GAME THEORETIC APPROACHES TO COMMUNICATION OVER MIMO INTERFERENCE CHANNELS IN THE PRESENCE OF A MALICIOUS JAMMER

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To my brother, Chris, for teaching me more than I ever knew.

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> "Our science, which we loved above everything, had brought us together. It appeared to us as a flowering garden. In this garden there were well-worn paths where one might look around at leisure and enjoy one-self without effort, especially at the side of a congenial companion. But we also liked to seek out hidden trails and discovered many an unexpected view which was pleasing to our eyes; and when the one pointed it out to the other, and we admired it together, our joy was complete."
>
> –*David Hilbert*

I have had the great fortune of enjoying many companions in my journey through the garden. Some have shown me their favorite plots and encouraged me to cultivate my own. Others have permitted me to lead them about. Many more had only a passing interest in the garden; nonetheless they looked after me as I traipsed about.

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Jason Marden has—inexplicably—believed in me since we met in 2006. He has provided me with countless opportunities throughout my graduate career, and I will forever be grateful for his mentorship.

I consider myself immensely fortunate to have friends too numerous to acknowledge by name. My extended ohana is filled with wonderful and thoughtful people who have played a large part in shaping me into the person I am today.

My family was rightfully dubious of my decision to continue my studies in Hawai'i. If you ask, they will gladly tell you how proud they are of me. Allow me to counter with how proud I am of them: Their patience, support, and love have never wavered in the face of unimaginable setbacks. Thank you Pam, Don, and Ann.

Lastly, I ponder where I would be without my wife, Serena. She has been by my side through my darkest hours and during my shining moments. I can only hope that in the coming decades I can similarly be there for her.

ABSTRACT

This dissertation considers a system consisting of self-interested users of a common multiple-input multiple-output (MIMO) channel and a jammer wishing to reduce the total capacity of the channel. In this setting, two games are constructed that model different system-level objectives. In the first—called "utility games"—the users maximize the mutual information between their transmitter and their receiver subject to a power constraint. In the other (termed "cost games"), the users minimize power subject to an information rate floor. A duality is established between the equilibrium strategies in these two games, and it is shown that Nash equilibria always exist in utility games. Via an exact penalty approach, a modified version of the cost game also possesses an equilibrium. Additionally, multiple equilibria may exist in utility games, but with mild assumptions on users' own channels and the jammer-user channels, systems with no user-user interference, there can be at most one Nash equilibrium where a user transmits on all of its subchannels. A similar but weaker result is also found for channels with limited amounts of user-user interference. Two distributed update processes are proposed: gradient-play and best-response. The performance of these algorithms are compared via software simulation. Finally, previous results on network-level improvement via stream control are shown to carry over when a jammer is introduced.

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NOMENCLATURE

x^*	Complex conjugate of x
$\mathrm{Re}[x]$	Real part of complex x
$\mathrm{Im}[x]$	Imaginary part of complex x
$\jmath$	The imaginary unit: $\jmath^2 = -1$
$x^\intercal$	Transpose of x
$A^{-\intercal}$	Transpose of A^{-1}
$x^\dagger$	Complex conjugate transpose of x
$\mathrm{diag}(x)$	Diagonal matrix formed from vector x
$\mathrm{diag}(A)$	Vector of diagonal elements of square matrix A
$\mathrm{Tr}\, A$	Trace of matrix A
$\mid A \mid$	Determinant of matrix A
$A \otimes B$	Kronecker product of A and B
$\mathrm{vec}(A)$	(Complex) Vectorization operator on matrix A: The column vector formed by stacking the columns of A from left to right
$\overrightarrow{A}$	Real vectorization operator on matrix A: The column vector formed by stacking the columns of $\mathrm{Re}\, A$ and then stacking the columns of $\mathrm{Im}\, A$
$\mathrm{CRI}(A)$	Real and imaginary blocking of the complex matrix A: $\begin{bmatrix} \mathrm{Re}\, A & -\mathrm{Im}\, A \\ \mathrm{Im}\, A & \mathrm{Re}\, A \end{bmatrix}$
$\\|A\\|$	Frobenius norm for matrix A: $\sqrt{\mathrm{Tr}(A^\dagger A)}$
$\\|x\\|_2$	Euclidean 2-norm for column vector x: $\sqrt{x^\dagger x}$
$\mathbf{E}\, x$	Expected value of x
$\lambda_i(A)$	The i-th largest eigenvalue of Hermitian matrix A
$\overline{\lambda}(A)$	The largest eigenvalue of Hermitian matrix A
$\underline{\lambda}(A)$	The least eigenvalue of Hermitian matrix A

$\overline{\sigma}(A)$	The largest singular value of A
$\underline{\sigma}(A)$	The smallest singular value of A
$A \sim B$	Matrices A and B are similar
$\log x$	Base e logarithm of x
$(a)^+$	$\max\{0, a\}$
$f : A \rightrightarrows B$	The correspondence f maps subsets of A to subsets of B
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{R}^n$	Euclidean space of dimension n
$\mathbb{R}^n_-$	Nonpositive orthant in $\mathbb{R}^n$
$\mathbb{R}^n_+$	Nonnegative orthant in $\mathbb{R}^n$
$\mathbb{R}^n_{++}$	Strictly positive orthant in $\mathbb{R}^n$
$\mathbb{C}^{n \times m}$	Space of n by m matrices with complex entries
$\mathbb{H}^n$	Space of Hermitian matrices of dimension n
$\mathbb{H}^n_+$	Cone of Hermitian positive semidefinite matrices of dimension n
$\mathbb{H}^n_{++}$	Cone of Hermitian positive definite matrices of dimension n
$\mathcal{N}(\mu, \Sigma)$	The μ-mean Gaussian distribution with covariance Σ
$\mathcal{O}(g(x))$	Collection of terms whose magnitudes are bounded by a constant times $g(x)$ for x sufficiently close to zero
$x > y$	For $x, y \in \mathbb{R}^n$, $x_i > y_i$ for all $i = 1, \ldots, n$
$A \succeq B$	$A - B$ is positive semidefinite
$A \succ B$	$A - B$ is positive definite
I	Identity matrix of appropriate dimension
$\mathbb{1}$	Column vector of all ones of appropriate dimension: $(1, \ldots, 1)^\intercal$
$-k$	When k belongs to a finite set of indices, the indices other than k

CHAPTER 1
INTRODUCTION

For as long as humans have had the ability to communicate, others have desired to block that communication. Through the millennia it has become clear that knowledge is power, and the ability to interfere with the transmission of information can give one a leg up on the competition.

This dissertation aims to model and predict a sensible outcome when a jammer wishes to disrupt a particular form of communication: multiple users transmitting through a common multiple-input multiple-output (MIMO) channel. In so doing optimal strategies and performance lower bounds are found for both users and jammers. We make no assumptions about who is on what "side" or what is being communicated. We simply analyze a modern incarnation of an ancient problem. Who gets to say what? Can they be stopped?

The contents of this dissertation are not necessarily presented chronologically; rather, they have been arranged to give continuity to the exposition. Chapter 1 lays out the motivation and sets the stage for the work. It then aims to give a sense of the relevant mathematical tools used. Chapter 2 builds on this foundation by describing the games used to model the systems under consideration and provides conditions under which these games have nice properties. Chapter 3 contains both theoretical and numerically simulated results on how the agents in these games might reach a steady state. Finally, chapter 4 summarizes the novel results and suggests areas in which they can be extended.

The background section in this chapter contains many concepts that are not due to this author. To help clarify as much as possible what is novel work and what has been reprinted here for the sake of coherence, all results that are not due to the author are labeled "Propositions." Their proofs have been included where instructive and omitted otherwise. The main contributions of this dissertation are contained in the various "Theorems" and supporting "Lemmas."

1.1 Motivation

Every day, we transmit vital information through media in which there is interference. The interference may be naturally occurring , but increasingly it is a result of the transmissions of others. Cell phone communication, satellite transmissions, and many ad-hoc network configurations transmit over these so-called interference channels. Even wire-line media—such as DSL—experiences cross-talk between wire strands [21]. Sharing a channel means that our transmissions are subject to myriad pitfalls including eavesdropping, spoofing, and outright jamming.

It is in this context that we present a methodology based on game theory to plan for interference not only from self-interested neighbors but also from a malicious jammer. The theory of games enables prediction and control of steady states in systems comprising self-interested agents with no centralized authority.

Many domains of communication meet this criterion. For example, one approach to open-access spectrum sharing is a selfish distributed power allocation scheme [20]. Additionally, due to significant latency, distributed control of satellite communications has been under development for quite some time [48]. The survey [7] also describes the application of game theory to multiple layers of modern wireless networks.

This dissertation is the nexus of two areas of game theoretic power control in wireless networks. On one hand, [23] considers an interference channel with orthogonal tones in the presence of a jammer. On the other hand, [2] treats the full (nonorthogonal) MIMO Gaussian interference channel problem without a jammer. As far as we know, this work is the first to consider both the full MIMO Gaussian interference channel and a jammer.

1.2 Literature Review

Wireless communication networks are proliferating and connecting an increasing number of devices. On each device, the tendency is toward a growing quantity of transmit and receive antennas; the resulting increase in information packed into a MIMO channel demands intelligent control. Alongside this increase in information capacity comes a higher risk of malicious agents to disrupt the data flow.

This work builds on research published in the fields of game theory, wireless MIMO communication, and network information theory. The existence of equilibrium points in finite non-cooperative games was established by Nash [35,36]. Glicksberg proved the existence of Nash equilibrium points in *mixed* strategies in n-person infinite games with continuous payoffs and convex strategy spaces in [22]. The result used heavily in this work is due to Rosen in [42], where concavity of every agent's payoff function with respect to its own action is shown to be a sufficient condition for existence of Nash equilibria in pure strategies. Also in [42], Rosen defines a property of games called "diagonally strictly concave" which is sufficient to show that a game has a unique Nash equilibrium.

The communication theory employed in this work builds on the concept of information theory due to Shannon in [46] and the resulting field [9].

Cellular telephone networks have fostered much work in optimizing wireless channels that are many-to-one [17], and one-to-many [41]. Demirkol and Ingram [11–13] have developed centralized algorithms to maximize total capacity in networks of MIMO communicators. The question of whether users of the interference channel can cooperate in order to improve their performance is pertinent here. That is, if communicators are transmitting on frequencies that interfere with each other, perhaps they can recognize that it might be in their best interests to coordinate their efforts. To this end, conditions for uniqueness of the Nash bargaining solution appear in [8] and dynamic bargaining processes are presented in [31]. Our version of cooperative control to optimize system-level performance comes in the form of an extension of the stream control approach in [2] and appears in Section 3.3.

Accounting for malicious jammers is a recent development in networked communications. Originating with a personal single-input single-output (SISO) channel and a single jammer, Medard [34] derived the capacity for this two-player model. In cellular networks, equilibrium existence results have been obtained for games between the cell tower and a jammer [15, 44]. In [29], a saddle point is found for a game between a jammer and a MIMO communicator. More recently, existence results have been found for multiplayer games between cellular users and a jammer [45]. These works—especially the cellular network ones—rely heavily on the many-to-one or one-to-many paradigms where joint encoding or decoding may be used to strengthen results.

The direct ancestors of this dissertation are the papers that combine elements from all of the above areas. Arslan, et al. [2] proved existence and gave conditions for uniqueness of Nash equilibria in a multiplayer MIMO game between users with no jammer. Also, [2] proposes a negotiated stream control approach in an effort to maximize total network capacity. Later, these authors built on the results of [39], again in the absence of a jammer, to give conditions for equilibrium existence in the dual game where MIMO transmit power is minimized subject to communication rate requirements [3, 4].

Finally, the orthogonal-tone multiplayer game with a jammer was shown to possess equilibria in [23]. The research proposed herein extends this work to the general MIMO case.

1.3 Background

In this section, we provide a broad introduction into the mathematical and engineering concepts used in this dissertation. For more in-depth treatments, the reader is referred to [19, 38] for game theory, [9] for information theory, and [52] for communication theory.

1.3.1 Fundamentals of Game Theory

Game theory is the mathematical study of systems of self-interested agents. Many classes of games have been identified so that their properties can be enumerated clearly.

As illustrated in Fig. 1.1, in this work we consider *strategic games*: those in which agents select their strategies simultaneously. Within strategic games we are interested in *multiplayer* systems (those with more than two agents) with *infinite* strategy sets (e.g. continua).

Components of a Game

Games of this class consist of three components: a set of agents (also called “players”), strategies available to each agent, and an objective function for each agent.

The games we consider comprise a finite set of p agents. These agents are identified by integers and are collectively known as the player set $\mathcal{P} = \{1, 2, \ldots, p\}$.

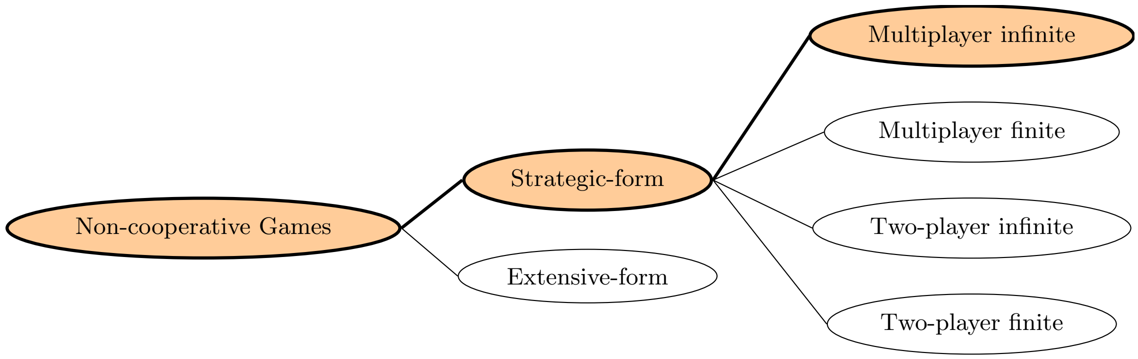


Figure 1.1: A partial taxonomy of games. The class of games utilized in this dissertation is shown as shaded.

Each agent possesses a set of available actions. The strategy set for agent k is denoted $\mathcal{S}_k$. In general, these sets can be of any cardinality, but as mentioned above we will be considering infinite strategy sets in this work, namely, subsets of Euclidean vector spaces. We will often find cause to refer to the Cartesian product of all of the agents' strategy sets,

$$\mathcal{S} = \prod_{k \in \mathcal{P}} \mathcal{S}_k, \tag{1.1}$$

the set of *joint strategies*.

The final necessary component of a game is an objective function for each agent. Each agent k attempts to optimize $f_k : \mathcal{S} \to \mathbb{R}$ but is only able to control its own action. This partial control is essential to the theory of games. As such, we will often refer to a joint action $s \in \mathcal{S}$ as the concatenation of two elements

$$s = (s_k, s_{-k}), \tag{1.2}$$

where s_k is the strategy employed by agent k and s_{-k} is the collection of strategies employed by the agents in $\mathcal{P} \backslash \{k\}$.

By combining these three components, the shorthand notation $\Gamma(\mathcal{P}, \mathcal{S}, \{f_k\})$ refers to the game with player set $\mathcal{P}$, joint strategy set $\mathcal{S}$, and set of objective functions $\{f_k\}$.

Solution Concepts

There are many notions for what it means to "solve" a game. In this work we will utilize Nash equilibria [35, 36] and later generalized Nash equilibria [10].

The essence of a Nash equilibrium is that no self-interested agent would unilaterally change its strategy if it cannot improve its objective. Likewise, no self-interested agent would fail to optimize its objective if given the chance. We formalize this concept with the definition of an equilibrium in non-cooperative games due to Nash [35].

Definition 1.1 (Nash equilibrium) A Nash equilibrium of $\Gamma(\mathcal{P}, \mathcal{S}, \{f_k\})$, where agents attempt to maximize their objectives, is a joint action $\hat{s} \in \mathcal{S}$ such that for all $k \in \mathcal{P}$,

$$f_k(\hat{s}_k, \hat{s}_{-k}) \geq f_k(s_k, \hat{s}_{-k}) \quad \forall s_k \in \mathcal{S}_k. \tag{1.3}$$

The definition is similar (with the inequality flipped) in games where agents attempt to minimize their objectives.

The concept of Nash equilibrium is not applicable when the strategies available to an agent depend on the actions of others. In this case, the strategies available to agent k when the other agents play s_{-k} are known as its *feasible set* which is denoted $\mathcal{S}_k(s_{-k})$. One can still imagine a joint strategy and associated feasible sets in which no agent can unilaterally improve. This scenario is known as a generalized Nash equilibrium.

Definition 1.2 (Generalized Nash equilibrium) A generalized Nash equilibrium of the maximization game $\Gamma(\mathcal{P}, \mathcal{S}, \{f_k\})$ is a joint action $\hat{s} \in \mathcal{S}(\hat{s}) = \prod_{k \in \mathcal{P}} \mathcal{S}_k(\hat{s}_{-k})$ such that for all $k \in \mathcal{P}$,

$$f_k(\hat{s}_k, \hat{s}_{-k}) \geq f_k(s_k, \hat{s}_{-k}) \quad \forall s_k \in \mathcal{S}_k(\hat{s}_{-k}). \tag{1.4}$$

The definition is similar (with the inequality flipped) in games where agents attempt to minimize their objectives.

Observe that when the agents' strategy sets are decoupled, generalized Nash equilibria reduce to Nash equilibria of Definition 1.1.

An important concept related to unilateral improvement is that of best-response.

Definition 1.3 (Best-response) The best-response correspondence for agent k in the maximization game $\Gamma(\mathcal{P}, \mathcal{S}, \{f_k\})$ is a point-to-set mapping $\mathrm{BR}_k : \mathcal{S}_{-k} \rightrightarrows \mathcal{S}_k$ where

$$\mathcal{S}_{-k} = \prod_{j \in \mathcal{P} \setminus \{k\}} \mathcal{S}_j \tag{1.5}$$

and

$$\mathrm{BR}_k(s_{-k}) = \operatorname*{arg\,max}_{s_k \in \mathcal{S}_k(s_{-k})} f_k(s_k, s_{-k}). \tag{1.6}$$

The definition is similar (with $\arg\max$ replaced by $\arg\min$) in minimization games.

This definition seems quite natural in that if agent k finds itself faced with the opponents' action s_{-k}, the most immediate response would be to try to optimize its objective subject to this state of the world.

Individual agents' best-responses can be concatenated to form the joint best-response $\mathrm{BR} : \mathcal{S} \rightrightarrows \mathcal{S}$ where

$$\mathrm{BR}(s) = \begin{pmatrix} \mathrm{BR}_1(s_{-1}) \\ \mathrm{BR}_2(s_{-2}) \\ \vdots \\ \mathrm{BR}_p(s_{-p}) \end{pmatrix}. \tag{1.7}$$

The joint best-response is relevant to the discussion of Nash equilibria because any Nash equilibrium must be a fixed point of $\mathrm{BR}(\cdot)$. That is, $\hat{s}$ is a (generalized) Nash equilibrium of a game if and only if

$$\hat{s} \in \mathrm{BR}(\hat{s}). \tag{1.8}$$

With this observation, we can now point out that not all games possess Nash equilibria.

Example 1.1 (Leader-follower game) Consider the game with two players: the leader (player 1) and the follower (player 2). Both players have the identical strategy sets $\mathcal{S}_1 = \mathcal{S}_2 = [0, 1] \subset \mathbb{R}$. The leader wants to maximize the distance between itself and the follower: $f_1(s_1, s_2) = |s_1 - s_2|$. The follower, on the other hand, wants to maximize the opposite: $f_2(s_1, s_2) = -|s_1 - s_2|$.

Qualitatively, the best response of the leader will be to go as far away from the follower as possible. Conversely, given the position of the leader, the follower will elect to go exactly there. Indeed, these correspondences can be made explicit:

$$\mathrm{BR}_1(s_2) = \begin{cases} 1, & s_2 < 0.5 \\ \{0, 1\}, & s_2 = 0.5 \\ 0, & s_2 > 0.5 \end{cases} \tag{1.9a}$$

$$\mathrm{BR}_2(s_1) = s_1 \tag{1.9b}$$

These correspondences are plotted in Fig. 1.2. The relevant observation to make is that the graphs of these two correspondences never intersect. That is, there is no $(\hat{s}_1, \hat{s}_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ such that $(\hat{s}_1, \hat{s}_2) \in \mathrm{BR}(\hat{s}_1, \hat{s}_2)$.

Thus, the leader-follower game does not possess a Nash equilibrium[1].

Example 1.1 is important because it illustrates that even "small" games with simple strategy sets may not possess a Nash equilibrium. In general, there is no reason to believe that larger games with more complicated strategy sets should either.

So under what conditions are Nash equilibria guaranteed to exist? Rosen [42] described a class called *concave games* that have the existence property.

[1]It should be pointed out that the leader-follower game has no Nash equilibria in *pure* strategies. As pure strategies are the sole focus of this dissertation, we omit references to solutions in mixed strategies [19].

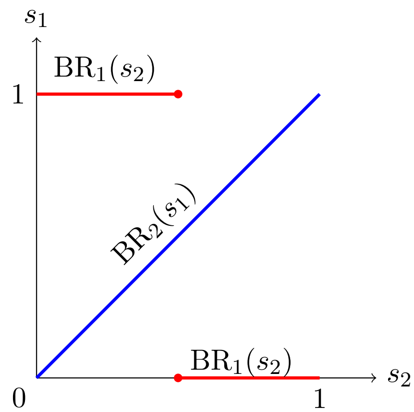


Figure 1.2: Best-response correspondences for both the leader and the follower plotted on the same axes. If $s_2 = 0.5$, then the leader is indifferent between 0 and 1, i.e. $\mathrm{BR}_1(0.5) = \{0, 1\}$.

Proposition 1.1 (Adapted from Theorem 1 in [42]) *The p-player game*

$$\Gamma(\mathcal{P}, \mathcal{S}, \{f_k\}),$$

where

- $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_p$*,*
- $\mathcal{S}_k \; \forall k \in \mathcal{P}$ *is a convex, closed, and bounded subset of* $\mathbb{R}^{N_k}$ *for some positive integer* N_k*, and*
- $f_k(s_k, s_{-k})$ *is continuous in* (s_k, s_{-k}) *and concave in* s_k *for all* $k \in \mathcal{P}$*,*

possesses a Nash equilibrium.

To investigate when concave games have a unique equilibrium, Rosen went on to develop the concept of *diagonally strictly concave* payoffs. He showed that it is a sufficient condition for uniqueness in concave games.

For a vector of nonnegative weights $r \in \mathbb{R}^p_+$, denote the weighted sum of payoffs in a concave game by

$$\sigma(s, r) = \sum_{k \in \mathcal{P}} r_k f_k(s). \tag{1.10}$$

Definition 1.4 (Pseudogradient) The pseudogradient $g(s, r)$ of $\sigma(s, r)$ is the vector of gradients of each agent's payoff function with respect to its own strategy:

$$g(s, r) = \begin{pmatrix} r_1 \nabla_{s_1} f_1(s) \\ r_2 \nabla_{s_2} f_2(s) \\ \vdots \\ r_p \nabla_{s_p} f_p(s) \end{pmatrix}. \tag{1.11}$$

Definition 1.5 (Diagonally strictly concave) The function $\sigma(\cdot, r)$ is diagonally strictly concave for fixed $r \in \mathbb{R}^p_+$ if for every $s^1 \neq s^0 \in \mathcal{S}$

$$(s^0 - s^1)^\intercal g(s^1, r) + (s^1 - s^0)^\intercal g(s^0, r) > 0. \tag{1.12}$$

This definition is critical in confirming when a concave game has a unique equilibrium. However, checking whether a game satisfies Definition 1.5 can be an onerous task. Fortunately, Rosen also provides a sufficient condition for diagonal strict concavity that is usually much easier to verify.

Proposition 1.2 (Adapted from Theorem 6 in [42]) *A sufficient condition that the function $\sigma(s, r)$ be diagonally strictly concave for $s \in \mathcal{S}$ and fixed $r \in \mathbb{R}^p_{++}$ is*

$$G(s, r) + G(s, r)^\intercal \prec 0 \quad \forall s \in \mathcal{S}, \tag{1.13}$$

where $G(s, r)$ is the Jacobian of $g(s, r)$:

$$G(s, r) = \begin{bmatrix} \frac{\partial g(s,r)}{\partial s_1} & \frac{\partial g(s,r)}{\partial s_2} & \cdots & \frac{\partial g(s,r)}{\partial s_p} \end{bmatrix}. \tag{1.14}$$

The full name "Jacobian of the pseudogradient" is cumbersone, so we elect to use the nonstandard term *pseudohessian* to refer to the $G(s, r)$ matrix. We will often make use of the pseudogradient of a game and its Jacobian without the reference to a general weight vector, r. To simplify notation, we adopt the convention that the omission of a second argument implies

$$g(s) = g(s, \mathbb{1}) \tag{1.15a}$$

$$G(s) = G(s, \mathbb{1}). \tag{1.15b}$$

Furthermore, we will often need to refer to the the symmetrized version of the pseudohessian which is denoted

$$\bar{G}(s) = G(s) + G^\intercal(s). \tag{1.16}$$

With these classifications, the background material on game theory can conclude with Rosen's result on uniqueness.

Proposition 1.3 (Adapted from Theorem 2 in [42]) *If $\sigma(s, r)$ is diagonally strictly concave for some $r \in \mathbb{R}^p_{++}$, then the equilibrium point guaranteed by Proposition 1.1 is unique.*

1.3.2 Fundamentals of Communication and Information Theories

The theories of communication and information are intimately intertwined. We largely lump them together, but a convenient distinction is that communication engineering deals with the "how" we transmit, and information theory is the study of "what" we transmit. A quick review of the sliver of these fields used in this work follows.

Information and Entropy

In order to quantify information and optimize its communication, we rely on the framework of information theory developed by Shannon [46].

Unless otherwise specified, log refers to the base e logarithm and information entropy is measured in nats. Let x be a vector-valued continuous random variable with support set $\mathcal{X}$.

Definition 1.6 (Entropy) The entropy of a continuous random variable x with density $p(\cdot)$ is

$$h(x) = -\int_{\mathcal{X}} p(\xi) \log p(\xi) \, d\xi. \tag{1.17}$$

The entropy of a random variable is a quantification of its uncertainty. When considering the relative uncertainty between two distributions, relative entropy is the appropriate metric.

Definition 1.7 (Relative entropy) The relative entropy between two densities $p(\cdot)$ and $q(\cdot)$ of a random variable x is

$$D(p||q) = \int p(\xi) \log \frac{p(\xi)}{q(\xi)} \, d\xi, \tag{1.18}$$

where the integral is performed over the support of x under p and we define $0 \log \frac{0}{0} = 0$.

The definition of entropy can be extended to cover joint and conditional random variables.

Definition 1.8 (Joint entropy) The joint entropy of n random variables $x_1, x_2, \ldots, x_n$ with support set $\mathcal{X}^n$ and joint distribution $p^n(\cdot)$ is

$$h(x_1, x_2, \ldots, x_n) = -\int_{\mathcal{X}^n} p^n(\xi) \log p^n(\xi) \, d\xi. \tag{1.19}$$

Definition 1.9 (Conditional entropy) For two random variables x and y with supports $\mathcal{X}$ and $\mathcal{Y}$, joint distribution $p_{x,y}(\cdot,\cdot)$, and conditional distribution $p_{x|y}(\cdot|\cdot)$, the conditional entropy of x on y is

$$h(x|y) = -\int_{\mathcal{X}\times\mathcal{Y}} p_{x,y}(\xi,\gamma) \log p_{x|y}(\xi|\gamma) \, d\xi \, d\gamma. \tag{1.20}$$

Finally, the reduction in uncertainty of one random variable based on the knowledge of another is termed their mutual information.

Definition 1.10 (Mutual information) The mutual information between two random variables x and y with joint distribution $p_{x,y}(\cdot,\cdot)$ and marginals $p_x(\cdot)$ and $p_y(\cdot)$ is

$$I(x;y) = \int p_{x,y}(\xi,\gamma) \log \frac{p_{x,y}(\xi,\gamma)}{p_x(\xi) p_y(\gamma)} \, d\xi \, d\gamma, \tag{1.21}$$

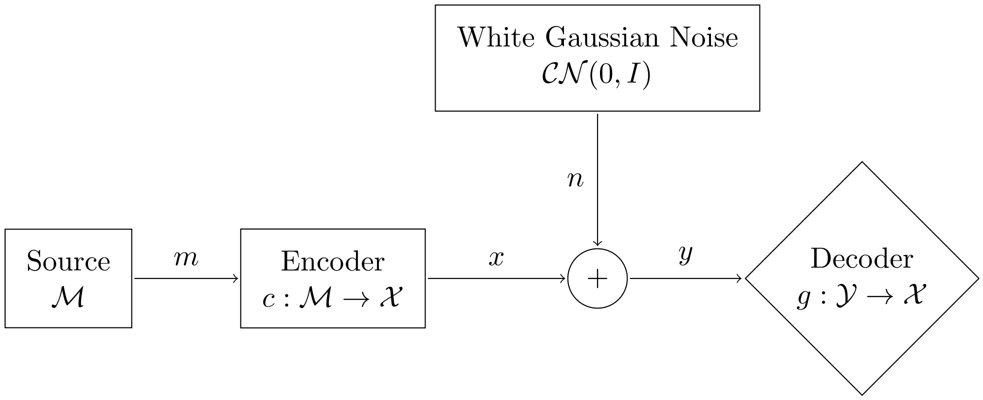


Figure 1.3: The AWGN channel model. A message m is encoded and enters the channel as transmission x where it encounters additive Gaussian noise n. The decoder acts on received signal y.

where the integral is performed over the joint support.

It is often useful to express mutual information as the difference in entropies. Using the definitions above, one can see that

$$I(x; y) = h(x) - h(x|y) \tag{1.22}$$

$$= h(y) - h(y|x) \tag{1.23}$$

$$= h(x) + h(y) - h(x, y) \tag{1.24}$$

$$= D\left(p_{x,y}(\cdot, \cdot) || p_x(\cdot) p_y(\cdot)\right). \tag{1.25}$$

A helpful interpretation of (1.22) is that $I(x; y)$ represents the reduction in uncertainty in x due to knowing y.

SISO Communication over Gaussian Channels

The additive white Gaussian noise (AWGN) channel model is used to model the sending of information from a transmitter to a receiver. As illustrated in Fig. 1.3, the model consists of a message *source*, $\mathcal{M}$, which periodically emits a message $m \in \mathcal{M}$ according to a distribution $\pi_{\mathcal{M}}(\cdot)$. The only assumptions we place on $\mathcal{M}$ and $\pi_{\mathcal{M}}(\cdot)$ are that they conform to the Source-Channel Coding Theorem [9, Theorem 7.13.1]. This allows us, without loss of optimality, to consider only the task of encoding for the AWGN channel and leave the job of source coding (compression) to someone else.

Before message m is transmitted, it is *encoded* with the mapping $c : \mathcal{M} \to \mathcal{X} \subseteq \mathbb{C}^n$. The choice of $c(\cdot)$ and the distribution of the vectors x in the codebook $\mathcal{X}$ it sends through the channel are critical to the performance of the system, and will be discussed in the MIMO subsection.

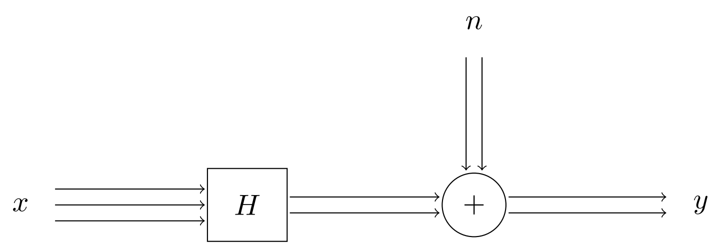


Figure 1.4: The MIMO AWGN channel model. In this instance, $N_t = 3$ and $N_r = 2$.

MIMO Communication over Gaussian Channels

When the transmitter and receiver have multiple antennae, the channel is said to be multiple-input multiple-output (MIMO). The multiple paths that information streams can travel are referred to as subchannels. When there are an equal number of receivers as transmitters and when there is no interference between subchannels, they are orthogonal.

In general, however, the channel inputs may interfere with each other (constructively or destructively). Therefore the single-user time-invariant memoryless MIMO AWGN channel is modeled as the linear system

$$y = Hx + n. \tag{1.26}$$

As in the SISO channel, n is an additive complex random vector of dimension N_r. The (i, j) component of $H \in \mathbb{C}^{N_r \times N_t}$ represents the gain (or attenuation) and phase shift between transmission x_j and received signal y_i. The channel is presented pictorially in Fig. 1.4.

In order to derive an optimal codebook for the MIMO channel, we first present the result that the entropy of any random vector with a given covariance is maximized when that vector has a Gaussian distribution.

First, a lemma.

Lemma 1.4 *Let p and q be two probability density functions on the complex random n-vector x such that $\int xp(x)\,dx = \int xq(x)\,dx = 0$ and $\int x_i x_j^* p(x)\,dx = \int x_i x_j^* q(x)\,dx$ for all $1 \leq i, j \leq n$. Also, let A be a quadratic form on $\mathbb{C}^n$ (i.e. $A : \mathbb{C}^n \to \mathbb{R}$ such that $A(x) = x^\dagger Ax + c$, $c \in \mathbb{R}$). Then*

$$\int A(x)p(x)\,dx = \int A(x)q(x)\,dx. \tag{1.27}$$

PROOF Expand the first integral:

$$\int A(x)p(x)\,dx = \int p(x)\left(x^\dagger Ax + c\right) dx \tag{1.28}$$

$$= \int p(x)\left(c + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i A_{ij} x_j^*\right) dx \tag{1.29}$$

$$= c\int p(x)\,dx + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\int p(x)x_i x_j^*\,dx \tag{1.30}$$

$$= c\int q(x)\,dx + \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\int q(x)x_i x_j^*\,dx \tag{1.31}$$

$$= \int A(x)q(x)\,dx. \tag{1.32}$$

□

Proposition 1.5 (Gaussian maximizes entropy) *If the random vector $x \in \mathbb{C}^n$ has zero mean and covariance $K = \mathbf{E}\,xx^\dagger$, then*

$$h(x) \le \frac{1}{2}\log\left((2\pi e)^n \det K\right) \tag{1.33}$$

with equality if and only if $x \sim \mathcal{CN}(0, K)$.

PROOF (FROM [9] AND [37]) Let $\phi_K(\cdot)$ be the $\mathcal{CN}(0, K)$ probability density function. Now, to prove the claim, recognize that $\log\phi_K(x)$ is a quadratic form as in Lemma 1.4:

$$\phi_K(x) = \frac{e^{-\frac{1}{2}x^\dagger K^{-1}x}}{\sqrt{(2\pi)^n \det K}} \tag{1.34}$$

$$\log\phi_K(x) = -\frac{1}{2}x^\dagger K^{-1}x - \frac{1}{2}\log\left((2\pi)^n \det K\right), \tag{1.35}$$

which is in the form $x^\dagger Ax + c$. Let $\hat{\phi}(\cdot)$ be any density that satisfies the statistical requirements: $\int\hat{\phi}(x)x\,dx = 0$ and $\int\hat{\phi}(x)x_i x_j^*\,dx = K_{ij} = \mathbf{E}\,x_i x_j^*$ for all $1 \le i, j \le n$. By Jensen's inequality,

$$0 \le D(\hat{\phi}||\phi_K) \tag{1.36}$$

$$= \int\hat{\phi}(x)\log\frac{\hat{\phi}(x)}{\phi_K(x)}\,dx \tag{1.37}$$

$$= \int\hat{\phi}(x)\log\hat{\phi}(x) - \int\hat{\phi}(x)\log\phi_K(x)\,dx \tag{1.38}$$

$$= -h(\hat{\phi}) - \int\phi_K(x)\log\phi_K(x)\,dx \tag{1.39}$$

$$= -h(\hat{\phi}) + h(\phi_K), \tag{1.40}$$

where (1.39) is a direct application of Lemma 1.4. Thus no zero-mean distribution with covariance K can beat $\mathcal{CN}(0, K)$ in terms of entropy. □

Proposition 1.5 appears in [9, Theorem 8.6.5] for real random variables and was generalized to the complex domain by Neeser and Massey [37]. Lemma 1.4 is referenced in [9] but is not explicitly proven.

In the interest of compactness, we will often refer to complex random variables as Gaussian as shorthand for circularly symmetric complex Gaussian.

Proposition 1.5 is critical in the development of the communication model used in this work. In particular, it is used in Proposition 1.6 to show that when faced with Gaussian noise, there is no better codebook than a Gaussian. Conversely, Proposition 1.6 also indicates that the noise—or more generally any source of interference—cannot be a larger source of additive disruption than if it too is Gaussian.

Proposition 1.6 (from [9, pp. 298]) *Given a MIMO additive noise channel with the following constraints:*

$$\mathbf{E}\, x = 0 \in \mathbb{C}^N \qquad \mathbf{E}\, xx^\dagger = K_x \tag{1.41a}$$

$$\mathbf{E}\, n = 0 \in \mathbb{C}^N \qquad \mathbf{E}\, nn^\dagger = K_n, \tag{1.41b}$$

and x independent of n, then the distributions $\hat{x} \sim \mathcal{CN}(0, K_x)$ and $\hat{n} \sim \mathcal{CN}(0, K_n)$ represent a saddle-point equilibrium of the game where player 1 picks the distribution of x to maximize $I(x; x+n)$ and player 2 picks the distribution of n to minimize $I(x; x+n)$.

Proof Let $n = \hat{n}$. The difference that player 1 sees if it switches away from $\hat{x}$ to x is

$$\begin{aligned} I(x; x+\hat{n}) - I(\hat{x}; \hat{x}+\hat{n}) &= h(x+\hat{n}) - h(\hat{n}) - h(\hat{x}+\hat{n}) + h(\hat{n}) && (1.42)\\ &= h(x+\hat{n}) - h(\hat{x}+\hat{n}) && (1.43)\\ &\leq 0, && (1.44) \end{aligned}$$

where (1.44) is a direct application of Proposition 1.5. That is, $\hat{x} + \hat{n}$ is Gaussian with zero mean and variance $K_x + K_n$, so no other distribution for $x + \hat{n}$ can have entropy larger than $h(\hat{x} + \hat{n})$.

Now consider the payoff to player 2 in response to $\hat{x}$:

$$\begin{aligned} I(\hat{x}; \hat{x}+n) &= h(\hat{x}+n) - h(\hat{x}+n|\hat{x}) && (1.45)\\ &= h(\hat{x}+n) - h(n), && (1.46) \end{aligned}$$

where the first line is simply the definition of mutual information, and the second is due to the independence of n and $\hat{x}$.

The entropy power inequality [49] states that for any two independent random vectors x and n,

$$e^{\frac{2}{N}h(x+n)} \geq e^{\frac{2}{N}h(x)} + e^{\frac{2}{N}h(n)}, \tag{1.47}$$

with equality if x and n are Gaussian. Equation (1.47) is equivalent to

$$h(x+n) \geq \frac{N}{2}\log\left(e^{\frac{2}{N}h(x)} + e^{\frac{2}{N}h(n)}\right). \tag{1.48}$$

Applying the entropy power inequality, we can continue from (1.46):

$$\begin{aligned}
h(\hat{x}+n) - h(n) &\geq \frac{N}{2}\log\left(e^{\frac{2}{N}h(\hat{x})} + e^{\frac{2}{N}h(n)}\right) - h(n) && (1.49)\\
&= \frac{N}{2}\left[\log\left(e^{\frac{2}{N}h(\hat{x})} + e^{\frac{2}{N}h(n)}\right) - \log e^{\frac{2}{N}h(n)}\right] && (1.50)\\
&= \frac{N}{2}\log\left(1 + \frac{e^{\frac{2}{N}h(\hat{x})}}{e^{\frac{2}{N}h(n)}}\right) && (1.51)\\
&\geq \frac{N}{2}\log\left(1 + \frac{e^{\frac{2}{N}h(\hat{x})}}{e^{\frac{2}{N}h(\hat{n})}}\right) && (1.52)\\
&= \frac{N}{2}\log\left(e^{\frac{2}{N}h(\hat{x})} + e^{\frac{2}{N}h(\hat{n})}\right) - h(\hat{n}) && (1.53)\\
&= h(\hat{x}+\hat{n}) - h(\hat{n}) && (1.54)\\
&= I(\hat{x};\hat{x}+\hat{n}), && (1.55)
\end{aligned}$$

where (1.52) must be true because the function

$$g_a(\xi) = \frac{N}{2}\log\left(1 + \frac{a}{e^{\frac{2}{N}\xi}}\right) \tag{1.56}$$

is nonincreasing for all $a \geq 0$ and $N \geq 1$ as illustrated in Fig. 1.5.

Therefore player 2 can also do no better than the Gaussian distribution. □

Proposition 1.6 justifies the use of the AWGN channel model throughout this dissertation. That is, if the covariance-limited signal is independent from the covariance-limited Gaussian noise, then all codebooks are dominated by Gaussian ones. Likewise, Gaussian codebooks face their largest challenge when confronted with additive Gaussian noise.

The *capacity*—the maximum amount of information that can travel through a channel with arbitrarily small probability of decoding error—was derived by Telatar [51].

Proposition 1.7 (Adapted from [51]) *The capacity of the AWGN channel* (1.26) *where the transmitter is restricted by an average power constraint* $\bar{c}$ *is achieved by a zero-mean Gaussian*

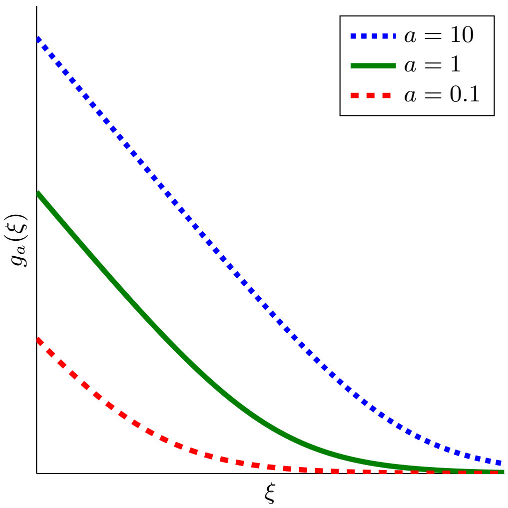


Figure 1.5: The monotonic function $g_a(\cdot)$ as in (1.56) parameterized by various $a \in \mathbb{R}_+$ with $N = 2$.

codebook with a covariance that is unitarily similar to

$$\tilde{Q} = \operatorname{diag}\left(\left[\left(\mu - \lambda_1(H^\dagger H)\right)^+, \cdots, \left(\mu - \lambda_{N_t}(H^\dagger H)\right)^+\right]\right), \tag{1.57}$$

where $\mu > 0$ *is chosen to satisfy* $\operatorname{Tr}\tilde{Q} = \bar{c}$*. The capacity achieved is*

$$\sum_{i=1}^{N_t} \left(\log(\mu\lambda_i(H^\dagger H))\right)^+ . \tag{1.58}$$

Proposition 1.7 complements Proposition 1.6 by assuring that when faced with an AWGN channel, the transmitter can achieve its maximum mutual information (with a Gaussian codebook). Additionally, Proposition 1.7 exhibits a salient feature in the transmitter's optimization problem: the selection of $\tilde{Q}$ is done via a water-filling process. That is, the convex maximization over the cone of all $N_t \times N_t$ positive semidefinite covariance matrices reduces to a scalar optimization problem.

MIMO Interference Channels

As a final layer in the foundation, we must model multiple users communicating through the same MIMO channel. That is, multiple copies of Fig. 1.4 all transmitting into the same medium and interfering with each other. We refer to this situation as a MIMO interference channel; in the literature it is sometimes also called the multi-user MIMO (MU-MIMO) model.

Figure 1.6 is a graphical representation of the interference channel. Each x_k is an encoded vector transmission intended to be decoded from received vector y_k. The matrices $H_{k,k}$ each correspond to matrix H in (1.26). The matrices $H_{j,k}$, $j \neq k$ are the the gains on transmission x_j seen as

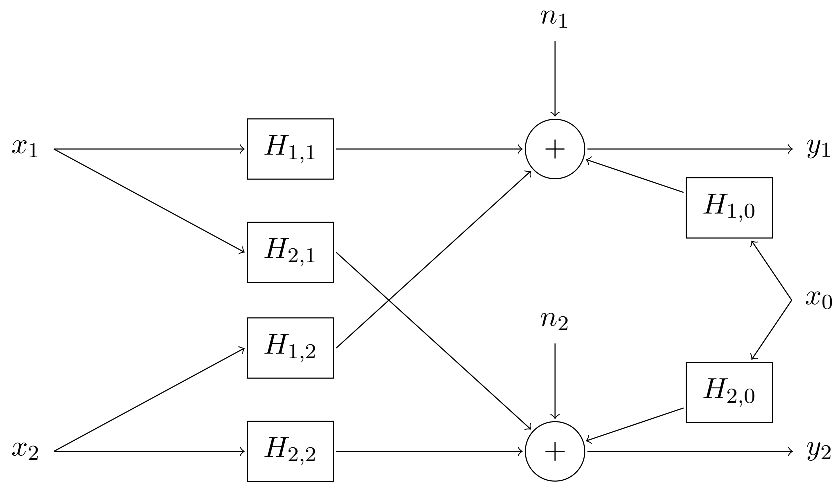


Figure 1.6: The MIMO interference channel with two users and one jammer.

interference by user k.

The noise vectors n_k are all assumed to be mutually independent with circularly symmetric Gaussian distributions. Via normalization, we can take $\mathbf{E}\, n_k n_k^\dagger = I$.

Each user's received signal is then

$$y_k = H_{k,k} x_k + \sum_{j \in \mathcal{P} \setminus \{k\}} H_{k,j} x_j + n_k. \tag{1.59}$$

Figure 1.6 is a pictorial representation of signal flow in a two-user-plus-jammer system. Note that there is no modeled receiver for the jammer's transmissions (x_0), and that the jammer's transmission is not customized for each user. Furthermore, in this model, the received signals y_1 and y_2 are decoded separately. We do not consider the joint decoding scheme in this work.

CHAPTER 2
GAMES ON MIMO INTERFERENCE CHANNELS

For users of an interference channel, there are at least two possible paradigms of operation. One can imagine users selecting codes to achieve the maximum amount of information through the channel under the constraint that the codebook they employ is subject to an average power constraint. This seems plausible if the power source at the transmitter is rechargeable and there is little or no on-board memory storage. Then it would be in the transmitter's best interest to pass on any data collected as quickly as possible. On the other hand, it is completely reasonable that users may wish to minimize their power consumption so long as their information can get through the channel at an acceptable rate. Such a paradigm might come into play in the context of remote sensing devices with finite power banks and uniform data transmissions.

2.1 Dual Games

We model the paradigms mentioned above as dual games to reflect their distinct but related structures and objectives.

We consider a network comprising r self-interested users and a single malicious jammer. We identify the users with the integers $1, \ldots, r$, and collectively refer to the set of users as

$$\mathcal{R} = \{1, \ldots, r\}. \tag{2.1}$$

The jammer is identified as agent 0, and the collection of all $p = r + 1$ agents in the system is the set

$$\mathcal{P} = \{0\} \cup \mathcal{R}. \tag{2.2}$$

Each agent comprises one transmit/receive pair—except for the jammer, for which it is not necessary to model a receiver—as in Fig. 1.6. Without loss of generality, let each agent possess N_t and N_r transmit and receive antennas, respectively.

In light of the results in Section 1.3.2, the strategy set for each agent $k \in \mathcal{P}$ is the cone of positive semidefinite matrices

$$\mathcal{S}_k = \mathbb{H}_+^{N_t} \tag{2.3}$$

representing covariance matrices for their zero-mean Gaussian codebooks, and the joint strategy set is

$$\mathcal{S} = \mathcal{S}_0 \times \cdots \times \mathcal{S}_r. \tag{2.4}$$

To apply Rosen's theorems on concave games played over real strategy sets, we will frequently view N_t-dimensional Hermitian matrices as real $2N_t^2$ vectors using the reshaping for $X \in \mathbb{H}^{N_t}$,

$$\overrightarrow{X} = \begin{bmatrix} \operatorname{vec}(\operatorname{Re} X) \\ \operatorname{vec}(\operatorname{Im} X) \end{bmatrix} \in \mathbb{R}^{2N_t^2}, \tag{2.5}$$

where $\operatorname{vec}(\cdot)$ is the standard vectorization operator that vertically concatenates the columns of its argument.

Additionally, let each user k have a utility function equal to the mutual information between its transmitter and receiver, which, for the Gaussian interference channel, can be derived from (1.59) to be

$$u_k(Q) = \log \left| \left(I + R_k^{-1/2}(Q) H_{k,k} Q_k H_{k,k}^{\dagger} R_k^{-1/2}(Q) \right) \right|, \tag{2.6}$$

where $R_k(\cdot)$ is the noise-plus-interference from agents other than k:

$$R_k(Q) = I + \sum_{j \in \mathcal{P} \setminus \{k\}} H_{k,j} Q_j H_{k,j}^{\dagger}. \tag{2.7}$$

Similarly, each user has a cost function

$$c_k(Q_k) = \operatorname{Tr} Q_k \tag{2.8}$$

that represents the power needed to employ a Gaussian codebook with covariance $Q_k \succeq 0$.

Before proceeding, we identify some properties of the utility function $u_k(\cdot)$ that will be useful later. First, $u_k(Q_k, Q_{-k})$ is continuous in both arguments (i.e. it is continuous throughout $\mathcal{S}$). It is also well-known to be concave on $\mathcal{S}_k$ and evaluates to zero whenever $Q_k = 0 \in \mathbb{H}^{N_t}$, the "zero action." Further, $u_k(\alpha Q_k, Q_{-k})$ is monotonic in α according to the following lemma.

Lemma 2.1 *For any $Q \in \mathcal{S}$ and any $k \in \mathcal{R}$ the function $f(\alpha) = u_k(\alpha Q_k, Q_{-k})$ is monotonically nondecreasing for $\alpha \in \mathbb{R}_+$. If $Q_k \succ 0$ and $\operatorname{rank} H_{k,k} = N_t$, this property can be strengthened to strictly increasing.*

Proof Pick an arbitrary $Q \in \mathcal{S}$, $k \in \mathcal{R}$, and real $\alpha > \beta$. An equivalent formulation of the utility in (2.6) is

$$u_k(Q) = \log |A_k(Q)| - \log |R_k(Q)| \tag{2.9}$$

where

$$A_k(Q) = R_k(Q) + H_{k,k} Q_k H_{k,k}^{\dagger}. \tag{2.10}$$

Since $R_k(Q)$ is invariant to scaling on Q_k, this formulation of the utility allows us to write

$$f(\alpha) - f(\beta) = \log |A_k(\alpha Q_k, Q_{-k})| - \log |A_k(\beta Q_k, Q_{-k})|. \tag{2.11}$$

Thus the question of the sign of $f(\alpha) - f(\beta)$ reduces to comparing the product of the eigenvalues of two A_k matrices. However, since

$$\begin{aligned} A_k(\alpha Q_k, Q_{-k}) - A_k(\beta Q_k, Q_{-k}) &= (\alpha - \beta) H_{k,k} Q_k H_{k,k}^\dagger & (2.12)\\ &\succeq 0, & (2.13) \end{aligned}$$

Weyl's inequality [6, Theorem 8.4.9] tells us that $\lambda_i(A_k(\alpha Q_k, Q_{-k})) \geq \lambda_i(A_k(\beta Q_k, Q_{-k}))$ for all $i = 1, \ldots, N_t$ and therfore $f(\cdot)$ is nondecreasing.

Under the stronger conditions, $H_{k,k} Q_k H_{k,k}^\dagger \succ 0$ and Weyl's inequality guarantees a strict inequality $\lambda_i(A_k(\alpha Q_k, Q_{-k})) > \lambda_i(A_k(\beta Q_k, Q_{-k}))$ for at least one $i = 1, \ldots, N_t$ which ensures $f(\cdot)$ is increasing. □

2.1.1 Utility Games

In the paradigm where users strive to maximize their link's mutual information subject to a power constraint $\bar{c} \in \mathbb{R}_+^p$, the feasible strategy set for user $k \in \mathcal{R}$ is

$$\mathcal{U}_k(\bar{c}_k) = \{Q_k \in \mathcal{S}_k : c_k(Q_k) \leq \bar{c}_k\}. \tag{2.14}$$

The jammer, however, strives to minimize the total mutual information across the network. This is equivalent to maximizing

$$J^u(Q) = -\sum_{k \in \mathcal{R}} u_k(Q) \tag{2.15}$$

subject to a feasible strategy set

$$\begin{aligned} \mathcal{U}_0(\bar{c}_0) &= \{Q_0 \in \mathcal{S}_0 : p^u(Q_0) \leq \bar{c}_0\}, & (2.16)\\ p^u(Q_0) &= \mathrm{Tr}(Q_0). & (2.17) \end{aligned}$$

Note that the agents' strategy sets are decoupled in this paradigm so that the joint strategy set

$$\mathcal{U}(\bar{c}) = \prod_{k \in \mathcal{P}} \mathcal{U}_k(\bar{c}_k) \tag{2.18}$$

is invariant to the strategies chosen by the agents.

Using the terminology of Section 1.3.1, we define the *utility game* with constraint vector $\bar{c}$ as the tuple

$$\Gamma^u(\bar{c}) = (\mathcal{P}, \mathcal{U}(\bar{c}), \{J^u(Q), u_1(Q), \ldots, u_r(Q)\}) . \tag{2.19}$$

This notation may at first seem cumbersome (especially the incongruous notation for the jammer). However, it allows us to speak of utility games where *all* agents are maximizers rather than making exceptions for the jammer in later discussion.

2.1.2 Cost Games

In the dual system where users wish to minimize their power consumption while maintaining minimum rates $\bar{u} \in \mathbb{R}_+^r$, the feasible strategy sets are now coupled vis-a-vis the actions of the other transmitters in the channel. We write the feasible strategy set for user $k \in \mathcal{R}$ as

$$\mathcal{C}_k(\bar{u}_k, Q_{-k}) = \{Q_k \in \mathcal{S}_k : u_k(Q_k, Q_{-k}) \geq \bar{u}_k\}. \tag{2.20}$$

The role of the jammer in this system bears some consideration. In constructing this dual game, it is natural to "flip" the jammer's objective and constraint functions as we have done for the other channel users. However, studies of radio-frequency jammers indicate that, at least in the IEEE 802.11 Wi-Fi [25] and terrestrial multi-hop network [47] application domains, jammers are more concerned with rate reduction than power conservation. As such, the jammer's objective is to again minimize

$$J^c(Q) = -J^u(Q) \tag{2.21}$$

within the feasible strategy set

$$\mathcal{C}_0(\bar{u}_0) = \{Q_0 \in \mathcal{S}_0 : p^c(Q_0) \geq \bar{u}_0\}, \tag{2.22}$$

$$p^c(Q_0) = -p^u(Q_0) \leq 0. \tag{2.23}$$

Unlike in the utility game, the users' strategy sets are coupled through their actions. That is, the joint strategy set

$$\mathcal{C}(\bar{u}, Q) = \mathcal{C}_0(\bar{u}_0) \times \prod_{k \in \mathcal{R}} \mathcal{C}_k(\bar{u}_k, Q_{-k}) \tag{2.24}$$

is not invariant to the strategies the agents employ.

Given a vector of minimum constraints, $\bar{u} \in \mathbb{R}_- \times \mathbb{R}_+^r$, we denote the *cost game* as the tuple

$$\Gamma^c(\bar{u}) = (\mathcal{P}, \mathcal{C}(\bar{u}, Q), \{J^c(Q), c_1(Q_1), \ldots, c_r(Q_r)\}) . \tag{2.25}$$

2.2 Existence of Equilibria

As noted in Example 1.1, not all games possess a Nash equilibrium. In this section, we explore under what conditions the utility and cost games have equilibria. To this end, define the (possibly empty) set of Nash equilibria of $\Gamma^u(\bar{c})$ and $\Gamma^c(\bar{u})$ as

$$\mathcal{E}^u(\bar{c}) = \left\{ \hat{Q} \in \mathcal{S} : \begin{array}{ll} u_k(\hat{Q}) \geq u_k(Q_k, \hat{Q}_{-k}) \; \forall Q_k \in \mathcal{U}_k(\bar{c}_k), & k = 1, \ldots, r, \\ J^u(\hat{Q}) \geq J^u(Q_0, \hat{Q}_{-0}) \; \forall Q_0 \in \mathcal{U}_0(\bar{c}_0) & \end{array} \right\} \tag{2.26}$$

$$\mathcal{E}^c(\bar{u}) = \left\{ \hat{Q} \in \mathcal{S} : \begin{array}{ll} c_k(\hat{Q}_k) \leq c_k(Q_k) \; \forall Q_k \in \mathcal{C}_k(\bar{u}_k, \hat{Q}_{-k}), & k = 1, \ldots, r, \\ J^c(\hat{Q}) \leq J^c(Q_0, \hat{Q}_{-0}) \; \forall Q_0 \in \mathcal{C}_0(\bar{u}_o) & \end{array} \right\}, \tag{2.27}$$

respectively. In both paradigms, the notation $\Gamma^u(\cdot)$ and $\Gamma^c(\cdot)$ obscures the channel behavior on which the game is played. This is a shortcoming of the compact game notation. In later sections we will place restrictions on the channel matrices $\{H_{k,j}\}_{(k,j)\in\mathcal{R}\times\mathcal{P}}$ in order to prove properties of the game comprised of objective functions that are dependent on these channel matrices.

Theorem 2.2 *For any vector of costs, $\bar{c} \in \mathbb{R}_+^p$, the utility game $\Gamma^u(\bar{c})$ possesses a Nash equilibrium.*

Proof For all $k \in \mathcal{P}$, $\bar{c}_k \geq 0$ implies that the strategy set $\mathcal{U}_k(\bar{c}_k) \subset \mathcal{S}_k$ is compact, convex, and nonempty. As observed above, $u_k(Q_k, Q_{-k})$ is continuous in Q and concave in Q_k for all $k \in \mathcal{R}$. As the sum of continuous functions, $J^u(Q_0, Q_{-0})$ is also continuous in Q. We now consider the concavity of $J^u(Q_0, Q_{-0})$ with respect to Q_0. For each $k \in \mathcal{R}$, rewrite user k's utility as in (2.9) and let

$$f_k(t) = u_k(Q_0 + t\Delta Q_0, Q_{-0}) \tag{2.28}$$

for some $Q_0 \in \mathcal{U}_0(\bar{c}_0)$, some $\Delta Q_0 \neq 0 \in \mathbb{H}^{N_t}$, and $t \in [0, \varepsilon]$ such that $Q_0 + t\Delta Q_0 \succeq 0$. Substituting $V_k = H_{k,0}\Delta Q_0 H_{k,0}^\dagger$ into (2.28) yields

$$\begin{aligned} f_k(t) &= \log|A_k + tV_k| - \log|R_k + tV_k| && (2.29) \\ &= \log\left|I + tA_k^{-1/2}V_kA_k^{-1/2}\right| + \log|A_k| - \log\left|I + tR_k^{-1/2}V_kR_k^{-1/2}\right| - \log|R_k|, && (2.30) \end{aligned}$$

where $R_k = I + \sum_{j\in\mathcal{P}\setminus\{k\}} H_{k,j}Q_jH_{k,j}^\dagger$ and $A_k = R_k + H_{k,k}Q_kH_{k,k}^\dagger$. For the terms that include t, expand the determinants in terms of their eigenvalues:

$$f_k(t) = \sum_{i=1}^{N_t} \log\left(1 + t\,\lambda_i\left(A_k^{-1/2}V_kA_k^{-1/2}\right)\right) - \log\left(1 + t\,\lambda_i\left(R_k^{-1/2}V_kR_k^{-1/2}\right)\right) + \log|A_k| - \log|R_k|. \tag{2.31}$$

In order to determine its concavity, the first and second derivatives of f_k are

$$f'_k(t) = \sum_{i=1}^{N_t} \frac{\lambda_i\left(A_k^{-1/2}V_kA_k^{-1/2}\right)}{1+t\,\lambda_i\left(A_k^{-1/2}V_kA_k^{-1/2}\right)} - \frac{\lambda_i\left(R_k^{-1/2}V_kR_k^{-1/2}\right)}{1+t\,\lambda_i\left(R_k^{-1/2}V_kR_k^{-1/2}\right)} \tag{2.32}$$

$$f''_k(t) = \sum_{i=1}^{N_t} \frac{-\lambda_i^2\left(A_k^{-1/2}V_kA_k^{-1/2}\right)}{\left[1+t\,\lambda_i\left(A_k^{-1/2}V_kA_k^{-1/2}\right)\right]^2} + \frac{\lambda_i^2\left(R_k^{-1/2}V_kR_k^{-1/2}\right)}{\left[1+t\,\lambda_i\left(R_k^{-1/2}V_kR_k^{-1/2}\right)\right]^2}. \tag{2.33}$$

The second derivative at $t = 0$ is

$$f''_k(0) = -\operatorname{Tr}\left(A_k^{-1/2}V_kA_k^{-1}V_kA_k^{-1/2}\right) + \operatorname{Tr}\left(R_k^{-1/2}V_kR_k^{-1}V_kR_k^{-1/2}\right). \tag{2.34}$$

By expanding and using the circular property of the trace, this is equivalent to

$$\begin{aligned} f''_k(0) = \operatorname{Tr}\left(R_k^{-1/2}V_kR_k^{-1}V_kR_k^{-1/2}\right) - \operatorname{Tr}\left(R_k^{-1/2}V_kA_k^{-1}V_kR_k^{-1/2}\right) + \\ \operatorname{Tr}\left(A_k^{-1/2}V_kR_k^{-1}V_kA_k^{-1/2}\right) - \operatorname{Tr}\left(A_k^{-1/2}V_kA_k^{-1}V_kA_k^{-1/2}\right). \end{aligned} \tag{2.35}$$

Finally, these terms can be combined and compared:

$$f''_k(0) = \operatorname{Tr}\left(R_k^{-1/2}V_k\left(R_k^{-1}-A_k^{-1}\right)V_kR_k^{-1/2}\right) + \operatorname{Tr}\left(A_k^{-1/2}V_k\left(R_k^{-1}-A_k^{-1}\right)V_kA_k^{-1/2}\right) \tag{2.36}$$

$$\geq 0, \tag{2.37}$$

where (2.36) can be seen by noting that for positive definite A_k and R_k, $A_k \succeq R_k$ if and only if $A_k^{-1} \preceq R_k^{-1}$ and therefore $R_k^{-1} - A_k^{-1} \succeq 0$. The nonnegativity of $f''_k(0)$ demonstrates that for all $k \in \mathcal{R}$, u_k is convex with respect to Q_0. Since $J^u(Q_0, Q_{-0})$ is the negative sum of functions that are convex in Q_0, we have that J^u is concave with respect to Q_0. Thus the existence result for concave games in Proposition 1.1 applies. □

The same results apply if the jammer instead employs a weighted sum of the individual users' mutual information as its utility.

Corollary 2.3 *The existence results of Theorem 2.2 hold if the jammer's utility is defined as a weighted sum of the users' utilities:*

$$J_\nu^u(Q) = -\sum_{k\in\mathcal{R}} \nu_k u_k(Q), \tag{2.38}$$

where the weights $\nu_k \geq 0$ *for all* $k \in \mathcal{R}$.

The existence of Nash equilibria in utility games without jammers first appeared in [2, Proposition 3.1]. The result was also shown in [23] when all agents (including the jammer) are restricted to orthogonal subchannel strategies (diagonal positive semidefinite matrices). The novelty of Theorem 2.2 lies in the merging of these two settings.

We have shown that $\mathcal{E}^u(\bar{c})$, the set of joint actions that constitute a Nash equilibrium of $\Gamma^u(\bar{c})$, is nonempty for all $\bar{c} \in \mathbb{R}^p_+$. The analogous question for $\mathcal{E}^c(\bar{u})$ is more complex. In this case the strategy sets $\mathcal{C}_k(\bar{u}_k), k \in \mathcal{R}$, depend on other agents' actions, and are in general not bounded, so the proof method for Theorem 2.2 no longer applies. Fortunately, the duality result in [4] can be extended to accommodate the presence of a jammer.

Theorem 2.4 *Fix $\bar{Q} \in \mathcal{S}$ and define $\bar{u}$ and $\bar{c}$ as*

$$\bar{u} = \left(p^c(\bar{Q}_0), u_1(\bar{Q}), \ldots, u_r(\bar{Q})\right) \tag{2.39a}$$

$$\bar{c} = \left(p^u(\bar{Q}_0), c_1(\bar{Q}_1), \ldots, c_r(\bar{Q}_r)\right). \tag{2.39b}$$

Then

$$\bar{Q} \in \mathcal{E}^c(\bar{u}) \iff \bar{Q} \in \mathcal{E}^u(\bar{c}). \tag{2.40}$$

Proof First, suppose $\bar{Q} \notin \mathcal{E}^u(\bar{c})$. Then there are two cases:

(i) for at least one user $k \in \mathcal{R}$, there is some $\hat{Q}_k \in \mathcal{S}_k$ such that $u_k(\hat{Q}_k, \bar{Q}_{-k}) > u_k(\bar{Q})$ and $c_k(\hat{Q}_k) \leq \bar{c}_k$ or

(ii) there is some $\hat{Q}_0 \in \mathcal{S}_0$ such that $J^u(\hat{Q}_0, \bar{Q}_{-0}) > J^u(\bar{Q})$ and $p^u(\hat{Q}_0) \leq \bar{c}_0$.

If (i) is the case for user $k \in \mathcal{R}$, then the strict inequality guarantees that $\hat{Q}_k \neq 0$. Thus for some scaling factor $0 < \alpha < 1$, $c_k(\alpha\hat{Q}_k) < c_k(\bar{Q}_k)$ and $u_k(\alpha\hat{Q}_k, \bar{Q}_{-k}) \geq \bar{u}_k$ which implies $\bar{Q}$ cannot be an equilibrium of the cost game: $\bar{Q} \notin \mathcal{E}^c(\bar{u})$.

Otherwise, (ii) is the case and by definition $J^c(\hat{Q}_0, \bar{Q}_{-0}) < J^c(\bar{Q})$. Likewise, $p^u(\hat{Q}_0) \leq \bar{c}_0$ implies $p^c(\hat{Q}_0) \geq -\bar{c}_0 = \bar{u}_0$. Thus $\bar{Q} \notin \mathcal{E}^c(\bar{u})$.

For the converse, suppose $\bar{Q} \notin \mathcal{E}^c(\bar{u})$. Again there are two cases:

(i) for at least one user $k \in \mathcal{R}$, there is some $\hat{Q}_k \in \mathcal{S}_k$ such that $c_k(\hat{Q}_k) < c_k(\bar{Q}_k)$ and $u_k(\hat{Q}_k, \bar{Q}_{-k}) \geq \bar{u}_k$ or

(ii) there is some $\hat{Q}_0 \in \mathcal{S}_0$ such that $J^c(\hat{Q}_0, \bar{Q}_{-0}) < J^c(\bar{Q})$ and $p^c(\hat{Q}_0) \geq \bar{u}_0$.

If (i) is the case for user $k \in \mathcal{R}$, then by Lemma 2.1 there is a scaling factor $\alpha > 1$ such that $c_k(\alpha\hat{Q}_k) \leq c_k(\bar{Q}_k) = \bar{c}_k$ and $u_k(\alpha\hat{Q}_k, \bar{Q}_{-k}) > u_k(\bar{Q}) = \bar{u}_k$ which implies $\bar{Q}$ cannot be an equilibrium of the utility game: $\bar{Q} \notin \mathcal{E}^u(\bar{c})$.

Otherwise, (ii) is the case and by definition $J^u(\hat{Q}_0, \bar{Q}_{-0}) > J^u(\bar{Q})$ and $p^u(\hat{Q}_0) \leq -\bar{u}_0 = \bar{c}_0$. Again this implies $\bar{Q} \notin \mathcal{E}^u(\bar{c})$. □

The existence result in Theorem 2.4 is not as strong as that in Theorem 2.2. In particular, it cannot be said that $\Gamma^c(\bar{u})$ has a generalized Nash equilibrium for all $\bar{u} \in \mathbb{R}_- \times \mathbb{R}_+^r$. What can be concluded, however, is that some cost games do indeed possess equilibria. Specifically, equilibria exist in cost games constrained by a $\bar{u}$—as defined in (2.39a)—that can be achieved at the equilibrium of $\Gamma^u(\bar{c})$ for some $\bar{c} \in \mathbb{R}_+^p$.

2.3 Uniqueness of Equilibria

The classical tool for showing uniqueness of equilibrium in concave games is to employ Rosen's sufficient condition for diagonal strict concavity. His sufficient condition requires the symmetrized pseudohessian matrix be negative definite when evaluated at any point in the joint action set. The utility game does not meet this (rather restrictive) sufficient condition. Furthermore, even utility games with "nice" channel structure—invertible user and jammer-user matrices with no inter-user interference—may fail to meet Definition 1.5.

Example 2.1 Consider the utility game with $r = 2$ and channel matrices

$$H_{1,1} = H_{2,2} = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$H_{1,0} = H_{2,0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$H_{1,2} = H_{2,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let the power budgets be $\bar{c}_1 =$, $\bar{c}_2 = 6$ and $\bar{c}_0 = 2$. If joint action Q^0 comprises the agent actions

$$Q_1^0 = \begin{bmatrix} 0.0940 & -0.0082 + 0.0025\jmath \\ -0.0082 - 0.0025\jmath & 0.0008 \end{bmatrix}$$
$$Q_2^0 = \begin{bmatrix} 4.4269 & 1.7820 - 1.8599\jmath \\ 1.7820 + 1.8599\jmath & 1.4988 \end{bmatrix}$$
$$Q_0^0 = \begin{bmatrix} 0.0950 & -0.1567 + 0.1078\jmath \\ -0.1567 - 0.1078\jmath & 0.3808 \end{bmatrix},$$

and joint action Q^1 comprises the agent actions

$$Q_1^1 = \begin{bmatrix} 0.1130 & 0.0153 + 0.1142\jmath \\ 0.0153 - 0.1142\jmath & 0.1175 \end{bmatrix}$$
$$Q_2^1 = \begin{bmatrix} 0.9980 & 0.2625 + 0.2251\jmath \\ 0.2625 - 0.2251\jmath & 0.1198 \end{bmatrix}$$
$$Q_0^1 = \begin{bmatrix} 0.8083 & -0.5297 - 0.7751\jmath \\ -0.5297 + 0.7751\jmath & 1.0904 \end{bmatrix},$$

then the check on diagonal strict concavity is $\left(\overrightarrow{Q^1} - \overrightarrow{Q^0}\right)^\intercal g(Q^0) + \left(\overrightarrow{Q^0} - \overrightarrow{Q^1}\right)^\intercal g(Q^1) = -1.1367$, which violates Definition 1.5.

Under certain circumstances, however, utility games can be shown to adhere to the spirit of Rosen's uniqueness proof. That is, under certain conditions, the set of joint actions in a utility game that satisfy the KKT optimality conditions outlined in Rosen's proof is a singleton. To see under which conditions this claim is true, we must devote exposition to the structure of the pseudohessian matrix in the context of the utility game.

Defined for a general game in (1.14), the pseudohessian matrix $G(Q)$ for a utility game is composed of blocks $G_{k,j}(Q)$ where

$$G_{k,j}(Q) = \nabla_{\overrightarrow{Q_j}} \nabla_{\overrightarrow{Q_k}} u_k(Q), \quad \forall (k,j) \in \mathcal{R} \times \mathcal{P}, \tag{2.41a}$$
$$G_{0,j}(Q) = \nabla_{\overrightarrow{Q_j}} \nabla_{\overrightarrow{Q_0}} J^u(Q), \quad \forall j \in \mathcal{P}. \tag{2.41b}$$

From an analytical point of view, these blocks fall into five regions according to their subscripts and by extension their location within the pseudohessian. We now proceed to derive the second-order derivatives that populate the pseudohessian and refer to the numbered regions in Fig. 2.1 to describe their placement in the large matrix.

To accomplish this we first need expressions for the components of the pseudogradient. First consider $\mathcal{D}u_k(Q; Z_k)$, the directional derivative of user k's utility when its own action deviates in the direction of Hermitian matrix Z_k:

$$\mathcal{D}u_k(Q; Z_k) = \frac{d}{dt} u_k(Q_k + tZ_k, Q_{-k}) \Big|_{t=0} \tag{2.42}$$
$$= \frac{d}{dt} \log \left| A_k(Q) + tH_{k,k} Z_k H_{k,k}^\dagger \right| \Big|_{t=0} \tag{2.43}$$
$$= \mathrm{Tr}\, A_k^{-1}(Q) H_{k,k} Z_k H_{k,k}^\dagger \tag{2.44}$$
$$= \mathrm{Tr}\, H_{k,k}^\dagger A_k^{-1}(Q) H_{k,k} Z_k. \tag{2.45}$$

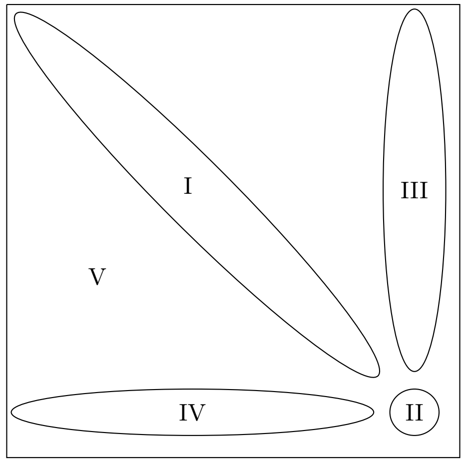


Figure 2.1: A graphical representation of the pseudohessian, $G(Q)$. In region I are the Hessian matrices of the form $\nabla_{\overrightarrow{Q_k}} \nabla_{\overrightarrow{Q_k}} u_k(Q)$, $k \in \mathcal{R}$. Region II is the (0,0) block: $\nabla_{\overrightarrow{Q_0}} \nabla_{\overrightarrow{Q_0}} J^u(Q)$. Region III contains the blocks of deriviatives of the pseudogradient with respect to the jammer's action: $\nabla_{\overrightarrow{Q_0}} \nabla_{\overrightarrow{Q_k}} u_k(Q)$ Region IV contains the blocks of derivatives of the jammer's pseudogradient with respect to user k's action: $\nabla_{\overrightarrow{Q_k}} \nabla_{\overrightarrow{Q_0}} J^u(Q)$. Finally, region V (both above and below the diagonal) contains blocks of the form $\nabla_{\overrightarrow{Q_j}} \nabla_{\overrightarrow{Q_k}} u_k(Q)$, $j \neq k \in \mathcal{R}$.

We use Proposition A.1 to identify

$$\nabla_{\overrightarrow{Q}_k} u_k(Q) = \overrightarrow{H_{k,k}^\dagger A_k^{-1}(Q) H_{k,k}}. \tag{2.46}$$

The last pseudogradient component is identified from the directional derivative of the jammer's utility with respect to its own action:

$$\begin{aligned}
\mathcal{D}J^u(Q; Z_0) &= \left.\frac{d}{dt} J^u(Q_0 + tZ_0, Q_{-0})\right|_{t=0} && (2.47)\\
&= -\sum_{k\in\mathcal{R}} \frac{d}{dt}\left[\log\left|A_k(Q) + tH_{k,0}Z_0H_{k,0}^\dagger\right| - \log\left|R_k(Q) + tH_{k,0}Z_0H_{k,0}^\dagger\right|\right]_{t=0} && (2.48)\\
&= -\sum_{k\in\mathcal{R}} \operatorname{Tr} A_k^{-1}(Q)H_{k,0}Z_0H_{k,0}^\dagger - \operatorname{Tr} R_k^{-1}(Q)H_{k,0}Z_0H_{k,0}^\dagger && (2.49)\\
&= \operatorname{Tr}\left(\sum_{k\in\mathcal{R}} H_{k,0}^\dagger \left(R_k^{-1}(Q) - A_k^{-1}(Q)\right) H_{k,0}\right) Z_0. && (2.50)
\end{aligned}$$

As a gradient, this is written

$$\nabla_{\overrightarrow{Q}_0} J^u(Q) = \sum_{k\in\mathcal{R}} \overrightarrow{H_{k,0}^\dagger \left(R_k^{-1}(Q) - A_k^{-1}(Q)\right) H_{k,0}}. \tag{2.51}$$

We now commence with calculating the blocks within the regions of the pseudohessian in Fig. 2.1. The k-th row in regions I, III, and V contains derivatives of the k-th block element of the pseudogradient with respect to Q_k, Q_0, and $Q_j, j \in \mathcal{R}\backslash\{k\}$, respectively. We can generally solve for the block in position (k, i) for $k \in \mathcal{R}$ and $i \in \mathcal{P}$ by first calculating the directional derivative of (2.45) when Q_i is perturbed in the direction Z_i:

$$\mathcal{D}\left(\mathcal{D}u_k(Q; Z_k); Z_i\right) = \frac{d}{dt} \operatorname{Tr} H_{k,k}^{\dagger} \left(A_k(Q) + tH_{k,i}Z_iH_{k,i}^{\dagger}\right)^{-1} H_{k,k}Z_k\bigg|_{t=0} \tag{2.52}$$

$$= -\operatorname{Tr} H_{k,k}^{\dagger}A_k^{-1}(Q)H_{k,i}Z_iH_{k,i}^{\dagger}A_k^{-1}(Q)H_{k,k}Z_k \tag{2.53}$$

$$= -\left(\overrightarrow{Z_k}\right)^{\mathsf{T}} \begin{bmatrix} \operatorname{Re}\left(C_{k,i}^{*} \otimes C_{k,i}\right) & -\operatorname{Im}\left(C_{k,i}^{*} \otimes C_{k,i}\right) \\ \operatorname{Im}\left(C_{k,i}^{*} \otimes C_{k,i}\right) & \operatorname{Re}\left(C_{k,i}^{*} \otimes C_{k,i}\right) \end{bmatrix} \overrightarrow{Z_i}, \tag{2.54}$$

where $C_{k,i} = H_{k,k}^{\dagger}A_k^{-1}(Q)H_{k,i}$ for any $i \in \mathcal{P}$ and where (2.54) follows from use of Lemma A.2. The $2N_t^2$-square matrix is cumbersome to write, so we abbreviate the act of splitting a complex matrix into its real and imaginary blocks as

$$\operatorname{CRI}(X) = \begin{bmatrix} \operatorname{Re} X & -\operatorname{Im} X \\ \operatorname{Im} X & \operatorname{Re} X \end{bmatrix}. \tag{2.55}$$

Using this notation, the i-th block entry in the k-th block row of the pseudohessian can be written

$$G_{k,i}(Q) = -\operatorname{CRI}\left(\left(H_{k,k}^{\dagger}A_k^{-1}(Q)H_{k,i}\right)^{*} \otimes \left(H_{k,k}^{\dagger}A_k^{-1}(Q)H_{k,i}\right)\right). \tag{2.56}$$

The last block on the diagonal, region II, contains the second deriviatives of the jammer's utility with respect to its own action. The directional derivative of (2.50) when Q_0 is perturbed in the direction Z_0 is[1]

$$\mathcal{D}\left(\mathcal{D}J^u(Q; Z_0); Z_0\right) = -\sum_{k \in \mathcal{R}} \operatorname{Tr} H_{k,0}^{\dagger} \left(R_k^{-1}H_{k,0}Z_0H_{k,0}^{\dagger}R_k^{-1} - A_k^{-1}H_{k,0}Z_0H_{k,0}^{\dagger}A_k^{-1}\right) H_{k,0}Z_0. \tag{2.57}$$

Here every term in the summation conains two traces of the form in Lemma A.2.

$$\begin{aligned} &\mathcal{D}\left(\mathcal{D}J^u(Q; Z_0); Z_0\right) = \\ &\qquad -\left(\overrightarrow{Z_0}\right)^{\mathsf{T}} \Bigg[\sum_{k \in \mathcal{R}} \operatorname{CRI}\left(\left(H_{k,0}^{\dagger}R_k^{-1}H_{k,0}\right)^{*} \otimes \left(H_{k,0}^{\dagger}R_k^{-1}H_{k,0}\right)\right) \\ &\qquad\qquad - \operatorname{CRI}\left(\left(H_{k,0}^{\dagger}A_k^{-1}H_{k,0}\right)^{*} \otimes \left(H_{k,0}^{\dagger}A_k^{-1}H_{k,0}\right)\right)\Bigg] \overrightarrow{Z_0}. \end{aligned} \tag{2.58}$$

[1]To save space, we will occasionally write $A_k = A_k(Q)$ and likewise for $R_k = R_k(Q)$. In all cases, however, the A_k and R_k matrices are functions of the joint action.

We extract from this the $(0,0)$ block of the pseudohessian:

$$G_{0,0}(Q) = -\sum_{k\in\mathcal{R}} \operatorname{CRI}\left(\left(H_{k,0}^\dagger R_k^{-1} H_{k,0}\right)^* \otimes \left(H_{k,0}^\dagger R_k^{-1} H_{k,0}\right)\right) - \operatorname{CRI}\left(\left(H_{k,0}^\dagger A_k^{-1} H_{k,0}\right)^* \otimes \left(H_{k,0}^\dagger A_k^{-1} H_{k,0}\right)\right). \quad (2.59)$$

In region IV the j-th block column contains derivatives of the jammer's component of the pseudogradient with respect to user j's action. The directional derivative of (2.50) with Q_j perturbed in the direction Z_j is

$$\mathcal{D}\left(\mathcal{D}J^u(Q;Z_0);Z_j\right) = \frac{d}{dt}\operatorname{Tr}\left(\sum_{\substack{k\in\mathcal{R}\\k\neq j}} H_{k,0}^\dagger\left(\left(R_k + tH_{k,j}Z_jH_{k,j}^\dagger\right)^{-1} - \left(A_k + tH_{k,j}Z_jH_{k,j}^\dagger\right)^{-1}\right)H_{k,0}Z_0\right)\Bigg|_{t=0} + \frac{d}{dt}\operatorname{Tr} -H_{j,0}^\dagger\left(A_j + tH_{j,j}Z_jH_{j,j}^\dagger\right)^{-1}H_{j,0}Z_0\Bigg|_{t=0}. \quad (2.60)$$

After differentiating we get

$$\mathcal{D}\left(\mathcal{D}J^u(Q;Z_0);Z_j\right) = -\sum_{\substack{k\in\mathcal{R}\\k\neq j}} \operatorname{Tr} H_{k,0}^\dagger\left(R_k^{-1}H_{k,j}Z_jH_{k,j}^\dagger R_k^{-1} - A_k^{-1}H_{k,j}Z_jH_{k,j}^\dagger A_k^{-1}\right)H_{k,0}Z_0 + \operatorname{Tr} H_{j,0}^\dagger A_j^{-1}H_{j,j}Z_jH_{j,j}^\dagger A_j^{-1}H_{j,0}Z_0. \quad (2.61)$$

Lemma A.2 turns this into

$$\mathcal{D}\left(\mathcal{D}J^u(Q;Z_0);Z_j\right) = \left(\overrightarrow{Z_0}\right)^\intercal \Bigg[\operatorname{CRI}\left(\left(H_{j,j}^\dagger A_j^{-1} H_{j,0}\right)^\intercal \otimes \left(H_{j,0}^\dagger A_j^{-1} H_{j,j}\right)\right) - \sum_{\substack{k\in\mathcal{R}\\k\neq j}} \operatorname{CRI}\left(\left(H_{k,j}^\dagger R_k^{-1} H_{k,0}\right)^\intercal \otimes \left(H_{k,0}^\dagger R_k^{-1} H_{k,j}\right)\right) - \operatorname{CRI}\left(\left(H_{k,j}^\dagger A_k^{-1} H_{k,0}\right)^\intercal \otimes \left(H_{k,0}^\dagger A_k^{-1} H_{k,j}\right)\right)\Bigg] \overrightarrow{Z_j}, \quad (2.62)$$

The j-th block column of region IV of the pseudohessian is then

$$G_{0,j}(Q) = \mathrm{CRI}\left(\left(H_{j,j}^{\dagger}A_j^{-1}H_{j,0}\right)^{\mathsf{T}} \otimes \left(H_{j,0}^{\dagger}A_j^{-1}H_{j,j}\right)\right)$$
$$- \sum_{\substack{k\in\mathcal{R}\\k\neq j}} \mathrm{CRI}\left(\left(H_{k,j}^{\dagger}R_k^{-1}H_{k,0}\right)^{\mathsf{T}} \otimes \left(H_{k,0}^{\dagger}R_k^{-1}H_{k,j}\right)\right) - \mathrm{CRI}\left(\left(H_{k,j}^{\dagger}A_k^{-1}H_{k,0}\right)^{\mathsf{T}} \otimes \left(H_{k,0}^{\dagger}A_k^{-1}H_{k,j}\right)\right). \tag{2.63}$$

With the pseudohessian fleshed out, we can now consider the conditions under which the size of a utility game's set of equilibria can be characterized. There is a natural split in this analysis between the case were users do not interfere with each other—termed *zero cross-talk*—and the case in which each users sees all three flavors of interference: ambient, competitive, and malicious.

2.3.1 Zero Cross-Talk

We define a zero cross-talk channel to be one which for every $k \in \mathcal{R}$,

$$H_{k,j} = 0 \quad \forall j \in \mathcal{R}\backslash\{k\}. \tag{2.64}$$

In games played on these channels, each user sees ambient noise as well as interference from the jammer but is unaffected by the other users. While highlighting the differences, the proof of uniqueness in the utility game follows Rosen's uniqueness proof closely.

Theorem 2.5 *In addition to being a zero cross-talk channel, assume the channel matrices meet the following conditions for all $k \in \mathcal{R}$:*

$$\operatorname{rank} H_{k,k} = N_t, \tag{2.65a}$$
$$\operatorname{rank} H_{k,0} = N_t. \tag{2.65b}$$

Then for any $\bar{c}$ the set of equilibria of the utility game $\Gamma^u(\bar{c})$ will have one of the following properties:

1. *$\mathcal{E}^u(\bar{c})$ is a singleton, $\hat{Q}$, in which $\hat{Q}_k \succ 0$ for at least one $k \in \mathcal{R}$, or*
2. *$\mathcal{E}^u(\bar{c})$ may contain multiple equilibria, but none of those equilibria will have positive definite user actions.*

Proof Any user $k \in \mathcal{R}$ with $\bar{c}_k = 0$ possess a singleton strategy set—the zero matrix—and has no effect on the utility of any other agent. If $\bar{c}_0 = 0$, then the result follows from the jammer-free case in [2, Proposition 3.1]. Therefore without loss of generality we consider games with strictly positive constraints: $\bar{c} \in \mathbb{R}^p_{++}$.

To classify the size of $\mathcal{E}^u(\bar{c})$, we first assume that there are two Nash equilibria, Q^0 and Q^1, and then proceed to show that if any of the user equilibrium actions is positive definite we arrive at a contradiction. On the other hand, if no user action is positive definite, no contradiction occurs and multiple equilibria may exist.

To proceed with the proof by contradiction, suppose Q^0 possesses a nonempty set of users with positive definite equilibrium strategies:

$$\mathcal{I}^0 = \left\{k \in \mathcal{R} : Q_k^0 \succ 0\right\} \neq \emptyset. \tag{2.66}$$

The KKT conditions at equilibrium imply the existence of nonnegative multipliers $\mu^0, \nu^0, \mu^1, \nu^1 \in \mathbb{R}_+^p$ such that for all $k \in \mathcal{P}$

$$\mu_k^0 \, \underline{\lambda}(Q_k^0) = 0 \tag{2.67a}$$

$$\nu_k^0 \left(\bar{c}_k - \mathrm{Tr}(Q_k^0)\right) = 0 \tag{2.67b}$$

$$\mu_k^1 \, \underline{\lambda}(Q_k^1) = 0 \tag{2.67c}$$

$$\nu_k^1 \left(\bar{c}_k - \mathrm{Tr}(Q_k^1)\right) = 0, \tag{2.67d}$$

where $\underline{\lambda}(X)$ is the least eigenvalue of $X \in \mathbb{H}^{N_t}$. Rosen's uniqueness proof places Lagrange multipliers on each of the concave constraint functions on the real vector actions. In the problem considered here we place them on the minimum eigenvalue and trace functions (reflecting the positive semidefiniteness and power constraint on the Hermitian matrix actions).

At each equilibrium the stationarity conditions can be written for all $k \in \mathcal{R}$

$$\nabla_{\overrightarrow{Q_k}} u_k(Q^0) + \mu_k^0 \nabla_{\overrightarrow{Q_k}} \underline{\lambda}(Q_k^0) + \nu_k^0 \nabla_{\overrightarrow{Q_k}} \left(\bar{c}_k - \mathrm{Tr}(Q_k^0)\right) = 0 \tag{2.68a}$$

$$\nabla_{\overrightarrow{Q_k}} u_k(Q^1) + \mu_k^1 \nabla_{\overrightarrow{Q_k}} \underline{\lambda}(Q_k^1) + \nu_k^1 \nabla_{\overrightarrow{Q_k}} \left(\bar{c}_k - \mathrm{Tr}(Q_k^1)\right) = 0 \tag{2.68b}$$

and

$$\nabla_{\overrightarrow{Q_0}} J^u(Q^0) + \mu_0^0 \nabla_{\overrightarrow{Q_0}} \underline{\lambda}(Q_0^0) + \nu_0^0 \nabla_{\overrightarrow{Q_0}} \left(\bar{c}_0 - \mathrm{Tr}(Q_0^0)\right) = 0 \tag{2.69a}$$

$$\nabla_{\overrightarrow{Q_0}} J^u(Q^1) + \mu_0^1 \nabla_{\overrightarrow{Q_0}} \underline{\lambda}(Q_0^1) + \nu_0^1 \nabla_{\overrightarrow{Q_0}} \left(\bar{c}_0 - \mathrm{Tr}(Q_0^1)\right) = 0. \tag{2.69b}$$

We proceed by left-multiplying each (2.68a) and (2.69a) by the corresponding $\left(\overrightarrow{Q_k^1} - \overrightarrow{Q_k^0}\right)^{\mathsf{T}}$, left-multiplying each (2.68b) and (2.69b) by the corresponding $\left(\overrightarrow{Q_k^0} - \overrightarrow{Q_k^1}\right)^{\mathsf{T}}$, and summing these terms over all agents. This results in the sum

$$\beta + \gamma = 0 \tag{2.70}$$

where[2]

$$\beta = \left(\overrightarrow{Q^1} - \overrightarrow{Q^0}\right)^{\mathsf{T}} g(Q^0) + \left(\overrightarrow{Q^0} - \overrightarrow{Q^1}\right)^{\mathsf{T}} g(Q^1) \tag{2.71}$$

[2]What follows is a slight abuse of the real vectorization notation. When the real vectorization operation is written over a joint action, we take this to mean the vertical concatenation of each of the real vectorizations of the agent

and

$$\gamma = \sum_{k\in\mathcal{P}} \left(\overrightarrow{Q_k^1} - \overrightarrow{Q_k^0}\right)^{\mathsf{T}} \left[\mu_k^0 \nabla_{\overrightarrow{Q_k}} \underline{\lambda}(Q_k^0) + \nu_k^0 \nabla_{\overrightarrow{Q_k}} \left(\bar{c}_k - \mathrm{Tr}(Q_k^0)\right)\right] + \left(\overrightarrow{Q_k^0} - \overrightarrow{Q_k^1}\right)^{\mathsf{T}} \left[\mu_k^1 \nabla_{\overrightarrow{Q_k}} \underline{\lambda}(Q_k^1) + \nu_k^1 \nabla_{\overrightarrow{Q_k}} \left(\bar{c}_k - \mathrm{Tr}(Q_k^1)\right)\right]. \quad (2.72)$$

Trace is linear in the space of Hermitian matrices and the Courant-Fischer Theorem [27, Theorem 4.26] characterizes the minimum eigenvalue as a concave functional on that space. Therefore (2.72) can be bounded by

$$\gamma \geq \sum_{k\in\mathcal{P}} \mu_k^0 \left[\underline{\lambda}(Q_k^1) - \underline{\lambda}(Q_k^0)\right] + \mu_k^1 \left[\underline{\lambda}(Q_k^0) - \underline{\lambda}(Q_k^1)\right] + \nu_k^0 \left[\left(\bar{c}_k - \mathrm{Tr}(Q_k^1)\right) - \left(\bar{c}_k - \mathrm{Tr}(Q_k^0)\right)\right] + \nu_k^1 \left[\left(\bar{c}_k - \mathrm{Tr}(Q_k^0)\right) - \left(\bar{c}_k - \mathrm{Tr}(Q_k^1)\right]\right). \quad (2.73)$$

By appealing to the KKT complementary slackness conditions (2.67), many zero terms drop out and (2.73) simplifies to

$$\begin{aligned} \gamma &\geq \sum_{k\in\mathcal{P}} \mu_k^0 \underline{\lambda}(Q_k^1) + \mu_k^1 \underline{\lambda}(Q_k^0) + \nu_k^0 \left(\bar{c}_k - \mathrm{Tr}(Q_k^1)\right) + \nu_k^1 \left(\bar{c}_k - \mathrm{Tr}(Q_k^0)\right) && (2.74) \\ &\geq 0, && (2.75) \end{aligned}$$

where the final inequality comes from observing that all of the factors in all of the terms in (2.74) are nonnegative.

To complete the contradiction, we endeavor to show that $\beta > 0$, which would imply that (2.70) is impossible. Define $Q(\theta) = \theta Q^1 + (1-\theta)Q^0$ for $0 \leq \theta \leq 1$. Using the Jacobian of the pseudogradient and the Chain Rule, we can write

$$\begin{aligned} \frac{dg(Q(\theta))}{d\theta} &= G(Q(\theta))\frac{dQ(\theta)}{d\theta} && (2.76) \\ &= G(Q(\theta))\left(\overrightarrow{Q^1} - \overrightarrow{Q^0}\right). && (2.77) \end{aligned}$$

---

actions:

$$\overrightarrow{Q} = \begin{bmatrix} \overrightarrow{Q_1} \\ \vdots \\ \overrightarrow{Q_0} \end{bmatrix}.$$

By the Fundamental Theorem of Calculus, this becomes

$$g(Q^1) - g(Q^0) = \int_0^1 G(Q(\theta)) \left(\overrightarrow{Q^1} - \overrightarrow{Q^0} \right) d\theta. \tag{2.78}$$

To write this in the form of β in (2.71), multiply both sides by $\left(\overrightarrow{Q^0} - \overrightarrow{Q^1} \right)^{\intercal}$:

$$\left(\overrightarrow{Q^0} - \overrightarrow{Q^1} \right)^{\intercal} g(Q^1) + \left(\overrightarrow{Q^1} - \overrightarrow{Q^0} \right)^{\intercal} g(Q^0) = -\int_0^1 \left(\overrightarrow{Q^1} - \overrightarrow{Q^0} \right)^{\intercal} G(Q(\theta)) \left(\overrightarrow{Q^1} - \overrightarrow{Q^0} \right) d\theta. \tag{2.79}$$

The integrand can be written as a real quadratic form by splitting G and adding its transpose:

$$\left(\overrightarrow{Q^0} - \overrightarrow{Q^1} \right)^{\intercal} g(Q^1) + \left(\overrightarrow{Q^1} - \overrightarrow{Q^0} \right)^{\intercal} g(Q^0) = \\ -\frac{1}{2} \int_0^1 \left(\overrightarrow{Q^1} - \overrightarrow{Q^0} \right)^{\intercal} \left[G(Q(\theta)) + G^{\intercal}(Q(\theta)) \right] \left(\overrightarrow{Q^1} - \overrightarrow{Q^0} \right) d\theta. \tag{2.80}$$

Thus it suffices to show that $\bar{G}(Q(\theta)) = G(Q(\theta)) + G^{\intercal}(Q(\theta))$ is a negative real quadratic form over vectors drawn from $\{\overrightarrow{Z} : Z \in \prod_{k \in \mathcal{P}} \mathbb{H}^{N_t}\}$[3] for all $0 < \theta < 1$.

Under zero cross-talk many of the pseudohessian blocks can be significantly simplified. Principally, all blocks in region V are zero. Further, all the terms in the summation in (2.63) disappear. This results in $G_{0,j}(Q(\theta)) = -G^{\intercal}_{j,0}(Q(\theta))$ for all $j \in \mathcal{R}$ and thus all off-diagonal blocks of the symmetrized pseudohessian are zero. Since the symmetrized pseudohessian is block diagonal in the zero cross-talk case, it would suffice to show that each of its diagonal blocks is negative definite with respect to Hermitian matrices for all $0 < \theta < 1$.

The blocks in region I all have the form (2.56) with $i = k$. Since $A_k^{-1}(Q)$ is positive definite for all $Q \in \mathcal{S}$, pre- and post-multiplication by the full rank $H_{k,k}$ preserves its definiteness. As such, (2.54) will be strictly negative for all Hermitian Z_k.

To analyze the definiteness of the region II block, we return to (2.57) and isolate the $\hat{k}$ term for some $\hat{k} \in \mathcal{I}^0$:

$$-\operatorname{Tr}\left[R_{\hat{k}}^{-1}(Q(\theta)) \tilde{Z}_0 R_{\hat{k}}^{-1}(Q(\theta)) - A_{\hat{k}}^{-1}(Q(\theta)) \tilde{Z}_0 A_{\hat{k}}^{-1}(Q(\theta)) \right] \tilde{Z}_0 = \\ \operatorname{Tr} A_{\hat{k}}^{-1/2}(Q(\theta)) \tilde{Z}_0 A_{\hat{k}}^{-1}(Q(\theta)) \tilde{Z}_0 A_{\hat{k}}^{-1/2}(Q(\theta)) \\ - \operatorname{Tr} R_{\hat{k}}^{-1/2}(Q(\theta)) \tilde{Z}_0 R_{\hat{k}}^{-1}(Q(\theta)) \tilde{Z}_0 R_{\hat{k}}^{-1/2}(Q(\theta)), \tag{2.81}$$

[3]While accurate, this terminolgoy is cumbersome. Henceforth we will shorten this to "negative definite with respect to Hermitian matrices" to strike a balance between accuracy and readability.

where we have substituted the Hermitian $\tilde{Z}_0$ for $H_{k,0}Z_0H_{k,0}^{\dagger}$. By adding and subtracting

$$\operatorname{Tr} A_{\hat{k}}^{-1/2}(Q(\theta))\tilde{Z}_0 R_{\hat{k}}^{-1}(Q(\theta))\tilde{Z}_0 A_{\hat{k}}^{-1/2}(Q(\theta)), \tag{2.82}$$

(2.81) can be written

$$\begin{aligned}\operatorname{Tr} A_{\hat{k}}^{-1/2}(Q(\theta))\tilde{Z}_0 \left(A_{\hat{k}}^{-1}(Q(\theta)) - R_{\hat{k}}^{-1}(Q(\theta))\right) \tilde{Z}_0 A_{\hat{k}}^{-1/2}(Q(\theta)) \\ + \operatorname{Tr} R_{\hat{k}}^{-1/2}(Q(\theta))\tilde{Z}_0 \left(A_{\hat{k}}^{-1}(Q(\theta)) - R_{\hat{k}}^{-1}(Q(\theta))\right) \tilde{Z}_0 R_{\hat{k}}^{-1/2}(Q(\theta)).\end{aligned} \tag{2.83}$$

By assumption, the action of user $\hat{k} \in \mathcal{I}^0$ is positive definite for all $0 \leq \theta < 1$. Since rank $H_{k,k} = N_t$ for all users, we have $A_{\hat{k}}(Q(\theta)) \succ R_{\hat{k}}(Q(\theta))$ over that same range of θ. This ordering is reversed for their inverses: $A_{\hat{k}}(Q(\theta)) \prec R_{\hat{k}}(Q(\theta))$, so (2.83) is strictly negative. Thus there must be at least one term in (2.57) that is negative. The other terms are at most zero (replace the strict Löwner orderings with weak ones in the above argument), so we can conclude that $\left(\overrightarrow{Z_0}\right)^{\mathsf{T}} G_{0,0}(Q(\theta))\overrightarrow{Z_0} < 0$ for all $0 \leq \theta < 1$. This is not a proof of the negative definiteness of $G_{0,0}(Q(\theta))$. However, we have shown that all of the diagonal blocks of the pseudohessian are negative definite with respect to Hermitian matrices, and this is enough to show that the integrand of (2.80) is strictly negative.

In summary, if the equilibrium assumed to have a positive definite user action does exist, then along the line connecting that equilibrium and any other possible equilibrium the integrand in (2.80) is strictly negative. This implies that (2.70) is impossible so the conclusion must be that either no other equilibria exist, or no equilibrium has positive definite user actions. □

2.3.2 Nonzero Cross-Talk

We now endeavor to generalize Theorem 2.5 to instances with nonzero cross-talk between the users. It is useful to view the pseudohessian in these games, $\bar{G}_{\mathrm{CT}}(Q)$, as a perturbation of the pseudohessian in the zero cross-talk scenario, $\bar{G}_{\mathrm{ZCT}}(Q)$:

$$\bar{G}_{\mathrm{CT}}(Q) = \bar{G}_{\mathrm{ZCT}}(Q) + \tilde{G}(Q). \tag{2.84}$$

That is, for any collection of channel matrices the pseudohessian in the utility game played on those channels can be decomposed into two terms: one that does not depend on the inter-user interferences and one that does.

The cross-talk channel matrices do not change regions I and II of the pseudohessian, so the diagonal blocks of the perturbation

$$\tilde{G}_{k,k}(Q) = 0 \quad \forall k \in \mathcal{P}. \tag{2.85}$$

Conversely, all of the blocks in region V come from the perturbation. We use (2.56) twice (once with (k,j) and once with (j,k)) along with Lemma A.7 to write the (k,j)-block of the perturbation pseudohessian for $(k,j) \in \mathcal{R} \times \mathcal{R} \backslash \{k\}$ as

$$\tilde{G}_{k,j}(Q) = -\operatorname{CRI}\left(H_{k,k}^{\intercal} \otimes H_{k,k}^{\dagger}\right) \operatorname{CRI}\left(A_k^{-\intercal}(Q) \otimes A_k^{-1}(Q)\right) \operatorname{CRI}\left(H_{k,j}^{*} \otimes H_{k,j}\right) - \operatorname{CRI}\left(H_{j,k}^{\intercal} \otimes H_{j,k}^{\dagger}\right) \operatorname{CRI}\left(A_j^{-\intercal}(Q) \otimes A_j^{-1}(Q)\right) \operatorname{CRI}\left(H_{j,j}^{*} \otimes H_{j,j}\right) \quad (2.86)$$

In region III, the jammer's column of the k-th row has the form $\bar{G}_{\mathrm{CT}_{k,0}}(Q) = G_{k,0}(Q) + G_{0,k}^{\intercal}(Q)$. Recalling (2.56) and (2.63), we write[4]

$$\begin{aligned} \bar{G}_{\mathrm{CT}_{k,0}}(Q) = &- \operatorname{CRI}\left(\left(H_{k,k}^{\dagger} A_k^{-1} H_{k,0}\right)^{*} \otimes \left(H_{k,k}^{\dagger} A_k^{-1} H_{k,0}\right)\right) \\ &+ \left[\operatorname{CRI}\left(\left(H_{k,k}^{\dagger} A_k^{-1} H_{k,0}\right)^{\intercal} \otimes \left(H_{k,0}^{\dagger} A_k^{-1} H_{k,k}\right)\right)\right]^{\intercal} \\ &- \sum_{\substack{j \in \mathcal{R} \\ j \neq k}} \left[\operatorname{CRI}\left(\left(H_{j,k}^{\dagger} R_j^{-1} H_{j,0}\right)^{\intercal} \otimes \left(H_{j,0}^{\dagger} R_j k^{-1} H_{j,k}\right)\right)\right. \\ &\left. - \operatorname{CRI}\left(\left(H_{j,k}^{\dagger} A_j^{-1} H_{j,0}\right)^{\intercal} \otimes \left(H_{j,0}^{\dagger} A_j^{-1} H_{j,k}\right)\right)\right]^{\intercal}. \end{aligned} \quad (2.87)$$

Just as they did in the zero cross-talk case, the first two terms in (2.87) cancel (which implies $\bar{G}_{\mathrm{ZCT}_{k,0}}(Q) = 0$ for all $k \in \mathcal{R}$). The remaining summation is new in the nonzero cross-talk case and can be written

$$\bar{G}_{\mathrm{CT}_{k,0}}(Q) = \tilde{G}_{k,0}(Q) \quad (2.88)$$

$$= -\sum_{\substack{j \in \mathcal{R} \\ j \neq k}} \operatorname{CRI}\left(H_{j,0}^{\intercal} \otimes H_{j,0}^{\dagger}\right) \operatorname{CRI}\left(R_j^{-\intercal} \otimes R_j^{-1} - A_j^{-\intercal} \otimes A_j^{-1}\right) \operatorname{CRI}\left(H_{j,k}^{*} \otimes H_{j,k}\right). \quad (2.89)$$

Finally, as the pseudohessian is real symmetric, $\bar{G}_{\mathrm{CT}_{k,0}}(Q) = \bar{G}_{\mathrm{CT}_{0,k}}^{\intercal}(Q)$ for all $k \in \mathcal{R}$.

In what follows, we will bound the effect of the user-user interference channels with the largest maximum singular value of those channel matrices:

$$\overline{\sigma}_c = \max_{\substack{(k,j) \in \mathcal{R} \times \mathcal{R} \\ k \neq j}} \overline{\sigma}(H_{k,j}). \quad (2.90)$$

Theorem 2.6 *For any $\bar{c} \in \mathbb{R}_{+}^{p}$, assume the channel matrices of the utility game $\Gamma(\bar{c})$ obey (2.65). For any $\epsilon > 0$, there is a $\hat{\sigma}_c > 0$ such that if $\overline{\sigma}_c < \hat{\sigma}_c$, there can be at most one equilibrium Q with the property that $\underline{\lambda}(Q_k) > \epsilon$ for some $k \in \mathcal{R}$.*

Proof Assume that there are two Nash equilibria, Q^0 and Q^1, that have nonempty sets of users with minimum eigenvalues greater than ϵ. Again define $Q(\theta) = \theta Q^1 + (1-\theta) Q^0$ for $0 \le \theta \le 1$.

[4]For notational clarity, the dependence of $A_k(\cdot)$ and $R_k(\cdot)$ on the joint action $Q \in \mathcal{U}(\bar{c})$ has been suppressed.

Based on the argument in Theorem 2.5, $\bar{G}_{\mathrm{ZCT}}(Q(\theta))$ must be negative definite with respect to Hermitian matrices for all $0 \leq \theta \leq 1$. That is, for all $Z \in \prod_{k\in\mathcal{P}} \mathbb{H}^{N_t}$,

$$\left(\overrightarrow{Z}\right)^{\mathsf{T}} \bar{G}_{\mathrm{ZCT}}(Q(\theta))\overrightarrow{Z} < 0. \tag{2.91}$$

To extend this statement to the nonzero cross-talk pseudohessian requires

$$\left(\overrightarrow{Z}\right)^{\mathsf{T}} \bar{G}_{\mathrm{CT}}(Q(\theta))\overrightarrow{Z} = \left(\overrightarrow{Z}\right)^{\mathsf{T}} \bar{G}_{\mathrm{ZCT}}(Q(\theta))\overrightarrow{Z} + \left(\overrightarrow{Z}\right)^{\mathsf{T}} \tilde{G}(Q(\theta))\overrightarrow{Z} \tag{2.92}$$

be strictly negative for θ throughout $(0, 1)$, thereby leading to a contradiction. To this end, if we could bound the maximum singular value of $\tilde{G}(Q(\theta))$, then we could bound $\left(\overrightarrow{Z}\right)^{\mathsf{T}} \tilde{G}(Q(\theta))\overrightarrow{Z}$, provided Z is drawn from a compact set.

In his work on block matrices, Feingold provides [16, Theorem 2] which bounds the maximum singular value of a matrix by the sum of the maximum singular values of the off-diagonal blocks. In the context of the proof at hand where all of the diagonal blocks of the perturbation pseudohessian are zero, we have

$$\overline{\sigma}(\tilde{G}(Q(\theta))) \leq \max_{k\in\mathcal{P}} \sum_{j\in\mathcal{P}\backslash\{k\}} \overline{\sigma}(\tilde{G}_{k,j}(Q(\theta))). \tag{2.93}$$

For row $k \in \mathcal{R}$ and column $j \in \mathcal{R}\backslash\{k\}$, start with (2.86) and apply the triangle inequality, Corollary A.9, and Lemma A.6 to get

$$\begin{aligned}\overline{\sigma}(\tilde{G}_{k,j}(Q(\theta))) \leq{}& \overline{\sigma}\left(H_{k,k}^{\mathsf{T}} \otimes H_{k,k}^{\dagger}\right) \overline{\sigma}\left(A_k^{-\mathsf{T}}(Q(\theta)) \otimes A_k^{-1}(Q(\theta))\right) \overline{\sigma}\left(H_{k,j}^{*} \otimes H_{k,j}\right) \\ &+ \overline{\sigma}\left(H_{j,k}^{\mathsf{T}} \otimes H_{j,k}^{\dagger}\right) \overline{\sigma}\left(A_j^{-\mathsf{T}}(Q(\theta)) \otimes A_j^{-1}(Q(\theta))\right) \overline{\sigma}\left(H_{j,j}^{*} \otimes H_{j,j}\right).\end{aligned} \tag{2.94}$$

The maximum singular value of a Kronecker product is the product of the maximum singular values of its factors (see, for example, Proposition A.5), so this bound is equivalent to

$$\overline{\sigma}(\tilde{G}_{k,j}(Q(\theta))) \leq \overline{\sigma}^2\left(H_{k,k}\right) \overline{\lambda}^2\left(A_k^{-1}(Q(\theta))\right) \overline{\sigma}^2\left(H_{k,j}\right) + \overline{\sigma}^2\left(H_{j,k}\right) \overline{\lambda}^2\left(A_j^{-1}(Q(\theta))\right) \overline{\sigma}^2\left(H_{j,j}\right) \tag{2.95}$$

$$= \frac{\overline{\sigma}^2\left(H_{k,k}\right) \overline{\sigma}^2\left(H_{k,j}\right)}{\underline{\lambda}^2\left(A_k(Q(\theta))\right)} + \frac{\overline{\sigma}^2\left(H_{j,j}\right) \overline{\sigma}^2\left(H_{j,k}\right)}{\underline{\lambda}^2\left(A_j(Q(\theta))\right)} \tag{2.96}$$

$$\leq \overline{\sigma}^2\left(H_{k,k}\right) \overline{\sigma}^2\left(H_{k,j}\right) + \overline{\sigma}^2\left(H_{j,j}\right) \overline{\sigma}^2\left(H_{j,k}\right), \tag{2.97}$$

where the last line follows by observing that for all users and all joint actions, $\underline{\lambda}(A_k(Q)) \geq 1$.

The jammer's block in row $k \in \mathcal{R}$ is (2.89). We use the same procedure to bound its maximum singular value:

$$\overline{\sigma}(\tilde{G}_{k,0}(Q(\theta))) \leq \sum_{\substack{j\in\mathcal{R}\\ j\neq k}} \overline{\sigma}^2\left(H_{j,0}\right)\overline{\sigma}\left(R_j^{-\intercal}(Q(\theta))\otimes R_j^{-1}(Q(\theta)) - A_j^{-\intercal}(Q(\theta))\otimes A_j^{-1}(Q(\theta))\right)\overline{\sigma}^2\left(H_{j,k}\right). \tag{2.98}$$

Note that $R_j^{-\intercal}(Q)\otimes R_j^{-1}(Q)$ and $A_j^{-\intercal}(Q)\otimes A_j^{-1}(Q)$ are Hermitian positive definite for all $Q \in \mathcal{U}(\bar{c})$ and all $j \in \mathcal{R}$. Lemma A.10 bounds the maximum singular value of their difference by $\max\{\overline{\lambda}(R_j^{-\intercal}(Q)\otimes R_j^{-1}(Q)), \overline{\lambda}(A_j^{-\intercal}(Q)\otimes A_j^{-1}(Q))\}$. Thus the size of the block $\tilde{G}_{k,0}(Q(\theta))$ can be bounded by

$$\begin{aligned}
\overline{\sigma}(\tilde{G}_{k,0}(Q(\theta))) &\leq \sum_{\substack{j\in\mathcal{R}\\ j\neq k}} \overline{\sigma}^2\left(H_{j,0}\right)\overline{\sigma}^2\left(H_{j,k}\right)\max\left\{\overline{\lambda}^2(R_j^{-1}(Q(\theta))), \overline{\lambda}^2(A_j^{-1}(Q(\theta)))\right\} && (2.99)\\
&= \sum_{\substack{j\in\mathcal{R}\\ j\neq k}} \frac{\overline{\sigma}^2\left(H_{j,0}\right)\overline{\sigma}^2\left(H_{j,k}\right)}{\underline{\lambda}^2(R_j(Q(\theta)))} && (2.100)\\
&\leq \sum_{\substack{j\in\mathcal{R}\\ j\neq k}} \overline{\sigma}^2\left(H_{j,0}\right)\overline{\sigma}^2\left(H_{j,k}\right), && (2.101)
\end{aligned}$$

where the last line follows because $\underline{\lambda}(R_j(Q)) \geq 1$ for all users.

The inequalities (2.97) and (2.101) give bounds on the off-diagonal blocks of the perturbation pseudohessian that are independent of the joint actions Q^0 and Q^1. If, as supposed, Q^0 and Q^1 both have at least one user with minimum eigenvalue greater than ϵ, then we must have

$$\max_{0\leq\theta\leq 1} \left(\overrightarrow{Q^1} - \overrightarrow{Q^0}\right)^{\intercal} \bar{G}_{\mathrm{ZCT}}(Q(\theta)) \left(\overrightarrow{Q^1} - \overrightarrow{Q^0}\right) < 0. \tag{2.102}$$

The difference $\left\|\overrightarrow{Q^1} - \overrightarrow{Q^0}\right\|$ is bounded because each joint action is drawn from a compact strategy set; this bound can be taken as independent of Q^0 and Q^1. Thus the quantity

$$\left(\overrightarrow{Q^1} - \overrightarrow{Q^0}\right)^{\intercal} \tilde{G}(Q(\theta)) \left(\overrightarrow{Q^1} - \overrightarrow{Q^0}\right) \tag{2.103}$$

can be restricted in magnitude without knowledge of Q^0 and Q^1 by appropriate choice of $\hat{\sigma}_c$. This guarantees $\bar{G}_{\mathrm{CT}}(Q(\theta))$ remains negative definite with respect to Hermitian matrices for all $0 \leq \theta \leq 1$, which produces a contradiction on the assumption that two Nash equilibria exist with user actions more than ϵ away from singularity. Therefore at most one such Nash equilibrium can exist. □

The results of Theorem 2.5 and Theorem 2.6 are consistent with results in the literature. In [23], the authors observe that for their generalized iterative water-filling algorithm to converge in a jammed multi-user MIMO network (limited to orthogonal tones), the users should not concentrate their power on a few of their tones. The following examples demonstrate the value of each of the assumptions in Theorem 2.5.

Example 2.2 Consider a zero cross-talk system with two users each with two antennae and all agents restricted to unit power. Let the channel matrices be

$$H_{1,1} = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H_{2,2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.9 \end{bmatrix}, \tag{2.104}$$

$$H_{1,0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H_{2,0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.105}$$

Clearly, these jammer channel matrices violate assumption (2.65b). Regardless of the jammer action, the users' optimal actions are to put all of their power on their stronger subchannel:

$$\hat{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.106}$$

However, given these user actions, the jammer cannot affect the capacity of the network: it can only jam the subchannels that each user is not using. Thus any feasible action is an equilibrium action for the jammer.

We now make a few comments about the results set out in Theorem 2.5. A sufficient condition to guarantee that at least one user has a positive definite equilibrium strategy is that the power budget for at least one user must be sufficiently large:

$$\bar{c}_i \geq \hat{c}, \tag{2.107}$$

for at least one $i \in \mathcal{R}$. The sufficiency can be seen by appealing to the water-filling optimization process. Regardless of other users' cross-talk, jammer interference, or ambient noise, a user's optimal power allocation will be distibutred over all subchannels if the volume of water that user "pours" (which is proportional to their power budget) is sufficiently large. The sufficient lower bound $\hat{c}$ will depend on the various channel gain matrices in the game as well as the power budgets of the other agents. Thus one can guarantee a unique equilibrium in a utility game with conditions only on the primitives of that game (channel gain matrices and agent power budgets).

If we increase the power budget to one of the users in Example 2.2, then a uniqueness of the equilibrium can be established.

Example 2.3 Take the game from Example 2.2 and change $\bar{c}_1$ from 1 to 200. This is roughly the smallest power level that allows user 1's water level to spill onto its second subchannel (see Fig.

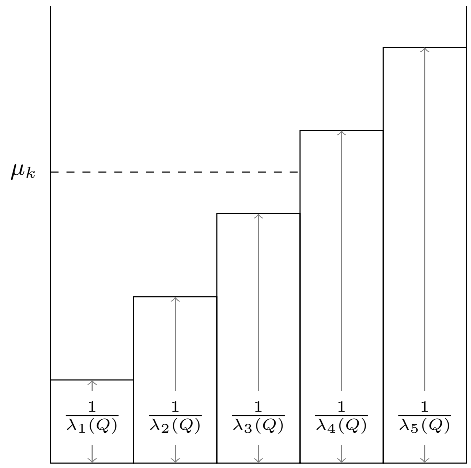


Figure 2.2: Illustration of multi-user MIMO water-filling from user k's perspective. The column heights are the reciprocals of the diagonal elements of the $D_k(Q)$ matrix defined in (2.109). The "water level," μ_k is the unique scalar that causes the volume of water above the columns and below the dashed line to be $\bar{c}_k$ (assuming columns of unit width).

2.2). Now the unique equilibrium is

$$\hat{Q}_1 = \begin{bmatrix} 199.3446 & 0 \\ 0 & 0.6554 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{Q}_0 = \begin{bmatrix} 0.9996 & 0 \\ 0 & 0.0004 \end{bmatrix}. \tag{2.108}$$

An additional factor to consider when evaluating a cost game for uniqueness is the size of the jammer's channel gain matrices, $\{H_{k,0}\}, k \in \mathcal{R}$. One would expect that as the norms of these matrices diminish, the jammer's ability to achieve meaninful interference would similarly decay. In the limit, we should see the game approach the jammer-free case. The norms of the jammer channel gain matrices need not necessarily be pushed to zero, because diminishing these matrices will have a similar effect as increasing users' power budgets.

The water-filling best response of user $k \in \mathcal{R}$ to a joint action Q can best been seen in terms of the eigendecomposition of the effective channel gain:

$$H_{k,k}^{\dagger} R_k^{-1}(Q) H_{k,k} = U_k(Q) D_k(Q) U_k^{\dagger}(Q), \tag{2.109}$$

where $D_k(Q)$ is a diagonal matrix of eigenvalues. With this notation, user k's best response is the water-filling operation

$$\mathrm{BR}_k(Q_{-k}) = U_k(Q) \left(\mu_k I - D_k^{-1}(Q)\right)^+ U_k^{\dagger}(Q), \tag{2.110}$$

which is illustrated in Fig. 2.2.

If $H_{k,0}$ shrinks in norm, this has the effect of inflating the eigenvalues of $H_{k,k}^{\dagger} R_k^{-1} H_{k,k}$, which in turn diminishes the heights of the columns in Fig. 2.2. User k's power budget is now more likely to cover with positive depth the columns in its vessel.

CHAPTER 3
DYNAMICS AND CONVERGENCE

The salient feature of modeling the jammed interference channel as a game is that the optimization decisions are distributed to the agents themselves; no centralized authority is present. Along these lines, in this chapter we present methods by which the agents may iteratively update their actions so that they reach an equilibrium state. Features of merit of these dynamic processes include the extent of the distribution of optimization, the tendency of the process to converge to an equilibrium state, and the time required to reach that equilibrium state.

One dynamic process is said to be more distributed than another if it requires fewer assumptions of common knowledge amongst the agents. We present no proofs of convergence—these are planned for future work—but we do include extensive simulation results that support the notion that these algorithms converge from all feasible initial joint actions and a wide variety of channel matrices. Lastly, we compare the speed of each algorithm by averaging run times of the simulations for a number of antenna array sizes and user population sizes.

Best-response is a widely employable method that can be applied to both the utility and cost games (albeit with some restrictions on the cost game). Alternatively, gradient-play is proposed as a computationally friendly process on the compact convex strategy sets of the utility game. Finally, a distributed stream control process designed to improve the overall network performance is considered in the presence of the jammer.

All of the algorithms in this chapter are intentionally presented as non-terminating. The intention is that these dynamic processes are run indefinitely but that agent actions converge to a steady state eventually. Of course, when these algorithms were simulated—using software presented in Appendix B—termination conditions were introduced to detect when steady state was reached.

3.1 Best-Response Dynamics

The update process known as best-response dynamics is well suited to find the equilbrium in both utility and cost games. For user $k \in \mathcal{R}$, the *best-response correspondence* in $\Gamma^u(\bar{c})$ is

$$\mathrm{BR}_k^u(Q_{-k}) = \underset{Q_k \in \mathcal{U}_k(\bar{c}_k)}{\arg\max}\ u_k(Q_k, Q_{-k}). \tag{3.1}$$

Likewise, in $\Gamma^c(\bar{u})$, the best-response correspondence for user $k \in \mathcal{R}$ is

$$\mathrm{BR}_k^c(Q_{-k}) = \underset{Q_k \in \mathcal{C}_k(\bar{u}_k, Q_{-k})}{\arg\min}\ c_k(Q_k). \tag{3.2}$$

Analogously, the best-response correspondences for the jammer are

$$\mathrm{BR}_0^u(Q_{-0}) = \underset{Q_0 \in \mathcal{U}_0(\bar{c}_0)}{\arg\max}\; J^u(Q_0, Q_{-0}) \tag{3.3}$$

$$\mathrm{BR}_0^c(Q_{-0}) = \underset{Q_0 \in \mathcal{C}_0(\bar{u}_0)}{\arg\min}\; J^c(Q_0, Q_{-0}). \tag{3.4}$$

Recall that $J^c(Q) = -J^u(Q)$ and that in both utility and cost games the jammer endeavors to minimize the sum of the mutual information of all users.

The joint best-response correspondence for the utility and cost games are $\mathrm{BR}^u(\cdot)$ and $\mathrm{BR}^c(\cdot)$, respectively, as defined in (1.7). Although not explicit in this notation, each best-response correspondence depends on its game's constraint vector: $\bar{c}$ and $\bar{u}$, respectively.

An implementation of the best-response process for $\Gamma^u(\bar{c})$ is presented in Algorithm 1, below. Each agent's update is tempered by the parameter $\alpha(t)$, which represents the agents' willingness to optimize as time goes on. It's complement, $1 - \alpha(t)$, can be thought of as the inertia experienced by each agent when it is tasked with updating. These parameters are chosen such that

$$0 < \alpha(t) < 1, \quad \forall t, \tag{3.5a}$$

$$\lim_{t \to \infty} \alpha(t) = 0, \tag{3.5b}$$

$$\sum_{t=1}^{\infty} \alpha(t) = \infty, \tag{3.5c}$$

and serve to smooth out the trajectories.

Algorithm 1 Best-response dynamics for $\Gamma^u(\bar{c})$

```
for k = {0, ..., r} do
  Initialize Q_k(0) ∈ U_k(c̄_k)
end for
t = 1
while t ≥ 1 do
  for k = {0, ..., r} do
    Q_k(t+1) ∈ (1 − α(t))Q_k(t) + α(t) BR^u_k(Q_{−k}(t))
  end for
  t = t + 1
end while
```

For user $k \in \mathcal{R}$, the best-response correspondence in principle requires knowledge of Q_{-k} and $H_{k,j}$ for all $j \in \mathcal{P}$. However, in practice it is sufficient for user k to measure the total interference it encounters, $R_k(Q)$, as none of the other agents' individual actions or channel matrices appear separately when user k optimizes its utility.

Figure 3.1 shows the mutual information trajectories for best-response dynamics in a sample

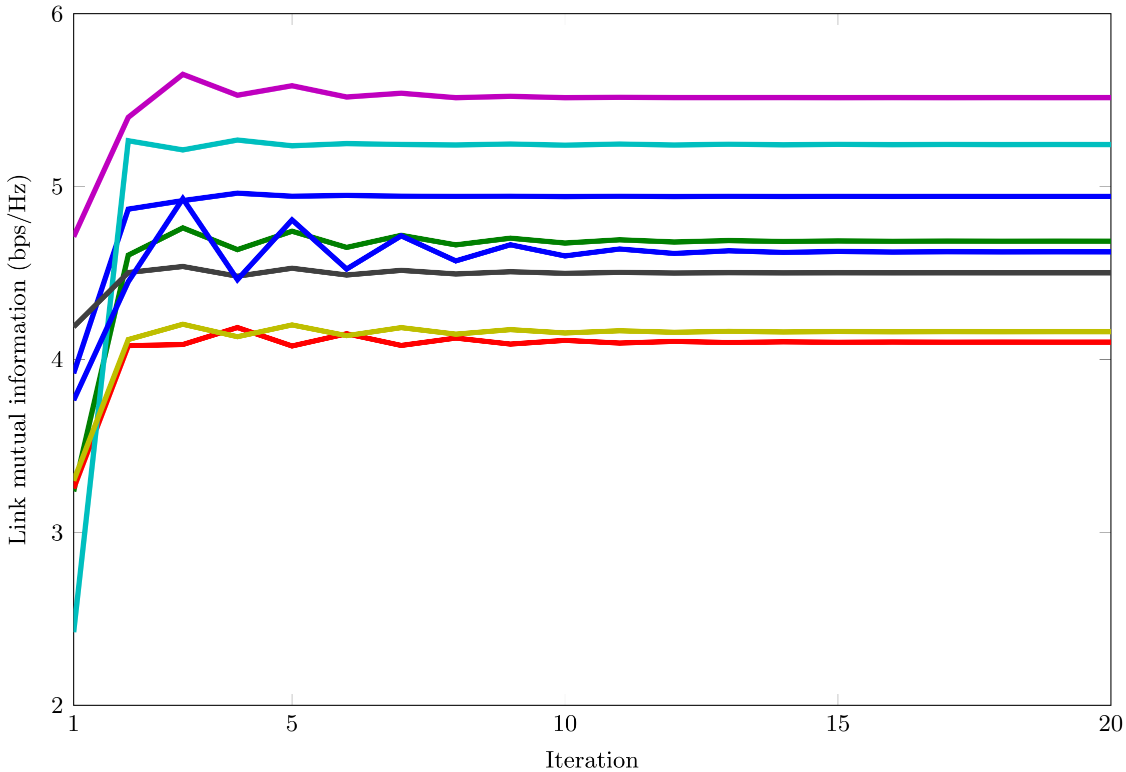


Figure 3.1: Best-response trajectories for $\Gamma^u(\bar{c})$ where $\bar{c}_k = 10$ for all $k \in \mathcal{P} = \{0, 1, \dots, 8\}$ and random channel matrices in $\mathbb{C}^{4\times 4}$. All user channel matrices were scaled such that $\|H_{k,k}\| = 20\,\mathrm{dB}$ for all $k \in \mathcal{R}$, and all interference channel matrices were scaled such that $\|H_{j,k}\| = 10\,\mathrm{dB}$ for all $j \neq k \in \mathcal{R} \times \mathcal{P}$. All agents, including the jammer (not shown), converged to their equilibrium strategy within 20 iterations.

utility game.

Whereas the utility game best-response dynamics appear to converge for all power budgets, the cost minimization game does not share this good fortune. For arbitrary target utility levels, the existence of an equilibrium cannot be guaranteed even for small games.

Example 3.1 Consider the cost game with $\bar{u} = (-1, 9, 6, 3)$ and channel matrices as follows:

$$H_{1,1} = H_{2,2} = H_{3,3} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$H_{1,2} = H_{1,3} = H_{2,3} = \frac{1}{2} \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}$$
$$H_{2,1} = H_{3,1} = H_{3,2} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ j & 1 \end{bmatrix}$$
$$H_{1,0} = H_{2,0} = H_{3,0} = \begin{bmatrix} 2 & j \\ 1-j & 1 \end{bmatrix}.$$

The users' trajectories under a best-response process are shown in Fig. 3.2. The top plot conveys the perils of unbounded strategy sets: the users iteratively increase their transmission power without bound in response to increased powers of the other agents. The bottom plot shows the trend that no finite power levels are enough for the users to achieve their utility goals. Even—as it appears for user 3 with constraint $\bar{u}_3 = 3$ shown in red in this example—when the utility levels asymptotically approach their constraint levels, the power grows without bound.

The jammer is present in this example, but the phenomenon of ever-increasing power levels was also observed in cost games without a jammer in [4].

By modifying the objectives of the users to include penalties for missing their target utilities, equilibrium existence can be recovered. For any target utility levels $\bar{u} \in \mathbb{R}_- \times \mathbb{R}_+^r$ and any vector of weights $w \in \mathbb{R}_+^r$, we define the *weighted cost minimization* game $\Gamma^w(\bar{u})$ as one in which each user $k \in \mathcal{R}$ aims to optimize

$$\min_{Q_k \in \mathcal{S}_k} c_k(Q_k) + w_k \left(\bar{u}_k - u_k(Q_k, Q_{-k})\right)^+. \tag{3.6}$$

The analogous best-response correspondence[1] for this game is

$$\mathrm{BR}_k^w(Q_{-k}) = \operatorname*{arg\,min}_{Q_k \in \mathcal{S}_k} c_k(Q_k) + w_k \left(\bar{u}_k - u_k(Q_k, Q_{-k})\right)^+. \tag{3.7}$$

Note that in $\Gamma^w(\bar{u})$ the jammer's objective and constraint is identical to that in $\Gamma^c(\bar{u})$ because the jammer's feasible strategy set is already decoupled. Indeed, the jammer is still power constrained and interested only in minimizing the total system capacity. That is, $\mathrm{BR}_0^w(\cdot) = \mathrm{BR}_0^c(\cdot)$.

[1] Again, the dependence of BR_k^w on $\bar{u}$, while important, is suppressed to minimize notational clutter.

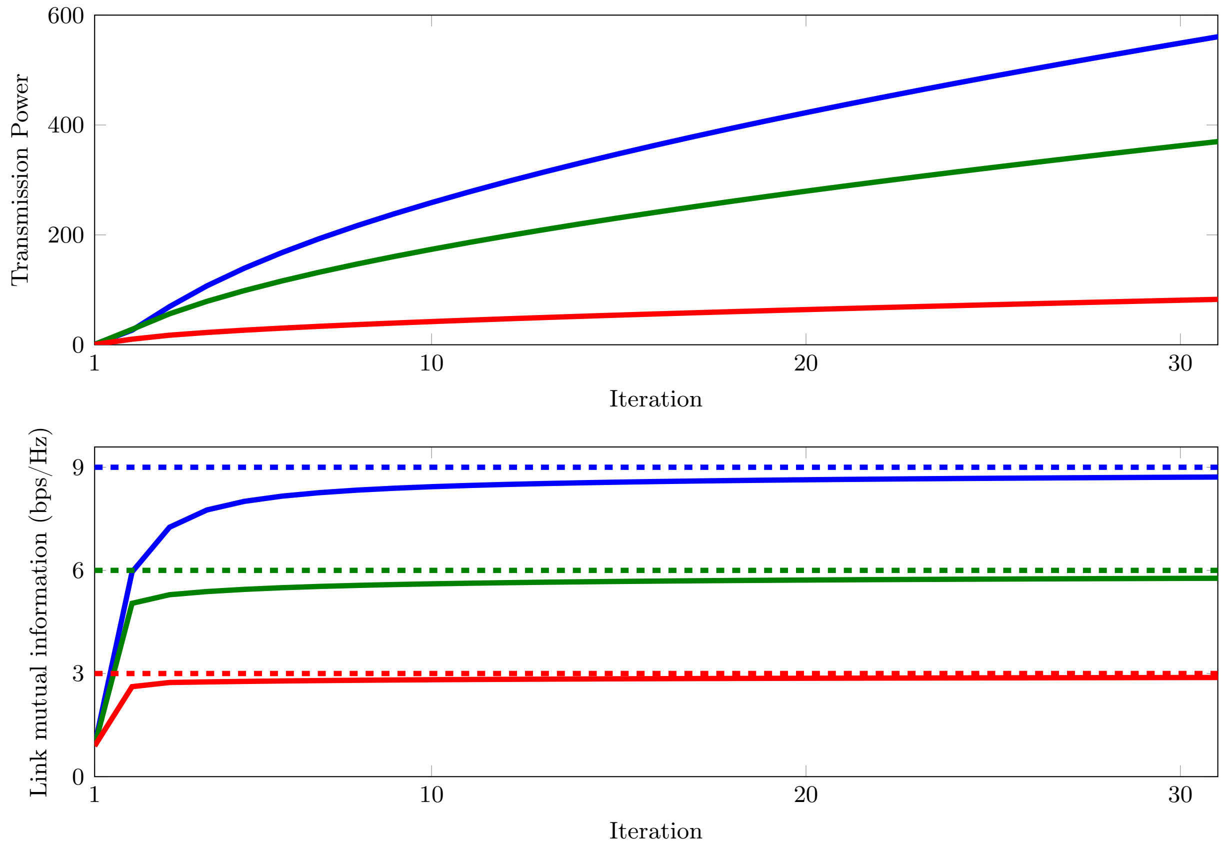


Figure 3.2: Non-convergent cost game under best-response dynamics. The cost and utility trajectories are plotted for the three users in the system described in Example 3.1. Dashed lines indicate each user's mandated utility minimum (which none achieved).

The strategy sets of the users in the weighted best-response game are unbounded, but nevertheless Theorem 3.1, below, provides for the existence of a joint strategy $\hat{Q} \in \mathcal{S}$ such that

$$\mathrm{BR}^w(\hat{Q}) = \hat{Q}, \tag{3.8}$$

where $\mathrm{BR}^w(\cdot)$ is the joint best-response correspondence as in (1.7), for any reasonable vectors of constraints and weights.

Theorem 3.1 *For any $\bar{u} \in \mathbb{R}_- \times \mathbb{R}_+^r$ and any $w \in \mathbb{R}_+^r$, $\Gamma^w(\bar{u})$ possesses a Nash equilibrium.*

Proof For any user $k \in \mathcal{R}$, playing the zero matrix guarantees a cost of $w_k \bar{u}_k$. Thus for every $k \in \mathcal{R}$ and any $Q_{-k} \in \mathcal{S}_{-k}$,

$$Q_k' \in \mathrm{BR}_k^w(Q_{-k}) \implies c_k(Q_k') \leq w_k \bar{u}_k. \tag{3.9}$$

For the jammer, the Weierstrass Theorem guarantees that for all $Q_{-0} \in \mathcal{S}_{-0}$,

$$\mathrm{BR}_0^w(Q_{-0}) = \underset{Q_0 \in \mathcal{C}_0(\bar{u}_0)}{\arg\min} \sum_{k \in \mathcal{R}} u_k(Q_0, Q_{-0}) \tag{3.10}$$

will achieve its minimum on the compact convex set $\mathcal{C}_0(\bar{u}_0)$.

Taken together these results imply that the joint best response maps $\mathcal{S}$ to $\bar{\mathcal{S}}^w$, where

$$\bar{\mathcal{S}}^w = \mathcal{C}_0(\bar{u}_0) \times \bar{\mathcal{S}}_1^w \times \cdots \times \bar{\mathcal{S}}_r^w, \tag{3.11}$$

$$\bar{\mathcal{S}}_k^w = \mathrm{cc}\left(\{Q_k \in \mathcal{S}_k : c_k(Q_k) \leq w_k \bar{u}_k\}\right), \tag{3.12}$$

and $\mathrm{cc}(\cdot)$ denotes the closure of the convex hull.

The restriction of $\Gamma^w(\bar{u})$ to be played on $\bar{\mathcal{S}}^w$ is a concave game played on compact, convex strategy sets and thus possesses a Nash equilibrium (Proposition 1.1). Since the image of $\mathcal{S}$ under $\mathrm{BR}^w(\cdot)$ is contained in $\bar{\mathcal{S}}^w$, any equilibrium of this restricted game must also be an equilibrium of the unrestricted game, $\Gamma^w(\bar{u})$. □

Theorem 3.1 appears for general cost-minimizers in [4, Proposition 7]. However, the presence of a utility-minimizing jammer is novel in this context.

Even games with no generalized Nash equilibria, such as the one in Example 3.1, can exhibit convergent best-response behavior when considered as a weighted cost game. Once the cost game in Example 3.1 is augmented with weight vector $w = (20, 20, 20)$, the best-response trajectories of this game appear in Fig. 3.3.

For a constraint vector $\bar{u}$ and weight vector w, the set of Nash equilibria of the weighted cost game $\Gamma^w(\bar{u})$ is

$$\mathcal{E}^w(\bar{u}) = \left\{\hat{Q} \in \mathcal{S} : \mathrm{BR}^w(\hat{Q}) = \hat{Q}\right\}. \tag{3.13}$$

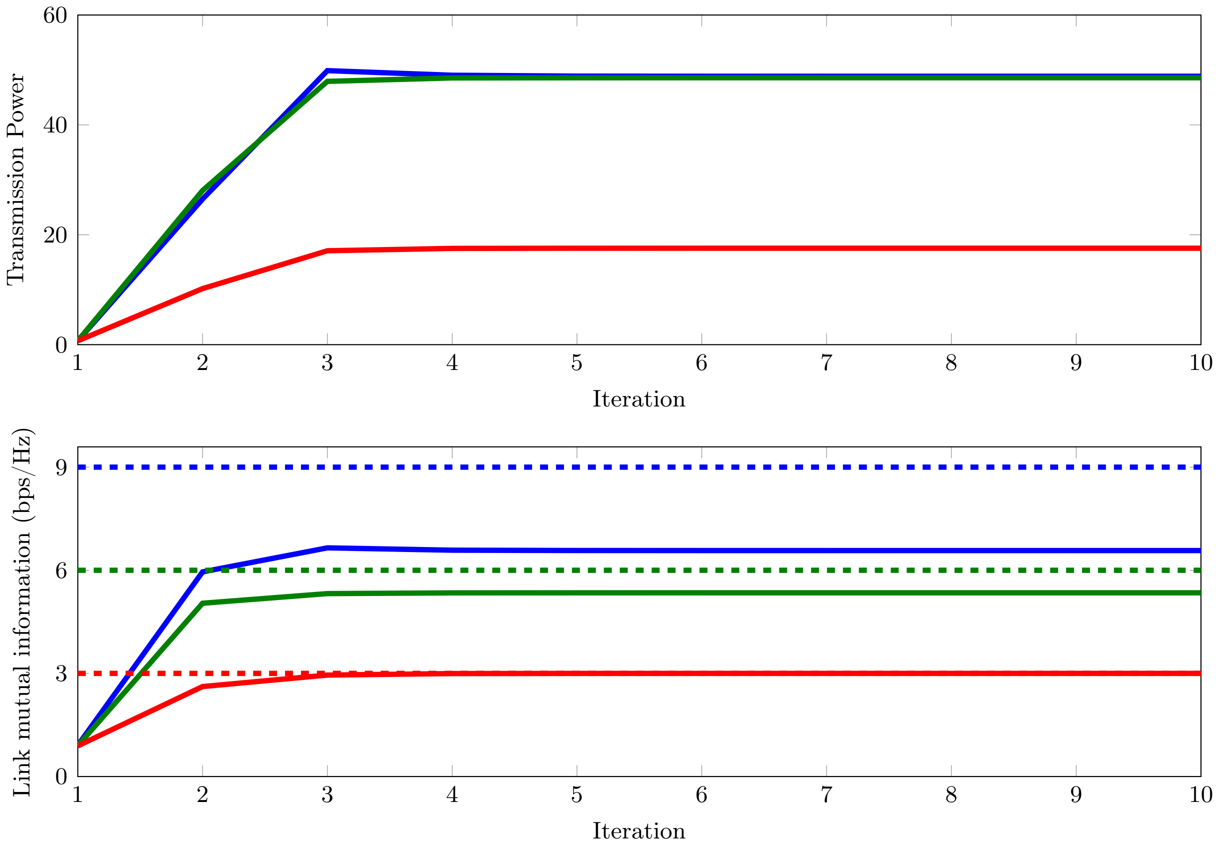


Figure 3.3: Convergent weighted cost game under best-response dynamics. Under the same utility requirements as Example 3.1, users now minimize a weighted sum of their cost and utility. For the uniform weight vector $w = (20, 20, 20)$, all agents converged to equilibrium strategies within five iterations. As expected, however, not all of the users are transmitting at their mandated minimum rate (shown as dashed lines) at equilibrium.

The intent of the weighted cost game is to soften the possibly unobtainable constraint on each user. However, if an equilibrium can be reached in the weighted cost game that satisfies the utility constraints, that strategy is also an equilibrium of the original unweighted cost game.

Theorem 3.2 *For any $\bar{u} \in \mathbb{R}_- \times \mathbb{R}_+^r$ and any $w \in \mathbb{R}_+^r$, if an equilibrium strategy of $\Gamma^w(\bar{u})$ satisfies all the minimum utility requirements, it is also an equilibrium of $\Gamma^c(\bar{u})$:*

$$\left\{ \hat{Q} \in \mathcal{E}^w(\bar{u}) : u_k(\hat{Q}) \geq \bar{u}_k \forall k \in \mathcal{R} \right\} \subseteq \mathcal{E}^c(\bar{u}) \tag{3.14}$$

Proof First note that if $\hat{Q} \in \mathcal{E}^w(\bar{u})$, then $J^c(\hat{Q}) \leq J^c(Q_0, \hat{Q}_{-0})$ for all $Q_0 \in \mathcal{C}_0(\bar{u}_0)$, which is the same requirement on $\hat{Q}_0$ to be a Nash equilibrium strategy for the jammer in $\Gamma^c(\bar{u})$.

The proof for the users follows the method of [4, Proposition 8]. Let $\hat{Q}$ be an equilibrium strategy in $\Gamma^w(\bar{u})$ as in the left-hand side of (3.14). Then for all $k \in \mathcal{R}$ and all $Q_k \in \mathcal{S}_k$,

$$c_k(\hat{Q}_k) = c_k(\hat{Q}_k) + w_k \left(\bar{u}_k - u_k(\hat{Q}) \right)^+ \tag{3.15}$$

$$\leq c_k(Q_k) + w_k \left(\bar{u}_k - u_k(Q_k, \hat{Q}_{-k}) \right)^+, \tag{3.16}$$

where (3.15) follows from the supposition $\bar{u}_k - u_k(\hat{Q}) \leq 0$, and (3.16) follows by virtue of $\hat{Q}$ being a Nash equilibrium of the weighted cost game.

If there exists some $Q'_k \in \mathcal{C}_k(\bar{u}_k, \hat{Q}_{-k})$, then substituting into (3.16) implies $c_k(\hat{Q}_k) \leq c_k(Q'_k)$. However, $\hat{Q}_k$ meets this requirement for all $k \in \mathcal{R}$, so $\hat{Q}$ must be an equilibrium of the unweighted cost game $\Gamma^c(\bar{u})$. □

As a final connection between the sets of equilibria for the cost and weighted cost games, Proposition 9 in [4] can be extended to be valid in the presence of a jammer. Recall that for real vectors x and y, we use $x > y$ to mean every element of x is larger than the corresponding element of y.

Theorem 3.3 *For any $\bar{u} \in \mathbb{R}_- \times \mathbb{R}_+^r$ and any bounded set $\mathcal{B} \subset \mathcal{S}$, there is some $\bar{w} \in \mathbb{R}_+^r$ such that for all $w > \bar{w}$, $\mathcal{E}^c(\bar{u}) \cap \mathcal{B} = \mathcal{E}^w(\bar{u}) \cap \mathcal{B}$.*

Proof The proof approach is to show that Proposition 9 from [4] applies. The requirements on a jammer's equilibrium action are identical between the cost and weighted cost games.

In order to apply Proposition 9 from [4], it is sufficient to show that for all $k \in \mathcal{R}$ and all $Q_{-k} \in \mathcal{S}_{-k}$, $c_k(\cdot)$ and $u_k(\cdot, Q_{-k})$ are differentiable everywhere in $\mathcal{S}_k$ and that the gradients $\nabla_{\overrightarrow{Q_k}} c_k$ and $\nabla_{\overrightarrow{Q_k}} u_k$ are continuous in $\mathcal{S}$ (per [4, Remark 3]).

The requirements on $c_k(\cdot)$ are certainly satisfied due to its linearity in Q_k and invariance to Q_{-k}.

Regarding the differentiability of $u_k(\cdot, Q_{-k})$ on all of $\mathcal{S}_k$, we first endeavor to show that $u_k(\cdot, Q_{-k})$ is defined on an open set containing $\mathcal{S}_k$. Consider a point outside but arbitrarily close to $\mathcal{S}_k$:

$$Q'_k = Q_k + \epsilon B, \tag{3.17}$$

where $Q_k \in \mathcal{S}_k$, $B \in \mathbb{H}^{N_t}$, and $\epsilon > 0$ is a small constant. Certainly if $Q_k' \in \mathcal{S}_k$, then $u_k(Q_k', Q_{-k})$ is well-defined. The interesting case is when Q_k' lies just outside $\mathcal{S}_k$; in this instance user k's utility function can be evaluated as[2]

$$u_k(Q_k', Q_{-k}) = \log \left| I + R_k^{-1/2} H_{k,k} \left(Q_k + \epsilon B\right) H_{k,k}^\dagger R_k^{-1/2} \right| \tag{3.18}$$

$$= \log \left| I' + \epsilon B' \right|, \tag{3.19}$$

where $I' = I + R_k^{-1/2} H_{k,k} Q_k H_{k,k}^\dagger R_k^{-1/2} \succeq I$ and $B' = R_k^{-1/2} H_{k,k} B H_{k,k}^\dagger R_k^{-1/2}$. By judicious use of matrix differentiation (as in [50]), (3.19) can be written

$$u_k(Q_k', Q_{-k}) = \log \left(|I'| \left[1 + \epsilon \operatorname{Tr}\left(I'^{-1} B' \right) + \mathcal{O}(\epsilon^2) \right] \right), \tag{3.20}$$

which is well-defined for sufficiently small $\epsilon > 0$. Thus $u_k(\cdot, Q_{-k})$ is differentiable on an open set containing $\mathcal{S}_k$ with differential (2.45) and gradient (2.46). The gradient $\nabla_{\overrightarrow{Q_k}} u_k(\cdot)$ is continuous on $\mathcal{S}$ because $A_k(Q) \succeq I$ is bounded strictly away from singularity for all $k \in \mathcal{R}$ and all $Q \in \mathcal{S}$. □

With the existence of equilibria in arbitrarily weighted cost games established, we now propose a version of best-response dynamics for these games in Algorithm 2.

Algorithm 2 Best-Response Dynamics for $\Gamma^w(\bar{u})$

Initialize $Q_0(0) \in \mathcal{C}_0(\bar{u}_0)$
for $k = \{1, \ldots, r\}$ **do**
  Initialize $Q_k(0) \in \bar{\mathcal{S}}_k^w$
end for
$t = 1$
while $t \geq 1$ **do**
  for $k = \{0, \ldots, r\}$ **do**
    $Q_k(t+1) \in (1 - \alpha(t)) Q_k(t) + \alpha(t) \, \mathrm{BR}_k^w(Q_{-k}(t))$
  end for
  $t = t + 1$
end while

Although conditions that guarantee covergence of the weighted cost game have not been found, simulations suggest that these conditions are not uncommon. We now consider the tradeoffs involved with choosing penalty weights. On one hand, the larger any user's weight is, the more incentive that user has to truly meet its utility requirement. On the other hand, there may be situations in which a user's utility requirement is just unrealistic, and an exorbitant penalty weight will only drive its cost through the roof without producing the desired outcome.

To examine this issue, Fig. 3.4 shows an average performance analysis for random weighted cost games. In particular, it can be seen that while increasing the penalty weights on the users

[2]Here the dependence of R_k on Q has been suppressed for notational simplicity.

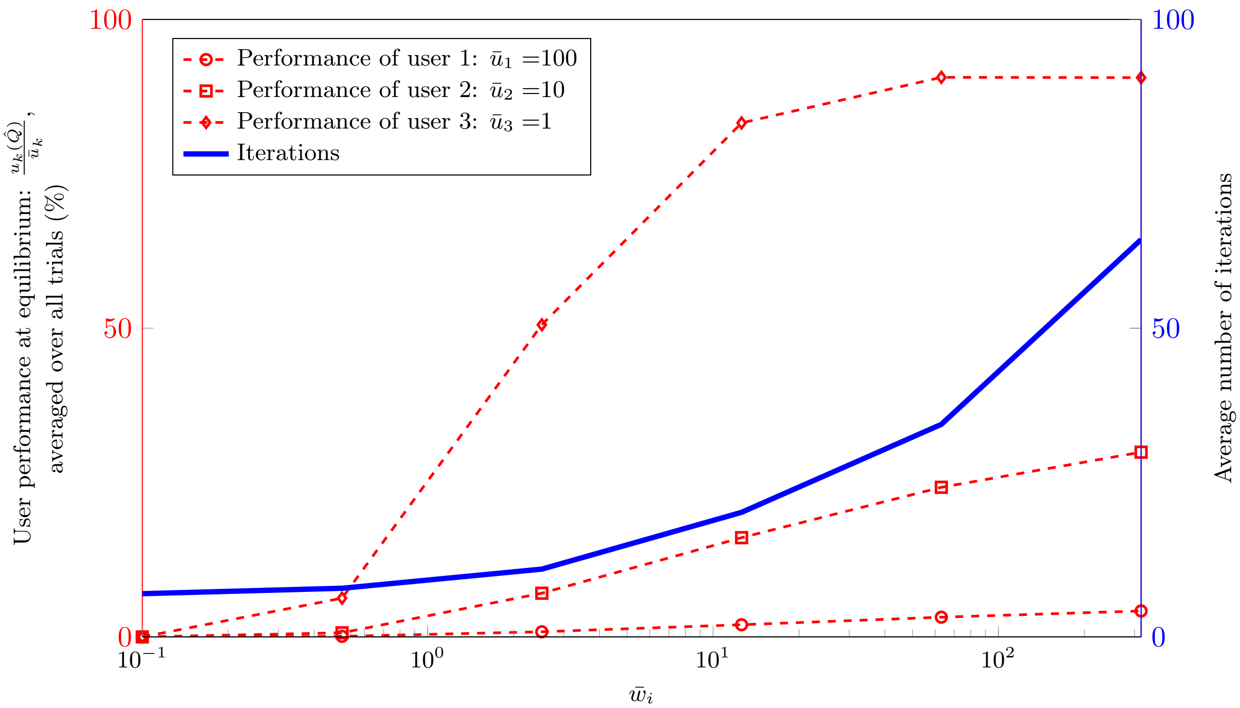


Figure 3.4: Average performance of BR^w plotted against weight size. For each of over 270 trials, random channel matrices were generated for a three-user-plus-jammer system with $N_t = N_r = 2$. For each trial, best-response dynamics were simulated at various uniform weight vectors $w = \bar{w}\mathbb{1}$.

does indeed drive them closer to their mandated utility levels, it has the side effect of significantly extending the number of iterations required to converge to equilibrium. Furthermore, this increase in penalty weighting has the largest effect on users with relatively low minimum utility requirements. The users with relatively large utility requirements see the smallest gains while still paying the price for longer convergence times.

3.2 Gradient-Play Dynamics

Similar to gradient-ascent optimization algorithms, the update process known as gradient-play dynamics has each agent update its action in the local direction of greatest gain given all other agents' actions.

As with any gradient method, it is critical to keep the trajectory within the set of feasible strategies. To this end, the following quantities are used in the discrete time gradient-play dynamics described in Algorithm 3, below:

- $\bar{\gamma}$ is the step size in the direction of steepest gradient ascent.
- $\Pi_A[x]$ is the projection of the vector $x \in B$ onto the convex set $A \subseteq B$.

Just as in best-response dynamics, the presence of inertia parameters serves to temper each agent's optimal update and smoothes out the trajectories.

Algorithm 3 Gradient-play dynamics for $\Gamma^u(\bar{c})$

for $k = \{0, 1, \ldots, r\}$ **do**
  Initialize $Q_k(0) \in \mathcal{U}_k(\bar{c}_k)$
end for
$t = 1$
while $t \geq 1$ **do**
  for $k = \{1, \ldots, r\}$ **do**
    $Q_k(t+1) = (1 - \alpha(t))Q_k(t) + \alpha(t)\Pi_{\mathcal{U}_k(\bar{c}_k)}\left[Q_k(t) + \bar{\gamma}\nabla_{Q_k} u_k(Q(t))\right]$
  end for
  $Q_0(t+1) = (1 - \alpha(t))Q_0(t) + \alpha(t)\Pi_{\mathcal{U}_0(\bar{c}_0)}\left[Q_0(t) + \bar{\gamma}\nabla_{Q_0} J^u(Q(t))\right]$
  $t = t + 1$
end while

The information requirements for the gradient-play algorithm are identical to those for best-response. At each time step user $k \in \mathcal{R}$ must measure or otherwise learn the composite interference it is experiencing: $R_k(Q(t))$.

Guarantees of convergence of the gradient-play algorithm in the cost game have not yet been proven. However, Monte Carlo simulation suggests that gradient-play is a time-efficient method of achieving a Nash equilibrium in the cost game.

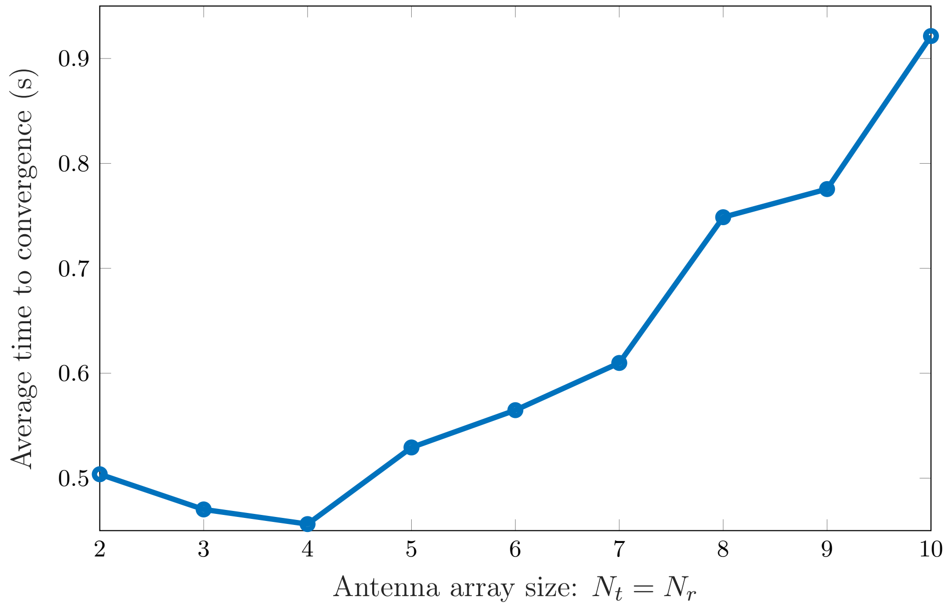


Figure 3.5: Monte Carlo gradient-play.

In support of this conclusion simulations were run that varied the number of antennae ($N_r = N_t$) from 2 to 10 with 1000 trials at each level. In each trial, the channel matrices for 10 users and a jammer were randomly generated and then scaled so that $\|H_{k,k}\| = 10$ and $\|H_{j,k}\| = 1$ for all $j \neq k \in \mathcal{P}$; additionally, the power budget for each agent was randomly generated between 1 and 100. In all trials, the willingness to optimize parameter evolved as $\alpha(t) = \frac{1}{t}$, and the step size was set to $\bar{\gamma} = 0.005$ for all agents. The gradient-play algorithm was implemented in the Matlab R2016a language (see Appendix B.4) on a 2 GHz Intel Core i5 processor. The average time to convergence for each MIMO antenna array size is shown in Fig. 3.5.

The performance of the gradient-play method is quite impressive. The derivatives calculated at each step are easily implementable matrix equations given in (2.46) for user $k \in \mathcal{R}$ and (2.51) for the jammer. Additionally, each projection operation involves an $VDV^\dagger$ spectral decomposition and a single-variable monotonic function root search. That is, the projection of $X \in \mathbb{H}^{N_t}$ onto $\mathcal{U}_k(\bar{c}_k)$, denoted $\Pi_{\mathcal{U}_k(\bar{c}_k)}[X]$, can be obtained in this way:

- Decompose $X = V\,\mathrm{diag}(\lambda_1, \ldots, \lambda_{N_t})V^\dagger$ for unitary V.
- Find $\mu \in \mathbb{R}$ such that $\sum_{i=1}^{N_t}(\lambda_i + \mu)^+ = \bar{c}_k$.
- Set the new eigenvalues as $\rho_i = (\lambda_i + \mu)^+,\ i = 1, \ldots, N_t$.
- Recompose $\Pi_{\mathcal{U}_k(\bar{c}_k)}[X] = V\,\mathrm{diag}(\rho_1, \ldots, \rho_{N_t})V^\dagger$.

Figure 3.6 shows user mutual information trajectories for a large sample system under gradient-play dynamics.

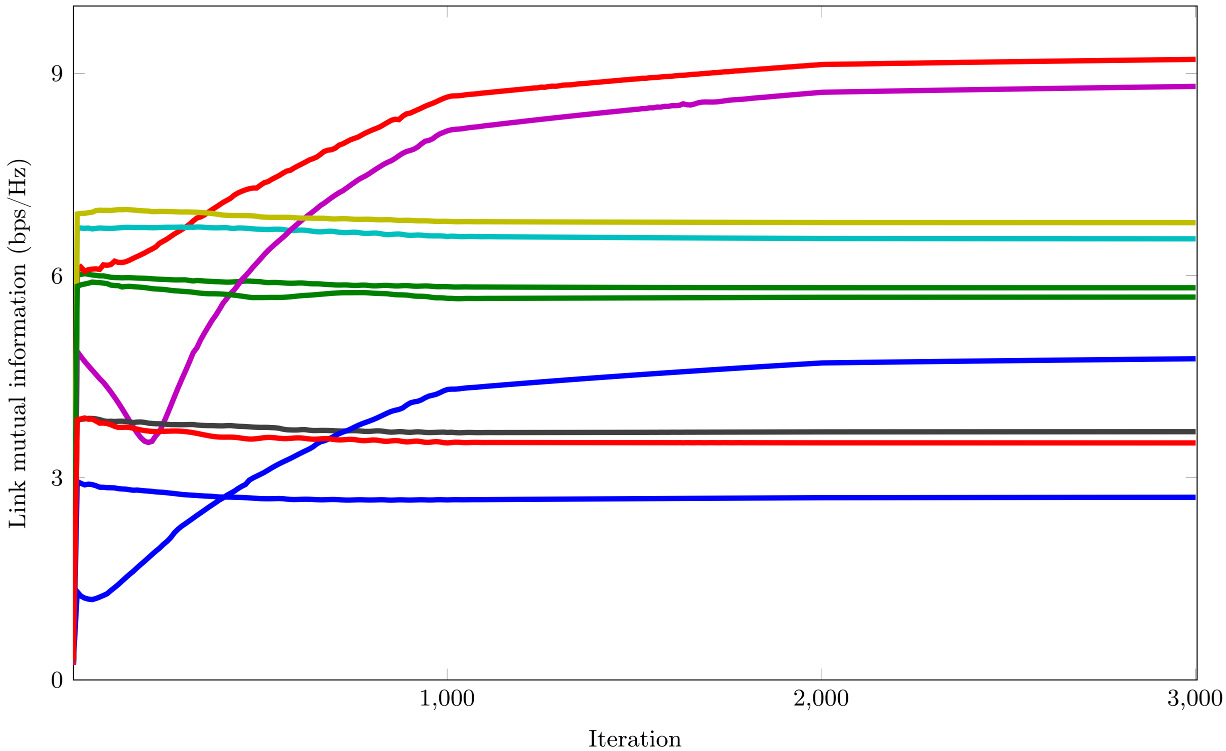


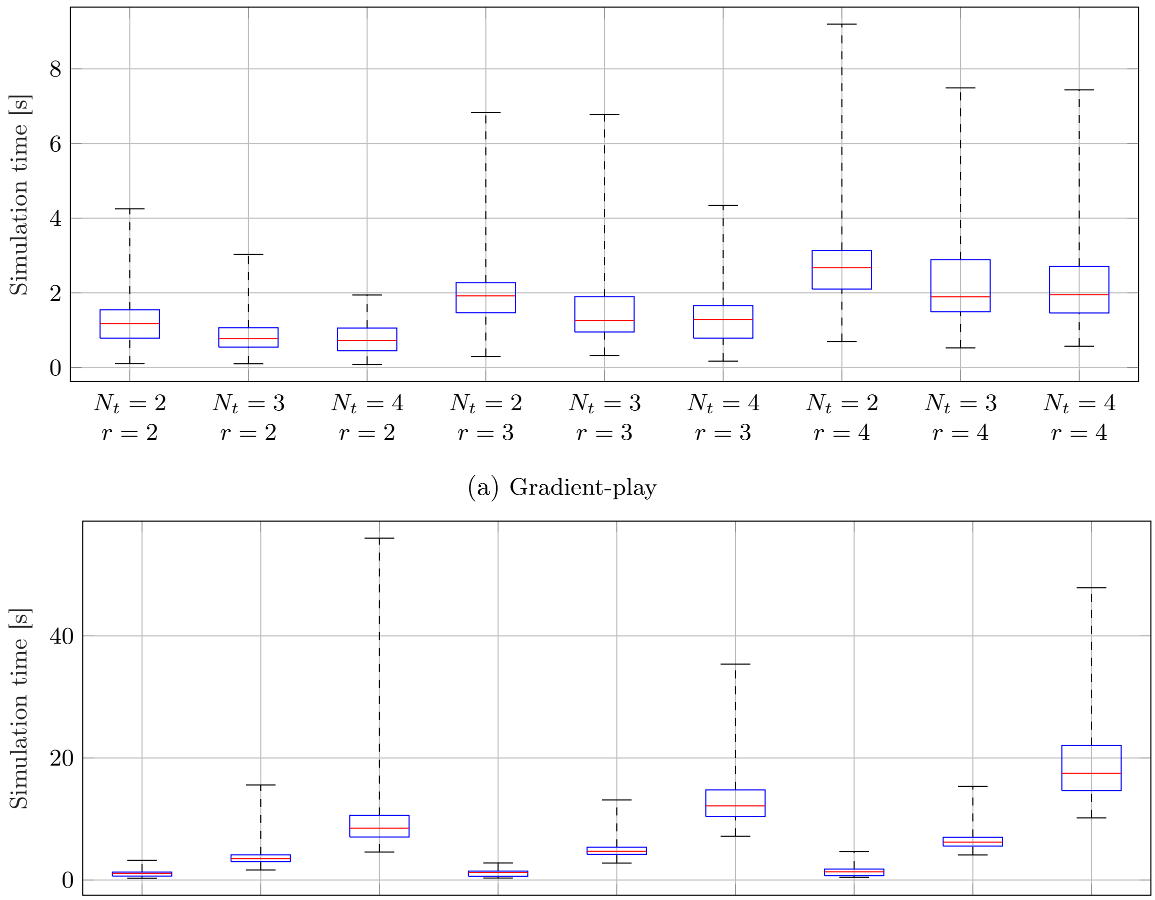
Figure 3.6: Sample trajectories for users' mutual information under the gradient-play process. In this sample, 10 users and one jammer each have $N_t = N_r = 10$ antennae. Power limits for each agent were chosen uniformly randomly in $[1, 100]$. Channel matrices were also chosen randomly. Once chosen, the channel matrices were scaled so that $\|H_{k,k}\| = 10$ and $\|H_{j,k}\| = 1$ for all $j \neq k \in \mathcal{P}$. The willingness to optimize parameter evolved according to

$$\alpha(t) = \begin{cases} t^{-0.3}, & 1 \leq t \leq 1000 \\ t^{-0.5}, & 1000 < t \leq 2000 \\ t^{-0.7}, & 2000 < t \leq 3000 \\ t^{-0.9}, & t > 3000 \end{cases}$$

The gradient step size was $\bar{\gamma} = 0.005$.

Since both best-response and gradient-play dynamics have identical distributed information requirements and simulations suggest both converge to the unique Nash equilibrium in the utility game when it exists, we can compare these processes on execution time. The question is: for a given utility game with a unique equilibrium, which process provides faster convergence?

We offer an empirical statistical answer to this question with the data summarized in Fig. 3.7. The two independent variables are the MIMO array size (in each case constraining $N_t = N_r$) and the number of users in the game (in each case including an additional jamming agent). The first conclusion to draw from this data is that the gradient-play process tends to be around an order of magnitude faster than the best-response process. This is not surprising due to the relatively simple manipulations in Algorithm 3 as opposed to the nonconvex optimization required at each time step for each agent in Algorithm 1.



(a) Gradient-play

(b) Best-response

Figure 3.7: Computer simulation computation time comparison between gradient-play and best-response dynamics. Columns are first grouped by number of users, r, and then sorted by the antenna array size with $N_t = N_r$ in all cases. Each column condenses approximately 400 trials into a box-and-whisker plot. For each trial, the same randomly generated zero cross-talk utility game was solved independently by each algorithm.

3.3 Optimization via Stream Control

The equilibrium reached in the utility game may not truly maximize the network's capacity in the presence of a jammer. The equilibrium strategies of all agents are merely guaranteed to be "unilaterally optimal." That is, a global planner may be able to coordinate the users' strategies to achieve higher total mutual information [14, 43]. However doing so would negate the distributed robustness inherent in the self-interested decision makers model (i.e. a game).

However, in [2], the authors demonstrate that a "pre-game" in which users adapt [54] the number of MIMO subchannels they employ [12,13] can be effective in raising the total network performance. This pre-game negotiation is against the jammer's interests, but we can model the jammer as a somewhat unwilling participant.

Theorem 3.4 *The decentralized stream control described in [2] (i.e. spatial adaptive play) enjoys the same convergence results when a jammer is introduced.*

Proof The jammer enters the stream negotiation game as player 0, but its strategy set in this game—the number of streams it selects to employ—is the singleton N_t. This is equivalent to saying that the jammer will always transmit (in a unilaterally optimal way) on all of its possible streams regardless of what the other players choose.

The jammer never changes its action, so its actions in the stream control game converge immediately. As in the jammer-free case, the users observe interference from the various other agents and continue to negotiate and converge in the presence of the jammer's interference. □

The assumption that the jammer is strictly self-interested is in keeping with the model used throughout this dissertation. Furthermore, it would be inappropriate to model the jammer as a willing participant in any kind of coordination routine due to the adversarial nature of the user versus jammer relationship. One can of course imagine situations in which the jammer might intentionally select a suboptimal strategy (perhaps by reducing its streams) with the aim to induce the stream control negotiations to settle on a globally suboptimal stream reduction strategy. However, this is not in keeping with the model employed throughout this dissertation, and any investigation into such meta-strategies will have to wait for future work.

To demonstrate the usefulness of the stream control approach, Fig. 3.8 shows average gains in network capacity over the typical full-stream approach. With the jammer present in both scenarios, the distributed steam control method consistently increased the system's performance. Furthermore, the spatial adaptive play algorithm for the stream control pre-game succeeded in locating the optimal reduction in streams for all users in the vast majority of trials.

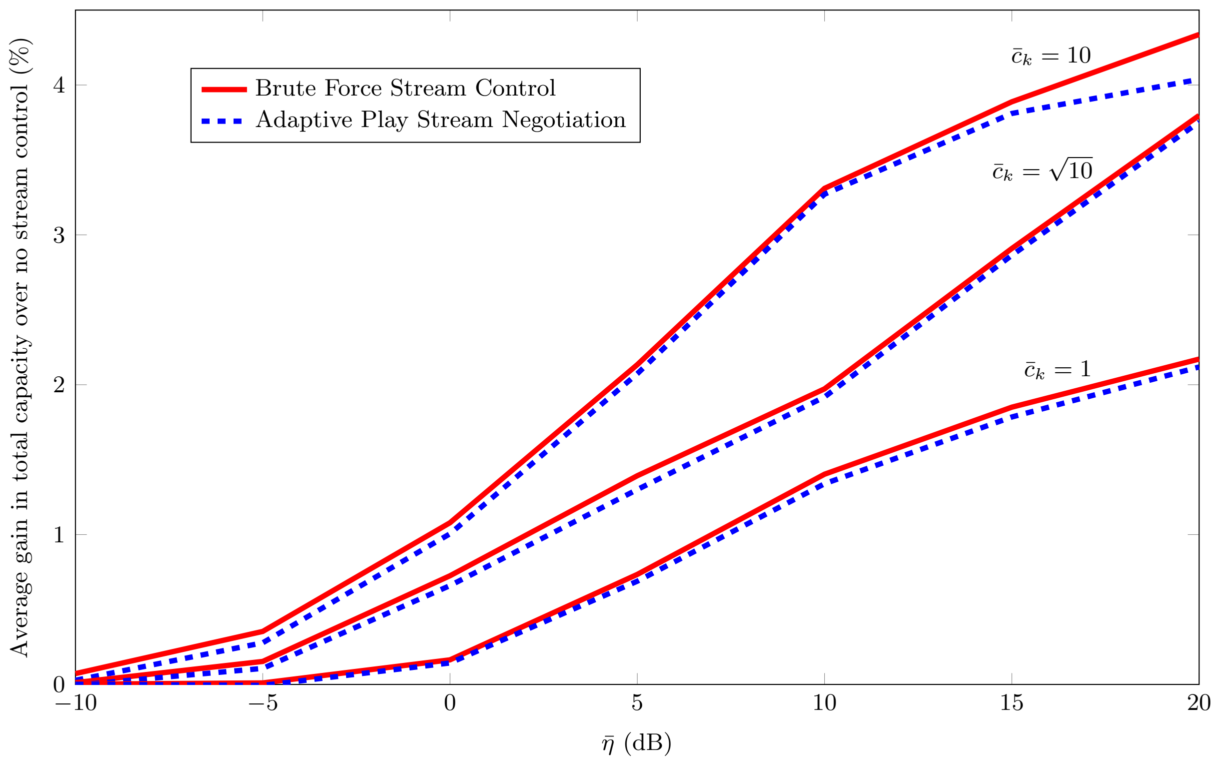


Figure 3.8: On average, stream control via negotiation performs very well relative to brute-force optimization. The results here are averaged over 250 trials with $r = 4$, $N_t = N_r = 2$, and $|\bar{c}_k|$ identical for all $k \in \mathcal{P}$. In each trial, $\|H_{k,k}\|$ was normalized to 1 for all users, and the interference gain matrices were scaled so that $\|H_{j,k}\| = \bar{\eta}$.

CHAPTER 4
CONCLUSION

This dissertation examines the question "What can be done when a jammer is present in a MIMO Gaussian interference channel?" from a game theoretic perspective. After setting up the preliminary game and information theoretic scaffolding, we modeled two paradigms under which users of the channel might operate. In utility games, users strive to maximize the information transmitted across their link in the network subject to a power constraint. In cost games, users have the dual objective to minimize their power subject to an information rate minimum. In both cases, the jammer appears as a minimizer of total information in the network subject to a power constraint.

The salient features of game theoretic analysis are the supposition of self-interest by all agents and the absence of a centralized authority. Both of these properties are vital when modeling malicious out-of-network attackers. The absence of centralized authority does not imply unpredictability. Via minor assumptions on rationality, we have described several methods by which self-interested agents might converge upon a steady state in which no one has incentive to deviate.

Implicit in all of these processes is that if any agent in the system "goes rogue" and deviates from its prescribed update, there is no special clause needed in the algorithms; the other agents use their self-interest to adapt optimally.

4.1 Summary of Novel Results

This dissertation sits at the confluence of game theoretic treatments of the MU-MIMO system with Hermitian action sets and MIMO jammers with orthogonal tones. The novel results herein include showing that Nash equilibria exist in all utility games and some cost games. In those cost games where an equilibrium does not exist, we can shift strategy constraints into objectives via an exact penalty approach. The resulting game is then guaranteed to possess an equilibrium.

The relevance of a Nash equilibrium in the systems under consideration should not be underestimated. If the agents—including the jammer—measure performance and cost as we have modeled, then at an equilibrium, no one can do better by unilaterally changing their strategy. That is to say, for an equilibrium strategy $\hat{Q}$ for either the cost or utility game, we can guarantee *at least* $\log_2 e \sum_{k\in\mathcal{R}} u_k(\hat{Q})$ bps/Hz of capacity in the network with costs *no greater* than $\operatorname{Tr}\hat{Q}_k$ for each $k \in \mathcal{R}$.

We also demonstrate that under certain constraints on the network structure, the set of equilibria in the utility game can be further characterized. In the absence of user-user interference, there may be multiple equilibria; but if any equilibrium has a nonsingular user action, then it is the unique equilibrium of the game. When cross-talk is allowed in the network, the uniqueness result weakens. Similar results exist in the literature [2, 4], but we have shown to what extent the presence of a jammer disrupts previous findings.

In all games that meet the network requirements for uniqueness, simulations suggest that equilibrium can be reached in the limit via a best-response process. Additionally, the closed, convex strategy sets in the utility game admit gradient-play as a viable equilibrium-reaching method.

Finally, the presence of a jammer does not disrupt the stream-control meta-game from [2]. Users can negotiate stream reduction while under the influence of jamming interference as a practical method to boost total network performance.

4.2 Future Work

Extending this work is possible in a number of areas. Some involve modification of the game structure and others would result from changes to the information and communication structures.

The immediate next step is to establish conditions under which the various dynamic update processes are guaranteed to converge to an equilibrium. Convergence results in the literature assume that the underlying game possesses a unique equilibrium. This uniqueness is achieved by enforcing small interference gains. The question of convergence in the absence of absolute uniqueness remains open. It is unclear whether trajectories might oscillate between multiple equilibria or if perhaps the feasible regions could be carved up into joint basins of attraction for each equilibrium.

Fundamental to the analysis in this dissertation was the assumption that utility and cost games are separate—but related—entities. However, there is nothing preventing, in theory, multiple users wishing to maximize utility transmitting over the same channel in which other users wish to minimize their power. It would be interesting to investigate what duality relationship this hybrid game would have.

From the outset, we have assumed that the virtues of decentralized control outweigh the drawbacks. While recent work in similar networks without jammers suggest that little is lost by shifting power control to the users [5], there are still more questions to be answered. For example, how does the price of anarchy—a measure of the worst-case system performance at equilibrium relative to best-case centralized optimization [40]—change in the presence of jammers? It is possible that decentralized control is more robust to attacks than a centralized scheme. Quantifying this resiliency is a major area of future work.

Tightly coupled with a price of anarchy analysis is the consideration of how the performance of the system changes with the addition of users and jammers. Do the losses introduced by decentralized control decrease (in a per capita sense) as the number of users increases? On the other hand, it would be interesting to investigate whether a single, powerful jammer has a greater network effect than several weaker ones. Recent work on deriving closed-form expressions for the price of anarchy in games as a function of the number of players indicates this is a promising field of study [33].

In terms of the network model, we have made many implicit assumptions that have simplified the analysis. Foremost among these is the time-invariance of the channel matrices. A major component of wireless network analysis is accounting for fading channels and other randomness in the network. Extending the analysis in this dissertation to fading channels would increase its applicability.

We have also assumed that each agent k has knowledge of the interference gains $H_{k,j}$ from all other agents $j \in \mathcal{P}$. In an ad hoc network this may be unrealistic (especially in the presence of potentially stealthy jammers), but it may be possible to estimate the interference channels stochastically as in [28] or perhaps induce the agents to learn the channel structure through repeated interactions in the spirit of fictitious play [18] or regression [53].

Finally, the issue of synchronization seems to always be a sticking point in algorithmic game theory. The concerns are valid, especially considering that communication over physical channels may involve significant transmission delays. The best-response algorithms are relatively robust to asynchronicities. That is, agents guarantee themselves a better payoff every time they update.

APPENDIX A
MATRIX LEMMAS

Throughout the text, many of the proofs and derivations rely on ancillary properties of matrices. Including these lemmas in the body of the text would unnecessarily interrupt flow. They are collected here and referenced in the text when required.

The trace of the product of two Hermitian matrices is real, and thus can be written as the product of real vectorizations.

Proposition A.1 (from [1]) *Let A and B be Hermitian matrices of dimension n.*

$$\operatorname{Tr} AB = (\overrightarrow{A})^{\mathsf{T}} \overrightarrow{B}. \tag{A.1}$$

To explicitly derive the various gradients of (2.46) and (2.51), we require the following result on Kronecker arithmetic and the CRI$(\cdot)$ operator introduced in (2.55).

Lemma A.2 *Let X and Y be Hermitian matrices of dimension n, and let C and D be complex $n \times n$ matrices such that* $\operatorname{Tr} CXDY$ *is real. Then*

$$\operatorname{Tr} CXDY = \left(\overrightarrow{Y}\right)^{\mathsf{T}} \operatorname{CRI}\left(D^{\mathsf{T}} \otimes C\right) \overrightarrow{X}. \tag{A.2}$$

Proof Start with the popular four-way trace result for complex vectorization [26, Lemma 2.14]:

$$\operatorname{Tr} CXDY = (\operatorname{vec}(Y^{\mathsf{T}}))^{\mathsf{T}} M \operatorname{vec} X, \tag{A.3}$$

where $M = D^{\mathsf{T}} \otimes C$. Split all matrices into their real and imaginary parts:

$$\begin{aligned} \operatorname{Tr} CXDY &= (\operatorname{vec}(\operatorname{Re} Y^{\mathsf{T}} + \jmath \operatorname{Im} Y^{\mathsf{T}}))^{\mathsf{T}} (\operatorname{Re} M + \jmath \operatorname{Im} M) \operatorname{vec}(\operatorname{Re} X + \jmath \operatorname{Im} X) && \text{(A.4)} \\ &= \left[(\operatorname{vec} \operatorname{Re} Y)^{\mathsf{T}} - \jmath (\operatorname{vec} \operatorname{Im} Y)^{\mathsf{T}}\right] (\operatorname{Re} M + \jmath \operatorname{Im} M) \left[\operatorname{vec} \operatorname{Re} X + \jmath \operatorname{vec} \operatorname{Im} X\right], && \text{(A.5)} \end{aligned}$$

where in the last line we have utilized the Hermitian symmetry of Y. Distributing the multiplication yields

$$\begin{aligned} \operatorname{Tr} CXDY = {} & (\operatorname{vec} \operatorname{Re} Y)^{\mathsf{T}} \operatorname{Re} M \operatorname{vec} \operatorname{Re} X + (\operatorname{vec} \operatorname{Im} Y)^{\mathsf{T}} \operatorname{Im} M \operatorname{vec} \operatorname{Re} X \\ & - (\operatorname{vec} \operatorname{Re} Y)^{\mathsf{T}} \operatorname{Im} M \operatorname{vec} \operatorname{Im} X + (\operatorname{vec} \operatorname{Im} Y)^{\mathsf{T}} \operatorname{Re} M \operatorname{vec} \operatorname{Im} X + \jmath (*), \end{aligned} \tag{A.6}$$

where $(*)$ stands for a collection of terms that must evaluate to zero based on the assumed realness of the trace. The remaining terms can be factored out as

$$\operatorname{Tr} CXDY = \begin{bmatrix} \operatorname{vec} \operatorname{Re} Y \\ \operatorname{vec} \operatorname{Im} Y \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \operatorname{Re} M & -\operatorname{Im} M \\ \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \begin{bmatrix} \operatorname{vec} \operatorname{Re} X \\ \operatorname{vec} \operatorname{Im} X \end{bmatrix}, \tag{A.7}$$

which is what is claimed in (A.2). □

Corollary A.3 *Let X and Y be Hermitian matrices of dimension n, and let C, D, C', and D' be complex $n \times n$ matrices such that $\operatorname{Tr} CXDY + \operatorname{Tr} C'XD'Y$ is real. Then*

$$\operatorname{Tr} CXDY + \operatorname{Tr} C'XD'Y = \left(\overrightarrow{Y}\right)^{\mathsf{T}} \left[\operatorname{CRI}\left(D^{\mathsf{T}} \otimes C\right) + \operatorname{CRI}\left(D'^{\mathsf{T}} \otimes C'\right)\right] \overrightarrow{X}. \tag{A.8}$$

Lemma A.4 *For any matrices A, B, C, and D for which the products AC and BD are defined,*

$$\operatorname{CRI}\left(AC \otimes BD\right) = \operatorname{CRI}(A \otimes B)\operatorname{CRI}(C \otimes D). \tag{A.9}$$

Proof Apply the popular Kronecker factoring property [32, Section 2.2] to each block:

$$\operatorname{CRI}\left(AC \otimes BD\right) = \begin{bmatrix} \operatorname{Re}\left[(A \otimes B)(C \otimes D)\right] & -\operatorname{Im}\left[(A \otimes B)(C \otimes D)\right] \\ \operatorname{Im}\left[(A \otimes B)(C \otimes D)\right] & \operatorname{Re}\left[(A \otimes B)(C \otimes D)\right] \end{bmatrix}. \tag{A.10}$$

We can write the two relevant quantities

$$\operatorname{Re}\left[(A \otimes B)(C \otimes D)\right] = \operatorname{Re}(A \otimes B)\operatorname{Re}(C \otimes D) - \operatorname{Im}(A \otimes B)\operatorname{Im}(C \otimes D), \tag{A.11a}$$

$$\operatorname{Im}\left[(A \otimes B)(C \otimes D)\right] = \operatorname{Re}(A \otimes B)\operatorname{Im}(C \otimes D) + \operatorname{Im}(A \otimes B)\operatorname{Re}(C \otimes D), \tag{A.11b}$$

which is exactly what results from the matrix product $\operatorname{CRI}(A \otimes B)\operatorname{CRI}(C \otimes D)$. □

We will also rely heavily on the following result pertaining to the spectrum of a Kronecker product.

Proposition A.5 (Adapted from [30, Theorem 13.10]) *Let X and Y be arbitrary matrices with singular value decompositions $U_X \Sigma_X V_X^\dagger$ and $U_Y \Sigma_Y V_Y^\dagger$, respectively. Then*

$$(U_X \otimes U_Y)(\Sigma_X \otimes \Sigma_Y)(V_X \otimes V_Y)^\dagger \tag{A.12}$$

is a singular value decomposition of $X \otimes Y$, up to a reordering of diagonal elements of $\Sigma_X \otimes \Sigma_Y$.

In particular, Proposition A.5 dictates that all of the eigenvalues of the Kronecker product of two Hermitian matrices X and Y must lie within $[\underline{\lambda}(X)\,\underline{\lambda}(Y), \overline{\lambda}(X)\,\overline{\lambda}(Y)] \subset \mathbb{R}$. Additionally, when a matrix is expanded to its real and imaginary blocks via the $\operatorname{CRI}(\cdot)$ operation, its singular values are preserved.

Lemma A.6 *If X has the singular value decomposition $X = U\Sigma V^\dagger$, then its real and imaginary block form has the singular value decomposition*

$$\mathrm{CRI}(X) = \begin{bmatrix} \operatorname{Re} U & \operatorname{Im} U \\ \operatorname{Im} U & -\operatorname{Re} U \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} \begin{bmatrix} \operatorname{Re} V^\intercal & \operatorname{Im} V^\intercal \\ \operatorname{Im} V^\intercal & -\operatorname{Re} V^\intercal \end{bmatrix}. \tag{A.13}$$

Principally, the singular values of $\mathrm{CRI}(X)$ *are identical to those of* X.

PROOF From its singular value decomposition, the real part of X can be written

$$\operatorname{Re} X = \operatorname{Re}\left[(\operatorname{Re} U + \jmath \operatorname{Im} U)\,\Sigma\,(\operatorname{Re} V^\intercal - \jmath \operatorname{Im} V^\intercal)\right] \tag{A.14}$$

$$= \operatorname{Re}\left[(\operatorname{Re} U\Sigma + \jmath \operatorname{Re} U\Sigma)\,(\operatorname{Re} V^\intercal - \jmath \operatorname{Im} V^\intercal)\right] \tag{A.15}$$

$$= (\operatorname{Re} U)\,\Sigma\,(\operatorname{Re} V^\intercal) + (\operatorname{Im} U)\,\Sigma\,(\operatorname{Im} V^\intercal)\,. \tag{A.16}$$

Likewise its imaginary part is

$$\operatorname{Im} X = (\operatorname{Im} U)\,\Sigma\,(\operatorname{Re} V^\intercal) - (\operatorname{Re} U)\,\Sigma\,(\operatorname{Im} V^\intercal)\,. \tag{A.17}$$

Placing (A.16) and (A.17) into the approriate blocks of $\mathrm{CRI}\, X$ verifies that (A.13) is true. What remains is to show that (A.13) is a valid singular value decomposition. To that end, we demonstrate that if U is unitary then $\begin{bmatrix} \operatorname{Re} U & \operatorname{Im} U \\ \operatorname{Im} U & -\operatorname{Re} U \end{bmatrix}$ is also unitary (real orthogonal). Since U is unitary we have

$$I = UU^\dagger \tag{A.18}$$

$$= (\operatorname{Re} U + \jmath \operatorname{Im} U)\,(\operatorname{Re} U^\intercal - \jmath \operatorname{Im} U^\intercal) \tag{A.19}$$

$$= \operatorname{Re} U \operatorname{Re} U^\intercal + \operatorname{Im} U \operatorname{Im} U^\intercal + \jmath\,(-\operatorname{Re} U \operatorname{Im} U^\intercal + \operatorname{Im} U \operatorname{Re} U^\intercal)\,. \tag{A.20}$$

This must mean $\operatorname{Re} U \operatorname{Re} U^\intercal + \operatorname{Im} U \operatorname{Im} U^\intercal = I$ and $-\operatorname{Re} U \operatorname{Im} U^\intercal + \operatorname{Im} U \operatorname{Re} U^\intercal = 0$. We can now conclude

$$\begin{bmatrix} \operatorname{Re} U & \operatorname{Im} U \\ \operatorname{Im} U & -\operatorname{Re} U \end{bmatrix} \begin{bmatrix} \operatorname{Re} U & \operatorname{Im} U \\ \operatorname{Im} U & -\operatorname{Re} U \end{bmatrix}^\dagger = \begin{bmatrix} \operatorname{Re} U & \operatorname{Im} U \\ \operatorname{Im} U & -\operatorname{Re} U \end{bmatrix} \begin{bmatrix} \operatorname{Re} U^\intercal & \operatorname{Im} U^\intercal \\ \operatorname{Im} U^\intercal & -\operatorname{Re} U^\intercal \end{bmatrix} \tag{A.21}$$

$$= \begin{bmatrix} \operatorname{Re} U \operatorname{Re} U^\intercal + \operatorname{Im} U \operatorname{Im} U^\intercal & \operatorname{Re} U \operatorname{Im} U^\intercal - \operatorname{Im} U \operatorname{Re} U^\intercal \\ -\operatorname{Re} U \operatorname{Im} U^\intercal + \operatorname{Im} U \operatorname{Re} U^\intercal & \operatorname{Re} U \operatorname{Re} U^\intercal + \operatorname{Im} U \operatorname{Im} U^\intercal \end{bmatrix} \tag{A.22}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{A.23}$$

A similar procedure holds for V. □

Lemma A.7 *For any complex matrix, X,*

$$(\mathrm{CRI}\, X)^{\mathsf{T}} = \mathrm{CRI}(X^{\dagger}). \tag{A.24}$$

Proof By definition,

$$\mathrm{CRI}\left(X^{\dagger}\right) = \begin{bmatrix} \mathrm{Re}(X^{\dagger}) & -\mathrm{Im}(X^{\dagger}) \\ \mathrm{Im}(X^{\dagger}) & \mathrm{Re}(X^{\dagger}) \end{bmatrix} \tag{A.25}$$

$$= \begin{bmatrix} \mathrm{Re}(X^{\mathsf{T}}) & \mathrm{Im}(X^{\mathsf{T}}) \\ -\mathrm{Im}(X^{\mathsf{T}}) & \mathrm{Re}(X^{\mathsf{T}}) \end{bmatrix} \tag{A.26}$$

$$= (\mathrm{CRI}\, X)^{\mathsf{T}}. \tag{A.27}$$

□

The proof of Theorem 2.6 relies heavily on bounding maximum singular values of products of matrices. The following results will aid in those bounds.

Lemma A.8 *For all compatible matrices A and B, $\underline{\sigma}(AB) \geq \underline{\sigma}(A)\,\underline{\sigma}(B)$.*

Proof The Courant-Fischer Theorem gives us a variational characterization of the minimum singular value:

$$\underline{\sigma}(AB) = \inf_{\|x\|_2=1} \|ABx\|_2 \tag{A.28}$$

$$= \inf_{\|x\|_2=1} \|U_A\Sigma_A V_A^{\dagger} U_B \Sigma_B V_B^{\dagger} x\|_2 \tag{A.29}$$

where $A = U_A\Sigma_A V_A^{\dagger}$ and $B = U_B\Sigma_B V_B^{\dagger}$ are singular value decompositions. If V_B is unitary then $\|V_B^{\dagger}x\|_2 = 1$. For any compatible vector y we have $\|\Sigma_B y\|_2 \geq \|\,\underline{\sigma}(B)Iy\|_2 = \underline{\sigma}(B)\|y\|_2$. Therefore, $\|\Sigma_B V_B^{\dagger}x\| \geq \underline{\sigma}(B)$. V_A and U_B are unitary so left-multiplying by them does not change a vector's norm. A similar argument as above shows that for any compatible vector y we have $\|\Sigma_A y\|_2 \geq \underline{\sigma}(A)\|y\|_2$. Lastly, multiplying by U_A does not change the norm so we have for all unit-norm vectors x, $\|U_A\Sigma_A V_A^{\dagger} U_B\Sigma_B V_B^{\dagger}x\|_2 \geq \underline{\sigma}(A)\,\underline{\sigma}(B)$. The infimum over all such x must also obey this lower bound. □

Very similar reasoning can be applied to upper bound the maximum singular value of a product.

Corollary A.9 *For all compatible matrices A and B, $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\,\overline{\sigma}(B)$.*

On occasion, we will need to bound the maximum eigenvalue of the difference of two positive semidefinite matrices. With no other information, the following lemma provides a crude bound

Lemma A.10 *If $X, Y \in \mathbb{H}_+^n$ then $\left|\overline{\lambda}(X-Y)\right| \leq \max\{\overline{\lambda}(X), \overline{\lambda}(Y)\}$.*

Proof Write the eigendecomposition of $X = V\Lambda V^{-1}$. Then

$$\overline{\lambda}(X)I - X = \overline{\lambda}(X)V\Lambda V^{-1} - V\Lambda V^{-1} \tag{A.30}$$

$$= V(\overline{\lambda}(X)I - \Lambda)V^{-1}, \tag{A.31}$$

which implies $\overline{\lambda}(X)I - X \succeq 0$. A similar statment can be made for Y: $-\overline{\lambda}(Y)I \preceq -Y$. By supposition we have $X - Y \succeq Y$ and $X - Y \preceq X$. Therefore we can assemble the following in increasing Löwner orer: $-\overline{\lambda}(Y)I \preceq -Y \preceq X - Y \preceq X \preceq \overline{\lambda}(X)I$. This implies that the spectrum of $X - Y$ is entirely contained in the closed interval $[-\overline{\lambda}(Y), \overline{\lambda}(X)]$. □

APPENDIX B
SIMULATION SOFTWARE

The simulations that test and verify the results in this dissertation were written in the Matlab language and have been tested to run on Matlab versions from 7.1 to R2016a. This appendix contains the various custom scripts, classes, and functions written by the author. Where possible, built-in Matlab dependencies have been highlighted.

B.1 MIMOgame.m

This custom class represents a single instance of a MU-MIMO game as studied in this dissertation. It's non-native dependencies include

- `BRu.m` in Section B.2
- `BRw.m` in Section B.3
- `GradPlay.m` in Section B.4

```
classdef MIMOgame < matlab.mixin.Copyable
    % A class for MIMO Gaussian IC games.
    %
    % Constructor: G = MIMOGame(L, jam, Nt, Nr)
    % L     Number of regular users
    % jam   1 for jammer, 0 for non—jammer
    % Nt    Number of transmission antennae
    % Nr    Number of receiver antennae
    properties
        % L is the number of regular players.
        L; jam;
        Nt; Nr;
        cbar;
        ubar;
        w;
        H;
        hasSolution = false;
        Q0;
        Qstar;
        PlayHistory;
```

```
end % properties
methods
    % Constructor
    function G = MIMOgame(NumUsers, JammerYN, Nt, Nr, varargin)
        G.L = NumUsers;
        G.jam = JammerYN;
        G.Nt = Nt;
        G.Nr = Nr;
        if nargin == 5
            G.cbar = varargin{1};
        end
    end % MIMOgame  constructor
    % Channel Generator
    function generateH(G, varargin)
        % Randomly generate channel matrices (populates MIMOgame.H).
        %
        % First optional argument: Overwrite with impunity [Boolean]
        % [Default = false]
        % Second optional argument: Zero crosstalk [Boolean] [Default =
        % false]
        overwrite_with_impunity = false;
        zero_crosstalk_bool = false;
        if nargin >= 2
            overwrite_with_impunity = varargin{1};
            if nargin == 3
                zero_crosstalk_bool = varargin{2};
            else
                error('MIMOgame:generateH', 'Invalid number of input arguments.'
                    );
            end
        end
        overwrite_approval = true;
        if (~overwrite_with_impunity && ~isempty(G.H))
            overwrite_approval = input('The game already has a channel matrix
               defined.  Are you sure you want to overwrite it?  (This warning
               can be skipped by setting the second argument to "1".)','s');
        end
```

```
    if (~overwrite_approval || strcmpi(overwrite_approval, 'no'))
        disp('MIMOgame:generateH operation cancelled.');
        return
    else
        % OK let's do this.
        for ii = 1:G.L+G.jam
            for jj = 1:G.L+G.jam
                if zero_crosstalk_bool && ii~=jj && jj ~= G.L+G.jam && ii ~=
                     G.L+G.jam
                    G.H{ii}{jj} = zeros(G.Nr,G.Nt);
                else
                    G.H{ii}{jj} = randn(G.Nr,G.Nt) + 1j*randn(G.Nr,G.Nt);
                end
            end
        end
    end
end % generateH
% Action generator
function generateQ(G)
    % Randomly generate initial actions (populates MIMOgame.Q0).  Actions
       will be
    % normalized to satisfy power limits in G.cbar, if defined.
    for m = 1:G.L+G.jam
        G.Q0{m} = (randn(G.Nt,G.Nt) + 1j * randn(G.Nt,G.Nt)) / G.Nt;
        G.Q0{m} = G.Q0{m} * G.Q0{m}';
        % If no power limits are set, do not normalize.
        if ~isempty(G.cbar) && trace(G.Q0{m}) > G.cbar(m)
            G.Q0{m} = G.cbar(m)*G.Q0{m}/trace(G.Q0{m});
        end
    end
end % generateQ
% Utility
% Clear properties
function clearProp(G, varargin)
    % Clear the properties listed as arguments.  A single property
    % can be the sole argument (as a string).  Otherwise, multiple
    % properties can be cleared by listing their names as strings,
```

```
    % then placing those strings in a cell array.
    if isa(varargin{1},'char')
        % clear the property labled in varargin{1}
        G.clearMe(varargin{1});
    elseif isa(varargin{1},'cell')
        % iterate through cell and delete the properties
        el = length(varargin{1});
        for c = 1:el
            G.clearMe(varargin{1}{c});
        end
    else
        return;
    end
end % clearProp
%% Solver
function BRu(G,varargin)
    if ~isempty(varargin)
        [cap, Qstar, conv, cap_hist] = BRu(G.L,G.cbar,G.H,G.Q0,varargin{1});
    else
        [cap, Qstar, conv, cap_hist] = BRu(G.L,G.cbar,G.H,G.Q0);
    end
    G.hasSolution = conv;
    G.Qstar = Qstar;
    G.PlayHistory = cap_hist;
end % BRu
function BRw(G,varargin)
    if ~isempty(varargin)
        [cap, Qstar, conv, cap_hist] = BRw(G.L,G.ubar,G.w,G.H,G.Q0,varargin
            {1});
    else
        [cap, Qstar, conv, cap_hist] = BRu(G.L,G.ubar,G.w,G.H,G.Q0);
    end
    G.hasSolution = conv;
    G.Qstar = Qstar;
    G.PlayHistory = cap_hist;
end % BRw
function GP(G,varargin)
```

```
    %tol = 1e-5;
    %alphafun = @(t) 1/t;
    if ~isempty(varargin)
        st = varargin{1};
        %fn = fieldnames(st);
        %optionalinputs = structvars(length(fn),st);
        %eval(optionalinputs);
        [cap, Qstar, conv, cap_hist] = GradPlay(G.L,G.cbar,G.H,G.Q0,st);
    else
        [cap, Qstar, conv, cap_hist] = GradPlay(G.L,G.cbar,G.H,G.Q0);
    end
    G.hasSolution = conv;
    G.Qstar = Qstar;
    G.PlayHistory = cap_hist;
end % GP
%% Helpers
function clearMe(G, st)
    try
        if isa(eval(['G.' st]),'cell')
            eval(['G.' st '={};']);
        else
            eval(['G.' st '=[];']);
        end
    catch
        warning('MIMOgame:clearProp',['Warning: No such property "' st '"
            found']);
        return;
    end
end % clearMe
function Q = ActionGen(G,k)
    % Generate a random action for player k \in [0,L]
    % Generate a random PSD matrix.
    X = rand(G.Nt);
    Q = X*X';
    % If it violates the power constraint, normalize it down.
    switch k
        case 0
```

```
            if trace(Q) > G.P0
                Q = G.P0/trace(Q) * Q;
            end
        otherwise
            if trace(Q) > G.P(k)
                Q = G.P(k)/trace(Q) * Q;
            end
    end
end % ActionGen method
function d = ActionDelta(G)
    try
        lastact = G.PlayHistory(:,end);
        prioract = G.PlayHistory(:,end-1);
        d = 0;
        for k = 1:length(lastact)
            d = d + norm(lastact{k} - prioract{k});
        end
    catch ME
        id = ME.identifier;
        lastSeg = regexp(id,'\:','split');
        if strcmp(lastSeg{end}, 'UndefinedFunction')
            d = realmax;
        elseif strcmp(lastSeg{end}, 'badsubscript')
            d = realmax;
        else
            rethrow(ME);
        end
    end
end % ActionDelta method
function bool = ActionCheck(G,Q,k)
    % Check validity of action Q for player k
    bool = true;
    E = eig(Q);
    bool = bool && all(real(E) == E);
    bool = bool && all(E >= 0);
    if k == 0
        bool = bool && (trace(Q) <= G.P0);
```

```
            else
                bool = bool && (trace(Q) <= G.P(k));
            end
        end % ActionCheck method
    end % methods
end % classdef
```

B.2 BRu.m

This custom function implements the best-response dynamics in the utility game described in Algorithm 1. Its non-native dependencies include

- `WF.m` in Section B.5
- `cal_capacity.m` in Section B.6
- `jammerBR.m` in Section B.7
- `row.m` in Section B.8

```
% [cap, Qstar, conv, cap_hist] = BRu(L,cbar,H,Q,<struct>)
%
% INPUTS
% L     — Number of users.
% cbar  — Vector of cost constraints.  cbar(L+1) is the jammer's
% constraint, if present.
% H     — Cell array of cell arrays of channel matrices.
% Q     — Initial joint action.
%
% OPTIONAL INPUT
% struct
%   error_tol   — Tolerance for judging convergence of actions.  Default =
%   0.01.
%   error_window— Number of iterations algorithm must be within tolerance
%   in order for convergence to be concluded.  Default = 10.
%   plot_flag   — Boolean regarding whether to output trajectory plot.
%   Default = false.
%   randomize_update — Boolean indicating whether within each iteration the
%   order of player updates should be random (true) or in ascending numeric
```

```
%   order (false).  Default = false.
%
% OUTPUTS
% cap       — Vector of utilities at the end of the process.
% Qstar     — Cell array of strategies at the end of the process.
% conv      — Boolean regarding whether the process converged or not.
% hist      — Structure
%   cap     — Array with utilities for users at each time step.
%   cond    — Array with condition numbers on the actions of each agent at
%   each time step.
%   action  — Cell array with actions at each time step.
%
function [cap, Qstar, conv, hist] = BRu(L,cbar,H,Q,varargin)
%% Flag if jammer is present.
if length(Q) ~= L
    jam = 1;
else
    jam = 0;
end
%% Algorithm parameters.
dT = 1000;   % Time limit
T0 = 1;
Tf = T0+dT;
ew = 10;    % Default error window
eth = 0.01; % Default error tolerance
player_order = 1:L+jam; % Default player update order.
if ~isempty(varargin)
    if isfield(varargin{1},'error_window')
        ew = varargin{1}.error_window;
    end
    if isfield(varargin{1},'error_tol')
        eth = varargin{1}.error_tol;
    elseif isfield(varargin{1},'error_tolerance')
        eth = varargin{1}.error_tolerance;
    end
    if isfield(varargin{1},'randomize_update')
        player_order = randperm(L+jam);
```

```
    end
end
%% Initialize the history.
hist = struct('cap',{},'cond',{},'action',{});
hist(1).cap = cal_capacity(L,H,Q);
hist(1).cond = cellfun(@cond, Q);
hist(1).action = Q;
%% Run BR process.
for e = 0.1:0.2:1
    for t = T0:Tf
        inert = 1 * (1 - 1 / (t ^ e));
        for m = player_order
            if m == L+jam
                Q{L+jam} = inert * Q{L+jam} + (1-inert) * jammerBR(L,cbar(L+1),H,Q);
            else
                Q{m} = inert * Q{m} + (1-inert) * WF(m,L,cbar,H,Q);
            end
        end
        hist(end+1).cap = cal_capacity(L,H,Q);
        hist(end).cond = cellfun(@cond,Q);
        hist(end).action = Q;
        if t > ew && norm(max([hist(:,t-ew+1:t).cap],[],2) ...
                - min([hist(:,t-ew+1:t).cap],[],2)) < eth
            conv = 1;
            Qstar = Q;
            cap = hist(end).cap;
            if ~isempty(varargin) && isfield(varargin{1},'plot_flag')
                figure; plot([hist.cap]');
                xlabel('Iteration'); ylabel('Capacity');
                title({'Mutual information trajectories for users. ';...
                    ['Condition of final action matrices: ',num2str(row(round(hist(
                        end).cond)))]...
                    });
            end
            return;
        end
    end
```

```
T0 = Tf+1;
Tf = T0+dT;
end
%% Fallback for processes that exceed time limit T.
disp('nonconvergent iteWF')
conv = 0;
end % —end BR—
```

B.3 BRw.m

This custom function implements the best-response dynamics in the weighted cost game described in Algorithm 2. Its non-native dependencies include

- `CVX`, a package for specifying and solving convex programs [24]
- `cal_capacity.m` in Section B.6
- `cal_power.m` in Section B.9
- `calc_R.m` in Section B.10
- `jammerBR.m` in Section B.7
- `row.m` in Section B.8

```
% [util, Qstar, conv, obj_hist] = BRwPlus(L,ubar,w,H,Q,<param.struct>)
% Parameter Field  ———  Possible values; Description
% plot                  true, {false}
%                       Plot trajectories?
% verbose               true, {false}
%                       Print after every iteration?
% time                  <positive integer> {1000}
%                       Iteration limit per alpha exponent
% exponent              <vector of positive fractions> {[.5]}
%                       After each segment of iterations defined by TIME expires,
    the
%                       exponent on the inertia term is incremented to the next one
%                       in EXPONENT.
% window                <positive integer> {5}
%                       Error window
```

```
% tol                   <positive float> {0.01}
%                       Error tolerance for overall convergence.
function [util, Qstar, conv, obj_hist] = BRw(L,ubar,w,H,Q,varargin)
if ~isempty(varargin)
    params = varargin{1};
end
%% Flag if jammer is present.
if length(Q) ~= L
    jam = 1;
else
    jam = 0;
end
%% Game parameters
[Nr, Nt] = size(H{1}{1});
%% Algorithm parameters.
exponent = 0.5;
dT = 1000;   % Time limit
ew = 5;    % Error window
eth = 0.01; % Error tolerance
if exist('params','var')
    if isfield(params,'time')
        dT = params.time;
    end
    if isfield(params,'exponent')
        exponent = params.exponent;
    end
    if isfield(params,'window')
        ew = params.window;
    end
    if isfield(params,'tol')
        eth = params.tol;
    end
end
T0 = 1;
Tf = T0+dT;
%% Initialize the objective history.
obj_hist = cal_power(L,Q);
```

```
util_hist = cal_capacity(L,H,Q);
objective = zeros(L,1);
%% Run BR process
for e = exponent
    for t = T0 : Tf
        inert = 1*(1-1/(t^e));
        for m = 1:L
            R = calc_R(L,H,Q,m);
            Ri = inv(R{m});
            cvx_begin sdp quiet
            variable X(Nt,Nt) hermitian
            dual variable Y
            minimize( trace(X) + w(m)*pos(ubar(m) - log2(exp(1))*log_det(eye(Nt) + X
                 * H{m}{m}' * Ri * H{m}{m}) ) )
            Y: X>=0
            cvx_end
            if isempty(strfind(cvx_status,'Solved'))
                conv = 0;
                Qstar = Q;
                util = util_hist(:,end);
                if exist('params','var') && isfield(params,'plot') && params.plot
                    visual_output(conv,L,ubar,w,obj_hist,util_hist,t)
                end
                disp(['When trying to find the best response of player ',num2str(m),
                    ' to '])
                celldisp(Q);
                disp('CVX resolved the following:');
                disp(cvx_status);
                return;
            end
            Q{m} = inert * Q{m} + (1-inert) * X;
        end
        if jam
            Q{m+1} = inert * Q{m+1} + (1-inert) * jammerBR(L,-ubar(L+1),H,Q);
        end
        objective = cal_power(L,Q);
        obj_hist = [obj_hist objective];
```

```
        util_hist = [util_hist, cal_capacity(L,H,Q)];
        if t > ew && norm(max(obj_hist(:,t-ew+1:t),[],2) - min(obj_hist(:,t-ew+1:t)
            ,[],2)) < eth
            conv = 1;
            Qstar = Q;
            util = util_hist(:,end);
            if exist('params','var') && isfield(params,'plot') && params.plot
                visual_output(conv,L,ubar,w,obj_hist,util_hist,t);
            end
            return;
        elseif exist('params','var') && isfield(params,'verbose') && params.verbose
            disp(['Iteration: ', num2str(t)]);
            disp(['Time is ', datestr(clock,16)]);
        end
    end
    T0 = Tf+1;
    Tf = T0+dT;
end
%% Non-convergence
conv = 0;
Qstar = Q;
util = util_hist(:,end);
if exist('params','var') && isfield(params,'plot') && params.plot
    visual_output(conv,L,ubar,w,obj_hist,util_hist,t)
end
%% Nested functions

end % -end BRc-

%% Subfunctions
function visual_output(flag,L,ubar,w,obj_hist,util_hist,finalT)
    figure;
    subplot(2,1,1);
    plot(obj_hist');
    title({'Power Trajectories for the Weighted Cost Game'; ['$\bar{u} = [$',num2str
       (row(ubar)),'$]$, $w = [$',num2str(row(w)),'$]$']});
    xlabel('Iteration'); ylabel('Transmission Power');
```

```
    axis([1, finalT, 0, 1.1*max(max(obj_hist))]);
    subplot(2,1,2);
    plot(util_hist');
    hold on;
    line(repmat([1;finalT],1,L),repmat(row(ubar(1:L)),2,1),'LineStyle','--');
    title('Mutual Information Trajectories');
    xlabel('Iteration'); ylabel('Link Mutual Information');
    axis([1, finalT, 0, 1.1*max([max(max(util_hist)), max(ubar)])]);
    if flag
        disp('The best-response dynamics converged');
    else
        disp('The best-response dynamics did NOT converge.');
    end
end
```

B.4 GradPlay.m

This cusom function implements the gradient-play dynamics described in Algorithm 3. Its non-native dependencies include

- `cal_capacity.m` in Section B.6
- `calc_R.m` in Section B.10
- `row.m` in Section B.8

```
% [cap, Qstar, conv, hist] = GradPlay(L,cbar,H,Q,<struct>)
%
% INPUTS
% L     - Number of users.
% cbar  - Vector of cost constraints.  cbar(L+1) is the jammer's
% constraint, if present.
% H     - Cell array of cell arrays of channel matrices.
% Q     - Initial joint action.
%
% OPTIONAL INPUT
% struct
%   plot    - Boolean flag whether to plot trajectories.  Default = false.
%
```

```
% OUTPUTS
% cap       — Vector of utilities at the end of the process.
% Qstar     — Cell array of strategies at the end of the process.
% conv      — Boolean regarding whether the process converged or not.
% hist      — Structure
%   cap     — Array with utilities for users at each time step.
%   cond    — Array with condition numbers on the actions of each agent at
%   each time step.
%   action  — Cell array with actions at each time step.
%
function [cap, Qstar, conv, hist] = GradPlay(L,cbar,H,Q,varargin)
if ~isempty(varargin)
    params = varargin{1};
end
%% Flag if jammer is present.
if length(Q) ~= L
    jam = 1;
else
    jam = 0;
end
%% Algorithm parameters
gammabar = 0.5;
dT = 1000;   % Time limit
T0 = 1;
Tf = T0+dT;
ew = 10;    % Error window
eth = 0.001; % Error tolerance
%% Initialize the history.
hist = struct('cap',{},'cond',{},'action',{});
hist(1).cap = cal_capacity(L,H,Q);
hist(1).cond = cellfun(@cond, Q);
hist(1).action = Q;
%% Run Gradient Play process
% Larger exponents (e) produce smaller inertias (larger willingness to optimize) at
% any given time.
Nt = size(Q{1},1);
for e = .01:.02:.5
```

```
    for t = T0:Tf
        gamma = gammabar;
        inert = 1 * (1 - 1 / (t ^ e));
        R = calc_R(L,H,Q);
        for m = 1:L
            grad{m} = H{m}{m}'*((R{m}+H{m}{m}*Q{m}*H{m}{m}')\eye(Nt))*H{m}{m}/log(2)
                ;
            gradnorm(m,t) = norm(grad{m});
            Q{m} = inert*Q{m} + (1-inert)*proj(Q{m}+gamma*grad{m},m);
        end
        if jam
            m=L+1;
            grad{m} = pseudogradientJammer();
            gradnorm(m,t) = norm(grad{m});
            Q{m} = inert*Q{m} + (1-inert)*proj(Q{m}+gamma*grad{m},m);
        end
        cap = cal_capacity(L,H,Q);
        hist(end+1).cap = cal_capacity(L,H,Q);
        hist(end).cond = cellfun(@cond,Q);
        hist(end).action = Q;
        if t > ew && ...
                norm(max([hist(t-ew+1:t).cap],[],2) - min([hist(t-ew+1:t).cap],[],2)
                    ) < eth && ...
                all(max(gradnorm(:,t-ew+1:t),[],2)-min(gradnorm(:,t-ew+1:t),[],2) <
                    eth)
            conv = 1;
            Qstar = Q;
            if exist('params','var') && isfield(params,'plot') && params.plot
                visual_output(conv,cbar,[hist.cap],t);
            end
            return;
        end
    end
    T0 = Tf+1;
    Tf = T0+dT;
end
%% Fallback for processes that exceed time limit T.
```

```
disp('nonconvergent GradPlay')
conv = 0;
Qstar = 0;
if exist('params','var') && isfield(params,'plot') && params.plot
    visual_output(conv,cbar,[hist.cap],t);
end
%% Nested functions
    function P = proj(F,k)
        % Project F onto strategy set of user k.
        [V,D] = eig(F);
        D = real(D);
        if trace(D)<=cbar(k)
            P = V*D*V'; return;
        end
        [lam,ind] = sort(diag(D),'descend');
        mu = (cbar(k)-transpose(cumsum(lam)))./(1:1:Nt);
        M = repmat(lam,1,Nt)+repmat(mu,Nt,1);
        M = M.*(M>0);
        f = sum(M);
        [mi,f_ind] = min(abs(f-cbar(k)));
        if mi > 1e4*eps
            warning('GradPlay:proj:musearch','The search for the best eigenvalue
                offset is out of tolerance.  Be careful.');
        end
        mustar = mu(f_ind);
        rho = lam+mustar*ones(size(lam));
        rho = rho.*(rho>0);
        P = V(:,ind)*diag(rho)*V(:,ind)';
    end % end proj
    function dJ = pseudogradientJammer()
        z = L+1;
        Nt = size(Q{z},1);
        dJ = 0;
        for k = 1:L
            Inter = 0;
            for j = setdiff(1:L,k)
                Inter = Inter + H{k}{j}*Q{j}*H{k}{j}';
```

```
            end
            First = eye(Nt) + H{k}{z}*Q{z}*H{k}{z}' + Inter + H{k}{k}*Q{k}*H{k}{k}';
            Second = eye(Nt) + H{k}{z}*Q{z}*H{k}{z}' + Inter;
            dJ = dJ+ H{k}{z}'*(First\eye(Nt) - Second\eye(Nt))*H{k}{z};
        end
        dJ = dJ/-log(2);
    end % end pseudogradientJammer
end

%% Subfunctions
function visual_output(flag,cbar,obj_hist,finalT)
figure;
plot(obj_hist');
title({'Gradient Play Rate Trajectories for the Utility Game'; ['$\bar{c} = [$',
    num2str(row(cbar)),'$]$']});
xlabel('Iteration'); ylabel('Link Mutual Information');
axis([1, finalT, 0, 1.1*max(max(obj_hist))]);
if flag
    disp('The gradient-play dynamics converged');
else
    disp('The gradient-play dynamics did NOT converge.');
end
end % end visual_output
```

B.5 WF.m

This custom function implements the water-filling optimization procedure for each user given the actions of the other agents. It is adapted from the procedure mentioned in [2].

```
function Qbr = WF(player,L,cbar,H,Q)
m = player;
jam = length(Q)-L;
R = eye(size(H{m}{m},1));
for k = 1:L+jam
    R = R + (k ~= m) * H{m}{k} * Q{k} * H{m}{k}';
end
[Ru,Rs,Rv] = svd(R);
[U,S,V] = svd(Ru * diag(diag(Rs).^-0.5) * Rv' * H{m}{m});
```

```
lambda = nonzeros(diag(S)) .^ (-2);
ns = length(lambda);
if ns == 1
    alpha = cbar(m);
elseif ns > 1
    % Find the first index such that when mu is set to that lambda, the sum total
    % water exceeds the limit.
    idx = find(cbar(m) < [lambda(2:ns) .* (1:ns-1)' ...
        - tril(ones(ns - 1,ns - 1)) * lambda(1:ns - 1); inf],1);
    % Overestimate the area: power + sum of the heights.
    % Then divide by the number of columns to get an overestimate of their proper
       water depths.
    % Perform max-with-zero operation to knock off any creepers.
    % (Don't actually ever calculate the water depth, mu!)
    alpha = max((cbar(m) + sum(lambda(1:idx))) / idx - lambda,0);
end
alpha = [alpha; zeros(size(H{m}{m},2) - length(alpha),1)];
Qbr = V * diag(alpha) * V';
```

B.6 cal_capacity.m

This subfunction evaluates the capacity (utility) of all users in the game.

```
% cal_capacity
%
% K. Clay McKell
% UH Manoa Electrical Engineering
% Wireless Power Simulation
%
% Calculate the mutual information for links in an interference channel.
%
% [ cap ] = cal_capacity( L, H, Q, [player_ids] )
%
% INPUTS
% L     The number of players in the game.
% H     2-d cell array of the channel gain matrices.
% Q     1-d cell array of player actions (Gaussian covariance matrices).
%
```

```
% OPTIONAL INPUTS
% player_ids    Array containing indices of the players for which you wish to
% calculate the capacity.  Default = 1:L.
%
% OUTPUTS
% cap   Array containing the mutual information of each link (in bits/s/Hz).  If any
% player was excluded (by specifying player_ids), then its capacity is reported as
% zero (0).
%
% VERSION HISTORY
%
function cap = cal_capacity(L,H,Q,varargin)
%% Flag if jammer is present.
if length(Q) ~= L
    jam = 1;
else
    jam = 0;
end
%% Check if we are only reporting for a single player.
p = 1:L;
if nargin == 4
    p = varargin{1};
end
%% Carry on.
R = cell(L,1);
cap = zeros(L,1);
for m = 1:L
    R{m} = eye(size(H{m}{m},1));
    for k = 1:L+jam
        R{m} = R{m} + (k ~= m) * H{m}{k} * Q{k} * H{m}{k}';
    end
end
for m = p
    cap(m) = real(log2(det(H{m}{m} * Q{m} * H{m}{m}' + R{m})) - log2(det(R{m})));
end
```

B.7 jammerBR.m

This subfunction evaluates the jammer's best response to a set of user actions. In general, a closed-form expression for the solution to this problem does not exist. The built-in Matlab function `fmincon` is used to numerically approximate the optimal solution. The non-native dependencies of `jammerBR` include

- `devec.m` in Section B.11
- `revec.m` in Section B.12

```
function Q0br = jammerBR(L,pow_limit,H,Q)
% Jammer's Best Response is a nonlinear optimization.
Nt = size(Q{L+1},1);
opts = optimset('Algorithm', 'interior-point', ...
    'MaxFunEvals', max([3000, Nt^Nt*1000]), ...
    'Diagnostics', 'off', ...
    'Display', 'off');
xbr = fmincon(@(x)jammerUtil(x), revec(Q{L+1}), ...
    [ones(1,Nt) zeros(1,Nt*(Nt-1))], pow_limit, ...
    [], [], ...
    [], [], ...
    @(x)nlconstraint(x), ...
    opts);
Q0br = devec(xbr);

    function U = jammerUtil(x)
        % Nested Function: Calculate jammer's utility for vectorized action x.
        R = cell(L,1);
        U = 0;
        for kk = 1:L
            R{kk} = eye(size(H{kk}{kk},1));
            for jj = 1:L+1
                R{kk} = R{kk} + (jj~=kk) * H{kk}{jj} * Q{jj} * H{kk}{jj}';
            end
            R{kk} = R{kk} + H{kk}{L+1} * devec(x) * H{kk}{L+1}';
            u = real(log2(det(H{kk}{kk} * Q{kk} * H{kk}{kk}' + R{kk})) - log2(det(R{
                kk})));
            U = U + u;
```

```
        end
    end % —end jammerUtil—

    function [c ceq] = nlconstraint(x)
        % Nested Function: Calculate the eigenvalues of matrix—ed action vector x
        % for use in nonlinear constraint parameter.
        Q0 = devec(x);
        c = —1*eig(Q0);
        ceq = [];
    end % —end nlconstraint—
end % —end jammerBR—
```

B.8 row.m

This simple function ensures its argument is a row vector.

```
function rowv = row(v)
%
% K. Clay McKell
% 17 December 2009 — Initial release.
% 14 November 2009 v2.0 — Use transpose instead of conjugate transpose.
%
% ROW
% rowv = row(v)
% Ensure a vector is a row vector.
%
% INPUT:
% v     Vector (1—dimensional array) that will be output as a row vector.
%
% OUTPUT:
% rowv  Vector v in row format.
%
s = size(v);
if min(s) ~= 1
    error('Input is not a vector');
elseif s(1) ~= 1
    rowv = transpose(v);
else
```

```
    rowv=v;
end
```

B.9 cal_power.m

This subfunction evaluates the power used (cost) of all users in the game.

```
function cost = cal_power(L,Q)
cost = zeros(L,1);
for m = 1:L
    cost(m) = trace(Q{m});
end
```

B.10 calc_R.m

This subfunction evaluates the noise-plus-interference matrix $R_k(Q)$.

```
function R = calc_R(L,H,Q,varargin)
%% Flag if jammer is present
if length(Q) ~= L
    jam = 1;
else
    jam = 0;
end
%% Check which players we are calculating for
p=1:L;
if nargin == 4
    p = varargin{1};
end
R = cell(length(p),1);
for m = p
    R{m} = eye(size(H{m}{m},1));
    for k = 1:L+jam
        R{m} = R{m} + (k ~= m) * H{m}{k} * Q{k} * H{m}{k}';
    end
end
```

B.11 devec.m

This helper function converts the independent entries of a complex Hermitian matrix into a real column vector. Its only non-native dependency is `row` in Section B.8.

```
function M = devec(v)
    % v is partitioned into Diagonal, Upper Triangular Real, and Upper Triangular
        Imaginary parts:
    % v = [vdiag; vreals; vimags].
    % vdiag contains the L real diagonal components.
    % vreals and vimags contains the L(L-1)/2 STRICTLY off-diagonal components of
        the
    % upper triangular part of Q:
    % vreals = Re([Q_12; ...; Q_1L; Q_23; ...; Q_2L; ...; Q_(L-1,L)]).
    n = length(v);
    L = sqrt(n);
    vdiag = v(1:L);
    vreal = v(L+1:L+L*(L-1)/2);
    UTreal = detri(vreal);
    vimag = v(1+L+L*(L-1)/2:end);
    UTimag = detri(vimag);
    M = diag(vdiag) + UTreal + 1j*UTimag + ctranspose(UTreal + 1j*UTimag);
end

function UT = detri(v)
    % Turn v into a strictly upper triangular matrix.
    m = length(v);
    L = (1+sqrt(1+8*m))/2;
    UT = zeros(L);
    for ii = 1:L-1
        UT(ii,ii+1:L) = row(v(1:L-ii));
        v = v(L+1-ii:end);
    end
end
```

B.12 revec.m

This helper function converts the independent entries of a complex Hermitian matrix from a real column vector to a square complex matrix.

```
function v = revec(X)
L = size(X,1);
d = diag(X);
im = imag(X);
re = real(X);
n = L*(L-1)/2;
vim = zeros(n,1);
vre = vim;
count = 0;
for r = 1:L-1
    for c = r+1: L
        count = count+1;
        vim(count) = im(r,c);
        vre(count) = re(r,c);
    end
end
v = [d; vre; vim];
```

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