Feigin, M. and Vrabec, M. (2019) Intertwining operator for AG2 Calogero-Moser-Sutherland system. *Journal of Mathematical Physics*, 60(7), 073503.
(doi: 10.1063/1.5090274)

There may be differences between this version and the published version. You are advised to consult the publisher's version if you wish to cite from it.

http://eprints.gla.ac.uk/188844/

Deposited on 21 June 2019

Enlighten – Research publications by members of the University of Glasgow
http://eprints.gla.ac.uk

Intertwining operator for AG_2 Calogero–Moser–Sutherland system

Misha Feigin, Martin Vrabec

School of Mathematics and Statistics, University of Glasgow, UK *

Abstract

We consider generalised Calogero–Moser–Sutherland quantum Hamiltonian H associated with a configuration of vectors AG_2 on the plane which is a union of A_2 and G_2 root systems. The Hamiltonian H depends on one parameter. We find an intertwining operator between H and the Calogero–Moser–Sutherland Hamiltonian for the root system G_2. This gives a quantum integral for H of order 6 in an explicit form thus establishing integrability of H.

1 Introduction

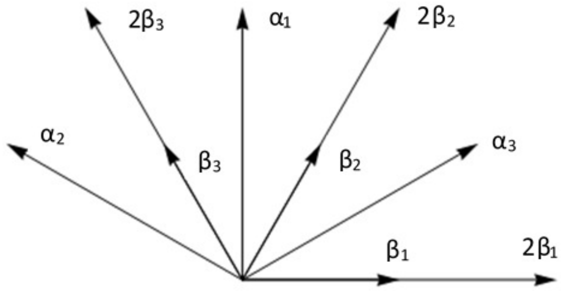
The study of Calogero–Moser–Sutherland (CMS) integrable systems goes back to the works [1] – [3]. Olshanetsky and Perelomov introduced generalised CMS systems related to root systems of Weyl groups [4], which includes the non-reduced root system BC_n. The corresponding Hamiltonians are closely related to radial parts of Laplace-Beltrami operators on symmetric spaces [5], [6]. In the case of root system G_2 the rational version of the corresponding CMS system was considered earlier by Wolfes [7]. A uniform proof of integrability for all root sysyems via trigonometric version of Dunkl operators was given by Heckman in [8]. Another more involved proof was provided earlier by Opdam in [9]. In the case of integer values of coupling parameters these CMS systems admit additional quantum integrals and they are algebraically integrable as it was established by Chalykh, Styrkas and Veselov in [10] (see also [11]).

It was found by Chalykh, Veselov and one of the authors in [12], [13] that there are integrable generalisations of CMS type quantum systems which correspond to special configurations of vectors generalising root systems. Examples of such configurations include deformations of the root systems A_n and C_n. These examples are related to symmetric superspaces, [14]–[17], and to special representations of Cherednik algebras [18]. The corresponding configurations of vectors have to satisfy so-called locus conditions [19]. It is expected that there are very few such configurations, but they are not classified yet. We refer to [20] for a survey of results on locus configurations and integrability of rational, trigonometric and elliptic generalised CMS systems (see also [21] and reference therein for the elliptic case).

The work [22] of Fairley and one of the authors deals with a class of trigonometric locus configurations on the plane. In the process of classification of such configurations a new locus configuration AG_2 was found in [22] (see also [23] where this configuration appears as well in different but related context of WDVV equations). This configuration of vectors with multiplicities depends on one integer parmeter m. Being a locus configuration, it follows from the general results of Chalykh [20] that the corresponding generalised CMS operator H has an intertwining

*Emails: Misha.Feigin@glasgow.ac.uk, martinvrabec222@gmail.com

Figure 1: A positive half of AG_2 system.



relation with the Laplacian. This implies integrability and, moreover, algebraic integrability of the Hamiltonian H with $m \in \mathbb{Z}$. The latter means existence of some additional quantum integrals; see [11], [10] for more details on algebraic integrability including precise definition and examples of intertwining operators.

Let us describe the configuration of vectors AG_2 in detail, following [22]. This is a non-reduced configuration consisting of the union of root systems G_2 and A_2. A positive half of this configuration is shown on Figure 1.

The short vectors from the root system G_2 are denoted as $\pm\beta_1$, $\pm\beta_2$, $\pm\beta_3$, and the long vectors from the root system G_2 are denoted as $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$. Let $\langle\cdot,\cdot\rangle$ be the standard Euclidean inner product on the plane. Then the ratio $\frac{\langle\alpha_i,\alpha_i\rangle}{\langle\beta_i,\beta_i\rangle} = 3$ for any i. The vectors from the additional root system A_2 are $\pm 2\beta_1$, $\pm 2\beta_2$, $\pm 2\beta_3$. Configuration AG_2 consists of 18 vectors $\pm\beta_i, \pm 2\beta_i, \pm\alpha_i, i = 1, 2, 3$. Note that the numbering of α's is chosen in such a way that $\langle\alpha_i, \beta_i\rangle = 0$ for all $i = 1, 2, 3$.

Note that

$$2\beta_2 - \alpha_1 = \alpha_1 - 2\beta_3 = \beta_1,\ 2\beta_3 - \alpha_2 = \alpha_2 + 2\beta_1 = \beta_2,\ 2\beta_2 - \alpha_3 = \alpha_3 - 2\beta_1 = \beta_3,$$

and

$$\alpha_3 - \beta_2 = \beta_3 - \alpha_2 = \beta_2 - \beta_3 = \beta_1, \text{ and } \alpha_1 - \beta_2 = \beta_3\,.$$

The configuration AG_2 is invariant under the G_2 Weyl group action and it belongs to a two-dimensional lattice spanned by β_1 and α_2. However, it is not a root system. In order to see this let us consider, for instance, vectors β_1, β_2 which are symmetric about the line orthogonal to vector $2\beta_3$. We have $\beta_2 - \beta_1 = k \cdot 2\beta_3$, where $k = \frac{1}{2}$ is not an integer so the crystallographic condition fails.

Let H_0 be the CMS Hamiltonian for the root system G_2 with multiplicities for the long and short roots m and $3m$, respectively. And let H be the Hamiltonian of the generalised CMS system under study, associated to the above collection of vectors AG_2 where α_i, β_i and $2\beta_i$ have multiplicities m, $3m$ and 1, respectively, where we use conventions for non-reduced systems coming from theory of symmetric spaces (see e.g. [16]). More precisely,

$$H_0 = -\Delta + \sum_{i=1}^{3} \big(v_i(x) + u_i(x)\big)\,,$$

$$H = -\Delta + \sum_{i=1}^{3} \big(v_i(x) + \widetilde{u}_i(x)\big)\,, \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian, and for $x = (x_1, x_2) \in \mathbb{C}^2$ we have

$$v_i(x) = \frac{m(m+1)\langle \alpha_i, \alpha_i \rangle}{\sinh^2 \langle \alpha_i, x \rangle},$$
$$u_i(x) = \frac{3m(3m+1)\langle \beta_i, \beta_i \rangle}{\sinh^2 \langle \beta_i, x \rangle}\,, \tag{1.2}$$

and

$$\begin{aligned}\widetilde{u}_i(x) &= \frac{9m(m+1)\langle \beta_i, \beta_i \rangle}{\sinh^2 \langle \beta_i, x \rangle} + \frac{2\langle 2\beta_i, 2\beta_i \rangle}{\sinh^2 \langle 2\beta_i, x \rangle} \\ &= \frac{(3m+1)(3m+2)\langle \beta_i, \beta_i \rangle}{\sinh^2 \langle \beta_i, x \rangle} - \frac{2\langle \beta_i, \beta_i \rangle}{\cosh^2 \langle \beta_i, x \rangle}.\end{aligned}$$

In addition, we introduce the following notation for the difference $\widetilde{u}_i(x) - u_i(x)$:

$$\widehat{u}_i(x) := \widetilde{u}_i(x) - u_i(x) = \frac{2(3m+1)\langle \beta_i, \beta_i \rangle}{\sinh^2 \langle \beta_i, x \rangle} - \frac{2\langle \beta_i, \beta_i \rangle}{\cosh^2 \langle \beta_i, x \rangle}. \tag{1.3}$$

Let ∂_i denote the partial derivative $\frac{\partial}{\partial x_i}$. For any vector (or a vector field) $\gamma = \big(\gamma^{(1)}, \gamma^{(2)}\big) \in \mathbb{C}^2$, we will write ∂_γ for the directional derivative operator $\gamma^{(1)}\partial_1 + \gamma^{(2)}\partial_2$. In particular, if ϕ is a scalar field on the plane and $\nabla(\phi) = \big(\partial_1(\phi), \partial_2(\phi)\big)$ is its gradient, then by $\partial_{\nabla(\phi)}$ we will mean $\partial_1(\phi)\partial_1 + \partial_2(\phi)\partial_2$.

In this paper we establish an intertwining relation between the Hamiltonian H and the integrable Hamiltonian H_0 of the CMS system associated with the root system G_2. This relation is valid for any value of the parameter m which is allowed to be non-integer. This leads to integrability of H for any m thus generalizing integrability for integer m known from [22], [20]. We also find the intertwining operator $\mathcal{D}$ of order 3 in an explicit form. This, in turn, gives quantum integral of H of order 6. We note that direct application of results of [20] in the case of integer m leads to a higher order intertwiner and a higher order integral of H. The degree 6 for the integral of H is expected to be minimal possible. Indeed, it follows from [24] that for generic m an independent integral for the rational version of H with constant highest term has to be of degree at least 6 since such highest term should be G_2-invariant.

The intertwining operator $\mathcal{D}$ has the form

$$\mathcal{D} = \partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3} + \sum_{\sigma \in A_3} f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}} + \sum_{i=1}^{3} g_i \partial_{\beta_i} + h\,, \tag{1.4}$$

where $A_3 = \{id,\ (1,2,3),\ (1,3,2)\}$ is the alternating group on 3 elements, and f_i, g_i $(i = 1,2,3)$ and h are some functions which we specify explicitly. We will use the notation $\sum_\sigma$ throughout the paper as a shorthand for the cyclic sum $\sum_{\sigma \in A_3}$. To be more precise, we will prove the following main theorem.

Theorem 1. *There exists a third-order differential operator $\mathcal{D}$ of the form* (1.4) *such that*

$$H\mathcal{D} = \mathcal{D}H_0. \tag{1.5}$$

We obtain quantum integrability of H and a quantum integral of motion as a direct corollary by making use of a general statement from [25]. Let us recall the notion of the formal adjoint differential operator D^* for a differential operator D. It can be defined by the relations $\partial_i^* = -\partial_i$, $f^* = f$ for any function f, and $(AB)^* = B^*A^*$ for any differential operators A, B.

Theorem 2. *Let $\mathcal{D}$ satisfy* (1.5) *and let $\mathcal{D}^*$ be the formal adjoint of $\mathcal{D}$. Let I be any differential operator such that the commutator $[I, H_0] = 0$. Then $\mathcal{D}I\mathcal{D}^*$ commutes with H. In particular, $[\mathcal{D}\mathcal{D}^*, H] = 0$.*

Indeed, taking the formal adjoint of the relation (1.5) gives $\mathcal{D}^*H = H_0\mathcal{D}^*$. Hence

$$H\mathcal{D}I\mathcal{D}^* = \mathcal{D}H_0I\mathcal{D}^* = \mathcal{D}IH_0\mathcal{D}^* = \mathcal{D}I\mathcal{D}^*H.$$

Note that for integer m the operator H_0 is algebraically integrable as the commutative ring of quantum integrals is larger than the ring of G_2–invariants [11], [10]. Therefore this gives a way to see algebraic integrability of the operator H for integer m (see also [20]). We also note that in the rational limit the operator $\mathcal{D}\mathcal{D}^*$ reduces to a quantum integral for the rational CMS system associated with the root system G_2 with multiplicities m and $3m+1$ for the long and short roots, respectively.

The structure of the paper is as follows. We collect some preliminary trigonometric identities associated with vectors from the configuration AG_2 in Section 2. We introduce all the coefficients of the intertwining operator (1.4) in Section 3, where we also establish some preliminary results on these coefficients. In Section 4 we prove the main Theorem 1 on the intertwining relation. We present results on the rational limit in Section 5. We outline some future directions in Section 6.

2 Preliminary trigonometric identities

In this section we collect some trigonometric identities involving vectors from the configuration AG_2. We will use these identities later in the proof of the intertwining relation (1.5) in Section 4.

One can choose a coordinate system where the vectors take the form $\beta_1 = \omega\big(\sqrt{2}, 0\big)$, $\beta_2 = \omega\big(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\big)$, $\beta_3 = \omega\big(-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\big)$, $\alpha_1 = \omega\big(0, \sqrt{6}\big)$, $\alpha_2 = \omega\big(-\frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\big)$ and $\alpha_3 = \omega\big(\frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\big)$ for some non-zero $\omega \in \mathbb{C}$. We will use inner products between the vectors but not the specific coordinates of the vectors.

Remark 1. *In most cases we state only one particular form of each identity, but other variants can be obtained by rotating or scaling the vectors. More precisely, the relevant transformations will be the replacement of β_i with $2\beta_i$, and the two rotations by $\frac{\pi}{3}$, clockwise and anti-clockwise. These rotations can alternatively be defined by the following replacement rules: $\beta_1 \to -\beta_3$, $\beta_2 \to \beta_1$, $\beta_3 \to \beta_2$, $\alpha_1 \to \alpha_3$, $\alpha_2 \to \alpha_1$, $\alpha_3 \to -\alpha_2$, and $\beta_1 \to \beta_2$, $\beta_2 \to \beta_3$, $\beta_3 \to -\beta_1$, $\alpha_1 \to \alpha_2$, $\alpha_2 \to -\alpha_3$, $\alpha_3 \to \alpha_1$, respectively.*

The vectors α_i and β_i in the configuration AG_2 satisfy the following trigonometric identities, where we omit the argument $x = \big(x_1, x_2\big)$. Thus we write $\coth\beta_i$ for $\coth\langle\beta_i, x\rangle$, etc.

Lemma 2.1. *We have*

$$\sum_{1\le j<k\le 3} \langle\beta_j, \beta_k\rangle \coth\beta_j \coth\beta_k = \omega^2 \, . \tag{2.1}$$

Proof. By a difference of cotangents formula and the fact that $\beta_2 - \beta_3 = \beta_1$, two terms in the sum become

$$\omega^2 \coth\beta_1\big(\coth\beta_2 - \coth\beta_3\big) = -\frac{\omega^2\cosh\beta_1}{\sinh\beta_2\sinh\beta_3} \, . \tag{2.2}$$

By expanding $\cosh\beta_1$ in terms of β_2 and β_3, we can rearrange the right-hand side of (2.2) further as

$$-\frac{\omega^2\cosh\beta_2\cosh\beta_3 - \omega^2\sinh\beta_2\sinh\beta_3}{\sinh\beta_2\sinh\beta_3} = -\omega^2\coth\beta_2\coth\beta_3 + \omega^2 \, ,$$

as required. □

It will be convenient to use the following notation throughout the paper:

$$X = \omega^2(\sinh\beta_1 \sinh\beta_2 \sinh\beta_3)^{-1}\,, \tag{2.3}$$

$$Y = \omega^2(\sinh 2\beta_1 \sinh 2\beta_2 \sinh 2\beta_3)^{-1}\,. \tag{2.4}$$

Here are some identities involving these functions.

Lemma 2.2. *We have*

$$\sum_\sigma \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)}\rangle \coth\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(2)} \sinh^2\beta_{\sigma(3)}} = -2X\,. \tag{2.5}$$

Proof. Multiplying equality (2.1) by $2\omega^{-2}$ and regrouping terms as in the proof of Lemma 2.1, we get

$$\begin{aligned} 2 &= \coth\beta_1\big(\coth\beta_2 - \coth\beta_3\big) + \coth\beta_2\big(\coth\beta_1 + \coth\beta_3\big) + \coth\beta_3\big(\coth\beta_2 - \coth\beta_1\big) \\ &= -\frac{\cosh\beta_1}{\sinh\beta_2\sinh\beta_3} + \frac{\cosh\beta_2}{\sinh\beta_1\sinh\beta_3} - \frac{\cosh\beta_3}{\sinh\beta_1\sinh\beta_2}\,. \end{aligned} \tag{2.6}$$

The statement follows by dividing (2.6) by $\sinh\beta_1 \sinh\beta_2 \sinh\beta_3$. ☐

There is also the following version of Lemma 2.2 involving the function tanh rather than coth.

Lemma 2.3. *We have*

$$-\frac{1}{2}\sum_\sigma \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)}\rangle \tanh\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(2)} \sinh^2\beta_{\sigma(3)}} = X + 4Y\,. \tag{2.7}$$

Proof. Note the relation $\tanh z = \coth z - (\sinh z \cosh z)^{-1}$ valid for all $z \in \mathbb{C}$. Hence by Lemma 2.2 we get

$$\begin{aligned} \sum_\sigma \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)}\rangle \tanh\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(2)} \sinh^2\beta_{\sigma(3)}} &= \sum_\sigma \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)}\rangle \coth\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(2)} \sinh^2\beta_{\sigma(3)}} - \sum_\sigma \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)}\rangle}{\sinh\beta_{\sigma(1)}\cosh\beta_{\sigma(1)}\sinh^2\beta_{\sigma(2)} \sinh^2\beta_{\sigma(3)}} \\ &= -2X - \frac{\sum_{1\le j<k\le 3}\langle\beta_j,\beta_k\rangle \coth\beta_j\coth\beta_k}{\sinh\beta_1\cosh\beta_1\sinh\beta_2\cosh\beta_2\sinh\beta_3\cosh\beta_3}\,. \end{aligned}$$

The result follows by applying Lemma 2.1. ☐

Lemma 2.4. *We have*

$$-X\sum_{i=1}^{3}\coth\beta_i\partial_{\beta_i} = \sum_\sigma \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)}\rangle}{\sinh^2\beta_{\sigma(2)} \sinh^2\beta_{\sigma(3)}}\partial_{\beta_{\sigma(1)}}\,. \tag{2.8}$$

Proof. Let us replace $\partial_{\beta_1} = \partial_{\beta_2} - \partial_{\beta_3}$ in (2.8). Then the coefficient of ∂_{β_2} in the left-hand side equals

$$-X\big(\coth\beta_1 + \coth\beta_2\big) = -\frac{\omega^2\sinh(\beta_1+\beta_2)}{\sinh^2\beta_1\sinh^2\beta_2\sinh\beta_3}\,.$$

Then on the right-hand side of equality (2.8) the coefficient at ∂_{β_2} is

$$\frac{\omega^2}{\sinh^2\beta_2\sinh^2\beta_3} - \frac{\omega^2}{\sinh^2\beta_1\sinh^2\beta_3} = -\frac{\omega^2(\sinh^2\beta_2 - \sinh^2\beta_1)}{\sinh^2\beta_1\sinh^2\beta_2\sinh^2\beta_3} = -\frac{\omega^2\sinh(\beta_1+\beta_2)}{\sinh^2\beta_1\sinh^2\beta_2\sinh\beta_3}$$

as $\sinh^2\beta_2 - \sinh^2\beta_1 = \sinh(\beta_1+\beta_2)\sinh(\beta_2-\beta_1)$. Similarly, the coefficient at ∂_{β_3} matches too. ☐

As an immediate corollary of Lemma 2.4 we get the following statement.

Corollary 2.5. *We have*

$$\partial_{\nabla(X)} = \sum_\sigma \frac{\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle}{\sinh^2\beta_{\sigma(2)}\sinh^2\beta_{\sigma(3)}}\partial_{\beta_{\sigma(1)}}\,,$$

and

$$\partial_{\nabla(Y)} = \sum_\sigma \frac{2\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle}{\sinh^2 2\beta_{\sigma(2)}\sinh^2 2\beta_{\sigma(3)}}\partial_{\beta_{\sigma(1)}}\,.$$

□

The following lemma can be proven by a straightforward calculation with the help of Lemma 2.1.

Lemma 2.6. *The following two equalities hold:*

$$\Delta(X) = 4\omega^2\left(2+\sum_{j=1}^3\frac{1}{\sinh^2\beta_j}\right)X\,, \tag{2.9}$$

$$\Delta(Y) = 16\omega^2\left(2+\sum_{j=1}^3\frac{1}{\sinh^2 2\beta_j}\right)Y\,. \tag{2.10}$$

Lemma 2.7. *The following identities hold:*

$$-\left(\frac{1}{\cosh^2\beta_2}+\frac{1}{\cosh^2\beta_3}\right)\frac{1}{\sinh^2\alpha_1}+2(\tanh\beta_2+\tanh\beta_3)\frac{\coth\alpha_1}{\sinh^2\alpha_1} = \frac{1}{\cosh^2\beta_2\cosh^2\beta_3}\,, \tag{2.11}$$

$$\left(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\right)\frac{1}{\sinh^2\alpha_1}+2(\coth\beta_2+\coth\beta_3)\frac{\coth\alpha_1}{\sinh^2\alpha_1} = \frac{1}{\sinh^2\beta_2\sinh^2\beta_3}\,, \tag{2.12}$$

$$-\left(\frac{1}{\cosh^2\beta_2}+\frac{1}{\cosh^2\beta_3}\right)\frac{1}{\sinh^2\beta_1}+2(\tanh\beta_2-\tanh\beta_3)\frac{\coth\beta_1}{\sinh^2\beta_1} = \frac{1}{\cosh^2\beta_2\cosh^2\beta_3}\,, \tag{2.13}$$

$$\left(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\right)\frac{1}{\sinh^2\beta_1}+2(\coth\beta_2-\coth\beta_3)\frac{\coth\beta_1}{\sinh^2\beta_1} = \frac{1}{\sinh^2\beta_2\sinh^2\beta_3}\,. \tag{2.14}$$

Proof. Since $\beta_2+\beta_3=\alpha_1$, we have

$$\tanh\beta_2+\tanh\beta_3 = \tanh\alpha_1(1+\tanh\beta_2\tanh\beta_3)\,.$$

Therefore the left-hand side of the relation (2.11) multiplied by $\sinh^2\alpha_1$ takes the form

$$-\frac{1}{\cosh^2\beta_2}-\frac{1}{\cosh^2\beta_3}+2(1+\tanh\beta_2\tanh\beta_3) = (\tanh\beta_2+\tanh\beta_3)^2 = \frac{\sinh^2(\beta_2+\beta_3)}{\cosh^2\beta_2\cosh^2\beta_3}\,,$$

which implies the relation (2.11). The equalities (2.12) – (2.14) can be proved by following a similar sequence of steps, using in addition that $\coth x+\coth y=\tanh(x+y)(1+\coth x\coth y)$ for $x,y\in\mathbb{C}$. □

Several other identities can be derived from Lemma 2.7, which we put in Lemmas 2.8 and 2.9 below.

Lemma 2.8. *The following relation is satisfied:*

$$\left(\frac{\coth\beta_3}{\sinh^2\beta_3}-\frac{\coth\beta_2}{\sinh^2\beta_2}\right)\frac{1}{\sinh^2\alpha_1}+\left(\frac{1}{\sinh^2\beta_3}-\frac{1}{\sinh^2\beta_2}\right)\frac{\coth\alpha_1}{\sinh^2\alpha_1}=\frac{\coth\beta_3-\coth\beta_2}{\sinh^2\beta_2\sinh^2\beta_3}\,. \quad (2.15)$$

Proof. By multiplying the relation (2.12) by $\coth\beta_3-\coth\beta_2$, and then using that $\coth^2\beta_3-\coth^2\beta_2=\sinh^{-2}\beta_3-\sinh^{-2}\beta_2$, we get

$$\left(\frac{\coth\beta_3-\coth\beta_2}{\sinh^2\beta_2}+\frac{\coth\beta_3-\coth\beta_2}{\sinh^2\beta_3}\right)\frac{1}{\sinh^2\alpha_1}+2\left(\frac{1}{\sinh^2\beta_3}-\frac{1}{\sinh^2\beta_2}\right)\frac{\coth\alpha_1}{\sinh^2\alpha_1}=\frac{\coth\beta_3-\coth\beta_2}{\sinh^2\beta_2\sinh^2\beta_3}\,. \quad (2.16)$$

Comparing relations (2.16) and (2.15), it remains to prove that

$$\left(\frac{\coth\beta_3}{\sinh^2\beta_2}-\frac{\coth\beta_2}{\sinh^2\beta_3}\right)\frac{1}{\sinh^2\alpha_1}+\left(\frac{1}{\sinh^2\beta_3}-\frac{1}{\sinh^2\beta_2}\right)\frac{\coth\alpha_1}{\sinh^2\alpha_1}=0\,. \quad (2.17)$$

Note that since $\alpha_1=\beta_2+\beta_3$ we get

$$\frac{\coth\alpha_1-\coth\beta_2}{\sinh^2\beta_3}-\frac{\coth\alpha_1-\coth\beta_3}{\sinh^2\beta_2}=-\frac{\sinh\beta_3}{\sinh^2\beta_3\sinh\alpha_1\sinh\beta_2}+\frac{\sinh\beta_2}{\sinh\beta_3\sinh\alpha_1\sinh^2\beta_2}=0\,,$$

which implies that relation (2.17) holds as required. □

Lemma 2.9. *The following identity holds:*

$$\sum_\sigma\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\left(\frac{1}{\sinh^2\beta_{\sigma(2)}}+\frac{1}{\sinh^2\beta_{\sigma(3)}}\right)\frac{\coth\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(1)}}=2\left(2+\sum_{j=1}^3\frac{1}{\sinh^2\beta_j}\right)X\,. \quad (2.18)$$

Proof. Let us multiply the identity (2.14) in Lemma 2.7 by $\coth\beta_1$. It follows that

$$\left(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\right)\frac{\coth\beta_1}{\sinh^2\beta_1}=\frac{\coth\beta_1}{\sinh^2\beta_2\sinh^2\beta_3}-2(\coth\beta_2-\coth\beta_3)\frac{\coth^2\beta_1}{\sinh^2\beta_1}$$
$$=\frac{\coth\beta_1}{\sinh^2\beta_2\sinh^2\beta_3}+\frac{2\coth^2\beta_1}{\sinh\beta_1\sinh\beta_2\sinh\beta_3}=\frac{\coth\beta_1}{\sinh^2\beta_2\sinh^2\beta_3}+2\omega^{-2}\left(1+\frac{1}{\sinh^2\beta_1}\right)X\,. \quad (2.19)$$

We obtain two more variants of the relation (2.19) by applying $\pm\frac{\pi}{3}$ rotations and interchanging the β's accordingly (see Remark 1). By adding together the resulting three equalities, we get, with use of Lemma 2.2, that the left-hand side of the identity (2.18) equals

$$\sum_\sigma\frac{\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\coth\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(2)}\sinh^2\beta_{\sigma(3)}}+2\left(3+\sum_{j=1}^3\frac{1}{\sinh^2\beta_j}\right)X=4X+\left(\sum_{j=1}^3\frac{2}{\sinh^2\beta_j}\right)X\,.$$

□

Lemma 2.10. *The following identity holds:*

$$\sum_\sigma\langle\alpha_{\sigma(2)},\alpha_{\sigma(3)}\rangle\left(\frac{1}{\sinh^2\alpha_{\sigma(2)}}-\frac{1}{\sinh^2\alpha_{\sigma(3)}}\right)\frac{\coth\alpha_{\sigma(1)}}{\sinh^2\alpha_{\sigma(1)}}=0\,. \quad (2.20)$$

Proof. Let us multiply both sides of (2.20) by $-\frac{1}{3}\sinh^2\alpha_1\sinh^2\alpha_2\sinh^2\alpha_3$. We need to prove that

$(\sinh^2\alpha_3 - \sinh^2\alpha_2)\coth\alpha_1 - (\sinh^2\alpha_1 - \sinh^2\alpha_3)\coth\alpha_2 - (\sinh^2\alpha_2 - \sinh^2\alpha_1)\coth\alpha_3 = 0.$ We have

$$(\sinh^2\alpha_3 - \sinh^2\alpha_2)\coth\alpha_1 = \sinh(\alpha_3-\alpha_2)\cosh(\alpha_2+\alpha_3) = \frac{1}{2}\sinh(2\alpha_3) - \frac{1}{2}\sinh(2\alpha_2)\,, \quad (2.21)$$

since $\alpha_1 = \alpha_2+\alpha_3$. By applying the rotations by $\pm\frac{\pi}{3}$ (see Remark 1), we obtain from (2.21) expressions for $(\sinh^2\alpha_1 - \sinh^2\alpha_3)\coth\alpha_2$ and $(\sinh^2\alpha_2 - \sinh^2\alpha_1)\coth\alpha_3$, which imply the statement. □

3 The intertwining operator

In this section we define the intertwining operator $\mathcal{D}$ given by formula (1.4), that is we define the corresponding functions f_i, g_i and h. We also find the gradient and Laplacian of these functions in a few lemmas in this section. This information is then used in Section 4 to prove the intertwining relation (1.5).

We start with the following general lemma.

Lemma 3.1. *For any single-variable function F and vectors α, β, γ such that $\langle\gamma,\gamma\rangle \neq 0$ we have*

$$\partial_\alpha\partial_\beta\big(F(\langle\gamma,x\rangle)\big) = \frac{\langle\gamma,\alpha\rangle\langle\gamma,\beta\rangle}{\langle\gamma,\gamma\rangle}\Delta\big(F(\langle\gamma,x\rangle)\big)\,.$$

Proof. By the chain rule of differentiation, $\Delta(F(\langle\gamma,x\rangle)) = \langle\gamma,\gamma\rangle F''(\langle\gamma,x\rangle)$, while $\partial_\alpha\partial_\beta(F(\langle\gamma,x\rangle)) = \langle\gamma,\alpha\rangle\langle\gamma,\beta\rangle F''(\langle\gamma,x\rangle)$, where F'' denotes the second derivative of the function F. □

In the expression (1.4) for the operator $\mathcal{D}$, let

$$f_j = -(3m+1)\langle\beta_j,\beta_j\rangle\coth\beta_j - \langle\beta_j,\beta_j\rangle\tanh\beta_j, \quad (3.1)$$

where $j = 1,2,3$.

In the next lemma we calculate the gradient and Laplacian of the functions f_j.

Lemma 3.2. *The functions f_j defined by expression* (3.1), *$j = 1,2,3$, satisfy the following relations:*

(1) $\nabla(f_j) = \frac{1}{2}\widehat{u}_j\beta_j$, *(or, equivalently,* $\partial_{\nabla(f_j)} = \frac{1}{2}\widehat{u}_j\partial_{\beta_j}$*),*

(2) $-\Delta(f_j) + \widehat{u}_j f_j = \partial_{\beta_j}(u_j)$.

Proof. Part (1) follows from the equality

$$\partial_i(f_j) = \left(\frac{(3m+1)\langle\beta_j,\beta_j\rangle}{\sinh^2\beta_j} - \frac{\langle\beta_j,\beta_j\rangle}{\cosh^2\beta_j}\right)\beta_j^{(i)} = \frac{1}{2}\widehat{u}_j\beta_j^{(i)}\,,$$

$i = 1,2$, where $\beta_j = (\beta_j^{(1)},\beta_j^{(2)})$ and we used the definition (1.3).

To establish property (2) we note that

$$\Delta(f_j) = -\frac{2(3m+1)\langle\beta_j,\beta_j\rangle^2\coth\beta_j}{\sinh^2\beta_j} + \frac{2\langle\beta_j,\beta_j\rangle^2\tanh\beta_j}{\cosh^2\beta_j}\,.$$

Expanding and simplifying the product $\widehat{u}_j f_j$ yields

$$\widehat{u}_j f_j = -\frac{2(3m+1)^2\langle\beta_j,\beta_j\rangle^2\coth\beta_j}{\sinh^2\beta_j} + \frac{2\langle\beta_j,\beta_j\rangle^2\tanh\beta_j}{\cosh^2\beta_j}\,.$$

Therefore,

$$-\Delta(f_j) + \widehat{u}_j f_j = -\frac{6m(3m+1)\langle\beta_j,\beta_j\rangle^2 \coth\beta_j}{\sinh^2\beta_j} = \partial_{\beta_j}(u_j)\,,$$

by relation (1.2), as required. □

For each $j = 1, 2, 3$, let g_j in the operator (1.4) be defined by

$$g_j = g_j^{(\mathrm{I})} + g_j^{(\mathrm{II})} + g_j^{(\mathrm{III})}\,, \tag{3.2}$$

where

$$g_j^{(\mathrm{I})} = \prod_{k\neq j} f_k\,, \tag{3.3}$$

$$g_j^{(\mathrm{II})} = -\frac{\prod_{k\neq j}\langle\alpha_j,\beta_k\rangle}{\langle\alpha_j,\alpha_j\rangle} v_j\,, \tag{3.4}$$

$$g_j^{(\mathrm{III})} = -\frac{\prod_{k\neq j}\langle\beta_j,\beta_k\rangle}{\langle\beta_j,\beta_j\rangle} u_j\,, \tag{3.5}$$

or, more explicitly,

$$g_1 = f_2 f_3 - \frac{9m(m+1)\omega^4}{\sinh^2\alpha_1} + \frac{3m(3m+1)\omega^4}{\sinh^2\beta_1}\,,$$

$$g_2 = f_1 f_3 + \frac{9m(m+1)\omega^4}{\sinh^2\alpha_2} - \frac{3m(3m+1)\omega^4}{\sinh^2\beta_2}\,,$$

$$g_3 = f_1 f_2 - \frac{9m(m+1)\omega^4}{\sinh^2\alpha_3} + \frac{3m(3m+1)\omega^4}{\sinh^2\beta_3}\,.$$

In the next lemma we find gradients of the functions $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$.

Lemma 3.3. *The functions $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$ defined by formulas* (3.4) *and* (3.5) *satisfy the following relations for all $\sigma \in A_3$:*

(1) $2\partial_{\nabla(g^{(\mathrm{II})}_{\sigma(1)})} = -\partial_{\beta_{\sigma(2)}}\big(v_{\sigma(1)}\big)\partial_{\beta_{\sigma(3)}} - \partial_{\beta_{\sigma(3)}}\big(v_{\sigma(1)}\big)\partial_{\beta_{\sigma(2)}}\,,$

(2) $2\partial_{\nabla(g^{(\mathrm{III})}_{\sigma(1)})} = -\partial_{\beta_{\sigma(2)}}\big(u_{\sigma(1)}\big)\partial_{\beta_{\sigma(3)}} - \partial_{\beta_{\sigma(3)}}\big(u_{\sigma(1)}\big)\partial_{\beta_{\sigma(2)}}\,.$

Proof. We give proof for $\sigma = id$, the other cases are analogous. In the right-hand side of part (1) we have

$$-\partial_{\beta_2}(v_1)\partial_{\beta_3} - \partial_{\beta_3}(v_1)\partial_{\beta_2} = -\langle\alpha_1,\beta_2\rangle v_1'\partial_{\beta_3} - \langle\alpha_1,\beta_3\rangle v_1'\partial_{\beta_2} = -3v_1'\omega^2(\partial_{\beta_3}+\partial_{\beta_2}) = -3v_1'\omega^2\partial_{\alpha_1}\,,$$

where $v_1'(x) = \frac{dV}{dz}|_{z=\langle\alpha_1,x\rangle}$, $V(z) = m(m+1)\langle\alpha_1,\alpha_1\rangle\sinh^{-2} z$. And in the left-hand side of relation (1) we get

$$2\partial_{\nabla(g_1^{(\mathrm{II})})} = -2\frac{\langle\alpha_1,\beta_2\rangle\langle\alpha_1,\beta_3\rangle}{\langle\alpha_1,\alpha_1\rangle}\partial_{\nabla(v_1)} = -2\frac{\langle\alpha_1,\beta_2\rangle\langle\alpha_1,\beta_3\rangle}{\langle\alpha_1,\alpha_1\rangle}v_1'\partial_{\alpha_1} = -3\omega^2 v_1'\partial_{\alpha_1}\,,$$

so the two sides are equal. The proof of part (2) is similar. □

It will be useful to combine gradients of functions $g_j^{(\mathrm{I})}$, $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$ as in the following lemma.

Lemma 3.4. *Functions $g_j^{(\mathrm{I})}$, $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$ defined by formulas* (3.3) – (3.5) *satisfy the following relations:*

(1) $2\sum_{i=1}^{3} \partial_{\nabla(g_i^{(\mathrm{I})})}\partial_{\beta_i} = \sum_{\sigma}\Big(\sum_{j\neq\sigma(1)} \widehat{u}_j\Big) f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\,,$

(2)
$$2\sum_{i=1}^{3} \partial_{\nabla(g_i^{(\mathrm{II})})}\partial_{\beta_i} = -\sum_{\sigma}\Big(\partial_{\beta_{\sigma(2)}}\big(v_{\sigma(1)}\big)\partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}}\big(v_{\sigma(1)}\big)\partial_{\beta_{\sigma(2)}}\Big)\partial_{\beta_{\sigma(1)}}$$
$$= -\sum_{\sigma}\partial_{\beta_{\sigma(1)}}\Big(\sum_{j\neq\sigma(1)} v_j\Big)\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}} = -\sum_{\sigma}\partial_{\beta_{\sigma(1)}}\Big(\sum_{j=1}^{3} v_j\Big)\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\,,$$

(3)
$$2\sum_{i=1}^{3} \partial_{\nabla(g_i^{(\mathrm{III})})}\partial_{\beta_i} = -\sum_{\sigma}\Big(\partial_{\beta_{\sigma(2)}}\big(u_{\sigma(1)}\big)\partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}}\big(u_{\sigma(1)}\big)\partial_{\beta_{\sigma(2)}}\Big)\partial_{\beta_{\sigma(1)}}$$
$$= -\sum_{\sigma}\partial_{\beta_{\sigma(1)}}\Big(\sum_{j\neq\sigma(1)} u_j\Big)\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\,.$$

Proof. In order to prove part (1), we note that by the definition (3.3) we have

$$\begin{aligned} 2\sum_{i=1}^{3} \partial_{\nabla(g_i^{(\mathrm{I})})}\partial_{\beta_i} &= 2\sum_{\sigma}\partial_{\nabla(f_{\sigma(2)}f_{\sigma(3)})}\partial_{\beta_{\sigma(1)}} \\ &= 2\sum_{\sigma} f_{\sigma(3)}\partial_{\nabla(f_{\sigma(2)})}\partial_{\beta_{\sigma(1)}} + 2\sum_{\sigma} f_{\sigma(2)}\partial_{\nabla(f_{\sigma(3)})}\partial_{\beta_{\sigma(1)}}\,. \end{aligned} \tag{3.6}$$

Substituting the result of Lemma 3.2 part (1) for $\partial_{\nabla(f_j)}$ into the expression (3.6) we obtain

$$\sum_{\sigma} f_{\sigma(3)}\widehat{u}_{\sigma(2)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(1)}} + \sum_{\sigma} f_{\sigma(2)}\widehat{u}_{\sigma(3)}\partial_{\beta_{\sigma(3)}}\partial_{\beta_{\sigma(1)}}\,,$$

which equals the right-hand side of (1).

The equalities (2) and (3) follow from Lemma 3.3. □

In the next lemma we deal with combining gradients of functions f_j and $g_j^{(\mathrm{I})}$, $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$.

Lemma 3.5. *The functions $g_j^{(\mathrm{I})}$, $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$ defined by formulas* (3.3) – (3.5) *satisfy also the following relations:*

(1) $\sum_{i=1}^{3}\langle\nabla(f_i),\nabla(g_i^{(\mathrm{I})})\rangle = \frac{1}{2}\sum_{\sigma}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\widehat{u}_{\sigma(2)}\widehat{u}_{\sigma(3)}f_{\sigma(1)}\,,$

(2) $\sum_{i=1}^{3}\langle\nabla(f_i),\nabla(g_i^{(\mathrm{II})})\rangle = 0\,,$

(3) $2\sum_{i=1}^{3}\langle\nabla(f_i),\nabla(g_i^{(\mathrm{III})})\rangle = \sum_{i=1}^{3}\widehat{u}_i\partial_{\beta_i}(g_i^{(\mathrm{III})})\,.$

Proof. The left-hand side of (1) can be expanded using the product rule as

$$\sum_{i=1}^{3}\langle\nabla(f_i),\nabla(g_i^{(\mathrm{I})})\rangle = \sum_{\sigma}\langle\nabla f_{\sigma(1)},\nabla(f_{\sigma(2)}f_{\sigma(3)})\rangle$$

$$= \sum_\sigma \langle \nabla(f_{\sigma(1)}), \nabla(f_{\sigma(2)})\rangle f_{\sigma(3)} + \sum_\sigma \langle \nabla(f_{\sigma(1)}), \nabla(f_{\sigma(3)})\rangle f_{\sigma(2)}$$
$$= 2\sum_\sigma \langle \nabla(f_{\sigma(2)}), \nabla(f_{\sigma(3)})\rangle f_{\sigma(1)}\,,$$

and the result follows by an application of Lemma 3.2 part (1).

The relation (2) holds because $\nabla(f_i)$ is proportional to β_i, while $\nabla(g_i^{(\mathrm{II})})$ is proportional to α_i , and $\langle \alpha_i, \beta_i\rangle = 0$, for all $i = 1, 2, 3$.

Further, we have by Lemma 3.2 part (1) that

$$2\sum_{i=1}^3 \langle \nabla(f_i), \nabla(g_i^{(\mathrm{III})})\rangle = \sum_{i=1}^3 \widehat{u}_i \langle \beta_i, \nabla(g_i^{(\mathrm{III})})\rangle = \sum_{i=1}^3 \widehat{u}_i \partial_{\beta_i}(g_i^{(\mathrm{III})})\,,$$

which proves identity (3). □

In the next Lemmas 3.6 and 3.7 we calculate and rearrange Laplacians of functions $g_j^{(\mathrm{I})}$, $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$.

Lemma 3.6. *The functions $g_j^{(\mathrm{II})}$ and $g_j^{(\mathrm{III})}$ defined by formulas* (3.4) *and* (3.5) *satisfy the following relations for all $i = 1, 2, 3$:*

(1) $\Delta(g_i^{(\mathrm{II})}) = -\prod_{k\neq i} \partial_{\beta_k} v_i = -\prod_{k\neq i} \partial_{\beta_k} \sum_{j=1}^3 v_j$,

(2) $\Delta(g_i^{(\mathrm{III})}) = -\prod_{k\neq i} \partial_{\beta_k} u_i\,.$

Proof. Statement (1) follows by formula (3.4) and Lemma 3.1. Similarly, property (2) follows directly from Lemma 3.1 and formula (3.5). □

Lemma 3.7. *Functions $g_j^{(\mathrm{I})}$, $g_j^{(\mathrm{II})}$, $g_j^{(\mathrm{III})}$ defined by formulas* (3.3) – (3.5) *satisfy the following relations:*

(1) $$\sum_{i=1}^3 \Delta(g_i^{(\mathrm{I})})\partial_{\beta_i} = \sum_{i=1}^3 \Big(\sum_{j\neq i} \widehat{u}_j\Big) g_i^{(\mathrm{I})}\partial_{\beta_i} + \frac{1}{2}\sum_\sigma \langle \beta_{\sigma(2)}, \beta_{\sigma(3)}\rangle \widehat{u}_{\sigma(2)}\widehat{u}_{\sigma(3)}\partial_{\beta_{\sigma(1)}}$$
$$- \sum_\sigma f_{\sigma(1)}\Big(\partial_{\beta_{\sigma(2)}}(u_{\sigma(2)})\partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}}(u_{\sigma(3)})\partial_{\beta_{\sigma(2)}}\Big)\,,$$

(2) $$\sum_{i=1}^3 \Delta(g_i^{(\mathrm{II})})\partial_{\beta_i} = -\sum_\sigma \partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\Big(\sum_{j=1}^3 v_j\Big)\partial_{\beta_{\sigma(1)}}\,,$$

(3) $$\sum_{i=1}^3 \Delta(g_i^{(\mathrm{III})})\partial_{\beta_i} = -\sum_\sigma \partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}(u_{\sigma(1)})\partial_{\beta_{\sigma(1)}}\,.$$

Proof. Let us first consider $\Delta(g_1^{(\mathrm{I})})$. By Lemma 3.2, we have

$$\begin{aligned}\Delta(g_1^{(\mathrm{I})}) &= \Delta(f_2 f_3) = \Delta(f_2) f_3 + 2\langle \nabla(f_2), \nabla(f_3)\rangle + \Delta(f_3) f_2\\ &= \Big(\widehat{u}_2 f_2 - \partial_{\beta_2}(u_2)\Big) f_3 + \frac{1}{2}\widehat{u}_2\widehat{u}_3\langle \beta_2, \beta_3\rangle + \Big(\widehat{u}_3 f_3 - \partial_{\beta_3}(u_3)\Big) f_2\\ &= \Big(\sum_{j\neq 1}\widehat{u}_j\Big) g_1^{(\mathrm{I})} + \frac{1}{2}\widehat{u}_2\widehat{u}_3\langle \beta_2, \beta_3\rangle - \big(\partial_{\beta_2}(u_2) f_3 + \partial_{\beta_3}(u_3) f_2\big)\,.\end{aligned} \tag{3.7}$$

By multiplying (3.7) by ∂_{β_1}, and adding it with similar expressions for $\Delta(g_2^{(\mathrm{I})})\partial_{\beta_2}$ and $\Delta(g_3^{(\mathrm{I})})\partial_{\beta_3}$, we obtain property (1).

Properties (2) and (3) follow from Lemma 3.6 parts (1) and (2), respectively, by multiplying these equalities by ∂_{β_i} and summing them up over $i = 1, 2, 3$. □

Let h in the operator (1.4) be defined by

$$h = h^{(\mathrm{I})} + h^{(\mathrm{II})} + h^{(\mathrm{III})} + h^{(\mathrm{IV})}\,, \tag{3.8}$$

where

$$h^{(\mathrm{I})} = f_1 f_2 f_3\,, \tag{3.9}$$

$$h^{(\mathrm{II})} = \sum_{i=1}^{3} f_i\big(g_i^{(\mathrm{II})} + g_i^{(\mathrm{III})}\big)\,, \tag{3.10}$$

$$h^{(\mathrm{III})} = \sum_{i=1}^{3} \partial_{\beta_i}(g_i^{(\mathrm{III})}) = -\sum_{i=1}^{3} \frac{\prod_{k\neq i}\langle \beta_i, \beta_k\rangle}{\langle \beta_i, \beta_i\rangle}\partial_{\beta_i}(u_i) \tag{3.11}$$

$$= -\frac{12m(3m+1)\omega^6}{\sinh^2\beta_1}\coth\beta_1 + \frac{12m(3m+1)\omega^6}{\sinh^2\beta_2}\coth\beta_2 - \frac{12m(3m+1)\omega^6}{\sinh^2\beta_3}\coth\beta_3\,,$$

$$h^{(\mathrm{IV})} = -3m(3m+1)\omega^{-2}\prod_{i=1}^{3}\langle \beta_i, \beta_i\rangle X - 4(3m+1)\omega^{-2}\prod_{i=1}^{3}\langle \beta_i, \beta_i\rangle Y. \tag{3.12}$$

In the next Lemmas 3.8, 3.9 we calculate gradients and Laplacians of the functions $h^{(\mathrm{I})}$, $h^{(\mathrm{II})}$, $h^{(\mathrm{III})}$.

Lemma 3.8. *The functions $h^{(\mathrm{I})}$, $h^{(\mathrm{II})}$, $h^{(\mathrm{III})}$ defined by formulas* (3.9) – (3.12) *satisfy the following relations:*

(1) $2\partial_{\nabla(h^{(\mathrm{I})}+h^{(\mathrm{II})})} = \sum_{i=1}^{3} \widehat{u}_i g_i \partial_{\beta_i} - \sum_{\sigma} f_{\sigma(1)}\Big(\partial_{\beta_{\sigma(2)}}\big(v_{\sigma(1)}+u_{\sigma(1)}\big)\partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}}\big(v_{\sigma(1)}+u_{\sigma(1)}\big)\partial_{\beta_{\sigma(2)}}\Big)$,

(2) $2\partial_{\nabla(h^{(\mathrm{III})})} = -\sum_{\sigma}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\Big(\sum_{j\neq\sigma(1)} u_j\Big)\partial_{\beta_{\sigma(1)}}$.

Proof. We have that

$$\partial_j(h^{(\mathrm{I})}) = \partial_j(f_1)f_2f_3 + \partial_j(f_2)f_1f_3 + \partial_j(f_3)f_1f_2 = \sum_{i=1}^{3}\partial_j(f_i)g_i^{(\mathrm{I})}\,,$$

therefore by Lemma 3.2 part (1),

$$2\partial_{\nabla(h^{(\mathrm{I})})} = \sum_{i=1}^{3} 2g_i^{(\mathrm{I})}\partial_{\nabla(f_i)} = \sum_{i=1}^{3}\widehat{u}_i g_i^{(\mathrm{I})}\partial_{\beta_i}\,. \tag{3.13}$$

On the other hand,

$$2\partial_{\nabla(h^{(\mathrm{II})})} = 2\sum_{i=1}^{3}\partial_{\nabla\left(f_i(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})})\right)} = 2\sum_{i=1}^{3}\big(g_i^{(\mathrm{II})} + g_i^{(\mathrm{III})}\big)\partial_{\nabla(f_i)} + 2\sum_{i=1}^{3} f_i\Big(\partial_{\nabla(g_i^{(\mathrm{II})})} + \partial_{\nabla(g_i^{(\mathrm{III})})}\Big)\,. \tag{3.14}$$

By Lemma 3.2 part (1) and Lemma 3.3 we can rearrange the expression (3.14) as

$$\sum_{i=1}^{3}\widehat{u}_i\big(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}\big)\partial_{\beta_i}-\sum_{\sigma} f_{\sigma(1)}\Big(\partial_{\beta_{\sigma(2)}}\big(v_{\sigma(1)}+u_{\sigma(1)}\big)\partial_{\beta_{\sigma(3)}}+\partial_{\beta_{\sigma(3)}}\big(v_{\sigma(1)}+u_{\sigma(1)}\big)\partial_{\beta_{\sigma(2)}}\Big). \tag{3.15}$$

The statement (1) follows by adding up equalities (3.13) and (3.15).

In the right-hand side of statement (2), the coefficient at ∂_{β_1} is equal to

$$-\partial_{\beta_2}\partial_{\beta_3}(u_2+u_3)=-24m(3m+1)\omega^6\left(\frac{2\coth^2\beta_2}{\sinh^2\beta_2}+\frac{1}{\sinh^4\beta_2}+\frac{2\coth^2\beta_3}{\sinh^2\beta_3}+\frac{1}{\sinh^4\beta_3}\right). \tag{3.16}$$

In the left-hand side of statement (2) one can check that $2\partial_{\nabla(h^{(\mathrm{III})})}$ is equal to

$$24m(3m+1)\omega^6\times$$
$$\left(\left(\frac{2\coth^2\beta_1}{\sinh^2\beta_1}+\frac{1}{\sinh^4\beta_1}\right)\partial_{\beta_1}-\left(\frac{2\coth^2\beta_2}{\sinh^2\beta_2}+\frac{1}{\sinh^4\beta_2}\right)\partial_{\beta_2}+\left(\frac{2\coth^2\beta_3}{\sinh^2\beta_3}+\frac{1}{\sinh^4\beta_3}\right)\partial_{\beta_3}\right). \tag{3.17}$$

Let us substitute in expression (3.17) $\partial_{\beta_1}=\partial_{\beta_2}-\partial_{\beta_3}$, $\partial_{\beta_2}=\partial_{\beta_1}+\partial_{\beta_3}$, and $\partial_{\beta_3}=\partial_{\beta_2}-\partial_{\beta_1}$. Then one can see that the coefficient at ∂_{β_1} equals expression (3.16). Similarly, the coefficients at ∂_{β_2} and ∂_{β_3} also match on both sides of equality (2). □

Lemma 3.9. *Functions $h^{(\mathrm{I})}$, $h^{(\mathrm{II})}$, $h^{(\mathrm{III})}$ given by formulas* (3.9) – (3.11) *satisfy the following relations:*

(1) $$\Delta(h^{(\mathrm{I})}+h^{(\mathrm{II})})=\sum_{i=1}^{3}\widehat{u}_i f_i g_i-\sum_{i=1}^{3}\partial_{\beta_i}(u_i)g_i+\frac{1}{2}\sum_{\sigma}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\widehat{u}_{\sigma(2)}\widehat{u}_{\sigma(3)}f_{\sigma(1)}$$
$$+\sum_{i=1}^{3}\widehat{u}_i\partial_{\beta_i}\big(g_i^{(\mathrm{III})}\big)-\sum_{\sigma}f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\big(v_{\sigma(1)}+u_{\sigma(1)}\big),$$

(2) $$\Delta(h^{(\mathrm{III})})=-\partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}\left(\sum_{j=1}^{3}u_j\right).$$

Proof. Firstly, by Lemma 3.2 part (2) and Lemma 3.5 part (1) we have

$$\begin{aligned}\Delta(h^{(\mathrm{I})})=\Delta(f_1f_2f_3)&=\sum_{i=1}^{3}\Delta(f_i)g_i^{(\mathrm{I})}+\sum_{i=1}^{3}\langle\nabla(f_i),\nabla(g_i^{(\mathrm{I})})\rangle\\&=\sum_{i=1}^{3}\widehat{u}_i f_i g_i^{(\mathrm{I})}-\sum_{i=1}^{3}\partial_{\beta_i}(u_i)g_i^{(\mathrm{I})}+\frac{1}{2}\sum_{\sigma}\widehat{u}_{\sigma(2)}\widehat{u}_{\sigma(3)}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle f_{\sigma(1)}\,.\end{aligned} \tag{3.18}$$

Secondly, by Lemma 3.2 part (2), by Lemma 3.5 parts (2) and (3), and Lemma 3.6 we have

$$
\begin{aligned}
\Delta(h^{(\mathrm{II})}) &= \Delta\Big(\sum_{i=1}^{3} f_i\big(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}\big)\Big)\\
&= \sum_{i=1}^{3}\Delta(f_i)\big(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}\big)+2\sum_{i=1}^{3}\langle\nabla(f_i),\nabla\big(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}\big)\rangle+\sum_{i=1}^{3} f_i\Delta\big(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}\big)\\
&= \sum_{i=1}^{3}\widehat{u}_i f_i\big(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}\big)-\sum_{i=1}^{3}\partial_{\beta_i}(u_i)\big(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}\big)\\
&\quad+\sum_{i=1}^{3}\widehat{u}_i\partial_{\beta_i}\big(g_i^{(\mathrm{III})}\big)-\sum_{\sigma} f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\big(v_{\sigma(1)}+u_{\sigma(1)}\big)\,.
\end{aligned}
\tag{3.19}
$$

The statement (1) follows by adding the equalities (3.18) and (3.19).

By Lemma 3.1 for any j we have

$$
\Delta\Big(-\frac{\prod_{k\neq j}\langle\beta_j,\beta_k\rangle}{\langle\beta_j,\beta_j\rangle}\partial_{\beta_j}(u_j)\Big)=-\Big(\prod_{k\neq j}\partial_{\beta_k}\Big)\partial_{\beta_j}(u_j)=-\partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}(u_j)\,. \tag{3.20}
$$

We get result (2) by summing equalities (3.20) over $j=1,2,3$. □

4 Proof of the intertwining relation

Let $A=A(x,\partial_1,\partial_2)$ be a differential operator of order N. Then A can be represented as

$$
A=\sum_{k=0}^{N}A^{(k)}\,,\quad \text{with } A^{(k)}=\sum_{i+j=k}a_{ij}(x)\partial_1^i\partial_2^j
$$

for some functions $a_{ij}(x)$ so $A^{(k)}$ denotes the k-th order part of A. That is $A^{(k)}$ is the sum of all terms in A that contain exactly k derivatives when all the derivatives are put on the right.

Both operators $H\mathcal{D}$ and $\mathcal{D}H_0$ have order 5. It is easy to see that the respective terms of orders 5 and 4 in both operators are the same. We are going to show that this is also true for lower orders.

Proposition 4.1. *The third order terms in the intertwining relation* (1.5) *satisfy*

$$
(H\mathcal{D})^{(3)}=(\mathcal{D}H_0)^{(3)}\,.
$$

Proof. We have

$$
(H\mathcal{D}-\mathcal{D}H_0)^{(3)}=-2\sum_{\sigma}\partial_{\nabla(f_{\sigma(1)})}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}+\Big(\sum_{j=1}^{3}\widehat{u}_j\Big)\partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}\,.
$$

By Lemma 3.2 part (1) we get

$$
\sum_{\sigma}\partial_{\nabla(f_{\sigma(1)})}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}=\frac{1}{2}\sum_{\sigma}\widehat{u}_{\sigma(1)}\partial_{\beta_{\sigma(1)}}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}=\frac{1}{2}\Big(\sum_{j=1}^{3}\widehat{u}_j\Big)\partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}\,,
$$

and the statement follows. □

Proposition 4.2. *The second order terms in the intertwining relation* (1.5) *satisfy*

$$(H\mathcal{D})^{(2)} = (\mathcal{D}H_0)^{(2)}\,.$$

Proof. We have that

$$\begin{aligned}(H\mathcal{D} - \mathcal{D}H_0)^{(2)} = &-\sum_{\sigma} \Delta(f_{\sigma(1)})\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}} - 2\sum_{i=1}^{3} \partial_{\nabla(g_i)}\partial_{\beta_i} \\ &+ \sum_{\sigma}\sum_{j=1}^{3} \widehat{u}_j f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}} - \sum_{\sigma}\partial_{\beta_{\sigma(1)}}\Big(\sum_{j=1}^{3}(v_j+u_j)\Big)\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\,,\end{aligned}$$

which is zero by applying Lemma 3.2 part (2) and adding equalities from all three parts of Lemma 3.4. □

The next lemma will be useful for dealing with the first and zero order terms in the intertwining relation.

Lemma 4.3. *For any* $\sigma \in A_3$*,*

$$\begin{aligned}&-\frac{1}{2}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\widehat{u}_{\sigma(2)}\widehat{u}_{\sigma(3)} + \Big(\sum_{j\neq\sigma(1)}\widehat{u}_j\Big)\big(g^{(\mathrm{II})}_{\sigma(1)} + g^{(\mathrm{III})}_{\sigma(1)}\big) - \Big(f_{\sigma(2)}\partial_{\beta_{\sigma(3)}} + f_{\sigma(3)}\partial_{\beta_{\sigma(2)}}\Big)(v_{\sigma(1)} + u_{\sigma(1)}) \\ &\quad = -\frac{48m(3m+1)\omega^4\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle}{\sinh^2\beta_{\sigma(2)}\sinh^2\beta_{\sigma(3)}} - \frac{128(3m+1)\omega^4\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle}{\sinh^2 2\beta_{\sigma(2)}\sinh^2 2\beta_{\sigma(3)}}\,. \qquad (4.1)\end{aligned}$$

Proof. Firstly we let $\sigma = id$. By the identities (2.11) and (2.12) in Lemma 2.7 we get

$$\Big(\sum_{j\neq 1}\widehat{u}_j\Big)g_1^{(\mathrm{II})} - \big(f_2\partial_{\beta_3} + f_3\partial_{\beta_2}\big)(v_1) = -36m(m+1)\omega^6\bigg(\frac{3m+1}{\sinh^2\beta_2\sinh^2\beta_3} + \frac{1}{\cosh^2\beta_2\cosh^2\beta_3}\bigg)\,. \qquad (4.2)$$

Similarly, by the identities (2.13) and (2.14) in Lemma 2.7 we get

$$\Big(\sum_{j\neq 1}\widehat{u}_j\Big)g_1^{(\mathrm{III})} - \big(f_2\partial_{\beta_3} + f_3\partial_{\beta_2}\big)(u_1) = 12m(3m+1)\omega^6\bigg(\frac{3m+1}{\sinh^2\beta_2\sinh^2\beta_3} + \frac{1}{\cosh^2\beta_2\cosh^2\beta_3}\bigg)\,. \qquad (4.3)$$

Note that the sum of expressions (4.2), (4.3) divided by ω^6 together with the term $-\frac{1}{2}\widehat{u}_2\widehat{u}_3$ divided by ω^4 equals

$$\begin{aligned}&-\frac{48m(3m+1)}{\sinh^2\beta_2\sinh^2\beta_3} - 8(3m+1)\bigg(\frac{1}{\sinh^2\beta_2} - \frac{1}{\cosh^2\beta_2}\bigg)\bigg(\frac{1}{\sinh^2\beta_3} - \frac{1}{\cosh^2\beta_3}\bigg) \\ &= -\frac{48m(3m+1)}{\sinh^2\beta_2\sinh^2\beta_3} - \frac{128(3m+1)}{\sinh^2 2\beta_2\sinh^2 2\beta_3}\,,\end{aligned}$$

which is the right-hand side of the equality (4.1) divided by ω^6 as required. The cases $\sigma \neq id$ follow from versions of (2.11) – (2.14) obtained by rotating vectors (see Remark 1). □

Proposition 4.4. *The first order terms in the intertwining relation* (1.5) *satisfy*

$$(H\mathcal{D})^{(1)} = (\mathcal{D}H_0)^{(1)}\,.$$

Proof. We have that

$$(H\mathcal{D}-\mathcal{D}H_0)^{(1)} = -\sum_{i=1}^{3}\Delta(g_i)\partial_{\beta_i} - 2\partial_{\nabla(h)} + \sum_{i=1}^{3}\Big(\sum_{j=1}^{3}\widehat{u}_j\Big)g_i\partial_{\beta_i} - \sum_{\sigma}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\Big(\sum_{j=1}^{3}(v_j+u_j)\Big)\partial_{\beta_{\sigma(1)}}$$
$$-\sum_{\sigma} f_{\sigma(1)}\Big(\partial_{\beta_{\sigma(2)}}\Big(\sum_{j=1}^{3}(v_j+u_j)\Big)\partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}}\Big(\sum_{j=1}^{3}(v_j+u_j)\Big)\partial_{\beta_{\sigma(2)}}\Big). \tag{4.4}$$

We substitute the expression for $\sum_{i=1}^{3}\Delta(g_i)\partial_{\beta_i}$ from Lemma 3.7 and the expression for $\partial_{\nabla(h^{(\mathrm{I})}+h^{(\mathrm{II})}+h^{(\mathrm{III})})}$ from Lemma 3.8 into the formula (4.4). Then the expression (4.4) can be rearranged as

$$\begin{aligned}&-\frac{1}{2}\sum_{\sigma}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\widehat{u}_{\sigma(2)}\widehat{u}_{\sigma(3)}\partial_{\beta_{\sigma(1)}} + \sum_{\sigma}\Big(\sum_{j\neq\sigma(1)}\widehat{u}_j\Big)\big(g^{(\mathrm{II})}_{\sigma(1)}+g^{(\mathrm{III})}_{\sigma(1)}\big)\partial_{\beta_{\sigma(1)}}\\ &-\sum_{\sigma}\Big(f_{\sigma(2)}\partial_{\beta_{\sigma(3)}} + f_{\sigma(3)}\partial_{\beta_{\sigma(2)}}\Big)(v_{\sigma(1)}+u_{\sigma(1)})\partial_{\beta_{\sigma(1)}} - 2\partial_{\nabla(h^{(\mathrm{IV})})}\,,\end{aligned} \tag{4.5}$$

which is zero by Lemma 4.3 and Corollary 2.5. □

The following lemma is needed in order to consider the zero order terms in the intertwining relation.

Lemma 4.5. *The zero order terms satisfy*

$$(H\mathcal{D}-\mathcal{D}H_0)^{(0)} = A+B+C+D\,,$$

where

$$A = \sum_{i=1}^{3}\Big(\sum_{j\neq i}\widehat{u}_j\Big)\partial_{\beta_i}\Big(g_i^{(\mathrm{III})}\Big) - \sum_{\sigma} f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\Big(\sum_{j\neq\sigma(1)}u_j\Big)\,, \tag{4.6}$$

$$B = -\frac{1}{2}\sum_{\sigma}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\widehat{u}_{\sigma(2)}\widehat{u}_{\sigma(3)}f_{\sigma(1)} + \sum_{\sigma}\Big(\sum_{j\neq\sigma(1)}\widehat{u}_j\Big)\big(g^{(\mathrm{II})}_{\sigma(1)}+g^{(\mathrm{III})}_{\sigma(1)}\big)f_{\sigma}(1)\,, \tag{4.7}$$

$$C = \Big(\sum_{j=1}^{3}\widehat{u}_j\Big)h^{(\mathrm{IV})} - \Delta(h^{(\mathrm{IV})})\,, \tag{4.8}$$

$$D = -\sum_{i=1}^{3} g_i\partial_{\beta_i}\Big(\sum_{j\neq i}(v_j+u_j)\Big)\,. \tag{4.9}$$

Moreover, the term D can be rearranged as $D = D_1 + D_2$, where

$$D_1 = -\sum_{i=1}^{3}(g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})})\partial_{\beta_i}\Big(\sum_{j\neq i}(v_j+u_j)\Big)\,, \tag{4.10}$$

$$D_2 = -\sum_{\sigma} f_{\sigma(1)}\Big(f_{\sigma(2)}\partial_{\beta_{\sigma(3)}} + f_{\sigma(3)}\partial_{\beta_{\sigma(2)}}\Big)\big(v_{\sigma(1)}+u_{\sigma(1)}\big)\,. \tag{4.11}$$

Proof. We have

$$(H\circ\mathcal{D})^{(0)}-(\mathcal{D}\circ H_0)^{(0)} = -\Delta(h)+\Big(\sum_{j=1}^{3}\widehat{u}_j\Big)h-\partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}\Big(\sum_{j=1}^{3}u_j\Big)$$
$$-\sum_{\sigma}f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\Big(v_{\sigma(1)}+\sum_{j=1}^{3}u_j\Big) \tag{4.12}$$
$$-\sum_{i=1}^{3}g_i\partial_{\beta_i}\Big(\sum_{j\neq i}v_j+\sum_{j=1}^{3}u_j\Big).$$

By putting in the results of Lemma 3.9, the expression (4.12) takes the required form $A+B+C+D$. By expanding $g_i=g_i^{(\mathrm{II})}+g_i^{(\mathrm{III})}+g_i^{(\mathrm{I})}$, we also have that

$$D=D_1-\sum_{\sigma}f_{\sigma(2)}f_{\sigma(3)}\partial_{\beta_{\sigma(1)}}\Big(\sum_{j\neq\sigma(1)}(v_j+u_j)\Big)=D_1+D_2$$

as required. □

In the next Lemmas 4.6 – 4.9 we rearrange the expressions for the zero order terms A, B, C, D. Namely, we rewrite these terms explicitly as functions of β_j.

Lemma 4.6. *The function A given by expression* (4.6) *can be rearranged as follows:*

$$\frac{A}{48\omega^6}=2m(3m+1)^2X-m(3m+1)\sum_{\sigma}\frac{\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\coth\beta_{\sigma(1)}}{\cosh^2\beta_{\sigma(2)}\cosh^2\beta_{\sigma(3)}}$$
$$+3m^2(3m+1)\Big(\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j}\Big)X+8m(3m+1)\Big(\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j}\Big)Y \tag{4.13}$$
$$+24m(3m+1)Y.$$

Proof. Consider the term involving $g_1^{(\mathrm{III})}$ in $\sum_{i=1}^{3}\Big(\sum_{j\neq i}\widehat{u}_j\Big)\partial_{\beta_i}\Big(g_i^{(\mathrm{III})}\Big)$. It gives

$$\frac{1}{48\omega^8}\Big(\sum_{j\neq1}\widehat{u}_j\Big)\partial_{\beta_1}\Big(g_1^{(\mathrm{III})}\Big)=-m(3m+1)^2\Big(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\Big)\frac{\coth\beta_1}{\sinh^2\beta_1}$$
$$+m(3m+1)\Big(\frac{1}{\cosh^2\beta_2}+\frac{1}{\cosh^2\beta_3}\Big)\frac{\coth\beta_1}{\sinh^2\beta_1}. \tag{4.14}$$

Now consider the terms involving u_1 in $-\sum_\sigma f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}}\Big(\sum_{j\neq\sigma(1)}u_j\Big)$. They produce

$$-\frac{1}{48\omega^8}\big(f_2\partial_{\beta_3}\partial_{\beta_1}(u_1)+f_3\partial_{\beta_1}\partial_{\beta_2}(u_1)\big)=$$
$$=-2m(3m+1)^2\big(\coth\beta_2-\coth\beta_3\big)\frac{\coth^2\beta_1}{\sinh^2\beta_1}-2m(3m+1)\big(\tanh\beta_2-\tanh\beta_3\big)\frac{\coth^2\beta_1}{\sinh^2\beta_1}$$
$$+m(3m+1)\Big((3m+1)\big(\coth\beta_3-\coth\beta_2\big)+\big(\tanh\beta_3-\tanh\beta_2\big)\Big)\frac{1}{\sinh^4\beta_1}. \tag{4.15}$$

It follows from Lemma 2.7 (namely, equalities (2.13) and (2.14) multiplied by $\coth\beta_1$) that the sum of the right-hand side of equality (4.14) with the first two terms in the right-hand side of equality (4.15) equals

$$-m(3m+1)^2\frac{\coth\beta_1}{\sinh^2\beta_2\sinh^2\beta_3}-m(3m+1)\frac{\coth\beta_1}{\cosh^2\beta_2\cosh^2\beta_3}\,, \tag{4.16}$$

Adding expression (4.16) with analogous ones coming from the terms $g_2^{(\mathrm{III})}$, u_2, and $g_3^{(\mathrm{III})}$, u_3 in the left-hand side of equality (4.13) we get the 1st line of the right-hand side of equality (4.13) by Lemma 2.2.

Now we rearrange the other terms in the right-hand side of equality (4.15). We have that

$$\begin{aligned}
&m(3m+1)\Big((3m+1)\big(\coth\beta_3-\coth\beta_2\big)+\big(\tanh\beta_3-\tanh\beta_2\big)\Big)\frac{1}{\sinh^4\beta_1}\\
&=m(3m+1)\left(\frac{3m\sinh\beta_1}{\sinh\beta_2\sinh\beta_3}+\frac{\sinh\beta_1}{\sinh\beta_2\sinh\beta_3}-\frac{\sinh\beta_1}{\cosh\beta_2\cosh\beta_3}\right)\frac{1}{\sinh^4\beta_1}\\
&=\frac{3m^2(3m+1)\omega^{-2}}{\sinh^2\beta_1}X+\frac{m(3m+1)\cosh\beta_1}{\sinh^3\beta_1\sinh\beta_2\sinh\beta_3\cosh\beta_2\cosh\beta_3}\\
&=\frac{3m^2(3m+1)\omega^{-2}}{\sinh^2\beta_1}X+\frac{m(3m+1)(1+\sinh^2\beta_1)}{\sinh^3\beta_1\sinh\beta_2\sinh\beta_3\cosh\beta_1\cosh\beta_2\cosh\beta_3}\\
&=\frac{3m^2(3m+1)\omega^{-2}}{\sinh^2\beta_1}X+\frac{8m(3m+1)\omega^{-2}}{\sinh^2\beta_1}Y+8m(3m+1)\omega^{-2}Y\,.
\end{aligned} \tag{4.17}$$

Similarly, terms from the right-hand side of a version of equality (4.15) for $g_2^{(\mathrm{III})}$, u_2, and $g_3^{(\mathrm{III})}$, u_3, add up to

$$3m^2(3m+1)\omega^{-2}\left(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\right)X+8m(3m+1)\omega^{-2}\left(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\right)Y+16m(3m+1)\omega^{-2}Y\,. \tag{4.18}$$

The sum of terms in the last line of (4.17) together with (4.18) equals the terms in the 2nd and 3rd line in the right-hand side of equality (4.13). □

Lemma 4.7. *Consider the functions B and D_2 given by* (4.7) *and* (4.11). *We have*

$$\begin{aligned}
\frac{B+D_2}{64\omega^6}&=-3m(3m+1)(3m+2)X-12m(3m+1)Y-16(3m+1)Y\\
&\quad+\sum_\sigma\frac{12m(3m+1)\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\coth\beta_{\sigma(1)}}{\sinh^2 2\beta_{\sigma(2)}\sinh^2 2\beta_{\sigma(3)}}\,.
\end{aligned} \tag{4.19}$$

Proof. As a consequence of Lemma 4.3 we have

$$\frac{B+D_2}{64\omega^6}=-\sum_\sigma\frac{3m(3m+1)\omega^{-2}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle}{4\sinh^2\beta_{\sigma(2)}\sinh^2\beta_{\sigma(3)}}f_{\sigma(1)}-\sum_\sigma\frac{2(3m+1)\omega^{-2}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle}{\sinh^2 2\beta_{\sigma(2)}\sinh^2 2\beta_{\sigma(3)}}f_{\sigma(1)}\,. \tag{4.20}$$

We substitute f_j given by (3.1) into the first sum in the relation (4.20), and we substitute

$$f_j=-2\omega^2(3m\coth\beta_j+2\coth 2\beta_j)\,,$$

$j=1,2,3$, in the second sum in the relation (4.20). By Lemma 2.2, as well as its version with β_j replaced with $2\beta_j$, and by Lemma 2.3, we can rearrange the right-hand side of (4.20) into the required form. □

Lemma 4.8. *The function C given by* (4.8) *can be rearranged as follows:*

$$\begin{aligned}\frac{C}{32\omega^6} &= -9m^2(3m+1)\Big(\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j}\Big)X + 3m(3m+1)\Big(2+\sum_{j=1}^{3}\frac{1}{\cosh^2\beta_j}\Big)X \\ &\quad - 12m(3m+1)\Big(\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j}\Big)Y + 32(3m+1)Y\,.\end{aligned} \tag{4.21}$$

Proof. By Lemma 2.6, we have that

$$-\frac{\Delta(h^{(\mathrm{IV})})}{32\omega^6} = 3m(3m+1)\Big(2+\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j}\Big)X + 16(3m+1)\Big(2+\sum_{j=1}^{3}\frac{1}{\sinh^2 2\beta_j}\Big)Y\,. \tag{4.22}$$

The product $\frac{1}{32\omega^6}(\sum_{j=1}^3 \widehat{u}_j)h^{(\mathrm{IV})}$ can be rearranged as

$$\begin{aligned}&-\Big(3m\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j} + \sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j} - \sum_{j=1}^{3}\frac{1}{\cosh^2\beta_j}\Big)3m(3m+1)X \\ &-\Big(3m\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j} + 4\sum_{j=1}^{3}\frac{1}{\sinh^2 2\beta_j}\Big)4(3m+1)Y\,.\end{aligned} \tag{4.23}$$

The equality (4.21) follows by multiplying (4.23) out and combining it with (4.22). □

Lemma 4.9. *The function D_1 given by* (4.10) *can be rearranged as*

$$D_1 = 144m^2(3m+1)\omega^6\Big(2+\sum_{j=1}^{3}\frac{1}{\sinh^2\beta_j}\Big)X\,. \tag{4.24}$$

Proof. Note that Lemma 2.10 can be restated in the following form:

$$\sum_{i=1}^{3} g_i^{(\mathrm{II})}\partial_{\beta_i}\Big(\sum_{j\neq i} v_j\Big) = 0\,.$$

Consider the terms in $-\sum_{i=1}^3 g_i^{(\mathrm{III})}\partial_{\beta_i}\Big(\sum_{j\neq i} v_j\Big)$ that involve v_1, that is $j=1$. They equal

$$108m^2(m+1)(3m+1)\omega^8\Big(\frac{1}{\sinh^2\beta_3}-\frac{1}{\sinh^2\beta_2}\Big)\frac{\coth\alpha_1}{\sinh^2\alpha_1}\,. \tag{4.25}$$

Now let us look at terms in $-\sum_{i=1}^3 g_i^{(\mathrm{II})}\partial_{\beta_i}\Big(\sum_{j\neq i} u_j\Big)$ that involve $g_1^{(\mathrm{II})}$. These terms are equal to

$$108m^2(m+1)(3m+1)\omega^8\Big(\frac{\coth\beta_3}{\sinh^2\beta_3}-\frac{\coth\beta_2}{\sinh^2\beta_2}\Big)\frac{1}{\sinh^2\alpha_1}\,. \tag{4.26}$$

By Lemma 2.8 the sum of expressions (4.25) and (4.26) equals

$$108m^2(m+1)(3m+1)\omega^8\frac{\coth\beta_3-\coth\beta_2}{\sinh^2\beta_2\sinh^2\beta_3}\,. \tag{4.27}$$

We also have that $-\sum_{i=1}^{3} g_i^{(\mathrm{III})}\partial_{\beta_i}\Big(\sum_{j\neq i} u_j\Big)$ is equal to

$$36m^2(3m+1)^2\omega^8\left(\frac{\coth\beta_2-\coth\beta_3}{\sinh^2\beta_2\sinh^2\beta_3}-\frac{\coth\beta_1+\coth\beta_3}{\sinh^2\beta_1\sinh^2\beta_3}+\frac{\coth\beta_2-\coth\beta_1}{\sinh^2\beta_1\sinh^2\beta_2}\right). \tag{4.28}$$

By adding expression (4.27) and the first term of expression (4.28) we get

$$72m^2(3m+1)\omega^8\frac{\coth\beta_3-\coth\beta_2}{\sinh^2\beta_2\sinh^2\beta_3}. \tag{4.29}$$

By grouping similarly the remaining terms in the left-hand side of identity (4.24) and by using variants of Lemma 2.8 obtained by rotating β's by $\pm\frac{\pi}{3}$ (see Remark 1), we get that the left-hand side of (4.24) can be rearranged as

$$\begin{aligned}&72m^2(3m+1)\omega^8\left(\frac{\coth\beta_3-\coth\beta_2}{\sinh^2\beta_2\sinh^2\beta_3}+\frac{\coth\beta_1+\coth\beta_3}{\sinh^2\beta_1\sinh^2\beta_3}+\frac{\coth\beta_1-\coth\beta_2}{\sinh^2\beta_1\sinh^2\beta_2}\right)\\&=72m^2(3m+1)\omega^6\sum_{\sigma}\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\left(\frac{1}{\sinh^2\beta_{\sigma(2)}}+\frac{1}{\sinh^2\beta_{\sigma(3)}}\right)\frac{\coth\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(1)}}.\end{aligned}$$

The result follows by Lemma 2.9. □

Proposition 4.10. *The zero order terms satisfy*

$$(H\mathcal{D})^{(0)}=(\mathcal{D}H_0)^{(0)}.$$

Proof. By Lemmas 4.5 – 4.9 we have

$$\begin{aligned}\frac{(H\mathcal{D}-\mathcal{D}H_0)^{(0)}}{48m(3m+1)\omega^6}&=2\left(\sum_{j=1}^{3}\frac{1}{\cosh^2\beta_j}\right)X-2X+8Y-\sum_{\sigma}\frac{\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\coth\beta_{\sigma(1)}}{\cosh^2\beta_{\sigma(2)}\cosh^2\beta_{\sigma(3)}}\\&\quad+\sum_{\sigma}\frac{16\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\coth\beta_{\sigma(1)}}{\sinh^2 2\beta_{\sigma(2)}\sinh^2 2\beta_{\sigma(3)}}.\end{aligned} \tag{4.30}$$

Let us replace $\sinh^{-2}2\beta_{\sigma(2)}\sinh^{-2}2\beta_{\sigma(3)}$ in the last sum in (4.30) with

$$\frac{1}{16}\left(\frac{1}{\sinh^2\beta_{\sigma(2)}}-\frac{1}{\cosh^2\beta_{\sigma(2)}}\right)\left(\frac{1}{\sinh^2\beta_{\sigma(3)}}-\frac{1}{\cosh^2\beta_{\sigma(3)}}\right).$$

By using Lemma 2.2 the right-hand side of (4.30) can be rewritten as $E+F$, where

$$E=-4X+8Y \tag{4.31}$$

and

$$F=2\left(\sum_{j=1}^{3}\frac{1}{\cosh^2\beta_j}\right)X-\sum_{\sigma\in S_3}\frac{\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\coth\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(2)}\cosh^2\beta_{\sigma(3)}}, \tag{4.32}$$

where the summation is over the symmetric group. Let us collect terms with $\cosh^{-2}\beta_1$ in F. We have

$$\left(2X+\frac{\omega^2\coth\beta_2}{\sinh^2\beta_3}-\frac{\omega^2\coth\beta_3}{\sinh^2\beta_2}\right)\frac{1}{\cosh^2\beta_1}$$

$$= \left(\frac{\sinh(\beta_2-\beta_1)+\sinh\beta_1\cosh\beta_2}{\sinh\beta_3} + \frac{\sinh(\beta_1+\beta_3)-\sinh\beta_1\cosh\beta_3}{\sinh\beta_2}\right)\frac{X}{\cosh^2\beta_1}$$
$$= \left(\frac{\sinh\beta_2}{\sinh\beta_3}+\frac{\sinh\beta_3}{\sinh\beta_2}\right)\frac{X}{\cosh\beta_1} = \omega^2\left(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\right)\frac{\tanh\beta_1}{\sinh^2\beta_1}\,. \tag{4.33}$$

Note that by multiplying the relation (2.14) in Lemma 2.7 through by $\tanh\beta_1$ we rearrange (4.33) as

$$\left(\frac{1}{\sinh^2\beta_2}+\frac{1}{\sinh^2\beta_3}\right)\frac{\tanh\beta_1}{\sinh^2\beta_1} = \frac{\tanh\beta_1}{\sinh^2\beta_2\sinh^2\beta_3} - \frac{2(\coth\beta_2-\coth\beta_3)}{\sinh^2\beta_1} = \frac{\tanh\beta_1}{\sinh^2\beta_2\sinh^2\beta_3}+2\omega^{-2}X\,. \tag{4.34}$$

Similarly, we collect and rearrange terms in F with $\cosh^{-2}\beta_2$ and $\cosh^{-2}\beta_3$. Then by using variants of the identity (4.34) obtained by rotating β's (see Remark 1) we get

$$F = \sum_\sigma \frac{\langle\beta_{\sigma(2)},\beta_{\sigma(3)}\rangle\tanh\beta_{\sigma(1)}}{\sinh^2\beta_{\sigma(2)}\sinh^2\beta_3} + 6X = 4X - 8Y$$

by Lemma 2.3. Hence $E+F=0$ as required. □

5 Rational limit

In the rational limit $\omega\to 0$ the operator H_0 takes the form

$$H_0^r = -\Delta + \sum_{i=1}^3\left(\frac{m(m+1)\langle\tilde\alpha_i,\tilde\alpha_i\rangle}{\langle\tilde\alpha_i,x\rangle^2} + \frac{3m(3m+1)\langle\tilde\beta_i,\tilde\beta_i\rangle}{\langle\tilde\beta_i,x\rangle^2}\right), \tag{5.1}$$

where vectors $\tilde\alpha_i$, $\tilde\beta_i$ can be taken as the original vectors α_i,β_i with any fixed non-zero value of ω. The Hamiltonian H in the rational limit becomes

$$H^r = -\Delta + \sum_{i=1}^3\left(\frac{m(m+1)\langle\tilde\alpha_i,\tilde\alpha_i\rangle}{\langle\tilde\alpha_i,x\rangle^2} + \frac{(3m+1)(3m+2)\langle\tilde\beta_i,\tilde\beta_i\rangle}{\langle\tilde\beta_i,x\rangle^2}\right). \tag{5.2}$$

And the explicit form of the intertwining operator $\mathcal{D}$ in the rational limit is the operator $\mathcal{D}^r = \lim_{\omega\to 0}\omega^{-3}\mathcal{D}$ which takes the form

$$\begin{aligned}
\mathcal{D}^r = {} & \partial_{\tilde\beta_1}\partial_{\tilde\beta_2}\partial_{\tilde\beta_3} - \sum_\sigma \frac{(3m+1)\langle\tilde\beta_{\sigma(1)},\tilde\beta_{\sigma(1)}\rangle}{\langle\tilde\beta_{\sigma(1)},x\rangle}\partial_{\tilde\beta_{\sigma(2)}}\partial_{\tilde\beta_{\sigma(3)}} + \sum_\sigma \frac{(3m+1)^2\langle\tilde\beta_{\sigma(1)},\tilde\beta_{\sigma(1)}\rangle^2}{\langle\tilde\beta_{\sigma(2)},x\rangle\langle\tilde\beta_{\sigma(3)},x\rangle}\partial_{\tilde\beta_{\sigma(1)}} \\
& - \sum_{i=1}^3\left(\frac{m(m+1)\prod_{k\neq i}\langle\tilde\alpha_i,\tilde\beta_k\rangle}{\langle\tilde\alpha_i,x\rangle^2} + \frac{3m(3m+1)\prod_{k\neq i}\langle\tilde\beta_i,\tilde\beta_k\rangle}{\langle\tilde\beta_i,x\rangle^2}\right)\partial_{\tilde\beta_i} \\
& + \sum_{i=1}^3 \frac{9m(m+1)(3m+1)\prod_{k=1}^3\langle\tilde\beta_i,\tilde\beta_k\rangle}{\langle\tilde\beta_i,x\rangle^3} + \sum_{i=1}^3\frac{m(m+1)(3m+1)\langle\tilde\beta_i,\tilde\beta_i\rangle\prod_{k\neq i}\langle\tilde\alpha_i,\tilde\beta_k\rangle}{\langle\tilde\beta_i,x\rangle\langle\tilde\alpha_i,x\rangle^2} \\
& - \frac{3(3m+1)(6m^2+6m+1)\prod_{i=1}^3\langle\tilde\beta_i,\tilde\beta_i\rangle}{2\langle\tilde\beta_1,x\rangle\langle\tilde\beta_2,x\rangle\langle\tilde\beta_3,x\rangle}.
\end{aligned} \tag{5.3}$$

Theorem 3. *The operators defined by formulas* (5.1), (5.2) *and* (5.3) *satisfy the intertwining relation*

$$H^r\mathcal{D}^r = \mathcal{D}^r H_0^r. \tag{5.4}$$

Similarly to the trigonometric case we derive quantum integrals in the factorised form, following [25]:

$$[H^r, \mathcal{D}^r D^{r*}] = 0, \quad [H_0^r, \mathcal{D}^{r*}\mathcal{D}^r] = 0.$$

Remark 2. *The operator H^r is the ordinary G_2 Calogero–Moser operator with multiplicities m, $3m+1$. Therefore the operator D^r is the rational version of the corresponding Opdam's shift operator in a suitable gauge for the G_2-orbit containing $\tilde{\beta}_i$ [26]. Hence the operator D^r can also be constructed via the product of the corresponding (rational) Dunkl operators $\nabla_{\tilde{\beta}_1}\nabla_{\tilde{\beta}_2}\nabla_{\tilde{\beta}_3}$ as it was demonstrated by Heckman for any root system in [27].*

6 Concluding remarks

We established integrability of the CMS system associated with the collection of vectors AG_2 and an arbitrary value of the parameter m. This configuration of vectors is interesting as it is an example of a slightly weakened notion of a root system. Indeed, the configuration is invariant under the Weyl group G_2 and the root vectors belong to the invariant lattice but the crystallgraphic condition between the root vectors is no longer satisfied. This makes it harder to study the corresponding CMS system as, for instance, we could not define (trigonometric) Dunkl operators with good properties for the model AG_2. Nonetheless integrability property appears to be present.

There are a number of further questions about this system. Firstly, it is natural to consider elliptic version and investigate its integrability. Secondly, it would be interesting to clarify whether the classical analogue of the sytem is integrable. In the case of root system G_2 Lax pairs for the corresponding CMS model were constructed in [28], [29] (see also [30]), which may be a starting point for approaching classical AG_2 CMS system. Another approach could be to investigate classical version of the quantum integral $\mathcal{D}\mathcal{D}^*$. On the other hand let us consider the operator $\hbar^2 H$ and take the limit $\hbar \to 0, m \to \infty$ such that $\hbar m \to const$. It is easy to see that the resulting classical Hamiltonian is the ordinary G_2 Hamiltonian. This suggests that the classical analogue of H where potential is the same as in the quantum case may be non-integrable.

Thirdly, it would be interesting to investigate bispectrality of the considered Hamiltonian H. More specifically, existence of the intertwining operator $\mathcal{D}$ implies that for integer m the Hamiltonian H has Baker–Akhiezer eigenfunction

$$\psi(k,x) = \mathcal{D}\phi(k,x), \quad H\psi(k,x) = (k_1^2 + k_2^2)\psi(k,x),$$

where $\phi(k,x)$ is the Baker–Akhiezer function for G_2 CMS system [11, 10], and $k = (k_1, k_2)$ is the spectral parameter. Bispectral dual Hamiltonian, if exists, would be an operator of Ruijsenaars–Macdonald type acting in k-variables of $\psi(k,x)$ so that $\psi(k,x)$ is its eigenfunction. In the root system case and for type A deformed CMS system such type of bispectrality is established in [31] (see also [32] for other examples).

We hope to return to some of these questions soon.

Acknowledgments. M.F. is very grateful to Oleg Chalykh for useful discussions and comments. We are grateful to the London Mathematical Society for the support through Undergraduate Research Bursary scheme which enabled us to carry out the main part of the work in summer 2018. M.V. also acknowledges matched funding from the School of Mathematics and Statistics, University of Glasgow.

References

[1] Calogero, F. "Solution of the one-dimensional n-body problem with quadratic and/or inversely quadratic pair potential", *J. Math. Phys.* **12**, (1971) pp. 419–436.

[2] Sutherland, B. "Exact results for a quantum many-body problem in one dimension", II. *Phys. Rev.* **A5**, (1972) pp. 1372–1376.

[3] Moser, J. "Three integrable Hamiltonian systems connected with isospectral deformations", *Adv. Math.* **16**, (1975) pp. 197–220.

[4] Olshanetsky, M. A.; Perelomov, A. M. "Completely integrable Hamiltonian systems connected with semisimple Lie algebras", *Inventiones mathematicae* **37(2)**, (1976) pp. 93–108.

[5] Olshanetsky, M. A.; Perelomov, A. M. "Quantum systems related to root systems and radial parts of Laplace operators", *Func. Anal. Appl.* **12**, (1978) pp. 121–128.

[6] Berezin, F. A.; Pokhil, G. P.; Finkelberg, V. M. "Schrödinger equation for a system of one-dimensional particles with point interaction", *Vestnik MGU* (1964).

[7] Wolfes, J. "On the three-body linear problem with three-body interaction", *J. Math. Phys.*, **15**, 1420 (1974)

[8] Heckman, G. J. "An elementary approach to the hypergeometric shift operators of Opdam", *Invent. Math*, **103** (1991), no. 2, 341–350.

[9] Opdam, E. M. "Root systems and hypergeometric functions. IV", Compositio Math. 67 (1988), no. 2, 191–209.

[10] Chalykh, O. A.; Styrkas, K. L.; Veselov, A. P. "Algebraic integrability for the Schrödinger equation and finite reflection groups", *Theoret. and Math. Phys.*, vol. **94** (1993), no. 2, pp. 182–197.

[11] Chalykh, O. A.; Veselov, A. P. "Commutative rings of partial differential operators and Lie algebras", *Comm. Math. Phys.*, vol. **126** (1990), no. 3, pp. 597–611.

[12] Veselov, A. P.; Feigin, M. V.; Chalykh, O. A. "New integrable deformations of the Calogero–Moser quantum problem", *Russ. Math. Surv.*, (1996) **51** 573.

[13] Chalykh, O.; Feigin, M.; Veselov, A. "New integrable generalizations of Calogero–Moser quantum problem", *J. Math. Phys.* **39**, 695 (1998).

[14] Sergeev, A. "Superanalogs of the Calogero operators and Jack polynomials", *J. Nonlinear Math. Phys.* **8** (2001), no. 1, pp. 59-–64.

[15] Sergeev, A. N. "The Calogero operator and Lie superalgebras", *Theoret. and Math. Phys.* **131** (2002), no. 3, pp. 747—764.

[16] Sergeev, A. N.; Veselov, A. P. "Deformed quantum Calogero–Moser problems and Lie superalgebras", *Comm. Math. Phys.* **245** (2004), no. 2, pp. 249—278.

[17] Sergeev, A. N.; Veselov, A. P. "Symmetric Lie superalgebras and deformed quantum Calogero–Moser problems", *Adv. Math.* **304** (2017), pp. 728—768.

[18] Feigin, M. "Generalized Calogero–Moser systems from rational Cherednik algebras", *Selecta Math. (N.S.)* **18**, (2012), no. 1, pp. 253-–281.

[19] Chalykh, O. A.; Feigin, M. V.; Veselov, A. P. “Multidimensional Baker-Akhiezer functions and Huygens’ principle", *Comm. Math. Phys.* **206** (1999), no. 3, pp. 533-–566.

[20] Chalykh, O. “Algebro-geometric Schrödinger operators in many dimensions", *Phil. Trans. R. Soc. A* **366** (2008), pp. 947–971.

[21] Chalykh, O.; Etingof, P.; Oblomkov, A. “Generalized Lamé operators", *Commun. Math. Phys.* **239** (2003), pp. 115–153.

[22] Fairley, A.; Feigin, M. “Trigonometric planar real locus configurations", in preparation.

[23] Feigin, M. “Trigonometric solutions of WDVV equations and generalized Calogero–Moser–Sutherland systems", *SIGMA* **5** (2009), 088.

[24] Taniguchi, K. “On the symmetry of commuting differential operators with singularities along hyperplanes", *Intern. Math. Res. Notices* **36** (2004), pp. 1845–1867.

[25] Chalykh, O.A. “Additional integrals of the generalized quantum Calogero–Moser problem", *Theoret. and Math. Phys.*, vol. **109** (1996), no. 1, pp. 1269–1273.

[26] Opdam, E. “Root systems and hypergeometric functions III", *Compositio Mathematica*, **67**, no 1 (1988), pp. 21–49.

[27] Heckman, G. “A remark on the Dunkl differential-difference operators", Barker, B; Sally, P (ed.), Proceedings of the Conference on Harmonic Analysis on Reductive Groups, Birkhäuser Progr. in Math. (1991) pp. 181–191.

[28] D’Hoker, E.; Phong, D. H. “Calogero–Moser Lax Pairs with Spectral Parameter for General Lie Algebras", *Nucl. Phys.*, **B530** (1998), pp. 537–610.

[29] Bordner, A. J.; Saski, R. “Calogero–Moser Models III: Elliptic Potentials and Twisting", *Prog. Theor. Phys.*, **101** (1999), pp. 799–829.

[30] Fring, A.; Manojlovic, N. “G(2)-Calogero–Moser Lax operators from reduction", *J. Nonlin. Math. Phys.*, **13** (2006), pp 467–478.

[31] Chalykh, O.A. “Bispectrality for the quantum Ruijsenaars model and its integrable deformation", *J. Math. Phys.* 41 (2000), no. 8, 5139–5167.

[32] Feigin M. “Bispectrality for deformed Calogero–Moser–Sutherland systems", *J. Nonlin. Math. Phys.* 12 (2005), suppl. 2, 95–136.