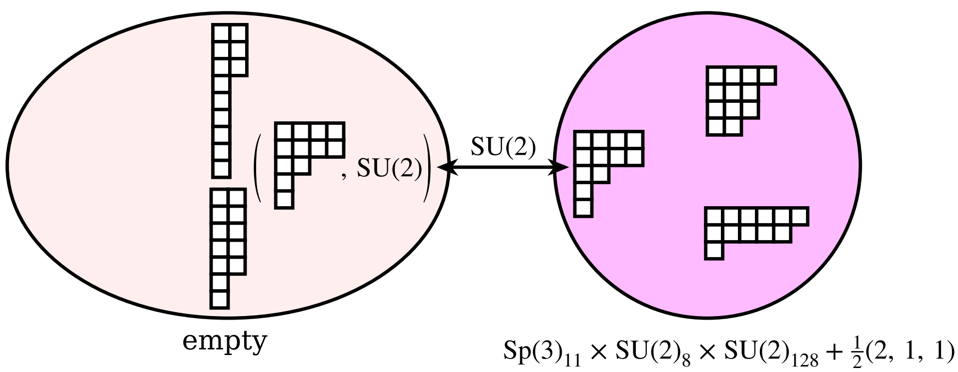
3.5.1.3 $Spin(11) + \frac{3}{2}(32) + 3(11)$



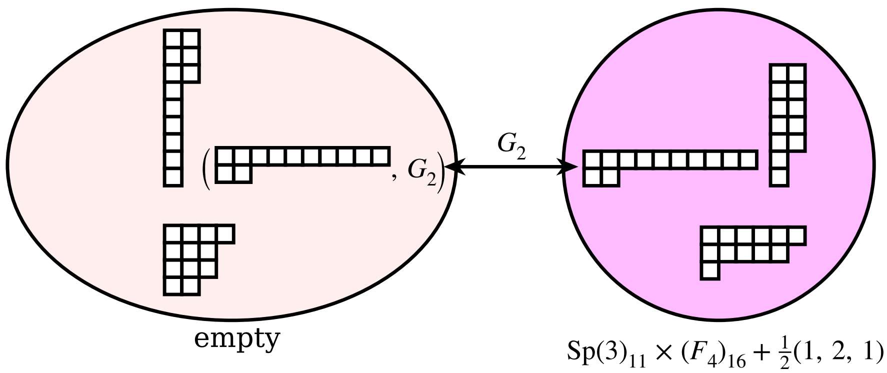
$$. \tag{3.20}$$

The S-dual theories are an $SU(2)$ gauging of the

$Sp(3)_{11} \times SU(2)_8 \times SU(2)_{128}$ SCFT $+ 1(1)$

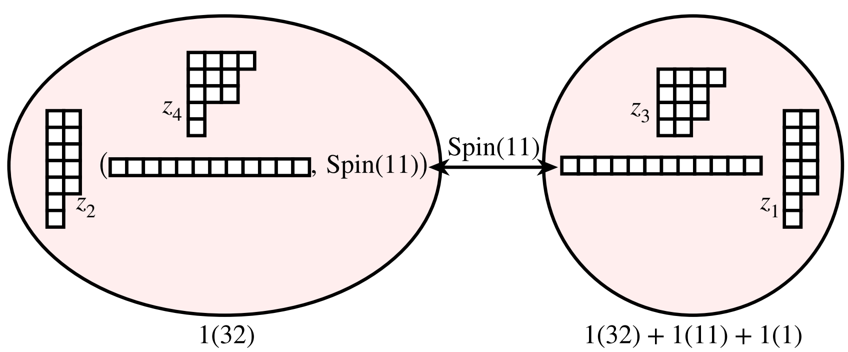


and a G_2 gauging[6] of the $Sp(3)_{11} \times (F_4)_{16}$ SCFT $+1(1)$



.

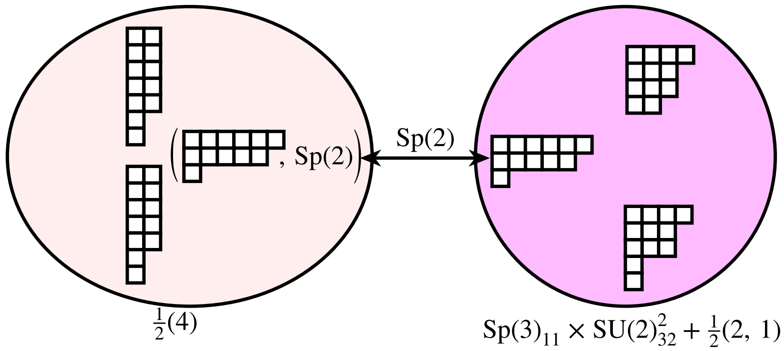
3.5.1.4 $Spin(11) + 2(32) + 1(11)$



. (3.21)

The S-dual theory is an $Sp(2)$ gauging of the $Sp(3)_{11} \times SU(2)^2_{32}$ SCFT $+\frac{1}{2}(4)+$ $1(1)$

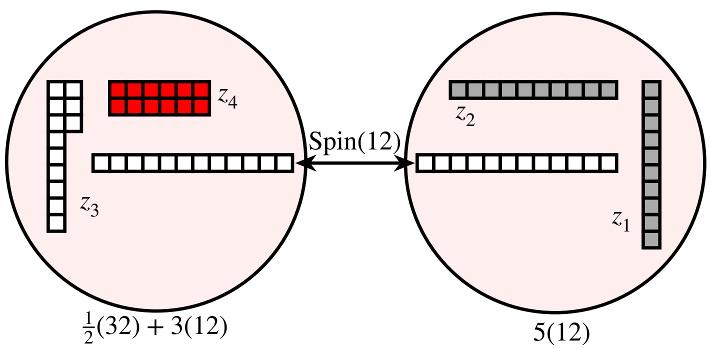
[6]Note that, here, we use the Lie algebra embedding, $(\mathfrak{f}_4)_k \supset (\mathfrak{g}_2)_k \times \mathfrak{su}(2)_{8k}$.



.

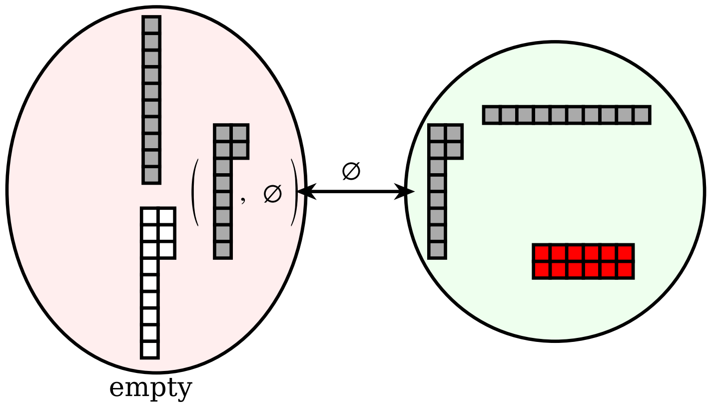
3.5.2 $Spin(12)$

3.5.2.1 $Spin(12) + \frac{1}{2}(32) + 8(12)$

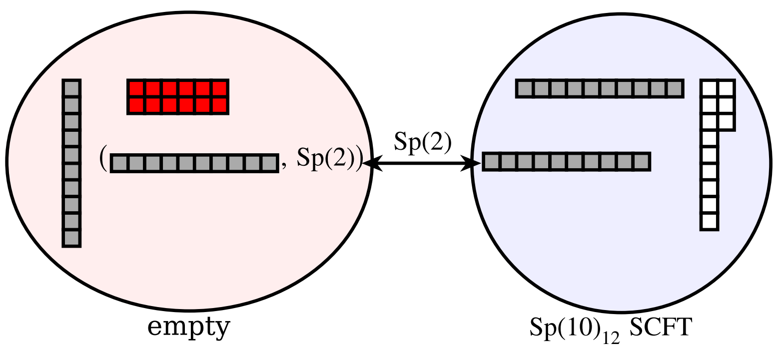


. (3.22)

The other degenerations involve an gauge theory fixture



and an $Sp(2)$ gauging of the $Sp(10)_{12}$ SCFT



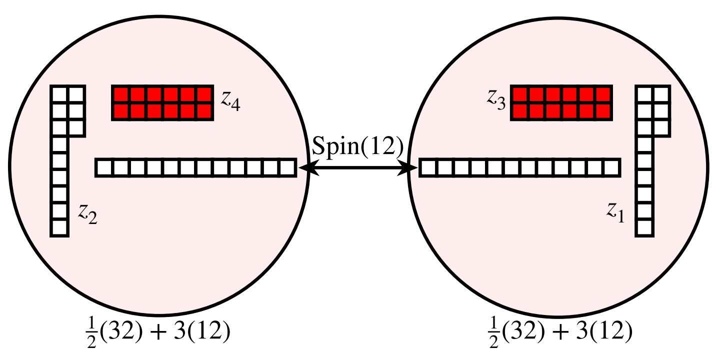
.

The invariant k-differentials for (3.22) are given by

$$\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{13} z_{24} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\\
\phi_4(z) &= \frac{\left[u_4\,(z-z_3) z_{24} - \frac{1}{4} u_2^2\,(z-z_2) z_{34}\right] z_{13} z_{24}^2 (dz)^4}{(z-z_1)(z-z_2)^3(z-z_3)^2(z-z_4)^3}\\
\phi_6(z) &= \frac{\left[u_6\,(z-z_4) z_{13} + 2\tilde{u}(z-z_3) z_{14}\right] z_{23} {z_{24}}^4 (dz)^6}{(z-z_1)(z-z_2)^5(z-z_3)^2(z-z_4)^5}\\
\phi_8(z) &= \frac{u_8\, z_{13} z_{23} z_{24}^6 (dz)^8}{(z-z_1)(z-z_2)^7(z-z_3)^2(z-z_4)^6}\\
\phi_{10}(z) &= \frac{u_{10}\, z_{13} z_{23} {z_{24}}^8 (dz)^{10}}{(z-z_1)(z-z_2)^9(z-z_3)^2(z-z_4)^8}\\
\tilde{\phi}(z) &= \frac{\tilde{u} {z_{14}}^{1/2} {z_{24}}^{9/2} z_{23} (dz)^6}{(z-z_1)^{1/2}(z-z_2)^{11/2}(z-z_3)(z-z_4)^5} \quad .
\end{aligned} \tag{3.23}$$

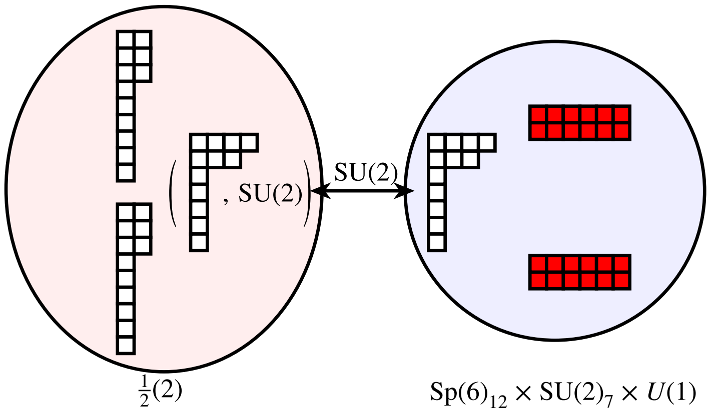
For (3.18), they are as above, but with $\tilde{u} \equiv 0$.

3.5.2.2 $Spin(12) + 1(32) + 6(12)$



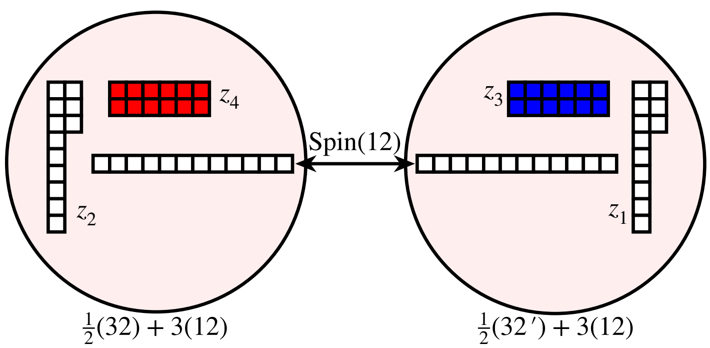
. (3.24)

The S-dual theory is an $SU(2)$ gauging of the $Sp(6)_{12} \times SU(2)_7 \times U(1)$ SCFT + $\frac{1}{2}(2)$



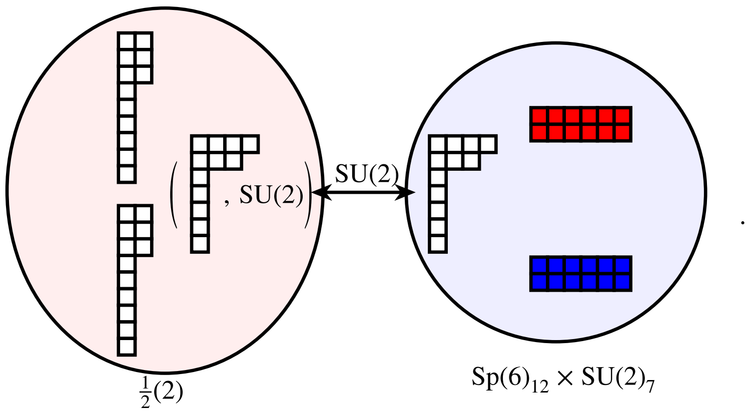
.

3.5.2.3 $Spin(12) + \frac{1}{2}(32) + \frac{1}{2}(32') + 6(12)$



. (3.25)

The S-dual is an $SU(2)$ gauging of the $Sp(6)_{12} \times SU(2)_7$ SCFT $+ \frac{1}{2}(2)$

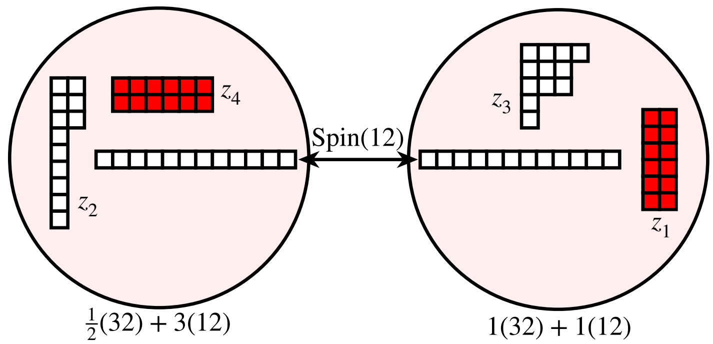


The invariant k-differentials for (3.24) and (3.25) are

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{[u_4\,(z-z_1)(z-z_2)z_{34} + \frac{1}{4}u_2^2\,((z-z_2)(z-z_3)z_{14}}{(z-z_1)^2(z-z_2)^2(z-z_3)^3(z-z_4)^3} \\
&\quad \underline{-(z-z_1)(z-z_4)z_{23})]z_{12}z_{34}^2(dz)^4} \\
\phi_6(z) &= \frac{[u_6\,(z-z_3)(z-z_4)z_{12} + 2\tilde{u}((z-z_1)(z-z_4)z_{23}}{(z-z_1)^2(z-z_2)^2(z-z_3)^5(z-z_4)^5} \\
&\quad \underline{\mp(z-z_2)(z-z_3)z_{14})]z_{12}{z_{34}}^4(dz)^6} \\
\phi_8(z) &= \frac{u_8\, z_{12}^2 z_{34}^6 (dz)^8}{(z-z_1)^2(z-z_2)^2(z-z_3)^6(z-z_4)^6} \\
\phi_{10}(z) &= \frac{u_{10}\, {z_{12}}^2 {z_{34}}^8 (dz)^{10}}{(z-z_1)^2(z-z_2)^2(z-z_3)^8(z-z_4)^8} \\
\tilde{\phi}(z) &= \frac{\tilde{u}\, z_{12}{z_{34}}^5(dz)^6}{(z-z_1)(z-z_2)(z-z_3)^5(z-z_4)^5} \quad .
\end{aligned}
\tag{3.26}
$$

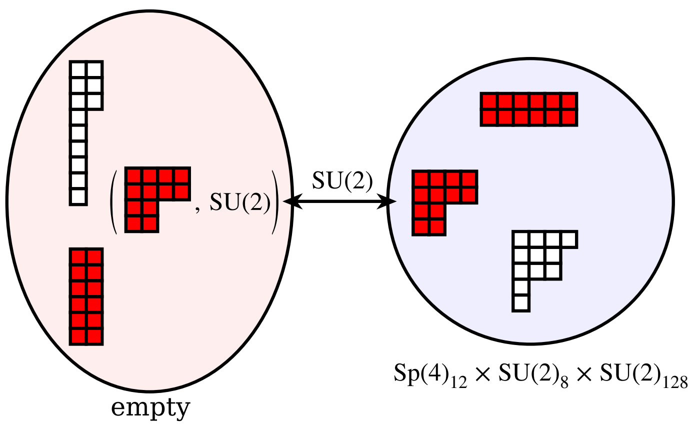
where the upper/lower sign in the expression for ϕ_6 is for (3.24)/(3.25), respectively. The invariant k-differentials for (3.19) are as above, but with $\tilde{u} \equiv 0$.

3.5.2.4 $Spin(12) + \frac{3}{2}(32) + 4(12)$

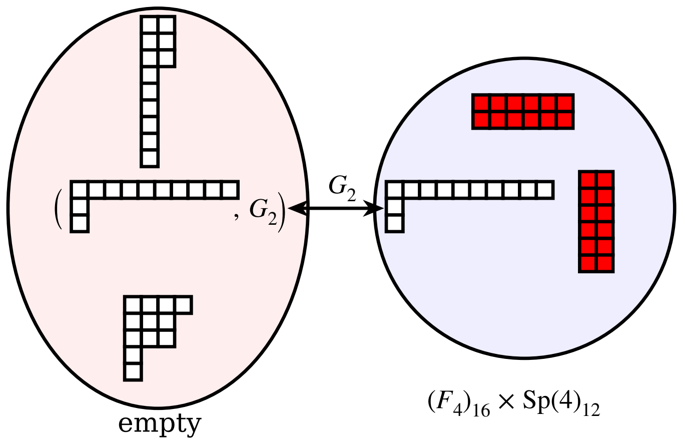


$$.\tag{3.27}$$

The S-dual theories are an $SU(2)$ gauging of the $Sp(4)_{12} \times SU(2)_8 \times SU(2)_{128}$ SCFT



and a G_2 gauging of the $(F_4)_{16} \times Sp(4)_{12}$ SCFT



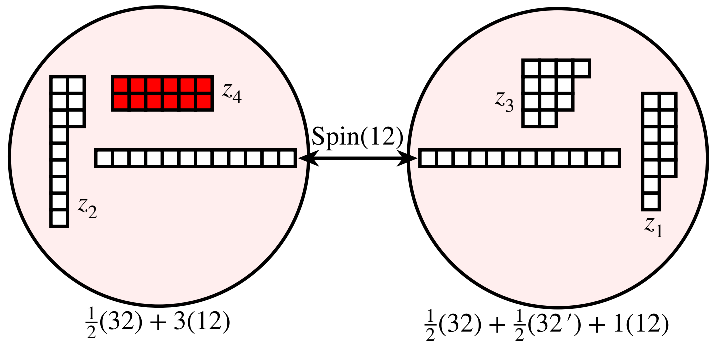
.

The invariant k-differentials for (3.27) are given by

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2 z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{[u_4(z-z_2)z_{14} + \frac{1}{4}u_2{}^2(z-z_4)z_{12}]z_{12}z_{34}{}^2(dz)^4}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^3} \\
\phi_6(z) &= \frac{[u_6(z-z_1)(z-z_4)z_{23} - 2\tilde{u}(z-z_1)(z-z_2)z_{34}}{(z-z_1)^3(z-z_2)^2(z-z_3)^4(z-z_4)^5} \\
&\quad \underline{+(2\tilde{u} + \tfrac{1}{4}u_2u_4)(z-z_3)(z-z_4)z_{12}]z_{12}z_{14}z_{34}{}^3(dz)^6} \\
\phi_8(z) &= \frac{[u_8(z-z_1)z_{34} + (\frac{1}{4}u_4{}^2 + \tilde{u}u_2)(z-z_4)z_{13}]z_{14}z_{12}{}^2z_{34}{}^4(dz)^8}{(z-z_1)^4(z-z_2)^2(z-z_3)^5(z-z_4)^6} \\
\phi_{10}(z) &= \frac{[u_{10}(z-z_1)z_{34} + \tilde{u}u_4(z-z_4)z_{13}]z_{12}{}^2z_{14}{}^2z_{34}{}^5(dz)^{10}}{(z-z_1)^5(z-z_2)^2(z-z_3)^6(z-z_4)^8} \\
\tilde{\phi}(z) &= \frac{\tilde{u}z_{12}z_{14}{}^2z_{34}{}^3(dz)^6}{(z-z_1)^3(z-z_2)(z-z_3)^3(z-z_4)^5} \quad .
\end{aligned}
\tag{3.28}
$$

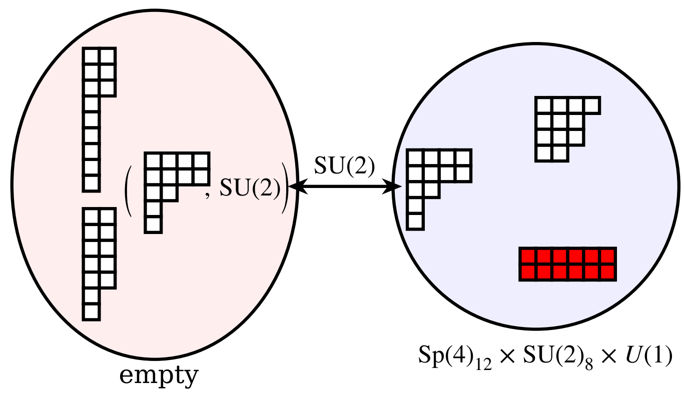
For (3.20), they are as above, but with $\tilde{u} \equiv 0$.

3.5.2.5 $Spin(12) + 1(32) + \frac{1}{2}(32') + 4(12)$

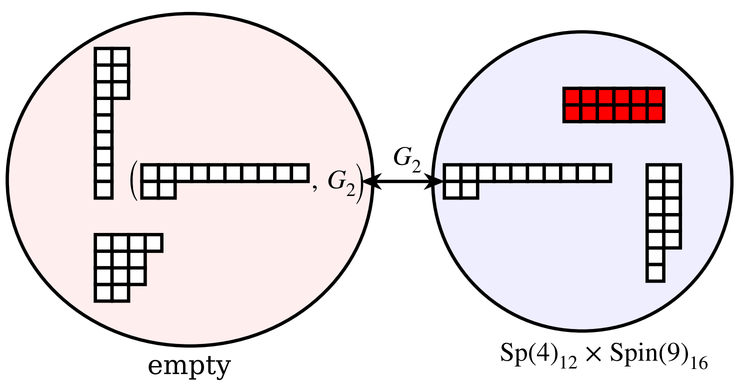


$$. \tag{3.29}$$

The S-dual theories are an $SU(2)$ gauging of the $Sp(4)_{12} \times SU(2)_8 \times U(1)$ SCFT



and a G_2 gauging of the $Sp(4)_{12} \times Spin(9)_{16}$ SCFT



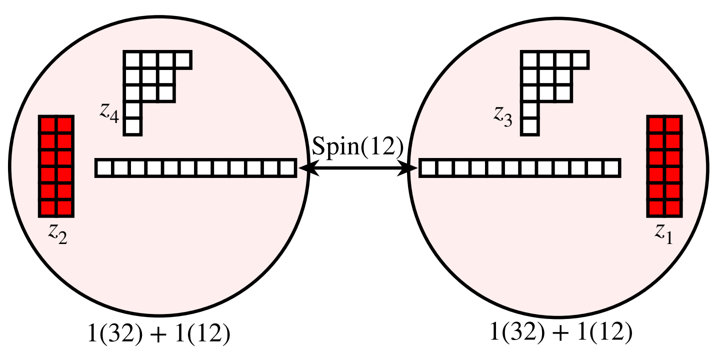
.

The invariant k-differentials for (3.29) are given by

$$\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{\left[u_4\,(z-z_2)z_{14} + \frac{1}{4}u_2^2\,(z-z_4)z_{12}\right] z_{12} z_{34}^2 (dz)^4}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^3} \\
\phi_6(z) &= \frac{[u_6(z-z_1)(z-z_4)z_{23} - 2\tilde{u}(z-z_1)(z-z_2)z_{34}}{(z-z_1)^3(z-z_2)^2(z-z_3)^4(z-z_4)^5} \\
&\quad \underline{+\tfrac{1}{4}u_2 u_4 (z-z_3)(z-z_4)z_{12}] z_{12} z_{14} {z_{34}}^3 (dz)^6} \\
\phi_8(z) &= \frac{\left[u_8\,(z-z_1)z_{34} + \frac{1}{4}u_4^2\,(z-z_4)z_{13}\right] z_{14} z_{12}^2 z_{34}^4 (dz)^8}{(z-z_1)^4(z-z_2)^2(z-z_3)^5(z-z_4)^6} \\
\phi_{10}(z) &= \frac{u_{10} {z_{12}}^2 {z_{14}}^2 {z_{34}}^6 (dz)^{10}}{(z-z_1)^4(z-z_2)^2(z-z_3)^6(z-z_4)^8} \\
\tilde{\phi}(z) &= \frac{\tilde{u} z_{12} z_{14} {z_{34}}^4 (dz)^6}{(z-z_1)^2(z-z_2)(z-z_3)^4(z-z_4)^5} \quad .
\end{aligned} \tag{3.30}$$

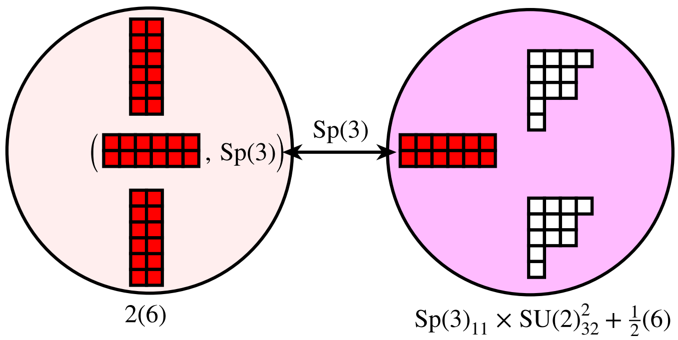
For (3.20), they are as above, but with $\tilde{u} \equiv 0$ (note that (3.30) and (3.28) become equal at $\tilde{u} = 0$).

3.5.2.6 $Spin(12) + 2(32) + 2(12)$



$$. \tag{3.31}$$

The S-dual theory is an $Sp(3)$ gauging of the $Sp(3)_{11} \times SU(2)^2_{32}$ SCFT $+ \frac{5}{2}(6)$

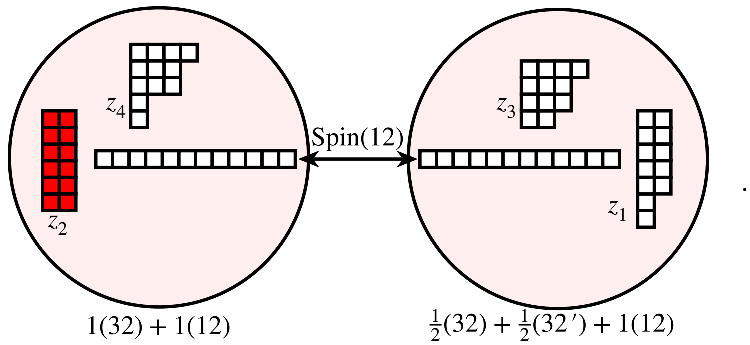


.

The invariant k-differentials for (3.31) are given by

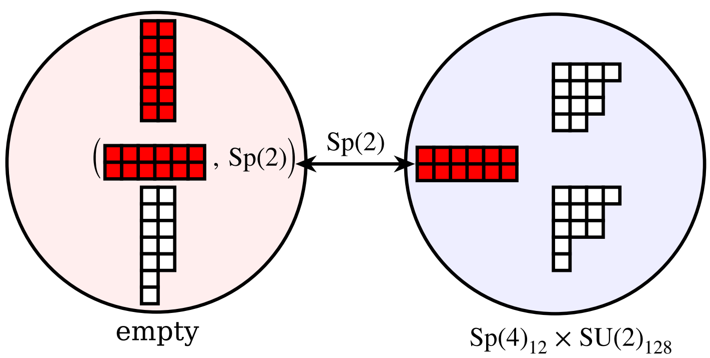
$$
\begin{aligned}
\phi_2(z) &= \frac{u_2 z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{u_4 {z_{12}}^2 {z_{34}}^2 (dz)^4}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^2} \\
\phi_6(z) &= \frac{[u_6(z-z_1)(z-z_2)z_{34} - (2\tilde{u} + \frac{1}{2}u_2(u_4 - \frac{1}{4}{u_2}^2))(z-z_1)(z-z_4)z_{23}}{(z-z_1)^3(z-z_2)^3(z-z_3)^4(z-z_4)^4} \\
&\quad \underline{+(2\tilde{u} + \tfrac{1}{2}u_2(u_4 - \tfrac{1}{4}{u_2}^2))(z-z_2)(z-z_3)z_{14}]{z_{12}}^2{z_{34}}^3(dz)^6} \\
\phi_8(z) &= \frac{[u_8(z-z_1)(z-z_2)z_{34} - (\frac{1}{4}(u_4 - \frac{1}{4}{u_2}^2)^2 + \tilde{u}u_2)(z-z_1)(z-z_4)z_{23}}{(z-z_1)^4(z-z_2)^4(z-z_3)^5(z-z_4)^5} \\
&\quad \underline{+(\tfrac{1}{4}(u_4 - \tfrac{1}{4}{u_2}^2)^2 + \tilde{u}u_2)(z-z_2)(z-z_3)z_{14}]{z_{12}}^3{z_{34}}^4(dz)^8} \\
\phi_{10}(z) &= \frac{[u_{10}(z-z_1)(z-z_2)z_{34} - \tilde{u}(u_4 - \frac{1}{4}{u_2}^2)(z-z_1)(z-z_4)z_{23}}{(z-z_1)^5(z-z_2)^5(z-z_3)^6(z-z_4)^6} \\
&\quad \underline{+\tilde{u}(u_4 - \tfrac{1}{4}{u_2}^2)(z-z_2)(z-z_3)z_{14}]{z_{12}}^4{z_{34}}^5(dz)^{10}} \\
\tilde{\phi}(z) &= \frac{\tilde{u}{z_{12}}^3{z_{34}}^3(dz)^6}{(z-z_1)^3(z-z_2)^3(z-z_3)^3(z-z_4)^3} \quad .
\end{aligned}
\tag{3.32}
$$

3.5.2.7 $Spin(12) + \frac{3}{2}(32) + \frac{1}{2}(32') + 2(12)$

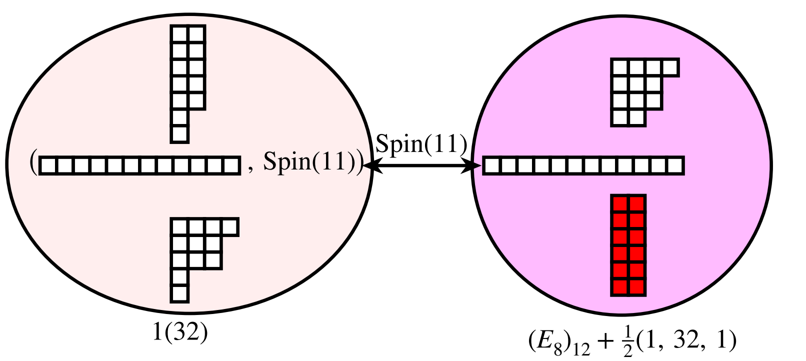


(3.33)

The S-dual theories are an $Sp(2)$ gauging of the $Sp(4)_{12} \times SU(2)_{128}$ SCFT



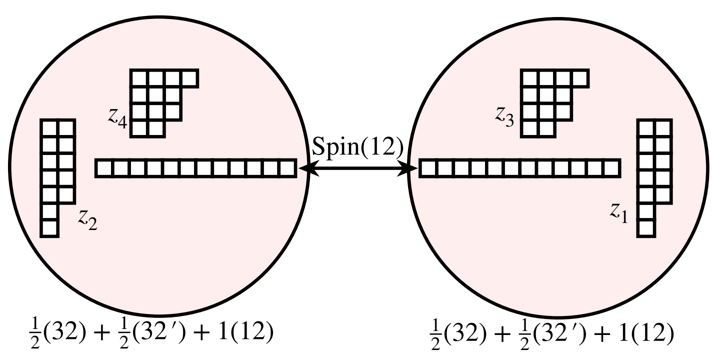
and a $Spin(11)$ gauging of the $(E_8)_{12}\,\mathrm{SCFT} + \frac{3}{2}(32)$



The invariant k-differentials for (3.33) are given by

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2 z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{u_4 {z_{12}}^2 {z_{34}}^2 (dz)^4}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^2} \\
\phi_6(z) &= \frac{\big[u_6(z-z_1)(z-z_2)z_{34} + (2\tilde{u} - \frac{1}{2}u_2(u_4 - \frac{1}{4}{u_2}^2))(z-z_1)(z-z_4)z_{23}}{(z-z_1)^3(z-z_2)^3(z-z_3)^4(z-z_4)^4} \\
&\qquad \underline{+\tfrac{1}{2}u_2(u_4 - \tfrac{1}{4}{u_2}^2)(z-z_2)(z-z_3)z_{14}\big]{z_{12}}^2{z_{34}}^3(dz)^6} \\
\phi_8(z) &= \frac{\big[u_8(z-z_1)(z-z_2)z_{34} - (\frac{1}{4}(u_4 - \frac{1}{4}{u_2}^2)^2 - \tilde{u}u_2)(z-z_1)(z-z_4)z_{23}}{(z-z_1)^4(z-z_2)^4(z-z_3)^5(z-z_4)^5} \\
&\qquad \underline{+\tfrac{1}{4}(u_4 - \tfrac{1}{4}{u_2}^2)^2(z-z_2)(z-z_3)z_{14}\big]{z_{12}}^3{z_{34}}^4(dz)^8} \\
\phi_{10}(z) &= \frac{[u_{10}(z-z_2)z_{34} + \tilde{u}(u_4 - \frac{1}{4}u_2^2)(z-z_4)z_{23}]{z_{12}}^4{z_{34}}^5(dz)^{10}}{(z-z_1)^4(z-z_2)^5(z-z_3)^6(z-z_4)^6} \\
\tilde{\phi}(z) &= \frac{\tilde{u} z_{23} {z_{12}}^2 {z_{34}}^3 (dz)^6}{(z-z_1)^2(z-z_2)^3(z-z_3)^4(z-z_4)^3} \quad .
\end{aligned}
\tag{3.34}
$$

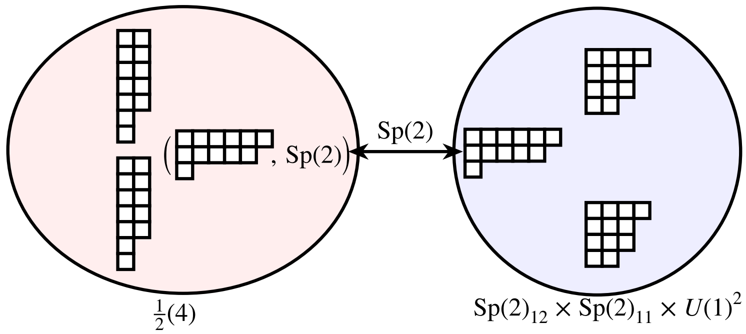
3.5.2.8 $Spin(12) + 1(32) + 1(32') + 2(12)$



. (3.35)

The S-dual theory is an $Sp(2)$ gauging of the $Sp(2)_{12} \times Sp(2)_{11} \times U(1)^2$ SCFT +

$\frac{1}{2}(4)$



The invariant k-differentials for (3.35) are given by

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34} (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{u_4\, z_{12}^2 z_{34}^2 (dz)^4}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^2} \\
\phi_6(z) &= \frac{\left[u_6\,(z-z_1)(z-z_2)z_{34} - \frac{1}{2}u_2(u_4-\frac{1}{4}u_2^2)((z-z_1)(z-z_3)z_{24} -(z-z_2)(z-z_4)z_{13})\right] z_{12}^2 z_{34}^3 (dz)^6}{(z-z_1)^3(z-z_2)^3(z-z_3)^4(z-z_4)^4} \\
\phi_8(z) &= \frac{\left[u_8\,(z-z_1)(z-z_2)z_{34} - \frac{1}{4}(u_4-\frac{1}{4}u_2^2)^2((z-z_1)(z-z_3)z_{24} -(z-z_2)(z-z_4)z_{13})\right] z_{12}^3 z_{34}^4 (dz)^8}{(z-z_1)^4(z-z_2)^4(z-z_3)^5(z-z_4)^5} \\
\phi_{10}(z) &= \frac{u_{10} {z_{12}}^4 {z_{34}}^6 (dz)^{10}}{(z-z_1)^4(z-z_2)^4(z-z_3)^6(z-z_4)^6} \\
\tilde{\phi}(z) &= \frac{\tilde{u} {z_{12}}^2 {z_{34}}^4 (dz)^6}{(z-z_1)^2(z-z_2)^2(z-z_3)^4(z-z_4)^4} \quad .
\end{aligned}
\tag{3.36}
$$

For (3.21), they are as above, but with $\tilde{u} \equiv 0$. As before, (3.32),(3.34) and (3.36) become identical when you set $\tilde{u} = 0$.

3.5.2.9 More Spinors

We cannot obtain

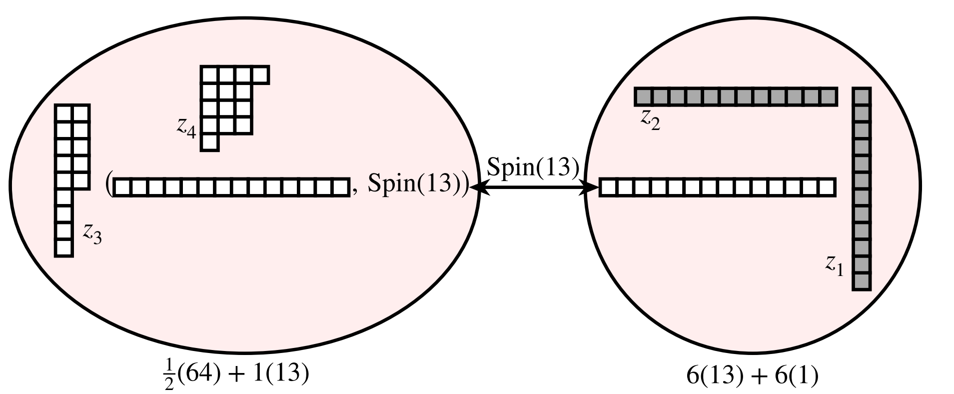
- $Spin(12) + \frac{5}{2}(32)$
- $Spin(12) + 2(32) + \frac{1}{2}(32')$
- $Spin(12) + \frac{3}{2}(32) + 1(32')$

from compactifying the D_6 theory.

3.6 $Spin(13)$ and $Spin(14)$ Gauge Theories

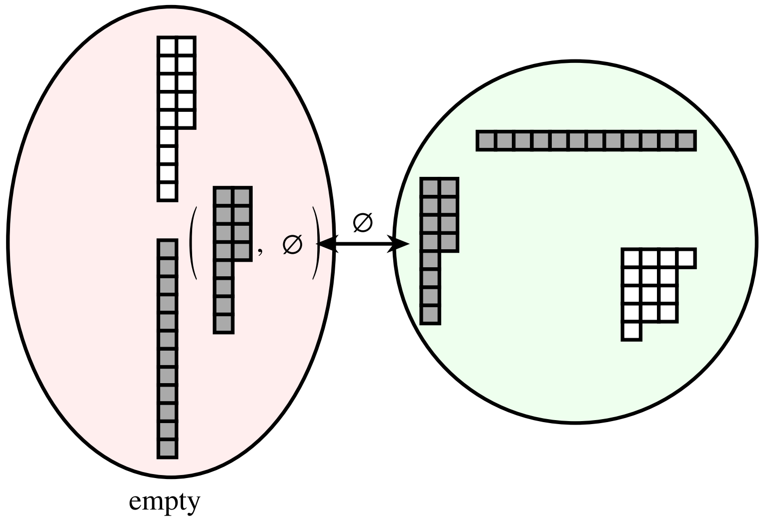
Here, we work in the D_7 theory.

3.6.1 $Spin(13) + \frac{1}{2}(64) + 7(13)$

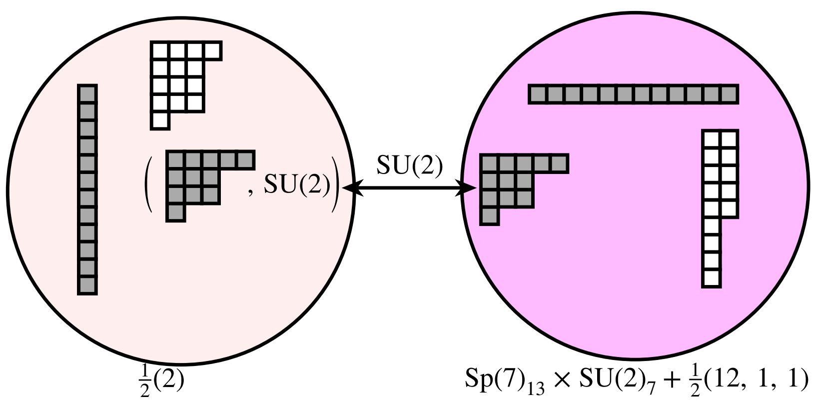


(3.37)

Over the other degenerations, we have a gauge theory fixture



and an $SU(2)$ gauging of the $Sp(7)_{13} \times SU(2)_7\text{SCFT} + \frac{1}{2}(2) + 6(1)$



.

The invariant k-differentials for (3.37) are given by

$$\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{13}z_{24}(dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_4(z) &= \frac{u_4\, z_{13}z_{23}{z_{24}}^2(dz)^4}{(z-z_1)(z-z_2)^3(z-z_3)^2(z-z_4)^2} \\
\phi_6(z) &= \frac{[u_6(z-z_3)z_{12}-\frac{1}{2}u_2(u_4-\frac{1}{4}{u_2}^2)(z-z_2)z_{13}]z_{23}z_{34}{z_{24}}^3(dz)^6}{(z-z_1)(z-z_2)^5(z-z_3)^3(z-z_4)^4} \\
\phi_8(z) &= \frac{[u_8(z-z_3)z_{12}-\frac{1}{4}(u_4-\frac{1}{4}{u_2}^2)^2(z-z_2)z_{13}]z_{34}{z_{23}}^2{z_{24}}^4(dz)^8}{(z-z_1)(z-z_2)^7(z-z_3)^4(z-z_4)^5} \qquad (3.38)\\
\phi_{10}(z) &= \frac{u_{10}\, z_{13}z_{24}{z_{23}}^3{z_{24}}^5(dz)^{10}}{(z-z_1)(z-z_2)^9(z-z_3)^4(z-z_4)^6} \\
\phi_{12}(z) &= \frac{u_{12}\, z_{13}z_{24}{z_{23}}^3{z_{24}}^7(dz)^{12}}{(z-z_1)(z-z_2)^{11}(z-z_3)^4(z-z_4)^8}
\end{aligned}$$

and

$$\tilde{\phi}(z) = 0 \quad .$$

3.6.2 More Spinors

We cannot obtain

- $Spin(13) + 1(64) + 3(13)$
- $Spin(14) + 1(64) + 4(14)$

from compactifying the D_7 theory.

3.7 Higher N?

For the "missing" theories of §3.5.2.9 and §3.6.2, we might hope to find realizations in the higher D_N or A_{2N-1} theories. It is easy to see that is no help. The key realization is that we need a candidate free-field fixture, consisting of three regular punctures. One of these punctures must be a full puncture.

In the D_N theory, the full puncture, $[1^{2N}]$, has a $Spin(2N)_{4(N-1)}$ flavour symmetry. The free fields transform as some representation of $Spin(2N)$ which reproduce the level $k = 4(N-1)$. If the representation should *happen* to decompose correctly under a $Spin(12)$ (*mutatis mutandis* for a $Spin(13)$ or $Spin(14)$) subgroup, then we would have a chance to build a realization of one of our missing gauge theories.

- For the $Spin(12)$ theories of §3.5.2.9, we could note that the 64 of $Spin(14)$ decomposes as $1(32)+1(32')$. But getting the right level would require a puncture with level $k = 32$, whereas the full puncture of the D_7 theory has only $k = 24$.

- For the $Spin(13)$ and $Spin(14)$ theories of §3.6.2, going to higher D_N could only produce the 64 with multiplicity > 1, which also does not help.

In the twisted sector of the A_{2N-1} theory, the full puncture has $Spin(2N+1)_{2(2N-1)}$ flavour symmetry.

- For the $Spin(12)$ theories of §3.5.2.9, we need k to be a multiple of 8, so none of these are satisfactory.
- For the $Spin(13)$ and $Spin(14)$ theories of §3.6.2, we need k to be a multiple of 4, which also does not work.

What about the exceptional (2,0) theories? E_7 and E_8 contain our desired gauge groups as subgroups. But neither the 56 of E_7, nor the 248 of E_8 decompose correctly to provide candidate free field fixtures with one full puncture (and two other regular punctures).

So it appears that the missing theories of §3.5.2.9 and §3.6.2, are not realizable as compactifications of the $(2, 0)$ theory.

Chapter 4

The E_6 Theory

In this chapter, we extend our classification program to the (2,0) theory of type E_6 [1]. There is no known construction of the E_6 theory as a low-energy theory of a stack of M5 branes, as was the case for the A- and D-series. Rather, the only known construction is as a compactification of IIB string theory on a K3 manifold at an E_6 singularity [50]. Still, computations are possible because the the 4D $\mathcal{N} = 2$ compactification of the E_6 theory is controlled by a Hitchin system [6] with gauge group E_6.

As a byproduct, we realize E_6 gauge theory with matter in the 4(27), as well as F_4 gauge theory with matter in the 3(26), as compactifications of the E_6 (2,0) theory on a 4-punctured sphere. The Seiberg-Witten solution to the E_6 gauge theory, with $N_f \leq 3$ 27s, appeared first in [51]. Our solution to the superconformal F_4 gauge theory is new.

[1]This chapter is based on [49].

4.1 The E_6 Theory

4.1.1 The Hitchin system

The Coulomb branch of the 4D $\mathcal{N} = 2$ theories obtained from the compactification of the 6D (2,0) theory of type E_6 on a Riemann surface C is described by the Hitchin equations on C with complexified gauge group E_6 [6]. We may also include codimension-two defects of the (2,0) theory localized at points on C; we refer to these as "punctures". A class of punctures is classified by nilpotent orbits (or, equivalently, by embeddings of $\mathfrak{sl}(2)$) in the complexified Lie algebra $\mathfrak{e}_6$ [7]. One of the main points of the construction is that a number of physical properties of the 4D theories can be computed directly from geometric properties of the nilpotent orbits that label the punctures on C, without any detailed knowledge of the (2,0) theory.

A puncture labeled by a nilpotent orbit $\mathcal{O}$, and located at $z = 0$ on C, corresponds to a local boundary condition for the Higgs field,

$$\Phi(z) = \frac{X}{z} + \dots \tag{4.1}$$

where Φ is a holomorphic 1-form on C that takes values in $\mathfrak{e}_6$ and transforms in the adjoint representation of the gauge group, X is a representative of the nilpotent orbit $d(\mathcal{O})$ in $\mathfrak{e}_6$, and $\dots$ represents a generic regular function of z taking values in $\mathfrak{e}_6$. Here, $d(\mathcal{O})$ is the image of $\mathcal{O}$ under the Lusztig-Spaltenstein map d [14, 5, 7]. Representatives of all nilpotent orbits in $\mathfrak{e}_6$ can be found in [52], and a diagram specifying the action of d, as well as other properties of the $\mathfrak{e}_6$ orbits, are collected in Appendix C of [7] (taken from [53, 54]) When d is not injective, we distinguish different punctures with the same $d(\mathcal{O})$ by their Sommers-Achar group $\mathcal{C}(\mathcal{O})$ [7], which is a discrete subgroup of E_6, imposing gauge invariance of Φ under the action of $\mathcal{C}(\mathcal{O})$.

As in our previous papers, we call $\mathcal{O}$, which labels the puncture, the *Nahm pole*, and $d(\mathcal{O})$, which appears in the Hitchin system boundary condition, the *Hitchin pole*. The physical properties of a puncture labeled by $\mathcal{O}$ will be directly related to geometric properties of the orbits $\mathcal{O}$ and $d(\mathcal{O})$, and the discrete group $\mathcal{C}(\mathcal{O})$.

Unlike classical Lie algebras, there is no natural parameterization of the nilpotent orbits of exceptional Lie algebras in terms of partitions or Young diagrams. Instead, the notation due to Bala and Carter is standard in the representation theory literature. This notation has been briefly discussed in previous works [7, 21, 33], but, for completeness, we review it in Appendix B.1, and discuss how to extract relevant information from it.

4.1.2 k-differentials

The low-energy solution of the 4D $\mathcal{N} = 2$ theory is encoded in the Seiberg-Witten curve, which is given by the spectral curve of the Hitchin system, i.e., by the characteristic polynomial for the Higgs field Φ, in representation R of $\mathfrak{e}_6$:

$$\Sigma_R : \det_R(\Phi - \lambda I) = \lambda^d + \lambda^{d-2} s_2 + \lambda^{d-3} s_3 + \cdots + s_d = 0,$$

where $d = \dim R$ and the λ^{d-1} is zero because $\mathrm{Tr}(\Phi) = 0$. Different choices of R will yield different curves Σ_R. However, as discussed in [55], the physical information that one can extract from them is the same.

For a choice of R, let s_k be the coefficient of $\lambda^{\dim(R)-k}$, for $k = 0, 1, 2, \ldots, \dim(R)$. ($s_0$ and s_1 are trivial – they are 1 and 0, respectively.) The $s_k(z)$ are holomorphic k-differentials on C (with poles at the punctures), and can be expressed as polynomials in the trace invariants $P_k = \mathrm{Tr}(\Phi^k)$. Notice that both the s_k and the P_k are dependent on the representation R.

On the other hand, we are actually interested in the Casimirs of Φ, which are the independent k-differentials providing the gauge-invariant information contained in Φ. For a Lie algebra $\mathfrak{g}$, the number of Casimirs is equal to the rank of $\mathfrak{g}$, and their scaling dimensions are the exponents (minus 1) of $\mathfrak{g}$. Unlike the s_k or the P_k, the Casimirs encode the non-redundant gauge-invariant information in Φ.

In our previous papers [3, 5, 4, 17], the Lie algebra was of classical type, and R was always chosen to be the smallest non-trivial representation (the fundamental for A_{N-1}, or the vector for D_N). In such cases, the coefficients s_k directly provide a basis for the Casimirs of Φ. For example, for $\mathfrak{g} = A_{N-1}$, we have $N - 1$ Casimirs, of dimensions $2, 3, 4, \ldots, N$. These dimensions match precisely those of the non-trivial coefficients s_k if R is chosen to be the fundamental representation. Thus, the s_k can be taken to be the Casimirs of A_{N-1}. Similarly, for $\mathfrak{g} = D_N$, the N Casimirs have degrees $2, 4, 6, \ldots, 2N-2; N$. If R is the vector representation, the s_k with k odd vanish, and the non-trivial coefficients are $s_2, s_4, \ldots, s_{2N-2}, s_{2N}$. Here, s_{2N} is the square of the Pfaffian, $s_{2N} = \tilde{s}^2$, and so $\tilde{s}$ has dimension N. Thus, as before, the $s_2, s_4, \ldots, s_{2N-2}; \tilde{s}$ provide a basis of Casimirs of D_N.

But if for A_{N-1} and D_N we had chosen R to be, say, the adjoint representation, then, for large enough N, the s_k would not have given directly the Casimirs, but instead a lot of redundant information. For example, for A_5, there are five Casimirs, with dimensions $2, 3, 4, 5, 6$. However, we have 34 non-trivial coefficients s_k, with dimensions $2, 3, 4, \ldots, 35$. These s_k are polynomials in the five Casimirs.

For $\mathfrak{j} = \mathfrak{e}_6$, the Casimirs have degrees 2,5,6,8,9,12. In our computations, we have chosen R to be the *adjoint* representation of $\mathfrak{e}_6$, as it is readily available in the form of structure constants; we used those from the computer algebra system GAP 4 [56]. Instead of trying to compute the 78 coefficients s_k, we focus directly on the trace invariants P_k for values of k only as large as needed to extract the Casimirs. For the adjoint representation of $\mathfrak{e}_6$, the P_k vanish for k odd, and are non-trivial for k even, except for $P_4 = \frac{1}{32}(P_2)^2$. Also, as we will see below, to extract the Casimirs, we only need to consider the P_k for $k = 2, 6, 8, 10, 12, 14$. From the P_k, one can construct a less-redundant basis,

$$
\begin{aligned}
\phi_2 =& \tfrac{1}{48} P_2 \\
\phi_6 =& \tfrac{1}{24}\left(P_6 - \tfrac{7}{4608}(P_2)^3\right) \\
\phi_8 =& \tfrac{1}{30}\left(P_8 - \tfrac{2}{9}P_6 P_2 + \tfrac{155}{663552}(P_2)^4\right) \\
\phi_{10} =& -\tfrac{1}{105}\left(P_{10} - \tfrac{17}{96}P_8 P_2 + \tfrac{77}{6912}P_6 (P_2)^2 - \tfrac{427}{63700992}(P_2)^5\right) \\
\phi_{12} =& \tfrac{1}{155}(P_{12} - \tfrac{107}{504}P_{10}P_2 + \tfrac{515}{32256}P_8(P_2)^2 - \tfrac{41}{108}(P_6)^2 + \tfrac{295}{497664}P_6(P_2)^3 \\
& - \tfrac{5669}{9172942848}(P_2)^6) \\
\phi_{14} =& \tfrac{1}{4389}\left(P_{14} - \tfrac{3479}{14880}P_{12}P_2 + \tfrac{61391}{3214080}P_{10}(P_2)^2 - \tfrac{539}{2160}P_8 P_6 - \tfrac{139733}{617103360}P_8(P_2)^3\right. \\
& \left. + \tfrac{165781}{4821120}(P_6)^2 P_2 - \tfrac{3488947}{44431441920}P_6(P_2)^4 + \tfrac{19596907}{409480168734720}(P_2)^7\right)
\end{aligned}
$$

This basis was constructed so that it reduces the constraints in our punctures to a minimum. In particular, the pole coefficients for the minimal puncture have no redundancies; that is, the ϕ_k are such that it not be possible to reduce their pole orders in z further by a change of basis, for z a local coordinate centered at the minimal puncture. The ϕ_k basis also makes apparent how the Casimirs of degree 5 and 9 appear. Specifically, ϕ_{10} and ϕ_{14} factor,

$$
\begin{aligned}
\phi_{10} &\equiv (\phi_5)^2, \\
\phi_{14} &\equiv \phi_5 \phi_9
\end{aligned}
$$

These relations define the odd-degree differentials ϕ_5 and ϕ_9 (up to a sign, which flips under the $\mathbb{Z}_2$ outer automorphism of E_6). So, we can declare the k-differentials $\{\phi_2, \phi_5, \phi_6, \phi_8, \phi_9, \phi_{12}\}$ to be our basis of $\mathfrak{e}_6$ Casimirs. In the following, by the ϕ_k, we will refer to the Casimirs, and ignore the auxiliary differentials ϕ_{10} and ϕ_{14}.

As for the Seiberg-Witten curve, to write it explicitly, we need to know how the 78 coefficients s_k depend on the six Casimirs ϕ_k. Instead, it is much simpler to write down the (representation independent) Seiberg-Witten geometry, given by an ALE fibration over C, and which equivalently describes the low-energy solution of 4D $\mathcal{N} = 2$ theories, but directly using the Casimirs [44, 57]. Let us briefly review that construction.

4.1.3 ALE geometry

The 4D $\mathcal{N} = 2$ SCFT constructed from the compactification of a 6D (2,0) theory of type J (where J is of A-D-E type) on the Riemann surface C can also be obtained, in a dual manner, from IIB string theory on a non-compact Calabi-Yau threefold, locally given by an ALE fibration over C of type J [44, 57]. For $\mathfrak{e}_6$, the threefold is realized as the hypersurface

$$\begin{aligned} X_{\vec{u}} =& \big\{ 0 = w^2 + x^3 + y^4 + \epsilon_2(z) x y^2 + \epsilon_5(z) x y + \epsilon_6(z) y^2 + \epsilon_8(z) x + \epsilon_9(z) y \\ & + \epsilon_{12}(z) \big\} \subset \mathrm{tot}(K_C^6 \oplus K_C^4 \oplus K_C^3) \end{aligned}$$

where the $\epsilon_k(z)$ are k-differentials on C [33] (in the "Katz-Morrison basis" [58]), related to our $\phi_k(z)$ by

$$
\begin{aligned}
\epsilon_2 &= \tfrac{1}{2}\phi_2 \\
\epsilon_5 &= \tfrac{1}{6}\phi_5 \\
\epsilon_6 &= \tfrac{1}{72}(-3\phi_2^3 + 2\phi_6) \\
\epsilon_8 &= \tfrac{1}{144}(-3\phi_2^4 + 4\phi_2\phi_6 - \phi_8) \\
\epsilon_9 &= \tfrac{1}{72}(-\phi_2^2\phi_5 + 4\phi_9) \\
\epsilon_{12} &= \tfrac{1}{5184}(4\phi_{12} + 6\phi_2^6 - 12\phi_2^3\phi_6 + 4\phi_6^2 + 3\phi_2^2\phi_8)
\end{aligned}
$$

The $\phi_k(z)$, in turn, depend on the Coulomb branch parameters, $\vec{u}$, as we determine below.

The Seiberg-Witten solution is obtained by computing the periods of the holomorphic 3-form, Ω, over a symplectic basis of (rational) 3-cycles on $X_{\vec{u}}$ which are locally of the form of a 2-sphere in the fiber times a curve on C. In the conformal case (which will be our focus in this paper), many of these cycles will necessarily be noncompact (the curve on C being a open curve, stretching between punctures). But, precisely for the parabolic case (where the Higgs field $\Phi(z)$ has simple poles at the punctures, with nilpotent residues), the singularity is integrable, and the requisite periods of Ω are finite.

In our realization of F_4 gauge theory in §4.4.1.2, the differentials $\phi_5(z)$ and $\phi_9(z)$ vanish identically. In this case, the Calabi-Yau, $X_{\vec{u}}$, has a holomorphic involution, $y \to -y$, under which $\Omega \to -\Omega$. The 3-cycles which give the Seiberg-Witten solution are the anti-invariant cycles and the periods of Ω over those cycles are finite, despite the slightly singular nature of $X_{\vec{u}}$ itself.

4.1.4 Puncture properties

We describe below how to compute the properties of a puncture. There is a systematic way to compute every property of the puncture, except for the constraints, so it is easiest to compute the other properties first, and use them to guess the constraints. Below, let $\mathcal{O}$ be the Nahm nilpotent orbit that labels a given puncture, and $\mathfrak{su}(2)_{\mathcal{O}}$ the associated $\mathfrak{su}(2)$ embedding in $\mathfrak{e}_6$.

4.1.4.1 Flavour groups

The Lie algebra $\mathfrak{f}$ of the flavour group $F = F_{\mathcal{O}}$ is the centralizer of $\mathfrak{su}(2)_{\mathcal{O}}$ in $\mathfrak{e}_6$. A list of the centralizers for each $\mathcal{O}$ can be found in Table 14 of [7], taken originally from [59].

The levels of the simple, nonabelian factors $\mathfrak{f}_i$ in $\mathfrak{f}$ follow from the decomposition of the adjoint of $\mathfrak{e}_6$ under $\mathfrak{su}(2) \times \mathfrak{f}$. These decompositions can be deduced from the Bala-Carter label for $\mathcal{O}$, and are summarized in the table in Appendix B.1.

Let the decomposition of the 78 be

$$\mathfrak{e}_6 = \bigoplus_n V_n \otimes R_{n,i}$$

where V_n is the n-dimensional irrep of $\mathfrak{su}(2)$ (denoted by "n" in the table) and $R_{n,i}$ is the corresponding (reducible) representation of $\mathfrak{f}_i$. Let $l_{n,i}$ be the index of $R_{n,i}$. Then, the level of $\mathfrak{f}_i$ is $k_i = \sum_n l_{n,i}$.

For example, consider the $3A_1$ puncture, which has $\mathfrak{f} = \mathfrak{su}(3) \times \mathfrak{su}(2)$.

From the table in Appendix B.1, we have, for $\mathfrak{f}_1 = \mathfrak{su}(3)$,

$$R_1 = 8 + 3(1), \quad R_2 = 2(8), \quad R_3 = 8 + 1, \quad R_4 = 2(1),$$

and so the level is $k_{\mathfrak{su}(3)} = 4l_8 = 24$. Similarly, for $\mathfrak{f}_2 = \mathfrak{su}(2)$, we have

$$R_1 = 3 + 8(1), \quad R_2 = 8(2), \quad R_3 = 9(1), \quad R_4 = 2,$$

and thus the level is $k_{\mathfrak{su}(2)} = l_3 + 9l_2 = 4 + 9 \times 1 = 13$.

4.1.4.2 δn_h and δn_v

The effective number of hyper- and vector multiplets, δn_h and δn_v, can be computed using the formulas in eq. (3.19) of [7]. Basically, given $\mathcal{O}$, one needs to know how $\mathfrak{e}_6$ decomposes into eigenspaces of the Cartan element of $\mathfrak{su}(2)_{\mathcal{O}}$.

Here, let us recast those formulas in terms of the *weighted Dynkin diagram* for $\mathcal{O}$, which can be found in Table 14 of [7]. Let $\vec{x}$ be the six-dimensional vector consisting of the labels of the weighted Dynkin diagram for $\mathcal{O}$. Now, for each root α of E_6, let $\vec{k}$ be a six-dimensional vector consisting of the (integer) components of α in any basis of simple roots. The "Weyl vector" is $\vec{W} = \frac{1}{2}\sum_{\vec{k}\geq 0}\vec{k}$, where the sum is over positive roots. Let n_0 and $n_{1/2}$ be the number of roots α that satisfy $(\vec{x}/2)\cdot\vec{k} = 0$ and $(\vec{x}/2)\cdot\vec{k} = 1/2$, respectively. (The dot product is Euclidean.) In this notation, the formulas in eq. (3.19) of [7] are:

$$
\begin{aligned}
n_h(\vec{x}) &= 8\left(\tfrac{1}{12}h^\vee(E_6)\dim(E_6) - \frac{1}{2}\vec{W}\cdot\vec{x}\right) + \frac{1}{2}n_{1/2} \\
n_v(\vec{x}) &= 8\left(\tfrac{1}{12}h^\vee(E_6)\dim(E_6) - \frac{1}{2}\vec{W}\cdot\vec{x}\right) - \frac{1}{2}n_0
\end{aligned}
\tag{4.2}
$$

where $h^\vee(E_6) = 12$ denotes the dual Coxeter number of E_6.

For example, for $\vec{x} = \vec{0}$, corresponding to the maximal puncture, one has that the adjoint of E_6 decomposes trivially into singlets of $\mathfrak{su}(2)$, $78 \to 78(1)$, so $n_0(\vec{0}) = \dim(E_6) - \mathrm{rank}(E_6)$, and $n_{1/2}(\vec{0}) = 0$. Thus,

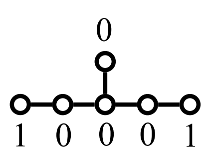
$$
\begin{aligned}
n_h(\vec{0}) &= \tfrac{2}{3}h^\vee(E_6)\dim(E_6) = 624 \\
n_v(\vec{0}) &= \tfrac{2}{3}h^\vee(E_6)\dim(E_6) - \frac{1}{2}(\dim(E_6) - \mathrm{rank}(E_6)) = 588
\end{aligned}
$$

As a self-consistency check, recall that the complex dimension $\dim_{\mathbb{C}}(\mathcal{O})$ of the orbit $\mathcal{O}$ (seen as a manifold) is related linearly to the difference $n_h - n_v$. Specifically, $n_h - n_v = C - \frac{1}{2}\dim_{\mathbb{C}}(\mathcal{O})$, where $C = n_h(\vec{0}) - n_v(\vec{0}) = \frac{1}{2}(\dim(E_6) - \mathrm{rank}(E_6))$. In other words,

$$
\dim_{\mathbb{C}}(\mathcal{O}) = \dim(E_6) - \mathrm{rank}(E_6) - (n_{1/2} + n_0), \tag{4.3}
$$

The dimensions of the nilpotent orbits of E_6 are listed in Table 14 of [7].

For a non-trivial example, consider the puncture $2A_1$, which has weighted Dynkin diagram



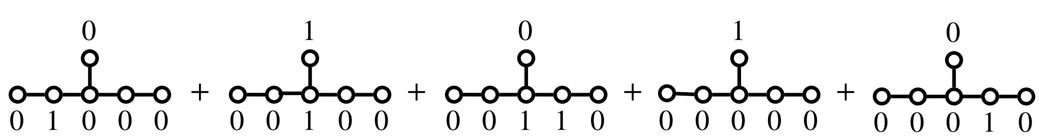
that is, $\vec{x} = (1, 0, 0, 0, 1; 0)$. One finds $\vec{W} = (8, 15, 21, 15, 8; 11)$, $n_{1/2} = 16$, $n_0 = 24$. Thus, $n_h(2A_1) = 568$ and $n_v(2A_1) = 548$, and one indeed checks (4.3) for $\dim_{\mathbb{C}}(2A_1) = 32$.

4.1.4.3 Pole structures

The "pole structure" is the set of leading pole orders $\{p_2, p_5, p_6, p_8, p_9, p_{12}\}$ in the expansion of the Casimirs ϕ_k in a coordinate z centered at the puncture, $\phi_k(z) \sim 1/z^{p_k}$.

To compute the pole structure, we need a representative of the Hitchin nilpotent orbit $d(\mathcal{O})$. A table of representatives of all nilpotent orbits of E_6 can be found in Table 2 of [52]. In this table, a nilpotent representative is given by a sum of weighted Dynkin diagrams, and each weighted Dynkin diagram represents an element in the root vector space of $\mathfrak{e}_6$ for a positive root α, where α is such that its components in a basis of simple roots are given by the labels of the Dynkin diagram. The nilpotent representative is the sum of these root-vector space elements. This procedure is most easily understood in terms of an example.

Take, for instance, $\mathcal{O} = D_4(a_1)$. The Hitchin orbit, given by the Spaltenstein dual, is the same, $d(\mathcal{O}) = \mathcal{O} = D_4(a_1)$. This orbit has a nilpotent representative X given by a sum of five elements [52],



The five summands above represent arbitrary non-zero elements X_{α_i} ($i = 1, \ldots, 5$) in the root vector spaces for the positive roots

$$\begin{aligned} \alpha_1 &= s_2, & \alpha_4 &= s_6, \\ \alpha_2 &= s_3 + s_6, & \alpha_5 &= s_4, \\ \alpha_3 &= s_3 + s_4, \end{aligned}$$

respectively, where $\{s_1, \ldots, s_5; s_6\}$ is a basis of simple roots of E_6. So, $X = X_{\alpha_1} + \cdots + X_{\alpha_5}$. Fortunately, GAP4 provides a Chevalley basis for the adjoint representation of $\mathfrak{e}_6$, so it is trivial to find elements X_{α_i}. Once we know X, we compute $\Phi(z)$ using X as the residue in (4.1), then the Casimir k-differentials ϕ_k as in §5.1.2, and we finally find the pole structure $\{1, 3, 4, 6, 6, 9\}$ for the $D_4(a_1)$ puncture. (Actually, there are *three* orbits, $D_4(a_1)$, $A_3 + A_1$ and $2A_2 + A_1$, that map under Spaltenstein to $D_4(a_1)$, so we have three punctures with the same pole structure. However, the other properties of these punctures are different.)

4.1.4.4 Constraints

The constraints for some E_6 punctures are, in some cases, much less obvious than those in the A_{N-1} and D_N series. The guiding quantities to find constraints are δn_v and the (complex) dimension of the Hitchin nilpotent orbit, d. These are, respectively, the graded and ungraded local contributions to the Coulomb branch.

Let us be specific. Let z be a local coordinate on C centered at the puncture, and let $c_l^{(k)}$ be the coefficient of z^{-l} in the expansion of $\phi_k = \phi_k(z)$

in z. Recall that, in the notation of our previous papers, a "c-constraint" is a polynomial relation among coefficients $c_l^{(k)}$ (of homogeneous bi-degree in both k and l). On the other hand, an "a-constraint" is a relation that defines a new quantity, $a_l^{(k)}$, of dimension k, in terms of the $c_l^{(k)}$. Only the $c_l^{(k)}$ with $l > 0$ parameterize the Hitchin nilpotent orbit [21]. In the absence of constraints, all the $c_l^{(k)}$ with $0 < l \leq p_k$ are independent, so their total number, $\sum p_k$, should be equal to the dimension of the Hitchin nilpotent orbit. Thus, if there are no constraints, $\sum p_k = d$. A c-constraint reduces the total number of independent parameters by one, whereas an a-constraint does not affect this number. So, one should have:

$$\sum p_k - (\text{number of c-constraints}) = d$$

Hence, d tells us how many c-constraints exist. On the other hand, the graded sum of the parameters, that is, the result of adding $(2k-1)$ for each parameter of degree k (in the presence of "a"-constraints, k is not restricted to the degrees of the Casimirs), should be equal to n_v. An a-constraint replaces a parameter of a certain degree k by another one of a different degree $k' < k$. So, to get precisely n_v, one must take into account all a-constraints and c-constraints.

4.1.4.5 Puncture collisions

Suppose we have two punctures on a plane, so the Higgs field has two simple poles with residues X_1 and X_2. Near each puncture, the Higgs field Φ looks like eq. (4.1). In the limit where the two punctures collide, the Higgs field has one simple pole with residue $X = X_1 + X_2$ (by the residue theorem applied to the sphere that bubbles off), which corresponds to a new puncture. Generically, X will be mass deformed. The mass deformations are interpreted as VEVs of the scalars in the gauge multiplet associate to the factor in the gauge group which becomes weakly coupled in the collision limit. One can also study this degeneration by computing the Casimirs ϕ_k from the Higgs field before taking the collision limit.

Alternatively, one can bypass the Higgs field, and study the collision directly with the ϕ_k, by writing a generic k-differential with poles at the positions of the two punctures (given by their pole structures), and imposing at each pole the constraints of the corresponding puncture. Then, taking the collision limit, the pole structure and constraints of the resulting puncture on the plane arise naturally.

As an example, let us see that the collision of two D_5 punctures on a plane produces an $Sp(2)$ gauge group, gauged off an A_3 puncture. Let us write generic Casimirs for the collision, taking the D_5 punctures to be at $z = 0$ and $z = x$:

$$\phi_2(z) = \frac{u_2 + zv_2 + z(z-x)P_2(z)}{z(z-x)}$$
$$\phi_5(z) = \frac{u_5 + zv_5 + z(z-x)w_5 + z^2(z-x)P_5(z)}{z^2(z-x)^2}$$
$$\phi_6(z) = \frac{u_6 + zv_6 + z(z-x)w_6 + z^2(z-x)P_6(z)}{z^3(z-x)^3}$$
$$\phi_8(z) = \frac{u_8 + zv_8 + z(z-x)w_8 + z^2(z-x)y_8 + z^2(z-x)^2P_8(z)}{z^4(z-x)^4}$$
$$\phi_9(z) = \frac{u_9 + zv_9 + z(z-x)w_9 + z^2(z-x)P_9(z)}{z^4(z-x)^4}$$
$$\phi_{12}(z) = \frac{u_{12} + zv_{12} + z(z-x)w_{12} + z^2(z-x)y_{12} + z^2(z-x)^2P_{12}(z)}{z^6(z-x)^6},$$

where $P_2(z), P_5(z), \ldots, P_{12}(z)$ denote regular functions in z. To solve the constraints at each D_5 puncture, we introduce new parameters s_4 and t_4 of dimension four, and write:

$$\begin{aligned}
u_6 &= \frac{3s_4u_2}{4}, & v_6 &= \frac{3}{2}(t_4u_2 + s_4v_2 + t_4v_2x), \\
u_8 &= 3s_4^2, & v_8 &= 3(2s_4t_4 + t_4^2x), \\
u_9 &= -\frac{s_4u_5}{4}, & v_9 &= -\frac{1}{4}(t_4u_5 + s_4v_5 + t_4v_5x), \\
u_{12} &= \frac{3s_4^3}{2}, & v_{12} &= \frac{3}{2}t_4(3s_4^2 + 3s_4t_4x + t_4^2x^2), \\
w_{12} &= \frac{3}{4}(3s_4t_4^2 + s_4w_8 + 2t_4^3x), & y_{12} &= -\frac{3}{4}(t_4^3 - t_4w_8 - s_4y_8 - t_4y_8x)
\end{aligned}$$

In the collision limit, $x \to 0$, the new puncture appears at $z = 0$. The expansion in z of the Casimirs in this limit is:

$$
\begin{aligned}
\phi_2(z) =& \frac{u_2}{z^2} + \frac{v_2}{z} + \dots \\
\phi_5(z) =& \frac{u_5}{z^4} + \dots \\
\phi_6(z) =& \frac{3s_4 u_2}{2z^6} + \frac{3(t_4 u_2 + s_4 v_2)}{2z^5} + \frac{w_6}{z^4} + \dots \\
\phi_8(z) =& \frac{3s_4^2}{z^8} + \frac{6s_4 t_4}{z^7} + \frac{w_8}{z^6} + \dots \\
\phi_9(z) =& -\frac{s_4 u_5}{4z^8} - \frac{(t_4 u_5 + s_4 v_5)}{4z^7} + \frac{w_9}{z^6} + \dots \\
\phi_{12}(z) =& \frac{3s_4^3}{2z^{12}} + \frac{9s_4^2 t_4}{2z^{11}} + \frac{3(3s_4 t_4^2 + s_4 w_8)}{4z^{10}} - \frac{3(t_4^3 - t_4 w_8 - s_4 y_8)}{4z^9} + \dots,
\end{aligned}
$$

where the ... indicate less singular terms in z. So, u_2 and s_4 can be interpreted as the VEVs of Coulomb branch parameters (of degree two and four) of the gauge group (which, with a little more work, can be checked to be $Sp(2)$). In the limit $u_2, s_4 \to 0$, we obtain the Casimirs for the massless puncture, with pole orders $\{1,4,4,6,7,9\}$, and with constraints

$$
\begin{aligned}
c_7^{(9)} =& \frac{1}{2}\tilde{t}_4 u_5 \\
c_9^{(12)} =& 6\tilde{t}_4^3 - \frac{3}{2} w_8 \tilde{t}_4,
\end{aligned}
$$

where $\tilde{t}_4 \equiv -t_4/2$. Thus, we get precisely the pole structure and constraints of the A_3 puncture.

4.1.5 Global symmetries and the superconformal index

4.1.5.1 Cataloguing fixtures using the superconformal index

For the E_6 theory, we find 880 fixtures with three regular punctures which correspond to interacting SCFTs, possibly with additional decoupled hypermultiplets. Each of these SCFTs has a manifest global symmetry group, which is given by the product of the flavor symmetry groups of the three punctures. This global symmetry group may, in general, become enhanced to a larger group.

To determine the global symmetry group and number of free hypermultiplets for each of these fixtures, we use the superconformal index [22, 23, 24, 25, 26]. The superconformal index of E-type class $\mathcal{S}$ theories has not yet been systematically studied. However, since the methods used for A- and D-type theories generalize to any root system, we assume the superconformal index[2] for a fixture in the E_6 theory takes the usual form

$$\mathcal{I}(\mathbf{a}_i, \tau) = \mathcal{A}(\tau) \sum_{\lambda} \frac{\prod_{i=1}^{3} \mathcal{K}(\mathbf{a}_i) P^{\lambda}(\mathbf{a}_i|\tau)}{P^{\lambda}(\mathbf{a}_{\text{triv}}|\tau)} \tag{4.4}$$

where

- The sum is over λ labeling the highest weights of finite-dimensional irreducible representations of $\mathfrak{e}_6$.
- The $P^{\lambda}(\mathbf{a}_i|\tau)$ are Hall-Littlewood polynomials, defined for a general root system by

[2]In what follows we will consider the Hall-Littlewood limit of the index [24], which depends on one superconformal fugacity, τ.

$$P^{\lambda} = W^{-1}(\tau) \sum_{w \in W} w \left(e^{\lambda} \prod_{\alpha \in R^+} \frac{1 - \tau^2 e^{-\alpha}}{1 - e^{-\alpha}} \right),$$

$$W(\tau) = \sqrt{\sum_{\substack{w \in W \\ w\lambda = \lambda}} \tau^{2\ell(w)}}$$

where R^+ denotes the set of positive roots, W the Weyl group, and $\ell(w)$ the length of the Weyl group element w.

- $\mathbf{a}_i \equiv \{e^{\alpha}\}_{\alpha \in R^+}$ denotes a set of flavor fugacities dual to the Cartan subalgebra of the flavor symmetry of the i^{th} puncture. $\mathbf{a}_{\text{triv}}$ denotes the set of fugacities dual to the Cartan of the embedded $\mathfrak{su}(2)$ of the trivial puncture.

- The $\mathcal{K}$-factors are discussed in [24, 60, 27, 26]. We will not need their detailed form for our purposes.

- $\mathcal{A}(\tau)$ is an overall, flavor fugacity independent normalization.

Consider a fixture corresponding to an interacting SCFT, with global symmetry G_{global}, plus free hypermultiplets transforming in a representation R of a flavor symmetry F. Let $G_{\text{fixt}} \equiv G_{\text{global}} \times F$ denote the global symmetry of the fixture. As discussed in [27], the number of free hypers in the fixture and the global symmetry of the fixture can be read off from the first two non-trivial terms in the Taylor expansion of the index. Schematically, this is given by

$$\mathcal{I} = 1 + \chi_{\text{F}}^{R} \tau + \chi_{G_{\text{fixt}}}^{adj} \tau^2 + \dots \tag{4.5}$$

where χ^R_{F} is the Weyl character of R and $\chi^{adj}_{G_{\mathrm{fixt}}}$ is the character of the adjoint representation of G_{fixt}, where both of these representations are viewed as reducible representations of the manifest symmetry algebra. By Taylor expanding $\mathcal{I}_{\mathrm{free}} = PE[\tau\chi^R_{\mathrm{F}}]$ (where PE denotes the Plethystic exponential) and removing the contribution of the free hypermultiplets in (4.5), we arrive at

$$\begin{aligned}\mathcal{I}_{\mathrm{SCFT}} &= \mathcal{I}/\mathcal{I}_{\mathrm{free}}\\ &= 1 + \chi^{adj}_{G_{\mathrm{global}}}\tau^2 + \ldots\end{aligned}$$

from which we can read off the global symmetry of the interacting SCFT.

4.1.5.2 Computing the expansion of the index

In (5.2) the term in the sum coming from the trivial representation of $\mathfrak{e}_6$ gives, to second order in τ, [27]

$$\mathcal{I} = 1 + \chi^{adj}_{G_{\mathrm{manifest}}}\tau^2 + \cdots$$

encoding the manifest global symmetry group. The global symmetry group of the fixture is enhanced if there are terms contributing at order τ^2 coming from the sum over $\lambda > 0$.

To order τ^2, (5.2) simplifies to [3]

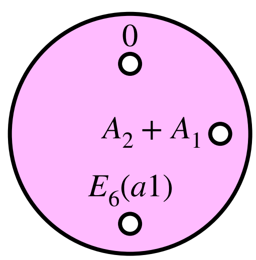
[3]Since the theories considered here are all "good" or "ugly" (in the sense of [61]), the lowest possible contribution from the sum over $\lambda > 0$ is at order τ (see [27] for a discussion of the superconformal index in the context of the good/ugly/bad trichotomy of 4d $\mathcal{N} = 2$ theories). From (4.6), we see that $\mathcal{A}(\tau)$ and $\mathcal{K}(\mathbf{a}_i)$ are both $1 + \mathcal{O}(\tau^2)$, so we can set them both to one in the order τ^2 approximation. We have also used the fact that $P^\lambda = \chi^\lambda + \mathcal{O}(\tau^2)$.

$$\mathcal{I} = 1 + \chi^{adj}_{G_{\text{manifest}}} \tau^2 + [\sum_{\lambda>0} \frac{\prod_{i=1}^{3} \chi^{\lambda}(\mathbf{a_i}|\tau)}{\chi^{\lambda}(\mathbf{a}_{\text{triv}}|\tau)}]_{\mathcal{O}(\tau^2)} \tag{4.6}$$

To compute (4.6), we consider each $\mathfrak{e}_6$ representation in the sum to be a reducible representation of $\mathfrak{su}(2) \times \mathfrak{f}$ and plug in the corresponding character expansion, where the embedded $\mathfrak{su}(2)$ has fugacity τ. The decomposition of any $\mathfrak{e}_6$ representation in terms of $\mathfrak{su}(2) \times \mathfrak{f}$ representations can be obtained using the projection matrices listed in Appendix E.2.

Of the 881 fixtures involving three regular punctures, we find that 1 is a free-field fixture, 60 are mixed fixtures and another 134 are interacting fixtures with an enhanced global symmetry group. We list these in the tables below. For the remaining 686 interacting fixtures, the global symmetry group is the manifest one.

As an example, consider the fixture



.

The manifest global symmetry is $(E_6)_{24} \times SU(3)_{12} \times U(1)$. The contributions at order τ^2 come from the sum over the 27, $\overline{27}$, 78, 351, $\overline{351}$, $351'$, $\overline{351}'$, and 650 of $\mathfrak{e}_6$. The expansion of the superconformal index is given by [4]

[4]For simplicity, we write the dimension to stand for the character of the corresponding representation. The subscript is the $U(1)$ weight.

$$
\begin{aligned}
\mathcal{I} = 1 &+ \{(27,1)_1 + (\overline{27},1)_{-1}\}\tau + \{(1,1)_0 + (78,1)_0 + \\
&(650,1)_0 + (27,1)_{-2} + (351',1)_{-2} + (\overline{27},1)_2 + \\
&(\overline{351}',1)_2 + (78,1)_0 + (1,8)_0 + (27,\overline{3})_0 + (\overline{27},3)_0\}\tau^2 + \ldots
\end{aligned}
$$

Due to the order τ term, this is a mixed fixture, with 27 free hypermultiplets transforming in the fundamental representation of E_6. The index of these free hypers is given by

$$
\begin{aligned}
\mathcal{I}_{\text{free}} &= PE[\tau\{(27,1)_1 + (\overline{27},1)_{-1}\}] \\
&= 1 + \{(27,1)_1 + (\overline{27},1)_{-1}\}\tau + \\
&\{(1,1)_0 + (78,1)_0 + (650,1)_0 + (27,1)_{-2} + (351',1)_{-2} + (\overline{27},1)_2 \\
&+ (\overline{351}',1)_2\}\tau^2 + \ldots
\end{aligned}
$$

The index of the underlying SCFT is then

$$
\begin{aligned}
\mathcal{I}_{\text{SCFT}} &= \mathcal{I}/\mathcal{I}_{\text{free}} \\
&= 1 + \{(78,1)_0 + (1,8)_0 + (27,\overline{3})_0 + (\overline{27},3)_0\}\tau^2
\end{aligned}
$$

We recognize the coefficient of τ^2 as the character of the adjoint representation of E_8. Computing the other numerical invariants of the fixture, we find that this is the $(E_8)_{12}$ theory of Minahan and Nemeschansky [62] with 27 additional free hypermultiplets.

4.1.6 Levels of enhanced global symmetry groups

Since the superconformal index gives the branching rule for the adjoint representation of G_{global} under the subgroup G_{manifest}, it most cases it is straightforward to determine the level of each factor in G_{global} from those of G_{manifest}: If $H_{k'}$ is a subgroup of G_k, then k is given by [2]

$$k = \frac{k'}{I_{H \hookrightarrow G}}$$

where $I_{H \hookrightarrow G}$ is the index of the embedding of H in G.

There are two cases which require a little more work. The first is when a manifest $U(1)$ becomes enhanced to $SU(2)$. Since we do not know how to assign a level to a $U(1)$ flavor symmetry (which would require a precise understanding of how the generator is normalized), we cannot immediately determine the level of the enhanced $SU(2)$ from the index.

The second case is when some factor H_k in G_{manifest} is embedded diagonally as

$$H_k \hookrightarrow H_{k_1} \times H_{k_2}.$$

Since the only embedding of H in itself has index one, in this case, all we know is that $k_1 + k_2 = k$.

If any of these remain as factors in G_{global} (that is, if they do not combine with some other factor, with known level, to enhance G_{global}), we cannot determine their levels from the index, and must determine them using an S-duality. To do so, we look for a 4-punctured sphere for which the SCFT appears in some degeneration, with H_{k_i} in the centralizer of subgroup of G_{global} being weakly gauged.

Unfortunately, there are a few such fixtures for which no puncture can be gauged (some of these can still be gauged in the twisted sector, which will be discussed in §4.3). For these, we do not have a way to determine the levels. In the end, there are two interacting fixtures whose levels we cannot completely determine.

4.2 Tinkertoys

4.2.1 Regular punctures

The pole structure $\{p_2, p_5, p_6, p_8, p_9, p_{12}\}$ of a puncture at $z = 0$ will be the leading pole orders in z of the differentials $\phi_k(z)$ for $k = 2, 5, 6, 8, 9, 12$. Notice that in some cases there are constraints, not just on the coefficient of this leading singularity, but also on *subleading* terms in the Laurent expansion of the k-differentials.

Table 4.1: Untwisted regular punctures

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
0	E_6	$\{1,4,5,7,8,11\}$	-	$(E_6)_{24}$	$(624, 588)$
A_1	$E_6(a_1)$	$\{1,4,5,7,8,10\}$	-	$SU(6)_{18}$	$(590, 565)$
$2A_1$	D_5	$\{1,4,5,7,7,10\}$	-	$Spin(7)_{16}$ $\times U(1)$	$(568, 548)$
$3A_1$ ($\underline{ns}$)	$(E_6(a_3), \mathbb{Z}_2)$	$\{1,4,5,6,7,10\}$	-	$SU(3)_{24}$ $\times SU(2)_{13}$	$(549, 533)$

Table 4.1: Untwisted regular punctures

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
A_2	$E_6(a_3)$	$\{1,4,5,6,7,10\}$	$c_{10}^{(12)} = -\left(c_5^{(6)}\right)^2 + \left(a_5^{(6)}\right)^2$	$SU(3)_{12}^2$	$(536, 521)$
$A_2 + A_1$	$D_5(a_1)$	$\{1,4,5,6,7,9\}$	-	$SU(3)_{12} \times U(1)$	$(523, 510)$
$2A_2$	D_4	$\{1,3,5,6,6,9\}$	-	$(G_2)_{12}$	$(496, 484)$
$A_2 + 2A_1$	$A_4 + A_1$	$\{1,4,4,6,7,9\}$	-	$SU(2)_{54} \times U(1)$	$(510, 499)$
A_3	A_4	$\{1,4,4,6,7,9\}$	$c_7^{(9)} = \frac{1}{2} c_4^{(5)} a_3^{(4)}$ $c_9^{(12)} = 6\left(a_3^{(4)}\right)^3 - \frac{3}{2} c_6^{(8)} a_3^{(4)}$	$Sp(2)_{10} \times U(1)$	$(476, 466)$
$2A_2 + A_1$ ($\underline{ns}$)	$(D_4(a_1), S_3)$	$\{1,3,4,6,6,9\}$	-	$SU(2)_{26}$	$(482, 473)$
$A_3 + A_1$ ($\underline{ns}$)	$(D_4(a_1), \mathbb{Z}_2)$	$\{1,3,4,6,6,9\}$	$c_9^{(12)} = a_3^{(4)}\left(\frac{16}{9}\left(a_3^{(4)}\right)^2 - c_6^{(8)}\right)$	$SU(2)_9 \times U(1)$	$(465, 457)$

Table 4.1: Untwisted regular punctures

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
$D_4(a_1)$	$D_4(a_1)$	$\{1,3,4,6,6,9\}$	$c_6^{(8)} = \frac{4}{3}\Big(\big(a_3^{(4)}\big)^2 + 3\big(a_3'^{(4)}\big)^2\Big)$ $c_9^{(12)} = \frac{4}{9}a_3^{(4)}\Big(\big(a_3^{(4)}\big)^2 - 9\big(a_3'^{(4)}\big)^2\Big)$	$U(1)^2$	$(456, 449)$
A_4	A_3	$\{1,3,4,6,6,9\}$	$c_6^{(8)} = 3\big(a_3^{(4)}\big)^2$ $c_6^{(9)} = \frac{1}{4}c_3^{(5)}a_3^{(4)}$ $c_9^{(12)} = -\frac{3}{2}\big(a_3^{(4)}\big)^3$ $c_8^{(12)} = -\frac{3}{4}a_3^{(4)}c_5^{(8)}$	$SU(2)_8 \times U(1)$	$(408, 402)$

Table 4.1: Untwisted regular punctures

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
D_4	$2A_2$	$\{1,3,4,5,6,8\}$	$c_5^{(8)} = -4c_4^{(6)}c_1^{(2)} + 4c_3^{(5)}a_2^{(3)} - 2\left(a_2^{(3)}\right)^2 c_1^{(2)}$ $c_6^{(9)} = -\frac{1}{12}a_2^{(3)}\left(c_4^{(6)} + \frac{1}{2}\left(a_2^{(3)}\right)^2\right)$ $c_8^{(12)} = -\left(c_4^{(6)} + \frac{1}{2}\left(a_2^{(3)}\right)^2\right)\cdot \cdot\left(c_4^{(6)} - \frac{3}{2}\left(a_2^{(3)}\right)^2\right)$ $c_7^{(12)} = -12c_5^{(9)}a_2^{(3)} - 2c_4^{(6)}c_3^{(6)} - c_3^{(6)}\left(a_2^{(3)}\right)^2$	$SU(3)_{12}$	$(368, 362)$
$A_4 + A_1$	$A_2 + 2A_1$	$\{1,3,4,5,5,7\}$	-	$U(1)$	$(400, 395)$
$D_5(a_1)$	$A_2 + A_1$	$\{1,3,4,5,5,7\}$	$c_4^{(6)} = -\frac{1}{8}\left(a_2^{(3)}\right)^2$ $c_5^{(8)} = 2c_3^{(5)}a_2^{(3)}$ $c_7^{(12)} = -6c_5^{(9)}a_2^{(3)}$	$U(1)$	$(355, 351)$
A_5 ($\underline{ns}$)	$(A_2, \mathbb{Z}_2)$	$\{1,2,4,4,4,6\}$	-	$SU(2)_7$	$(335, 331)$

Table 4.1: Untwisted regular punctures

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
$E_6(a_3)$	A_2	$\{1,2,4,4,4,6\}$	$c_4^{(6)} = \left(a_2^{(3)}\right)^2$	none	$(328, 325)$
D_5	$2A_1$	$\{1,2,3,4,4,6\}$	$c_3^{(6)} = \frac{3}{2} c_1^{(2)} a_2^{(4)}$ $c_4^{(8)} = 3\left(a_2^{(4)}\right)^2$ $c_4^{(9)} = -\frac{1}{4} a_2^{(4)} c_2^{(5)}$ $c_6^{(12)} = \frac{3}{2}\left(a_2^{(4)}\right)^3$ $c_5^{(12)} = \frac{3}{4} c_3^{(8)} a_2^{(4)}$	$U(1)$	$(240, 238)$
$E_6(a_1)$	A_1	$\{1,1,2,2,2,3\}$	-	none	$(168, 167)$

Note that there is a special piece, consisting of three punctures: $2A_2 + A_1$, $A_3 + A_1$ and the special puncture $D_4(a_1)$. For $2A_2 + A_1$, the Sommers-Achar group is the nonabelian group, S_3. It acts on $a^{(4)}, a'^{(4)}$ as

$$\begin{pmatrix} a^{(4)} \\ a'^{(4)} \end{pmatrix} \to \gamma \begin{pmatrix} a^{(4)} \\ a'^{(4)} \end{pmatrix}$$

for

$$\gamma \in \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} -1 & -3 \\ 1 & -1 \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} -1 & 3 \\ -1 & -1 \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} -1 & -3 \\ -1 & 1 \end{pmatrix}, \tfrac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \}$$

For A_3+A_1, the Sommers-Achar group is the $\mathbb{Z}_2$ subgroup of S_3, generated by $a'^{(4)} \to -a'^{(4)}$. For $D_4(a_1)$, the Sommers-Achar group is of course trivial, so that both $a^{(4)}, a'^{(4)}$ survive as Coulomb branch parameters.

4.2.2 Free-field fixtures

We denote a 3-punctured sphere, in the tables below, by listing the Bala-Carter labels of the three punctures. For the free-field fixtures, one of the punctures is an irregular puncture[5] (in the sense used in our previous papers), which we denote[6] by the pair, $(\mathcal{O}, G_k)$, where $\mathcal{O}$ is the regular puncture obtained as the OPE of the two regular punctures which collide, and this fixture is attached to the rest of the surface via a cylinder

$$(\mathcal{O}, G_k) \xleftrightarrow{\;G\;} \mathcal{O}$$

with gauge group $G \subset F \subset E_6$. Here, F is the flavour symmetry group of the puncture, $\mathcal{O}$, and the levels are such that G has vanishing β-function.

Table 4.2: Free field fixtures

#	Fixture	n_h	Representation
1	$E_6(a_1)$, $E_6(a_1)$, $(A_5, SU(2)_1)$	1	$\frac{1}{2}(2)$
2	$E_6(a_1)$, D_5, $(A_4, SU(2)_0)$	0	empty

[5] Or, in the case of fixture 13, a full puncture, corresponding to the trivial orbit, 0.

[6] For brevity, we will often omit the level, k, when denoting an irregular puncture.

Table 4.2: Free field fixtures

#	Fixture	n_h	Representation
3	$E_6(a_1)$ $E_6(a_3)$ $(2A_2, SU(3)_0)$	0	empty
4	$E_6(a_1)$ A_5 $(2A_2, (G_2)_4)$	7	$\frac{1}{2}(2,7)$
5	$E_6(a_1)$ $D_5(a_1)$ $(A_2 + A_1, SU(3)_0)$	0	empty
6	$E_6(a_1)$ $A_4 + A_1$ $(2A_1, (G_2)_0)$	0	empty
7	$E_6(a_1)$ D_4 $(A_2, SU(3)_0)$	0	empty
8	$E_6(a_1)$ A_4 $(2A_1, Spin(7)_4)$	8	$\frac{1}{2}(2,8)$
9	$E_6(a_1)$ $D_4(a_1)$ $(0, Spin(8)_0)$	0	empty
10	$E_6(a_1)$ $A_3 + A_1$ $(0, Spin(9)_4)$	9	$\frac{1}{2}(2,9)$
11	$E_6(a_1)$ $2A_2 + A_1$ $(0, (F_4)_{12})$	26	$\frac{1}{2}(2,26)$
12	$E_6(a_1)$ A_3 $(0, Spin(10)_8)$	20	$\frac{1}{2}(4,10)$
13	$E_6(a_1)$ $A_2 + 2A_1$ 0	54	$(2,27)$
14	D_5 D_5 $(A_3, Sp(2)_2)$	4	$1(4)$
15	D_5 $E_6(a3)$ $(2A_1, SU(4)_0)$	0	empty

Table 4.2: Free field fixtures

#	Fixture	n_h	Representation
16	D_5 A_5 $(2A_1, Spin(7)_4)$	7	$\frac{1}{2}(2,7)$
17	D_5 $D_5(a_1)$ $(A_1, SU(5)_2)$	5	$1(5)$
18	D_5 A_4+A_1 $(0, Spin(10)_8)$	16	$1(16)$
19	D_5 D_4 $(A_1, SU(6)_6)$	18	$3(6)$

4.2.3 Interacting fixtures with one irregular puncture

In the tables below, n_d is the number of Coulomb branch parameters of degree d. The total Coulomb branch dimension is $\sum_d n_d$ and the effective number of vector multiplets is $n_v = \sum_d (2d-1)n_d$.

Table 4.3: Interacting fixtures with one irregular puncture

Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	Theory
$E_6(a_1)$ $2A_2$ $(0,(F_4)_{12})$	$(0,0,0,0,1,0,0,0)$	$(40,11)$	$(E_8)_{12}$ SCFT
D_5 A_4 $(0, Spin(10)_8)$	$(0,0,1,0,0,0,0,0)$	$(24,7)$	$(E_7)_8$ SCFT
$E_6(a_3)$ $E_6(a_3)$ $(0,(F_4)_{12})$	$(0,2,0,0,0,0,0,0)$	$(32,10)$	$[(E_6)_6\text{ SCFT}]^2$
$E_6(a_3)$ A_5 $(0,(F_4)_{12})$	$(0,1,0,0,1,0,0,0)$	$(39,16)$	$(E_6)_{12} \times SU(2)_7$ SCFT
A_5 A_5 $(0,(F_4)_{12})$	$(0,0,0,0,2,0,0,0)$	$(46,22)$	$(F_4)_{12} \times SU(2)_7^2$ SCFT

The $(E_6)_{12}\times SU(2)_7$ and $(F_4)_{12}\times SU(2)_7^2$ first appeared in [5], as fixtures in the untwisted D_4 theory.

4.2.4 Interacting fixtures with enhanced global symmetry

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	G_k
1	$E_6(a_1)$ A_2 0	$(0,0,0,0,2,0,0,0)$	$(80,22)$	$[(E_8)_{12}$ SCFT$]^2$
2	$E_6(a_1)$ $3A_1$ 0	$(0,0,0,0,1,0,0,1)$	$(93,34)$	$(E_8)_{24}\times SU(2)_{13}$
3	$E_6(a_1)$ $2A_1$ 0	$(0,0,0,0,1,1,0,1)$	$(112,49)$	$(E_7)_{24}\times Spin(7)_{16}$
4	$E_6(a_1)$ A_1 A_1	$(0,0,0,0,1,1,1,0)$	$(100,43)$	$SU(12)_{18}$
5	D_5 $D_4(a_1)$ 0	$(0,0,3,0,0,0,0,0)$	$(72,21)$	$[(E_7)_8$ SCFT$]^3$
6	D_5 A_3+A_1 0	$(0,0,2,0,0,1,0,0)$	$(81,29)$	$[(E_7)_8$ SCFT$]$ $\times[(E_7)_{16}$ $\times SU(2)_9$ SCFT$]$
7	D_5 $2A_2+A_1$ 0	$(0,0,1,0,0,1,0,1)$	$(98,45)$	$(E_7)_{24}\times SU(2)_{26}$
8	D_5 A_3 0	$(0,0,2,1,0,1,0,0)$	$(92,38)$	$[(E_7)_8$ SCFT$]$ $\times[(E_6)_{16}\times Sp(2)_{10}$ $\times U(1)$ SCFT$]$
9	D_5 $2A_2$ 0	$(0,0,1,0,1,1,0,1)$	$(112,56)$	$(E_7)_{24}\times (G_2)_{12}$
10	D_5 A_2+2A_1 A_1	$(0,0,1,1,0,1,1,0)$	$(92,48)$	$SU(8)_{18}\times SU(2)_{36}\times U(1)$
11	D_5 A_2+A_1 A_1	$(0,0,1,1,1,1,1,0)$	$(105,59)$	$SU(7)_{18}\times SU(3)_{12}\times U(1)^2$
12	D_5 A_2 $2A_1$	$(0,0,1,1,2,1,0,0)$	$(96,53)$	$Spin(8)_{16}\times SU(4)_{12}^2\times U(1)$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	G_k
13	D_5 A_1 A_2	$(0,0,1,1,2,1,1,0)$	$(118,70)$	$SU(6)_{18} \times SU(3)_{12}^2 \times U(1)^2$
14	D_5 $3A_1$ $3A_1$	$(0,0,1,1,1,0,0,1)$	$(90,50)$	$SU(6)_{24} \times Sp(2)_{13}$
15	D_5 $2A_1$ $3A_1$	$(0,0,1,1,1,1,0,1)$	$(109,65)$	$Spin(7)_{16} \times SU(4)_{24} \times SU(2)_{13} \times U(1)$
16	D_5 $2A_1$ $2A_1$	$(0,0,1,1,1,2,0,1)$	$(128,80)$	$Spin(7)_{16}^2 \times SU(2)_{24} \times U(1)^2$
17	$E_6(a_3)$ 0 A_4+A_1	$(0,1,0,0,1,1,0,1)$	$(104,54)$	$(E_7)_{24}$
18	$E_6(a_3)$ 0 D_4	$(0,2,0,0,1,0,0,0)$	$(72,21)$	$[(E_8)_{12}$ SCFT$] \times [(E_6)_6$ SCFT$]^2$
19	$E_6(a_3)$ 0 A_4	$(0,1,1,0,1,1,0,1)$	$(112,61)$	$(E_7)_{24} \times SU(2)_8$
20	$E_6(a_3)$ A_2 $D_4(a_1)$	$(0,1,2,0,2,0,0,0)$	$(72,41)$	$Spin(8)_{12}^2 \times U(1)^2$
21	$E_6(a_3)$ $3A_1$ $D_4(a_1)$	$(0,1,2,0,1,0,0,1)$	$(85,53)$	$Spin(8)_{24} \times SU(2)_{13}$
22	$E_6(a_3)$ $2A_1$ $D_4(a_1)$	$(0,1,2,0,1,1,0,1)$	$(104,68)$	$Spin(7)_{16} \times SU(2)_{24}^3$
23	$E_6(a_3)$ A_2 A_3+A_1	$(0,1,1,0,2,1,0,0)$	$(81,49)$	$Spin(7)_{12}^2 \times SU(2)_9 \times U(1)$
24	$E_6(a_3)$ $3A_1$ A_3+A_1	$(0,1,1,0,1,1,0,1)$	$(94,61)$	$Spin(7)_{24} \times SU(2)_{13} \times SU(2)_9$
25	$E_6(a_3)$ $2A_1$ A_3+A_1	$(0,1,1,0,1,2,0,1)$	$(113,76)$	$Spin(7)_{16} \times SU(2)_{48} \times SU(2)_{24} \times SU(2)_9$
26	$E_6(a_3)$ A_2 $2A_2+A_1$	$(0,1,0,0,2,1,0,1)$	$(98,65)$	$(G_2)_{12}^2 \times SU(2)_{26}$
27	$E_6(a_3)$ $3A_1$ $2A_2+A_1$	$(0,1,0,0,1,1,0,2)$	$(111,77)$	$(G_2)_{24} \times SU(2)_{26} \times SU(2)_{13}$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	G_k
28	$E_6(a_3)$ $2A_1$ $2A_2+A_1$	$(0,1,0,0,1,2,0,2)$	$(130,92)$	$Spin(7)_{16} \times SU(2)_{26} \times SU(2)_{72}$
29	$E_6(a_3)$ A_2 A_3	$(0,1,1,1,2,1,0,0)$	$(92,58)$	$SU(4)_{12}^2 \times Sp(2)_{10} \times U(1)$
30	$E_6(a_3)$ $3A_1$ A_3	$(0,1,1,1,1,1,0,1)$	$(105,70)$	$SU(4)_{24} \times Sp(2)_{10} \times SU(2)_{13}$
31	$E_6(a_3)$ $2A_1$ A_3	$(0,1,1,1,1,2,0,1)$	$(124,85)$	$Spin(7)_{16} \times Sp(2)_{10} \times SU(2)_{24} \times U(1)$
32	$E_6(a_3)$ A_2+2A_1 A_2+2A_1	$(0,1,0,1,0,1,1,1)$	$(100,69)$	$SU(4)_{54} \times U(1)$
33	$E_6(a_3)$ A_2+A_1 A_2+2A_1	$(0,1,0,1,1,1,1,1)$	$(113,80)$	$SU(3)_{54} \times SU(3)_{12} \times U(1)$
34	$E_6(a_3)$ A_2 $2A_2$	$(0,1,0,0,3,1,0,1)$	$(112,76)$	$(G_2)_{12}^3$
35	$E_6(a_3)$ $3A_1$ $2A_2$	$(0,1,0,0,2,1,0,2)$	$(125,88)$	$(G_2)_{24} \times (G_2)_{12} \times SU(2)_{13}$
36	$E_6(a_3)$ $2A_1$ $2A_2$	$(0,1,0,0,2,2,0,2)$	$(144,103)$	$Spin(7)_{16} \times (G_2)_{12} \times SU(2)_{72}$
37	$E_6(a_3)$ A_2+A_1 A_2+A_1	$(0,1,0,1,2,1,1,1)$	$(126,91)$	$SU(3)_{12}^2 \times SU(2)_{24} \times U(1)$
38	A_5 0 A_4+A_1	$(0,0,0,0,2,1,0,1)$	$(111,60)$	$(E_7)_{24} \times SU(2)_7$
39	A_5 0 D_4	$(0,1,0,0,2,0,0,0)$	$(79,27)$	$[(E_8)_{12}$ SCFT$]$ $\times$ $[(E_6)_{12}$ $\times$ $SU(2)_7$ SCFT$]$
40	A_5 0 A_4	$(0,0,1,0,2,1,0,1)$	$(119,67)$	$(E_7)_{24} \times SU(2)_8 \times SU(2)_7$
41	A_5 A_2 $D_4(a_1)$	$(0,0,2,0,3,0,0,0)$	$(79,47)$	$Spin(8)_{12}^2 \times SU(2)_7$
42	A_5 $3A_1$ $D_4(a_1)$	$(0,0,2,0,2,0,0,1)$	$(92,59)$	$Spin(8)_{24} \times SU(2)_{13} \times SU(2)_7$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	G_k
43	A_5 $D_4(a_1)$ $2A_1$	$(0,0,2,0,2,1,0,1)$	$(111,74)$	$Spin(7)_{16}\times SU(2)^3_{24}\times SU(2)_7$
44	A_5 A_3+A_1 A_2	$(0,0,1,0,3,1,0,0)$	$(88,55)$	$Spin(7)^2_{12}\times SU(2)_9\times SU(2)_7$
45	A_5 A_3+A_1 $3A_1$	$(0,0,1,0,2,1,0,1)$	$(101,67)$	$Spin(7)_{24}\times SU(2)_{13}\times SU(2)_9\times SU(2)_7$
46	A_5 A_3+A_1 $2A_1$	$(0,0,1,0,2,2,0,1)$	$(120,82)$	$Spin(7)_{16}\times SU(2)_{48}$ $\times SU(2)_{24}\times SU(2)_9$ $\times SU(2)_7$
47	A_5 $2A_2+A_1$ A_2	$(0,0,0,0,3,1,0,1)$	$(105,71)$	$(G_2)^2_{12}\times SU(2)_{26}\times SU(2)_7$
48	A_5 $2A_2+A_1$ $3A_1$	$(0,0,0,0,2,1,0,2)$	$(118,83)$	$(G_2)_{24}\times SU(2)_{26}\times SU(2)_{13}\times SU(2)_7$
49	A_5 $2A_2+A_1$ $2A_1$	$(0,0,0,0,2,2,0,2)$	$(137,98)$	$Spin(7)_{16}\times SU(2)_{72}\times SU(2)_{26}\times SU(2)_7$
50	A_5 A_3 A_2	$(0,0,1,1,3,1,0,0)$	$(99,64)$	$SU(4)^2_{12}\times Sp(2)_{10}\times SU(2)_7$
51	A_5 A_3 $3A_1$	$(0,0,1,1,2,1,0,1)$	$(112,76)$	$SU(4)_{24}\times Sp(2)_{10}\times SU(2)_{13}\times SU(2)_7$
52	A_5 A_3 $2A_1$	$(0,0,1,1,2,2,0,1)$	$(131,91)$	$Spin(7)_{16}\times Sp(2)_{10}$ $\times SU(2)_{24}\times SU(2)_7$ $\times U(1)$
53	A_5 A_2+2A_1 A_2+2A_1	$(0,0,0,1,1,1,1,1)$	$(107,75)$	$SU(4)_{54}\times SU(2)_7\times U(1)$
54	A_5 A_2+2A_1 A_2+A_1	$(0,0,0,1,2,1,1,1)$	$(120,86)$	$SU(3)_{54}\times SU(3)_{12}\times SU(2)_7\times U(1)$
55	A_5 $2A_2$ A_2	$(0,0,0,0,4,1,0,1)$	$(119,82)$	$(G_2)^3_{12}\times SU(2)_7$
56	A_5 $2A_2$ $3A_1$	$(0,0,0,0,3,1,0,2)$	$(132,94)$	$(G_2)_{24}\times (G_2)_{12}\times SU(2)_{13}\times SU(2)_7$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	G_k
57	A_5 $2A_2$ $2A_1$	$(0,0,0,0,3,2,0,2)$	$(151,109)$	$Spin(7)_{16} \times (G_2)_{12} \times SU(2)_{72} \times SU(2)_7$
58	A_5 A_2+A_1 A_2+A_1	$(0,0,0,1,3,1,1,1)$	$(133,97)$	$SU(3)_{12}^2 \times SU(2)_{24} \times SU(2)_7 \times U(1)$
59	$D_5(a_1)$ $D_5(a_1)$ 0	$(0,2,0,1,0,0,1,0)$	$(86,36)$	$(E_7)_{18} \times (E_6)_6 \times U(1)$
60	$D_5(a_1)$ A_4+A_1 A_1	$(0,1,0,1,1,1,1,0)$	$(97,57)$	$SU(7)_{18} \times U(1)^2$
61	$D_5(a_1)$ D_4 0	$(0,2,0,1,1,0,1,0)$	$(99,47)$	$(E_6)_{18} \times (E_6)_6 \times SU(3)_{12} \times U(1)$
62	$D_5(a_1)$ A_4 A_1	$(0,1,1,1,1,1,1,0)$	$(105,64)$	$SU(7)_{18} \times SU(2)_8 \times U(1)^2$
63	$D_5(a_1)$ $D_4(a_1)$ A_2+2A_1	$(0,1,2,1,0,0,1,0)$	$(73,45)$	$SU(3)_{54-k-k'} \times SU(3)_k \times SU(3)_{k'} \times U(1)$
64	$D_5(a_1)$ $D_4(a_1)$ A_2+A_1	$(0,1,2,1,1,0,1,0)$	$(86,56)$	$SU(3)_{12} \times SU(2)_{18}^3 \times U(1)^3$
65	$D_5(a_1)$ $D_4(a_1)$ A_2	$(0,1,2,1,2,0,1,0)$	$(99,67)$	$SU(3)_{12}^2 \times U(1)^5$
66	$D_5(a_1)$ A_3+A_1 A_2+2A_1	$(0,1,1,1,0,1,1,0)$	$(82,53)$	$SU(3)_{54-k} \times SU(3)_k \times SU(2)_9 \times U(1)$
67	$D_5(a_1)$ A_3+A_1 A_2+A_1	$(0,1,1,1,1,1,1,0)$	$(95,64)$	$SU(3)_{12} \times SU(2)_{36} \times SU(2)_{18} \times SU(2)_9 \times U(1)^2$
68	$D_5(a_1)$ A_3+A_1 A_2	$(0,1,1,1,2,1,1,0)$	$(108,75)$	$SU(3)_{12}^2 \times SU(2)_9 \times U(1)^3$
69	$D_5(a_1)$ $2A_2+A_1$ A_2+2A_1	$(0,1,0,1,0,1,1,1)$	$(99,69)$	$SU(3)_{54} \times SU(2)_{26} \times U(1)$
70	$D_5(a_1)$ $2A_2+A_1$ A_2+A_1	$(0,1,0,1,1,1,1,1)$	$(112,80)$	$SU(3)_{12} \times SU(2)_{54} \times SU(2)_{26} \times U(1)$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	G_k
71	$D_5(a_1)$ A_3 A_2+2A_1	$(0,1,1,2,0,1,1,0)$	$(93,62)$	$SU(3)_{18}\times SU(2)_{36}\times Sp(2)_{10}\times U(1)^2$
72	$D_5(a_1)$ A_3 A_2+A_1	$(0,1,1,2,1,1,1,0)$	$(106,73)$	$SU(3)_{12}\times Sp(2)_{10}\times SU(2)_{18}\times U(1)^3$
73	$D_5(a_1)$ A_3 A_2	$(0,1,1,2,2,1,1,0)$	$(119,84)$	$SU(3)_{12}^2\times Sp(2)_{10}\times U(1)^3$
74	$D_5(a_1)$ A_2+2A_1 $2A_2$	$(0,1,0,1,1,1,1,1)$	$(113,80)$	$(G_2)_{12}\times SU(3)_{54}\times U(1)$
75	$D_5(a_1)$ $2A_2$ A_2+A_1	$(0,1,0,1,2,1,1,1)$	$(126,91)$	$(G_2)_{12}\times SU(3)_{12}\times SU(2)_{54}\times U(1)$
76	A_4+A_1 A_4+A_1 A_2	$(0,0,0,1,3,1,0,0)$	$(88,57)$	$SU(4)_{12}^2\times U(1)$
77	A_4+A_1 A_4+A_1 $3A_1$	$(0,0,0,1,2,1,0,1)$	$(101,69)$	$SU(4)_{24}\times SU(2)_{13}\times U(1)$
78	A_4+A_1 A_4+A_1 $2A_1$	$(0,0,0,1,2,2,0,1)$	$(120,84)$	$Spin(7)_{16}\times SU(2)_{24}\times U(1)^2$
79	A_4+A_1 D_4 $2A_1$	$(0,1,0,1,2,1,0,0)$	$(88,51)$	$Spin(8)_{16}\times SU(4)_{12}\times U(1)^2$
80	A_4+A_1 D_4 A_1	$(0,1,0,1,2,1,1,0)$	$(110,68)$	$SU(6)_{18}\times SU(3)_{12}\times U(1)^2$
81	A_4+A_1 A_4 A_2	$(0,0,1,1,3,1,0,0)$	$(96,64)$	$SU(4)_{12}^2\times SU(2)_8\times U(1)$
82	A_4+A_1 A_4 $3A_1$	$(0,0,1,1,2,1,0,1)$	$(109,76)$	$SU(4)_{24}\times SU(2)_{13}\times SU(2)_8\times U(1)$
83	A_4+A_1 A_4 $2A_1$	$(0,0,1,1,2,2,0,1)$	$(128,91)$	$Spin(7)_{16}\times SU(2)_8\times SU(2)_{24}\times U(1)^2$
84	A_4+A_1 $D_4(a_1)$ $D_4(a_1)$	$(0,0,4,0,1,0,0,0)$	$(64,39)$	$SU(2)_8^9$
85	A_4+A_1 $D_4(a_1)$ A_3+A_1	$(0,0,3,0,1,1,0,0)$	$(73,47)$	$SU(2)_{16}^3\times SU(2)_9\times SU(2)_8^3$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	G_k
86	A_4+A_1 $D_4(a_1)$ $2A_2+A_1$	$(0,0,2,0,1,1,0,1)$	$(90,63)$	$SU(2)_{26}\times SU(2)_{24}^3$
87	A_4+A_1 $D_4(a_1)$ A_3	$(0,0,3,1,1,1,0,0)$	$(84,56)$	$Sp(2)_{10}\times SU(2)_8^3\times U(1)^3$
88	A_4+A_1 $D_4(a_1)$ $2A_2$	$(0,0,2,0,2,1,0,1)$	$(104,74)$	$(G_2)_{12}\times SU(2)_{24}^3$
89	A_4+A_1 A_3+A_1 A_3+A_1	$(0,0,2,0,1,2,0,0)$	$(82,55)$	$SU(2)_{32}\times SU(2)_{16}^2\times SU(2)_9^2\times SU(2)_8^2$
90	A_4+A_1 A_3+A_1 $2A_2+A_1$	$(0,0,1,0,1,2,0,1)$	$(99,71)$	$SU(2)_{48}\times SU(2)_{26}\times SU(2)_{24}\times SU(2)_9$
91	A_4+A_1 A_3+A_1 A_3	$(0,0,2,1,1,2,0,0)$	$(93,64)$	$Sp(2)_{10}\times SU(2)_{16}\times SU(2)_9\times SU(2)_8\times U(1)^2$
92	A_4+A_1 A_3+A_1 $2A_2$	$(0,0,1,0,2,2,0,1)$	$(113,82)$	$(G_2)_{12}\times SU(2)_{48}\times SU(2)_{24}\times SU(2)_9$
93	A_4+A_1 $2A_2+A_1$ $2A_2+A_1$	$(0,0,0,0,1,2,0,2)$	$(116,87)$	$SU(2)_{72}\times SU(2)_{26}^2$
94	A_4+A_1 $2A_2+A_1$ A_3	$(0,0,1,1,1,2,0,1)$	$(110,80)$	$Sp(2)_{10}\times SU(2)_{26}\times SU(2)_{24}\times U(1)$
95	A_4+A_1 $2A_2+A_1$ $2A_2$	$(0,0,0,0,2,2,0,2)$	$(130,98)$	$(G_2)_{12}\times SU(2)_{72}\times SU(2)_{26}$
96	A_4+A_1 A_3 A_3	$(0,0,2,2,1,2,0,0)$	$(104,73)$	$Sp(2)_{10}^2\times SU(2)_8\times U(1)^3$
97	A_4+A_1 A_3 $2A_2$	$(0,0,1,1,2,2,0,1)$	$(124,91)$	$(G_2)_{12}\times Sp(2)_{10}\times SU(2)_{24}\times U(1)$
98	A_4+A_1 $2A_2$ $2A_2$	$(0,0,0,0,3,2,0,2)$	$(144,109)$	$(G_2)_{12}^2\times SU(2)_{72}$
99	D_4 D_4 0	$(0,2,0,1,2,0,1,0)$	$(112,58)$	$(E_6)_{18}\times (E_6)_6\times SU(3)_{12}^2$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	G_k
100	D_4 A_4 $2A_1$	$(0,1,1,1,2,1,0,0)$	$(96,58)$	$Spin(8)_{16} \times SU(4)_{12} \times SU(2)_8 \times U(1)^2$
101	D_4 A_4 A_1	$(0,1,1,1,2,1,1,0)$	$(118,75)$	$SU(6)_{18} \times SU(3)_{12} \times SU(2)_8 \times U(1)^2$
102	D_4 $D_4(a_1)$ A_2+2A_1	$(0,1,2,1,1,0,1,0)$	$(86,56)$	$SU(3)_{12} \times SU(2)^3_{18} \times U(1)^3$
103	D_4 $D_4(a_1)$ $2A_2$	$(0,1,2,0,2,0,0,0)$	$(72,41)$	$Spin(8)^2_{12} \times U(1)^2$
104	D_4 $D_4(a_1)$ A_2+A_1	$(0,1,2,1,2,0,1,0)$	$(99,67)$	$SU(3)^2_{12} \times U(1)^5$
105	D_4 $D_4(a_1)$ A_2	$(0,1,2,1,3,0,1,0)$	$(112,78)$	$SU(3)^3_{12} \times U(1)^4$
106	D_4 A_3+A_1 A_2+2A_1	$(0,1,1,1,1,1,1,0)$	$(95,64)$	$SU(3)_{12} \times SU(2)_{36}$ $\times SU(2)_{18} \times SU(2)_9$ $\times U(1)^2$
107	D_4 A_3+A_1 $2A_2$	$(0,1,1,0,2,1,0,0)$	$(81,49)$	$Spin(7)^2_{12} \times SU(2)_9 \times U(1)$
108	D_4 A_3+A_1 A_2+A_1	$(0,1,1,1,2,1,1,0)$	$(108,75)$	$SU(3)^2_{12} \times SU(2)_9 \times U(1)^3$
109	D_4 A_3+A_1 A_2	$(0,1,1,1,3,1,1,0)$	$(121,86)$	$SU(3)^3_{12} \times SU(2)_9 \times U(1)^2$
110	D_4 $2A_2+A_1$ $2A_2+A_1$	$(0,1,0,0,1,1,0,1)$	$(84,54)$	$(G_2)_{12} \times Sp(2)_{26}$
111	D_4 $2A_2+A_1$ $2A_2$	$(0,1,0,0,2,1,0,1)$	$(98,65)$	$(G_2)^2_{12} \times SU(2)_{26}$
112	D_4 A_3 A_2+2A_1	$(0,1,1,2,1,1,1,0)$	$(106,73)$	$Sp(2)_{10} \times SU(3)_{12}$ $\times SU(2)_{36} \times SU(2)_{18}$ $\times U(1)^2$
113	D_4 A_3 $2A_2$	$(0,1,1,1,2,1,0,0)$	$(92,58)$	$Spin(7)_{12} \times SU(4)_{12} \times Sp(2)_{10} \times U(1)$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	G_k
114	D_4 A_3 $A_2 + A_1$	$(0,1,1,2,2,1,1,0)$	$(119,84)$	$SU(3)_{12}^2 \times Sp(2)_{10} \times U(1)^3$
115	D_4 A_3 A_2	$(0,1,1,2,3,1,1,0)$	$(132,95)$	$SU(3)_{12}^3 \times Sp(2)_{10} \times U(1)^2$
116	D_4 $2A_2$ $2A_2$	$(0,1,0,0,3,1,0,1)$	$(112,76)$	$(G_2)_{12}^3$
117	A_4 A_4 A_2	$(0,0,2,1,3,1,0,0)$	$(104,71)$	$SU(4)_{12}^2 \times SU(2)_8^2 \times U(1)$
118	A_4 A_4 $3A_1$	$(0,0,2,1,2,1,0,1)$	$(117,83)$	$SU(4)_{24} \times SU(2)_{13} \times SU(2)_8^2 \times U(1)$
119	A_4 A_4 $2A_1$	$(0,0,2,1,2,2,0,1)$	$(136,98)$	$Spin(7)_{16} \times SU(2)_8^2 \times SU(2)_{24} \times U(1)^2$
120	A_4 $D_4(a_1)$ $D_4(a_1)$	$(0,0,5,0,1,0,0,0)$	$(72,46)$	$SU(2)_8^{10}$
121	A_4 $D_4(a_1)$ $A_3 + A_1$	$(0,0,4,0,1,1,0,0)$	$(81,54)$	$SU(2)_{16}^3 \times SU(2)_9 \times SU(2)_8^4$
122	A_4 $D_4(a_1)$ $2A_2 + A_1$	$(0,0,3,0,1,1,0,1)$	$(98,70)$	$SU(2)_{26} \times SU(2)_{24}^3 \times SU(2)_8$
123	A_4 $D_4(a_1)$ A_3	$(0,0,4,1,1,1,0,0)$	$(92,63)$	$Sp(2)_{10} \times SU(2)_8^4 \times U(1)^3$
124	A_4 $D_4(a_1)$ $2A_2$	$(0,0,3,0,2,1,0,1)$	$(112,81)$	$(G_2)_{12} \times SU(2)_8 \times SU(2)_{24}^3$
125	A_4 $A_3 + A_1$ $A_3 + A_1$	$(0,0,3,0,1,2,0,0)$	$(90,62)$	$SU(2)_{32} \times SU(2)_{16}^2 \times SU(2)_9^2 \times SU(2)_8^2$
126	A_4 $A_3 + A_1$ $2A_2 + A_1$	$(0,0,2,0,1,2,0,1)$	$(107,78)$	$SU(2)_{48} \times SU(2)_{26} \times SU(2)_{24} \times SU(2)_9 \times SU(2)_8$

Table 4.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	G_k
127	A_4 A_3+A_1 A_3	$(0,0,3,1,1,2,0,0)$	$(101,71)$	$Sp(2)_{10}\times SU(2)_{16}$ $\times SU(2)_9\times SU(2)_8^2$ $\times U(1)^2$
128	A_4 A_3+A_1 $2A_2$	$(0,0,2,0,2,2,0,1)$	$(121,89)$	$(G_2)_{12}\times SU(2)_{48}$ $\times SU(2)_{24}\times SU(2)_9$ $\times SU(2)_8$
129	A_4 $2A_2+A_1$ $2A_2+A_1$	$(0,0,1,0,1,2,0,2)$	$(124,94)$	$SU(2)_{72}\times SU(2)_{26}^2\times$ $SU(2)_8$
130	A_4 $2A_2+A_1$ A_3	$(0,0,2,1,1,2,0,1)$	$(118,87)$	$Sp(2)_{10}\times SU(2)_{26}$ $\times SU(2)_{24}\times SU(2)_8$ $\times U(1)$
131	A_4 $2A_2+A_1$ $2A_2$	$(0,0,1,0,2,2,0,2)$	$(138,105)$	$(G_2)_{12}\times SU(2)_{72}\times$ $SU(2)_{26}\times SU(2)_8$
132	A_4 A_3 A_3	$(0,0,3,2,1,2,0,0)$	$(112,80)$	$Sp(2)_{10}^2\times SU(2)_8^2\times$ $U(1)^3$
133	A_4 A_3 $2A_2$	$(0,0,2,1,2,2,0,1)$	$(132,98)$	$(G_2)_{12}\times Sp(2)_{10}$ $\times SU(2)_{24}\times SU(2)_8$ $\times U(1)$
134	A_4 $2A_2$ $2A_2$	$(0,0,1,0,3,2,0,2)$	$(152,116)$	$(G_2)_{12}^2\times SU(2)_{72}\times$ $SU(2)_8$

We were unable to determine the $SU(3)$ levels in fixtures 63 and 66.

4.2.5 Mixed fixtures

We find many "new" SCFTs in our list of mixed fixtures. For each fixture in the table below, we list the global symmetry group, the graded Coulomb branch dimensions, and the effective number of vector and hypermultiplets of the SCFT. The effective number of hypermultiplets, for the fixture as a whole, is the sum of the n_h listed in the table and the number of free hypermultiplets in the last column. When the hypermultiplets transform under the nonabelian part of the "manifest" global symmetry of the fixture, we list that representation. Otherwise, we just give their number.

All SCFTs in the list below are "new", except for the $(E_6)_6$ SCFT, the $(E_6)_{12} \times SU(2)_7$ SCFT, the $SU(4)^3_8$ SCFT, and the $(E_8)_{12}$ SCFT, which have previously appeared in the classification of the A- and D-series fixtures, and the $(E_7)_{16} \times SU(2)_9$ and $(G_2)_{12} \times Sp(2)_{26}$ SCFTs, which appeared above.

Table 4.5: Mixed fixtures

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	Theory
1	$E_6(a_1)$ A_2+A_1 $\quad 0$	$(0,0,0,0,1,0,0,0)$	$(40,11)$	$(E_8)_{12}$ SCFT $+ 1(1,27)$
2	$E_6(a_1)$ $3A_1$ $\quad A_1$	$(0,0,0,0,1,0,0,0)$	$(40,11)$	$(E_8)_{12}$ SCFT $+ \frac{1}{2}(1,2,1) + 1(3,1,6)$
3	$E_6(a_1)$ $2A_1$ $\quad A_1$	$(0,0,0,0,1,1,0,0)$	$(72,26)$	$Spin(20)_{16}$ SCFT $+ 1(6,1)$
4	D_5 $2A_2+A_1$ $\quad A_1$	$(0,0,1,0,0,1,0,0)$	$(57,22)$	$(E_7)_{16} \times SU(2)_9 + \frac{1}{2}(2,1) + 1(1,6)$
5	D_5 A_2+2A_1 $\quad 3A_1$	$(0,0,1,1,0,0,0,0)$	$(42,16)$	$SU(8)_{10} \times SU(3)_{12} + 1(2;3,1) + \frac{1}{2}(3;1,2)$
6	D_5 A_2+2A_1 $\quad 2A_1$	$(0,0,1,1,0,1,0,0)$	$(68,31)$	$(E_6)_{16} \times Sp(2)_{10} \times U(1) + 1(2,1)$

Table 4.5: Mixed fixtures

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	Theory
7	D_5 A_1 $2A_2$	$(0,0,1,0,1,1,0,0)$	$(72,33)$	$Spin(7)_{12} \times Spin(12)_{16}$ SCFT $+ 1(1,6)$
8	D_5 $3A_1$ A_2+A_1	$(0,0,1,1,1,0,0,0)$	$(60,27)$	$SU(8)_{12} \times SU(4)_{10} + \frac{1}{2}(1;1,2) + 1(1;3,1)$
9	D_5 $2A_1$ A_2+A_1	$(0,0,1,1,1,1,0,0)$	$(82,42)$	$Spin(10)_{16} \times SU(4)_{12}$ $\times SU(2)_{10} \times U(1)$ $+1$ free hyper
10	D_5 $3A_1$ A_2	$(0,0,1,1,2,0,0,0)$	$(76,38)$	$SU(6)^2_{12} \times SU(2)_{12} + \frac{1}{2}(1,1;1,2)$
11	$E_6(a_3)$ 0 $D_5(a1)$	$(0,2,0,0,0,0,0,0)$	$(32,10)$	$[(E_6)_6)\,\text{SCFT}]^2 + 1(27)$
12	$E_6(a3)$ A_1 A_4+A_1	$(0,1,0,0,1,1,0,0)$	$(64,31)$	$Spin(13)_{16} \times U(1) + 1(6)$
13	$E_6(a3)$ A_1 A_4	$(0,1,1,0,1,1,0,0)$	$(72,38)$	$Spin(12)_{16} \times SU(2)_8 \times U(1)$ SCFT $+ 1(1,6)$
14	$E_6(a3)$ A_2+2A_1 $D_4(a1)$	$(0,1,2,0,0,0,0,0)$	$(40,19)$	$SU(4)^3_8$ SCFT $+ 3(2)$
15	$E_6(a3)$ A_2+A_1 $D_4(a1)$	$(0,1,2,0,1,0,0,0)$	$(56,30)$	$Spin(8)_{12} \times SU(2)^3_8 \times U(1)^2 + 3$ free hypers
16	$E_6(a3)$ A_2+2A_1 A_3+A_1	$(0,1,1,0,0,1,0,0)$	$(51,27)$	$Sp(3)_9 \times SU(4)_{16} + 2(1,2)$
17	$E_6(a3)$ A_2+A_1 A_3+A_1	$(0,1,1,0,1,1,0,0)$	$(66,38)$	$Spin(7)_{12} \times Sp(2)_9$ $\times SU(2)_{32} \times U(1)$ $+2$ free hypers

Table 4.5: Mixed fixtures

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	Theory
18	$E_6(a3)$ A_2 + $2A_2+A_1$ $2A_1$	$(0,1,0,0,0,1,0,1)$	$(70,43)$	$Sp(3)_{26}+1(1,2)$
19	$E_6(a3)$ A_2+A_1 $2A_2+A_1$	$(0,1,0,0,1,1,0,1)$	$(84,54)$	$(G_2)_{12} \times Sp(2)_{26}$ + 1 free hyper
20	$E_6(a3)$ A_2+2A_1 A_3	$(0,1,1,1,0,1,0,0)$	$(64,36)$	$Sp(4)_{10} \times SU(2)^2_{16} \times U(1)^2+1(1,2)$
21	$E_6(a3)$ A_2+A_1 A_3	$(0,1,1,1,1,1,0,0)$	$(78,47)$	$SU(4)_{12} \times Sp(3)_{10} \times U(1)^2$ + 1 free hyper
22	$E_6(a3)$ $2A_2$ A_2+2A_1	$(0,1,0,0,1,1,0,1)$	$(84,54)$	$(G_2)_{12} \times Sp(2)_{26}+1(2,1)$
23	$E_6(a3)$ A_2+A_1 $2A_2$	$(0,1,0,0,2,1,0,1)$	$(98,65)$	$(G_2)^2_{12} \times SU(2)_{26}$ + 1 free hyper
24	A_5 0 $D_5(a_1)$	$(0,1,0,0,1,0,0,0)$	$(39,16)$	$(E_6)_{12} \times SU(2)_7$ SCFT + $1(1,27)$
25	A_5 A_1 A_4+A_1	$(0,0,0,0,2,1,0,0)$	$(71,37)$	$Spin(13)_{16} \times SU(2)_7$ + $1(1,6)$
26	A_5 A_1 A_4	$(0,0,1,0,2,1,0,0)$	$(79,44)$	$Spin(12)_{16} \times SU(2)_8$ $\times SU(2)_7$ SCFT $+1(1,1,6)$
27	A_5 A_2+2A_1 $D_4(a1)$	$(0,0,2,0,1,0,0,0)$	$(47,25)$	$Sp(2)^3_8 \times SU(2)_7+3(1,2)$
28	A_5 A_2+A_1 $D_4(a1)$	$(0,0,2,0,2,0,0,0)$	$(63,36)$	$Spin(8)_{12} \times SU(2)^3_8$ $\times SU(2)_7$ +3 free hypers

Table 4.5: Mixed fixtures

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h,n_v)	Theory
29	A_5 A_2+2A_1 A_3+A_1	$(0,0,1,0,1,1,0,0)$	$(58,33)$	$Sp(3)_9 \times Sp(2)_{16} \times SU(2)_7 + 2(1,1,2)$
30	A_5 A_2+A_1 A_3+A_1	$(0,0,1,0,2,1,0,0)$	$(73,44)$	$Spin(7)_{12} \times Sp(2)_9$ $\times SU(2)_{32} \times SU(2)_7$ +2 free hypers
31	A_5 A_2+ $2A_2+A_1$ $2A_1$	$(0,0,0,0,1,1,0,1)$	$(77,49)$	$Sp(3)_{26} \times SU(2)_7 + 1(1,1,2)$
32	A_5 A_2+A_1 $2A_2+A_1$	$(0,0,0,0,2,1,0,1)$	$(91,60)$	$(G_2)_{12} \times Sp(2)_{26} \times SU(2)_7 + 1$ free hyper
33	A_5 A_2+2A_1 A_3	$(0,0,1,1,1,1,0,0)$	$(71,42)$	$Sp(4)_{10} \times SU(2)_{32}$ $\times SU(2)_7 \times U(1)$ $+1(1,1,2)$
34	A_5 A_2+A_1 A_3	$(0,0,1,1,2,1,0,0)$	$(85,53)$	$SU(4)_{12} \times Sp(3)_{10}$ $\times SU(2)_7 \times U(1)$ +1 free hyper
35	A_5 $2A_2$ A_2+2A_1	$(0,0,0,0,2,1,0,1)$	$(91,60)$	$(G_2)_{12} \times Sp(2)_{26} \times SU(2)_7 + 1(1,2,1)$
36	A_5 A_2+A_1 $2A_2$	$(0,0,0,0,3,1,0,1)$	$(105,71)$	$(G_2)^2_{12} \times SU(2)_{26} \times SU(2)_7 + 1$ free hyper
37	$D_5(a1)$ $3A_1$ A_4+A_1	$(0,1,0,1,1,0,0,0)$	$(52,25)$	$SU(6)_{12} \times Spin(7)_{10} + \frac{1}{2}(1,2) + 1(3,1)$
38	$D_5(a1)$ $2A_1$ A_4+A_1	$(0,1,0,1,1,1,0,0)$	$(74,40)$	$Spin(10)_{16} \times SU(2)_{10}$ $\times SU(2)_{32} \times U(1)$ +1 free hyper

Table 4.5: Mixed fixtures

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	Theory
39	$D_5(a1)$ $3A_1$ A_4	$(0,1,1,1,1,0,0,0)$	$(60,32)$	$SU(5)_{12}\times SU(4)_{10}$ $\times SU(2)_8\times U(1)$ $+\frac{1}{2}(1;1,2)+1(1;3,1)$
40	$D_5(a1)$ $2A_1$ A_4	$(0,1,1,1,1,1,0,0)$	$(82,47)$	$Spin(10)_{16}\times SU(2)_8$ $\times SU(2)_{10}\times U(1)^2$ +1 free hyper
41	$D_5(a1)$ $2A_2+A_1$ $D_4(a1)$	$(0,1,2,0,0,0,0,0)$	$(40,19)$	$SU(4)_8^3$ SCFT + 1(2) + 3 free hypers
42	$D_5(a1)$ $2A_2$ $D_4(a1)$	$(0,1,2,0,1,0,0,0)$	$(56,30)$	$Spin(8)_{12}\times SU(2)_8^3$ $\times U(1)^2$ +3 free hypers
43	$D_5(a1)$ $2A_2+A_1$ A_3+A_1	$(0,1,1,0,0,1,0,0)$	$(51,27)$	$SU(4)_{16}\times Sp(3)_9$ + $\frac{1}{2}(2,1)$ + 2 free hypers
44	$D_5(a1)$ $2A_2$ A_3+A_1	$(0,1,1,0,1,1,0,0)$	$(66,38)$	$Spin(7)_{12}\times Sp(2)_9$ $\times SU(2)_{32}\times U(1)$ +2 free hypers
45	$D_5(a1)$ $2A_2$ + $2A_2+A_1$ A_1	$(0,1,0,0,0,1,0,1)$	$(70,43)$	$Sp(3)_{26}$ + 1 free hyper
46	$D_5(a1)$ A_3 $2A_2+A_1$	$(0,1,1,1,0,1,0,0)$	$(63,36)$	$Sp(3)_{10}\times SU(3)_{16}$ $\times SU(2)_9\times U(1)$ $+\frac{1}{2}(2,1)$ + 1 free hyper
47	$D_5(a1)$ $2A_2$ $2A_2+A_1$	$(0,1,0,0,1,1,0,1)$	$(84,54)$	$(G_2)_{12}\times Sp(2)_{26}$ + 1 free hyper

Table 4.5: Mixed fixtures

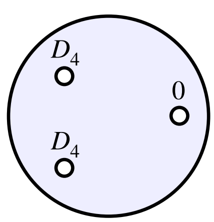
#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	Theory
48	$D_5(a1)$ $2A_2$ A_3	$(0,1,1,1,1,1,0,0)$	$(78,47)$	$Spin(7)_{12} \times Sp(3)_{10} \times U(1)^2 + 1$ free hyper
49	$D_5(a1)$ $2A_2$ $2A_2$	$(0,1,0,0,2,1,0,1)$	$(98,65)$	$(G_2)^2_{12} \times SU(2)_{26} +$ 1 free hyper
50	A_4+A_1 A_2+2A_1 A_4+A_1	$(0,0,0,1,1,1,0,0)$	$(60,35)$	$SU(3)_{32} \times Sp(3)_{10} + 1(2)$
51	A_4+A_1 A_2+A_1 A_4+A_1	$(0,0,0,1,2,1,0,0)$	$(74,46)$	$SU(4)_{12} \times SU(2)_{10} \times {SU(2)_{32}}^2 + 1$ free hyper
52	A_4+A_1 A_2+2A_1 A_4	$(0,0,1,1,1,1,0,0)$	$(68,42)$	$SU(3)_{32} \times Sp(2)_{10} \times SU(2)_8 \times U(1) + 1(1,2)$
53	A_4+A_1 A_2+A_1 A_4	$(0,0,1,1,2,1,0,0)$	$(82,53)$	$SU(4)_{12} \times SU(2)_{32}$ $\times SU(2)_{10}$ $\times SU(2)_8 \times U(1)$ $+1$ free hyper
54	D_4 $3A_1$ A_4+A_1	$(0,1,0,1,2,0,0,0)$	$(68,36)$	$SU(6)_{12} \times SU(3)^2_{12} + \frac{1}{2}(1;1,2)$
55	D_4 $3A_1$ A_4	$(0,1,1,1,2,0,0,0)$	$(76,43)$	$SU(6)_{12} \times SU(3)_{12}$ $\times SU(2)_8 \times U(1)$ $+\frac{1}{2}(1,1;1,2)$
56	D_4 $2A_2+A_1$ $D_4(a1)$	$(0,1,2,0,1,0,0,0)$	$(56,30)$	$Spin(8)_{12} \times SU(2)^3_8 + 1(1,2)$
57	D_4 $2A_2+A_1$ A_3+A_1	$(0,1,1,0,1,1,0,0)$	$(66,38)$	$Spin(7)_{12} \times Sp(2)_9$ $\times SU(2)_{16} \times U(1)$ $+\frac{1}{2}(1,1,2)$

Table 4.5: Mixed fixtures

#	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{12})$	(n_h, n_v)	Theory
58	D_4 A_3 $2A_2 + A_1$	$(0,1,1,1,1,1,0,0)$	$(77,47)$	$SU(4)_{12} \times Sp(2)_{10}$ $\times SU(2)_{16}$ $\times SU(2)_9 \times U(1)$ $+\frac{1}{2}(1,2,1)$
59	A_4 $A_2 + 2A_1$ A_4	$(0,0,2,1,1,1,0,0)$	$(76,49)$	$Sp(2)_{10} \times SU(2)_{32}$ $\times SU(2)_8^2 \times U(1)^2$ $+1(1,1,2)$
60	A_4 $A_2 + A_1$ A_4	$(0,0,2,1,2,1,0,0)$	$(90,60)$	$SU(4)_{12} \times SU(2)_{10}$ $\times SU(2)_8^2 \times U(1)^2$ +1 free hyper

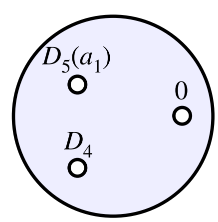
4.3 A Detour Through the Twisted Sector

There are several fixtures on our list, where the levels of the enhanced flavour symmetry group cannot be determined by considerations from the untwisted sector alone. For instance, consider the pair of fixtures,



$(E_6)_{18} \times (E_6)_6 \times \mathrm{SU}(3)^2_{12}$ SCFT

and

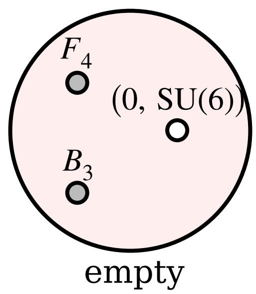


$(E_6)_{18} \times (E_6)_6 \times \mathrm{SU}(3)_{12} \times U(1)$ SCFT

In each case, only the diagonal $(E_6)_{24} \subset (E_6)_{24-k} \times (E_6)_k$ is manifest. Moreover, the only gaugings, available in the untwisted sector, have Abelian centralizers in $(E_6)_{24-k} \times (E_6)_k$, which makes determining the individual levels (as opposed to their sum) difficult.

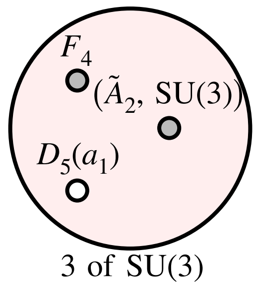
To fix the ambiguity, we need to make recourse to the $\mathbb{Z}_2$-twisted sector. While a full discussion of the $\mathbb{Z}_2$-twisted sector is beyond the scope of this paper, we will borrow a few results of that analysis, deferring a full discussion to a future paper.

The twisted punctures are labeled by nilpotent orbits in F_4. We will denote them by their Bala-Carter labels, and colour them grey. The empty fixture

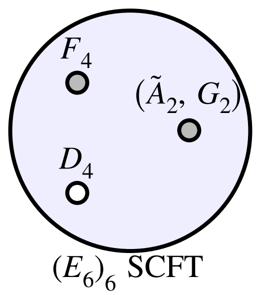


will allow us to gauge an $SU(6)_{24} \subset (E_6)_{24-k} \times (E_6)_k$. The centralizer is $SU(2)_{24-k} \times SU(2)_k$, from which we can read off the "missing" information about the levels.

We will also need the free-field fixture

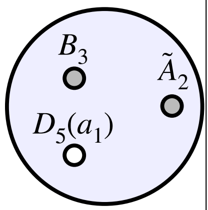
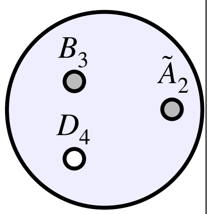


and the interacting fixture

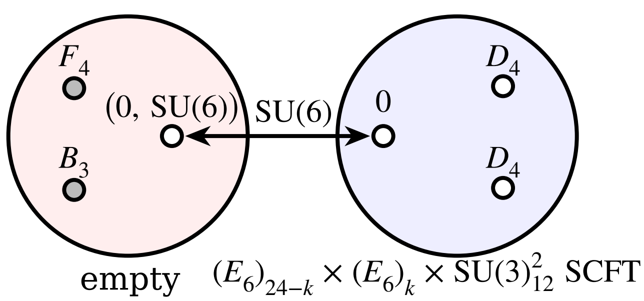


which is a realization of the $(E_6)_6$ SCFT. Finally, we will also need two "new" interacting fixtures

Table 4.6: Twisted interacting SCFTs

Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Global Symmetry
	$(0, 2, 1, 2, 1, 0, 1, 0)$	$(83, 63)$	$(G_2)_{10} \times SU(2)_{18} \times SU(2)_6 \times U(1)$
	$(0, 2, 1, 2, 2, 0, 1, 0)$	$(96, 74)$	$(G_2)_{10} \times SU(3)_{12} \times SU(2)_{18} \times SU(2)_6$

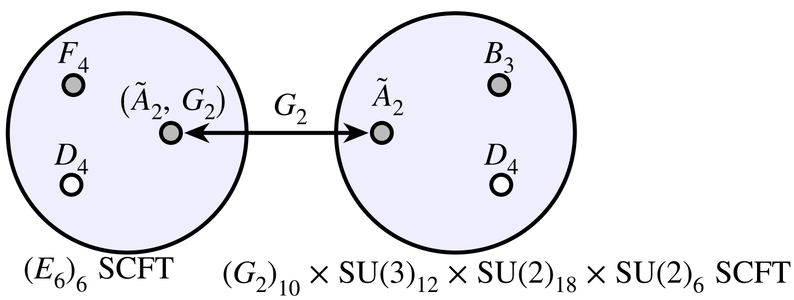
In both cases, all of the global symmetry *except* the $SU(2)_{18}$ is manifest (in particular, the $SU(2)_6$ is manifest). The 4-punctured sphere



has global symmetry

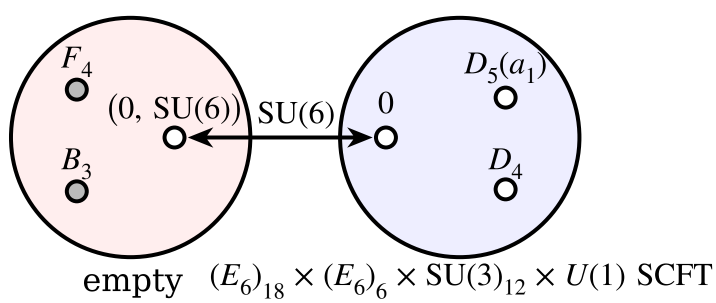
$$F = SU(3)_{12}^2 \times SU(2)_{24-k} \times SU(2)_k$$

The S-dual



manifestly has one of the $SU(2)$ levels as $k = 6$, which determines the other level to be 18.

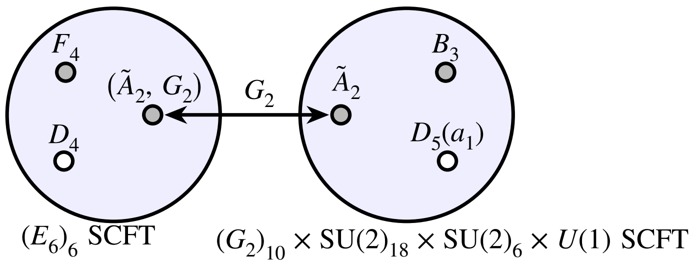
Similarly, for



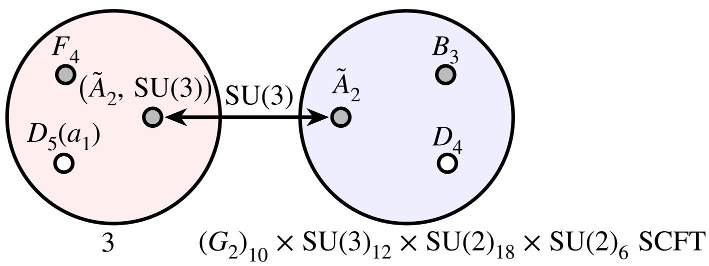
the global symmetry group is

$$F = SU(3)_{12} \times SU(2)_{24-k} \times SU(2)_k \times U(1)$$

Now there are two S-dual presentations of the theory:

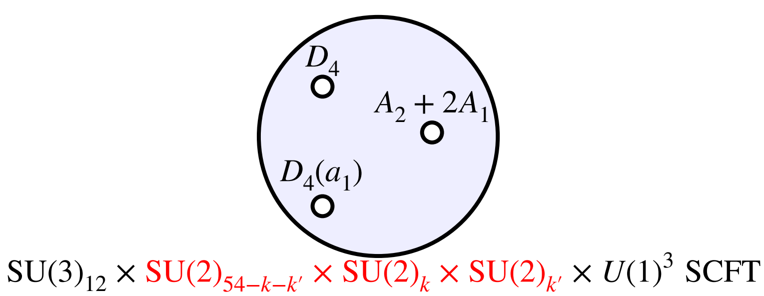


and



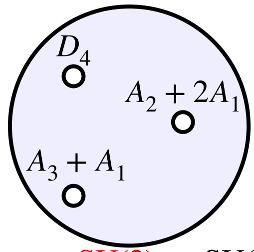
Again, the fact that one of the $SU(2)$ levels is manifest suffices to determine the other.

As another example, consider the pair of fixtures



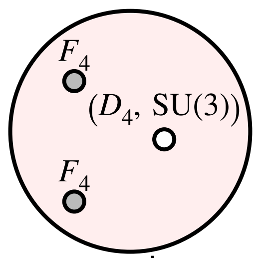
(4.7)

and



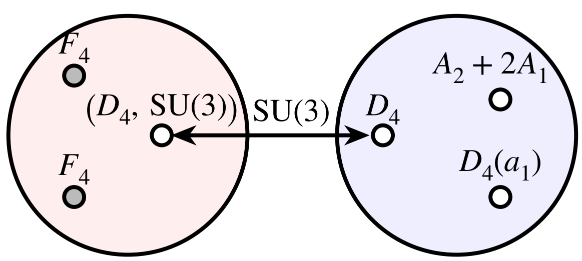
$$\mathrm{SU}(3)_{12} \times \mathrm{SU}(2)_{54-k} \times \mathrm{SU}(2)_k \times \mathrm{SU}(2)_9 \times U(1)^2 \text{ SCFT} \tag{4.8}$$

In each case, only the diagonal $SU(2)_{54}$ subgroup, of the indicated $SU(2)$s, is manifest. Moreover, these fixtures are not gaugeable within the untwisted theory. So there is no obvious way to determine the individual $SU(2)$ levels. Fortunately, the twisted sector provides the empty fixture

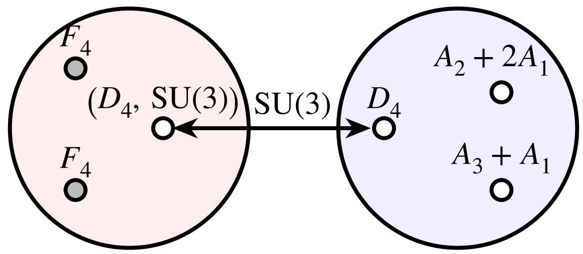


empty

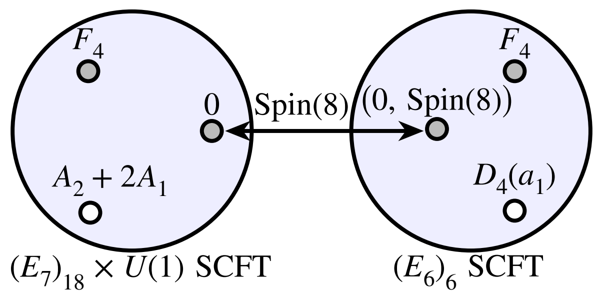
which allows us to gauge the $SU(3)_{12}$ symmetry of each of these fixtures:



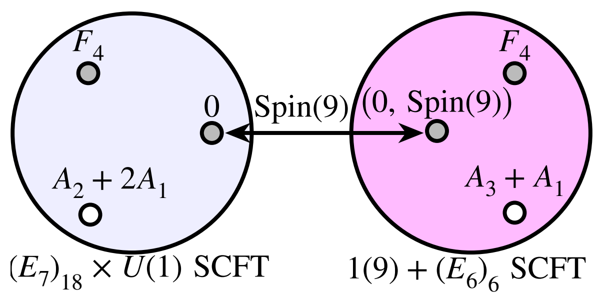
and



From the S-duals



and



and the Lie-algebra embeddings

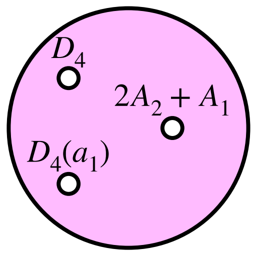
$$(\mathfrak{e}_7)_k \supset (\mathfrak{f}_4)_k \oplus \mathfrak{su}(2)_{3k}$$

$$(\mathfrak{e}_7)_k \supset \mathfrak{so}(9)_k \oplus \mathfrak{su}(2)_{2k} \oplus \mathfrak{su}(2)_k$$

$$(\mathfrak{e}_7)_k \supset \mathfrak{so}(8)_k \oplus \mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_k$$

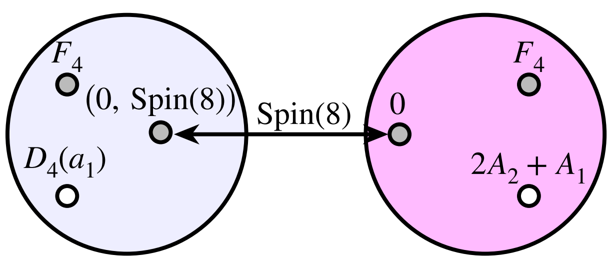
we determine the levels in (4.7) and (4.8) to be $k = k' = 18$.

Finally, let us turn to the mixed fixture



1(1, 2) +
$\mathrm{Spin}(8)_{12} \times \mathrm{SU}(2)_{24-k_1-k_2} \times \mathrm{SU}(2)_{k_1} \times \mathrm{SU}(2)_{k_2}$ SCFT

Only the diagonal $SU(2)_{24} \subset SU(2)_{24-k_1-k_2} \times SU(2)_{k_2} \times SU(2)_{k_2}$ is manifest. Gauging the $SU(3)_{12}$ symmetry of the D_4 puncture, as before, we find that the S-dual is a $Spin(8)$ gauge theory, with matter in the $1(8_v)+1(8_s)+1(8_c)+2(1)$, coupled to two copies of the $(E_6)_6$ SCFT.



$(E_6)_6$ SCFT　　1(26) + $(E_6)_6$ SCFT

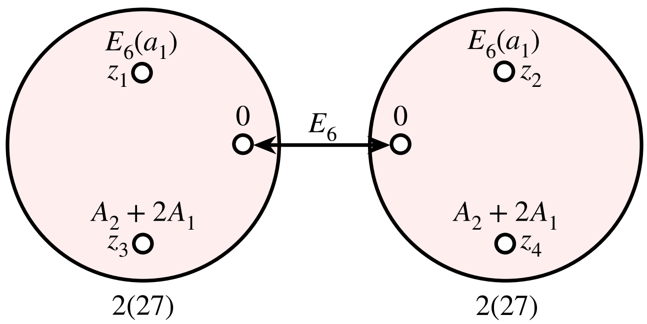
From this, we read off the levels of the three $SU(2)$s: $k_1 = k_2 = 24-k_1-k_2 = 8$.

4.4 Applications

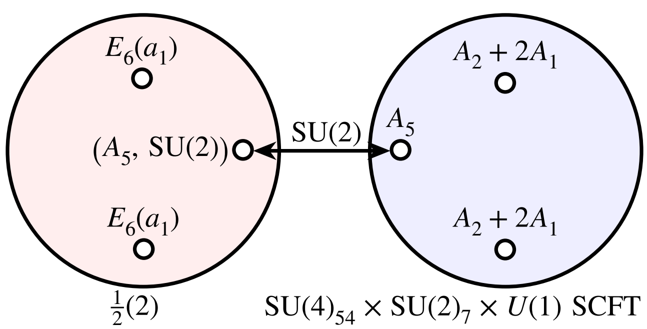
4.4.1 E_6 and F_4 gauge theory

4.4.1.1 $E_6 + 4(27)$

E_6 gauge theory, with four fundamental hypermultiplets, is superconformal. It is realized as the 4-punctured sphere



The S-dual theory is an $SU(2)$ gauging of the $SU(4)_{54} \times SU(2)_7 \times U(1)$ SCFT, with an additional half-hypermultiplet in the fundamental.



The k-differentials, which determine the Seiberg-Witten solution, are

$$
\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34}\, (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_5(z) &= \frac{u_5\, z_{12} z_{34}^4\, (dz)^5}{(z-z_1)(z-z_2)(z-z_3)^4(z-z_4)^4} \\
\phi_6(z) &= \frac{u_6\, z_{12}^2 z_{34}^4\, (dz)^6}{(z-z_1)^2(z-z_2)^2(z-z_3)^4(z-z_4)^4} \\
\phi_8(z) &= \frac{u_8\, z_{12}^2 z_{34}^6\, (dz)^8}{(z-z_1)^2(z-z_2)^2(z-z_3)^6(z-z_4)^6} \\
\phi_9(z) &= \frac{u_9\, z_{12}^2 z_{34}^7\, (dz)^9}{(z-z_1)^2(z-z_2)^2(z-z_3)^7(z-z_4)^7} \\
\phi_{12}(z) &= \frac{u_{12}\, z_{12}^3 z_{34}^9\, (dz)^{12}}{(z-z_1)^3(z-z_2)^3(z-z_3)^9(z-z_4)^9}
\end{aligned}
\tag{4.9}
$$

The gauge coupling, $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$, is determined by the $SL(2,\mathbb{C})$-invariant cross-ratio

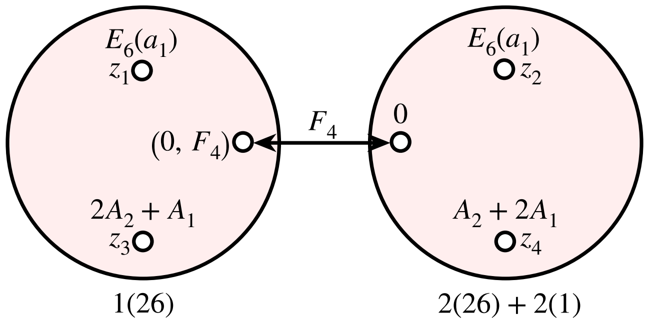
$$
f(\tau) \equiv -\frac{\theta_2^4(0,\tau)}{\theta_4^4(0,\tau)} = \frac{z_{13} z_{24}}{z_{14} z_{23}} \tag{4.10}
$$

and, for calculational purposes, it is usually convenient to use $SL(2,\mathbb{C})$ to fix $(z_1, z_2, z_3, z_4) = (0, \infty, f(\tau), 1)$ in (4.9).

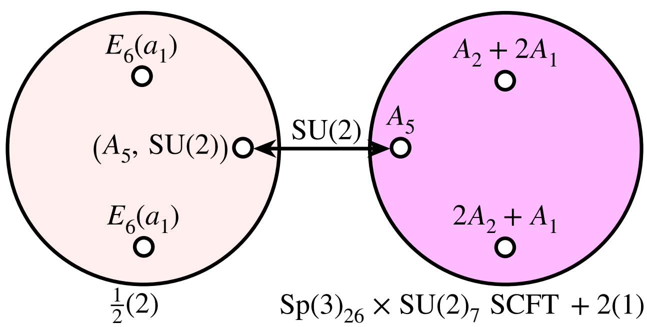
The solution to E_6 gauge theory with $N_f \leq 3$ fundamental hypermultiplets was first found in [51].

4.4.1.2 $F_4 + 3(26)$

F_4 gauge theory, with three fundamentals, is also superconformal. It is realized as



The S-dual theory is an $SU(2)$ gauging of the $Sp(3)_{26} \times SU(2)_7$ SCFT, with additional matter in the $\frac{1}{2}(2) + 2(1)$.



The nonzero k-differentials, which determine the Seiberg-Witten solution, are the same as in (4.9) but with $\phi_5(z) \equiv 0 \equiv \phi_9(z)$. The gauge coupling is again given by (5.17). Physically, this theory is obtained by Higgsing $E_6 \to F_4$, using one of the hypermultiplets in the 27.

In practice, given the solution to $E_6+4(27)$, the solution to $F_4+3(26)+2(1)$ is obtained by noting that

- There is a $\mathbb{Z}_2$ symmetry, $\sigma : (u_5, u_9) \mapsto (-u_5, -u_9)$, acting on the Coulomb branch of the $E_6 + 4(27)$.

- The Coulomb branch geometry of $F_4 + 3(26) + 2(1)$ is the geometry of the fixed-locus of σ.

4.4.2 Adding $(E_8)_{12}$ SCFTs

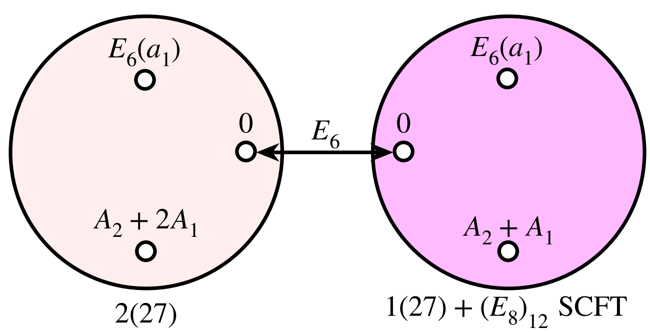
Starting with the $E_6 + 4(27)$ Lagrangian field theory, we can start replacing hypermultiplets in the 27 with copies of the $(E_8)_{12}$ SCFT. For n 27s and $4-n$ copies of the $(E_8)_{12}$ SCFT, the flavour symmetry group of the theory is

$$F = SU(3)_{12}^{4-n} \times U(n)_{54}$$

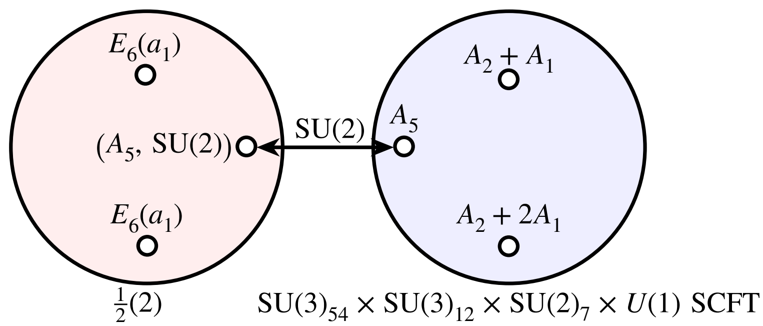
In each of these cases, the S-dual theory is an $SU(2)$ gauging of the $SU(3)_{12}^{4-n} \times SU(n)_{54} \times SU(2)_7 \times U(1)$ SCFT, with an additional half-hypermultiplet in the fundamental (the $U(1)$ is absent for $n = 0$).

4.4.2.1 $n = 3$

With one copy of the $(E_8)_{12}$ SCFT,



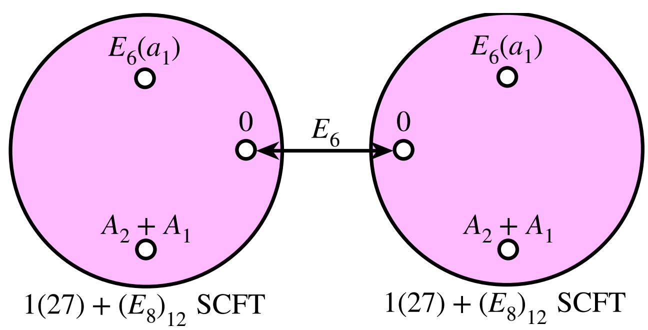
is dual to



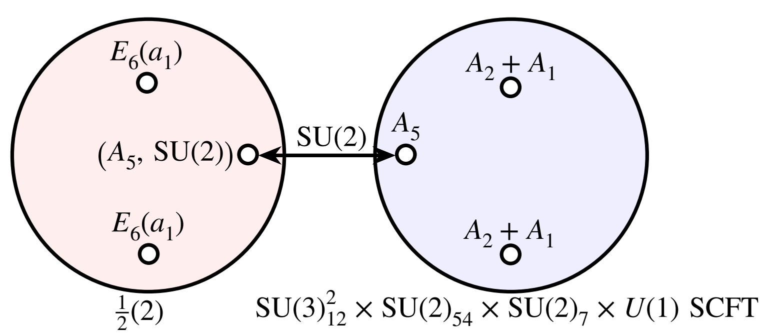
4.4.2.2 $n = 2$

With two copies of the $(E_8)_{12}$ SCFT, there are two possible realizations.

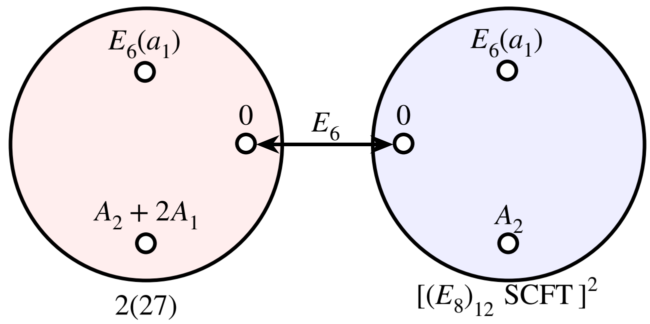
Either



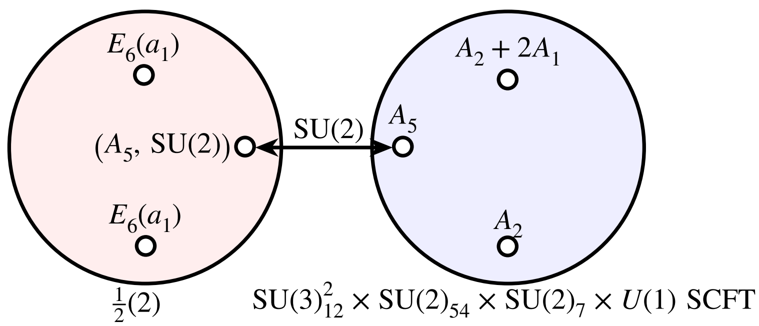
dual to



or



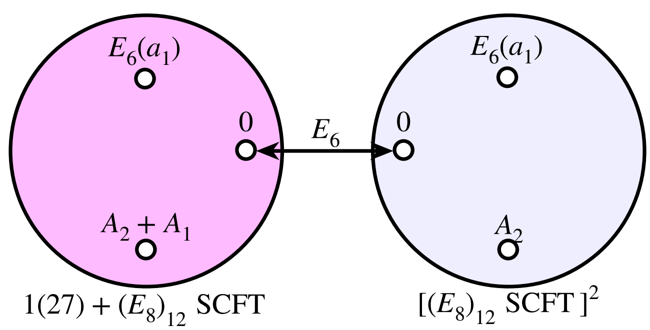
dual to



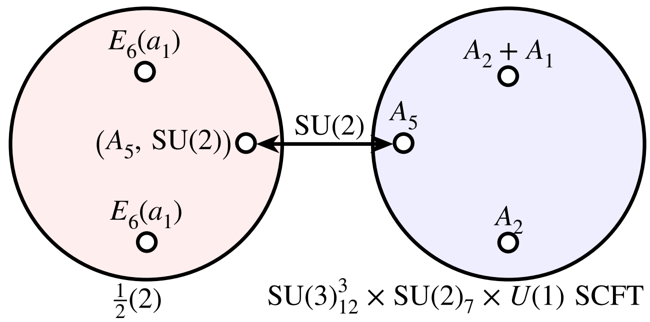
These give two, apparently distinct, realizations of the $SU(3)^2_{12}\times SU(2)_{54}\times SU(2)_7\times U(1)$ SCFT.

4.4.2.3 $n=1$

With three copies of the $(E_8)_{12}$ SCFT, we have

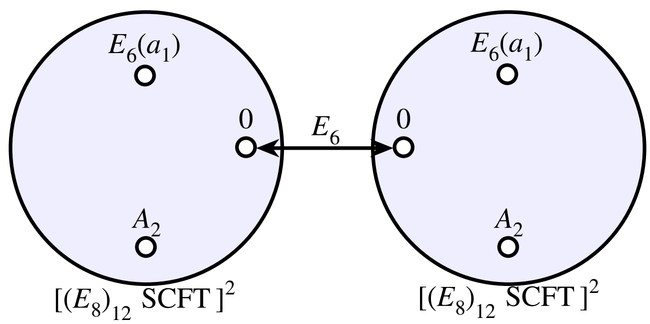


dual to

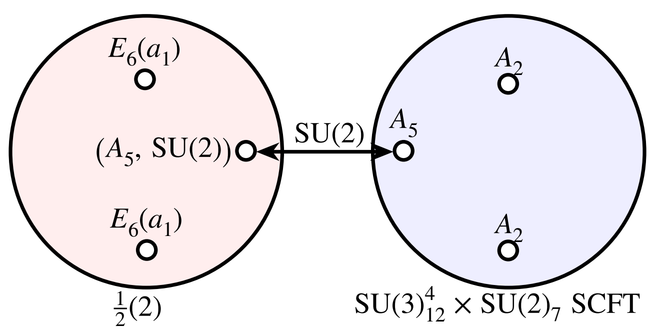


4.4.2.4 $n = 0$

Finally, the E_6 gauging of four copies of the $(E_8)_{12}$ SCFT,



is dual to



4.4.3 Connections with F-theory

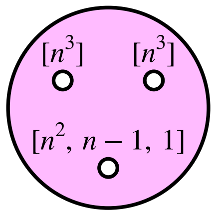
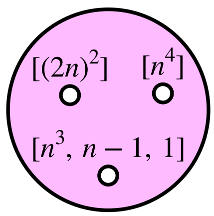
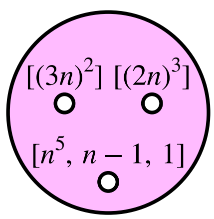
Placing n D3-branes at a IV^*, III^* or II^* singularity in F-Theory yields an $\mathcal{N}=2$ superconformal field theory on the world-volume of the D3-branes [47, 48]. For $n=1$ these are, respectively, the $(E_6)_6$, $(E_7)_8$ and $(E_8)_{12}$ superconformal field theories of Minahan and Nemenschansky [62]. For higher n, the properties of these SCFTs were computed in [46]. The results may be summarized as follows

Table 4.7: Properties of higher-rank Minahan-Nemeschansky SCFTs

F-Theory singularity	Flavour symmetry	Graded Coulomb branch dimensions	(n_h, n_v)
IV^*	$(E_6)_{6n} \times SU(2)_{(n-1)(3n+1)}$	$n_{3l}=1,\ l=1,2,\dots,n$	$\big(3n^2+14n-1, n(3n+2)\big)$
III^*	$(E_7)_{8n} \times SU(2)_{(n-1)(4n+1)}$	$n_{4l}=1,\ l=1,2,\dots,n$	$\big(4n^2+21n-1, n(4n+3)\big)$
II^*	$(E_8)_{12n} \times SU(2)_{(n-1)(6n+1)}$	$n_{6l}=1,\ l=1,2,\dots,n$	$\big(6n^2+35n-1, n(6n+5)\big)$

In [27], Gaiotto and Razamat proposed a realization of these $(n \geq 2)$ SCFTs as a mixed fixture, with one free hypermultiplet, in the A_{N-1} theory, for $N = 3n, 4n$ and $6n$, respectively.

Table 4.8: Realization of higher-rank Minahan-Nemeschansky SCFTs in A_{N-1} series

Theory	Fixture	Manifest flavour symmetry	Enhanced to
IV*		$SU(3)^2_{6n} \times SU(2)_{6n} \times U(1)^2$	$(E_6)_{6n} \times SU(2)_k + \frac{1}{2}(2)$
III*		$SU(2)_{8n} \times SU(4)_{8n} \times SU(3)_{8n} \times U(1)^2$	$(E_7)_{8n} \times SU(2)_k + \frac{1}{2}(2)$
II*		$SU(2)_{12n} \times SU(3)_{12n} \times SU(5)_{12n} \times U(1)^2$	$(E_8)_{12n} \times SU(2)_k + \frac{1}{2}(2)$

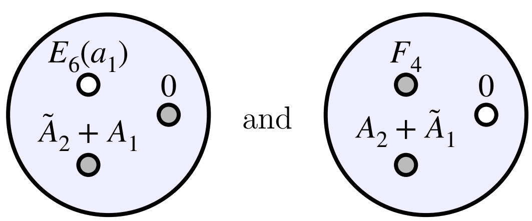
For $n = 2$, the $SU(2)$ flavour symmetry is manifest, and one readily verifies that it has the predicted level (given that the hypermultiplet transforms as $\frac{1}{2}(2)$ under the $SU(2)$). But, for $n \geq 3$, only the $U(1)$ Cartan is manifest and it is not easy to determine the level of the $SU(2)$.

We have, of course, numerous realizations of the $n = 1$ theories. But we also find examples of the higher-n theories

- We find the $n = 2$ IV* SCFT as one of our fixtures in §5.2.3 and as part of a product SCFT in fixture 39 of §5.2.4. It also appeared as an interacting fixture in the D_4 theory in [5].
- We find the $n = 2$ III* SCFT as mixed fixture 4 in §5.2.5 and as part of a product SCFT in fixture 6 of §5.2.4.
- We find the $n = 2$ II* SCFT as interacting fixture 2 in §5.2.4.
- We find the $n = 3$ III* SCFT as interacting fixture 7 in §5.2.4.

In particular, the latter gives a nice check of the $SU(2)$ level for $n = 3$.

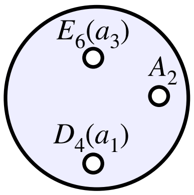
Further examples can be found in the $\mathbb{Z}_2$-twisted sector. Notably, the fixtures

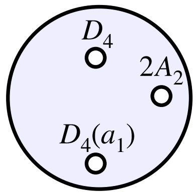


provide realizations, respectively, of the $n = 3, 4$ IV* SCFTs. Again, the $SU(2)$ levels agree with the predictions of [46]. Together with the above examples, these exhaust all the IV*, III* and II* theories with nonzero graded Coulomb branch dimensions in degrees ≤ 12.

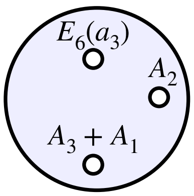
4.5 Isomorphic Theories

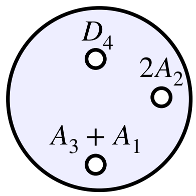
In our table of interacting fixtures with enhanced global symmetry in §5.2.4, we find several SCFTs which seem to be realized in more than one way. Most of these isomorphisms can be checked by various dualities. Some, however, cannot and we list them below.



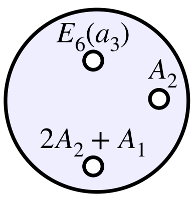
$\simeq$


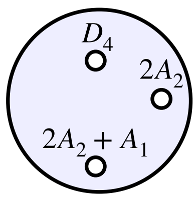
$\simeq$ $Spin(8)^2_{12} \times U(1)^2\,\mathrm{SCFT}$



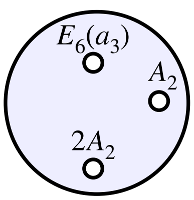
$\simeq$


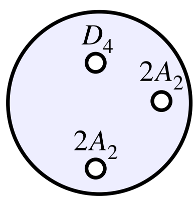
$\simeq$ $Spin(7)^2_{12} \times SU(2)_9 \times U(1)\,\mathrm{SCFT}$



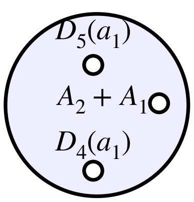
$\simeq$


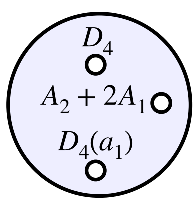
$\simeq$ $(G_2)^2_{12} \times SU(2)_{26}\,\mathrm{SCFT}$



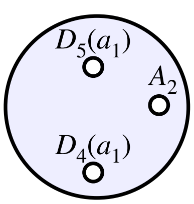
$\simeq$


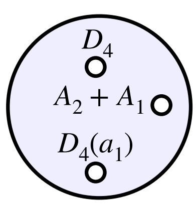
$\simeq$ $(G_2)^3_{12}\,\mathrm{SCFT}$



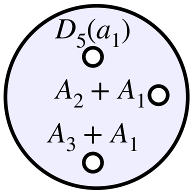
$\simeq$


$\simeq$ $SU(3)_{12} \times SU(2)^3_{18} \times U(1)^3\,\mathrm{SCFT}$

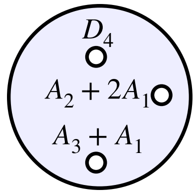


$\simeq$


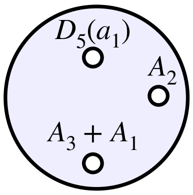
$\simeq$ $SU(3)^2_{12} \times U(1)^5\,\mathrm{SCFT}$



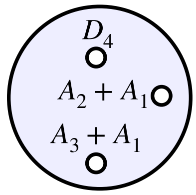
$\simeq$



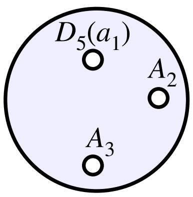
$$\simeq \quad SU(3)_{12}\times SU(2)_{36}\times SU(2)_{18} \times SU(2)_9\times U(1)^2 \text{ SCFT}$$



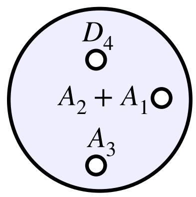
$\simeq$



$$\simeq \quad SU(3)_{12}^2\times SU(2)_9\times U(1)^3\,\text{SCFT}$$



$\simeq$



$$\simeq \quad SU(3)_{12}^2\times Sp(2)_{10}\times U(1)^3\,\text{SCFT}$$

It would be nice to check these conjectured isomorphisms by comparing the expansions of the superconformal indices for these pairs of fixtures to higher order in τ.

Chapter 5

The $\mathbb{Z}_2$-Twisted E_6 Theory

We now turn our attention to the theories obtained by compactifying the $(2,0)$ theory of type E_6 in the presence of punctures twisted by a $\mathbb{Z}_2$ outer-automorphism [1]. The twisted punctures are in 1-1 correspondence with embeddings $\rho : \mathfrak{su}(2) \hookrightarrow \mathfrak{f}_4$, and we label them by the Bala-Carter label of the corresponding nilpotent orbit. For a given puncture, we compute all the local properties which contribute to determining the $4D$ $\mathcal{N} = 2$ SCFT and record them in Table 5.2.1. We also determine a projection matrix implementing the branching rule under each embedding, which we use to compute the expansion of the superconformal index. These can be found in Appendix E.2.

5.1 The twisted E_6 theory

5.1.1 The Hitchin system

For a choice of Riemann surface C, the compactification of the 6D (2,0) theory of type E_6 on $\mathbb{R}^{3,1} \times C$ yields a 4D $\mathcal{N} = 2$ theory on $\mathbb{R}^{3,1}$. The $(2,0)$ theory of type E_6 has an outer-automorphism group which is isomorphic to $\mathbb{Z}_2$. This allows us to introduce a class of "twisted" punctures, around which

[1] This chapter is based on [63].

the fields on C undergo a monodromy by a non-trivial element of the outer-automorphism group[2]. The properties of these punctures are listed in table 5.2.1.

In [49] we studied the theories that arise from compactifying the E_6 $(2,0)$ on a Riemann surface with untwisted punctures. These punctures are classified by nilpotent orbits in the complexified Lie algebra $\mathfrak{e}_6$, and obey a Hitchin boundary condition of the form

$$\Phi(z) = \frac{A}{z} + \mathfrak{e}_6$$

where Φ is the Higgs field, z is a local coordinate on C such that the puncture is at $z = 0$, A is a nilpotent element in $\mathfrak{e}_6$, and $\mathfrak{e}_6$ in the boundary condition above denotes a generic element of $\mathfrak{e}_6$ (or a regular function of z taking values in $\mathfrak{e}_6$).

By contrast, twisted punctures are classified by nilpotent orbits in the complexified Lie algebra $\mathfrak{f}_4$, and obey a twisted boundary condition,

$$\Phi(z) = \frac{A}{z} + \frac{o_{-1}}{z^{1/2}} + \mathfrak{f}_4$$

Here, we have split $\mathfrak{e}_6$ into eigenspaces under the action of the $\mathbb{Z}_2$ outer-automorphism, as $\mathfrak{e}_6 = \mathfrak{f}_4 \oplus o_{-1}$, where $\mathfrak{f}_4$ (o_{-1}) is the even (odd) eigenspace.

[2]We also allow for the fields on C to undergo a monodromy upon traversing a homologically non-trivial cycle.

Also, A is a nilpotent element in $\mathfrak{f}_4$, and o_{-1} and $\mathfrak{f}_4$ above represent generic elements in the respective spaces.

5.1.2 k-differentials

We use the basis of E_6 Casimir k-differentials $\{\phi_2, \phi_5, \phi_6, \phi_8, \phi_9, \phi_{12}\}$ of our previous paper [49]. For the reader's convenience, we repeat here how to construct this basis in terms of the trace invariants $P_k = Tr(\Phi^k)$ for Φ in the adjoint representation of E_6.

$$\begin{aligned}
\phi_2 =& \tfrac{1}{48} P_2 \\
\phi_6 =& \tfrac{1}{24}\left(P_6 - \tfrac{7}{4608}(P_2)^3\right) \\
\phi_8 =& \tfrac{1}{30}\left(P_8 - \tfrac{2}{9}P_6P_2 + \tfrac{155}{663552}(P_2)^4\right) \\
\phi_{10} =& -\tfrac{1}{105}\left(P_{10} - \tfrac{17}{96}P_8P_2 + \tfrac{77}{6912}P_6(P_2)^2 - \tfrac{427}{63700992}(P_2)^5\right) \\
\phi_{12} =& \tfrac{1}{155}(P_{12} - \tfrac{107}{504}P_{10}P_2 + \tfrac{515}{32256}P_8(P_2)^2 - \tfrac{41}{108}(P_6)^2 + \tfrac{295}{497664}P_6(P_2)^3 \\
& - \tfrac{5669}{9172942848}(P_2)^6) \\
\phi_{14} =& \tfrac{1}{4389}\left(P_{14} - \tfrac{3479}{14880}P_{12}P_2 + \tfrac{61391}{3214080}P_{10}(P_2)^2 - \tfrac{539}{2160}P_8P_6 - \tfrac{139733}{617103360}P_8(P_2)^3\right. \\
& \left. + \tfrac{165781}{4821120}(P_6)^2P_2 - \tfrac{3488947}{44431441920}P_6(P_2)^4 + \tfrac{19596907}{409480168734720}(P_2)^7\right)
\end{aligned}$$

These relations define all the Casimirs except ϕ_5 and ϕ_9. These can be computed from ϕ_{10} and ϕ_{14}, which factorize as

$$\begin{aligned}
\phi_{10} &= \phi_5^2, \\
\phi_{14} &= \phi_5\phi_9.
\end{aligned}$$

Notice that the choice of sign of ϕ_5 determines also the sign of ϕ_9. This is precisely the action of the $\mathbb{Z}_2$ outer-automorphism of E_6 on the Casimir k-differentials,

$$\begin{aligned}
&\phi_5 \mapsto -\phi_5 \\
&\phi_9 \mapsto -\phi_9 \\
&\phi_k \mapsto \phi_k, \qquad k = 2, 6, 8, 12
\end{aligned}$$

So, we can expect that the leading pole orders of the ϕ_k for twisted punctures will be half-integer for $k = 5, 9$, and integer for $k = 2, 6, 8, 12$, corresponding to the orders of the Casimirs of F_4.

5.2 Tinkertoys

We find 2078 fixtures with 3 regular punctures, two twisted and one untwisted, which correspond to either an interacting SCFT, a mix of an interacting SCFT and free hypers, or a gauge theory. Of these, we find 1757 SCFTs without global symmetry enhancement, 122 SCFTs with enhanced global symmetry, 32 mixed fixtures, and 167 gauge theory fixtures.

Additionally, there are 23 fixtures with one irregular puncture: 15 free-field fixtures, 6 interacting fixtures, 1 mixed fixture, and 1 gauge theory fixture.

Below, we give tables of the twisted punctures and their properties, as well as tables of the twisted fixtures. For the mixed fixtures, we list $\{d_k\}$ and (n_h, n_v) of the interacting SCFT, and the representation of the free hypermultiplets. We do not list the fixtures without global symmetry enhancement, as their properties can be readily computed from the tables of punctures. Tables of untwisted punctures and fixtures can be found in our previous paper [49]. Following the conventions of that paper, in the tables we denote the Bala-Carter labels of twisted punctures by underlining them; in the figures, twisted punctures are denoted in gray.

5.2.1 Twisted punctures

Twisted punctures in the E_6 theory are labeled by nilpotent orbits in $\mathfrak{f}_4$, which we denote by the corresponding Bala-Carter label. As discussed in [49], the Bala-Carter notation provides a systematic way to label nilpotent orbits in any exceptional semisimple Lie algebra, and a concise review can be found in appendix A of [49]. Here we merely add that for the $\mathfrak{f}_4$ nilpotent orbits, components of the Levi subalgebra in the Bala-Carter label with (without) a tilde are constructed from the short (long) roots of $\mathfrak{f}_4$. (So, e.g., $A_2 + \widetilde{A}_1$ and $\widetilde{A}_2 + A_1$ represent different orbits.)

The pole structure of the k-differentials is denoted by $\{p_2, p_5, p_6, p_8, p_9, p_{12}\}$, and, for twisted punctures, p_5 are p_9 are half-integer. The contributions to the graded Coulomb branch dimensions are denoted by $\{d_2, d_3, d_4, d_5, d_6, d_8, d_9, d_{12}\}$, allowing for new Coulomb branch parameters (introduced by a-constraints) of dimensions 3 and 4, which are not degrees of E_6 Casimirs. The constraints are shown separately in Appendix D.1.

Table 5.1: Twisted regular punctures

Nahm orbit	Hitchin orbit	Pole structure	Coulomb branch contributions	Flavour group	$(\delta n_h, \delta n_v)$
$\underline{0}$	F_4	$\{1, \frac{9}{2}, 5, 7, \frac{17}{2}, 11\}$	$\{1, 0, 0, \frac{9}{2}, 5, 7, \frac{17}{2}, 11\}$	$(F_4)_{18}$	$(624, 601)$
$\underline{A_1}$	$(F_4(a_1), \mathbb{Z}_2)$	$\{1, \frac{9}{2}, 5, 7, \frac{15}{2}, 11\}$	$\{1, 0, 0, \frac{9}{2}, 5, 7, \frac{15}{2}, 11\}$	$Sp(3)_{13}$	$(599, 584)$
$\underline{\widetilde{A}_1}$	$F_4(a_1)$	$\{1, \frac{9}{2}, 5, 7, \frac{15}{2}, 11\}$	$\{1, 0, 0, \frac{9}{2}, 6, 7, \frac{15}{2}, 10\}$	$SU(4)_{12}$	$(584, 572)$
$\underline{A_1 + \widetilde{A}_1}$	$F_4(a_2)$	$\{1, \frac{9}{2}, 5, 7, \frac{15}{2}, 10\}$	$\{1, 0, 0, \frac{9}{2}, 5, 7, \frac{15}{2}, 10\}$	$SU(2)_{64} \times SU(2)_{10}$	$(570, 561)$
$\underline{A_2}$	B_3	$\{1, \frac{7}{2}, 5, 7, \frac{15}{2}, 10\}$	$\{1, 0, 0, \frac{7}{2}, 5, 7, \frac{15}{2}, 10\}$	$SU(3)_{16}$	$(560, 552)$
$\underline{\widetilde{A}_2}$	C_3	$\{1, \frac{9}{2}, 5, 7, \frac{15}{2}, 10\}$	$\{1, 1, 0, \frac{9}{2}, 5, 6, \frac{15}{2}, 9\}$	$(G_2)_{10}$	$(536, 528)$
$\underline{A_2 + \widetilde{A}_1}$	$(F_4(a_3), S_4)$	$\{1, \frac{7}{2}, 5, 6, \frac{15}{2}, 10\}$	$\{1, 0, 0, \frac{7}{2}, 5, 6, \frac{15}{2}, 10\}$	$SU(2)_{39}$	$(543, 537)$
$\underline{B_2}$	$(F_4(a_3), \mathbb{Z}_2 \times \mathbb{Z}_2)$	$\{1, \frac{7}{2}, 5, 6, \frac{15}{2}, 10\}$	$\{1, 1, 0, \frac{7}{2}, 6, 6, \frac{13}{2}, 9\}$	$SU(2)_7^2$	$(518, 513)$
$\underline{\widetilde{A}_2 + A_1}$	$(F_4(a_3), S_3)$	$\{1, \frac{7}{2}, 5, 6, \frac{15}{2}, 10\}$	$\{1, 1, 0, \frac{7}{2}, 5, 6, \frac{15}{2}, 9\}$	$SU(2)_{20}$	$(524, 519)$
$\underline{C_3(a_1)}$	$(F_4(a_3), \mathbb{Z}_2)$	$\{1, \frac{7}{2}, 5, 6, \frac{15}{2}, 10\}$	$\{1, 2, 0, \frac{7}{2}, 5, 6, \frac{13}{2}, 9\}$	$SU(2)_7$	$(511, 507)$
$\underline{F_4(a_3)}$	$F_4(a_3)$	$\{1, \frac{7}{2}, 5, 6, \frac{15}{2}, 10\}$	$\{1, 3, 0, \frac{7}{2}, 4, 6, \frac{13}{2}, 9\}$	$-$	$(504, 501)$
$\underline{B_3}$	A_2	$\{1, \frac{7}{2}, 4, 6, \frac{13}{2}, 9\}$	$\{1, 0, 1, \frac{7}{2}, 4, 5, \frac{11}{2}, 8\}$	$SU(2)_{24}$	$(440, 438)$
$\underline{C_3}$	$\widetilde{A}_2$	$\{1, \frac{7}{2}, 5, 6, \frac{15}{2}, 10\}$	$\{1, 1, 1, \frac{7}{2}, 4, 5, \frac{11}{2}, 7\}$	$SU(2)_6$	$(422, 420)$
$\underline{F_4(a_2)}$	$A_1 + \widetilde{A}_1$	$\{1, \frac{7}{2}, 4, 6, \frac{13}{2}, 9\}$	$\{1, 0, 1, \frac{7}{2}, 4, 5, \frac{11}{2}, 7\}$	$-$	$(416, 415)$

Table 5.1: Twisted regular punctures

Nahm orbit	Hitchin orbit	Pole structure	Coulomb branch contributions	Flavour group	$(\delta n_h, \delta n_v)$
$\underline{F_4(a_1)}$	$\widetilde{A}_1$	$\{1, \frac{7}{2}, 4, 6, \frac{13}{2}, 9\}$	$\{2, 1, 0, \frac{5}{2}, 4, 4, \frac{9}{2}, 6\}$	–	$(352, 352)$
$\underline{F_4}$	0	$\{1, \frac{5}{2}, 3, 4, \frac{9}{2}, 6\}$	$\{1, 1, 0, \frac{3}{2}, 2, 2, \frac{5}{2}, 3\}$	–	$(184, 185)$

There is a special piece consisting of *five* nilpotent orbits,

$$A_2 + \tilde{A}_1, \quad \tilde{A}_2 + A_1, \quad B_2, \quad C_3(a_1), \quad F_4(a_3).$$

The corresponding Hitchin boundary conditions are $(F_4(a_3), \Gamma)$, where the Sommers-Achar group, Γ, is a subgroup of S_4. The leading pole coefficients,

$$\begin{aligned} c_5^{(6)} &= -\left(6a^2 + 3a'^2 + a''^2\right) \\ c_{15/2}^{(9)} &= \tfrac{1}{3}(a + a')\left((2a - a')^2 - a''^2\right) \\ c_{10}^{(12)} &= 3a'^2(4a + a')^2 + 2(8a^2 - 12aa' + a'^2)a''^2 + \tfrac{1}{3}a''^4 - \tfrac{4}{3}\left(c_5^{(6)}\right)^2 \end{aligned} \tag{5.1}$$

are invariant under the S_4 action, $\left(\begin{smallmatrix} a \\ a' \\ a'' \end{smallmatrix}\right) \mapsto \gamma \left(\begin{smallmatrix} a \\ a' \\ a'' \end{smallmatrix}\right)$, generated by

$$\sigma_{12} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \sigma_{23} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \quad \sigma_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

- For the special orbit, $F_4(a_3)$, the Sommers-Achar group is trivial, and a, a', a'' are invariants.

- For $C_3(a_1)$, the Sommers-Achar group is the $\mathbb{Z}_2$ generated by σ_{34} and the invariants are a, a', a''^2.

- For B_2, the Sommers-Achar group is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by σ_{12}, σ_{34} and the invariants are $a + 2a'$, a''^2, $2a^2 + a'^2$.

- For $\tilde{A}_2 + A_1$, the Sommers-Achar group is the S_3 generated by σ_{23}, σ_{34} and the invariants are a, $3a'^2 + a''^2$, $c^{(9)}_{15/2}$.

- Finally, for $A_2 + \tilde{A}_1$, the Sommers-Achar group is the full S_4, and the invariants are $c^{(6)}_5$, $c^{(9)}_{15/2}$, $c^{(12)}_{10}$.

In §5.4, we will discover an action of this S_4 group on the Higgs branch of certain fixtures obtained by varying one of the punctures over this special piece.

5.2.2 Free-field fixtures

Table 5.2: Free-field fixtures

#	Fixture	n_h	Representation
1	$\frac{F_4}{\underline{F_4}}$ $(D_4, SU(3)_0)$	0	empty
2	$\frac{F_4}{\underline{F_4(a_1)}}$ $(A_2, SU(3)_0^2)$	0	empty
3	$\frac{F_4}{\underline{F_4(a_2)}}$ $(A_1, SU(6)_6)$	10	$\frac{1}{2}(20)$
4	$\frac{F_4}{\underline{B_3}}$ $(0, SU(6)_0)$	0	empty

Table 5.2: Free-field fixtures

#	Fixture	n_h	Representation
5	$\frac{F_4}{E_6(a_1)}$ $(\underline{F_4(a_1)},)$	0	empty
6	$\frac{F_4}{E_6(a_3)}$ $(\underline{C_3(a_1)}, SU(2)_1)$	1	$\frac{1}{2}(2)$
7	$\frac{F_4}{A_5}$ $(\underline{B_2}, SU(2)_1)$	1	$\frac{1}{2}(2)$
8	$\frac{F_4}{D_5(a_1)}$ $(\underline{\widetilde{A}_2}, SU(3)_2)$	1	$1(3)$
9	$\frac{F_4}{D_5}$ $(\underline{C_3}, SU(2)_2)$	2	$1(2)$
10	$\frac{F_4}{A_4+A_1}$ $(\underline{\widetilde{A}_1}, SU(3)_0)$	0	empty
11	$\frac{F_4}{A_4}$ $(\underline{\widetilde{A}_1}, SU(4)_4)$	8	$(2,4)$
12	$\frac{F_4(a_1)}{E_6(a_1)}$ $(\underline{B_2}, SU(2)_1^2)$	2	$\frac{1}{2}(2,1)+\frac{1}{2}(1,2)$
13	$\frac{F_4(a_2)}{E_6(a_1)}$ $(\underline{\widetilde{A}_1}, Sp(2)_0)$	0	empty
14	$\frac{C_3}{E_6(a_1)}$ $(\underline{\widetilde{A}_1}, SU(4)_4)$	6	$\frac{1}{2}(2,6)$
15	$\frac{B_3}{E_6(a_1)}$ $(\underline{A_1}, Sp(3)_3)$	9	$\frac{1}{2}(3,6)$

5.2.3 Interacting fixtures with one irregular puncture

Table 5.3: Interacting fixtures with one irregular puncture

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Theory
1	$\frac{F_4}{D_4}$ $(\underline{\widetilde{A}_2}, G_2)$	$(0,1,0,0,0,0,0,0)$	$(16,5)$	$(E_6)_6$ SCFT
2	$\frac{F_4}{D_4(a_1)}$ $(\underline{0}, Spin(8))$	$(0,1,0,0,0,0,0,0)$	$(16,5)$	$(E_6)_6$ SCFT
3	$\frac{F_4}{\underline{C_3}}$ $(A_1, SU(6)_6)$	$(0,1,0,0,0,0,0,0)$	$(16,5)$	$(E_6)_6$ SCFT
4	$\frac{F_4}{A_3}$ $(\underline{0}, Spin(9))$	$(0,1,0,1,0,0,0,0)$	$(36,14)$	$Spin(14)_{10} \times U(1)$ SCFT
5	$\frac{C_3}{D_5}$ $(\underline{0}, Spin(9))$	$(0,1,1,1,0,0,0,0)$	$(38,21)$	$Spin(9)_{10} \times SU(2)_6 \times U(1)$
6	$\frac{F_4(a_2)}{D_5}$ $(\underline{0}, Spin(9))$	$(0,0,1,1,0,0,0,0)$	$(32,16)$	$Spin(9)_{10} \times U(1)$

5.2.4 Interacting fixtures with enhanced global symmetry

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
1	$\frac{F_4}{0}$ $\underline{F_4(a_3)}$	$(0,4,0,0,0,0,0,0)$	$(64,20)$	$[(E_6)_6 \text{ SCFT}]^4$
2	$\frac{F_4}{0}$ $\underline{C_3(a_1)}$	$(0,3,0,0,1,0,0,0)$	$(71,26)$	$[(E_6)_6 \text{ SCFT}]^2 \times [(E_6)_{12} \times SU(2)_7 \text{ SCFT}]$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
3	$\frac{F_4}{0}\ \underline{\widetilde{A}_2 + A_1}$	$(0,2,0,0,1,0,1,0)$	$(84,38)$	$[(E_6)_6$ SCFT$] \times [(E_6)_{18} \times SU(2)_{20}$ SCFT$]$
4	$\frac{F_4}{0}\ \underline{B_2}$	$(0,2,0,0,2,0,0,0)$	$(78,32)$	$[(E_6)_{12} \times SU(2)_7$ SCFT$]^2$
5	$\frac{F_4}{0}\ \underline{\widetilde{A}_2}$	$(0,2,0,1,1,0,1,0)$	$(96,47)$	$[(E_6)_6$ SCFT$] \times [(E_6)_{18} \times (G_2)_{10}$ SCFT$]$
6	$\frac{F_4}{2A_1}\ \underline{A_2}$	$(0,1,0,0,1,1,0,0)$	$(64,31)$	$Spin(13)_{16} \times U(1)$
7	$\frac{F_4}{A_1}\ \underline{A_2}$	$(0,1,0,0,1,1,1,0)$	$(86,48)$	$(G_2)_{16} \times SU(6)_{18}$
8	$\frac{F_4}{2A_1}\ \underline{A_1 + \widetilde{A}_1}$	$(0,1,0,1,1,1,0,0)$	$(74,40)$	$Spin(10)_{16} \times SU(2)_{10} \times SU(2)_{32} \times U(1)$
9	$\frac{F_4}{A_1}\ \underline{A_1 + \widetilde{A}_1}$	$(0,1,0,1,1,1,1,0)$	$(96,57)$	$SU(6)_{18} \times SU(2)_{64-k} \times SU(2)_k \times SU(2)_{10}$
10	$\frac{F_4}{2A_1}\ \underline{\widetilde{A}_1}$	$(0,1,0,1,2,1,0,0)$	$(88,51)$	$Spin(8)_{16} \times SU(4)_{12} \times U(1)^2$
11	$\frac{F_4}{A_1}\ \underline{\widetilde{A}_1}$	$(0,1,0,1,2,1,1,0)$	$(110,68)$	$SU(6)_{18} \times SU(4)_{12} \times U(1)$
12	$\frac{F_4}{3A_1}\ \underline{A_1}$	$(0,1,0,1,1,0,0,1)$	$(84,48)$	$Sp(4)_{13} \times SU(3)_{24}$
13	$\frac{F_4}{A_2 + 2A_1}\ \underline{0}$	$(0,1,0,1,0,0,1,0)$	$(70,31)$	$(E_7)_{18} \times U(1)$
14	$\frac{F_4}{2A_2}\ \underline{0}$	$(0,1,0,0,1,0,0,0)$	$(56,16)$	$[(E_8)_{12}$ SCFT$] \times [(E_6)_6$ SCFT$]$
15	$\frac{F_4}{A_2 + A_1}\ \underline{0}$	$(0,1,0,1,1,0,1,0)$	$(83,42)$	$(E_6)_{18} \times SU(3)_{12} \times U(1)$
16	$\frac{F_4}{A_2}\ \underline{0}$	$(0,1,0,1,2,0,1,0)$	$(96,53)$	$(E_6)_{18} \times SU(3)^2_{12}$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
17	$\frac{F_4(a_2)}{A_2+2A_1}\ \underline{F_4(a_2)}$	$(0,0,2,2,1,1,1,0)$	$(94,75)$	$SU(2)_{54-k} \times SU(2)_k \times U(1)$
18	$\frac{F_4(a_2)}{2A_2}\ \underline{F_4(a_2)}$	$(0,0,2,1,2,1,0,0)$	$(80,60)$	$Spin(7)_{12} \times U(1)$
19	$\frac{F_4(a_2)}{A_2+A_1}\ \underline{F_4(a_2)}$	$(0,0,2,2,2,1,1,0)$	$(107,86)$	$SU(3)_{12} \times U(1)^2$
20	$\frac{F_4(a_2)}{A_2}\ \underline{F_4(a_2)}$	$(0,0,2,2,3,1,1,0)$	$(120,97)$	$SU(3)_{12}^2 \times U(1)$
21	$\frac{F_4(a_2)}{A_2+2A_1}\ \underline{C_3}$	$(0,1,2,2,1,1,1,0)$	$(100,80)$	$SU(2)_{36} \times SU(2)_{18} \times SU(2)_6 \times U(1)$
22	$\frac{F_4(a_2)}{2A_2}\ \underline{C_3}$	$(0,1,2,1,2,1,0,0)$	$(86,65)$	$Spin(7)_{12} \times SU(2)_6 \times U(1)$
23	$\frac{F_4(a_2)}{A_2+A_1}\ \underline{C_3}$	$(0,1,2,2,2,1,1,0)$	$(113,91)$	$SU(3)_{12} \times SU(2)_6 \times U(1)^2$
24	$\frac{F_4(a_2)}{A_2}\ \underline{C_3}$	$(0,1,2,2,3,1,1,0)$	$(126,102)$	$SU(3)_{12}^2 \times SU(2)_6 \times U(1)$
25	$\frac{F_4(a_2)}{D_4(a_1)}\ \underline{B_3}$	$(0,0,4,1,1,0,0,0)$	$(64,48)$	$SU(2)_8^3 \times U(1)^2$
26	$\frac{F_4(a_2)}{A_3+A_1}\ \underline{B_3}$	$(0,0,3,1,1,1,0,0)$	$(73,56)$	$SU(2)_{16} \times SU(2)_8 \times SU(2)_9 \times U(1)$
27	$\frac{F_4(a_2)}{A_3}\ \underline{B_3}$	$(0,0,3,2,1,1,0,0)$	$(84,65)$	$SU(2)_{16} \times SU(2)_8 \times Sp(2)_{10} \times U(1)$
28	$\frac{F_4(a_2)}{A_4+A_1}\ \underline{C_3(a_1)}$	$(0,2,1,1,2,1,0,0)$	$(79,63)$	$SU(2)_7 \times U(1)^3$
29	$\frac{F_4(a_2)}{A_4}\ \underline{C_3(a_1)}$	$(0,2,2,1,2,1,0,0)$	$(87,70)$	$SU(2)_8 \times SU(2)_7 \times U(1)^3$
30	$\frac{F_4(a_2)}{A_4+A_1}\ \underline{\widetilde{A}_2+A_1}$	$(0,1,1,1,2,1,1,0)$	$(92,75)$	$SU(2)_{20} \times U(1)^2$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
31	$\frac{F_4(a_2)}{A_4}$ $\underline{\widetilde{A}_2 + A_1}$	$(0,1,2,1,2,1,1,0)$	$(100,82)$	$SU(2)_{20} \times SU(2)_8 \times U(1)^2$
32	$\frac{F_4(a_2)}{A_4 + A_1}$ $\underline{B_2}$	$(0,1,1,1,3,1,0,0)$	$(86,69)$	$SU(2)_7^2 \times U(1)^2$
33	$\frac{F_4(a_2)}{A_4}$ $\underline{B_2}$	$(0,1,2,1,3,1,0,0)$	$(94,76)$	$SU(2)_8 \times SU(2)_7^2 \times U(1)^2$
34	$\frac{F_4(a_2)}{A_4 + A_1}$ $\underline{\widetilde{A}_2}$	$(0,1,1,2,2,1,1,0)$	$(104,84)$	$(G_2)_{10} \times U(1)^2$
35	$\frac{F_4(a_2)}{A_4}$ $\underline{\widetilde{A}_2}$	$(0,1,2,2,2,1,1,0)$	$(112,91)$	$(G_2)_{10} \times SU(2)_8 \times U(1)^2$
36	$\frac{F_4(a_2)}{E_6(a_3)}$ $\underline{A_2}$	$(0,1,1,0,1,1,0,0)$	$(56,38)$	$Spin(7)_{16} \times U(1)$
37	$\frac{F_4(a_2)}{A_5}$ $\underline{A_2}$	$(0,0,1,0,2,1,0,0)$	$(63,44)$	$(G_2)_{16} \times SU(2)_7 \times U(1)^2$
38	$\frac{F_4(a_2)}{D_5(a_1)}$ $\underline{A_2}$	$(0,1,1,1,1,1,1,0)$	$(83,64)$	$(G_2)_{16} \times U(1)$
39	$\frac{F_4(a_2)}{D_4}$ $\underline{A_2}$	$(0,1,1,1,2,1,1,0)$	$(96,75)$	$(G_2)_{16} \times SU(3)_{12}$
40	$\frac{F_4(a_2)}{E_6(a_3)}$ $\underline{A_1 + \widetilde{A}_1}$	$(0,1,1,1,1,1,0,0)$	$(66,47)$	$SU(2)_{64-k_1-k_2} \times SU(2)_{k_1} \times SU(2)_{k_2} \times SU(2)_{10}$
41	$\frac{F_4(a_2)}{A_5}$ $\underline{A_1 + \widetilde{A}_1}$	$(0,0,1,1,2,1,0,0)$	$(73,53)$	$SU(2)_{32}^2 \times SU(2)_{10} \times SU(2)_7$
42	$\frac{F_4(a_2)}{D_5(a_1)}$ $\underline{A_1 + \widetilde{A}_1}$	$(0,1,1,2,1,1,1,0)$	$(93,73)$	$SU(2)_{64-k} \times SU(2)_k \times SU(2)_{10} \times U(1)$
43	$\frac{F_4(a_2)}{D_4}$ $\underline{A_1 + \widetilde{A}_1}$	$(0,1,1,2,2,1,1,0)$	$(106,84)$	$SU(3)_{12} \times SU(2)_{64-k} \times SU(2)_k \times SU(2)_{10}$
44	$\frac{F_4(a_2)}{E_6(a_3)}$ $\underline{\widetilde{A}_1}$	$(0,1,1,1,2,1,0,0)$	$(80,58)$	$SU(4)_{12} \times U(1)^2$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
45	$\frac{F_4(a_2)}{A_5}\quad \underline{\widetilde{A}_1}$	$(0,0,1,1,3,1,0,0)$	$(87,64)$	$SU(4)_{12} \times SU(2)_7 \times U(1)$
46	$\frac{F_4(a_2)}{D_5(a_1)}\quad \underline{\widetilde{A}_1}$	$(0,1,1,2,2,1,1,0)$	$(107,84)$	$SU(4)_{12} \times U(1)^2$
47	$\frac{F_4(a_2)}{D_4}\quad \underline{\widetilde{A}_1}$	$(0,1,1,2,3,1,1,0)$	$(120,95)$	$SU(4)_{12} \times SU(3)_{12} \times U(1)$
48	$\frac{C_3}{A_2+2A_1}\quad \underline{C_3}$	$(0,2,2,2,1,1,1,0)$	$(106,85)$	$SU(2)_{36} \times SU(2)_{18} \times SU(2)_6^2 \times U(1)$
49	$\frac{C_3}{2A_2}\quad \underline{C_3}$	$(0,2,2,1,2,1,0,0)$	$(92,70)$	$Spin(7)_{12} \times SU(2)_6^2 \times U(1)$
50	$\frac{C_3}{A_2+A_1}\quad \underline{C_3}$	$(0,2,2,2,2,1,1,0)$	$(119,96)$	$SU(3)_{12} \times SU(2)_6^2 \times U(1)^2$
51	$\frac{C_3}{A_2}\quad \underline{C_3}$	$(0,2,2,2,3,1,1,0)$	$(132,107)$	$SU(3)_{12}^2 \times SU(2)_6^2 \times U(1)$
52	$\frac{C_3}{D_4(a_1)}\quad \underline{B_3}$	$(0,1,4,1,1,0,0,0)$	$(70,53)$	$SU(2)_8^3 \times SU(2)_6 \times U(1)^2$
53	$\frac{C_3}{A_3+A_1}\quad \underline{B_3}$	$(0,1,3,1,1,1,0,0)$	$(79,61)$	$SU(2)_9 \times SU(2)_{16} \times SU(2)_8 \times SU(2)_6 \times U(1)$
54	$\frac{C_3}{A_3}\quad \underline{B_3}$	$(0,1,3,2,1,1,0,0)$	$(90,70)$	$Sp(2)_{10} \times SU(2)_{16} \times SU(2)_8 \times SU(2)_6 \times U(1)$
55	$\frac{C_3}{A_4+A_1}\quad \underline{C_3(a_1)}$	$(0,3,1,1,2,1,0,0)$	$(85,68)$	$SU(2)_7 \times SU(2)_6 \times U(1)^3$
56	$\frac{C_3}{A_4}\quad \underline{C_3(a_1)}$	$(0,3,2,1,2,1,0,0)$	$(93,75)$	$SU(2)_8 \times SU(2)_7 \times SU(2)_6 \times U(1)^3$
57	$\frac{C_3}{A_4+A_1}\quad \underline{\widetilde{A}_2+A_1}$	$(0,2,1,1,2,1,1,0)$	$(98,80)$	$SU(2)_{20} \times SU(2)_6 \times U(1)^2$
58	$\frac{C_3}{A_4}\quad \underline{\widetilde{A}_2+A_1}$	$(0,2,2,1,2,1,1,0)$	$(106,87)$	$SU(2)_{20} \times SU(2)_8 \times SU(2)_6 \times U(1)^2$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
59	$\frac{C_3}{A_4 + A_1}\quad \underline{B_2}$	$(0,2,1,1,3,1,0,0)$	$(92,74)$	$SU(2)_7^2 \times SU(2)_6 \times U(1)^2$
60	$\frac{C_3}{A_4}\quad \underline{B_2}$	$(0,2,2,1,3,1,0,0)$	$(100,81)$	$SU(2)_8 \times SU(2)_7^2 \times SU(2)_6 \times U(1)^2$
61	$\frac{C_3}{A_4 + A_1}\quad \underline{\widetilde{A}_2}$	$(0,2,1,2,2,1,1,0)$	$(110,89)$	$(G_2)_{10} \times SU(2)_6 \times U(1)^2$
62	$\frac{C_3}{A_4}\quad \underline{\widetilde{A}_2}$	$(0,2,2,2,2,1,1,0)$	$(118,96)$	$(G_2)_{10} \times SU(2)_8 \times SU(2)_6 \times U(1)^2$
63	$\frac{C_3}{E_6(a_3)}\quad \underline{A_2}$	$(0,2,1,0,1,1,0,0)$	$(62,43)$	$Spin(7)_{16} \times SU(2)_6$
64	$\frac{C_3}{A_5}\quad \underline{A_2}$	$(0,1,1,0,2,1,0,0)$	$(69,49)$	$(G_2)_{16} \times SU(2)_7 \times SU(2)_6 \times U(1)$
65	$\frac{C_3}{D_5(a_1)}\quad \underline{A_2}$	$(0,2,1,1,1,1,1,0)$	$(89,69)$	$(G_2)_{16} \times SU(2)_6 \times U(1)$
66	$\frac{C_3}{D_4}\quad \underline{A_2}$	$(0,2,1,1,2,1,1,0)$	$(102,80)$	$(G_2)_{16} \times SU(3)_{12} \times SU(2)_6$
67	$\frac{C_3}{E_6(a_3)}\quad \underline{A_1 + \widetilde{A}_1}$	$(0,2,1,1,1,1,0,0)$	$(72,52)$	$SU(2)_{32} \times SU(2)_{16}^2 \times SU(2)_{10} \times SU(2)_6$
68	$\frac{C_3}{A_5}\quad \underline{A_1 + \widetilde{A}_1}$	$(0,1,1,1,2,1,0,0)$	$(79,58)$	$SU(2)_{32}^2 \times SU(2)_{10} \times SU(2)_7 \times SU(2)_6$
69	$\frac{C_3}{D_5(a_1)}\quad \underline{A_1 + \widetilde{A}_1}$	$(0,2,1,2,1,1,1,0)$	$(99,78)$	$SU(2)_{64-k} \times SU(2)_k \times SU(2)_{10} \times SU(2)_6 \times U(1)$
70	$\frac{C_3}{D_4}\quad \underline{A_1 + \widetilde{A}_1}$	$(0,2,1,2,2,1,1,0)$	$(112,89)$	$SU(3)_{12} \times SU(2)_{64-k} \times SU(2)_k \times SU(2)_{10} \times SU(2)_6$
71	$\frac{C_3}{E_6(a_3)}\quad \underline{\widetilde{A}_1}$	$(0,2,1,1,2,1,0,0)$	$(86,63)$	$SU(4)_{12} \times SU(2)_6 \times U(1)^2$
72	$\frac{C_3}{A_5}\quad \underline{\widetilde{A}_1}$	$(0,1,1,1,3,1,0,0)$	$(93,69)$	$SU(4)_{12} \times SU(2)_7 \times SU(2)_6 \times U(1)$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
73	$\frac{C_3}{D_5(a_1)}\quad\underline{\widetilde{A}_1}$	$(0,2,1,2,2,1,1,0)$	$(113,89)$	$SU(4)_{12} \times SU(2)_6 \times U(1)^2$
74	$\frac{C_3}{D_4}\quad\underline{\widetilde{A}_1}$	$(0,2,1,2,3,1,1,0)$	$(126,100)$	$SU(4)_{12} \times SU(3)_{12} \times SU(2)_6 \times U(1)$
75	$\frac{B_3}{D_5(a_1)}\quad\underline{F_4(a_3)}$	$(0,4,1,1,0,0,0,0)$	$(51,36)$	$SU(2)_6^4 \times U(1)^3$
76	$\frac{B_3}{D_4}\quad\underline{F_4(a_3)}$	$(0,4,1,1,1,0,0,0)$	$(64,47)$	$SU(3)_{12} \times SU(2)_6^4$
77	$\frac{B_3}{D_5(a_1)}\quad\underline{C_3(a_1)}$	$(0,3,1,1,1,0,0,0)$	$(58,42)$	$SU(2)_{12} \times SU(2)_6^2 \times SU(2)_7 \times U(1)^2$
78	$\frac{B_3}{D_4}\quad\underline{C_3(a_1)}$	$(0,3,1,1,2,0,0,0)$	$(71,53)$	$SU(3)_{12} \times SU(2)_{12} \times SU(2)_6^2 \times SU(2)_7$
79	$\frac{B_3}{D_5(a_1)}\quad\underline{\widetilde{A}_2 + A_1}$	$(0,2,1,1,1,0,1,0)$	$(71,54)$	$SU(2)_{20} \times SU(2)_{18} \times SU(2)_6 \times U(1)$
80	$\frac{B_3}{D_4}\quad\underline{\widetilde{A}_2 + A_1}$	$(0,2,1,1,2,0,1,0)$	$(84,65)$	$SU(3)_{12} \times SU(2)_{20} \times SU(2)_{18} \times SU(2)_6$
81	$\frac{B_3}{D_5(a_1)}\quad\underline{B_2}$	$(0,2,1,1,2,0,0,0)$	$(65,48)$	$SU(2)_7^2 \times SU(2)_{12}^2 \times U(1)^2$
82	$\frac{B_3}{D_4}\quad\underline{B_2}$	$(0,2,1,1,3,0,0,0)$	$(78,59)$	$SU(3)_{12} \times SU(2)_7^2 \times SU(2)_{12}^2$
83	$\frac{B_3}{E_6(a_3)}\quad\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,0,1,0,0,1)$	$(63,46)$	$SU(2)_k \times SU(2)_{39-k} \times SU(2)_{24}$
84	$\frac{B_3}{A_5}\quad\underline{A_2 + \widetilde{A}_1}$	$(0,0,1,0,2,0,0,1)$	$(70,52)$	$SU(2)_7 \times SU(2)_{26} \times SU(2)_{13} \times SU(2)_{24}$
85	$\frac{B_3}{E_6(a_3)}\quad\underline{\widetilde{A}_2}$	$(0,2,1,1,1,0,0,0)$	$(56,37)$	$Spin(7)_{10} \times SU(2)_{12} \times SU(2)_6^2$
86	$\frac{B_3}{A_5}\quad\underline{\widetilde{A}_2}$	$(0,1,1,1,2,0,0,0)$	$(63,43)$	$Spin(7)_{10} \times SU(2)_7 \times SU(2)_{12}^2$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
87	$\frac{B_3}{D_5(a_1)}$ $\underline{\widetilde{A}_2}$	$(0,2,1,2,1,0,1,0)$	$(83,63)$	$(G_2)_{10} \times SU(2)_{18} \times SU(2)_6 \times U(1)$
88	$\frac{B_3}{D_4}$ $\underline{\widetilde{A}_2}$	$(0,2,1,2,2,0,1,0)$	$(96,74)$	$(G_2)_{10} \times SU(3)_{12} \times SU(2)_{18} \times SU(2)_6$
89	$\frac{B_3}{E_6(a_3)}$ $\underline{A_2}$	$(0,1,1,0,1,1,0,1)$	$(80,61)$	$SU(3)_{16} \times SU(2)_{24} \times U(1)$
90	$\frac{B_3}{A_5}$ $\underline{A_2}$	$(0,0,1,0,2,1,0,1)$	$(87,67)$	$SU(3)_{16} \times SU(2)_7 \times SU(2)_{24} \times U(1)$
91	$\frac{B_3}{D_5}$ $\underline{0}$	$(0,0,2,1,0,0,0,0)$	$(56,23)$	$(E_7)_8 \times (F_4)_{10} \times U(1)$
92	$\frac{F_4(a_3)}{A_4+A_1}$ $\underline{F_4(a_2)}$	$(0,3,1,1,1,1,0,0)$	$(72,57)$	$U(1)^4$
93	$\frac{F_4(a_3)}{A_4}$ $\underline{F_4(a_2)}$	$(0,3,2,1,1,1,0,0)$	$(80,64)$	$SU(2)_8 \times U(1)^4$
94	$\frac{F_4(a_3)}{A_4+A_1}$ $\underline{C_3}$	$(0,4,1,1,1,1,0,0)$	$(78,62)$	$SU(2)_6 \times U(1)^4$
95	$\frac{F_4(a_3)}{A_4}$ $\underline{C_3}$	$(0,4,2,1,1,1,0,0)$	$(86,69)$	$SU(2)_8 \times SU(2)_6 \times U(1)^4$
96	$\frac{F_4(a_3)}{D_5}$ $\underline{A_2}$	$(0,3,1,0,0,1,0,0)$	$(56,37)$	$Spin(8)_{16} \times U(1)$
97	$\frac{F_4(a_3)}{D_5}$ $\underline{A_1+\widetilde{A}_1}$	$(0,3,1,1,0,1,0,0)$	$(66,46)$	$SU(2)_{16}^4 \times SU(2)_{10} \times U(1)$
98	$\frac{F_4(a_3)}{D_5}$ $\underline{\widetilde{A}_1}$	$(0,3,1,1,1,1,0,0)$	$(80,57)$	$SU(4)_{12} \times U(1)^4$
99	$\frac{F_4(a_3)}{E_6(a_1)}$ $\underline{0}$	$(0,3,0,0,0,0,0,0)$	$(48,15)$	$[(E_6)_6 \text{ SCFT}]^3$
100	$\frac{C_3(a_1)}{D_5}$ $\underline{A_2}$	$(0,2,1,0,1,1,0,0)$	$(63,43)$	$Spin(7)_{16} \times SU(2)_7 \times U(1)$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
101	$\frac{C_3(a_1)}{D_5}\ \underline{A_1+\widetilde{A}_1}$	$(0,2,1,1,1,1,0,0)$	$(73,52)$	$SU(2)_{32}\times SU(2)_{16}^2\times SU(2)_{10}\times SU(2)_7\times U(1)$
102	$\frac{C_3(a_1)}{D_5}\ \underline{\widetilde{A}_1}$	$(0,2,1,1,2,1,0,0)$	$(87,63)$	$SU(4)_{12}\times SU(2)_7\times U(1)^3$
103	$\frac{C_3(a_1)}{E_6(a_1)}\ \underline{0}$	$(0,2,0,0,1,0,0,0)$	$(55,21)$	$[(E_6)_6$ SCFT$]\times[(E_6)_{12}\times SU(2)_7$ SCFT$]$
104	$\frac{\widetilde{A}_2+A_1}{D_5}\ \underline{A_2}$	$(0,1,1,0,1,1,1,0)$	$(76,55)$	$(G_2)_{16}\times SU(2)_{20}\times U(1)$
105	$\frac{\widetilde{A}_2+A_1}{D_5}\ \underline{A_1+\widetilde{A}_1}$	$(0,1,1,1,1,1,1,0)$	$(86,64)$	$SU(2)_{48}\times SU(2)_{16}\times SU(2)_{10}\times SU(2)_{20}\times U(1)$
106	$\frac{\widetilde{A}_2+A_1}{D_5}\ \underline{\widetilde{A}_1}$	$(0,1,1,1,2,1,1,0)$	$(100,75)$	$SU(4)_{12}\times SU(2)_{20}\times U(1)^2$
107	$\frac{\widetilde{A}_2+A_1}{E_6(a_1)}\ \underline{0}$	$(0,1,0,0,1,0,1,0)$	$(68,33)$	$(E_6)_{18}\times SU(2)_{20}$ SCFT
108	$\frac{B_2}{D_5}\ \underline{A_2}$	$(0,1,1,0,2,1,0,0)$	$(70,49)$	$(G_2)_{16}\times SU(2)_7^2\times U(1)^2$
109	$\frac{B_2}{D_5}\ \underline{A_1+\widetilde{A}_1}$	$(0,1,1,1,2,1,0,0)$	$(80,58)$	$SU(2)_{32}^2\times SU(2)_{10}\times SU(2)_7^2\times U(1)$
110	$\frac{B_2}{D_5}\ \underline{\widetilde{A}_1}$	$(0,1,1,1,3,1,0,0)$	$(94,69)$	$SU(4)_{12}\times SU(2)_7^2\times U(1)^2$
111	$\frac{B_2}{E_6(a_1)}\ \underline{0}$	$(0,1,0,0,2,0,0,0)$	$(62,27)$	$[(E_6)_6$ SCFT$]\times[(F_4)_{12}\times SU(2)_7^2$ SCFT$]$
112	$\frac{A_2+\widetilde{A}_1}{D_5}\ \underline{A_2+\widetilde{A}_1}$	$(0,0,1,0,1,0,1,1)$	$(78,58)$	$Sp(2)_{39}\times U(1)$
113	$\frac{A_2+\widetilde{A}_1}{E_6(a_1)}\ \underline{A_1}$	$(0,0,0,0,1,0,0,1)$	$(62,34)$	$Sp(4)_{13}\times SU(2)_{26}$
114	$\frac{\widetilde{A}_2}{D_5}\ \underline{A_2}$	$(0,1,1,1,1,1,1,0)$	$(88,64)$	$(G_2)_{16}\times(G_2)_{10}\times U(1)$

Table 5.4: Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5,$ $n_6, n_8, n_9, n_{12})$	(n_h, n_v)	G_k
115	$\frac{\widetilde{A}_2}{D_5}$ $\underline{A_1+\widetilde{A}_1}$	$(0,1,1,2,1,1,1,0)$	$(98,73)$	$(G_2)_{10}\times SU(2)_{64-k}\times SU(2)_k$ $\times SU(2)_{10}\times U(1)$
116	$\frac{\widetilde{A}_2}{D_5}$ $\underline{\widetilde{A}_1}$	$(0,1,1,2,2,1,1,0)$	$(112,84)$	$SU(4)_{12}\times (G_2)_{10}\times U(1)^2$
117	$\frac{\widetilde{A}_2}{E_6(a_1)}$ $\underline{0}$	$(0,1,0,1,1,0,1,0)$	$(80,42)$	$(E_6)_{18}\times (G_2)_{10}$ SCFT
118	$\frac{A_2}{E_6(a_1)}$ $\underline{\widetilde{A}_1}$	$(0,0,0,0,2,1,0,0)$	$(64,37)$	$Spin(7)_{12}\times (G_2)_{16}\times U(1)$
119	$\frac{A_2}{E_6(a_1)}$ $\underline{A_1}$	$(0,0,0,0,1,1,0,1)$	$(79,49)$	$Sp(3)_{13}\times SU(3)_{16}\times U(1)$
120	$\frac{A_1+\widetilde{A}_1}{E_6(a_1)}$ $\underline{A_1+\widetilde{A}_1}$	$(0,0,0,1,1,1,0,0)$	$(60,35)$	$SU(4)_{32}\times Sp(2)_{10}$
121	$\frac{A_1+\widetilde{A}_1}{E_6(a_1)}$ $\underline{\widetilde{A}_1}$	$(0,0,0,1,2,1,0,0)$	$(74,46)$	$SU(4)_{12}\times SU(2)_{32}^2\times SU(2)_{10}$
122	$\frac{\widetilde{A}_1}{E_6(a_1)}$ $\underline{\widetilde{A}_1}$	$(0,0,0,1,3,1,0,0)$	$(88,57)$	$SU(4)_{12}^2\times U(1)$

5.2.5 Mixed fixtures

Table 5.5: Mixed fixtures

#	Fixture	$(n_2, n_3, n_4, n_5,$ $n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Theory
1	$\frac{F_4}{A_3+A_1}$ $(\underline{0}, Spin(9))$	$(0,1,0,0,0,0,0,0)$	$(16,5)$	$(E_6)_6$ SCFT $+\,1(9)$
2	$\frac{F_4}{2A_1}$ $\underline{A_2+\widetilde{A}_1}$	$(0,1,0,0,1,0,0,0)$	$(38,16)$	$(E_6)_{12}\times SU(2)_7+\frac{1}{2}(1,2)+\frac{1}{2}(7,2)$

Table 5.5: Mixed fixtures

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Theory
3	$\underline{F_4}$ $\underline{A_2 + \widetilde{A}_1}$ A_1	$(0,1,0,0,1,0,1,0)$	$(68,33)$	$(E_6)_{18} \times SU(2)_{20} + \frac{1}{2}(1,2)$
4	$\underline{F_4}$ $\underline{A_2}$ $3A_1$	$(0,1,0,0,1,0,0,0)$	$(39,16)$	$(E_6)_{12} \times SU(2)_7 + (1,2,3)$
5	$\underline{F_4}$ $\underline{A_1 + \widetilde{A}_1}$ $3A_1$	$(0,1,0,1,1,0,0,0)$	$(52,25)$	$SU(6)_{12} \times Spin(7)_{10} + \frac{1}{2}(1,2,3,1)$
6	$\underline{F_4}$ $\underline{\widetilde{A}_1}$ $3A_1$	$(0,1,0,1,2,0,0,0)$	$(68,36)$	$SU(6)_{12} \times SU(3)^2_{12} + \frac{1}{2}(1,2,1)$
7	$\underline{F_4}$ $\underline{A_1}$ $A_2 + 2A_1$	$(0,1,0,1,0,0,0,0)$	$(36,14)$	$Spin(14)_{10} \times U(1) + \frac{1}{2}(3,6)$
8	$\underline{F_4}$ $\underline{A_1}$ $A_2 + A_1$	$(0,1,0,1,1,0,0,0)$	$(55,25)$	$SU(9)_{12} \times U(1) + \frac{1}{2}(1,6)$
9	$\underline{F_4}$ $\underline{A_1}$ A_2	$(0,1,0,1,2,0,0,0)$	$(68,36)$	$SU(6)_{12} \times SU(3)^2_{12} + \frac{1}{2}(1,1,6)$
10	$\underline{F_4}$ $\underline{0}$ $2A_2 + A_1$	$(0,1,0,0,0,0,0,0)$	$(16,5)$	$(E_6)_6$ SCFT $+ \frac{1}{2}(26,2)$
11	$\underline{F_4(a_2)}$ $\underline{F_4(a_2)}$ $2A_2 + A_1$	$(0,0,2,1,1,1,0,0)$	$(65,49)$	$SU(2)_{25-k} \times SU(2)_k \times U(1) + \frac{1}{2}(2)$
12	$\underline{F_4(a_2)}$ $\underline{C_3}$ $2A_2 + A_1$	$(0,1,2,1,1,1,0,0)$	$(71,54)$	$SU(2)_{16} \times SU(2)_9 \times SU(2)_6 \times U(1) + \frac{1}{2}(2,1)$

Table 5.5: Mixed fixtures

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Theory
13	$\frac{F_4(a_2)}{E_6(a_3)}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,0,1,0,0,0)$	$(37,23)$	$Sp(2)_{12} \times SU(2)_7 \times SU(2)_6 + 2$
14	$\frac{F_4(a_2)}{A_5}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,0,1,0,2,0,0,0)$	$(45,29)$	$Sp(2)_7 \times SU(2)_7 \times SU(2)_{12}^2 + \frac{1}{2}(1,2)$
15	$\frac{F_4(a_2)}{D_5(a_1)}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,1,1,0,1,0)$	$(65,49)$	$SU(2)_{38-k} \times SU(2)_k \times U(1) + \frac{1}{2}(2)$
16	$\frac{F_4(a_2)}{D_4}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,1,2,0,1,0)$	$(78,60)$	$SU(3)_{12} \times SU(2)_{20} \times SU(2)_{18} + \frac{1}{2}(1,2)$
17	$\frac{C_3}{2A_2 + A_1}$ $\underline{C_3}$	$(0,2,2,1,1,1,0,0)$	$(77,59)$	$SU(2)_{16} \times SU(2)_9 \times SU(2)_6^2 \times U(1) + \frac{1}{2}(2,1,1)$
18	$\frac{C_3}{E_6(a_3)}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,2,1,0,1,0,0,0)$	$(43,28)$	$SU(2)_{12}^2 \times SU(2)_6^2 \times SU(2)_7 + (2,1)$
19	$\frac{C_3}{A_5}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,0,2,0,0,0)$	$(51,34)$	$Sp(2)_7 \times SU(2)_{24} \times SU(2)_7 \times SU(2)_6 + \frac{1}{2}(1,2,1)$
20	$\frac{C_3}{D_5(a_1)}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,2,1,1,1,0,1,0)$	$(71,54)$	$SU(2)_{20} \times SU(2)_{18} \times SU(2)_6 \times U(1) + \frac{1}{2}(2,1)$
21	$\frac{C_3}{D_4}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,2,1,1,2,0,1,0)$	$(84,65)$	$SU(3)_{12} \times SU(2)_{20} \times SU(2)_{18} \times SU(2)_6 + \frac{1}{2}(1,2,1)$

Table 5.5: Mixed fixtures

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Theory
22	$\frac{B_3}{E_6(a_3)}$ $\underline{\widetilde{A}_2 + A_1}$	$(0,2,1,0,1,0,0,0)$	$(43,28)$	$SU(2)_{12}^2 \times SU(2)_6^2 \times SU(2)_7 + \frac{1}{2}(2,1)$
23	$\frac{B_3}{A_5}$ $\underline{\widetilde{A}_2 + A_1}$	$(0,1,1,0,2,0,0,0)$	$(50,34)$	$SU(2)_7 \times SU(2)_{12}^3 \times SU(2)_7 + \frac{1}{2}(1,2,1)$
24	$\frac{F_4(a_3)}{D_5}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,3,1,0,0,0,0,0)$	$(36,22)$	$SU(2)_6^6 \times U(1) + \frac{3}{2}(2)$
25	$\frac{C_3(a_1)}{D_5}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,2,1,0,1,0,0,0)$	$(44,28)$	$Sp(2)_7 \times SU(2)_{12}^2$ $\times SU(2)_6 \times U(1)$ $+(2,1)$
26	$\frac{\widetilde{A}_2 + A_1}{D_5}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,0,1,0,1,0)$	$(58,40)$	$Sp(2)_{20} \times SU(2)_{18} \times U(1) + \frac{1}{2}(2,1)$
27	$\frac{\widetilde{A}_2 + A_1}{E_6(a_1)}$ $\underline{A_1}$	$(0,1,0,0,1,0,0,0)$	$(39,16)$	$(E_6)_{12} \times SU(2)_7 + \frac{1}{2}(6,1) + \frac{1}{2}(1,2)$
28	$\frac{B_2}{D_5}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,0,2,0,0,0)$	$(52,34)$	$Sp(2)_7^2 \times SU(2)_{24} \times U(1) + \frac{1}{2}(2,1,1)$
29	$\frac{A_2 + \widetilde{A}_1}{E_6(a_1)}$ $\underline{A_1 + \widetilde{A}_1}$	$(0,0,0,0,1,0,0,0)$	$(27,11)$	$Sp(5)_7 + \frac{1}{2}(3,1,2) + \frac{1}{2}(1,2,3)$
30	$\frac{A_2 + \widetilde{A}_1}{E_6(a_1)}$ $\underline{\widetilde{A}_1}$	$(0,0,0,0,2,0,0,0)$	$(46,22)$	$(F_4)_{12} \times SU(2)_7^2 + \frac{1}{2}(1,2)$
31	$\frac{\widetilde{A}_2}{D_5}$ $\underline{A_2 + \widetilde{A}_1}$	$(0,1,1,1,1,0,1,0)$	$(70,49)$	$SU(2)_{20} \times SU(2)_{18}$ $\times (G_2)_{10} \times U(1)$ $+\frac{1}{2}(2,1)$

Table 5.5: Mixed fixtures

#	Fixture	$(n_2, n_3, n_4, n_5,$ $n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Theory
32	$\underline{\widetilde{A}_2}$ $\underline{A_1}$ $E_6(a_1)$	$(0,1,0,1,1,0,0,0)$	$(52,25)$	$SU(6)_{12} \times Spin(7)_{10} + \frac{1}{2}(6,1)$
33	$\underline{A_2}$ $\underline{A_1 + \widetilde{A}_1}$ $E_6(a_1)$	$(0,0,0,0,1,1,0,0)$	$(49,26)$	$SU(6)_{16} \times SU(2)_9 + \frac{1}{2}(1,2,1)$

We note that for mixed fixture 22 on our list, the order q (equivalently, τ^2) term in the expansion of its superconformal index implies that the global symmetry is enhanced to $SU(2)_{19-k} \times SU(2)_k \times SU(2)_{24-k_1-k_2} \times SU(2)_{k_1} \times SU(2)_{k_2}$. Since we are not able to gauge any of the punctures, we cannot determine the levels k, k_1, k_2 using an S-duality.

However, by setting $k = 7, k_1 = k_2 = 6$, its properties agree with that of mixed fixture 18, up to the addition of a half-hypermultiplet. As further evidence, we have checked that the next non-trivial term in the expansion of the superconformal index is the same for each theory:

$$
\begin{aligned}
\mathcal{I}_{\#18} &= \mathcal{I}_{\#22} \times \mathcal{I}_{\text{free}} \\
&= (1 + 2q^{\frac{1}{2}} + 18q + 66q^{\frac{3}{2}} + \dots)(1 + 2q^{\frac{1}{2}} + 3q + 6q^{\frac{3}{2}} + \dots) \\
&= 1 + 4q^{\frac{1}{2}} + 25q + 114q^{\frac{3}{2}} + \dots
\end{aligned}
$$

Thus we conjecture that the SCFT realized by fixture 22 is the same as that of 18, and fill in the levels in the table above.

5.2.6 Gauge theory fixtures

Gauge theory fixtures are 3-punctured spheres with 1 or 2 insertions of $F_4(a_1)$. There are 167 such 3-punctured spheres involving three regular punctures and 1 involving an irregular puncture. Of the 167, all but 1 are resolved by replacing the $F_4(a_1)$ by the pair F_4, $E_6(a_1)$; that is, they can be thought-of as 4-punctured spheres in disguise. The remaining case involves *two* $F_4(a_1)$ punctures and is really a 5-punctured sphere in disguise. The two exceptional cases are listed in the table below. Note that the latter involves two decoupled copies of a theory to be discussed at greater length in §5.7.2.

Table 5.6: Gauge theory fixtures

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	Theory
1	$\frac{F_4(a_1)}{D_5}$ $\underline{(\widetilde{A}_1, SU(4)_4)}$	$(1,0,0,0,0,0,0,0)$	$SU(2)+4(2)$
2	$\frac{F_4(a_1)}{0}$ $\underline{F_4(a_1)}$	$(2,2,0,0,2,0,0,0)$	$\left[SU(3)+(E_8)_{12}\right]^2 \simeq \left[SU(2)+\frac{1}{2}(2) + (E_6)_{12}\times SU(2)_7\right]^2$

5.3 Global symmetries and the superconformal index

To determine the global symmetry of each SCFT and the number of free hypermultiplets for each fixture, we use the superconformal index [22, 23, 24, 25, 26]. This analysis was carried out for the untwisted E_6 fixtures in [49].

We leave many of the details to that paper, and the references therein.

5.3.1 Superconformal index for twisted fixtures

Following [17, 26, 64], we assume that the superconformal index for an E_6 fixture with a two twisted punctures and one untwisted puncture takes the usual form:

$$\mathcal{I}(\mathbf{a}_i;\tau) = \mathcal{A}(\tau) \sum_{(a_1,a_2,a_3,a_4)} \frac{\mathcal{K}(\mathbf{a}_1;\tau) P_{E_6}^{(a_1,a_2,a_3,a_2,a_1,a_4)}(\mathbf{a}_1;\tau) \prod_{i=2}^{3} \mathcal{K}(\mathbf{a}_i;\tau) P_{F_4}^{(a_1,a_2,a_3,a_4)}(\mathbf{a}_i;\tau)}{P_{E_6}^{(a_1,a_2,a_3,a_2,a_1,a_4)}(\mathbf{a}_{\mathrm{triv}};\tau)} \tag{5.2}$$

where the sum runs over finite-dimensional irreducible representations of F_4, and the Dynkin labels of each E_6 representation are determined by those of the corresponding F_4 representation, as indicated.

To obtain this formula, one can use the fact that, when C has genus zero, the Hall-Littlewood limit of the superconformal index coincides with the Coulomb branch Hilbert series of the $3d$ mirror of the $(2,0)$ theory on $C \times S^1$. For a fixture of type E_6 with twisted punctures, the 3d mirror is obtained by assigning the 3d $\mathcal{N}=4$ SCFT $T_{\tilde{\rho}}(F_4)$ to each twisted puncture $\tilde{\rho}$ and the SCFT $T_\rho(E_6)$ to the untwisted puncture ρ, and gauging the common centerless flavor symmetry $F_4/Z(F_4)$ [14, 12]. The Coulomb branch Hilbert series can then be computed following [65, 66], giving (5.2).

The Taylor expansion of the superconformal index is given by [27]

$$\mathcal{I}(\mathbf{a_i};\tau) = 1 + \tau\chi^{R}_{free}(\mathbf{a_i}) + \tau^2(\chi^{adj}_{free}(\mathbf{a_i}) + \chi^{adj}_{SCFT}(\mathbf{a_i})) + \dots$$

allowing us to read off the number of free hypermultiplets and the global symmetry group of the interacting SCFT for a given fixture. Examples of this type of calculation can be found in [17, 49, 27].

5.3.2 Higher-order expansion of the index

Computing the expansion of (5.2) to higher-order becomes very tedious due to the sum over the Weyl group in the definition of the Hall-Littlewood polynomials. We will therefore also be interested in the Schur limit of the superconformal index, where the Hall-Littlewood polynomials are replaced by characters of the corresponding representations[3]. For a twisted fixture, this is given by

$$\mathcal{I}(\mathbf{a}_i;q) = \prod_{j=2,5,6,8,9,12}(q^j;q)\sum_{(a_1,a_2,a_3,a_4)}\frac{\prod_{i=1}^{2}\mathcal{K}(\mathbf{a}_i)\chi^{(a_1,a_2,a_3,a_4)}_{F_4}(\mathbf{a}_i)\mathcal{K}(\mathbf{a}_3)\chi^{(a_1,a_2,a_3,a_2,a_1,a_4)}_{E_6}(\mathbf{a}_3)}{\chi^{(a_1,a_2,a_3,a_2,a_1,a_4)}_{E_6}(\mathbf{a}_{\text{triv}})} \tag{5.3}$$

where $(a;q) \equiv \prod_{j=0}^{\infty}(1-aq^j)$ is the q-Pochhammer symbol. We expand each character χ^λ in (5.3) in terms of $\mathfrak{su}(2)\times\mathfrak{f}$ characters as determined by the

[3]This limit corresponds to the $(0, q, t = q)$ slice in the space of superconformal fugacities [24].

$\mathfrak{su}(2)$ embedding which defines the puncture, where the $\mathfrak{su}(2)$ fugacity is set equal to $q^{\frac{1}{2}}$. Decomposing the adjoint representation as

$$\mathfrak{g} = \bigoplus_n V_n \otimes R_n$$

where V_n is the n-dimensional irrep of $\mathfrak{su}(2)$ and R_n is the corresponding representation of $\mathfrak{f}$, $\mathcal{K}(\mathbf{a})$ is defined by

$$\mathcal{K}(\mathbf{a}) = PE[\sum_n q^{\frac{n+1}{2}} \chi_{\mathfrak{f}}^{R_n}(\mathbf{a})],$$

where PE denotes the plethystic exponential. By their definitions, one can easily see that the coefficient of τ (τ^2) in the Hall-Littlewood index is the same as that of $q^{\frac{1}{2}}$ (q) in the Schur index (though the higher-order terms are different). However, while the Schur index removes the difficulty of explicitly summing over the Weyl group, we find that the number of terms in the sum in (5.3) grows very quickly at each order in $q^{\frac{1}{2}}$ and begins to involve large-dimensional representations of E_6, also making the calculation very tedious. Therefore, in most of the calculations that follow, we compute only the next 1-2 terms in the expansion of the Schur index. It would be very useful to find a more efficient way to explicitly calculate (5.2), (5.3).

5.4 Enhanced global symmetries: the Sommers-Achar group on the Higgs branch

Consider a family of fixtures where we keep two of the punctures fixed, and vary the third puncture over a special piece, $\{O\}$. Let O_s be the special puncture in this special piece. The Sommers-Achar group $C(O)$, for each of the punctures O in the special piece, is a subgroup of Lusztig's canonical quotient group, $\overline{A}(d(O)) \simeq S_n$. Let O_m be the puncture with the maximal Sommers-Achar group, i.e, the one whose Hitchin pole is $(d(O), S_n)$.

It frequently happens that, when $O = O_s$, a simple factor (associated to one of the *other* punctures, which we are holding fixed) in the manifest global symmetry of the fixture is enhanced as

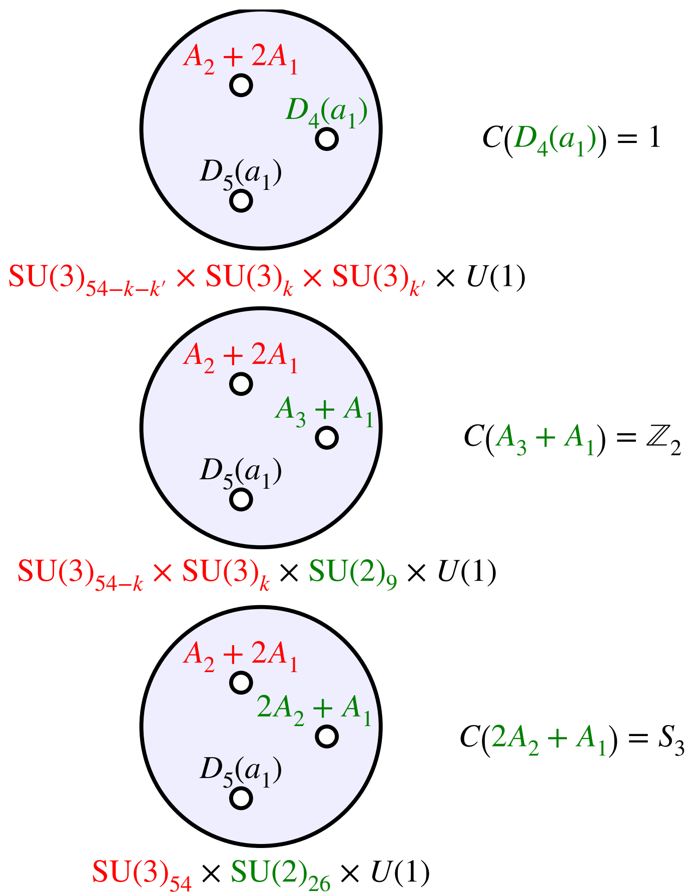
$$F_{kn} \to (F_k)^n$$

(There may, in addition, be further enhancements of the global symmetry but, by examining the fugacity-dependence of the superconformal index, we know unambiguously which ones are associated to the enhancement of F_{kn}.) When this enhancement takes place, for $O = O_s$, then, for $O = O_m$, the F_{kn} is unenhanced and, as O varies over the special piece, the enhancement is the subgroup of $(F_k)^n$ which is invariant under $C(O)$ acting by permutations of the n copies of F_k.

In particular, this gives an explicit action of the Sommers-Achar group on the holomorphic moment map operators, which are generators of the Higgs branch chiral ring. Heretofore, the Sommers-Achar group was purely a Coulomb-branch concept.

We found numerous examples of this in [49] and were able to verify, using various S-dualities (see, e.g., Section 4 of [49]) that the levels of the factors of F in the global symmetry behave as predicted by this permutation action.

One example eluded us there. We were unable to verify, using S-duality, the levels of the $SU(3)$s in the first two fixtures in



This example has an additional enhancement. As above, the manifest symmetry of the $A_2 + 2A_1$ puncture is enhanced

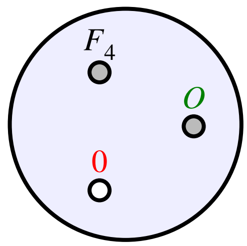
$$SU(2)_{54} \times U(1) \to SU(2)^3_{18} \times U(1)$$

with the further enhancement

$$SU(2)^3_{18} \times U(1)^3 \to SU(3)^3_{18}$$

Otherwise, this example fits the same pattern: the Sommers-Achar group, $C(O)$, acts by permutations on the $SU(3)^3_{18}$, and the global symmetry group of the fixture is the subgroup invariant under $C(O)$. That is, $k = k' = 18$.

The twisted sector of the E_6 theory provides further examples of this phenomenon. Perhaps the most striking example is the fixture



with an untwisted full puncture, a twisted simple puncture and a twisted puncture, O. As we let the puncture, O, vary over the special piece of $F_4(a_3)$, the $(E_6)_{24}$ symmetry of the 0 puncture is enhanced to (the $C(O)$-invariant subgroup of) $(E_6)_6^4$. The resulting SCFTs are products of the generalized E_6 Minahan-Nemeschansky SCFTs whose Higgs branches are the moduli space of l E_6 instantons, $M(E_6, l)^4$ [4].

Table 5.7: Fixtures obtained by varying over special piece of $F_4(a_3)$

#	O	$C(O)$	Theory	Higgs Branch	$\dim_{\mathbb{H}}$ Higgs	(n_h, n_v)
1	$\underline{F_4(a_3)}$	1	$[(E_6)_6 \text{ SCFT}]^4$	$M(E_6, 1)^4$	44	$(64, 20)$

[4] These SCFTs are realized in F-theory as the theory on l D3-branes coincident with a IV* singularity.

Table 5.7: Fixtures obtained by varying over special piece of $F_4(a_3)$

#	O	$C(O)$	Theory	Higgs Branch	$\dim_{\mathbb{H}}$ Higgs	(n_h, n_v)
2	$\underline{C_3(a_1)}$	$\mathbb{Z}_2$	$[(E_6)_6$ SCFT$]^2 \times [(E_6)_{12} \times SU(2)_7$ SCFT$]$	$M(E_6,1)^2 \times M(E_6,2)$	45	$(71, 26)$
3	$\underline{\widetilde{A_2}+A_1}$	S_3	$[(E_6)_6$ SCFT$] \times [(E_6)_{18} \times SU(2)_{20}$ SCFT$]$	$M(E_6,1) \times M(E_6,3)$	46	$(84, 38)$
4	$\underline{B_2}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[(E_6)_{12} \times SU(2)_7$ SCFT$]^2$	$M(E_6,2)^2$	46	$(78, 32)$
	$\underline{A_2+\widetilde{A_1}}$	S_4	$[(E_6)_{24} \times SU(2)_{39}$ SCFT$]$	$M(E_6,4)$	47	$(103, 56)$

Here the $(E_6)_{6l}$ global symmetry is realized as the E_6 global symmetry of $M(E_6, l)$. More subtle relations between instanton moduli spaces will be discussed below in §5.7.

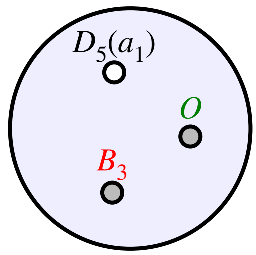
In this example, the global symmetry groups and the levels were all determined by S-duality. In other examples, S-duality determines some, but not all of the levels of the enhanced global symmetries, and we can use the action of the Sommers-Achar group on the Higgs branch to fill in the missing levels[5].

Two more sequences of fixtures, which have one puncture running over the special piece of $F_4(a_3)$, have global symmetry groups which are enhanced in this fashion, but levels we could not completely determine using S-duality[6]:

[5]The action of $C(O)$ on the Higgs branch of mixed fixtures is not so transparent.

[6]Interacting fixture 83 in the table above contains the puncture $\underline{A_2+\widetilde{A}_1}$, which is in the

In the case of



as O varies over the special piece of $F_4(a_3)$, the $SU(2)_{24}$ global symmetry of $\underline{B_3}$ is enhanced.

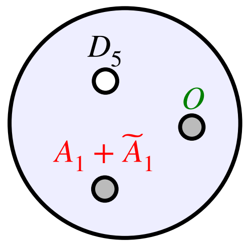
Table 5.8: More fixtures obtained by varying over the special piece of $F_4(a_3)$

#	O	$C(O)$	Global Symmetry
75	$\underline{F_4(a_3)}$	1	$SU(2)_{24-k_1-k_2-k_3} \times SU(2)_{k_1} \times SU(2)_{k_2}$ $\times SU(2)_{k_3} \times U(1)^3$
77	$\underline{C_3(a_1)}$	$\mathbb{Z}_2$	$SU(2)_{12} \times SU(2)_6^2 \times SU(2)_7 \times U(1)^2$
79	$\underline{\widetilde{A_2} + A_1}$	S_3	$SU(2)_{24-k} \times SU(2)_k \times SU(2)_{20} \times U(1)$
81	$\underline{B_2}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$SU(2)_{12}^2 \times SU(2)_7^2 \times U(1)$
	$\underline{A_2 + \widetilde{A_1}}$	S_4	$SU(2)_{24} \times SU(2)_{39} \times U(1)$

Filling in the missing levels, we find $k_1 = k_2 = k_3 = k = 6$.

special piece of $F_4(a_3)$. However, three of the other four fixtures related by varying over the special piece are *bad* (the other good fixture is mixed fixture 22). In particular, the fixture with the special puncture $F_4(a_3)$ is bad, so there is no enhancement of the form discussed above. Thus we don't know how to use this method to determine the levels for fixture 83.

Similarly, as O in



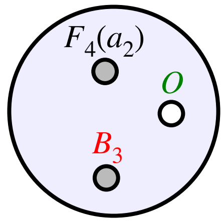
varies over the special piece of $F_4(a_3)$, the $SU(2)_{64}\times SU(2)_{10}$ global symmetry of $\underline{A_1+\widetilde{A}_1}$ is enhanced.

Table 5.9: More fixtures obtained by varying over the special piece of $F_4(a_3)$

#	O	$C(O)$	Global Symmetry
97	$\underline{F_4(a_3)}$	1	$SU(2)_{64-k_1-k_2-k_3}\times SU(2)_{k_1}\times SU(2)_{k_2}$ $\times SU(2)_{k_3}\times SU(2)_{10}\times U(1)$
101	$\underline{C_3(a_1)}$	$\mathbb{Z}_2$	$SU(2)_{32}\times SU(2)^2_{16}\times SU(2)_{10}\times SU(2)_7\times U(1)$
105	$\underline{\widetilde{A}_2+A_1}$	S_3	$SU(2)_{64-k}\times SU(2)_k\times SU(2)_{10}\times SU(2)_{20}\times U(1)$
109	$\underline{B_2}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$SU(2)^2_{32}\times SU(2)_{10}\times SU(2)^2_7\times U(1)$
	$\underline{A_2+\widetilde{A}_1}$	S_4	$SU(2)_{64}\times SU(2)_{10}\times SU(2)_{39}\times U(1)$

Again, we can fill in the missing levels: $k_1=k_2=k_3=k=16$.

Additionally, we find the following three fixture by varying over special piece of the untwisted puncture $D_4(a_1)$:



As O varies over the special piece of $D_4(a_1)$, the $SU(2)_{24}$ global symmetry of $\underline{B_3}$ is enhanced.

Table 5.10: Fixtures obtained by varying over the special piece of $D_4(a_1)$

#	O	$C(O)$	Global Symmetry
25	$D_4(a_1)$	1	$SU(2)_{24-k_1-k_2} \times SU(2)_{k_1} \times SU(2)_{k_2} \times U(1)^2$
26	$A_3 + A_1$	$\mathbb{Z}_2$	$SU(2)_{24-k} \times SU(2)_k \times SU(2)_9 \times U(1)$
	$2A_2 + A_1$	S_3	$SU(2)_{24} \times SU(2)_{26}$

We fill in the missing levels $k_1 = k_2 = k = 8$.

5.5 $R_{2,5}$

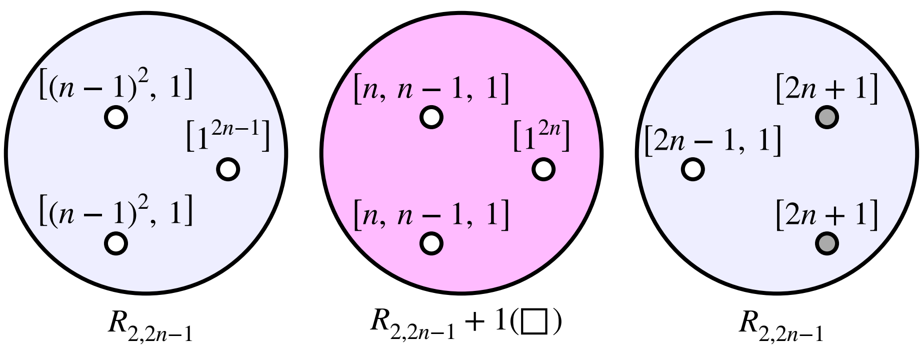
In [3], we introduced a series of $\mathcal{N} = 2$ SCFTs, which we dubbed $R_{2,2n-1}$. $R_{2,2n-1}$ has a $\big(Spin(4n+2)_{4n-2} \times U(1)\big)/\mathbb{Z}_2$ global symmetry (enhanced to $(E_6)_6$ for $n = 2$), central charges $(n_h, n_v) = \big(4n^2, (n-1)(2n+1)\big)$

and graded Coulomb branch dimensions $n_{2k-1} = 1$, for $k = 2, \ldots, n$.

These play an important role in the strong-coupling duals of various familiar gauge theories. Specifically

$$\begin{aligned} SU(2n-1) + 4(\Box) + 2\left(\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}\right) &\simeq Sp(n-1) + 1(\Box) + R_{2,2n-1} \\ SU(2n) + 4(\Box) + 2\left(\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}\right) &\simeq Sp(n) + 3(\Box) + R_{2,2n-1} \\ SU(2n) + 1\left(\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}\right) + 1(\Box\!\Box) &\simeq Spin(2n+1) + R_{2,2n-1} \end{aligned} \tag{5.4}$$

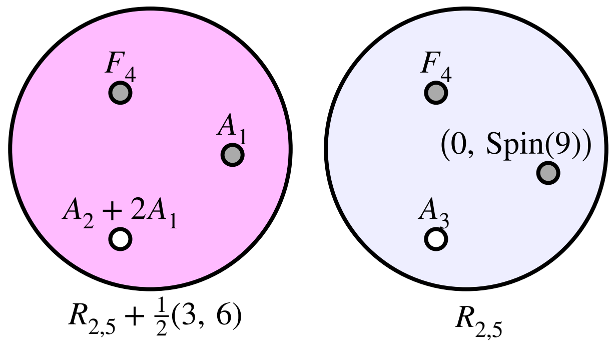
The realizations of the $R_{2,2n-1}$ are:



in the A_{2n-2}, A_{2n-1} and the twisted sector of the A_{2n-1} theory, respectively. These different realizations expose different manifest subalgebras[7] (respectively, $\mathfrak{su}(2n-1)_{4n-2} \times \mathfrak{su}(2)^2_{4n-2} \times \mathfrak{u}(1)^2$, $\mathfrak{su}(2n)_{4n-2} \times \mathfrak{u}(1)^4$ and $\mathfrak{so}(2n+1)^2_{4n-2} \times \mathfrak{u}(1)$) of the full global symmetry algebra of the $R_{2,2n-1}$ SCFT.

The twisted sector of the E_6 theory provides two new realizations of $R_{2,5}$:

[7]As symmetry groups, they are, respectively, $S\big(U(2n-1) \times U(2)^2\big)$, $S\big(U(2n) \times U(1)^3\big)$ and $\big(Spin(2n+1) \times Spin(2n+1) \times U(1)\big)/\mathbb{Z}_2$.

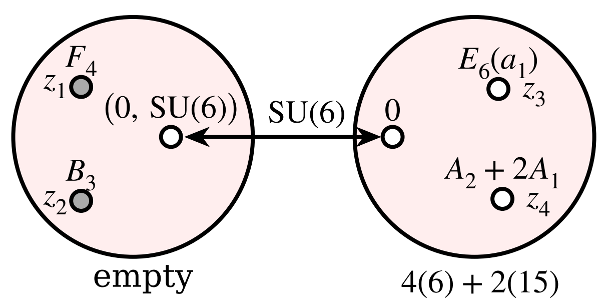


which expose a manifest $\mathfrak{sp}(3)_{10} \times \mathfrak{su}(2)_{30} \times \mathfrak{u}(1)$ or $\mathfrak{so}(9)_{10} \times \mathfrak{sp}(2)_{10} \times \mathfrak{u}(1)$ subalgebra, respectively, of the $\mathfrak{so}(14)_{10} \times \mathfrak{u}(1)$ global symmetry algebra of $R_{2,5}$.

The latter realization will be useful to us in §5.7.6. The former provides, among other things, another realization of the aforementioned duality

$$SU(6) + 4(6) + 2(15) \simeq Sp(3) + 3(6) + R_{2,5}$$

via the 4-punctured sphere



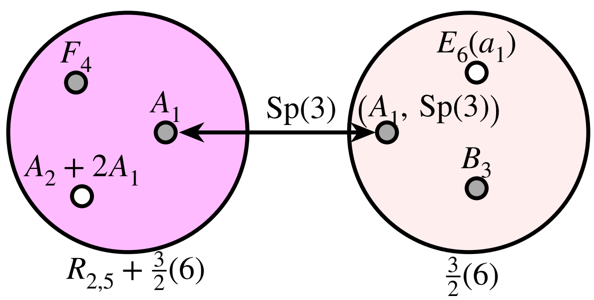
Here, the gauge coupling,

$$
f(\tau) \equiv -\frac{\theta_2^4(0,\tau)}{\theta_4^4(0,\tau)} = \frac{w-1}{w+1}
$$

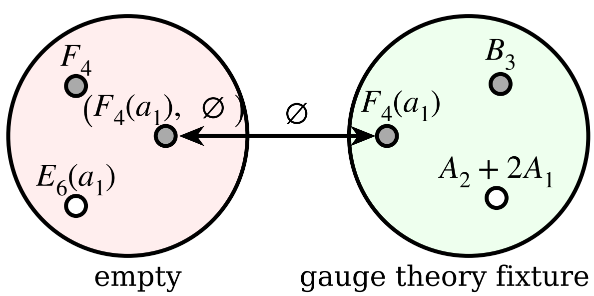
is a function on the double-cover of $\mathcal{M}_{0,4}$, where

$$
w^2 = x \equiv \frac{z_{13}z_{24}}{z_{14}z_{23}}
$$

so that $f(\tau) = 1$ at the degeneration



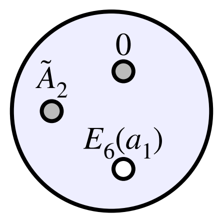
and the degeneration



corresponds to the interior point of the gauge theory moduli space, $f(\tau) = -1$.

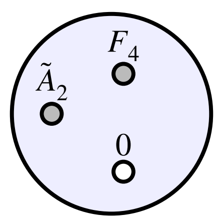
5.6 Product SCFTs

The $(E_6)_{18} \times (G_2)_{10}$ SCFT occurs twice on our list of interacting fixtures: once, by itself, in



$(E_6)_{18} \times (G_2)_{10}$ SCFT (5.5)

(interacting fixture 117) and once — we claim — as part of a product SCFT



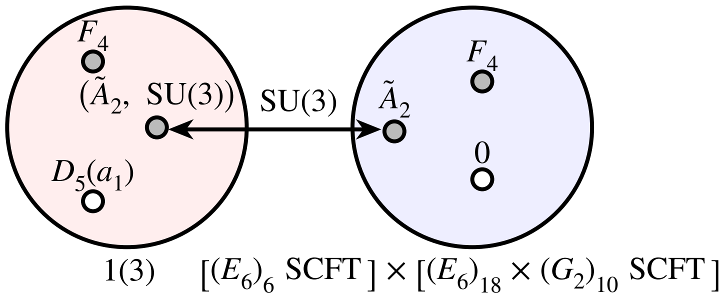
$[(E_6)_6$ SCFT $] \times [(E_6)_{18} \times (G_2)_{10}$ SCFT $]$ (5.6)

(interacting fixture 5). We can check the latter claim, explicitly, by comparing the SCI for (5.6) with the SCI for (5.5) and the (known) SCI for the $(E_6)_6$ SCFT.

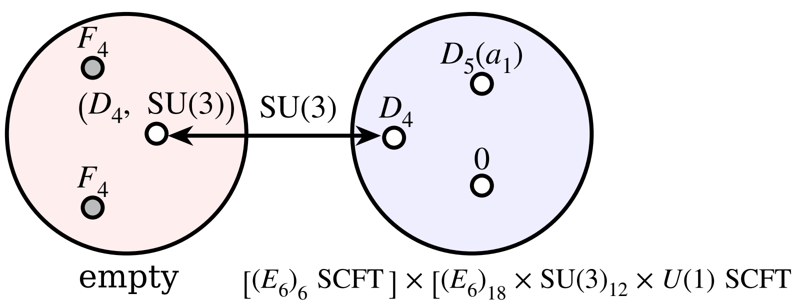
Indeed, we find that, to second order in q, we have

$$\begin{aligned}
\mathcal{I}_{\#5} &= 1 + 170q + 14601q^2 + \ldots \\
&= (1 + 92q + 4916q^2 + \ldots)(1 + 78q + 2509q^2 + \ldots) \\
&= (\mathcal{I}_{\#117} \times \mathcal{I}_{(E_6)_6 SCFT})|_{q^2}
\end{aligned}$$

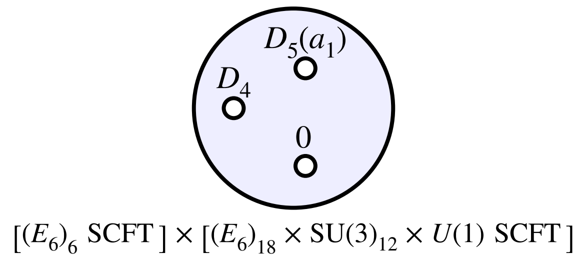
Having established that (5.6) is a product SCFT, we can apply that knowledge to deduce that other fixtures are also product SCFTs. For instance, consider the 4-punctured sphere



The $SU(3)$ gauges a subgroup of the $(G_2)_{10}$ symmetry of the $(E_6)_{18} \times (G_2)_{10}$ SCFT, leaving the $(E_6)_6$ SCFT decoupled. Taking the S-dual,



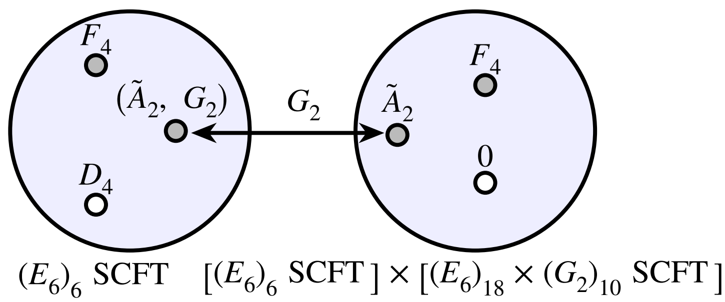
we conclude that fixture on the right also contains a decoupled $(E_6)_6$ SCFT and, hence, that



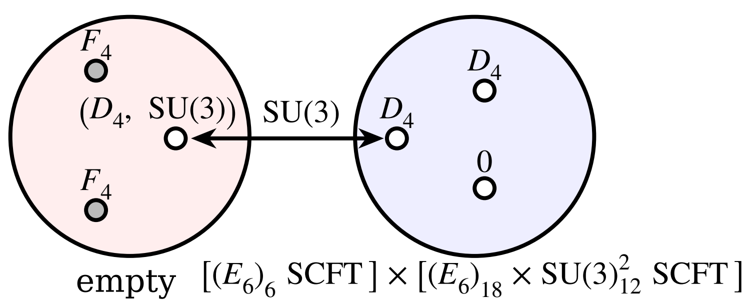
(5.7)

(interacting fixture 61 of [49]) is also a product SCFT.

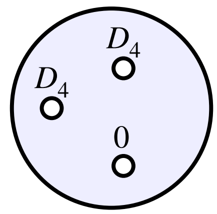
Similarly, in



the gauging of the G_2 symmetry leaves the $(E_6)_6$ SCFT decoupled. Hence, in the S-dual,



the fixture



$$[(E_6)_6 \text{ SCFT}] \times [(E_6)_{18} \times \mathrm{SU}(3)^2_{12} \text{ SCFT}] \tag{5.8}$$

(interacting fixture 99 of [49]) is, again, a product SCFT.

As a further check of these identifications, we can compare the SCIs for (5.7) and (5.8) with those of interacting fixtures 15 and 16 above, which directly realize, respectively, the $(E_6)_{18} \times SU(3)_{12} \times U(1)$ and $(E_6)_{18} \times SU(3)^2_{12}$ SCFTs.

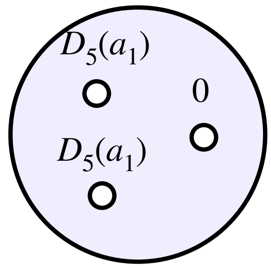
Indeed, we find that

$$\begin{aligned}
\mathcal{I}_{\#61} &= 1 + 165q + 164q^{\frac{3}{2}} + 13451q^2 + \dots \\
&= (1 + 87q + 164q^{\frac{3}{2}} + 4156q^2 + \dots)(1 + 78q + 2509q^2 + \dots) \\
&= (\mathcal{I}_{\#15} \times \mathcal{I}_{(E_6)_6 \text{ SCFT}})|_{q^2}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{\#99} &= 1 + 172q + 14886q^2 + \dots \\
&= (1 + 94q + 5045q^2 + \dots)(1 + 78q + 2509q^2 + \dots) \\
&= (\mathcal{I}_{\#16} \times \mathcal{I}_{(E_6)_6 \text{ SCFT}})|_{q^2}
\end{aligned}$$

Similarly, we can check that interacting fixture 59 of [49]



$[(E_6)_6 \text{ SCFT}] \times [(E_7)_{18} \times U(1) \text{ SCFT}]$

is a product SCFT by comparing the expansion of its SCI with that of interacting fixture 13 above, which directly realizes the $(E_7)_{18} \times U(1)$ SCFT. Indeed, one finds that

$$
\begin{aligned}
\mathcal{I}_{\#59} &= 1 + 212q + 112q^{\frac{3}{2}} + 22273q^2 + \ldots \\
&= (1 + 78q + 2509q^2 + \ldots)(1 + 134q + 112q^{\frac{3}{2}} + 9312q^2 + \ldots) \\
&= (\mathcal{I}_{(E_6)_6 \text{ SCFT}} \times \mathcal{I}_{\#13})|_{q^2}.
\end{aligned}
$$

Finally, we claim that interacting fixture 111 above is the product of the $(E_6)_6$ SCFT and the $(F_4)_{12} \times SU(2)^2_7$ SCFT. The latter previously appeared in our list of interacting fixtures for the D_4 theory [5] and appears in mixed fixture 30 above.

We find the expansion of the SCI for fixture 111 is given by

$$
\mathcal{I}_{\#111} = 1 + 136q + 104q^{\frac{3}{2}} + 9036q^2 + \ldots
$$

That of mixed fixture 30 reads

$$
\begin{aligned}
\mathcal{I}_{\#30} &= 1 + 2q^{\frac{1}{2}} + 61q + 226q^{\frac{3}{2}} + 2394q^2 + \ldots \\
&= (1 + 2q^{\frac{1}{2}} + 3q + 6q^{\frac{3}{2}} + 9q^2 + \ldots)(1 + 58q + 104q^{\frac{3}{2}} + 2003q^2 + \ldots) \\
&= (\mathcal{I}_{\text{free}} \times \mathcal{I}_{(F_4)_{12} \times SU(2)_7^2 SCFT})|_{q^2}
\end{aligned}
$$

Extracting the order q^2 expansion of the index of the $(F_4)_{12} \times SU(2)_7^2$ SCFT from the above, we see that

$$
\begin{aligned}
&\mathcal{I}_{(E_6)_6 SCFT} \times \mathcal{I}_{(F_4)_{12} \times SU(2)_7^2 SCFT} \\
&\quad = (1 + 78q + 2509q^2 + \ldots)(1 + 58q + 104q^{\frac{3}{2}} + 2003q^2 + \ldots) \\
&\quad = 1 + 136q + 104q^{\frac{3}{2}} + 9036q^2 + \ldots \\
&\quad = \mathcal{I}_{\#111}|_{q^2}
\end{aligned}
$$

5.7 Instanton moduli spaces

Let $M(G, k)$ denote the moduli space of k instantons on $\mathbb{R}^4$, for gauge group G[8] . $M(G, k)$ is a hyperKähler space of dimension

$$
\dim_{\mathbb{H}}\big(M(G, k)\big) = \kappa_G k - 1
$$

where κ_G is the dual Coxeter number and the "-1" is present because we have removed the overall translational degree of freedom.

For $k = 1$, $M(G, k)$ has hyperKähler isometry group G. In fact, $M(G, 1)$ is the minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$. For k 1, the hyperKähler isom-

[8] Equivalently, the moduli space of *framed* instantons on S^4.

etry group of $M(G,k)$ is $G \times SU(2)$. The origin of the additional $SU(2)$ is as follows[9] . While we've removed the translational symmetry of $\mathbb{R}^4$, the $SO(4) = \big(SU(2)\times SU(2)\big)/\mathbb{Z}_2$ rotational symmetry still acts on the space of instanton solutions. One of the $SU(2)$s acts by rotating the complex structures of $M(G,k)$ among themselves. The other $SU(2)$ preserves the quaternionic structure. For $k = 1$, it is easy to see that it acts trivially, whereas for $k > 1$ it acts nontrivially.

For the classical groups G, the ADHM construction [67] provides a realization of $M(G,k)$ as a hyperKähler quotient of a quaternionic *vector space*. When G is exceptional, no such construction exists. But (at least for low k) something almost as nice exists. Namely: the hyperKähler quotient $M(G,k)///H$, for H some subgroup of the isometry group of $M(G,k)$, has an alternative realization as a hyperKähler quotient either of a quaternionic vector space or of some other well-known hyperKähler space.

The first examples of this phenomenon come from the classic paper of Argyres-Seiberg [2]

$$\begin{gathered}\Big(M(E_6,1)\times\mathbb{H}^2\Big)///SU(2) \simeq \mathbb{H}^{18}///SU(3)\\ M(E_7,1)///SU(2) \simeq \mathbb{H}^{24}///Sp(2)\end{gathered} \tag{5.9}$$

They established something much stronger: the S-duality of a pair of $\mathcal{N} = 2$ supersymmetric quantum field theories. The Higgs branch of one theory is the

[9]We thank Andrew Neitzke for a discussion of this point.

LHS; the Higgs branch of the other is the RHS. Because the Higgs branch geometry is independent of the gauge coupling, the S-duality of the two theories implies that the two Higgs branches are isomorphic. An independent, non-trivial check on the first of these isomorphisms was performed in [68]. At the holomorphic-symplectic level, an axiomatization of this general construction is given in [69].

Further examples of such isomorphisms of hyperKähler quotients of instanton moduli spaces (implied, again, by the S-duality of the corresponding QFTs) appeared in our previous papers. In section 4.2.3 of [17], we found

$$M(E_8,1)///Sp(2) \simeq \mathbb{H}^{40}///Sp(3) \tag{5.10}$$

Here, the defining 6-dimensional representation and the 14-dimensional traceless 3-index antisymmetric tensor representation of $Sp(3)$ are both pseudo-real (have quaternionic structures) and hence induce, respectively, linear actions on $\mathbb{H}^3$ and $\mathbb{H}^7$. On the RHS of (5.10), we decompose $\mathbb{H}^{40}$ as 11 copies of the former and 1 copy of the latter. In the usual physics notation, we denote this by $\mathbb{H}^{40} \simeq \frac{11}{2}(6) \oplus \frac{1}{2}(14')$ ("11 half-hypermultiplets in the fundamental and 1 half-hypermultiplet in the $14'$ representation of $Sp(3)$"). Similarly, on the RHS of (5.9), we have $\mathbb{H}^{24} \simeq 6(4)$ ("6 full hypermultiplets in the fundamental representation of $Sp(2)$").

In section 4.1.3 of [38], we found

$$M(E_7,2)///G_2 \simeq \mathbb{H}^{57}///Spin(9) \tag{5.11}$$

where, on the RHS, $\mathbb{H}^{57}$ decomposes as the $3(16)+1(9)$ of $Spin(9)$.

In this section, we will demonstrate five new identities of this sort.

$$\Big(M(E_6,2)\times\mathbb{H}\Big)///SU(2) \simeq M(E_8,1)///SU(3) \tag{5.12}$$

$$M(E_6,2)///SU(3) \simeq \Big(M(E_6,1)\times M(E_6,1)\times\mathbb{H}^7\Big)///G_2 \tag{5.13}$$

$$M(E_7,3)///Spin(8) \simeq \Big(M(E_7,1)\times M(E_7,1)\times M(E_7,1)\times\mathbb{H}^{26}\Big)///F_4 \tag{5.14}$$

$$\begin{gathered}\Big(M(E_7,2)\times M(E_7,1)\Big)///Spin(8) \simeq \\ \Big(M(E_7,1)\times M(E_7,1)\times M(E_7,1)\times\mathbb{H}^9\Big)///Spin(9)\end{gathered} \tag{5.15}$$

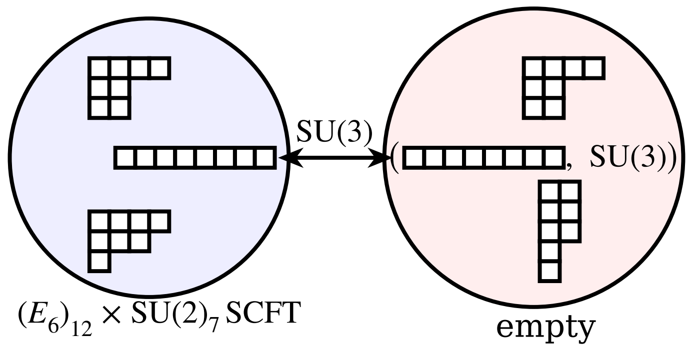
and

$$\Big(M(E_8,2)\times\mathbb{H}^{32}\Big)///Spin(12) \simeq \Big(M(E_8,1)\times M(E_8,1)\times\mathbb{H}^{45}\Big)///Spin(13) \tag{5.16}$$

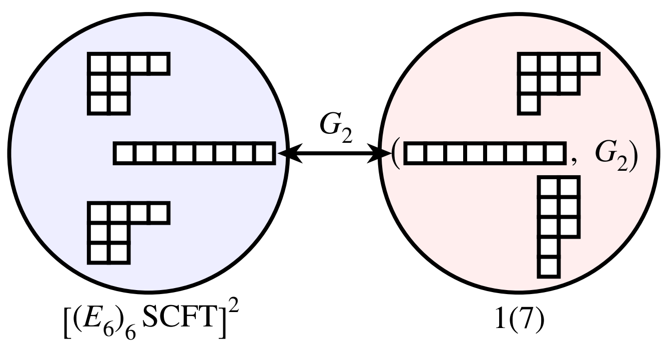
where, on the LHS, the two irreducible spinor representations of $Spin(12)$ are pseudoreal ($\mathbb{H}^{32} \simeq \frac{1}{2}(32)\oplus\frac{1}{2}(32'))$ and, on the RHS, we have $\mathbb{H}^{45} \simeq \frac{1}{2}(64)+1(13)$.

5.7.1 $M(E_6,2)///SU(3) \simeq \Big(M(E_6,1)\times M(E_6,1)\times\mathbb{H}^7\Big)///G_2$

(5.13) is realized in the untwisted D_4 theory by the 4-punctured sphere



which is an $SU(3)$ gauging of the $(E_6)_{12} \times SU(2)_7$ SCFT (whose Higgs branch is $M(E_6, 2)$). The S-dual theory

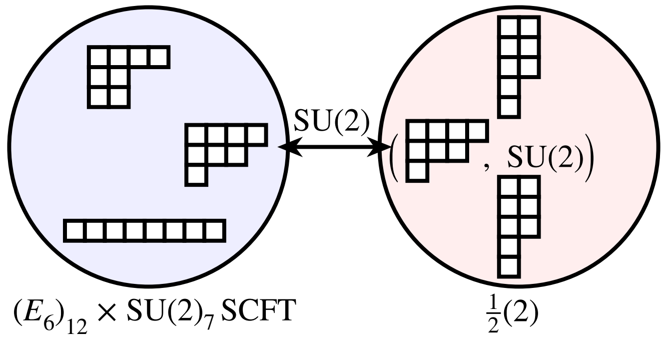


is a G_2 gauge theory coupled to two copies of the $(E_6)_6$ SCFT (whose Higgs branch is $M(E_6, 1)$) and one hypermultiplet in the 7.

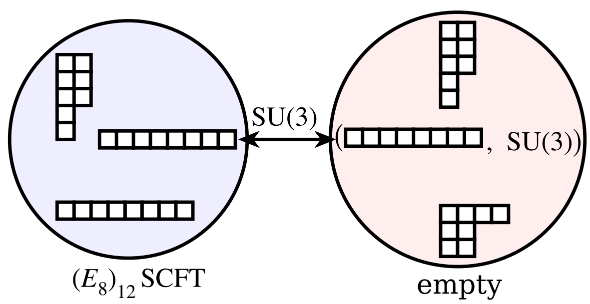
5.7.2 $\Big(M(E_6, 2) \times \mathbb{H}\Big)///SU(2) \simeq M(E_8, 1)///SU(3)$

Recall that, for $k > 1$, $M(E_n, k)$ has an $E_n \times SU(2)$ isometry group. (5.12) is unique among the examples listed here, in that on the LHS we use the $SU(2)$ action on $M(E_6, 2)$, which commutes with E_6 action, to perform the hyperKähler quotient.

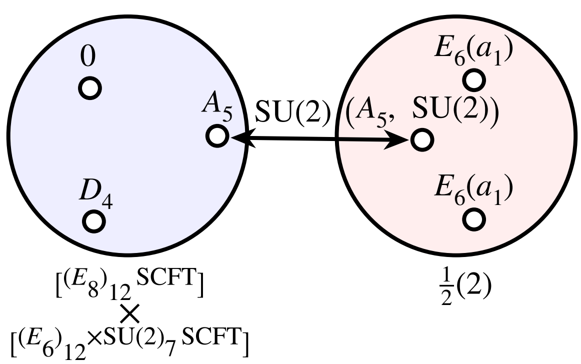
A realization in the D_4 theory is



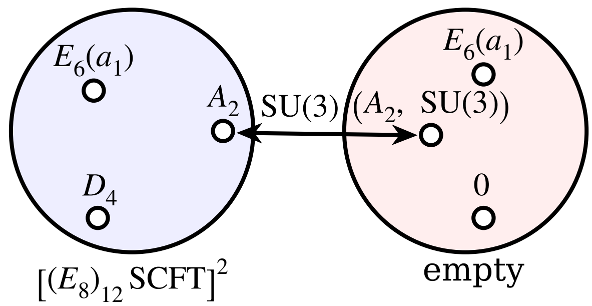
which is S-dual to



It is also realized in the untwisted E_6 theory as

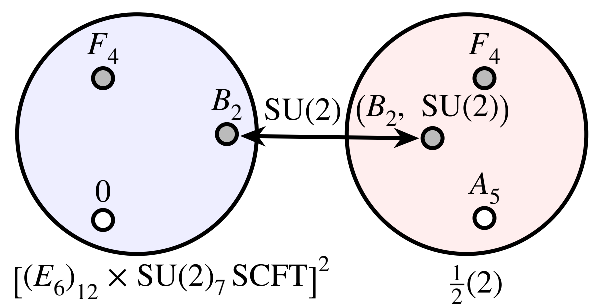


where the $SU(2)$ gauges the $SU(2)_7$ of the $(E_6)_{12} \times SU(2)_7$ SCFT and the $(E_8)_{12}$ SCFT is decoupled. The S-dual theory is

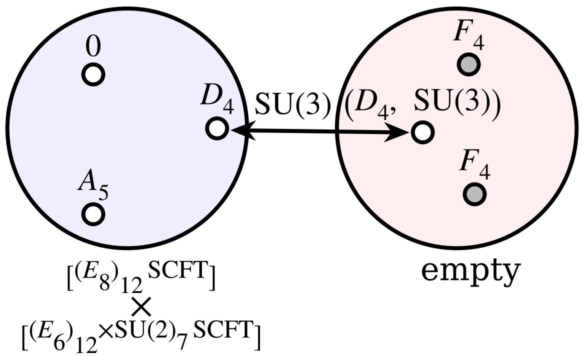


where we gauge an $SU(3)$ subgroup of one of the E_8s while the other $(E_8)_{12}$ SCFT is decoupled.

Another realization of (5.12) appears in the twisted sector of the E_6 theory. In

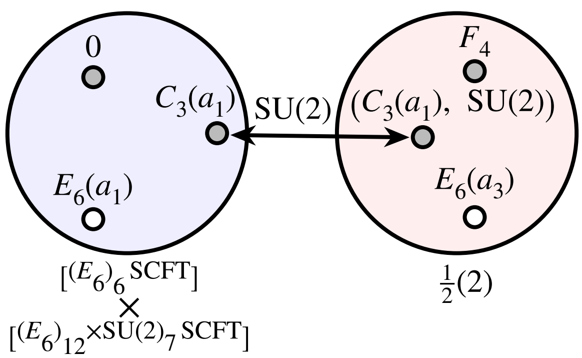


the $SU(2)$ gauges the $SU(2)_7$ of one of the $(E_6)_{12} \times SU(2)_7$ SCFTs, while the other $(E_6)_{12} \times SU(2)_7$ SCFT is decoupled. In the S-dual theory,

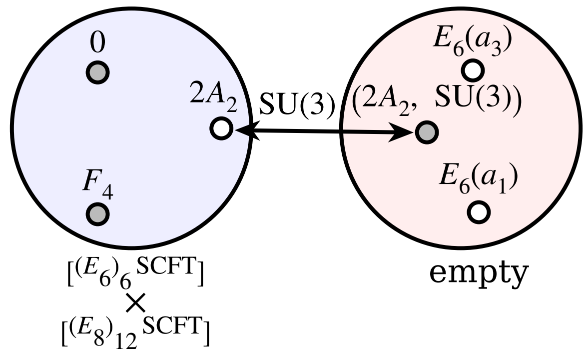


the $SU(3)$ gauges a subgroup of the E_8, while the $(E_6)_{12} \times SU(2)_7$ SCFT is decoupled.

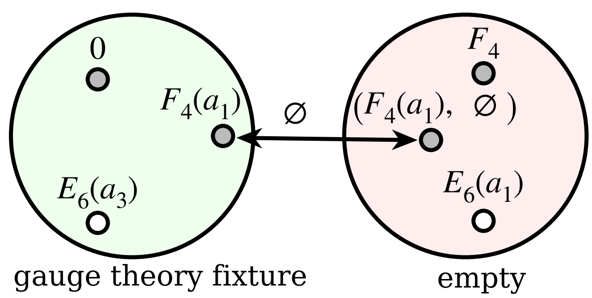
A third realization, in which an $(E_6)_6$ SCFT is decoupled throughout, is given by



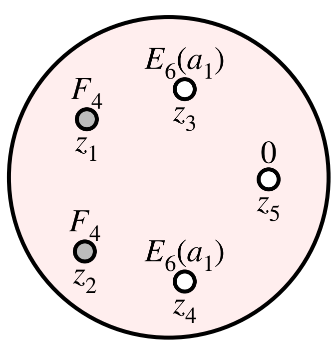
and has S-duals given by



and the gauge-theory fixture



Finally, the 5-punctured sphere



gives a realization of *two decoupled copies* of this theory. The gauge theory moduli space is a 4-fold branched cover of $\mathcal{M}_{0,4}$, with coordinates (y, w) given in terms of the cross-ratios as

$$y^2 = s_1 = \frac{z_{13}z_{25}}{z_{15}z_{23}}, \qquad w^2 = s_2 = \frac{z_{14}z_{25}}{z_{15}z_{24}}$$

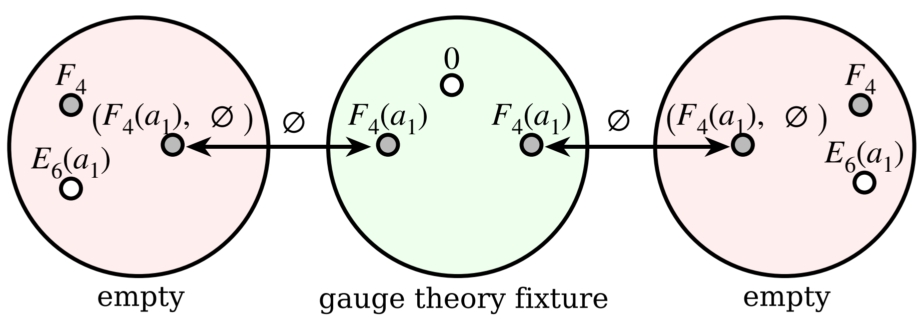
The gauge couplings are

$$f(\tau_1) = \frac{y-1}{y+1}\frac{w+1}{w-1}, \qquad f(\tau_2) = \frac{y-1}{y+1}\frac{w-1}{w+1}$$

where

$$f(\tau) \equiv -\frac{\theta_2^4(0,\tau)}{\theta_4^4(0,\tau)} \tag{5.17}$$

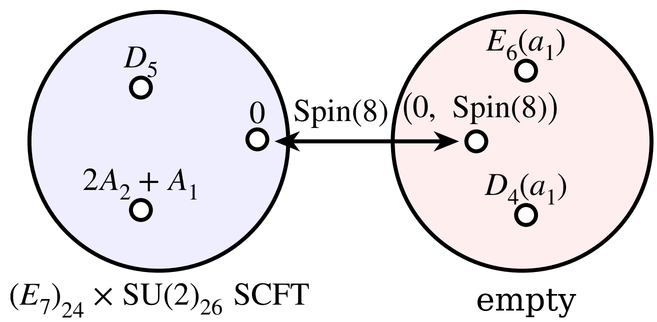
and $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$. In the limits $f(\tau) \to 0, \infty$, the $SU(3) + \left[(E_8)_{12}\right]$ description is weakly-coupled. For $f(\tau) \to 1$, the $SU(2) + \frac{1}{2}(2) + \left[(E_6)_{12} \times SU(2)_7\right]$ description is weakly-coupled. Over the degeneration



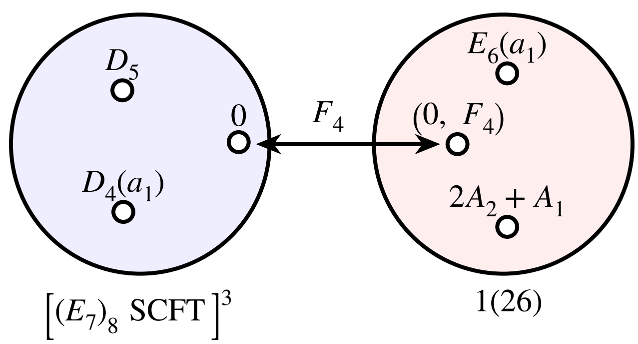
we have $(f(\tau_1), f(\tau_2)) \to (-1, -1)$ and both descriptions are strongly-coupled.

5.7.3 $M(E_7,3)///Spin(8) \simeq \Big(M(E_7,1)^3 \times \mathbb{H}^{26}\Big)///F_4$ and $\Big(M(E_7,2)\times M(E_7,1)\Big)///Spin(8) \simeq \Big(M(E_7,1)^3 \times \mathbb{H}^9\Big)///Spin(9)$

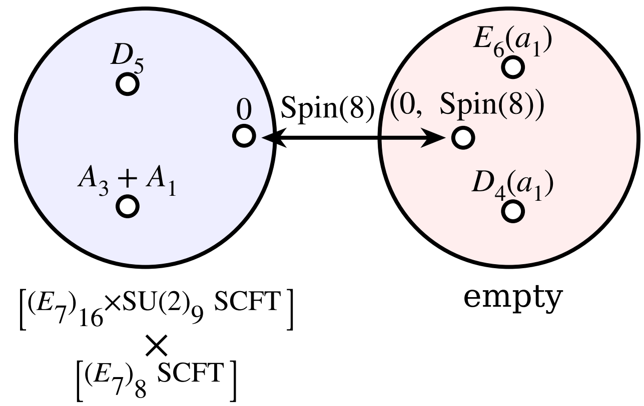
(5.14) and (5.15) both have realizations in the untwisted E_6 theory. The former is given by the duality between



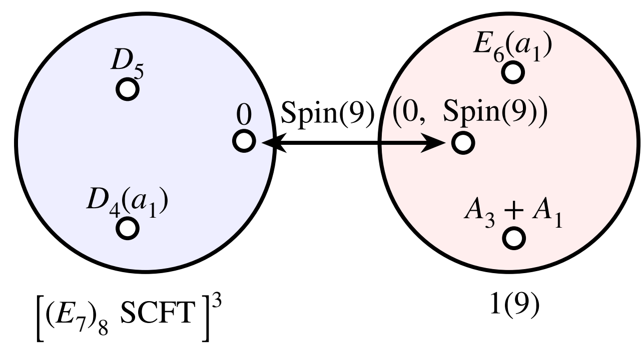
and



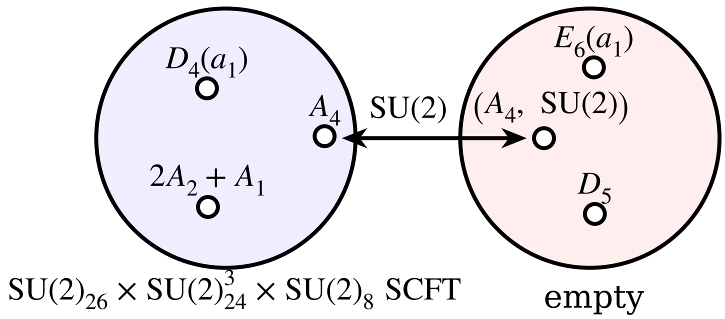
The latter is given by the duality between



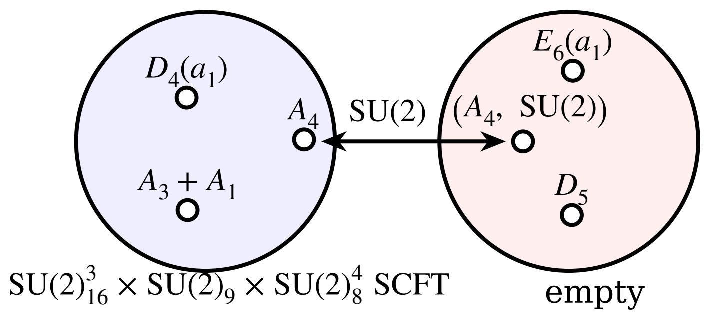
and



In both cases, unlike our previous examples, there is a third S-duality frame, respectively



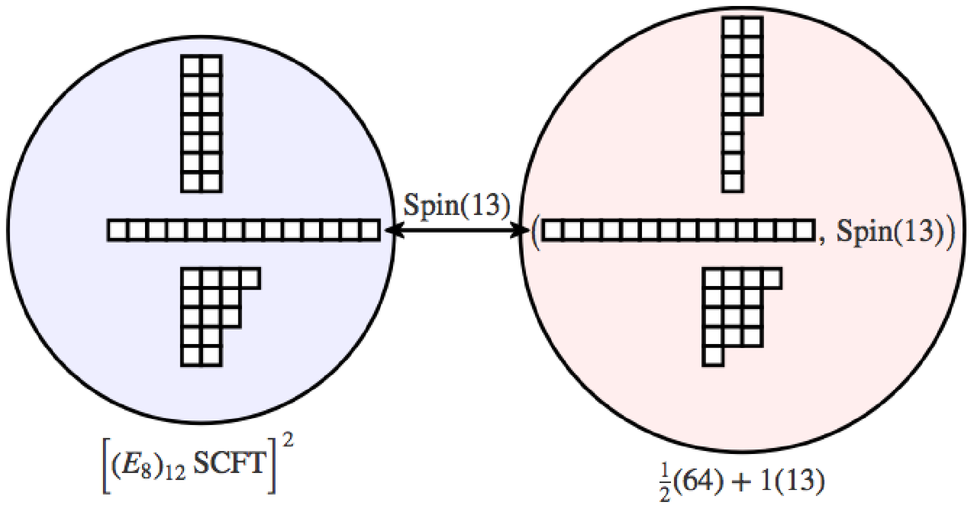
and



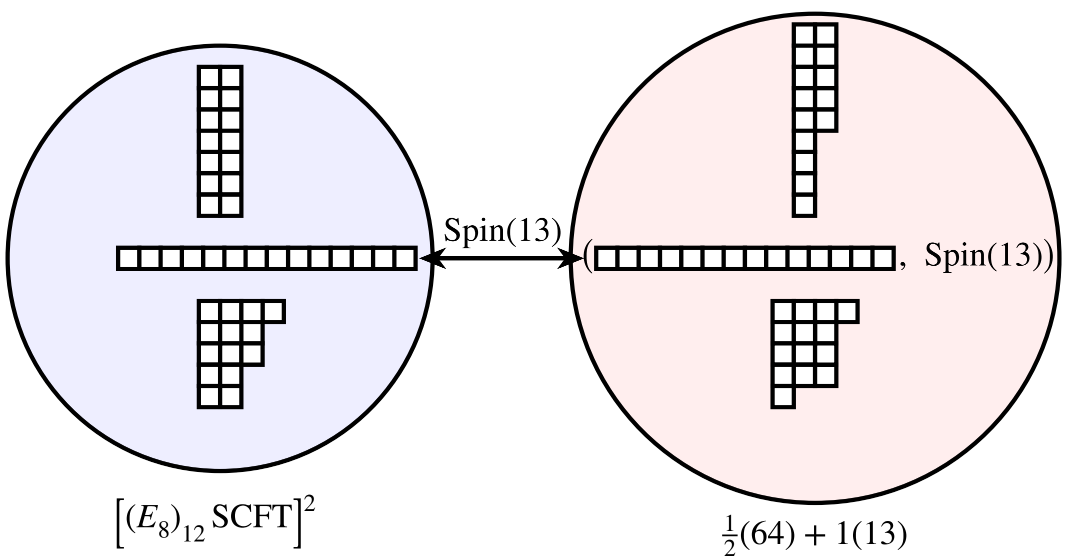
which are $SU(2)$ gaugings of some new non-Lagrangian SCFTs. Alas, since we don't have an independent construction of the Higgs branches of the latter theories, these isomorphisms don't shed much additional light on these instanton moduli spaces.

5.7.4 $(M(E_8, 2) \times \mathbb{H}^{32}) /// Spin(12) \simeq \big(M(E_8, 1) \times M(E_8, 1) \times \mathbb{H}^{45}\big) /// Spin(13)$

Turning to (5.16), there is a realization in the D_7 theory

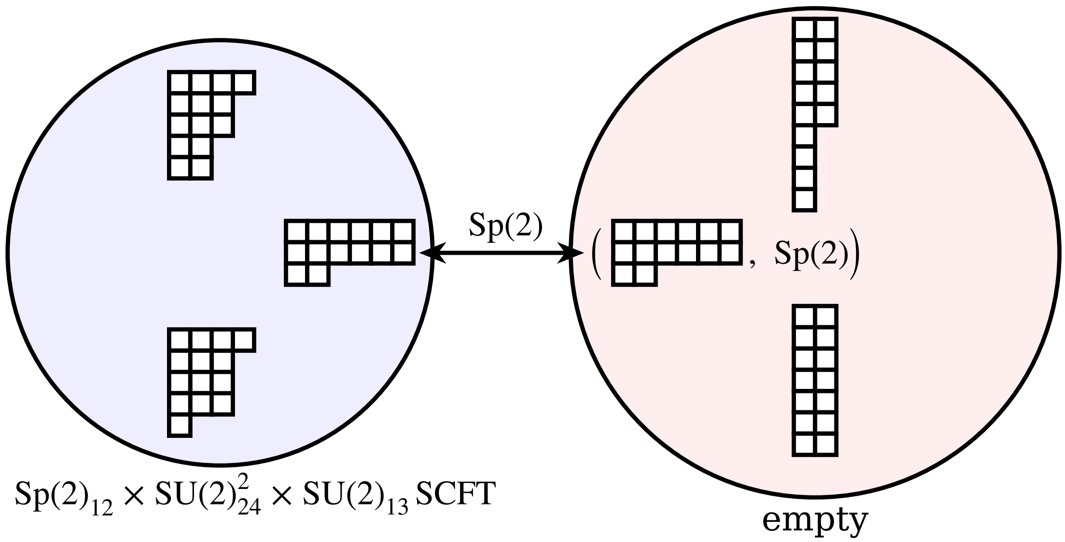


which has one S-dual presentation as



realizing the isomorphism of Higgs branches stated in (5.16).

This theory also has a third S-duality frame,



Alas, as in §5.7.3, we have no alternative construction of the Higgs branch of the $Sp(2)_{12} \times SU(2)_{24}^2 \times SU(2)_{13}$ SCFT, so we don't learn anything new from this duality.

5.7.5 Semi-simple quotients

In §5.7 we considered isomorphisms of hyperKähler quotients of the form

$$X_1///G_1 \simeq X_2///G_2 \tag{5.18}$$

where G_i is a *simple* subgroup of the group of hyperKähler isometries of X_i. Let H be the residual group of hyperKähler isometries of the quotient. Of course, we can further quotient both sides of (5.18) by a subgroup of H, but this would typically not yield anything new; all it would do is lose some of the information contained in (5.18).

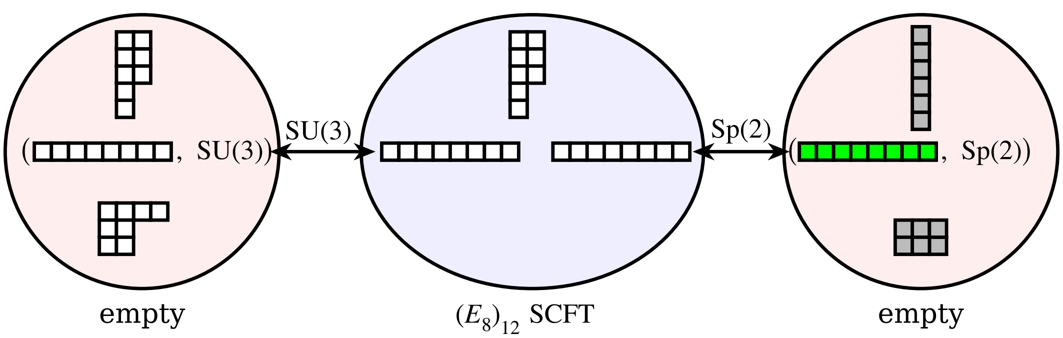
There are, however, exceptions. For instance, we can combine (5.10) with the first isomorphism in (5.9) to obtain

$$\begin{aligned} M(E_8,1)///SU(3)\times Sp(2) &\simeq \mathbb{H}^{40}///SU(3)\times Sp(3) \\ &\simeq \left(M(E_6,1)\times \mathbb{H}^{24}\right)///SU(2)\times Sp(3) \end{aligned}$$

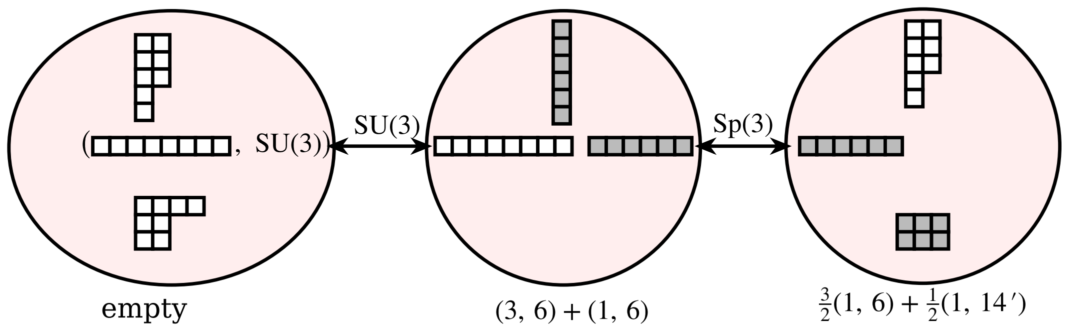
where

$$\begin{aligned} \mathbb{H}^{40} &= \tfrac{5}{2}(1,6)+\tfrac{1}{2}(1,14')+(3,6) \quad \text{of } SU(3)\times Sp(3) \\ \mathbb{H}^{24} &= \tfrac{5}{2}(1,6)+\tfrac{1}{2}(1,14')+(2,1) \quad \text{of } SU(2)\times Sp(3) \end{aligned}$$

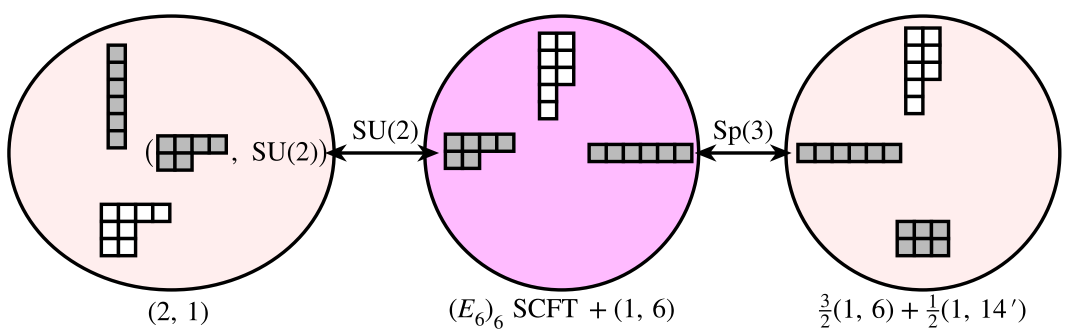
These isomorphisms are realized in the twisted D_4 theory, as the 5-punctured sphere



has, among its various other S-duality frames,



and



5.7.6 More isomorphisms among hyperKähler quotients

If we are are willing to venture a little further afield, we can find additional hyperKähler quotient identities satisfied by the $M(G,k)$. In §5.5, we recalled the $R_{2,2n-1}$ series of SCFTs. Let us denote the Higgs branch of $R_{2,2n-1}$ as $M_{2,2n-1}$. $M_{2,2n-1}$ has hyperKähler isometry group $Spin(4n+2)\times U(1)$ and dimension

$$\dim_{\mathbb{H}}(M_{2,2n-1}) = 2n^2 + n + 1$$

From the S-dualities in (5.4), certain hyperKähler quotients of $M_{2,2n-1}$ are isomorphic to hyperKähler quotients of quaternionic vector spaces

$$
\begin{gathered}
\left(M_{2,2n-1} \times \mathbb{H}^{2(n-1)}\right) ///Sp(n-1) \simeq \mathbb{H}^{(2n-1)(2n+2)} ///SU(2n-1) \\
\left(M_{2,2n-1} \times \mathbb{H}^{6n}\right) ///Sp(n) \simeq \mathbb{H}^{2n(2n+3)} ///SU(2n) \\
M_{2,2n-1} ///Spin(2n+1) \simeq \mathbb{H}^{4n^2} ///SU(2n)
\end{gathered}
$$

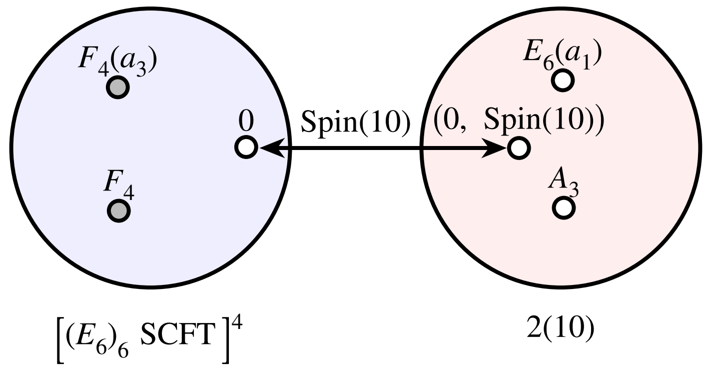
where, in the first two, the quaternionic vector space on the RHS transforms as $4(\square) + 2\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$ and, in the third, it transforms as $1\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + 1(\square\square)$.

This isn't quite enough information to reconstruct $M_{2,2n-1}$. But, with a certain poetic license, we can proceed as if we understand that hyperKähler space.

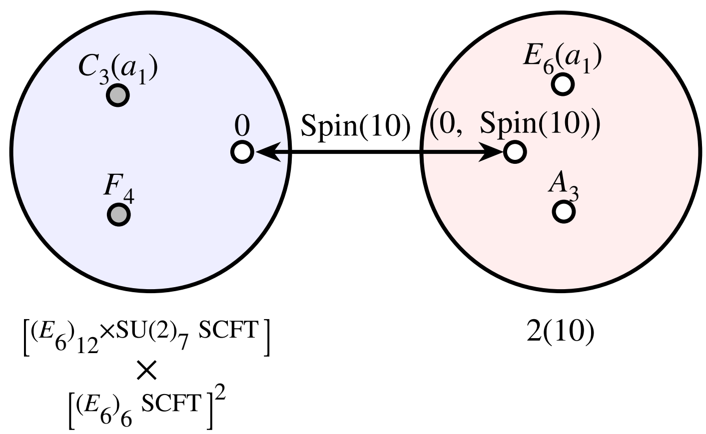
Using the realization of $M_{2,5}$ given in §5.5, we have the new isomorphisms

$$
\begin{aligned}
\left(M(E_6,1)^4 \times \mathbb{H}^{20}\right) ///Spin(10) &\simeq \left(M(E_6,1)^3 \times M_{2,5}\right) ///Spin(9) \\
\left(M(E_6,2) \times M(E_6,1)^2 \times \mathbb{H}^{20}\right) ///Spin(10) &\simeq (M(E_6,2) \times M(E_6,1) \\
&\quad \times M_{2,5}) ///Spin(9) \\
\left(M(E_6,3) \times M(E_6,1) \times \mathbb{H}^{20}\right) ///Spin(10) &\simeq \left(M(E_6,3) \times M_{2,5}\right) ///Spin(9)
\end{aligned}
\tag{5.19}
$$

from studying the 4-punctured spheres

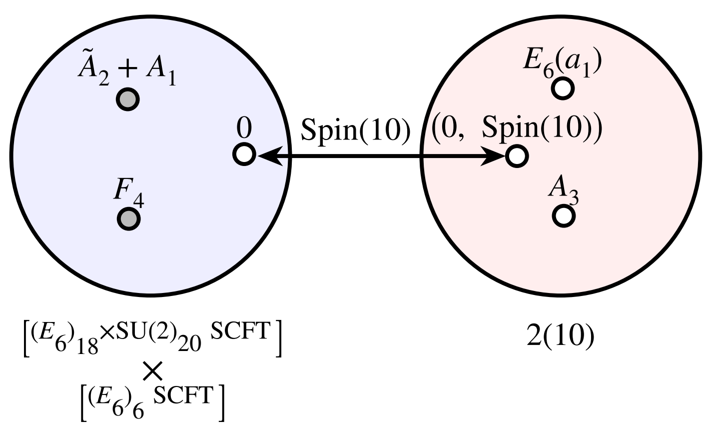


$F_4(a_3)$
F_4
0
Spin(10)
(0, Spin(10))
$E_6(a_1)$
A_3
$\left[(E_6)_6 \text{ SCFT}\right]^4$
2(10)



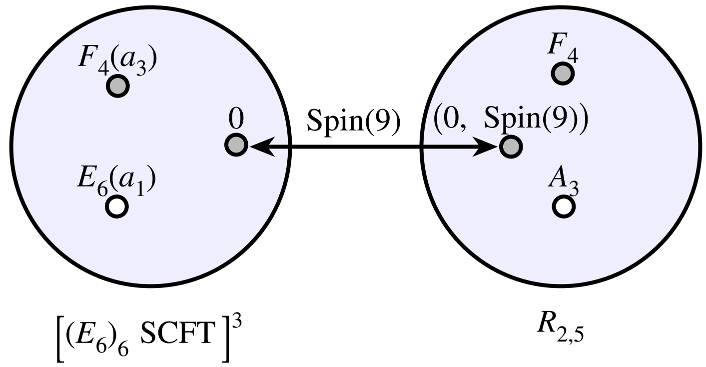
$C_3(a_1)$
F_4
0
Spin(10)
(0, Spin(10))
$E_6(a_1)$
A_3
$\left[(E_6)_{12}\times SU(2)_7 \text{ SCFT}\right]$
$\times$
$\left[(E_6)_6 \text{ SCFT}\right]^2$
2(10)

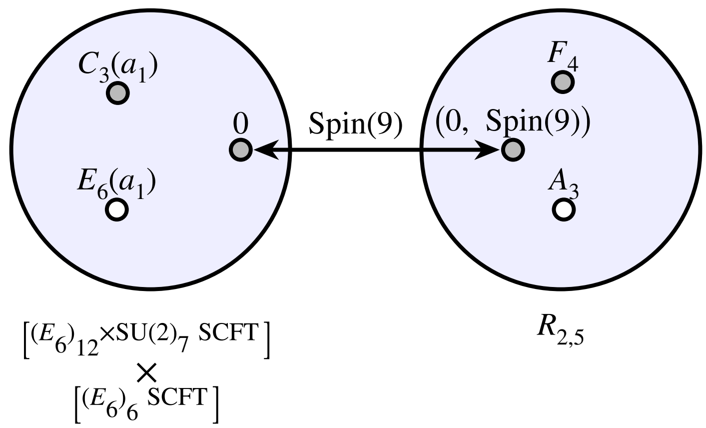
and



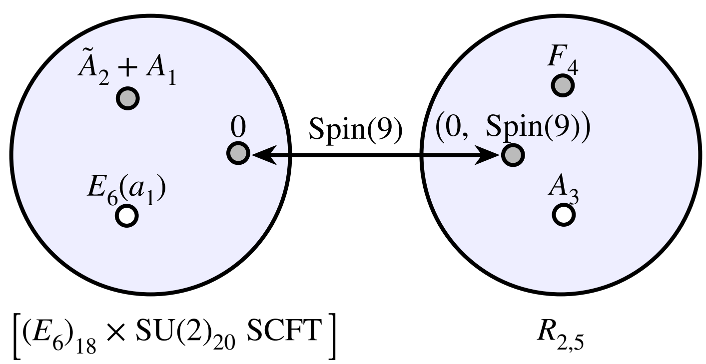
$\tilde{A}_2 + A_1$
F_4
0
Spin(10)
(0, Spin(10))
$E_6(a_1)$
A_3
$\left[(E_6)_{18}\times SU(2)_{20} \text{ SCFT}\right]$
$\times$
$\left[(E_6)_6 \text{ SCFT}\right]$
2(10)

which are, respectively, S-dual to





and



Note that:

- The three examples are related by allowing the twisted puncture in the upper left corner to vary over the special piece of $F_4(a_3)$. (As discussed above, this special piece consists of *five* nilpotent orbits. The other two involve theories whose Higgs branches are “new” hyperKähler spaces.)

- In each case, there’s a third S-duality frame, which we won’t write down, which is a gauge theory fixture.

5.8 Instanton moduli spaces as affine algebraic varieties

As mentioned above, $M(G,1)$ admits a uniform description as the minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$. For classical groups, G, the ADHM construction [67] gives a description of $M(G,k)$, for higher k, as a hyperKähler quotient. For exceptional G, a concrete description of the $M(G,k)$ for higher k is not known. However, in a series of papers [70, 71, 72], it was shown that the Hilbert series of $M(G,k)$ for $k>1$ and classical G can be written in terms of the root data of G alone. This provides a natural conjecture for the Hilbert series of $M(G,k)$ for exceptional G, which has been shown to pass many tests.

The Hilbert series contains all information about the ring of holomorphic functions on $M(G,k)$. From this information, in [72] the authors extracted the representations of the generators of $M(G,k)$ at each scaling dimension, and their lowest order chiral ring relations.

They conjectured that $M(G,k)$, as a complex variety, can be realized as an affine algebraic variety whose ring of functions has generators

$$\begin{aligned} M &\in (1;\mathrm{Adj}) \\ M_p &\in (p;\mathrm{Adj}) \\ P_p &\in (p+1;1) \end{aligned}$$

transforming in the indicated representations[10] of $SU(2)\times G$, for $p = 2,\ldots,k$.

These generators are subject to a set of polynomial relations. For $k=1$, the M_p and P_p are absent, and the only non-trivial relations are the celebrated Joseph relations

$$(M\otimes M)|_{\mathcal{I}_2} = 0 \tag{5.20}$$

where the reducible representation, $\mathcal{I}_2$, is defined through

$$Sym^2(\mathrm{Adj}) = V(2\alpha)\oplus\mathcal{I}_2$$

Here, $V(2\alpha)$ is the representation whose highest weight is twice the highest root. (5.20) gives a realization of $M(G,1)$ as an affine algebraic variety.

For $k=2,3$, the lowest-order relations are given in [72].

[10] We label irreducible representations of $SU(2)$ by their dimension. In what follows, it is convenient to realize the n-dimensional irrep as a rank-$(n-1)$ symmetric tensor, $\Phi_{(\alpha_1\alpha_2\ldots\alpha_{n-1})}$.

The isomorphisms discussed in §5.7 provide a strong test of this conjectured description of $M(G,k)$. Following [68], one can explicitly take the hyperKähler quotient on each side, and compare the gauge-invariant generators and relations.

As an illustrative example, we consider $(M(E_7,1)^3 \times \mathbb{H}^{26})///F_4 \simeq M(E_7,3)///Spin(8)$. We will not give a precise mapping of the generators, as the methods of [72] do not determine the constants appearing in the relations defining $M(E_7,3)$ but, up to a few unknown constants, we will be able to determine the form of the correspondence. The generators transform in representations of the $SU(2)_s \times SU(2)^3$ global symmetry. Moreover, there is an action of S_3 permuting the $SU(2)^3$. On the LHS, it acts by permuting the three $M(E_7,1)$s; on the RHS, it is the S_3 subgroup of E_7 which acts as triality on the $Spin(8) \subset E_7$. The generators of the ring of functions arrange themselves into representations of this S_3 action.

We first consider the proposed description of $M(E_7,3)$ above. Decomposing the 133 of E_7 under $SU(2)^3 \times Spin(8)$:

$$\begin{aligned} E_7 &\supset SU(2)^3 \times Spin(8) \\ 133 &= (3,1,1;1) + (1,3,1;1) + (1,1,3;1) + (1,1,1;28) \\ &\quad + (2,2,1;8_c) + (2,1,2;8_s) + (1,2,2;8_v) \end{aligned}$$

we have operators

Table 5.11: Generators of $M(E_7, 3)$

Order	Operator	Representation of $SU(2)_s \times SU(2)^3 \times Spin(8)$
2	$\Psi_{(\alpha\beta)}$	$(3; 1, 1, 1; 1)$
	J, K, L	$(1; 3, 1, 1; 1), (1; 1, 3, 1; 1), (1; 1, 1, 3; 1)$
	M, N, O	$(1; 2, 2, 1; 8_c), (1; 2, 1, 2; 8_s), (1; 1, 2, 2; 8_v)$
	P	$(1; 1, 1, 1; 28)$
3	$\Phi_{(\alpha\beta\gamma)}$	$(4; 1, 1, 1; 1)$
	$Q_\alpha, R_\alpha, S_\alpha$	$(2; 3, 1, 1; 1), (2; 1, 3, 1; 1), (2; 1, 1, 3; 1)$
	$T_\alpha, U_\alpha, V_\alpha$	$(2; 2, 2, 1; 8_c), (2; 2, 1, 2; 8_s), (2; 1, 2, 2; 8_v)$
	W_α	$(2; 1, 1, 1; 28)$
4	$X_{(\alpha\beta)}, Y_{(\alpha\beta)}, Z_{(\alpha\beta)}$	$(3; 3, 1, 1; 1), (3; 1, 3, 1; 1), (3; 1, 1, 3; 1)$
	$\widetilde{A}_{(\alpha\beta)}, \widetilde{B}_{(\alpha\beta)}, \widetilde{C}_{(\alpha\beta)}$	$(3; 2, 2, 1; 8_c), (3; 2, 1, 2; 8_s), (3; 1, 2, 2; 8_v)$
	$\widetilde{D}_{(\alpha\beta)}$	$(3; 1, 1, 1; 28)$

The lowest-order relation is at order 5, given by [72]

$$(JQ_\alpha + a_1 KR_\alpha + a_2 LS_\alpha + a_3 MT_\alpha + a_4 NU_\alpha + a_5 OV_\alpha + a_6 PW_\alpha)|_{(2;1,1,1;1)} = 0,$$

where the a_i are constants.

Let us now take the hyperKähler quotient by Spin(8). The F-term constraint is simply

$$P = 0.$$

So, the gauge-invariant operators are given by

Table 5.12: Generators of $M(E_7, 3)///Spin(8)$

Order	Operator	Representation of $SU(2)_s \times SU(2)^3$
2	$\Psi_{(\alpha\beta)}$	$(3; 1, 1, 1)$
	J, K, L	$(1; 3, 1, 1), (1; 1, 3, 1), (1; 1, 1, 3)$
3	$\Phi_{(\alpha\beta\gamma)}$	$(4; 1, 1, 1)$
	$Q_\alpha, R_\alpha, S_\alpha$	$(2; 3, 1, 1), (2; 1, 3, 1), (2; 1, 1, 3)$
4	$X_{(\alpha\beta)}, Y_{(\alpha\beta)}, Z_{(\alpha\beta)}$	$(3; 3, 1, 1), (3; 1, 3, 1), (3; 1, 1, 3)$
	M^2, N^2, O^2	$(1; 1, 1, 1) + (1; 3, 3, 1), (1; 1, 1, 1) + (1; 3, 1, 3), (1; 1, 1, 1) + (1; 1, 3, 3)$

subject to

$$(JQ_\alpha + a_1 KR_\alpha + a_2 LS_\alpha + a_3 MT_\alpha + a_4 NU_\alpha + a_5 OV_\alpha)|_{(2;1,1,1;1)} = 0. \quad (5.21)$$

Let's see how this structure is reproduced on the $M(E_7, 1)^3$ side. We first decompose the $(E_7)^3$ global symmetry under $SU(2)^3 \times (F_4)_{\text{diag}}$. Using the description of $M(E_7, 1)$ as the minimal nilpotent orbit in $\mathfrak{e}_7$, we have operators at order 2 in the 133, subject to the Joseph relations at order 4 in the $\mathcal{I}_2 = 1 + 1539$. These representations decompose under $SU(2) \times F_4$ as

$$E_7 \supset SU(2) \times F_4$$

$$133 = (3, 1) + (3, 26) + (1, 52)$$

$$1539 = (1, 1) + (1, 26) + (1, 324) + (3, 26) + (3, 273) + (3, 52) + (5, 1) + (5, 26)$$

Additionally, we have the generator of $\mathbb{H}^{26}$ at order 1. In total, we have the following generators of $M(E_7, 1)^3 \times \mathbb{H}^{26}$:

Table 5.13: Generators of $M(E_7, 1)^3 \times \mathbb{H}^{26}$

Order	Operator	Representation of $SU(2)_s \times SU(2)^3 \times (F_4)_{diag}$
1	v_α	$(2; 1, 1, 1; 26)$
2	A, B, C	$(1; 3, 1, 1; 1), (1; 1, 3, 1; 1), (1; 1, 1, 3; 1)$
	D, E, F	$(1; 3, 1, 1; 26), (1; 1, 3, 1; 26), (1; 1, 1, 3; 26)$
	G, H, I	$(1; 1, 1, 1; 52), (1; 1, 1, 1; 52), (1; 1, 1, 1; 52)$

subject to the Joseph relations at order 4.

To describe the hyperKähler quotient by $(F_4)_{\text{diag}}$, we impose the F-term constraints

$$G + H + I + (v_\alpha v_\beta)_{(1;1,1,1;52)} = 0$$

and form gauge-invariant generators. To order 4, these are given by:

Table 5.14: $(M(E_7,1)^3 \times \mathbb{H}^{26})///F_4$

Order	Operator	Representation of $SU(2)_s \times SU(2)^3$
2	$(v_\alpha v_\beta)_{(3;1,1,1;1)}$	$(3;1,1,1)$
	A, B, C	$(1;3,1,1;1), (1;1,3,1;1), (1;1,1,3;1)$
3	$(v_\alpha v_\beta v_\gamma)_{(4;1,1,1;1)}$	$(4;1,1,1)$
	$(Dv_\alpha)_{(2;3,1,1;1)}, (Ev_\alpha)_{(2;1,2,1;1)}, (Fv_\alpha)_{(2;1,1,3;1)}$	$(2;3,1,1), (2;1,3,1), (2;1,1,3)$
4	$((v_\alpha v_\beta)_{(3;1,1,1;26)}D)_{(3;3,1,1;1)}$, $((v_\alpha v_\beta)_{(3;1,1,1;26)}E)_{(3;1,3,1;1)}$, $((v_\alpha v_\beta)_{(3;1,1,1;26)}F)_{(3;1,1,3;1)}$	$(3;3,1,1), (3;1,3,1), (3;1,1,3)$
	$(D^2)_{(1;1+5,1,1;1)}, (E^2)_{(1;1,1+5,1;1)}, (F^2)_{(1;1,1,1+5;1)}$	$(1;1+5,1,1)$, $(1;1,1+5,1)$, $(1;1,1,1+5)$
	$(G^2)_{(1;1,1,1;1)}, (H^2)_{(1;1,1,1;1)}, (I^2)_{(1;1,1,1;1)}$	$(1;1,1,1), (1;1,1,1), (1;1,1,1)$
	$(DE)_{(1;3,3,1;1)}, (DF)_{(1;3,1,3;1)}, (EF)_{(1;1,3,3;1)}$	$(1;3,3,1), (1;3,1,3), (1;1,3,3)$
	$(GH)_{(1;1,1,1;1)}, (GI)_{(1;1,1,1;1)}, (HI)_{(1;1,1,1;1)}$	$(1;1,1,1), (1;1,1,1), (1;1,1,1)$

The gauge-invariant relations at order 4 are given by

$$(A^2 + c_1 D^2 + c_2 G^2)|_{(1;1,1,1)} = 0 \tag{5.22}$$

$$(B^2 + c_1 E^2 + c_2 H^2)|_{(1;1,1,1)} = 0 \tag{5.23}$$

$$(C^2 + c_1 F^2 + c_2 I^2)|_{(1;1,1,1)} = 0 \tag{5.24}$$

$$(A^2 + c_3 D^2)|_{(1;5,1,1)} = 0 \tag{5.25}$$

$$(B^2 + c_3 E^2)|_{(1;1,5,1)} = 0 \tag{5.26}$$

$$(C^2 + c_3 F^2)|_{(1;1,1,5)} = 0 \tag{5.27}$$

where the c_i are constants which can be fixed by evaluating a few points on the nilpotent orbit [68].

We see that the correspondence between the generators is given by[11]

Table 5.15: Correspondence between generators

$(M(E_7,1)^3 \times \mathbb{H}^{26})///F_4$	$M(E_7,3)///Spin(8)$
$(v_\alpha v_\beta)_{(3;1,1,1;1)}$	$\Psi_{(\alpha\beta)}$
A, B, C	J, K, L
$(v_\alpha v_\beta v_\gamma)_{(4;1,1,1;1)}$	$\Phi_{(\alpha\beta\gamma)}$
$((v_\alpha D)_{(2;3,1,1;1)}, (v_\alpha E)_{(2;1,3,1;1)}, (v_\alpha F)_{(2;1,1,3;1)}$	$Q_\alpha, R_\alpha, S_\alpha$

[11] We have multiple generators with the same quantum numbers, so, without knowing the constants a_i, the correspondence between these generators is only up to a permutation (or linear combination).

Table 5.15: Correspondence between generators

$(M(E_7,1)^3 \times \mathbb{H}^{26})///F_4$	$M(E_7,3)///Spin(8)$
$((v_\alpha v_\beta)_{(3;1,1,1;26)}D)_{(3;3,1,1;1)}$, $((v_\alpha v_\beta)_{(3;1,1,1;26)}E)_{(3;1,3,1;1)}$, $((v_\alpha v_\beta)_{(3;1,1,1;26)}F)_{(3;1,1,3;1)}$	$X_{(\alpha\beta)}, Y_{(\alpha\beta)}, Z_{(\alpha\beta)}$
$(GH)_{(1;1,1,1;1)}$, $(DE)_{(1;3,3,1;1)}$	$M^2_{(1;1,1,1;1)}$, $M^2_{(1;3,3,1;1)}$
$(GI)_{(1;1,1,1;1)}$, $(DF)_{(1;3,1,3;1)}$	$N^2_{(1;1,1,1;1)}$, $N^2_{(1;3,1,3;1)}$
$(HI)_{(1;1,1,1;1)}$, $(EF)_{(1;1,3,3;1)}$	$O^2_{(1;1,1,1;1)}$, $O^2_{(1;1,3,3;1)}$

The “extra” generators at order 4, $(D^2)_{(1;1+5,1,1;1)}$, $(E^2)_{(1;1,1+5,1;1)}$, $(F^2)_{(1;1,1,1+5;1)}$ and $(G^2)_{(1;1,1,1;1)}$, $(H^2)_{(1;1,1,1;1)}$, $(I^2)_{(1;1,1,1;1)}$, are removed from the chiral ring by the Joseph relations (5.22)-(5.27).

We find the order 5 relation (5.21) on the $M(E_7,1)^3$ side by adding the order 4 Joseph relations

$$(AD)_{(1;3,1,1;26)} + c_4(DG)_{(1;3,1,1;26)} = 0 \tag{5.28}$$

$$(BE)_{(1;1,3,1;26)} + c_4(EH)_{(1;1,3,1;26)} = 0 \tag{5.29}$$

$$(CF)_{(1;1,1,3;26)} + c_4(FI)_{(1;1,1,3;26)} = 0 \tag{5.30}$$

and contracting with v_α:

$$(vAD + vBE + vCF + c_4(vDG + vEH + vFI))_{(2;1,1,1;1)} = 0.$$

Following [72], one can extract the higher-order relations for $M(E_7, 3)$ and compare them with those on the $M(E_7, 1)^3$ side obtained from the remaining Joseph relations. It would be interesting to carry out this analysis for the other examples in §5.7 as well.

Chapter 6

A Family of Interacting SCFTs from the Twisted A_{2N} Series

In this chapter, we consider the strong-coupling limit of $SU(2N+1)$ gauge theory with hypermultiplets in the $1\left(\ydiagram{1,1}\right) + 1\left(\ydiagram{2}\right)$ [1]. We find the following S-duality for this theory

$$SU(2N+1) + 1\left(\ydiagram{1,1}\right) + 1\left(\ydiagram{2}\right) \simeq Sp(N) + R_{2,2N} \tag{6.1}$$

where $R_{2,2N}$, $N \geq 1$, belongs to a family of interacting SCFTs with the following graded Coulomb branch dimensions, trace anomaly coefficients, and global symmetry:

Table 6.1: $R_{2,2N}$ family of interacting SCFTs

	$\{d_2, d_3, d_4, d_5, \ldots, d_{2N}, d_{2N+1}\}$	(a, c)	G_{global}
$R_{2,2N}$	$\{0, 1, 0, 1, \ldots, 0, 1\}$	$\left(\frac{1+19N+14N^2}{24}, \frac{1+10N+8N^2}{12}\right)$	$Sp(2N)_{2N+2} \times U(1)$

[1]This chapter is based on [73].

The proposed duality (6.1) is analogous to the duality

$$SU(2N) + 1\left(\,\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}\,\right) + 1(\square\square) \simeq Spin(2N+1) + R_{2,2N-1}$$

discussed[2] in §3.5.4 of [7]. However the $R_{2,2N}$ series of SCFTs is *new*.

The strong-coupling limit of $SU(3) + 1(3) + 1(6)$ was considered by Argyres and Wittig in [15]. They conjectured that this theory is dual to an $SU(2)$ gauge theory with $n \leq 3$ fundamental hypermultiplets, coupled to a new rank 1 interacting SCFT. This S-duality alone does not fix n, and the properties of this theory have remained only partially-known. In [4], we gave a $6D$ realization of this S-duality, which is given by taking $N = 1$ in the figure below. From this construction, the properties of the holomorphic moment map operators for the flavor symmetry of the two twisted punctures imply that $n = 0$ [4, 61]. In the following, we will provide independent evidence that $n = 0$ using the superconformal index. In addition, we will use the index to determine the enhanced global symmetry of the SCFT, and generalize this duality to arbitrary N.

The main tool in our analysis is the Hall-Littlewood limit of the superconformal index [24, 25, 74]. In this limit, the superconformal index is

[2] The $R_{2,2N-1}$ series of SCFTs, which have global symmetry $Spin(4N+2)_{4N?2} \times U(1)$ also play a role in the dualities [3]

$$SU(2N-1) + 4(\square) + 2\left(\,\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}\,\right) \simeq Sp(N-1) + 1(\square) + R_{2,2N-1}$$
$$SU(2N) + 4(\square) + 2\left(\,\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}\,\right) \simeq Sp(N) + 3(\square) + R_{2,2N-1}$$

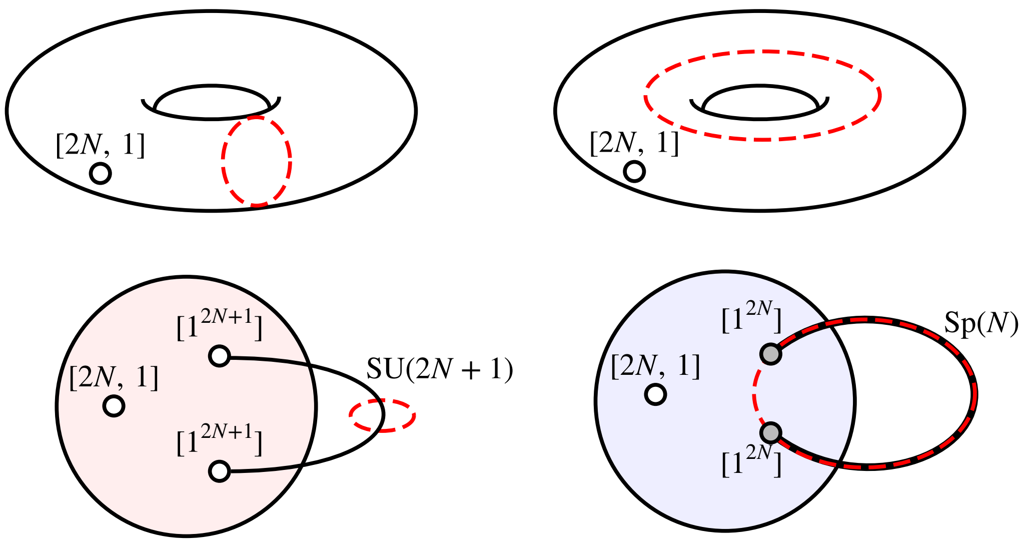
equivalent to the Coulomb branch Hilbert series of the $3D$ mirror of the $(2,0)$ theory on $C \times S^1$ [14, 27]. The $3D$ interpretation allows us to easily obtain the formula for the index of a fixture with twisted A_{2N} punctures, following [65, 66]. The result is (6.6).

To construct the $R_{2,2N}$ theories, we consider compactifications of the A_{2N} $(2,0)$ theory in the presence of $\mathbb{Z}_2$ outer-automorphism twists. As discussed in [75], these twists are particularly subtle, and we do not yet understand them well enough to attempt a systematic classification of the $4D$ theories which arise in this way. Nevertheless, our current level of understanding is sufficient to use them for our purposes here.

6.1 S-duality of $SU(2N+1) + \wedge^2(\square) + \mathbf{Sym}^2(\square)$

The $4D$ $\mathcal{N}=2$ $SU(2N+1)$ gauge theory with hypermultiplets in the $1\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) + 1(\square\square)$ can be constructed as follows [4]. The A_{2N} $(2,0)$ theory compactified on a fixture with two full punctures and one minimal puncture gives rise to a free bifundamental hypermultiplet of $SU(2N+1)$. One can gauge the diagonal $SU(2N+1)$ flavor symmetry of the two full punctures by connecting them by a cylinder with a $\mathbb{Z}_2$ twist line around it, to obtain a one-punctured torus, with a twist line around the a-cycle. This is shown on the left in the figure below. The $\mathbb{Z}_2$-twist line acts as complex conjugation on one of the $SU(2N+1)$ factors, giving rise to hypermultiplets in the tensor product representation $\square \otimes \square = \begin{smallmatrix}\square\\ \square\end{smallmatrix} + \square\square$.

The S-dual theory is obtained by exchanging the a- and b-cycles of the torus. The resulting theory can be seen to arise by compactifying the A_{2N} theory on a fixture with a minimal untwisted puncture and two full twisted punctures, and connecting the two twisted punctures by a cylinder with a twist line running along it. This is shown on the right in the figure below. We expect this to give rise to an $Sp(N)$ gauge theory coupled to an interacting SCFT, possibly with n additional hypermultiplets.



In the following, we will use the superconformal index to show that $n = 0$, and that the manifest $Sp(N)_{2N+2} \times Sp(N)_{2N+2} \times U(1)$ global symmetry of the interacting SCFT is enhanced to $Sp(2N)_{2N+2} \times U(1)$. The graded Coulomb branch dimensions and trace anomaly coefficients of the SCFT then follow from S-duality.

6.2 The Argyres-Wittig SCFT

6.2.1 Global symmetry enhancement

We begin by considering the case of $N = 1$. The fixture on the right in the figure above then has a manifest $SU(2)_4 \times SU(2)_4 \times U(1)$ subgroup of its global symmetry group, which could be enhanced. We will now use the superconformal index to show that this fixture contains an interacting SCFT with no additional free hypermultiplets, and that the global symmetry of the SCFT is enhanced to $Sp(2)_4 \times U(1)$.

We assume that the superconformal index for this fixture has the same general structure one always finds for a 3-punctured sphere, namely[3]

$$\mathcal{I}(\mathbf{a}_i;\tau) = \sum_{\lambda=0}^{\infty} \frac{\prod_{i=1}^{2} \mathcal{K}(\mathbf{a}_i) P^{\lambda}_{SU(2)}(\mathbf{a}_i;\tau) \mathcal{K}(\mathbf{a}_3) P^{(2\lambda,\lambda)}_{SU(3)}(\mathbf{a}_3;\tau)}{\mathcal{K}_\rho P^{(2\lambda,\lambda)}_{SU(3)}(\mathbf{a}_\rho;\tau)} \tag{6.2}$$

Assigning fugacities to each puncture, this becomes

$$\begin{aligned}\mathcal{I}(\mathbf{a}_i;\tau) &= \frac{(1-\tau^2)(1-\tau^4)(1-\tau^6)}{(1-\tau^2)^3(1-\tau^2 a_1^{\pm 2})(1-\tau^2 a_2^{\pm 2})(1-\tau^3 a_3^{\pm 3})(1-\tau^4)} \times \\ &\left(1+\tau^2+\sum_{\lambda=1}^{\infty} \frac{P^{\lambda}_{SU(2)}(a_1,a_1^{-1};\tau) P^{\lambda}_{SU(2)}(a_2,a_2^{-1};\tau) P^{(2\lambda,\lambda)}_{SU(3)}(a_3\tau, a_3\tau^{-1}, a_3^{-2};\tau)}{P^{(2\lambda,\lambda)}_{SU(3)}(\tau^2,1,\tau^{-2};\tau)}\right) \\ &= 1 + (1+\chi^{\mathbf{10}}_{Sp(2)}(a_1,a_2))\tau^2 + \dots\end{aligned}$$

where

[3]The various parts of this formula have been explained many times in the references above. See, e.g., [64].

$$\chi^{\mathbf{10}}_{Sp(2)}(a_1,a_2) = \chi^{\mathbf{3}}_{SU(2)}(a_1) + \chi^{\mathbf{3}}_{SU(2)}(a_2) + \chi^{\mathbf{2}}_{SU(2)}(a_1)\chi^{\mathbf{2}}_{SU(2)}(a_2). \tag{6.3}$$

The coefficient of τ is zero, indicating that the fixture contains no additional free hypermultiplets. The coefficient of τ^2 is therefore the character of the adjoint representation of the global symmetry group of the SCFT, which is enhanced to $Sp(2) \times U(1)$. Since the embedding (6.3) has index 1, the level of $Sp(2)$ is $k = 4$.

Setting $a_1 = a_2 = a_3 = 1$, we can sum (6.2) to obtain

$$\mathcal{I} = \frac{1 + 2\tau + 8\tau^2 + 20\tau^3 + 41\tau^4 + 62\tau^5 + 87\tau^6 + 96\tau^7 + \cdots (\text{palindrome}) \cdots + \tau^{14}}{(1-\tau)^8(1+\tau)^6(1+\tau+\tau^2)^4}.$$

This expression takes the expected form (see, e.g., [71]), and the order of the pole at $\tau = 1$ gives the complex dimension of the $4D$ Higgs branch, which agrees with the answer obtained from S-duality [3, 7]

$$\dim_{\mathbb{C}}\mathcal{H} = 48(c-a) = 48\left(\frac{19}{12} - \frac{17}{12}\right) = 8.$$

6.2.2 Argyres-Wittig duality

As a further check on the validity of our computation, we compare the index on both sides of the S-duality

$$SU(3) + 1(3) + 1(6) \simeq SU(2) + R_{2,2}$$

The $SU(3)$ theory is a Lagrangian theory, so its index is computed by the matrix integral [4]

$$
\begin{aligned}
\mathcal{I} &= \oint [d\mathbf{z}]_{SU(3)} PE[-\tau^2 \chi^{\mathbf{8}}_{SU(3)}(z_1,z_2)] PE[\tau \chi^{\mathbf{3}}_{SU(3)}(z_1,z_2)] PE[\tau \chi^{\overline{\mathbf{3}}}_{SU(3)}(z_1,z_2)] \\
&\quad \times PE[\tau \chi^{\mathbf{6}}_{SU(3)}(z_1,z_2)] PE[\tau \chi^{\overline{\mathbf{6}}}_{SU(3)}(z_1,z_2)] \\
&= \frac{1}{3!} \oint \frac{dz_1 dz_2}{(2\pi i)^2 z_1 z_2} (1-(z_1/z_2)^{\pm})(1-z_1^{\pm 2} z_2^{\pm})(1-z_1^{\pm} z_2^{\pm 2}) \\
&\quad \times \frac{(1-\tau^2)^2 (1-\tau^2 (z_1/z_2)^{\pm})(1-\tau^2 z_1^{\pm 2} z_2^{\pm})(1-\tau^2 z_1^{\pm} z_2^{\pm 2})}{(1-\tau z_1^{\pm})^2 (1-\tau z_2^{\pm})^2 (1-\tau (z_1 z_2)^{\pm})^2 (1-\tau z_1^{\pm 2})(1-\tau z_2^{\pm 2})(1-\tau (z_1 z_2)^{\pm 2})}.
\end{aligned}
$$

Summing the residues at the poles inside the unit circle (taking $|z_1| = |z_2| = 1, \tau < 1$), we arrive at

$$
\mathcal{I} = \frac{1+\tau^2+2\tau^3+\tau^4}{(1-\tau)^3(1+\tau)(1+\tau+\tau^2)^2}. \tag{6.4}
$$

On the $SU(2)$ side of the duality, the index is given by

$$
\mathcal{I} = \frac{1}{2!} \oint \frac{da}{2\pi i a} (1-a^{\pm 2}) \mathcal{I}^V_{SU(2)}(a) \mathcal{I}_{R_{2,2}}(a, a^{-1}, 1),
$$

where $\mathcal{I}^V_{SU(2)}(a)$ is the index of a free $SU(2)$ vector multiplet. The integrand has poles inside the unit circle at $a = 0, \pm\tau, \pm\tau^{3/2}$. Summing the residues, we reproduce (6.4). This gives strong evidence that (6.2) indeed gives the index of the Argyres-Wittig SCFT.

[4]For simplicity, we set the fugacities of the $U(1)^2$ flavor symmetry to 1.

6.2.3 Chiral ring

We can study the Hall-Littlewood chiral ring [76] of the Argyres-Wittig SCFT by taking the plethystic log of (6.2):

$$
\begin{aligned}
PL[\mathcal{I}] &= \tau^2(\chi^{\mathbf{10}}_{Sp(2)}(a_1,a_2)+1)+\tau^3(\chi^{\mathbf{5}}_{Sp(2)}(a_1,a_2)a_3^3+\chi^{\mathbf{5}}_{Sp(2)}(a_1,a_2)a_3^{-3}) \\
&- \tau^4(\chi^{\mathbf{5}}_{Sp(2)}(a_1,a_2)+1)-\tau^5((\chi^{\mathbf{10}}_{Sp(2)}(a_1,a_2)+\chi^{\mathbf{5}}_{Sp(2)}(a_1,a_2))a_3^3 \\
&+ (\chi^{\mathbf{10}}_{Sp(2)}(a_1,a_2)+\chi^{\mathbf{5}}_{Sp(2)}(a_1,a_2))a_3^{-3})-\dots
\end{aligned}
$$

where $\chi^{\mathbf{5}}(a_1,a_2)=1+\chi^{\mathbf{2}}_{SU(2)}(a_1)\chi^{\mathbf{2}}_{SU(2)}(a_2)$.

From this expression, we see that the chiral ring has generators at order 2 in the $\mathbf{10}_0+\mathbf{1}_0$ of $Sp(2)\times U(1)$ (which are the holomorphic moment map operators for the global symmetry), as well as order 3 generators in the $\mathbf{5}_3+\mathbf{5}_{-3}$. These generators are subject to relations at order 4 in the $\mathbf{5}_0+\mathbf{1}_0$, as well as higher order relations.

We can understand the two relations at order 4 as follows. In [76, 77], it was shown that any $4D$ $\mathcal{N}=2$ SCFT contains families of protected operators whose correlation functions possess the structure of a $2D$ chiral algebra. The existence of the chiral algebra structure leads to additional unitarity bounds on central charges in the $4D$ theory, and saturation of these bounds was shown to follow from Higgs branch chiral ring relations. The operators counted by the Hall-Littlewood superconformal index fall into this class of protected operators, and from table 3 in [77], we see that the level of the $Sp(2)$ factor in the global symmetry group saturates the unitarity bound, which follows from an order 4 relation in the **5**. It was also shown in [77] that an order 4 relation in the **1** implies that the $2D$ chiral algebra has a stress tensor given by the Sugawara construction, with central charge:

$$c_{2D} = \frac{k_{2D}\dim G_F}{k_{2D} + h^\vee} \tag{6.5}$$

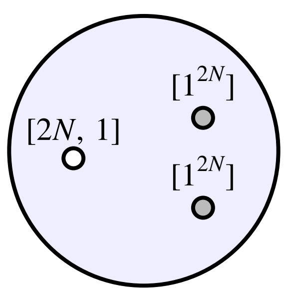
where the $2D$ central charges are related to those in $4D$ by $c_{2D} = -12c_{4D}$, $k_{2D} = -\frac{k_{4D}}{2}$.

Indeed, we find that

$$c_{4D} = -\frac{1}{12}\left(\frac{(-2)(10)}{-2+1} + 1\right) = \frac{19}{12}.$$

6.3 Higher N

We now consider the compactification of the $6D$ $(2,0)$ theory of type A_{2N} on



and use the superconformal index to argue that these fixtures describe the $R_{2,2N}$ family of interacting SCFTs, with no additional hypermultiplets.

The Hall-Littlewood index is given by

$$\mathcal{I}(\mathbf{x}_i;\tau) = \sum_{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0} \frac{\prod_{i=1}^{2} \mathcal{K}(\mathbf{x}_i) P_{Sp(N)}^{(\lambda_1,\ldots,\lambda_N)}(\mathbf{x}_i;\tau) \mathcal{K}(\mathbf{x}_3) P_{SU(2N+1)}^{(\lambda'_1,\ldots,\lambda'_N)}(\mathbf{x}_3;\tau)}{\mathcal{K}_\rho P_{SU(2N+1)}^{(\lambda'_1,\ldots,\lambda'_N)}(\mathbf{x}_\rho;\tau)} \tag{6.6}$$

where the sum runs over integers λ_i labeling $Sp(N)$ irreps, and the corresponding $SU(2N+1)$ representations are given by $(\lambda'_1,\ldots,\lambda'_N) = (2\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_N, \lambda_1, \lambda_1 - \lambda_N, \ldots, \lambda_1 - \lambda_2)$.

We can characterize these fixtures by the leading power of τ in the expansion of (6.6), as follows [27]: The term coming from taking $(\lambda_1,\ldots,\lambda_N) = (0,\ldots,0)$ in the sum is

$$1 + \tau^2(\chi_{Sp(N)}^{\mathbf{N(2N+1)}}(\mathbf{x}_1) + \chi_{Sp(N)}^{\mathbf{N(2N+1)}}(\mathbf{x}_2) + 1) + \mathcal{O}(\tau^4),$$

encoding the manifest $Sp(N)_{2N+2} \times Sp(N)_{2N+2} \times U(1)$ global symmetry. This global symmetry is enhanced if there are additional terms at order τ^2 coming from the sum over non-trivial representations. If there are also terms at order τ, then the fixture contains free hypermultiplets along with the interacting SCFT.

Following [27], we give the leading behavior of (6.6) in terms of a vector v, with components

$$v_i = (2N + 2 - 2i) + \sum_{\ell=1}^{3} d_i^{(\ell)} \tag{6.7}$$

where the first term is the leading power of τ from the denominator in (6.6), and the second term is the leading power of τ from the punctures. For the full twisted punctures, $d_i^{(\ell)} = 0$ for all i, while for the minimal untwisted puncture we have

$$d_i^{(3)} = \begin{cases} 2i - 2N - 1, & 1 \leq i \leq N, \\ 0, & i = N+1, \\ 2i - 2N - 3, & N+2 \leq i \leq 2N. \end{cases}$$

Equation (6.7) then gives

$$v_i = \begin{cases} 1, & 1 \leq i \leq N, \\ 0, & i = N+1, \\ -1, & N+2 \leq i \leq 2N. \end{cases}$$

The leading power in τ of (6.6) is therefore given by

$$\begin{aligned} v \cdot \lambda' &= (N+1)\lambda_1 + \lambda_2 + \cdots + \lambda_N + 0 - (N-1)\lambda_1 + \lambda_2 + \cdots + \lambda_N \\ &= 2(\lambda_1 + \lambda_2 + \cdots + \lambda_N) \end{aligned} \tag{6.8}$$

This is minimized by taking $(\lambda_1, \lambda_2, \ldots, \lambda_N) = (1, 0, \ldots, 0)$, which gives $v \cdot \lambda' = 2$. Thus, the fixture contains no free hypermultiplets. It is easy to check that the global symmetry is enhanced to $Sp(2N)_{2N+2} \times U(1)$ by the index 1 embedding

$$\begin{aligned} &Sp(2N) \supset Sp(N) \times Sp(N) \\ &\quad \mathbf{Adj} = (\mathbf{Adj}, \mathbf{1}) + (\mathbf{1}, \mathbf{Adj}) + (\mathbf{2N}, \mathbf{2N}) \end{aligned}$$

and the level of $Sp(2N)$ therefore saturates the unitarity bound in [77].

With $n = 0$, the S-duality is then given by (6.1), which implies that this SCFT has an effective number of vector and hypermultiplets given by $(n_h, n_v) = ((2N+1)^2, (2N+1)^2 - 1 - N(2N+1))$, which encode the trace anomaly coefficients $(a, c) = (\frac{n_h+5n_v}{24}, \frac{n_h+2n_v}{12}) = (\frac{1+19N+14N^2}{24}, \frac{1+10N+8N^2}{12})$. We see that the $4D$ central charge is given by the Sugawara construction:

$$\begin{aligned} c_{4D} &= -\frac{1}{12}\left(\frac{-(N+1)2N(4N+1)}{-(N+1)+2N+1}+1\right) \\ &= \frac{1+10N+8N^2}{12}. \end{aligned} \tag{6.9}$$

It is easy to check that the saturation of these unitarity bounds follows from chiral ring relations at order τ^4 in the $\mathbf{1}_0 + (\mathbf{\frac{1}{2}(4N+1)(4N-2)})_\mathbf{0}$ of $Sp(2N) \times U(1)$ by taking the plethystic log of (6.6).

A similar analysis can be carried-through for the $R_{2,2N-1}$ theories. As already pointed out in [77], the level of the $Spin(4N+2)_{4N-2}$ saturates the unitarity bound and the stress tensor of the 2D chiral algebra takes the Sugawara form, (6.5).

Appendices

Appendix A

Tables of $\mathbb{Z}_2$-Twisted D_N Punctures

A.1 D_5 Twisted Sector

Table A.1: D_5 Twisted Sector

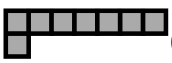
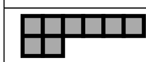
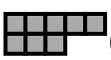
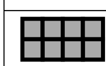
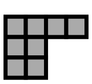
Nahm C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
	$[9]$	$\{1,3,5,7;\frac{9}{2}\}$	$-$	$Sp(4)_{10}$	$(240, \frac{449}{2})$
(ns)	$([7,1^2], \mathbb{Z}_2)$	$\{1,3,5,7;\frac{7}{2}\}$	$-$	$Sp(3)_9$	$(227, \frac{431}{2})$
	$[7,1^2]$	$\{1,3,5,7;\frac{7}{2}\}$	$c_7^{(8)} = (a_{7/2}^{(4)})^2$	$Sp(2)_8 \times U(1)$	$(216, \frac{415}{2})$
(ns)	$([5,3,1], \mathbb{Z}_2)$	$\{1,3,5,6;\frac{7}{2}\}$	$-$	$SU(2)_{32} \times SU(2)_7$	$(207, \frac{401}{2})$
	$[5,3,1]$	$\{1,3,5,6;\frac{7}{2}\}$	$c_5^{(6)} = (a_{5/2}^{(3)})^2$	$SU(2)_{16}^2$	$(200, \frac{389}{2})$
	$[5,2^2]$	$\{1,3,5,6;\frac{7}{2}\}$	$c_5^{(6)} = (a_{5/2}^{(3)})^2$ $c_6^{(8)} = 2a_{5/2}^{(3)}\tilde{c}_{7/2}^{(5)}$	$SU(2)_{10} \times SU(2)_6$	$(184, \frac{359}{2})$

Table A.1: D_5 Twisted Sector

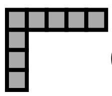
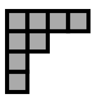
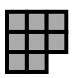
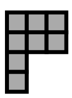
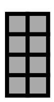
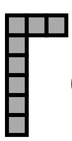
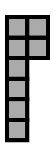
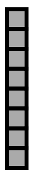
Nahm C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
 (ns)	$([5,1^4], \mathbb{Z}_2)$	$\{1,3,5,5;\frac{5}{2}\}$	$-$	$Sp(2)_7$	$(182, \frac{353}{2})$
	$[5,1^4]$	$\{1,3,5,5;\frac{5}{2}\}$	$c_5^{(6)} = (a_{5/2}^{(3)})^2$	$SU(2)_6$	$(174, \frac{341}{2})$
	$[3^3]$	$\{1,3,4,5;\frac{7}{2}\}$	$-$	$SU(2)_{10}$	$(178, \frac{349}{2})$
	$[3^2,1^3]$	$\{1,3,4,5;\frac{5}{2}\}$	$-$	$U(1)$	$(168, \frac{331}{2})$
	$[3,2^2,1^2]$	$\{1,3,4,5;\frac{5}{2}\}$	$c_3^{(4)} = (a_{3/2}^{(2)})^2$ $c_4^{(6)} = 2a_{3/2}^{(2)}a_{5/2}^{(4)}$ $c_5^{(8)} = (a_{5/2}^{(4)})^2$	$U(1)$	$(144, \frac{285}{2})$
 (ns)	$([3,1^6], \mathbb{Z}_2)$	$\{1,3,3,3;\frac{3}{2}\}$	$-$	$SU(2)_5$	$(117, \frac{231}{2})$
	$[3,1^6]$	$\{1,3,3,3;\frac{3}{2}\}$	$c_3^{(4)} = (a_{3/2}^{(2)})^2$	none	$(112, \frac{223}{2})$

Table A.1: D_5 Twisted Sector

Nahm C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
	$[1^9]$	$\{1, 1, 1, 1; \frac{1}{2}\}$	–	none	$(40, \frac{81}{2})$

A.2 D_6 Twisted Sector

Table A.2: D_6 Twisted Sector

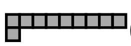
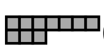
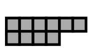
Nahm C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
	$[11]$	$\{1, 3, 5, 7, 9; \frac{11}{2}\}$	–	$Sp(5)_{12}$	$(440, \frac{831}{2})$
(ns)	$([9, 1^2], \mathbb{Z}_2)$	$\{1, 3, 5, 7, 9; \frac{9}{2}\}$	–	$Sp(4)_{11}$	$(424, \frac{809}{2})$
	$[9, 1^2]$	$\{1, 3, 5, 7, 9; \frac{9}{2}\}$	$c_9^{(10)} = (a_{9/2}^{(5)})^2$	$Sp(3)_{10} \times U(1)$	$(410, \frac{789}{2})$
(ns)	$([7, 3, 1], \mathbb{Z}_2)$	$\{1, 3, 5, 7, 8; \frac{9}{2}\}$	–	$Sp(2)_9 \times SU(2)_{40}$	$(398, \frac{771}{2})$
	$[7, 3, 1]$	$\{1, 3, 5, 7, 8; \frac{9}{2}\}$	$c_7^{(8)} = (a_{7/2}^{(4)})^2$	$SU(2)_{20}^2 \times SU(2)_8$	$(388, \frac{755}{2})$

Table A.2: D_6 Twisted Sector

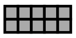
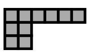
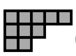
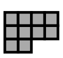
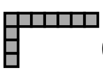
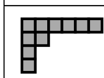
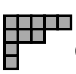
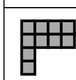
Nahm C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
	$[5^2, 1]$	$\{1, 3, 5, 6, 8; \frac{9}{2}\}$	$-$	$Sp(2)_{12}$	$(380, \frac{741}{2})$
	$[7, 2^2]$	$\{1, 3, 5, 7, 8; \frac{9}{2}\}$	$c_7^{(8)} = (a_{7/2}^{(4)})^2$ $c_8^{(10)} = 2a_{7/2}^{(4)}\tilde{c}_{9/2}^{(6)}$	$Sp(2)_8 \times SU(2)_{12}$	$(368, \frac{717}{2})$
(ns)	$([5, 3^2], \mathbb{Z}_2)$	$\{1, 3, 5, 6, 7; \frac{9}{2}\}$	$-$	$SU(2)_{12} \times SU(2)_7$	$(359, \frac{703}{2})$
	$[5, 3^2]$	$\{1, 3, 5, 6, 7; \frac{9}{2}\}$	$c_5^{(6)} = (a_{5/2}^{(3)})^2$	$SU(2)_{12} \times U(1)$	$(352, \frac{691}{2})$
(ns)	$([7, 1^4], \mathbb{Z}_2)$	$\{1, 3, 5, 7, 7; \frac{7}{2}\}$	$-$	$Sp(3)_9$	$(367, \frac{711}{2})$
	$[7, 1^4]$	$\{1, 3, 5, 7, 7; \frac{7}{2}\}$	$c_7^{(8)} = (a_{7/2}^{(4)})^2$	$Sp(2)_8$	$(356, \frac{695}{2})$
(ns)	$([5, 3, 1^3], \mathbb{Z}_2)$	$\{1, 3, 5, 6, 7; \frac{7}{2}\}$	$-$	$SU(2)_7 \times U(1)$	$(347, \frac{681}{2})$
	$[5, 3, 1^3]$	$\{1, 3, 5, 6, 7; \frac{7}{2}\}$	$c_5^{(6)} = (a_{5/2}^{(3)})^2$	$SU(2)_{32}$	$(340, \frac{669}{2})$

Table A.2: D_6 Twisted Sector

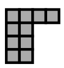
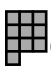
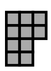
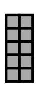
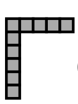
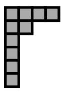
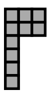
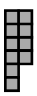
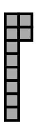
Nahm C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
	$[5, 2^2, 1^2]$	$\{1, 3, 5, 6, 7; \frac{7}{2}\}$	$c_5^{(6)} = (a_{5/2}^{(3)})^2$ $c_6^{(8)} = 2a_{5/2}^{(3)} a_{7/2}^{(5)}$ $c_7^{(10)} = (a_{7/2}^{(5)})^2$	$SU(2)_6 \times U(1)$	$(314, \frac{619}{2})$
(ns)	$([3^3, 1^2], \mathbb{Z}_2)$	$\{1, 3, 4, 5, 7; \frac{7}{2}\}$	$-$	$SU(2)_{11}$	$(319, \frac{629}{2})$
	$[3^3, 1^2]$	$\{1, 3, 4, 5, 7; \frac{7}{2}\}$	$c_7^{(10)} = (a_{7/2}^{(5)})^2$	$U(1)$	$(308, \frac{609}{2})$
	$[3, 2^4]$	$\{1, 3, 4, 5, 6; \frac{7}{2}\}$	$c_3^{(4)} = (a_{3/2}^{(2)})^2$ $c_4^{(6)} = 2a_{3/2}^{(2)} a_{5/2}^{(4)}$ $c_5^{(8)} = (a_{5/2}^{(4)})^2 + 2a_{3/2}^{(2)} \tilde{c}_{7/2}^{(6)}$ $c_6^{(10)} = 2a_{5/2}^{(4)} \tilde{c}_{7/2}^{(6)}$	$SU(2)_{12}$	$(256, \frac{507}{2})$
 (ns)	$([5, 1^6], \mathbb{Z}_2)$	$\{1, 3, 5, 5, 5; \frac{5}{2}\}$	$-$	$Sp(2)_7$	$(282, \frac{553}{2})$

Table A.2: D_6 Twisted Sector

Nahm C-partition	Hitchin B-partition	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
	$[5, 1^6]$	$\{1, 3, 5, 5, 5; \frac{5}{2}\}$	$c_5^{(6)} = (a_{5/2}^{(3)})^2$	$SU(2)_6$	$(274, \frac{541}{2})$
	$[3^2, 1^5]$	$\{1, 3, 4, 5, 5; \frac{5}{2}\}$	$-$	$U(1)$	$(268, \frac{531}{2})$
	$[3, 2^2, 1^4]$	$\{1, 3, 4, 5, 5; \frac{5}{2}\}$	$c_3^{(4)} = (a_{3/2}^{(2)})^2$ $c_4^{(6)} = 2a_{3/2}^{(2)}a_{5/2}^{(4)}$ $c_5^{(8)} = (a_{5/2}^{(4)})^2$	none	$(244, \frac{485}{2})$
 (ns)	$([3, 1^8], \mathbb{Z}_2)$	$\{1, 3, 3, 3, 3; \frac{3}{2}\}$	$-$	$SU(2)_5$	$(177, \frac{351}{2})$
	$[3, 1^8]$	$\{1, 3, 3, 3, 3; \frac{3}{2}\}$	$c_3^{(4)} = (a_{3/2}^{(2)})^2$	none	$(172, \frac{343}{2})$
	$[1^{11}]$	$\{1, 1, 1, 1, 1; \frac{1}{2}\}$	$-$	none	$(60, \frac{121}{2})$

Appendix B

Bala-Carter Notation

B.1 Bala-Carter Labels

In the twisted and untwisted sectors of the A and D series, punctures were in one-to-one correspondence with certain classes of partitions [4, 7, 1, 11]. The partition denotes how the fundamental representation (vector representation, in the case of $\mathfrak{so}(N)$) of $\mathfrak{g}$ [1] decomposes into representations of the corresponding (Nahm) $\mathfrak{su}(2)$. Moreover, one can also read off the centralizer, $\mathfrak{f}$, of $\mathfrak{su}(2)$ inside $\mathfrak{g}$, as well as the decomposition of the fundamental representation of $\mathfrak{g}$ under $\mathfrak{su}(2) \times \mathfrak{f}$, from the partition (see (2.7) in [7]). The decomposition under $\mathfrak{su}(2) \times \mathfrak{f}$ for each puncture is precisely the information needed to compute the flavour group levels in §4.1.4.1, as well as the expansion of the superconformal index in §5.3. In what follows, we will explain how these decompositions are obtained for the punctures in the $\mathfrak{e}_6$ theory.

In contrast to classical $\mathfrak{g}$, nilpotent orbits in the exceptional Lie algebras, which label our punctures, are not naturally classified by partitions. The theorem of Bala and Carter states that there is a one-to-one correspondence

[1]For untwisted (twisted) punctures in the A and D series, $\mathfrak{g}$ is of type A (B) and D (C), respectively.

between nilpotent orbits in $\mathfrak{g}$ and (conjugacy classes of) pairs $(\mathfrak{l}, O^{\mathfrak{l}})$ where $\mathfrak{l}$ is a Levi subalgebra [2] of $\mathfrak{g}$ and $O^{\mathfrak{l}}$ is a *distinguished* [3] nilpotent orbit in $\mathfrak{l}$. By the Jacobson-Morozov theorem, any representative X of $O^{\mathfrak{l}}$ embeds in a standard triple [4] $\{H, X, Y\} \subset \mathfrak{l}$, where $H \in \mathfrak{h}$. $\mathfrak{l}$ then has a decomposition into ad_H-eigenspaces

$$\mathfrak{l} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{l}_k$$

where $\mathfrak{l}_k = \{x \in \mathfrak{l} \mid [H, x] = kx\}$. Let $\mathfrak{l}' \equiv \mathfrak{l}_0$ and $\mathfrak{u}' \equiv \oplus_{0<k\in\mathbb{Z}} \mathfrak{l}_k$. Then, $\mathfrak{p} = \mathfrak{l}' + \mathfrak{u}'$ is a parabolic subalgebra of $\mathfrak{l}$, with explicit Levi decomposition into a Levi subalgebra $\mathfrak{l}'$ and the nilradical $\mathfrak{u}'$ of $\mathfrak{p}$. (Notice that the Cartan of $\mathfrak{l}$ is contained in $\mathfrak{l}'$, so $\operatorname{rank}(\mathfrak{l}') = \operatorname{rank}(\mathfrak{l})$.)

A nilpotent orbit in $\mathfrak{g}$ is then given the label $X_N(a_i)$, called the *Bala-Carter label*, where X_N is the Cartan type of the semisimple part of $\mathfrak{l}$, and i is the number of simple roots in $\mathfrak{l}'$. The case $i = 0$ is denoted just by X_N, and corresponds to the principal orbit in $\mathfrak{l}$, which is always distinguished.

There are 16 conjugacy classes of Levi subalgebras of E_6. These are specified by their semisimple parts: 0, A_1, $2A_1$, $3A_1$, A_2, $A_2 + A_1$, $2A_2$, A_2, $A_2 + 2A_1$, $A_3 + A_1$, D_4, A_4, $A_4 + A_1$, A_5, D_5, and E_6. Here, kA_N denotes the direct sum of k copies of A_N. The label 0 denotes the Cartan subalgebra,

[2] A Levi subalgebra $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{g}$ is a reductive subalgebra, $\mathfrak{l}$, containing the Cartan subalgebra, $\mathfrak{h}$, of $\mathfrak{g}$. See section 3.8 of [?] for an introduction.

[3] A nilpotent orbit, $\mathcal{O}$, in $\mathfrak{g}$ is *distinguished* if and only if the only Levi subalgebra of $\mathfrak{g}$, containing $\mathcal{O}$, is $\mathfrak{g}$ itself.

[4] Any $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}$ is spanned by a *standard triple* $\{H, X, Y\}$ of nonzero elements of $\mathfrak{g}$ satisfying the bracket relations $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$.

for which the only distinguished orbit is the zero orbit. For $\mathfrak{l}$ of classical type, distinguished orbits in $\mathfrak{l}$ are easily specified in terms of their partition: for $\mathfrak{l}$ of type A, the only distinguished orbit is the principal orbit (which, for A_{N-1}, has partition $[N]$), while for $\mathfrak{l}$ of type B, C, D, distinguished orbits are those for which the partition has no repeated parts. It was found by Bala and Carter that, for $\mathfrak{l}$ of type G_2, F_4, E_6, E_7, and E_8, there are $2, 4, 3, 6$, and 11 distinguished orbits, respectively.

The distinguished orbits in the Levi subalgebras listed above give rise to 21 nilpotent orbits in $\mathfrak{e}_6$. We list these in the table below, along with the centralizer, $\mathfrak{f}$, and the decomposition of the 27 and 78 of $\mathfrak{e}_6$ under $\mathfrak{su}(2) \times \mathfrak{f}$ [5]. But, before that, let us give a few examples of how to obtain the decomposition of the 27 for various embeddings.

First, consider $\mathfrak{l} = D_4$. In this case there are two distinguished orbits, with partitions [7,1] and [5,3], corresponding to nilpotent orbits D_4 and $D_4(a_1)$, respectively, in $\mathfrak{e}_6$. The first has centralizer $\mathfrak{su}(3)$ and the second, $\mathfrak{u}(1)^2$. We can obtain the decomposition of the 27 for each of these by embedding $\mathfrak{su}(2)$ in the $\mathfrak{so}(8)$ factor in $\mathfrak{so}(8) \times \mathfrak{u}(1)^2 \subset \mathfrak{so}(10) \times \mathfrak{u}(1) \subset \mathfrak{e}_6$. The 27 of $\mathfrak{e}_6$ decomposes

[5]The decomposition of the 27 determines a projection matrix, which can be used to obtain the decompositions of higher-dimensional representations. We list a projection matrix for each puncture in Appendix E.2. The decomposition of the 78 determines the levels of the flavor groups, as described in §4.1.4.1.

under $\mathfrak{so}(10) \times \mathfrak{u}(1)$ as

$$\mathfrak{e}_6 \supset \mathfrak{so}(10) \times \mathfrak{u}(1)$$
$$27 = 1_{-4} + 10_2 + 16_{-1}$$

The 10 and 16 of $\mathfrak{so}(10)$ decompose under $\mathfrak{so}(8) \times \mathfrak{u}(1)$ as

$$\mathfrak{so}(10) \supset \mathfrak{so}(8) \times \mathfrak{u}(1)$$
$$10 = 1_2 + 1_{-2} + (8_v)_0$$
$$16 = (8_s)_1 + (8_c)_{-1}$$

so we have

$$\mathfrak{e}_6 \supset \mathfrak{so}(8) \times \mathfrak{u}(1) \times \mathfrak{u}(1)$$
$$27 = 1_{0,-4} + 1_{2,2} + 1_{-2,2} + (8_v)_{0,2} + (8_s)_{1,-1} + (8_c)_{-1,-1}$$

For $D_4(a_1)$, we embed $\mathfrak{su}(2)$ in $\mathfrak{so}(8)$ by taking

$$\mathfrak{so}(8) \supset \mathfrak{su}(2)$$
$$8_{v,s,c} = 5 + 3$$

which gives

$$\mathfrak{e}_6 \supset \mathfrak{su}(2) \times \mathfrak{u}(1) \times \mathfrak{u}(1)$$
$$27 = 1_{0,-4} + 1_{2,2} + 1_{-2,2} + 3_{0,2} + 3_{1,-1} + 3_{-1,-1} + 5_{0,2} + 5_{1,-1} + 5_{-1,-1}$$

For D_4, we embed $\mathfrak{su}(2)$ in $\mathfrak{so}(8)$ by taking

$$\begin{aligned} \mathfrak{so}(8) &\supset \mathfrak{su}(2) \\ 8_{v,s,c} &= 7+1 \end{aligned}$$

which gives

$$\begin{aligned} &\mathfrak{e}_6 \supset \mathfrak{su}(2)\times\mathfrak{u}(1)\times\mathfrak{u}(1) \\ &27 = 1_{0,-4}+1_{2,2}+1_{-2,2}+1_{0,2}+1_{1,-1}+1_{-1,-1}+7_{0,2}+7_{1,-1}+7_{-1,-1} \end{aligned}$$

For this embedding, the $\mathfrak{u}(1)^2$ centralizer enhances to $\mathfrak{su}(3)$. To see this, we can make a change of basis so that the two $\mathfrak{u}(1)$ charges are given in terms of the old ones by

$$\begin{aligned} q_1' &= \frac{1}{2}(q_1+q_2) \\ q_2' &= \frac{1}{2}(q_1-q_2) \end{aligned}$$

Then the decomposition becomes

$$\begin{aligned} &\mathfrak{e}_6 \supset \mathfrak{su}(2)\times\mathfrak{u}(1)\times\mathfrak{u}(1) \\ &27 = 1_{-2,2}+1_{2,0}+1_{0,-2}+1_{1,-1}+1_{0,1}+1_{-1,0}+7_{1,-1}+7_{0,1}+7_{-1,0} \end{aligned}$$

where we recognize these $\mathfrak{u}(1)^2$ charges as the weights (in the Dynkin basis) of the 6 and $\overline{3}$ of $\mathfrak{su}(3)$. Thus, the decomposition of the 27 is given by

$$\begin{aligned} \mathfrak{e}_6 &\supset \mathfrak{su}(2)\times\mathfrak{su}(3) \\ 27 &= (1,6)+(7,\overline{3}) \end{aligned}$$

Now, consider $\mathfrak{l} = E_6$. There are three distinguished orbits in $\mathfrak{e}_6$, giving rise to nilpotent orbits $E_6, E_6(a_1)$, and $E_6(a_3)$. The decomposition of the 27 for each of these can be obtained by taking the principal embedding of $\mathfrak{su}(2)$ inside the maximal subalgebras $\mathfrak{f}_4$, $\mathfrak{sp}(4)$, and $\mathfrak{su}(3)$ of $\mathfrak{e}_6$ [6], respectively. We work out the decomposition for E_6 (the principal nilpotent orbit in $\mathfrak{e}_6$); the decompositions for $E_6(a_1)$ and $E_6(a_3)$ follow the same steps.

The 27 of $\mathfrak{e}_6$ decomposes under $\mathfrak{f}_4$ as

$$\begin{aligned} &\mathfrak{e}_6 \supset \mathfrak{f}_4 \\ &27 = 1 + 26 \end{aligned}$$

The principal embedding of $\mathfrak{su}(2)$ in $\mathfrak{f}_4$ is given by taking

$$\begin{aligned} &\mathfrak{f}_4 \supset \mathfrak{su}(2) \\ &26 = 9 + 17 \end{aligned}$$

so the decomposition of the 27 for E_6 is given by

$$\begin{aligned} &\mathfrak{e}_6 \supset \mathfrak{su}(2) \\ &27 = 1 + 9 + 17 \end{aligned}$$

To see which distinguished orbit corresponds to which $E_6(a_i)$, we need to count the number of simple roots in $\mathfrak{l}'$. To do that, we make recourse to the decomposition of the 78.

[6] One might wonder about the other maximal subalgebras of $\mathfrak{e}_6$. One finds that the principal embedding of $\mathfrak{su}(2)$ in $\mathfrak{su}(2) \times \mathfrak{su}(6)$ or $\mathfrak{su}(3) \times \mathfrak{g}_2$ again gives $E_6(a_3)$, in $\mathfrak{g}_2$ gives $E_6(a_1)$, in $\mathfrak{so}(10) \times \mathfrak{u}(1)$ gives D_5, and in $\mathfrak{su}(3)^3$ gives D_4.

- For the first case (embedding via $\mathfrak{f}_4$), the 78 decomposes as $3 + 9 + 11 + 15 + 17 + 23$. So $\dim(\mathfrak{l}') = \dim(\mathfrak{g}_0) = 6$, which is also equal to $\text{rank}(\mathfrak{l}') = \text{rank}(\mathfrak{e}_6)$. Thus $\mathfrak{l}'$ is just the Cartan subalgebra and this is the principal embedding, E_6.

- For the embedding via $\mathfrak{sp}(4)$, the 78 decomposes as $3+5+7+9+2(11)+15+17$, so we have $\dim(\mathfrak{l}') = 8$ and $\mathfrak{l}'$ must contain precisely one positive (hence, simple) root. Thus, this is $E_6(a_1)$.

- Finally, for the embedding via $\mathfrak{su}(3)$, the 78 decomposes as $3(3)+3(5)+2(7)+2(9)+2(11)$, so $\dim(\mathfrak{l}') = 12$ and $\mathfrak{l}'$ contains three simple roots. Hence, this is $E_6(a_3)$.

Appendix C

Embeddings of $SU(2)$ in E_6

C.1 Embeddings of $SU(2)$ in E_6

Here we give the decompositions of the 27 and the 78 under $\mathfrak{su}(2) \times \mathfrak{f}$ for the nilpotent orbits in $\mathfrak{e}_6$.

Table C.1: Embeddings of $SU(2)$ in E_6

Bala-Carter	$\mathfrak{f}$	27	78
0	$\mathfrak{e}_6$	$(1; 27)$	$(1; 78)$
A_1	$\mathfrak{su}(6)$	$(1; \overline{15}) + (2; 6)$	$(1; 35) + (2; 20) + (3; 1)$
$2A_1$	$\mathfrak{so}(7) \times \mathfrak{u}(1)$	$(1; 7_2 + 1_{-4}) + (2; 8_{-1}) + (3; 1_2)$	$(1; 1_0 + 21_0) + (2; 8_3 + 8_{-3}) + (3; 7_0 + 1_0)$
$3A_1$	$\mathfrak{su}(3) \times \mathfrak{su}(2)$	$(1; \overline{6}, 1) + (2; 3, 2) + (3; 3, 1)$	$(1; 8, 1) + (1; 1, 3) + (2; 8, 2) + (3; 1, 1) + (3; 8, 1) + (4; 1, 2)$
A_2	$\mathfrak{su}(3) \times \mathfrak{su}(3)$	$(1; 3, \overline{3}) + (3; 1, 3) + (3; \overline{3}, 1)$	$(1; 8, 1) + (1; 1, 8) + (3; 1, 1) + (3; 3, 3) + (3; \overline{3}, \overline{3}) + (5; 1, 1)$

Table C.1: Embeddings of $SU(2)$ in E_6

Bala-Carter	$\mathfrak{f}$	27	78
A_2+A_1	$\mathfrak{su}(3)\times\mathfrak{u}(1)$	$(1;3_2)+(2;3_{-1}+1_1)+(3;\overline{3}_0+1_{-2})+(4;1_1)$	$(1;8_0+1_0)+(2;3_1+\overline{3}_{-1}+1_{-3}+1_3)+(3;3_{-2}+\overline{3}_2+1_0+1_0)+(4;3_1+\overline{3}_{-1})+(5;1_0)$
$2A_2$	$\mathfrak{g}_2$	$(1;1)+(3;7)+(5;1)$	$(1;14)+(3;7+1)+(5;7+1)$
A_2+2A_1	$\mathfrak{su}(2)\times\mathfrak{u}(1)$	$(1;1_2+1_{-4})+(2;4_{-1})+(3;3_2)+(4;2_{-1})$	$(1;1_0+3_0)+(2;4_3+4_{-3})+(3;1_0+3_0+5_0)+(4;2_3+2_{-3})+(5;3_0)$
A_3	$\mathfrak{sp}(2)\times\mathfrak{u}(1)$	$(1;5_{-2}+1_4)+(4;4_1)+(5;1_{-2})$	$(1;10_0+1_0)+(3;1_0)+(4;4_3+4_{-3})+(5;5_0)+(7;1_0)$
$2A_2+A_1$	$\mathfrak{su}(2)$	$(1;1)+(2;2)+(3;3)+(4;2)+(5;1)$	$(1;3)+(2;4+2)+(3;3+1+1)+(4;2+2)+(5;3+1)+(6;2)$
A_3+A_1	$\mathfrak{su}(2)\times\mathfrak{u}(1)$	$(1;1_4+1_{-2})+(2;2_{-2})+(3;1_1)+(4;2_1)+(5;1_1+1_{-2})$	$(1;1_0+3_0)+(2;2)_0+(3;1_3+1_{-3}+1_0+1_0)+(4;2_3+2_{-3}+2_0)+(5;1_3+1_0+1_{-3})+(6;2)_0+(7;1_0)$

Table C.1: Embeddings of $SU(2)$ in E_6

Bala-Carter	$\mathfrak{f}$	27	78
$D_4(a1)$	$\mathfrak{u}(1) \times \mathfrak{u}(1)$	$1_{2,2} + 1_{0,-4} + 1_{-2,2} + 3_{1,-1} + 3_{0,2} + 3_{-1,-1} + 5_{1,-1} + 5_{0,2} + 5_{-1,-1}$	$1_{0,0} + 1_{0,0} + 3_{0,0} + 3_{2,0} + 3_{1,3} + 3_{1,-3} + 3_{0,0} + 3_{-2,0} + 3_{-1,-3} + 3_{-1,3} + 3_{0,0} + 5_{2,0} + 5_{1,3} + 5_{1,-3} + 5_{0,0} + 5_{-2,0} + 5_{-1,-3} + 5_{-1,3} + 7_{0,0} + 7_{0,0}$
A_4	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$(1;2_{-5}) + (3;1_{-2}) + (5;2_1 + 1_4) + (7;1_{-2})$	$(1;3_0 + 1_0) + (3;2_3 + 2_{-3} + 1_0) + (5;1_6 + 1_0 + 1_{-6}) + (7;2_3 + 2_{-3} + 1_0) + (9;1_0)$
D_4	$\mathfrak{su}(3)$	$(1;6) + (7;\overline{3})$	$(1;8) + (3;1) + (7;8) + (11;1)$
$A_4 + A_1$	$\mathfrak{u}(1)$	$2_{-5} + 3_{-2} + 4_1 + 5_4 + 6_1 + 7_{-2}$	$1_0 + 2_3 + 2_{-3} + 3_0 + 3_0 + 4_3 + 4_{-3} + 5_6 + 5_0 + 5_{-6} + 6_{-3} + 6_3 + 7_0 + 8_{-3} + 8_3 + 9_0$
$D_5(a1)$	$\mathfrak{u}(1)$	$1_{-4} + 2_{-1} + 3_2 + 6_{-1} + 7_2 + 8_{-1}$	$1_0 + 2_3 + 2_{-3} + 3_0 + 3_0 + 5_0 + 6_3 + 6_{-3} + 7_0 + 7_0 + 8_3 + 8_{-3} + 9_0 + 11_0$
A_5	$\mathfrak{su}(2)$	$(1;1) + (5;1) + (6;2) + (9;1)$	$(1;3) + (3;1) + (4;2) + (5;1) + (6;2) + (7;1) + (9;1) + (10;2) + (11;1)$
$E_6(a3)$	$-$	$1 + 5 + 5 + 7 + 9$	$3 + 3 + 3 + 5 + 5 + 5 + 7 + 7 + 9 + 9 + 11 + 11$
D_5	$\mathfrak{u}(1)$	$1_2 + 1_{-4} + 5_{-1} + 9_2 + 11_{-1}$	$1_0 + 3_0 + 5_3 + 5_{-3} + 7_0 + 9_0 + 11_3 + 11_0 + 11_{-3} + 15_0$
$E_6(a1)$	$-$	$5 + 9 + 13$	$3 + 5 + 7 + 9 + 11 + 11 + 15 + 17$
E_6	$-$	$1 + 9 + 17$	$3 + 9 + 11 + 15 + 17 + 23$

C.2 Projection Matrices

Our classification of interacting and mixed fixtures using the superconformal index, carried out in section 5.3, required that we know the decomposition of a number of higher-dimensional $\mathfrak{e}_6$ representations (and not just the 27 and the 78) under $\mathfrak{su}(2) \times \mathfrak{f}$. These are trivial to obtain using LieART [78], provided we know a projection matrix for each embedding [79, 78].

From the decomposition of the 27, listed in the table above, one obtains a projection matrix simply by defining a $6 \times \mathrm{rk}\,(\mathfrak{su}(2) \times \mathfrak{f})$ matrix, M, such that the LieART command

```
In[1]= Project[M,WeightSystem[Irrep[E6][1,0,0,0,0,0]]]
```

gives the corresponding $\mathfrak{su}(2) \times \mathfrak{f}$ weights. This projection matrix can then be used to obtain the decomposition of any $\mathfrak{e}_6$ irrep under $\mathfrak{su}(2) \times \mathfrak{f}$.

Below, we list a projection matrix for each embedding, following the conventions of LieART.

Table C.2: Projection Matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
A_1	$\mathfrak{su}(6)$	$\begin{pmatrix} -1 & -2 & -3 & -2 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Table C.2: Projection Matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
$2A_1$	$\mathfrak{so}(7) \times \mathfrak{u}(1)$	$\begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & -1 & -2 & 0 \end{pmatrix}$
$3A_1$	$\mathfrak{su}(3) \times \mathfrak{su}(2)$	$\begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{pmatrix}$
A_2	$\mathfrak{su}(3) \times \mathfrak{su}(3)$	$\begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -2 & -1 & -2 \end{pmatrix}$
$A_2 + A_1$	$\mathfrak{su}(3) \times \mathfrak{u}(1)$	$\begin{pmatrix} 3 & 5 & 7 & 5 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$
$2A_2$	$\mathfrak{g}_2$	$\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

Table C.2: Projection Matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
$A_2 + 2A_1$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 3 & 4 & 6 & 4 & 3 & 4 \\ 1 & 4 & 6 & 4 & 1 & 2 \\ -1 & -2 & 0 & 2 & 1 & 0 \end{pmatrix}$
A_3	$\mathfrak{sp}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 4 & 7 & 10 & 7 & 4 & 6 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \end{pmatrix}$
$2A_2 + A_1$	$\mathfrak{su}(2)$	$\begin{pmatrix} 4 & 6 & 9 & 6 & 4 & 5 \\ 0 & 2 & 3 & 2 & 0 & 1 \end{pmatrix}$
$A_3 + A_1$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 4 & 8 & 11 & 8 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 1 & 2 & 0 \end{pmatrix}$
$D_4(a1)$	$\mathfrak{u}(1) \times \mathfrak{u}(1)$	$\begin{pmatrix} 4 & 8 & 10 & 8 & 4 & 6 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$
A_4	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 6 & 10 & 12 & 10 & 6 & 6 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -3 & -2 & 2 & 0 \end{pmatrix}$

Table C.2: Projection Matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
D_4	$\mathfrak{su}(3)$	$\begin{pmatrix} 6 & 10 & 16 & 10 & 6 & 10 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}$
$A_4 + A_1$	$\mathfrak{u}(1)$	$\begin{pmatrix} 6 & 10 & 12 & 10 & 6 & 7 \\ -2 & -4 & -6 & -2 & 2 & -3 \end{pmatrix}$
$D_5(a1)$	$\mathfrak{u}(1)$	$\begin{pmatrix} 7 & 12 & 18 & 12 & 7 & 10 \\ -1 & -2 & 0 & 2 & 1 & 0 \end{pmatrix}$
A_5	$\mathfrak{su}(2)$	$\begin{pmatrix} 8 & 14 & 19 & 14 & 8 & 10 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
$E_6(a3)$	–	$\begin{pmatrix} 8 & 14 & 18 & 14 & 8 & 8 \end{pmatrix}$
D_5	$\mathfrak{u}(1)$	$\begin{pmatrix} 10 & 18 & 24 & 18 & 10 & 10 \\ -1 & -2 & 0 & 2 & 1 & 0 \end{pmatrix}$
$E_6(a1)$	–	$\begin{pmatrix} 12 & 22 & 30 & 22 & 12 & 16 \end{pmatrix}$
E_6	–	$\begin{pmatrix} 16 & 30 & 42 & 30 & 16 & 22 \end{pmatrix}$

As an example, let's work out the decomposition of the 51975 for the orbit $2A_2$. Running LieART, we obtain the decomposition with the following two lines of code:

```
In[1]= ProjectionMatrix[E6,ProductAlgebra[SU2,G2]]=
```

$$\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

```
In[2]= DecomposeIrrep[Irrep[E6][1,0,1,0,0,0],ProductAlgebra[SU2,G2]]
```

```
Out[2]=
```

$$\begin{gathered}
(1,1)+14(3,1)+10(5,1)+13(1,7)+13(7,1)+ \\
25(3,7)+5(9,1)+34(5,7)+4(11,1)+ \\
25(7,7)+9(1,14)+17(9,7)+16(3,14)+6(11,7)+ \\
22(5,14)+2(13,7)+15(7,14)+10(9,14)+3(11,14)+ \\
(13,14)+6(1,27)+25(3,27)+23(5,27)+21(7,27)+ \\
9(9,27)+4(11,27)+5(1,64)+12(3,64)+13(5,64)+ \\
9(7,64)+4(9,64)+(11,64)+4(1,77)+6(3,77)+ \\
2(3,77')+8(5,77)+(5,77')+4(7,77)+(7,77')+ \\
2(9,77)+(3,182)+(1,189)+2(3,189)+2(5,189)+ \\
(7,189)
\end{gathered}$$

This works for all of the orbits above, except for $D_4(a_1)$, as the LieART command "`DecomposeIrrep`" does not seem to work when the target subalgebra has more than one $\mathfrak{u}(1)$ factor. In this case, getting the decomposition is only slightly more complicated. For example, we obtain the decomposition of the 27 of E_6 as follows:

```
In[1]= ProjectionMatrix[D5,ProductAlgebra[D4,U1]]=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

```
In[2]= ProjectionMatrix[D4,ProductAlgebra[A1]]=
```

$$\begin{pmatrix} 4 & 6 & 4 & 4 \end{pmatrix}$$

```
In[3]= DecomposeIrrep[ DecomposeIrrep[ DecomposeIrrep[
      Irrep[E6][1,0,0,0,0,0], ProductAlgebra[D5, U1]],
      ProductAlgebra[D4, U1],1], ProductAlgebra[A1], 1]
Out[3]= (1)(2)(2)+(1)(0)(-4)+(1)(-2)(2)+(3)(1)(-1)+(3)(0)(2)
        +(3)(-1)(-1)+(5)(1)(-1)+(5)(0)(2)+(5)(-1)(-1)
```

Appendix D

Constraints for twisted E_6 punctures

D.1 Constraints

Table D.1: Constraints

Bala-Carter	New parameters	Constraints
$\widetilde{A}_1$	$h_6 \equiv \frac{a^{(6)}_{11/2}}{z^{11/2}}$	$\phi_{12} - h_6^2 \sim \frac{1}{z^{10}}$
$\widetilde{A}_2$	$h_3 \equiv \frac{a^{(3)}_{5/2}}{z^{5/2}}$	$\phi_8 - \phi_5 h_3 \sim \frac{1}{z^6}$ $\phi_{12} + \phi_6^2 + \frac{1}{16}\phi_6 h_3^2 + 3\phi_9 h_3 + \frac{1}{1024} h_3^4 \sim \frac{1}{z^9}$
B_2	$h_3 \equiv \frac{a^{(3)}_{5/2}}{z^{5/2}}$ $h_6 \equiv \frac{a^{(6)}_{5}}{z^{5}}$	$\phi_9 - \frac{1}{12} h_3(h_3^2 + h_6 + 2\phi_6) \sim \frac{1}{z^{13/2}}$ $\phi_{12} - h_3^2(h_3^2 - h_6) - \frac{1}{4}\left(h_3^2 - h_6 - 2\phi_6\right)^2 \sim \frac{1}{z^9}$
$\widetilde{A}_2 + A_1$	$h_3 \equiv \frac{a^{(3)}_{5/2}}{z^{5/2}}$	$\phi_{12} - 24\phi_9 h_3 + (\phi_6 + 2h_3^2)^2 \sim \frac{1}{z^9}$
$C_3(a_1)$	$h_3 \equiv \frac{a^{(3)}_{5/2}}{z^{5/2}}$ $h'_3 \equiv \frac{a'^{(3)}_{5/2}}{z^{5/2}}$	$\phi_9 - \frac{1}{12} h_3(h_3^2 + h'^2_3 + 2\phi_6) \sim \frac{1}{z^{13/2}}$ $\phi_{12} - h_3^2(h_3^2 - h'^2_3) - \frac{1}{4}\left(h_3^2 - h'^2_3 - 2\phi_6\right)^2 \sim \frac{1}{z^9}$

Table D.1: Constraints

Bala-Carter	New parameters	Constraints
$F_4(a_3)$	$h_3 \equiv \frac{a^{(3)}_{5/2}}{z^{5/2}}$ $h'_3 \equiv \frac{a'^{(3)}_{5/2}}{z^{5/2}}$ $h''_3 \equiv \frac{a''^{(3)}_{5/2}}{z^{5/2}}$	$\phi_6 + 2h_3^2 + h'^2_3 - 3h''^2_3 \sim \frac{1}{z^4}$ $\phi_9 - \frac{1}{3}h_3(h'^2_3 + 3h''^2_3) \sim \frac{1}{z^{13/2}}$ $\phi_{12} + 8h_3^2(h'^2_3 - 3h''^2_3) + \left(h'^2_3 + 3h''^2_3\right)^2 \sim \frac{1}{z^9}$
B_3	$h_4 \equiv \frac{a_3^{(4)}}{z^3}$	$\phi_8 - 48h_4^2 \sim \frac{1}{z^5}$ $\phi_9 - \phi_5 h_4 \sim \frac{1}{z^{11/2}}$ $\phi_{12} + 96h_4^3 \sim \frac{1}{z^8}$
C_3	$h_3 \equiv \frac{a^{(3)}_{5/2}}{z^{5/2}}$ $h_4 \equiv \frac{a_3^{(4)}}{z^3}$	$\phi_6 + 6h_3^2 \sim \frac{1}{z^4}$ $\phi_8 - 16\phi_2 h_3^2 - 8\phi_5 h_3 - 48h_4^2 \sim \frac{1}{z^5}$ $\phi_9 + \frac{1}{3}\phi_6 h_3 + \phi_5 h_4 - 2\phi_2 h_4 h_3 + \frac{2}{3}h_3^3 \sim \frac{1}{z^{11/2}}$ $\phi_{12} + \phi_6^2 + 24\phi_9 h_3 - 3\phi_8 h_4 - 12\phi_6\phi_2 h_4 + 4\phi_6 h_3^2$ $+24\phi_5 h_4 h_3 + 36\phi_2^2 h_4^2 - 24\phi_2 h_4 h_3^2 + 4h_3^4 + 48h_4^3 \sim \frac{1}{z^7}$
$F_4(a_2)$	$h_4 \equiv \frac{a_3^{(4)}}{z^3}$	$\phi_8 - 48h_4^2 \sim \frac{1}{z^5}$ $\phi_9 + \phi_5 h_4 \sim \frac{1}{z^{11/2}}$ $\phi_{12} + \phi_6^2 - 12h_4(\frac{1}{4}\phi_8 - 3\phi_2^2 h_4 + \phi_6\phi_2 - 4h_4^2) \sim \frac{1}{z^7}$

Table D.1: Constraints

Bala-Carter	New parameters	Constraints
$F_4(a_1)$	$h_2 \equiv \frac{a^{(2)}_{3/2}}{z^{3/2}}$ $h_3 \equiv \frac{a^{(3)}_2}{z^2}$ $h_6 \equiv \frac{a^{(6)}_{7/2}}{z^{7/2}}$	$\phi_5 - 2h_3h_2 \sim \frac{1}{z^{5/2}}$ $\phi_6 - 6\phi_2 h_2^2 - h_3^2 \sim \frac{1}{z^3}$ $\phi_8 + 4\phi_2 h_3^2 + 48h_2(h_6 - h_2^3) \sim \frac{1}{z^4}$ $\phi_9 + \phi_5 h_2^2 - h_6 h_3 \sim \frac{1}{z^{9/2}}$ $\phi_{12} - 3\phi_8 h_2^2 + 2\phi_6 h_3^2 - 12\phi_2 h_3^2 h_2^2 - 36h_6^2 - h_3^4 \sim \frac{1}{z^6}$
F_4	$h_3 \equiv \frac{a^{(3)}_{3/2}}{z^{3/2}}$	$\phi_5 - \phi_2 h_3 \sim \frac{1}{z^{3/2}}$ $\phi_6 + \frac{3}{2}\phi_2^3 + \frac{3}{2}h_3^2 \sim \frac{1}{z^2}$ $\phi_8 + 4\phi_6\phi_2 + 3\phi_2^4 - 4\phi_5 h_3 + 2\phi_2 h_3^2 \sim \frac{1}{z^2}$ $\phi_9 + \frac{1}{6}\phi_6 h_3 - \frac{1}{4}\phi_5\phi_2^2 + \frac{1}{4}\phi_2^3 h_3 + \frac{1}{12}h_3^3 \sim \frac{1}{z^{5/2}}$ $\phi_{12} + \phi_6^2 + 12\phi_9 h_3 + \phi_6 h_3^2 + \frac{3}{2}\phi_2^6 + 3\phi_6\phi_2^3$ $+\frac{3}{4}\phi_8\phi_2^2 - 3\phi_5\phi_2^2 h_3 + \frac{3}{2}\phi_2^3 h_3^2 + \frac{1}{4}h_3^4 \sim \frac{1}{z^3}$

Appendix E

Embeddings of $SU(2)$ in F_4

E.1 Appendix: Embeddings of $SU(2)$ in F_4

Table E.1: Embeddings of $SU(2)$ in F_4

Bala-Carter	$\mathfrak{f}$	26	52
A_1	$\mathfrak{sp}(3)$	$(2,6)+(1,14)$	$(3,1)+(2,14')+(1,21)$
$\widetilde{A}_1$	$\mathfrak{su}(4)$	$(2,4)+(2,\overline{4})+(1,6)$ $+(3,1)+(1,1)$	$(3,1)+(3,6)$ $+(2,4)+(2,\overline{4})+(1,15)$
$A_1+\widetilde{A}_1$	$\mathfrak{su}(2)\times\mathfrak{su}(2)$	$(1;5,1)+(2;3,2)+(3;3,1)$	$(1;3,1)+(1;1,3)+$ $(2;5,2)+(3;1,1)+$ $(3;5,1)+(4;1,2)$
A_2	$\mathfrak{su}(3)$	$(3,3)+(3,\overline{3})+(1,8)$	$(5,1)+(3,6)+(3,\overline{6})$ $+(3,1)+(1,8)$
$\widetilde{A}_2$	$\mathfrak{g}_2$	$(3,7)+(5,1)$	$(5,7)+(3,1)+(1,14)$
$A_2+\widetilde{A}_1$	$\mathfrak{su}(2)$	$(4,2)+(3,3)+$ $(2,4)+(1,1)$	$(5,3)+(4,2)+(3,5)$ $+(3,1)+(2,4)+(1,3)$
B_2	$\mathfrak{su}(2)\times\mathfrak{su}(2)$	$(5,1,1)+(4,2,1)+$ $(4,1,2)+(1,2,2)+$ $(1,1,1)$	$(7,1,1)+(5,2,2)+$ $(4,2,1)+(4,1,2)$ $+(3,1,1)+(1,3,1)$ $+(1,1,3)$
$\widetilde{A}_2+A_1$	$\mathfrak{su}(2)$	$(5,1)+(4,2)+$ $(3,3)+(2,2)$	$(6,2)+(5,3)+(4,2)+$ $2(3,1)+(2,4)+(1,3)$
$C_3(a_1)$	$\mathfrak{su}(2)$	$2(5,1)+(4,2)+(3,1)$ $+(2,2)+(1,1)$	$(7,1)+(6,2)+(5,1)+$ $2(4,2)+3(3,1)+(1,3)$

Table E.1: Embeddings of $SU(2)$ in F_4

Bala-Carter	$\mathfrak{f}$	26	52
$F_4(a_3)$	$-$	$3(5)+3(3)+2(1)$	$2(7)+4(5)+6(3)$
B_3	$\mathfrak{su}(2)$	$(1;5)+(7,3)$	$(1;3)+(3;1)+$ $(7;5)+(11;1)$
C_3	$\mathfrak{su}(2)$	$(9,1)+(6,2)+(5,1)$	$(11,1)+(10,2)+(7,1)$ $+(4,2)+(3,1)+(1,3)$
$F_4(a_2)$	$-$	$(9)+(7)+2(5)$	$2(11)+(9)+$ $(7)+(5)+3(3)$
$F_4(a_1)$	$-$	$(11)+(9)+(5)+(1)$	$(15)+2(11)+$ $(7)+(5)+(3)$
F_4	$-$	$(17)+(9)$	$(23)+(15)+(11)+(3)$

E.2 Projection matrices

Table E.2: Projection Matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
$\underline{A_1}$	$Sp(3)_{13}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$
$\underline{\widetilde{A}_1}$	$SU(4)_{12}$	$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

Table E.2: Projection Matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
$\underline{A_1 + \widetilde{A}_1}$	$SU(2)_{64} \times SU(2)_{10}$	$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 4 & 8 & 4 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix}$
$\underline{A_2}$	$SU(3)_{16}$	$\begin{pmatrix} 4 & 6 & 4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}$
$\underline{\widetilde{A}_2}$	$(G_2)_{10}$	$\begin{pmatrix} 4 & 8 & 6 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
$\underline{A_2 + \widetilde{A}_1}$	$SU(2)_{39}$	$\begin{pmatrix} 2 & 6 & 5 & 3 \\ 4 & 6 & 3 & 1 \end{pmatrix}$
$\underline{B_2}$	$SU(2)_7^2$	$\begin{pmatrix} 4 & 10 & 7 & 4 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$
$\underline{\widetilde{A}_2 + A_1}$	$SU(2)_{20}$	$\begin{pmatrix} 4 & 9 & 7 & 4 \\ 2 & 3 & 1 & 0 \end{pmatrix}$
$\underline{C_3(a_1)}$	$SU(2)_7$	$\begin{pmatrix} 5 & 11 & 8 & 4 \\ 1 & 1 & 0 & 0 \end{pmatrix}$
$\underline{F_4(a_3)}$	$-$	$\begin{pmatrix} 6 & 12 & 8 & 4 \end{pmatrix}$
$\underline{B_3}$	$SU(2)_{24}$	$\begin{pmatrix} 6 & 16 & 12 & 6 \\ 4 & 4 & 2 & 2 \end{pmatrix}$

Table E.2: Projection Matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
$\underline{C_3}$	$SU(2)_6$	$\begin{pmatrix} 9 & 19 & 14 & 8 \\ 1 & 1 & 0 & 0 \end{pmatrix}$
$\underline{F_4(a_2)}$	$-$	$\begin{pmatrix} 10 & 20 & 14 & 8 \end{pmatrix}$
$\underline{F_4(a_1)}$	$-$	$\begin{pmatrix} 14 & 26 & 18 & 10 \end{pmatrix}$
$\underline{F_4}$	$-$	$\begin{pmatrix} 22 & 42 & 30 & 16 \end{pmatrix}$

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