ON THE GENERALIZED WORD PROBLEM FOR FINITELY PRESENTED LATTICES

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ABSTRACT

The generalized word problem for a lattice $\mathbf{L}$ in a variety $\mathscr{V}$ asks if, given a finite subset $Y \subseteq L$ and an element $d \in L$, there is an algorithm to determine if d is in the subalgebra of $\mathbf{L}$ generated by Y. In [6], it was shown that the generalized word problem for finitely presented lattices is solvable. This algorithm, though effective, is potentially exponential. We present a polynomial time algorithm for the generalized word problem for free lattices, but explain the complications which can arise when trying to adapt this algorithm to the generalized word problem for finitely presented lattices. Though some of the results for free lattices are shown to transfer over for finitely presented lattices, we give a potential syntactic algorithm for the generalized word problem for finitely presented lattices. Finally, we give a new proof that the generalized word problem for finitely presented lattices is solvable, relying on the partial completion, $\mathrm{PC}(P)$, of a partially defined lattice P.

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CHAPTER 1
GENERAL LATTICE THEORY

We begin this chapter with some preliminary definitions which are necessary for the rest of the paper. We also introduce a useful construction of Alan Day which will be important for future results. The definitions and results presented in this chapter can be found in any standard introduction to lattice theory, such as [8].

1.1 Preliminary Definitions

Definition 1.1.1. An *order relation* on a set P is a binary relation which satisfies

1. $x \leq x$, for all $x \in P$ (reflexivity)
2. $x \leq y$ and $y \leq x$ imply $x = y$, for all x, $y \in P$ (anti-symmetry)
3. $x \leq y$ and $y \leq z$ imply $x \leq z$ for all x, y, $z \in P$ (transitivity)

An *ordered set* (also known as a *partially ordered set*) is a pair $\mathbf{P} = \langle P, \leq \rangle$, where P is a set and $\leq$ is an order relation on P. If $\langle P, \leq \rangle$ is an ordered set, the relation $\geq$ is defined by $x \geq y$ if and only if $y \leq x$. $\langle P, \geq \rangle$ is an ordered set known as the *dual* of $\langle P, \leq \rangle$.

Remark 1.1.2. Each concept and theorem about ordered sets has a dual obtained by reversing the roles of $\leq$ and $\geq$. In proofs we often use the phrase "by duality" to express the symmetry between $\leq$ and $\geq$. Of course $x < y$ if and only if $x \leq y$ and $x \neq y$, and $>$ is defined dually.

Definition 1.1.3. A relation which is reflexive and transitive, but not necessarily anti-symmetric is called a *quasiorder*. If $\leq$ is a quasiorder on S, $a \equiv b$ if $a \leq b$ and $b \leq a$ defines an equivalence relation on S. Then $\leq$ induces a natural partial order on $P/\equiv$.

Definition 1.1.4. A *chain* C in an ordered set $\mathbf{P}$ is a subset of P such that any two elements of C are comparable, i.e., if x and $y \in C$ then either $x \leq y$ or $y \leq x$. An *antichain* is a subset A of P such that no two elements of A are comparable.

Definition 1.1.5. Let S be a subset of P and $a \in P$. We say that a is the *least upper bound* of S if a is an *upper bound* for S, i.e., $s \leq a$ for all $s \in S$, and $a \leq b$ for any upper bound b of S. If it exists, we denote the least upper bound by $\bigvee S$. The dual concept is called the *greatest lower bound* and is denoted by $\bigwedge S$. If $S = \{a, b\}$ then $\bigvee S$ is denoted by $a \vee b$ and $\bigwedge S$ by $a \wedge b$. The terms *supremum* and *join* are also used for the least upper bound and *infimum* and *meet* are used for the greatest lower bound.

Definition 1.1.6. A *lattice* is an algebra $\mathbf{L} = \langle L, \vee, \wedge \rangle$, with two binary operations which are both idempotent, commutative, and associative, and satisfy the absorptive laws: For all $x,\ y \in L$,

$$x \vee (y \wedge x) = x \qquad \text{and} \qquad x \wedge (y \vee x) = x$$

Remark 1.1.7. If $\mathbf{L} = \langle L, \vee, \wedge \rangle$ is a lattice, we can define an order on L by $x \leq y$ if and only if $x \wedge y = x$. Under this order, $x \wedge y$ is the greatest lower bound of x and y, and $x \vee y$ is the least upper bound of x and y. Conversely, an ordered set $\langle L, \leq \rangle$ such that each pair of elements of L has both a greatest lower bound and a least upper bound defines a lattice. It is easy to see that the dual of a lattice $\langle L, \vee, \wedge \rangle$ is $\langle L, \wedge, \vee \rangle$.

Definition 1.1.8. An element a in a lattice $\mathbf{L}$ is *join irreducible* if $a = b \vee c$ implies that either $a = b$ or $a = c$. An element a is *completely join irreducible* if $a = \bigvee S$ implies $a \in S$. An element a is *join prime* if $a \leq b \vee c$ implies that either $a \leq b$ or $a \leq c$; it is *completely join prime* if $a \leq \bigvee S$ implies $a \leq s$ for some $s \in S$. Naturally *meet irreducible*, *completely meet irreducible*, *meet prime*, and *completely meet prime* are defined dually.

Remark 1.1.9. If a lattice has a least element, it is denoted by 0 and if it has a greatest element, it is denoted by 1. Note that, technically, 0 is join irreducible but not completely join irreducible. However, we will follow the long-standing convention that in a *finite* lattice, 0 is not regarded as join irreducible. (This is because we treat a finite lattice as a complete lattice.) Dually, in a finite lattice 1 is not considered to be meet irreducible. We let $\mathrm{J}(\mathbf{L})$ denote the join irreducible elements of $\mathbf{L}$ and $\mathrm{M}(\mathbf{L})$ denote the meet irreducible elements of $\mathbf{L}$.

Definition 1.1.10. If $a < b$ are elements in a lattice $\mathbf{L}$ and there is no $c \in L$ with $a < c < b$, then we say that *a is covered by b*, and we write $a \prec b$. In this situation we also say that *b covers a* and write $b \succ a$. In addition we say that b is an *upper cover* of a and that a is a *lower cover* of b.

Definition 1.1.11. If $a \leq b$, then we let b/a denote the interval $\{x : a \leq x \leq b\}$.

Definition 1.1.12. An *order ideal* in an ordered set $\mathbf{P}$ is a subset S of P such that whenever $a \leq b$ and $b \in S$ then $a \in S$. An *order filter* is defined dually. A subset I of a lattice $\mathbf{L}$ is called an *ideal* if it is an order ideal and it is closed under finite joins. A *filter* is defined dually. If $S \subseteq L$, then the ideal generated by S, the smallest ideal containing S, consists of all elements $a \in L$ such that $a \leq s_1 \vee \cdots \vee s_k$ for some $s_1, \ldots, s_k \in S$.

1.2 Day's Doubling Construction

We now give Alan Day's useful construction, which he introduced in [2].

Definition 1.2.1. Let L be a lattice. A subset C of L is *convex* if whenever a and b are in C and $a \leq c \leq b$, then $c \in C$.

Definition 1.2.2. Let C be a convex subset of a lattice $\mathbf{L}$ and let $L[C]$ be the disjoint union $(L - C) \cup (C \times 2)$. Order $L[C]$ by $x \leq y$ if one of the following holds.

1. $x, y \in L - C$ and $x \leq y$ holds in $\mathbf{L}$,
2. $x, y \in C \times 2$ and $x \leq y$ holds in $C \times 2$,
3. $x \in L - C$, $y = (u, i) \in C \times 2$, and $x \leq u$ holds in $\mathbf{L}$,
4. $x = (v, i) \in C \times 2$, $y \in L - C$, and $v \leq y$ holds in $\mathbf{L}$.

There is a natural map λ from $L[C]$ back onto L given by

$$\lambda(x) = \begin{cases} x & \text{if } x \in L - C \\ v & \text{if } x = (v, i) \in C \times 2. \end{cases} \tag{1.1}$$

The next theorem shows that, under this order, $L[C]$ is a lattice, denoted $\mathbf{L}[C]$.

Theorem 1.2.3. *Let C be a convex subset of a lattice $\mathbf{L}$. Then $\mathbf{L}[C]$ is a lattice and $\lambda\colon \mathbf{L}[C] \to \mathbf{L}$ is a lattice epimorphism.*

Proof. Routine calculations show that $\mathbf{L}[C]$ is a partially ordered set. Let $x_i \in L - C$ for $i = 1, \ldots, n$ and let $(u_j, k_j) \in C \times 2$ for $j = 1, \ldots, m$. Let $v = \bigvee x_i \vee \bigvee u_j$ in $\mathbf{L}$ and let $k = \bigvee k_j$ in $\mathbf{2}$; if $m = 0$, then let $k = 0$. Then in $\mathbf{L}[C]$,

$$x_1 \vee \cdots \vee x_n \vee (u_1, k_1) \vee \cdots \vee (u_m, k_m) = \begin{cases} v & \text{if } v \in L - C, \\ (v, k) & \text{if } v \in C. \end{cases} \tag{1.2}$$

To see this, let y be the right side of the above equation, i.e., let $y = v$ if $v \in L - C$ and $y = (v, k)$ if $v \in C$. It is easy to check that y is an upper bound for each x_i and each (u_j, k_j). Let z be another upper bound. First, suppose $z = (a, r)$ where $a \in C$. Since z is an upper bound, it follows from the definition of the ordering that $v \leq a$ and $k \leq r$, and this implies $y \leq z$. Thus, in this case, y is the least upper bound. Next, suppose $z \notin C$. Then $v \leq z$ and so $y \leq z$, which again makes y the least upper bound. The formula for meets is of course dual. Thus $\mathbf{L}[C]$ is a lattice.
Since

$$\begin{aligned} \lambda(x_1 \vee \cdots \vee x_n \vee (u_1, k_1) \vee \cdots \vee (u_m, k_m)) &= \begin{cases} \lambda(v) & \text{if } v \in L - C, \\ \lambda(v, k) & \text{if } v \in C. \end{cases} \\ &= v = x_1 \vee \cdots \vee x_n \vee u_1 \vee \cdots \vee u_m \\ &= \lambda(x_1) \vee \cdots \vee \lambda(x_n) \vee \lambda(u_1, k_1) \vee \cdots \vee \lambda(u_m, k_m) \end{aligned}$$

holds as well as its dual, λ is a homomorphism which is clearly onto $\mathbf{L}$. □

Corollary 1.2.4. *Let L be a lattice generated by a set X, and let C be a convex subset of L with $X \cap C = \emptyset$. Let s be a term with variables in X whose evaluation in L is v. Then the evaluation of s in $L[C]$ is v if $v \notin C$, and either $(v, 0)$ or $(v, 1)$ otherwise.*

Proof. We induct on the complexity of s: If s has complexity 0, then $s \in X$ and so $s^{\mathbf{L}[C]} = v$ since $X \cap C = \emptyset$. Now suppose that s has complexity greater than 0 and any term with complexity less than s whose evaluation in L is w evaluates in $L[C]$ to w if $w \notin C$, and either $(w, 0)$ or $(w, 1)$ otherwise. WLOG, assume $s = s_1 \vee \cdots \vee s_n \vee t_1 \vee \cdots \vee t_m$, where $s_i^{\mathbf{L}} \notin C$ for $i = 1, \ldots, n$ and $t_j^{\mathbf{L}} \in C$ for $j = 1, \ldots, m$. Since s_i and t_j all have complexity less than s for $1 \leq i \leq n$ and $1 \leq j \leq m$, $s_i^{\mathbf{L}[C]} = s_i^{\mathbf{L}}$ for $i = 1, \ldots, n$ and $t_j^{\mathbf{L}[C]} = (t_j^{\mathbf{L}}, k_j)$, for some $k_j \in 2$, for $j = 1, \ldots, m$. Referring to (1.2), we see that $s^{\mathbf{L}[C]}$ is v if $v \notin C$, and either $(v, 0)$ or $(v, 1)$ if $v \in C$, as desired. $\square$

CHAPTER 2
FREE LATTICES

We begin this chapter by defining lattice terms and free lattices, as well as discussing the connection between the two. We then develop some results which culminate in Whitman's solution to the word problem for free lattices. Finally, we conclude by defining the canonical form of a term and of an element in a free lattice, and discuss an important property of the latter. The reader may consult [5] for further details.

2.1 Introduction

Definition 2.1.1. We define *lattice terms* over a set X, and their associated *lengths* (or *ranks*), in the following way:
Each element of X is a term of length (or rank) 1. Terms of length (or rank) 1 are called *variables*. If $t_1, \ldots, t_n$ are terms of lengths (or ranks) $k_1, \ldots, k_n$, then $(t_1 \vee \cdots \vee t_n)$ and $(t_1 \wedge \cdots \wedge t_n)$ are terms with length (or rank) $1 + k_1 + \cdots + k_n$.

Remark 2.1.2. When we write a term we usually omit the outermost parentheses. Notice that if x, y, and $z \in X$ then

$$x \vee y \vee z \qquad x \vee (y \vee z) \qquad (x \vee y) \vee z$$

are all terms (which always represent the same element when interpreted in any lattice) but the length of $x \vee y \vee z$ is 4, while the other two terms are both of length 5. Thus our length function gives preference to the first expression, i.e., it gives preference to expressions where unnecessary parentheses are removed. Also note that the length of a term (when it is written with the outside parentheses) is the number of variables, counting repetitions, plus the number of pairs of parentheses (i.e., the number of left parentheses).

Definition 2.1.3. The *complexity*, or *depth*, of a term t the depth of its term tree; that is, t has depth 0 if $t \in X$, and if $t = t_1 \vee \cdots \vee t_n$ or $t = t_1 \wedge \cdots \wedge t_n$, where $n > 1$, then the complexity of t is one more than the maximum of the complexities of $t_1, \ldots, t_n$.

Definition 2.1.4. By the phrase '$t(x_1, \ldots, x_n)$ is a term' we mean that t is a term and $x_1, \ldots, x_n$ are (pairwise) distinct variables including all variables occurring in t. If $t(x_1, \ldots, x_n)$ is a term and $\mathbf{L}$ is a lattice, then $t^{\mathbf{L}}$ denotes the interpretation of t in $\mathbf{L}$, i.e., the induced n–ary operation on $\mathbf{L}$. If $a_1, \ldots, a_n \in L$, we will usually abbreviate $t^{\mathbf{L}}(a_1, \ldots, a_n)$ by $t(a_1, \ldots, a_n)$. Very often in the study of free lattices, we will be considering a lattice $\mathbf{L}$ with a specific generating set $\{x_1, \ldots, x_n\}$. In this case we will use $t^{\mathbf{L}}$ to denote $t^{\mathbf{L}}(x_1, \ldots, x_n)$.

Definition 2.1.5. If $s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ are terms and $\mathbf{L}$ is a lattice in which $s^{\mathbf{L}} = t^{\mathbf{L}}$ as functions, then we say the *equation* $s \approx t$ *holds* in $\mathbf{L}$.

Definition 2.1.6. Let $\mathbf{F}$ be a lattice and $X \subseteq F$. We say that $\mathbf{F}$ is *freely generated by* X if X generates $\mathbf{F}$ and every map from X into any lattice $\mathbf{L}$ extends to a lattice homomorphism of $\mathbf{F}$ into $\mathbf{L}$.

Since X generates $\mathbf{F}$, such an extension is unique. If follows easily that if $\mathbf{F}_1$ is freely generated by X_1 and $\mathbf{F}_2$ is freely generated by X_2 and $|X_1| = |X_2|$, then $\mathbf{F}_1$ and $\mathbf{F}_2$ are isomorphic Thus, if X is a set, a lattice freely generated by X is unique up to isomorphism. We will see that such a lattice always exists. It is referred to as the *free lattice over* X and is denoted $\mathbf{FL}(X)$.

If n is a cardinal number, $\mathbf{FL}(n)$ denotes a free lattice whose free generating set has size n.

To construct $\mathbf{FL}(X)$, let $\mathrm{T}(X)$ be the set of all terms over X. $\mathrm{T}(X)$ can be viewed as an algebra with two binary operations. Define an equivalence relation $\sim$ on $\mathrm{T}(X)$ by $s \sim t$ if and only if the equation $s \approx t$ holds in all lattices. It is not difficult to verify that $\sim$ restricted to X is the equality relation, that $\sim$ is a congruence relation on $\mathrm{T}(X)$, and that $\mathrm{T}(X)/\sim$ is a lattice freely generated by X, provided we identify each element of $x \in X$ with its $\sim$–class. This is the standard construction of free algebras.

This construction is much more useful if we have an effective procedure which determines, for arbitrary lattice terms s and t, if $s \sim t$. The problem of finding such a procedure is informally known as the *word problem* for free lattices.

Definition 2.1.7. If $w \in \mathbf{FL}(X)$, then w is an equivalence class of terms. Each term of this class is said to *represent* w and is called a *representative* of w. More generally, if $\mathbf{L}$ is a lattice generated by a set X, we say that a term $t \in \mathrm{T}(X)$ represents $a \in L$ if $t^{\mathbf{L}} = a$.

Definition 2.1.8. A *variety* is a class of algebras (such as lattices) closed under the formation of homomorphic images, subalgebras, and direct products. A variety is called *nontrivial* if it contains an algebra with more than one element.

By Birkhoff's Theorem (see, for example, [1]), varieties are *equational classes*, i.e., they are defined by the equations they satisfy. If $\mathscr{V}$ is a variety of lattices and X is a set, we denote the free algebra in $\mathscr{V}$ by $\mathbf{F}_{\mathscr{V}}(X)$ and refer to it as the *relatively free lattice* in $\mathscr{V}$ over X. If $\mathscr{L}$ is the variety of all lattices, then, in this notation, $\mathbf{F}_{\mathscr{L}}(X) = \mathbf{FL}(X)$. However, because of tradition, we will use $\mathbf{FL}(X)$ to denote the free lattice. The relatively free lattice $\mathbf{F}_{\mathscr{V}}(X)$ can be constructed in the same way as $\mathbf{FL}(X)$.

Notice that every nontrivial variety of lattices contains the two element lattice, which is denoted by $\mathbf{2}$.

Lemma 2.1.9. *Let $\mathscr{V}$ be a nontrivial variety of lattices and let $\boldsymbol{F}_{\mathscr{V}}(X)$ be the relatively free lattice in $\mathscr{V}$ over X. Then*

$$\bigwedge S \leq \bigvee T \text{ implies } S \cap T \neq \emptyset \text{ for each pair of finite subsets } S,\, T \subseteq X. \tag{†}$$

Proof. We shall prove the contrapositive of (†): Suppose that S and T are finite, disjoint subsets of X. As noted above, $\mathbf{2} \in \mathscr{V}$. Let f be the map from X to $\mathbf{2} = \{0,1\}$ which sends each $x \in S$ to 1 and all other x's to 0. By the defining property of free algebras, f can be extended to a homomorphism from $\mathbf{F}_{\mathscr{V}}(X)$ onto $\mathbf{2}$, which we also denote by f. Then $f(\bigwedge S) = 1 \nleq 0 = f(\bigvee T)$. Since f must be order-preserving, this implies that $\bigwedge S \nleq \bigvee T$, as desired. □

Lemma 2.1.10. *Let a be an element of a lattice $\boldsymbol{L}$ generated by a set X. Suppose that for every finite subset S of X,*

$$a \leq \bigvee S \quad \text{implies} \quad a \leq s \quad \text{for some } s \in S. \tag{‡}$$

Then (‡) holds for all finite subsets of L.

Proof. Let $\mathscr{K}$ be the collection of all sets U with $X \subseteq U \subseteq L$ such that (‡) holds for every finite subset S of U. We shall use Zorn's Lemma to show that $\mathscr{K}$ contains a maximal element:
First, $\mathscr{K}$ is a partially ordered set with respect to set inclusion, and by hypothesis, $X \in \mathscr{K}$. Let C be a nonempty chain in $\mathscr{K}$. We argue that $\bigcup C \in \mathscr{K}$: If S is a finite subset of $\bigcup C$, then as C is a chain and S is finite, there exists $U_S \in C$ such that $S \subseteq U_S$. Thus, (‡) holds for S, and so $\bigcup C \in \mathscr{K}$. Therefore, by Zorn's Lemma, there exists a maximal $U \in \mathscr{K}$.
Now, let $u, v \in U$. Then $U \cup \{u \wedge v\} \in \mathscr{K}$. To see this, suppose that $a \leq \bigvee S \vee (u \wedge v)$ for some finite $S \subseteq U$, but $a \nleq s$ for all $s \in S$. Then, since $a \leq \bigvee S \vee u$, (‡) implies $a \leq u$. Similarly, $a \leq v$ and so $a \leq u \wedge v$. But, as U is maximal in $\mathscr{K}$, $U = U \cup \{u \wedge v\}$, i.e. $u \wedge v \in U$.
Finally, it is trivial that $U \cup \{u \vee v\} \in \mathscr{K}$. So again by maximality of U, $U \cup \{u \vee v\} = U$, i.e. $u \vee v \in U$. Therefore, as U is a sublattice of L containing X, $L = U \in \mathscr{K}$. □

Lemma 2.1.11. *Let $\boldsymbol{L}$ be a lattice generated by a set X and let $a \in L$.*
Then

1. *If a is join prime, then $a = \bigwedge S$ for some finite subset $S \subseteq X$,*

2. *If a is meet prime, then $a = \bigvee S$ for some finite subset $S \subseteq X$.*

If X satisfies condition (†) above, then

3. *For every finite, nonempty subset $S \subseteq X$, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime,*

4. *If X is the disjoint union of Y and Z, and F is the filter of $\boldsymbol{L}$ generated by Y and I is the ideal generated by Z, then L is the disjoint union of F and I.*

Proof. Since **L** is generated by X, every element of L can be represented by a term with variables in X. It follows from this and an easy induction on the length of such a term that *if $X = Y \cup Z$, then $L = F \cup I$ where F is the filter generated by Y and I is the ideal generated by Z.*

To prove 1., let F be the filter generated by $Y = \{x \in X : a \leq x\}$ and let I be the ideal generated by $Z = \{x \in X : a \nleq x\}$. Since a is join prime, $a \notin I$, for otherwise a would be below some join of elements in Z and hence below one of the joinands, contradicting the fact that a is not below any element of Z. So, by the above observation, $a \in F$. This implies that $a \geq \bigwedge S$, for some finite $S \subseteq Y$. But every element of Y is above a; hence $a = \bigwedge S$, as desired. Of course 2. is proved dually.

Let T be a finite, nonempty subset of X and let $a = \bigwedge T$. If S is a finite subset of X such that $\bigwedge T = a \leq \bigvee S$, then by condition (†) there exists $s \in T \cap S$. In particular, $a = \bigwedge T \leq s$ for some $s \in S$. Thus, condition (‡) holds for all finite subsets S of X and hence $a = \bigwedge T$ is join prime by Lemma 2.1.10. By a similar argument, using the dual of Lemma 2.1.10, $\bigvee T$ is meet prime. Therefore, 3. holds.

For 4., we have already observed that $L = F \cup I$. If $F \cap I$ is nonempty, there would be finite subsets $S \subseteq Y \subseteq X$ and $T \subseteq Z \subseteq X$ with $\bigwedge S \leq \bigvee T$. But since (†) holds for X, $S \cap T \neq \emptyset$, contrary to $Y \cap Z \neq \emptyset$. □

Corollary 2.1.12. *Let $\mathscr{V}$ be a nontrivial variety of lattices and let $\boldsymbol{F}_{\mathscr{V}}(X)$ be the relatively free lattice in $\mathscr{V}$ over X. For each finite, nonempty subset S of X, $\bigwedge S$ is join prime and $\bigvee S$ is meet prime. In particular, every $x \in X$ is both join and meet prime. Moreover, if $x \leq y$ for x and $y \in X$, then $x = y$.*

Proof. By Lemma 2.1.9, X satisfies (†) and so the first assertion follows from 3. of Lemma 2.1.11. If we let $S = \{x\}$ for $x \in X$, then it immediately follows that $x = \bigwedge S$ is join prime and $x = \bigvee S$ is meet prime. Finally, if $x, y \in X$ such that $x \leq y$, $S = \{x\}$, and $T = \{y\}$, then $\bigwedge S \leq \bigvee T$ and so $\{x\} \cap \{y\} \neq \emptyset$ by Lemma 2.1.9, i.e. $x = y$ as desired. □

Corollary 2.1.13. *If **L** is a lattice generated by a set X which satisfies condition (†), then the following hold.*

1. *If Y generates **L** then $X \subseteq Y$.*
2. *Every automorphism of **L** is induced by a permutation of X.*

In particular, these statements hold for the relatively free lattice, $\boldsymbol{F}_{\mathscr{V}}(X)$, for any nontrivial variety of lattices $\mathscr{V}$. Moreover, the automorphism group of $\boldsymbol{F}_{\mathscr{V}}(X)$ is isomorphic to the full symmetric group on X.

Proof. By Lemma 2.1.11 3., each $x \in X$ is both join and meet prime, and hence both join and meet irreducible. Fix $x \in X$, and let t be a term representing x in the sublattice generated by

Y. We induct on the complexity of t to show that $x \in Y$: If t has complexity 0, then $x \in Y$. If $t = t_1 \vee \cdots \vee t_n$ and if an element of X can be represented by a term with complexity smaller than t is forced to also be in Y, then x being join irreducible allows x to be represented in the sublattice generated by Y by one of the joinands of t. Thus, $x \in Y$ by our inductive hypothesis. A dual argument will prove that $t = t_1 \wedge \cdots \wedge t_n$ also forces $x \in Y$. Therefore, 1. holds.
Now, let f be an automorphism of $\mathbf{L}$. Since $f(X)$ must generate $\mathbf{L}$, by 1. above $X \subseteq f(X)$. We now prove that (†) holds when X is replaced with $f(X)$:
Let S and T be finite subsets of $f(X)$ such that $\bigwedge S \leq \bigvee T$. Since f is an isomorphism, $f^{-1}(\bigwedge S) \leq f^{-1}(\bigvee T)$, that is $\bigwedge f^{-1}(S) \leq \bigvee f^{-1}(T)$, where $f^{-1}(S)$ and $f^{-1}(T)$ are finite subsets of X. Since X satisfies (†), $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T) \neq \emptyset$, and thus $S \cap T \neq \emptyset$.
Thus, $f(X)$ satisfies (†) and is a generating set for $\mathbf{L}$, so by a similar argument 1. must hold with $f(X)$ in place of X. Therefore, since X generates $\mathbf{L}$, $f(X) \subseteq X$, and hence f is a permutation of X. Since f is determined by where it sends X, we have proven 2.
Furthermore, by Lemma 2.1.9, 1. and 2. hold for $\mathbf{F}_{\mathscr{V}}(X)$. Finally, since every permutation of X induces an automorphism of $\mathbf{F}_{\mathscr{V}}(X)$, 2. gives us that the automorphism group of $\mathbf{F}_{\mathscr{V}}(X)$ is isomorphic to the full symmetric group on X. □

Theorem 2.1.14. *The free lattice* $\boldsymbol{FL}(X)$ *satisfies the following condition:*

(W) *If* $v = v_1 \wedge \cdots \wedge v_r \leq u_1 \vee \cdots \vee u_s = u$, *then either* $v_i \leq u$ *for some* i, *or* $v \leq u_j$ *for some* j.

Proof. Suppose $v = v_1 \wedge \cdots \wedge v_r \leq u_1 \vee \cdots \vee u_s = u$ but that $v_i \nleq u$ and $v \nleq u_j$ for all i and all j. If $v \leq x \leq u$ for some $x \in X$, then since x is meet prime, $v_i \leq x \leq u$ for some i, contrary to our assumption. Let I be the interval u/v and let $\mathbf{FL}(X)[I]$ be the lattice obtained by doubling I. By the above remarks, none of the generators is doubled. This implies that X is a subset of $\mathbf{FL}(X)[I]$ and so the identity map on X extends to a homomorphism $f : \mathbf{FL}(X) \to \mathbf{FL}(X)[I]$. Since $x \notin I$ for $x \in X$, $\lambda(x) = x$ for $x \in X$, where λ is the epimorphism defined by (1.1). Hence $\lambda(f(w)) = w$ for all $w \in \mathbf{FL}(X)$ and this implies $f(w) = w$ if $w \notin I$. Thus it follows from (1.2) and its dual that

$$\begin{aligned} f(v) = f(v_1) \wedge \cdots \wedge f(v_r) = v_1 \wedge \cdots \wedge v_r &= \left(v, \bigwedge \emptyset\right) = (v, 1) \\ &\nleq (u, 0) = \left(u, \bigvee \emptyset\right) = u_1 \vee \cdots \vee u_s \\ &= f(u_1) \vee \cdots \vee f(u_s) = f(u), \end{aligned}$$

contradicting the fact that $v \leq u$ and f is an order-preserving map. □

Definition 2.1.15. The condition (W) is known as *Whitman's condition.*

Remark 2.1.16. Note that Day's doubling is a procedure for correcting (W)–failures.

Corollary 2.1.17. *Every sublattice of a free lattice satisfies (W). Every element of a lattice which satisfies (W) is either join or meet irreducible.*

Theorems 2.1.12 and 2.1.14 combine to give a recursive procedure for deciding, for terms s and t, if $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$ known as Whitman's solution to the word problem. To test if $s \sim t$, the algorithm is used twice to check if both $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$ and $t^{\mathbf{FL}(X)} \leq s^{\mathbf{FL}(X)}$ hold. The following appears in [10].

Theorem 2.1.18. *If $s = s(x_1, \ldots, x_n)$ and $t = t(x_1, \ldots, x_n)$ are terms and $x_1, \ldots, x_n \in X$, then the truth of*

$$s^{\boldsymbol{FL}(X)} \leq t^{\boldsymbol{FL}(X)} \tag{$*$}$$

can be determined by applying the following rules.

1. *If $s = x_i$ and $t = x_j$, then $(*)$ holds if and only $x_i = x_j$.*
2. *If $s = s_1 \vee \cdots \vee s_k$ is a formal join then $(*)$ holds if and only if $s_i^{\boldsymbol{FL}(X)} \leq t^{\boldsymbol{FL}(X)}$ holds for all i.*
3. *If $t = t_1 \wedge \cdots \wedge t_k$ is a formal meet then $(*)$ holds if and only if $s^{\boldsymbol{FL}(X)} \leq t_i^{\boldsymbol{FL}(X)}$ holds for all i.*
4. *If $s = x_i$ and $t = t_1 \vee \cdots \vee t_k$ is a formal join, then $(*)$ holds if and only if $x_i \leq t_j^{\boldsymbol{FL}(X)}$ for some j.*
5. *If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and $t = x_i$, then $(*)$ holds if and only if $s_j^{\boldsymbol{FL}(X)} \leq x_i$ for some j.*
6. *If $s = s_1 \wedge \cdots \wedge s_k$ is a formal meet and $t = t_1 \vee \cdots \vee t_m$ is a formal join, then $(*)$ holds if and only if $s_i^{\boldsymbol{FL}(X)} \leq t^{\boldsymbol{FL}(X)}$ holds for some i, or $s^{\boldsymbol{FL}(X)} \leq t_j^{\boldsymbol{FL}(X)}$ holds for some j.*

Proof. Conditions 1., 4., and 5. hold by Corollary 2.1.12, while 2. and 3. are trivial. Theorem 2.1.14 shows that free lattices satisfy 6. It is easy to see that all possibilities are covered by 1.–6. and that each of these leads to a genuine reduction (except for 1., which gives the answer directly). □

2.2 Canonical Form in Free Lattices

In this section we show that each element w of a free lattice has a term of least rank representing it, unique up to commutativity.[1] This term is called the *canonical form* of w. The phrase "unique up to commutativity" can be made precise by defining *equivalent under commutativity* to be the equivalence relation, $s \equiv t$, given by recursively applying the following rules.

[1] Usually the canonical form is described as "unique up to commutativity and associativity." However, our definition of lattice terms and their ranks, given at the beginning of the previous section, imply that a term of minimal rank can be associated in only one way.

1. $s, t \in X$ and $s = t$.

2. $s = s_1 \vee \cdots \vee s_n$ and $t = t_1 \vee \cdots \vee t_n$ and there is a permutation σ of $\{1, \ldots, n\}$ such that $s_i \equiv t_{\sigma(i)}$ for all i.

3. The dual of 2. holds.

Theorem 2.2.6 below shows that if two terms both represent the same element of $\mathbf{FL}(X)$ and both have minimal rank among all such representatives, then they are equivalent under commutativity. We define $s \leq t$ for terms s and t to mean $s^{\mathbf{FL}(X)} \leq t^{\mathbf{FL}(X)}$. Note that this is only a quasiorder.

Definition 2.2.1. Let $\mathbf{L}$ be a lattice and let A and B be finite subsets of L. We say that A *join refines* B and we write $A \ll B$ if for each $a \in A$ there is a $b \in B$ with $a \leq b$. The dual notion is called *meet refinement* and is denoted $A \gg B$.

Remark 2.2.2. Note that $A \ll B$ does not imply $B \gg A$.

Lemma 2.2.3. *The join refinement relation has the following properties.*

1. *$A \ll B$ implies $\bigvee A \leq \bigvee B$.*

2. *The relation $\ll$ is a quasiorder on the finite subsets of L.*

3. *If $A \subseteq B$ then $A \ll B$.*

4. *If A is an antichain, $A \ll B$, and $B \ll A$, then $A \subseteq B$.*

5. *If A and B are antichains with $A \ll B$ and $B \ll A$, then $A = B$.*

6. *If $A \ll B$ and $B \ll A$, then A and B have the same set of maximal elements.*

Proof. 1. Since $A \ll B$, for each $a \in A$ there exists $b \in B$ such that $a \leq b$. So, for each $a \in A$, $a \leq \bigvee B$. Thus, $\bigvee A \leq \bigvee B$.

2. Let A, B be finite subsets of L. Since $a \leq a$ for all $a \in A$, $A \ll A$. Now assume $A \ll B$ and $B \ll C$, and let $a \in A$. If $b \in B$ such that $a \leq b$, and $c \in C$ such that $b \leq c$, then $a \leq c$. Thus, $A \ll C$.

3. If $a \in A$, then $a \in B$ such that $a \leq a$. Therefore, $A \ll B$.

4. Let $a \in A$. Since $A \ll B$, there exists $b \in B$ such that $a \leq b$. But since $B \ll A$, there exists $a_1 \in A$ such that $b \leq a_1$. But since $a \leq a_1$ and A is an antichain, $a = a_1 = b \in B$.

5. Since A is an antichain, $A \ll B$, and $B \ll A$, $A \subseteq B$ by 4. Similarly, since B is an antichain, $A \ll B$, and $B \ll A$, $B \subseteq A$ by 4. above. Therefore, $A = B$.

6. Let a be a maximal element of A. First, we argue that $a \in B$:
   Since $A \ll B$, there exists $b \in B$ such that $a \leq b$. But since $B \ll A$, there exists $a_1 \in A$ such that $b \leq a_1$. So as a is a maximal element of A, $a = a_1 = b \in B$.
   Next, we argue that a must be a maximal element of B:
   Let $b \in B$ such that $a \leq b$. Since $B \ll A$, there exists $a_1 \in A$ such that $b \leq a_1$. So since a is a maximal element of A, $a = a_1 = b$.
   A similar argument shows that a maximal element of B will be a maximal element of A, and hence A and B have the same set of maximal elements.

□

Remark 2.2.4. We use the term "join refinement" because if $u = \bigvee A = \bigvee B$ and $A \ll B$, then $u = \bigvee A$ is a better join representation of u than $u = \bigvee B$ in that its elements are further down in the lattice.

Lemma 2.2.5. *Let $t = t_1 \vee \cdots \vee t_n$, with $n > 1$, be a term such that*

1. *Each t_i is either in X or formally a meet,*
2. *If $t_i = \bigwedge t_{ij}$ then $t_{ij} \not\leq t$ for all j.*

If $s = s_1 \vee \cdots \vee s_m$ and $s \sim t$, then $\{t_1, \dots t_n\} \ll \{s_1, \dots, s_m\}$.

Proof. For each i we have $t_i \leq s_1 \vee \cdots \vee s_m$. Applying (W) if t_i is formally a meet and using join primality if $t_i \in X$, we conclude that either $t_i \leq s_j$ for some j, or $t_i = \bigwedge t_{ij}$ and $t_{ij} \leq s$ for some j. However, since $s \sim t$, the second case would imply $t_{ij} \leq t$, contrary to assumption 2. Hence in all cases there is a j such that $t_i \leq s_j$. Thus $\{t_1, \dots, t_n\} \ll \{s_1, \dots, s_m\}$. □

Theorem 2.2.6. *For each $w \in \boldsymbol{FL}(X)$ there is a term of minimal rank representing w, unique up to commutativity. This term is called the canonical form of w.*

Proof. Suppose that s and t are both terms of minimal rank that represent the same element w in $\mathbf{FL}(X)$. If either s or t is in X, then clearly $s = t$.
Suppose that $t = t_1 \vee \cdots \vee t_n$ and $s = s_1 \vee \cdots \vee s_m$. If some t_i is formally a join, we could lower the rank of t by removing the parentheses around t_i. Thus each t_i is not formally a join. If there is a t_i such that $t_i = \bigwedge t_{ij}$ and $t_{ij} \leq t$ for some j, then $t_i \leq t_{ij} \leq t$. In this case we could replace t_i with t_{ij} in t, producing a shorter term still representing w, which violates the minimality of the term t. Thus t satisfies the hypotheses of Lemma 2.2.5, whence $\{t_1, \dots, t_n\} \ll \{s_1, \dots, s_m\}$. By symmetry, $\{s_1, \dots, s_m\} \ll \{t_1, \dots, t_n\}$. Since both are antichains (by the minimality) they represent the same set of elements of $\mathbf{FL}(X)$. Thus $m = n$ and after renumbering $s_i \sim t_i$. Now by induction s_i and t_i are the same up to commutativity.
If $t = t_1 \vee \cdots \vee t_n$ and $s = s_1 \wedge \cdots \wedge s_m$, then (W) implies that either $t_i \sim t$ for some i or $s_j \sim s$

for some j, violating the minimality.

The remaining cases can be handled by duality. $\square$

Definition 2.2.7. A term is in *canonical form* if it is the canonical form of the element it represents.

Theorem 2.2.8. *A term $t = t_1 \vee \cdots \vee t_n$, with $n > 1$, is in canonical form if and only if*

1. *Each t_i is either in X or formally a meet,*
2. *Each t_i is in canonical form,*
3. *$t_i \not\leq t_j$ for all $i \neq j$ (the t_i's form an antichain),*
4. *If $t_i = \bigwedge t_{ij}$ then $t_{ij} \not\leq t$ for all j.*

A term $t = t_1 \wedge \cdots \wedge t_n$, with $n > 1$, is in canonical form if and only if the duals of the above conditions hold. A term $x \in X$ is always in canonical form.

Proof. All of these conditions are clearly necessary. For the converse we need to show that if t satisfies 1.–4. then it has minimal rank among the terms which represent the same element of $\mathbf{FL}(X)$ as t. Suppose that $s = s_1 \vee \cdots \vee s_m$ is a term of minimal rank representing the same element of $\mathbf{FL}(X)$ as t. Now using 1. and 4. for t, and the arguments of the last theorem for s, Lemma 2.2.5 yields

$$\{t_1, \ldots, t_n\} \ll \{s_1, \ldots, s_m\} \text{ and}$$
$$\{s_1, \ldots, s_m\} \ll \{t_1, \ldots, t_n\}.$$

Since both are antichains, we have that $n = m$ and after renumbering $s_i \sim t_i$, $i = 1, \ldots, n$. The proof can now easily be completed with the aid of induction. $\square$

Definition 2.2.9. Let $w \in \mathbf{FL}(X)$ be join reducible and suppose $t = t_1 \vee \cdots \vee t_n$ (with $n > 1$) is the canonical form of w. Let $w_i = t_i^{\mathbf{FL}(X)}$. Then $\{w_1, \ldots, w_n\}$ are called the *canonical joinands* of w. We also say $w = w_1 \vee \cdots \vee w_n$ *canonically* and that $w_1 \vee \cdots \vee w_n$ is the *canonical join representation* (or *canonical join expression*) of w. If w is join irreducible, we define the canonical joinands of w to be the set $\{w\}$. Of course the *canonical meet representation* and *canonical meetands* of an element in a free lattice are defined dually.

Definition 2.2.10. A join representation $a = a_1 \vee \cdots \vee a_n$ in an arbitrary lattice is said to be a *minimal (nonrefinable) join representation* if $a = b_1 \vee \cdots \vee b_m$ and $\{b_1, \ldots, b_m\} \ll \{a_1, \ldots, a_n\}$ imply $\{a_1, \ldots, a_n\} \subseteq \{b_1, \ldots, b_m\}$. Equivalently, a join representation $a = a_1 \vee \cdots \vee a_n$ is minimal if it is an antichain and nonrefinable, in the sense that whenever $a = b_1 \vee \cdots \vee b_m$ and $\{b_1, \ldots, b_m\} \ll \{a_1, \ldots, a_n\}$, then $\{a_1, \ldots, a_n\} \ll \{b_1, \ldots, b_m\}$.

Theorem 2.2.11. *Let* $w = w_1 \vee \cdots \vee w_n$ *canonically in* $\boldsymbol{FL}(X)$*. If also* $w = u_1 \vee \cdots \vee u_m$*, then*

$$\{w_1, \ldots, w_n\} \ll \{u_1, \ldots, u_m\}.$$

Thus $w = w_1 \vee \cdots \vee w_n$ *is the unique minimal join representation of* w*.*

Proof. Interpreting the terms of Lemma 2.2.5 in $\mathbf{FL}(X)$ immediately gives this result, since $w_1, \ldots, w_n$ is an antichain. □

CHAPTER 3
FINITELY PRESENTED LATTICES

We begin this chapter with giving important definitions for the theory of finitely presented lattices, followed by Dean's solution to the word problem for finitely presented lattices. We then define the canonical form in finitely presented lattices and give an important result characterizing the nonrefinable join representations of an element in a finitely presented lattice. These results can all be found in [7]. We finish the chapter by giving Freese and Nation's solution to the generalized word problem for finitely presented lattices, as well some important definitions for our own solution appearing in the following chapter.

3.1 Introduction

Definition 3.1.1. Let X be a set (of variables). A lattice *relation* is a formal expression of the form $s \approx t$, where s and t are terms with variables from X. We also consider $s \leq t$ to be a relation, which in lattices is obviously equivalent to $s \approx s \wedge t$. A *presentation* is a pair (X, R) where X is a set and R is a set of relations with variables from X. We say that (X, R) is a *finite presentation* if both X and R are finite.
A lattice F is the *lattice finitely presented by* (X, R) if there is a map $\varphi\colon X \to F$ such that F is generated by $\varphi(X)$, F satisfies the relations R under the substitution $x \mapsto \varphi(x)$, for $x \in X$, and F satisfies the following mapping property: if L is a lattice and $\psi\colon X \to L$ is a map such that L satisfies R under the substitution $x \mapsto \psi(x)$, then there is a homomorphism $f\colon F \to L$ such that $f\varphi(x) = \psi(x)$ for all $x \in X$.

Remark 3.1.2. Using the definition it is easy to see that the lattice finitely presented by (X, R) is unique up to isomorphism. This lattice is denoted $\text{Free}(X, R)$.

Remark 3.1.3. We verify that $\text{Free}(X)/\theta_R$, where θ_R is the congruence generated by R, is $\text{Free}(X, R)$, thereby showing the existence of $\text{Free}(X, R)$: Let $\eta_X : X \hookrightarrow \text{Free}(X)$ be the natural inclusion mapping and $q_{\theta_R} : \text{Free}(X) \to \text{Free}(X)/\theta_R$ be the natural projection mapping. Then, following with the notation of the definition above, we define $\varphi = q_{\theta_R}\eta_X$. Clearly, since $\text{Free}(X)$ is generated by X, $\text{Free}(X)/\theta_R$ is generated by $\varphi(X)$. Furthermore, since θ_R is the congruence generated by R, $\text{Free}(X)/\theta_R$ satisfies the relations R under the substitution $x \mapsto \varphi(x)$, for $x \in X$. Finally, let L be a lattice and $\psi\colon X \to L$ be a map such that L satisfies R under the substitution $x \mapsto \psi(x)$. Using the universal mapping property of the free lattice $\text{Free}(X)$, there exists a homomorphism $h : \text{Free}(X) \to L$ such that $\psi = h\eta_X$. Since $h : \text{Free}(X) \to L$ and $q_{\theta_R} : \text{Free}(X) \to \text{Free}(X)/\theta_R$ are homomorphisms, q_{θ_R} is onto, and $\theta_R = \ker(q_{\theta_R}) \subseteq \ker(h)$ (as L satisfies R under the substitution $x \mapsto \psi(x)$), there a homomorphism $f : \text{Free}(X)/\theta_R \to L$ such that $h = fq_{\theta_R}$ by the Second Homomorphism Theorem (see [1]). Notice also that $f\varphi = fq_{\theta_R}\eta_X = h\eta_X = \psi$, as desired.

Definition 3.1.4. A *partially defined lattice* is a partially ordered set $(P, \leq)$ together with two partial functions, $\bigvee$ and $\bigwedge$, from subsets of P into P such that if $p = \bigvee S$ then p is the least upper bound of S in $(P, \leq)$, and dually. We use $(P, \leq, \bigvee, \bigwedge)$ to denote this structure.

Remark 3.1.5. The defined joins and meets in a partially defined lattice are not restricted to be binary. So, for example, $d = a \vee b \vee c$ is allowed (assuming d is the least upper bound in P, of course), while $a \vee b$ may not be defined and may not even exist in P.

Remark 3.1.6. Given any finite lattice presentation there is a polynomial time algorithm to produce a finite partially defined lattice such that the finitely presented lattices generated by both are isomorphic. Consequently, in our study of finitely presented lattices we will study $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Rather than providing the details of the algorithm here, we refer the reader to [9] and presently discuss the following example of producing a finite partially defined lattice from a finite lattice presentation:
Consider the lattice presentation

$$\langle a, b, c, d, e \mid d \wedge e = (a \wedge b) \vee c \rangle.$$

Now, P will contain every generator of this presentation. However, for each subterm of a relation of the presentation which itself is not already a generator of the presentation, we will need to designate a new element of P. The following figure gives P along with its ordering:

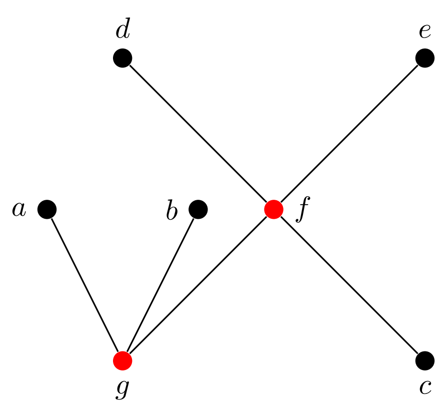

Figure 3.1: A new $(P, \leq)$

Finally, to complete the construction of the new finite partially defined lattice, we give the defined joins and meets:

$$f = c + g, \;\; f = de, \;\; g = ab$$

3.2 Dean's Theorem

The *word problem* for (X, R) is, given terms s and t with variables from X, to decide if the interpretations of s and t in $\mathrm{Free}(X, R)$ are equal. Equivalently, is $(s, t) \in \theta_R$? In this section, we provide one solution to the word problem.

Definition 3.2.1. An *ideal* I in a partially defined lattice $(P, \leq, \bigvee, \bigwedge)$ is a subset of P such that if $a \in I$ and $b \leq a$ then $b \in I$, and if $a_1, \ldots, a_k$ are in I and $a = \bigvee a_i$ is defined then $a \in I$.

Remark 3.2.2. It is worth pointing out that these two rules may have to be applied repeatedly to find the ideal generated by a set.

Remark 3.2.3. The set of all ideals of $(P, \leq, \bigvee, \bigwedge)$ including the empty ideal forms a lattice denoted $\mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge)$ or just $\mathrm{Idl}_0(P)$. The map $p \mapsto \mathrm{id}(p)$ embeds P into $\mathrm{Idl}_0(P)$, preserving the order (and its negation) and all the defined joins and meets. This is easy to see: if $a < b$ in P then $\mathrm{id}(a) \subsetneq \mathrm{id}(b)$, and if $a = a_1 \vee \cdots \vee a_k$ is a defined join then a is in the ideal I generated by the union of the $\mathrm{id}(a_i)$'s, whence it follows that $I = \mathrm{id}(a)$. If b is the greatest lower bound in $(P, \leq)$ of $\{a_1, \ldots, a_k\}$ then $\mathrm{id}(b) = \mathrm{id}(a_1) \cap \cdots \cap \mathrm{id}(a_k)$, so the meet relations are certainly preserved. Hence the map $p \mapsto \mathrm{id}(p)$ extends to a map

$$\psi\colon \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to \mathrm{Idl}_0(P, \leq, \bigvee, \bigwedge),$$

and this shows in particular that $(P, \leq)$ is embedded in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$.

Definition 3.2.4. If $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ we let

$$\underline{w} = \mathrm{id}_P(w) = \{a \in P : a \leq w\},$$

the ideal of P below w. Define $\overline{w}$ (the filter above w), $\mathrm{Fil}_1(P, \leq, \bigvee, \bigwedge)$, and $\psi^d : \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to \mathrm{Fil}_1(P, \leq, \bigvee, \bigwedge)$ dually. If $w_1, \ldots, w_k \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ let $\mathrm{id}_P(w_1, \ldots, w_k)$ be the ideal of $(P, \leq, \bigvee, \bigwedge)$ generated by $\underline{w_1} \cup \cdots \cup \underline{w_k}$ which of course is the ideal $\underline{w_1} \vee \cdots \vee \underline{w_k}$. The filter $\mathrm{fil}_P(w_1, \ldots, w_k)$ is defined dually.

Remark 3.2.5. One can show by induction on the rank of w that, for the map ψ above,

$$\psi(w) = \mathrm{id}_P(w) = \underline{w},$$

as follows: If w has rank 1, then $w \in P$ and hence $\psi(w) = \mathrm{id}(w) = \underline{w}$. Now, assume w has rank greater than 1, and that if $v \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ has rank less than w then $\psi(v) = \underline{v}$. WLOG, assume $w = w_1 \vee \cdots \vee w_k$. Since ψ is a homomorphism and $w_1, \ldots, w_k$ all have rank less than w, $\psi(w) = \psi(w_1) \vee \cdots \vee \psi(w_k) = \underline{w_1} \vee \cdots \vee \underline{w_k} = \underline{w_1 \vee \cdots \vee w_k} = \underline{w}$, as desired.
A dual argument shows that $\psi^d(w) = \mathrm{fil}_P(w) = \overline{w}$.

The following is Dean's solution to the word problem for finitely presented lattices (see [3]).

Theorem 3.2.6. *Let s and t be terms with variables in P. Then $s \leq t$ holds in Free$(P, \leq, \bigvee, \bigwedge)$ if and only if one of the following holds:*

(i) $s \in P$ *and* $t \in P$ *and* $s \leq t$ *in* $(P, \leq)$;

(ii) $s = s_1 \vee \cdots \vee s_k$ *and* $\forall\, i\ s_i \leq t$;

(iii) $t = t_1 \wedge \cdots \wedge t_k$ *and* $\forall\, j\ s \leq t_j$;

(iv) $s \in P$ *and* $t = t_1 \vee \cdots \vee t_k$ *and* $s \in id_P(\{t_1, \ldots, t_k\})$;

(v) $s = s_1 \wedge \cdots \wedge s_k$ *and* $t \in P$ *and* $t \in fil_P(\{s_1, \ldots, s_k\})$;

(vi) $s = s_1 \wedge \cdots \wedge s_k$ *and* $t = t_1 \vee \cdots \vee t_m$ *and* $\exists\, i\ s_i \leq t$ *or* $\exists\, j\ s \leq t_j$ *or* $\exists\, a \in P\ s \leq a \leq t$.

Proof. First, it is easy to see that all possibilities are covered by (i) – (vi) and that each of these leads to a genuine reduction (except for (i), which gives the answer directly).
Since $(P, \leq)$ is embedded in Free$(P, \leq, \bigvee, \bigwedge)$, if s and t are in P, then $s \leq t$ holds in Free$(P, \leq, \bigvee, \bigwedge)$ if and only if it holds in $(P, \leq)$.
Now, assume that (ii), (iii), (iv), (v), or (vi) hold. Clearly, if (ii), (iii) or (vi) hold, then $s \leq t$ holds in Free$(P, \leq, \bigvee, \bigwedge)$. A straightforward inductive argument shows that if (iv) or (v) holds then $s \leq t$ holds in Free$(P, \leq, \bigvee, \bigwedge)$. Therefore, any of (i) to (vi) implies $s \leq t$.
For the converse suppose $s \leq t$ holds in Free$(P, \leq, \bigvee, \bigwedge)$.
If $s = s_1 \vee \cdots \vee s_k$, then clearly (ii) holds. Similarly, if $t = t_1 \wedge \cdots \wedge t_k$, then (iii) immediately follows.
Now, assume $s \in P$ and $t = t_1 \vee \cdots \vee t_k$. Using the homomorphism ψ above

$$\begin{aligned}
\mathrm{id}_P(s) = \psi(s) \leq \psi(t) = \mathrm{id}_P(t) &= \mathrm{id}_P(\{t_1 \vee \cdots \vee t_k\}) \\
&= \mathrm{id}_P(t_1) \vee \cdots \vee \mathrm{id}_P(t_k) \\
&= \mathrm{id}_P(\{t_1, \ldots, t_k\}),
\end{aligned}$$

and so $s \in \mathrm{id}_P(\{t_1, \ldots, t_k\})$, proving (iv).
If $s = s_1 \wedge \cdots \wedge s_k$ and $t \in P$, using a dual argument with the dual homomorphism ψ^d, (v) easily follows.
Finally, suppose $s \leq t$ and $s = s_1 \wedge \cdots \wedge s_k$ and $t = t_1 \vee \cdots \vee t_m$ and, for a contradiction, that there is no i with $s_i \leq t$, no j with $s \leq t_j$ and no $a \in P$ with $s \leq a \leq t$. Let C be the interval $[s, t]$ and let $F_P[C]$ be the lattice with C doubled, where $F_P = \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Since $P \cap C = \emptyset$, $(P, \leq)$ is embedded in $F_P[C]$, and by (1.2) the image satisfies the join and meet relations. Hence there is a homomorphism $\varphi\colon F_P \twoheadrightarrow F_P[C]$. Let v be the interpretation of s in F_P and let u be the interpretation of t. By Corollary 1.2.4 the interpretation of s in $F_P[C]$ is either $(v, 0)$ or $(v, 1)$.

Since $s = s_1 \wedge \cdots \wedge s_k$ and each s_i is not in C, each s_i is above $(v, 1)$, i.e. $(v, 1) \leq s$ in $F_P[C]$. Thus, as either possible interpretation of s in $F_P[C]$ is below $(v, 1)$, it follows that the interpretation of s in $F_P[C]$ must be $(v, 1)$. By a similar argument, the interpretation of t in $F_P[C]$ must be $(u, 0)$. But then $\varphi(s) = (v, 1) \not\leq (u, 0) = \varphi(t)$, which is a contradiction since $s \leq t$. Therefore, (vi) holds. □

Remark 3.2.7. If there are no defined joins then $\mathrm{id}_P(\{t_1, \ldots, t_k\})$ is simply the set of elements below one of the t_i's. Hence condition (iv) of Dean's Theorem can be simplified to saying that if $s \in P$ and $t = t_1 \vee \cdots \vee t_k$, then $s \leq t$ if and only if $s \leq t_i$ for some i. In other words the elements of P are join prime. Also note that in this case condition (vi) simplifies to

$$s = s_1 \wedge \cdots \wedge s_k \text{ and } t = t_1 \vee \cdots \vee t_m \text{ implies } \exists\ i\ s_i \leq t \text{ or } \exists\ j\ s \leq t_j \tag{W}$$

which is just *Whitman's condition*.
If no joins and no meets are defined and the order on P is an antichain, then Dean's solution reduces to Whitman's solution to the word problem for free lattices.

Lemma 3.2.8. *If $x \in P$ and $x \leq t_1 \vee \cdots \vee t_n$ in Free$(P, \leq, \bigvee, \bigwedge)$ then there is a set $Y \subseteq P$ such that $Y \ll \{t_1, \ldots, t_n\}$ and $x \leq \bigvee Y$ in Free$(P, \leq, \bigvee, \bigwedge)$.*

Proof. By (iv) of Dean's Theorem, the hypotheses imply that x is in the ideal of $(P, \leq, \bigvee, \bigwedge)$ generated by $Y = \{y \in P : y \leq t_i \text{ for some } i\}$. Clearly $Y \ll \{t_1, \ldots, t_n\}$. The join of Y may not be defined in $(P, \leq, \bigvee, \bigwedge)$, but it is easy to see that every element of the ideal of $(P, \leq, \bigvee, \bigwedge)$ generated by Y, $\mathrm{id}_P(Y)$, is below $\bigvee Y$ in Free$(P, \leq, \bigvee, \bigwedge)$, and hence $x \leq \bigvee Y$. □

3.3 Canonical Form in Finitely Presented Lattices

Each element in a free lattice has a *canonical form*, that is a shortest term representing it, which is unique up to commutativity and associativity. This syntactical concept is closely related to the arithmetic of the free lattice. We will see that the elements of Free$(P, \leq, \bigvee, \bigwedge)$ also have a canonical form and that there is a nice connection between this form and the arithmetic of the finitely presented lattice. The canonical form presented here (taken from [4]) has the nice property that when applied to free lattices, it agrees with Whitman's.
As we mentioned above, the major difference between Dean's algorithm and Whitman's lies in conditions (iv), (v) and (vi). However if we are dealing with a certain kind of term, which we will call *adequate*, these difficult conditions can be replaced with the simple free lattice conditions.

Definition 3.3.1. Let $(P, \leq, \bigvee, \bigwedge)$ be a finite partially defined lattice. A term t with variables from P is called *adequate* if it is an element of P, or if $t = t_1 \vee \cdots \vee t_n$ is a formal join, each t_i is adequate, and if $p \leq t$ for $p \in P$ then $p \leq t_i$ for some i. If t is formally a meet the dual condition must hold.

Lemma 3.3.2. *Let s and t be adequate terms. Then $s \le t$ in Free$(P, \le, \bigvee, \bigwedge)$ if and only if $s \le t$ in Free$(P, \le)$.*

Proof. Note that $(P, \le)$ denotes P as a partially ordered set, with no nontrivial joins and meets defined. Thus, (i) of Dean's Theorem holds in Free$(P, \le, \bigvee, \bigwedge)$ if and only if (i) holds in Free$(P, \le)$. Furthermore, as (ii) and (iii) of Dean's Theorem hold in any lattice if $s \le t$, (ii) and (iii) hold in Free$(P, \le, \bigvee, \bigwedge)$ if and only (ii) and (ii) hold in Free$(P, \le)$, respectively.
If $s \in P$ and $t = t_1 \vee \cdots \vee t_k$ and (iv) of Dean's Theorem holds in Free$(P, \le, \bigvee, \bigwedge)$, then since t is adequate, $s \le t_i$ for some i. By Remark 3.2.7, this implies that (iv) of Dean's Theorem holds in Free$(P, \le)$. A dual argument shows that if $s = s_1 \wedge \cdots \wedge s_k$ and $t \in P$ and (v) of Dean's Theorem holds in Free$(P, \le, \bigvee, \bigwedge)$, it must hold in Free$(P, \le)$.
Finally, if $s = s_1 \wedge \cdots \wedge s_k$ and $t = t_1 \vee \cdots \vee t_m$ and (vi) of Dean's Theorem holds in Free$(P, \le, \bigvee, \bigwedge)$, then the case that $s \le a \le t$ for some $a \in P$ reduces to $s \le s_i \le a \le t_j \le t$ for some s_i and t_j as both s and t are adequate terms. Thus, (W) holds in Free$(P, \le)$, i.e. (vi) of Dean's Theorem holds for Free$(P, \le)$.
For the converse, it is easy to see from Remark 3.2.7 that if (iv), (v), or (vi) of Dean's Theorem hold in Free$(P, \le)$, they must hold in Free$(P, \le, \bigvee, \bigwedge)$, respectively. $\square$

Remark 3.3.3. An easy inductive argument on complexity of terms shows that for every element w of the lattice Free$(P, \le, \bigvee, \bigwedge)$ there is an adequate term representing w.

Remark 3.3.4. It follows from Corollary 2.1.12 and the definition of adequate that every term is adequate in the case of free lattices.

Theorem 3.3.5. *For each element of Free$(P, \le, \bigvee, \bigwedge)$ there is an adequate term of minimal rank representing it, and this term is unique up to commutativity.*

Proof. Suppose that s and t are both shortest adequate terms that represent the same element w in Free$(P, \le, \bigvee, \bigwedge)$. If either s or t is in P, then clearly $s = t$.
Observe that if $t = t_1 \vee \cdots \vee t_n$ and some t_i is formally a join, we could lower the rank of t by removing the parentheses around t_i. Since t_i is adequate, the resulting term would still adequately represent w. But this would violate the minimality of t. Thus we conclude that each t_i is not formally a join.
Suppose that $t = t_1 \vee \cdots \vee t_n$ and $s = s_1 \vee \cdots \vee s_m$. Then $t_i \le s_1 \vee \cdots \vee s_m$ for each t_i. This implies that either $t_i \le s_j$ for some j, or $t_i = \bigwedge t_{ij}$ and $t_{ij} \le s$ for some j, or there is an $x \in P$ with $t_i \le x \le s_1 \vee \cdots \vee s_m$. In the second case we have $t_i \le t_{ij} \le t$, and replacing t_i by t_{ij} in t produces a shorter term still representing w. It is easy to see that this term is still adequate, violating the minimality of the term t. If the third case holds then, by the adequacy of s, $x \le s_j$ for some j. Hence in all cases there is a j such that $t_i \le s_j$. Thus $\{t_1, \ldots, t_n\} \ll \{s_1, \ldots, s_m\}$. By symmetry, $\{s_1, \ldots, s_n\} \ll \{t_1, \ldots, t_m\}$. Since both are antichains (by the minimality) they represent the same

set of elements of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Thus $m = n$ and after renumbering $s_i \approx t_i$. Now by induction s_i and t_i are the same up to commutativity.
If $t = t_1 \vee \cdots \vee t_n$ and $s = s_1 \wedge \cdots \wedge s_m$, then, since neither s nor t is in P, (W) implies that either $t_i = t$ for some i or $s_j = s$ for some j, violating the minimality.
The remaining cases can be handled by duality. □

Definition 3.3.6. The shortest adequate term mentioned in the above theorem representing w, unique up to commutativity, is called the *canonical form* of w.

Remark 3.3.7. Examining the proof of this theorem we see that an adequate term $t = t_1 \vee \cdots \vee t_n$ is a minimal adequate term if every proper subterm is a minimal adequate term, the t_i's form an antichain, and if $t_i = \bigwedge_j t_{ij}$, then $t_{ij} \not\leq t$ for every j.

Theorem 3.3.8. *To put a term $t = t_1 \vee \cdots \vee t_n$ with $n > 1$ into canonical form, do the following.*

(a) (Remove unnecessary parentheses) For each i for which t_i is a formal join, replace t_i by its joinands. We still use $t_1, \ldots, t_n$ to denote the list of joinands.

(b) Put each of the t_i's into canonical form.

(c) Let T be the maximal elements of $\{t_1, \ldots, t_n\} \cup id_P(t)$.

(d) If $t_i \in T$ is a formal meet, $t_i = \bigwedge_j t_{ij}$, and $t_{ij} \leq t$ for some j, then replace t_i with t_{ij} in T.

(e) If $s_1, \ldots, s_m$ are the maximal elements of T, then the canonical form of t is $s_1 \vee \cdots \vee s_m$.

In free lattices the canonical form is associated with nonrefinable join representations, which in free lattices are unique. The next theorem will show that in a finitely presented lattice each element can have only finitely many nonrefinable join representations, and these can be easily found from the canonical form.

Definition 3.3.9. We define the *canonical join representation* of $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ to be $w_1 \vee \cdots \vee w_m$ if the canonical form of w is $t_1 \vee \cdots \vee t_m$ and the interpretation of t_i in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ is w_i. It is useful to separate out the elements of P in such a representation. Thus let

$$w = w_1 \vee \cdots \vee w_n \vee x_1 \vee \cdots \vee x_k \tag{3.1}$$

$$= \bigvee \bigwedge w_{ij} \vee \bigvee x_i \tag{3.2}$$

be the canonical join representation of w where $x_i \in P$, $i = 1, \ldots, k$, and the canonical meet representation of w_i is $w_i = \bigwedge w_{ij}$.

Remark 3.3.10. Note that an element $x \in P$ is join irreducible in $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ except when some $(z_1, \ldots, z_\ell, x) \in \bigvee$ is among the defining relations of $(P, \leq, \bigvee, \bigwedge)$ and $x \neq z_i$, $i = 1, \ldots, \ell$.

Theorem 3.3.11. *Let the canonical join representation for w be given by (3.1). Every join representation of w can be refined to a nonrefinable join representation of w. If $w = v_1 \vee \cdots \vee v_m$ in Free$(P, \leq, \bigvee, \bigwedge)$ then there exist $y_1, \ldots, y_r \in P$ such that*

$$w = w_1 \vee \cdots \vee w_n \vee y_1 \vee \cdots \vee y_r$$

and

$$\{w_1, \ldots, w_n, y_1, \ldots, y_r\} \ll \{v_1, \ldots, v_m\}.$$

Every nonrefinable join representation of w contains $\{w_1, \ldots, w_n\}$ and also contains every x_i which is join irreducible.

Proof. Assume $w = v_1 \vee \cdots \vee v_m$. Since, for fixed $i = 1, \ldots, n$,

$$w_i \leq v_1 \vee \cdots \vee v_m = w$$

we have that either (i) $w_i \leq v_j$ for some j, (ii) $w_{ij} \leq w$, or (iii) $w_i \leq x \leq w$ for some $x \in P$. If either (ii) or (iii) held, we could produce a shorter adequate term representing w, violating the minimality of the representation $w = w_1 \vee \cdots \vee w_n \vee x_1 \vee \cdots \vee x_k$. Hence (i) must hold.
Since, for $1 \leq i \leq k$, $x_i \leq v_1 \vee \cdots \vee v_m$, by Lemma 3.2.8 there is a set $\{z_{i1}, \ldots, z_{is}\} \subseteq P$ such that $x_i \leq z_{i1} \vee \cdots \vee z_{is}$ in Free$(P, \leq, \bigvee, \bigwedge)$ and

$$\{z_{i1}, \ldots, z_{is}\} \ll \{v_1, \ldots, v_m\}.$$

Hence if we let $\{y_1, \ldots y_r\}$ be the union of the z's obtained from all of the x_i's,

$$\{w_1, \ldots, w_n, y_1, \ldots, y_r\} \ll \{v_1, \ldots, v_m\}.$$

But then, $x_i \leq z_{i1} \vee \cdots \vee z_{is}$ gives us that $x_1 \vee \cdots \vee x_k \leq y_1 \vee \cdots \vee y_r$, and so since $\{w_1, \ldots, w_n, y_1, \ldots, y_r\} \ll \{v_1, \ldots, v_m\}$, $w = x_1 \vee \cdots \vee x_k \vee w_1 \vee \cdots \vee w_n \leq y_1 \vee \cdots \vee y_r \vee w_1 \vee \cdots \vee w_n \leq v_1 \vee \cdots \vee v_m = w$, i.e.

$$w = w_1 \vee \cdots \vee w_n \vee y_1 \vee \cdots \vee y_r.$$

This proves the first part of the theorem and also shows that every nonrefinable join representation of w must be a subset of $\{w_1, \ldots, w_n, y_1, \ldots, y_r\}$ for some $y_1, \ldots, y_r$ in P. Since $\{w_1, \ldots, w_n\} \ll \{v_1, \ldots, v_m\}$ by the argument at the beginning of this proof, no w_i can be omitted from $\{w_1, \ldots, w_n, y_1, \ldots, y_r\}$ if $w = v_1 \vee \cdots \vee v_m$ is a nonrefinable join representation since $\{w_1, \ldots, w_n\}$ forms an antichain. Hence every nonrefinable join representation of w has the form $\{w_1, \ldots, w_n, y_1, \ldots, y_r\}$ for some $y_1, \ldots, y_r$ in P.
This proves everything except the statement about the join irreducible x_i's. First we claim that *each x_i in (3.1) is a maximal element of $id_P(w)$*. If, on the other hand, $x_i < y \leq w$ for some $y \in P$,

we could replace x_i by y in (3.1). The resulting expression would still correspond to an adequate term, in violation of the uniqueness of the canonical form. Assume $\{v_1, \dots, v_m\}$ is a nonrefinable join representation of w. By Theorem 3.2.6, $x_i \le v_1 \vee \cdots \vee v_m$ means that x_i is in the ideal of P generated by $\bigcup_j \mathrm{id}_P(v_j)$. This ideal is obtained from this union by alternately taking joins of subsets of this union that are defined in $(P, \le, \bigvee, \bigwedge)$ and adding all elements in P less than something in the set. Obviously all such elements will be in $\mathrm{id}_P(w)$. But since x_i is a maximal element in $\mathrm{id}_P(w)$, the only way for a join of elements of P below w to contain (be greater than or equal to) x_i is for it to equal x_i. Thus, in the case that x_i is join irreducible, we must have $x_i \le v_j$ for some j. We have shown that $\{v_1, \dots, v_m\} = \{w_1, \dots, w_n, y_1, \dots, y_r\}$ for some y_j's. Since $x_i \le w_k$ would violate the canonical form (3.1) of w, we must have $x_i \le y_j$ for some j. But the maximality of x_i implies $x_i = y_j$, proving the last statement. $\square$

Remark 3.3.12. Notice that this proof shows that *every nonrefinable join representation of w refines the canonical join representation.*

3.4 The Generalized Word Problem and PC(P)

The beginning results in this section are largely based on [6].

Definition 3.4.1. We denote the join and meet closure of P in $\mathrm{Free}(P, \le, \bigvee, \bigwedge)$ by $P^\vee$ and $P^\wedge$, respectively. Furthermore, we let $L_0 = P^{\vee(\wedge\vee)^n}$ be the n-fold closure of $P^\vee$ under joins and meets, and $L_1 = P^{\wedge(\vee\wedge)^n}$ be the n-fold closure of $P^\wedge$ under joins and meets.

Remark 3.4.2. L_0 and L_1 are finite subsets of $\mathrm{Free}(P, \le, \bigvee, \bigwedge)$ closed under joins and meets, where L_0 possesses a least element and L_1 possesses a greatest element. Hence, both L_0 and L_1 are lattices. More specifically, L_0 is a join subsemilattice of $\mathrm{Free}(P, \le, \bigvee, \bigwedge)$ and L_1 is a meet subsemilattice of $\mathrm{Free}(P, \le, \bigvee, \bigwedge)$. If n is large enough these lattices will satisfy the relations of P and thus there are epimorphisms $f_0 : \mathrm{Free}(P, \le, \bigvee, \bigwedge) \to L_0$ and $f_1 : \mathrm{Free}(P, \le, \bigvee, \bigwedge) \to L_1$.

Definition 3.4.3. The epimorphisms f_0 and f_1 above are referred to as the standard epimorphism and the dual standard homomorphism, respectively.

Remark 3.4.4. It is easy to see that f_0 is the identity on L_0 and f_1 is the identity on L_1. Furthermore, we argue that $f_0(w) \le w$ and $w \le f_1(w)$ for all $w \in \mathrm{Free}(P, \le, \bigvee, \bigwedge)$: Let $\vee_i$ and $\wedge_i$ denote the operations of L_i $(i = 0, 1)$, and let $S_0 = \{y \in \mathrm{Free}(P, \le, \bigvee, \bigwedge) : f_0(y) \le y\}$. Note first that

$$a \wedge_0 b = \bigvee \{x \in L_0 : x \le a \text{ and } x \le b\} \le a \wedge b \text{ for } a, b \in L_0$$

$$a \vee_1 b = \bigwedge \{y \in L_1 : a \le y \text{ and } b \le y\} \ge a \vee b \text{ for } a, b \in L_1.$$

If $b, c \in S_0$, then

$$f_0(b \wedge c) = f_0(b) \wedge_0 f_0(c) \le f_0(b) \wedge f_0(c) \le b \wedge c,$$

so $b \wedge c \in S_0$. Similarly, $b \vee c \in S_0$. Thus S_0 is a sublattice of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ with $P \subseteq S_0$, and so $S_0 = \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. Therefore, $f_0(w) \leq w$, and by a dual argument, $w \leq f_1(w)$, for all $w \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$.

Lemma 3.4.5. *Let $f : \mathrm{Free}(P, \leq, \bigvee, \bigwedge) \to L_0 \times L_1$ be given by $f(w) = (f_0(w), f_1(w))$. If $w \in L_0 \cap L_1$, then $f^{-1}(f(w)) = \{w\}$.*

Proof. Since $w \in L_0 \cap L_1$, $f(w) = (w, w)$. So, if $u \in f^{-1}(f(w))$, then $f(u) = (w, w)$, and hence $w = f_0(u) \leq u$. Similarly, $u \leq f_1(u) = w$. Therefore, $f^{-1}(f(w)) = \{w\}$. □

Definition 3.4.6. The *generalized word problem* for a finitely presented algebra A asks if there is an algorithm to determine, for an arbitrary element $d \in A$ and a finite set $U = \{u_1, \ldots, u_k\}$ of elements of A, if d is in the subalgebra generated by U.

Theorem 3.4.7. *The generalized word problem for lattices is (uniformly) solvable.*

Proof. Let $d \in \mathrm{Free}(P, \leq, \bigvee, \bigwedge)$ and let f be the homomorphism onto the finite lattice $L_0 \times L_1$ described above. If n is chosen large enough so that $d \in L_0 \cap L_1 = P^{\vee(\wedge\vee)^n} \cap P^{\wedge(\vee\wedge)^n}$, by Lemma 3.4.5, $f^{-1}(f(d)) = \{d\}$. Now let $u_1, \ldots, u_k$ be elements of $\mathrm{Free}(P, \leq, \bigvee, \bigwedge)$. We claim d is in the sublattice generated by $u_1, \ldots, u_n$ if and only if $f(d)$ is in the sublattice generated by $f(u_1), \ldots, f(u_n)$: If the latter condition holds, then there is a term t such that

$$f(d) = t(f(u_1), \ldots, f(u_n)).$$

Since f is a homomorphism, $t(f(u_1), \ldots, f(u_n)) = f(t(u_1, \ldots, u_n))$. Thus, $d = t(u_1, \ldots, u_n)$ is in the sublattice generated by $u_1, \ldots, u_n$. The other direction is obvious. This construction is effective so the theorem follows from the claim. □

We now end this section by giving two important definitions which will be important for our new results.

Definition 3.4.8. An epimorphism $f : K \to L$ is called lower bounded if each element $x \in L$ has a least preimage. This least preimage, when it exists, is denoted $\beta_f(x)$ or just $\beta(x)$. Upper bounded is defined dually and the greatest preimage, when it exists, is denoted $\alpha_f(x)$ or just $\alpha(x)$. The map f is bounded if it is both upper and lower bounded.

Definition 3.4.9. Let $(P, \leq, \bigvee, \bigwedge)$ be a partially defined lattice. The partial completion of $(P, \leq, \bigvee, \bigwedge)$, denoted $\mathrm{PC}(P)$, is the sublattice of $\mathrm{Idl}_0(P) \times \mathrm{Fil}_1(P)$ generated by $\{(\mathrm{id}_P(p), \mathrm{fil}_P(p)) : p \in P\}$.

Remark 3.4.10. In working with $\mathrm{PC}(P)$, P is usually identified with $\{(\mathrm{id}_P(p), \mathrm{fil}_P(p)) : p \in P\}$.

CHAPTER 4
NEW RESULTS

We now further explore algorithms for the generalized word problem. Our first result shows that, for free lattices, there is a polynomial time algorithm for the generalized word problem. We begin with the following definition.

Definition 4.0.1. Let $\mathbf{L}$ be a lattice generated by X. Let Y be a subset of L and $t(X)$ be a lattice term. We say that Y *interlaces* t iff, for every branch of the term tree of t, there are nodes t' and t'', with t'' a child of t', such that there exists $y \in Y$ between $t'(X)$ and $t''(X)$.

Remark 4.0.2. Before we continue, let us take some time to clarify some aspects of what we have just defined:

First, if t is a variable, Y interlaces t iff $t(X) \in Y$.

Next, let us clarify other aspects of our definition by referring to the following term tree:

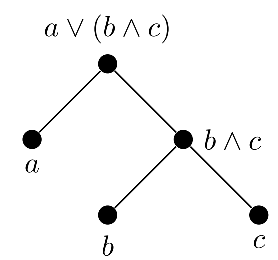

Figure 4.1: Interlacing clarification example

In the above term tree, we have three branches: one originates at $a \vee (b \wedge c)$ and ends at a; another originates at $a \vee (b \wedge c)$, passes through $b \wedge c$, and ends at b; a final branch originates at $a \vee (b \wedge c)$, passes through $b \wedge c$, and ends at c. Thus, by "branch of the term tree of t" in our definition above, in particular we mean a path along the term tree of t which originates at t.

Furthermore, in the above term tree, a is a child of $a \vee (b \wedge c)$, $b \wedge c$ is a child of $a \vee (b \wedge c)$, and b is a child of $b \wedge c$. However, b is not a child of $a \vee (b \wedge c)$. Thus, by "t'' a child of t' " in our definition above, we mean t'' is a *direct* descendant of t'.

Lastly, if we had a set Y which interlaced our term $a \vee (b \wedge c)$, we would be able to find $y \in Y$ between a and $a \vee (b \wedge c)$, i.e. $a \leq y \leq a \vee (b \wedge c)$ in the lattice $\mathbf{L}$. However, if we are able to find $\overline{y} \in Y$ between $b \wedge c$ and b, we of course would have $b \wedge c \leq \overline{y} \leq b$ in the lattice $\mathbf{L}$. Thus, by "there exists $y \in Y$ between $t'(X)$ and $t''(X)$" in our definition above, we mean either $t'(X) \leq y \leq t''(X)$ in $\mathbf{L}$ or $t''(X) \leq y \leq t'(X)$ in $\mathbf{L}$, depending on the ordering of $t'(X)$ and $t''(X)$ in $\mathbf{L}$.

We now present the main result that we will use in order to show that the generalized word problem for free lattices can be done in polynomial time.

Theorem 4.0.3. *Let $d \in \mathrm{FL}(X)$ and Y be a finite subset of $\mathrm{FL}(X)$.*

(a) If Y interlaces a term t representing d in $\mathrm{FL}(X)$, then $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$.

(b) If $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$, then Y interlaces the canonical form of d.

Proof. Suppose t is a term such that $d = t(X)$ and Y interlaces t. If $c(t) = 1$, $d \in Y$. Now, assume $c(t) > 1$, every term s with $c(s) < c(t)$ and Y interlacing s forces $s(X) \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$, and $t = t_1 \vee \cdots \vee t_r$. Fix t_i. We must either have some $y_i \in Y$ between $t_i(X)$ and $t(X)$ or Y interlaces t_i. If some $y_i \in Y$ is between $t_i(X)$ and $t(X)$, we can replace $t_i(X)$ with y_i in the expression of $t(X)$ and obtain another representation of d. If Y interlaces t_i, then by our inductive hypothesis, $t_i(X) \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. Let $B = \{b_1, \ldots, b_r\}$, where b_j is either $t_j(X)$ if $t_j(X) \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$ or is y_j if $t_j(X) \leq y_j \leq t(X)$. Then clearly $\bigvee t_j(X) \leq \bigvee B \leq t(X)$ and hence $d = \bigvee B \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. If $t = t_1 \wedge \cdots \wedge t_r$, then by duality we can also show $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. Therefore, by induction on $c(t)$, $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$.

Now suppose $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$, and let s be a term of minimal complexity with $d = s(Y)$.

First, if $d \in X$ we shall see that s can only be a variable: First, if s is in fact a variable, then clearly $d = s(Y) \in Y$. If $s = s_1 \vee \cdots \vee s_m$, then since $d \in X$ and hence join irreducible, d must be equal to one of the joinands of $s(Y)$, a contradiction to s being a term of minimal complexity representing d in $\mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. If $s = s_1 \wedge \cdots \wedge s_m$, then a dual argument will give another contradiction since $d \in X$ is meet irreducible. Therefore, $d \in Y$.

Now let $d = d_1 \vee \ldots \vee d_n$ canonically in $\mathrm{FL}(X)$. If s is a variable, then $d \in Y$ and we are done. Suppose for a contradiction that $s = s_1 \wedge \cdots \wedge s_m$. By (W), we would have that either $s_j(Y) \leq d$ or $s(Y) \leq d_k$. In the first case $d = s_j(Y)$, contrary to s being of minimal complexity. In the second case $d = d_k$, contrary to the canonical form of d. Thus, we must have $s = s_1 \vee \cdots \vee s_m$.

Fix d_i, where $1 \leq i \leq n$. Since $\{d_1, \ldots, d_n\} \ll \{s_1(Y), \ldots, s_m(Y)\}$ by Theorem 2.2.11, $d_i \leq s_j(Y)$ for some s_j. Since s is of minimal complexity, s_j cannot be a join and hence must either be a variable or $s_j = s_{j1} \wedge \ldots \wedge s_{jk}$. If s_j is a variable, $s_j(Y) \in Y$ and $s_j(Y) \in d/d_i$. If $s_j = s_{j1} \wedge \ldots \wedge s_{jk}$, (W) would guarantee that either $s_{jl}(Y) \leq d$ for some $1 \leq l \leq k$ or $s_j(Y) \leq d_p$ for some $1 \leq p \leq n$. In the first case, we could replace s_j with s_{jl} in s and obtain a term of lower complexity representing d, a contradiction. In the second case, $d_i \leq s_j(Y) \leq d_p$ forces $d_i = s_j(Y) \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. Since $c(d_i) < c(d)$, we can invoke induction to conclude that Y interlaces the canonical form of d_i, and hence for every branch of the term tree of $d = d_1 \vee \ldots \vee d_n$ containing d_i, there are nodes d_i' and d_i'' with d_i'' a child of d_i' such that there exists $y \in Y$ between d_i' and d_i''.

A duality argument completes the proof. □

Corollary 4.0.4. *Let $d \in \mathrm{Free}(P, \leq)$ and Y be a finite subset of $\mathrm{Free}(P, \leq)$.*

(a) If Y interlaces a term t representing d in $\mathrm{Free}(P, \leq)$, then $d \in \mathrm{Sg}_{\mathrm{Free}(P,\leq)}(Y)$.

(b) If $d \in \mathrm{Sg}_{\mathrm{Free}(P,\leq)}(Y)$, then Y interlaces the canonical form of d.

Corollary 4.0.5. *Let $d \in \mathrm{FL}(X)$ and Y be a finite subset of $\mathrm{FL}(X)$. Then $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$ iff there is a term $t(X)$ representing d in $\mathrm{FL}(X)$, i.e. $d = t(X)$, such that Y interlaces t.*

Given $d \in \mathrm{FL}(X)$, we can use Corollary 4.0.5 to write out a polynomial time algorithm to test if $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$:

1. First, test if $d \in Y$. If it is, $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$ and we are done.

2. At this point, we may sssume $d \notin Y$. If $d \in X$, as we saw in the proof of Theorem 4.0.5, $d \notin \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. Thus, we may assume that d is either canonically a join or a meet in $\mathrm{FL}(X)$. If $d = d_1 \vee \cdots \vee d_n$ canonically, for each branch of the term tree of $d = d_1 \vee \cdots \vee d_n$, test if the branch contains nodes d' and d'' with d'' a child of d' such that there exists $y \in Y$ between d' and d''. If this holds for every branch of the term tree $d = d_1 \vee \cdots \vee d_n$, then $d \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. A similar test would be applied if $d = d_1 \wedge \cdots \wedge d_m$ canonically.

3. If all of the tests above fail, then $d \notin \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$.

We would hope that a similar algorithm can be stated for finitely presented lattices, but the following example illustrates one issue that arises when trying to use our polynomial time algorithm for free lattices on finitely presented lattices:
Let $P = \{a, b, c, d\}$ with order given in Figure 4.2 and the single defined join $d = a + b$ and the single defined meet $b = cd$. Furthermore, take $Y = \{a, c\}$, its elements labeled in red in Figure 4.2.

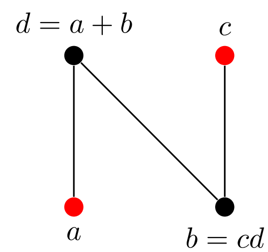

Figure 4.2: A nonconvergent example

Now, $d \notin Y$, nor is d canonically a join or a meet since it is in P. However, $d = a \vee b$ is a join representation to which we can attempt to apply the second step of our algorithm. We easily find an element of Y along the term tree branch from d to a (namely a itself), however we find none along the term tree branch from d to b. Nevertheless, keeping in mind that $b = c \wedge d$ is a meet representation for b, we can continue looking for an element of Y along the term tree branch from d to b. From b, we again can easily find an element of Y along one branch but not the other, namely the branch from b to d. However, when using again the join representation $d = a \vee b$, we arrive back at the problem with which we started, as we see in Figure 4.3 below.
Thus, our algorithm never converges for this example.

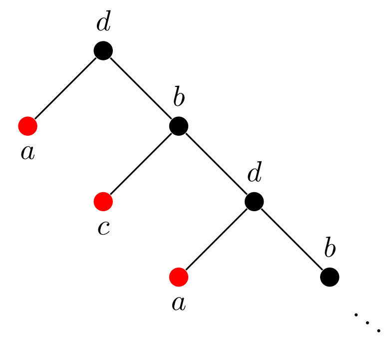

Figure 4.3: A neverending term tree

Even worse, the following example illustrates further issues that can arise for a syntactic algorithm for the generalized word problem for finitely presented lattices:
Let P be given by Figure 4.4 below, with the obvious order and no defined meets.

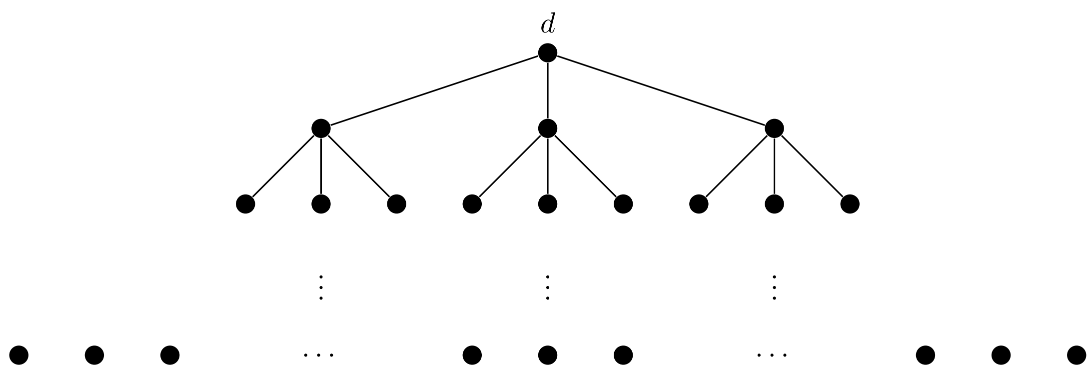

Figure 4.4: Many nonrefinable join representations

That is, d is the top element of P, with exactly three incomparable coatoms in P. Each of these coatoms is above exactly three elements, and these nine new elements are incomparable to each other. Continuing on, each element in a"level" of P is above exactly three elements, giving way to another "level" of incomparable elements of P.
Finally, we specify the defined joins in P in the following way: For each $p \in P$, except for the minimal elements of P, there are three elements directly below it in P; call them q, r, and s. P has defined joins $p = q + r = q + s = r + s$.
Now, if $n = \max(\text{depth}(d))$, then

$$|P| = 1 + 3 + 9 + \cdots + 3^n = \sum_{k=0}^{n} 3^k = \frac{1 - 3^{n+1}}{1 - 3} = \frac{3^{n+1} - 1}{2}.$$

We shall show that d in $(P, \leq, \bigvee, \bigwedge)$ has exponentially-many nonrefinable join representations:

First, there are $\binom{3}{2} = 3$ many ways to write d as a join of two coatoms of P. But then, for each of the two coatoms that we choose, there are 3 ways to write them as a join of two elements below them. Continuing on in this way, we see that are

$$3 \cdot 3^2 \cdot 3^{2^2} \cdot \dots \cdot 3^{2^{n-1}} = 3^{\sum_{k=0}^{n-1} 2^k} = 3^{2^n - 1}$$

nonrefinable join representations of d in $(P, \leq, \bigvee, \bigwedge)$. Since 3^{2^n-1} is exponential in $(3^{n+1}-1)/2$, the number of nonrefinable join representations of an element can be exponential in $|P|$. Hence, we see that an algorithm for the generalized word problem which requires us to search through the nonrefinable join representations of an element could be exponential.

Nevertheless, we wish to see if we can salvage a similar result for finitely presented lattices. The following theorem gives one similar result that can be carried over.

Theorem 4.0.6. *Let P be finite, $Y \subseteq F_P$, and $d \in \mathrm{Sg}_{F_P}(Y) - Y$. Let $d = w_1 \vee \dots \vee w_n \vee x_1 \vee \dots \vee x_k$ be the canonical join representation of $d \in F_P$, as given in Equation 3.1. Then $d/w_i \cap \mathrm{Sg}_{F_P}(Y) \neq \emptyset$ for $1 \leq i \leq n$.*

Proof. Let t be a term of minimal complexity representing $d \in \mathrm{Sg}_{F_P}(Y)$. Since $d \notin Y$ and d is canonically a join in F_P, a similar argument to that in Theorem 4.0.5 can be used to conclude that $t = t_1 \vee \dots \vee t_r$. As we saw in the proof of Theorem 3.3.11, there exists a nonrefinable join representation $d = w_1 \vee \dots \vee w_n \vee z_1 \vee \dots \vee z_l$ in F_P such that $\{w_1, \dots, w_n, z_1, \dots, z_l\} \ll \{t_1(Y), \dots, t_r(Y)\}$.
Fix w_i. There exists t_j such that $w_i \leq t_j(Y)$. Since t is of minimal complexity, t_j cannot be a join and hence must either be a variable or $t_j = t_{j1} \wedge \dots \wedge t_{jm}$. If t_j is a variable, $t_j(Y) \in Y$ and so $t_j(Y) \in d/w_i$. If $t_j = t_{j1} \wedge \dots \wedge t_{jm}$, (W) would guarantee either $t_{jp}(Y) \leq d$, $t_j(Y) \leq w_q$, or $t_j(Y) \leq x \leq d$ for some $x \in P$. In the first case, we could replace $t_j(Y)$ with $t_{jp}(Y)$ and obtain a term of lower complexity representing d, a contradiction. In the second case, $w_i \leq t_j(Y) \leq w_q$ forces $w_i = t_j(Y) \in \mathrm{Sg}_{F_P}(Y)$. If the last case holds, $x \leq d = w_1 \vee \dots \vee w_n \vee x_1 \vee \dots \vee x_k$ would give us that x is below one of the canonical joinands of d (by adequacy), and since $w_i \leq t_j(Y) \leq x$, we would again have that $w_i = t_j(Y) \in \mathrm{Sg}_{F_P}(Y)$. □

While a polynomial time algorithm may not exist for the generalized word problem for finitely presented lattices, we vie instead for a syntactic algorithm. We begin with the following Lemma.

Lemma 4.0.7. *Let $d = d_1 \vee \dots \vee d_n \in F_P$ be a nonrefinable join representation and, for some d_i, there exists $p \in P$ such that $d_i \leq p \leq d$. Then $d_i \in P$.*

Proof. Suppose, for a contraction, that $d_i \notin P$. Then $\bigvee \mathrm{id}_P(d_i) < d_i$, for if $\bigvee \mathrm{id}_P(d_i) = d_i$ we could replace d_i by the elements of $\mathrm{id}_P(d_i)$ and obtain a refinement of $\{d_1, \dots, d_n\}$, contradicting the fact that $\{d_1, \ldots, d_n\}$ forms a nonrefinable join representation of d. Furthermore, since $p \leq d = d_1 \vee \cdots \vee d_n$, $p \in \mathrm{id}_P(d_1, \ldots, d_n)$ by **Dean's Theorem** and hence

$$d_i \leq p \leq d_1 \vee \cdots \vee d_{i-1} \vee \left(\bigvee \mathrm{id}_P(d_i)\right) \vee d_{i+1} \vee \cdots \vee d_n \leq d.$$

But then $d = d_1 \vee \cdots \vee d_{i-1} \vee (\bigvee \mathrm{id}_P(d_i)) \vee d_{i+1} \vee \cdots \vee d_n$, contradicting the fact that $\{d_1, \ldots, d_n\}$ forms a nonrefinable join representation of d. Therefore, $d_i \in P$ as desired. $\square$

Theorem 4.0.8. *Let $d \in F_P$ and Y be a finite subset of F_P. Then $d \in Sg_{F_P}(Y)$ iff either $d \in Y$, there exists a nonrefinable join representation of d, call it $\{d_1, \ldots, d_n\}$, such that, for each d_i, either*

(a) $d/d_i \cap Y \neq \emptyset$,

(b) $d_i \in Sg_{F_P}(Y)$, or

(c) $d_i \in P$ and there exists $p \in P$ such that $d_i \leq p \leq d$ and $p/d_i \cap Sg_{F_P}(Y) \neq \emptyset$,

or there exists a non-upper refinable meet representation of d such that the duals of (a), (b), and (c) hold for the elements of this non-upper refinable representation.

Proof. First, suppose $d \in \mathrm{Sg}_{F_P}(Y)$. If $d \in Y$, then we are done. If d is a join in $\mathrm{Sg}_{F_P}(Y)$, let $t = t_1 \vee \cdots \vee t_m$ be a term of minimal complexity representing d in $\mathrm{Sg}_{F_P}(Y)$. Then, there exists a nonrefinable join representation $d_1 \vee \cdots \vee d_n = d$ such that $\{d_1, \ldots, d_n\} \ll \{t_1(Y), \ldots, t_m(Y)\}$ as in the proof of Theorem 3.3.11. Fix d_i, and let t_j be such that $d_i \leq t_j(Y)$.
Now, since t is of minimal complexity, $t_j(Y) \in Y$ or $t_j = t_{j1} \wedge \cdots \wedge t_{jl}$. If the former happens, $d/d_i \cap Y \neq \emptyset$. If the latter happens, we must either have that $t_{jk}(Y) \leq d$ for some t_{jk}, $t_j(Y) \leq d_k$ for some d_k, or there exists $p \in P$ such that $t_j(Y) \leq p \leq d$. In the first case, we could replace t_j with t_{jk} and obtain a term of lower complexity representing d in $\mathrm{Sg}_{F_P}(Y)$, a contradiction. In the second case, we would have $d_i = t_j(Y) \in \mathrm{Sg}_{F_P}(Y)$ since $\{d_1, \ldots, d_n\}$ forms an antichain. In the last case, Lemma 4.0.7 gives us that $d_i \in P$, and so $t_j(Y) \in p/d_i \cap \mathrm{Sg}_{F_P}(Y)$.
We now prove the converse: If $d \in Y$, then $d \in \mathrm{Sg}_{F_P}(Y)$. Now, suppose that $\{d_1, \ldots, d_n\}$ is a nonrefinable join representation of d such that, for each d_i, either (a), (b), or (c) above hold. If some $y_i \in Y$ is between d_i and d, we can replace d_i with y_i in the nonrefinable join representation for d and obtain another expression for d. If $d_i \in P$ and there exists $p \in P$ such that $d_i \leq p \leq d$ and $t_i(Y) \in p/d_i \cap \mathrm{Sg}_{F_P}(Y)$, we can similarly replace d_i with $t_i(Y)$ in the nonrefinable join representation for d and obtain another expression for d. Let $B = \{b_1, \ldots, b_n\}$, where b_i is y_i if $d/d_i \cap Y \neq \emptyset$, d_i if $d_i \in \mathrm{Sg}_{F_P}(Y)$, or $t_i(Y)$ if $d_i \in P$ and there exists $p \in P$ such that $d_i \leq p \leq d$ and $p/d_i \cap \mathrm{Sg}_{F_P}(Y) \neq \emptyset$. Then clearly $d_1 \vee \cdots \vee d_n \leq \bigvee B \leq d$ and hence $d = \bigvee B \in \mathrm{Sg}_{\mathrm{FL}(X)}(Y)$. Dually, if $\{d_1, \ldots, d_n\}$ is a non-upper refinable meet representation, we would again have $d \in \mathrm{Sg}_{F_P}(Y)$. $\square$

We now use the above result to provide a possible syntactic algorithm for the generalized word problem for finitely presented lattices: Suppose there is an oracle that can decide, for all $p, q \in P$

with $p \leq q$ if there exists $f \in \mathrm{Sg}_{F_P}(Y)$ such that $p \leq f \leq q$. Given $d \in F_P$, we can test if $d \in \mathrm{Sg}_{F_P}(Y)$ in the following way:

1. First, test if $d \in Y$. Since Y is finite, this can be completed in polynomial time.

2. If $d \notin Y$, then d is either a join or a meet in F_P.

   (a) If d is a join, for each nonrefinable join representation $\{d_1, \ldots, d_n\}$ of d, and for each joinand d_i, we test if one of the following holds for d_i:

      i. $d/d_i \cap Y \neq \emptyset$. Since Y is finite, and through use of **Dean's Theorem**, this can be done in polynomial time for fixed d_i.

      ii. $d_i \in \mathrm{Sg}_{F_P}(Y)$. Note that this step is a reduction since $c(d_i) < c(d)$.

      iii. $d_i \in P$ and there exists $p \in P$ such that $d_i \leq p \leq d$ and $p/d_i \cap \mathrm{Sg}_{F_P}(Y) \neq \emptyset$. Since P is finite, through the use of **Dean's Theorem** it is easy to test whether or not $d_i \in P$. If we in fact find that $d_i \in P$, for each $p \in d/d_i$, we use our oracle to test if there exists $f \in \mathrm{Sg}_{F_P}(Y)$ such that $d_i \leq f \leq p$.

   If we are able to find a nonrefinable join representation for d such that one of the above holds for each of the joinands, then $d \in \mathrm{Sg}_{F_P}(Y)$.

   (b) If d is a meet, for each non-upper refinable meet representation $\{d_1, \ldots, d_n\}$ of d, and for each meetand d_i, we test if one of the following holds for d_i:

      i. $d_i/d \cap Y \neq \emptyset$. Since Y is finite, and through use of **Dean's Theorem**, this can be done in polynomial time for fixed d_i.

      ii. $d_i \in \mathrm{Sg}_{F_P}(Y)$. Note that this step is a reduction since $c(d_i) < c(d)$.

      iii. $d_i \in P$ and there exists $p \in P$ such that $d \leq p \leq d_i$ and $d_i/p \cap \mathrm{Sg}_{F_P}(Y) \neq \emptyset$. Since P is finite, through the use of **Dean's Theorem** it is easy to test whether or not $d_i \in P$. If we in fact find that $d_i \in P$, for each $p \in d_i/d$, we use our oracle to test if there exists $f \in \mathrm{Sg}_{F_P}(Y)$ such that $p \leq f \leq d_i$.

   If we are able to find a non-upper refinable meet representation for d such that one of the above holds for each of the meetands, then $d \in \mathrm{Sg}_{F_P}(Y)$.

3. If none of the above holds for d, then $d \notin \mathrm{Sg}_{F_P}(Y)$.

We conclude this chapter by giving another proof that the generalized word problem for finitely presented lattices is solvable. The following results closely mirror those presented in Section 3.4, though using the tools defined at the end of Chapter 3.

Recall the homomorphism $\psi : F_P \to \mathrm{Idl}_0(P)$ defined by $\psi(w) = \mathrm{id}_P(w)$.We claim that ψ is lower bounded:
Let $I \in \mathrm{Idl}_0(P)$. Now, $\bigvee I$ may not exist in $(P, \leq, \bigvee, \bigwedge)$, but it certainly exists in F_P. We show

that $\bigvee I$ is the least element of $\psi^{-1}(P/I)$.
Suppose that $x \in F_P$ such that $x \leq \bigvee I$ and $\mathrm{id}_P(x) = \psi(x) \in P/I$. So for all $i \in I$, $i \in \mathrm{id}_P(x)$, i.e. $i \leq x$. Therefore $\bigvee I \leq x$, and so $x = \bigvee I$.
Therefore, ψ is lower bounded, and we note that $\beta_\psi(\mathrm{id}_P(p)) = p$ for $p \in P$. By duality, the homomorphism $\psi^d : F_P \to \mathrm{Fil}_1(P)$ is upper bounded and we note that $\alpha_{\psi^d}(\mathrm{fil}_P(p)) = p$ for all $p \in P$. Define $h(x) = (\psi(x), \psi^d(x))$ for all $x \in F_P$. Since ψ and ψ^d are homomorphisms, so is h. Furthermore, since F_P is generated by P, $h(F_P) = h(\mathrm{Sg}_{F_P}(P)) = \mathrm{Sg}_{\mathrm{Idl}_0(P)\times \mathrm{Fil}_1(P)}(h(P)) = \mathrm{Sg}_{\mathrm{Idl}_0(P)\times \mathrm{Fil}_1(P)}(\{(\mathrm{id}_P(p), \mathrm{fil}_P(p)) : p \in P\}) = \mathrm{PC}(P)$.

Lemma 4.0.9. *Let P be finite, $Y \subseteq F_P$, and $d \in P$. Then $d \in \mathrm{Sg}_{F_P}(Y)$ iff $h(d) \in \mathrm{Sg}_{\mathrm{PC}(P)}(h(Y))$.*

Proof. First, suppose $d \in \mathrm{Sg}_{F_P}(Y)$. Then there exists a term t such that $d = t^{F_P}(Y)$. Thus, as h is a homomorphism, $h(d) = t^{\mathrm{PC}(P)}(h(Y)) \in \mathrm{Sg}_{\mathrm{PC}(P)}(h(Y))$.
Now suppose that $h(d) \in \mathrm{Sg}_{\mathrm{PC}(P)}(h(Y))$. Thus, there is a term t such that $h(d) = t^{\mathrm{PC}(P)}(h(Y)) = h(t^{F_P}(Y))$. Since $d \in P$, $(\psi(t^{F_P}(Y)), \psi^d(t^{F_P}(Y)) = h(t^{F_P}(Y)) = h(d) = (\mathrm{id}_P(d), \mathrm{fil}_P(d))$, and so $\psi(t^{F_P}(Y)) = \mathrm{id}_P(d)$ and $\psi^d(t^{F_P}(Y)) = \mathrm{fil}_P(d)$. Therefore, $d = \beta_\psi(\mathrm{id}_P(d)) \leq t^{F_P}(Y) \leq \alpha_{\psi^d}(\mathrm{fil}_P(d)) = d$, and so $d = t^{F_P}(Y) \in \mathrm{Sg}_{F_P}(Y)$. $\square$

Since the map $h|_P : P \to \mathrm{PC}(P)$ is an isomorphism and it is customary to identify P and $\{(\mathrm{id}_P(p), \mathrm{fil}_P(p)) : p \in P\}$, we can rephrase Lemma 4.0.9 above as:

Let P be finite, $Y \subseteq F_P$, and $d \in P$. Then $d \in \mathrm{Sg}_{F_P}(Y)$ iff $d \in \mathrm{Sg}_{\mathrm{PC}(P)}(h(Y))$.

Lemma 4.0.10. *Let P be finite, $F_P = \langle P \mid r_1 = s_1, \ldots, r_m = s_m\rangle$, and $d \in F_P$. Then there exists a finite lattice B and an epimorphism $f : F_P \to B$ such that $f^{-1}(f(d)) = \{d\}$.*

Proof. Let t be a term representing d in F_P. Define $L_1 = \mathrm{PC}(P)^{(\vee\wedge)^l}$ and $L_2 = \mathrm{PC}(P)^{(\wedge\vee)^l}$, where $l + 1 > \max\{c(t), c(r_i), c(s_i) : 1 \leq i \leq m\}$. Now, L_1 is a finite join-subsemillatice of F_P and L_2 is a finite meet-subsemilattice of F_P. Since $l \geq 1$, the least element of F_P is in L_1 and the greatest element of F_P is in L_2, so L_1 and L_2 must both be lattices. Since L_1 and L_2 both satisfy the relations of F_P, there exist homomorphisms $f_1 : L_1 \to F_P$ and $f_2 : L_2 \to F_P$ such that $f_1 \mid_P = f_2 \mid_P = id_P$.
Let $\vee_i$ and $\wedge_i$ denote the operations of L_i $(i = 1, 2)$. Then $a \wedge_1 b = \bigvee(\downarrow a \cap \downarrow b) \leq a \wedge b$ for $a, b \in L_1$, $a \vee_2 b = \bigwedge(\uparrow a \cap \uparrow b) \geq a \vee b$ for $a, b \in L_2$, $a \vee_1 b = a \vee b$ for $a, b \in L_1$, and $a \wedge_2 b = a \wedge b$ for $a, b \in L_2$. Note that $f_1(a) = a$ for $a \in L_1$ and $f_2(b) = b$ for $b \in L_2$.
We argue that $f_1(a) \leq a$ for $a \in F_P$: Define $S_1 = \{y \in L : f_1(y) \leq y\}$. If $b, c \in S_1$, then we must have $b \wedge c \in S_1$ since $f_1(b \wedge c) = f_1(b) \wedge_1 f_1(c) \leq f_1(b) \wedge f_1(c) \leq b \wedge c$. Since we must also have $b \vee c \in S_1$, S_1 is a sublattice of F_P containing P, forcing $S_1 = F_P$. A dual argument shows that $f_2(a) \geq a$ for $a \in F_P$.
Define $f : F_P \to L_1 \times L_2$ by $f(a) = (f_1(a), f_2(a))$ for $a \in F_P$, and let $B = f(F_P)$. Since L_1 and L_2 are finite, B must also be finite. By the choice of l, $d \in L_1 \cap L_2$, and by what we have shown

above $f(d) = (f_1(d), f_2(d)) = (d, d)$. If $a \in f^{-1}(f(d))$, $(d, d) = f(d) = f(a) = (f_1(a), f_2(a))$. But then $d = f_1(a) \leq a \leq f_2(a) = d$, and so $f^{-1}(f(d)) = \{d\}$ as desired. □

Lemma 4.0.11. *Let P be finite, U be a sublattice of F_P, and $d \in F_P - U$. Then there is a finite lattice B and a homomorphism $f : F_P \to B$ such that $f(d) \notin f(U)$.*

Proof. Let B and $f : F_P \to B$ be as in the proof of Lemma 4.0.10. Suppose for a contradiction that $f(d) \in f(U)$. Then there exists $u \in U$ such that $f(u) = f(d)$. But by Lemma 4.0.10, $d = u \in U$, a contradiction. Therefore $f(d) \notin f(U)$, as desired. □

Theorem 4.0.12. *The generalized word problem for F_P is solvable.*

Proof. Let $d \in F_P$ and Y be a finite subset of F_P. Furthermore, let L_1, L_2, B, and $f : F_P \to B$ be as in the proof of Lemma 4.0.10. List all of the elements from $\mathrm{Sg}_{L_1 \times L_2}(f(Y))$, which is a sublattice of B; there will only be finitely many such elements since B is finite. Then, check to see if $f(d) \in \mathrm{Sg}_{L_1 \times L_2}(f(Y))$. Since the word problem for finitely presented lattices is solvable by **Dean's Theorem**, this process is recursive. If we find that $f(d) \in \mathrm{Sg}_{L_1 \times L_2}(f(Y))$ then $d \in \mathrm{Sg}_{F_P}(Y)$, for if $d \in \mathrm{Sg}_{F_P}(Y)$ then $f(d) \notin f(\mathrm{Sg}_{F_P}(Y)) = \mathrm{Sg}_{L_1 \times L_2}(f(Y))$ by Lemma 4.0.11. If we find instead that $f(d) \notin \mathrm{Sg}_{L_1 \times L_2}(f(Y))$, then of course $d \notin \mathrm{Sg}_{F_P}(Y)$. □

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