Use of the theory of dynamical systems for transient problems: application to the switching-on problem

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SUMMARY. This paper addresses the problem of dynamical systems with parameters varying in time (transient systems). A method to predict their behaviour is proposed. This class of transient systems can be seen as the evolution of an ordinary steady system into another ordinary steady system, for both of which the classical theory of dynamical systems holds. The evolution from a steady system to the other is driven by a transient force, which is regarded as a map between the two steady systems. We apply our method to a system which is subjected to a transient excitation, that is neither constant nor periodic, to simulate the effect of switching-on procedures.

1 INTRODUCTION

Dynamical systems are extremely pervasive in science and technology and are used to model natural and engineering phenomena. Very often mathematical models are autonomous, i.e. they do not entail an explicit dependence on time. Such a definition can be extended to include systems with periodic excitation as well, since time can be added as an extra-variable with constant time-derivative [1, 2]. In dissipative dynamical systems, which are of great use in engineering, it is possible to introduce a qualitative distinction between *steady states* and *transient states*. A steady state is characterised by recurrent behaviour, i.e. a particular point in the phase space is a steady state if the system, after sufficient time, returns arbitrarily close to the point [3]. This definition includes fixed points, limit cycles, quasi-periodic and chaotic steady states. A point in the phase space is a transient state if it is not a steady state. Both steady and transient states are typical of dissipative dynamical systems with time-invariant parameters and constant or periodic excitations, i.e. autonomous or periodically forced systems which therefore can be called 'steady systems'. It is apparent that many interesting phenomena cannot be described as steady systems since their explicit dependence on time is a crucial feature.

Transient systems, as opposed to steady systems, are the topic of a recent work by Galvanetto and Magri [4]. They are characterised by system and/or excitation parameters which change in time. Galvanetto and Magri [4], however, have studied only the effects of 'switching-off' and 'pure transient' forces. In the present paper, we limit our attention to a system which is transient because subjected to non-periodic excitations to simulate switching-on procedures.

2 A CLASSIFICATION OF TRANSIENT FORCES

We consider engineering systems which can be modelled with nonlinear ordinary differential equations, such as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \gamma), \tag{1}$$

where x is the state vector, γ is the set of system parameters, and t is the time. If the system is forced then an additional term appears in eq. (1):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, \gamma) + a \cdot p(t), \tag{2}$$

where a is the excitation amplitude and $p(t)$ is a normalised periodic function of time such that $p(t) = p(t + T)$, where T is the period. The unforced system is characterised by the set of system parameters, γ; whereas the forced system is additionally characterised by the excitation parameters such as a and those contained in the definition of $p(t)$. If both the system and excitation parameters are constant, the system is called a *steady system*. On the other hand, if the system parameters and/or the excitation parameters change in time, the system described by either eq. (1) or (2) is a *transient system*. In mechanics usual system parameters are mass, damping, stiffness; and excitation parameters can be the amplitude, frequency and phase of the excitation.

In this paper, we will assume that the system can operate in two steady conditions: one with no excitation, represented by eq. (1), and the second under the action of a periodic excitation, represented by eq. (2). Also, we will assume that in both conditions there exists only one *acceptable functioning mode*. For example, an acceptable functioning mode can be an attracting fixed point, when the excitation is not present, or a stable limit cycle, when the periodic excitation is present. Moreover, in the unforced system a *safe zone* exists around the stable fixed point, within its basin of attraction. The safe zone represents all configurations that the system can safely assume when in action and affected by perturbations or imperfections that keep it slightly away from the stable fixed point. Likewise, for a periodically forced system a *safe zone* exists around the stable limit cycle, within its basin of attraction. In this case, the safe zone represents all configurations that the system can safely assume when in action and affected by perturbations or imperfections that keep the system away from the stable limit cycle [5, 6].

We assume that the safe zones are limited sets so that they are contained by boundaries of finite length.

The idea of [4] and this work is that a transient problem can be regarded as a map between two steady systems: one is the initial system and the other the final system. We apply this idea to a case of transient problem in which the system parameters are constant and only the excitation amplitude, $a(t)$, can vary. The periodic function $p(t)$ is chosen to be a harmonic function with constant angular frequency, $p(t) = \cos(\omega t)$. Hence, the transition between the two steady systems is driven by a transient force with a known time-varying amplitude. In [4] only switching-off and transient forces were considered. In this paper we investigate switching-on forces (fig. 1) to extend the study by Galvanetto and Magri [4].

A *switching-on* transient force brings an unforced system to a forced condition in a finite time. This is the subject of the present study. During this time, the amplitude of the transient force varies from zero, at the beginning of the transient, to $a_0 \neq 0$, at the end of the transient. The switching-on problem is depicted in the sketch of fig. 1. The initial system is represented by eq. (1) and the final system by eq. (2). A *switching-off* transient force brings a forced system to an unforced condition in a finite time. During this time, the amplitude of the transient force varies from the value $a_0 \neq 0$, at the beginning of the transient, to zero, at the end of the transient. A *full*-transient force acts between two unforced systems for a finite time. During this time, the amplitude of the transient force varies continuously assuming the value 0 both at the beginning and the end of the transient. Both initial and final system are given by eq. (1). Similarly, the full-transient force can act also between two forced systems for a finite time. In this case the full-transient force can be viewed as a perturbation to the

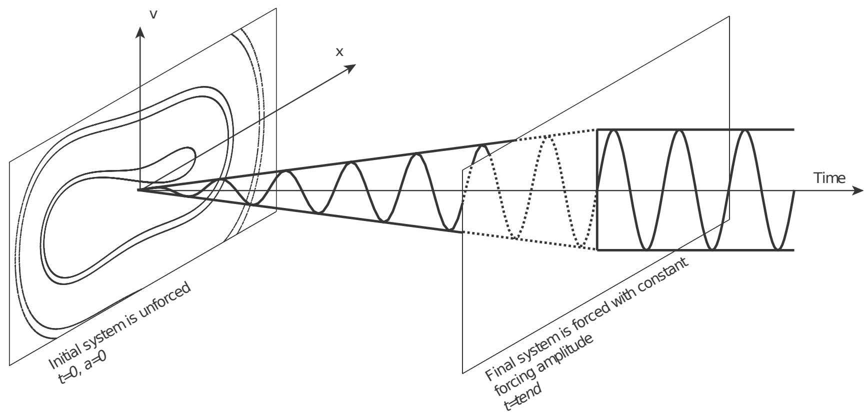

Figure 1: Sketch of a generic transient system with switching-on transient force. The stable manifolds of a saddle point are drawn in the phase plane of the unforced system.

steady excitation during which it can vary continuously assuming the value 0 both at the beginning and the end of the transient. In this case both initial and final system are given by eq. (2).

The transient problem is therefore characterised by a finite duration in time and it is bound by two steady systems. For switching-on transient forces as well as switching-off and transient forces [4], the theory of dynamical systems can be applied to the two steady systems, which exist before and after the transient, so that we can have a complete description of the dynamical features of the steady systems: attractors, basins of attraction, basin boundaries, etc.

In many engineering applications it is essential to determine if a transient between two steady conditions brings the system to an acceptable functioning mode or not. Therefore, the engineering point of view of a switching-on transient problem can be described in the following way: given an engineering system in a safe zone around a fixed point, if a driving switching-on force acts, will the system after the transient be operating in an acceptable condition? In other words, can we determine the set of initial conditions (at $t = 0$) which, under the action of the switching-on transient force, give origin to a trajectory ending up in the safe zone of the final system (for $t > t_{end}$, where t_{end} is the end of the transient duration)? The problem is illustrated in the sketch of fig. 2. In general, we can locate the steady states of the final system [7, 8], in particular its saddles (points or cycles) with the relevant stable manifolds. One of the stable steady states is acceptable and its safe region will therefore be inside its basin of attraction. To answer the afore-mentioned question, we have to integrate backward in time all points belonging to the boundary of the safe zone from t_{end} to $t = 0$ under the action of the time-varying-amplitude (transient) force. The image at $t = 0$ of the boundary of the safe zone of the final system (called *transformed boundary* in the remainder of the paper) will provide the boundary, in the initial condition plane ($t = 0$), of the set of safe points, called *safe transient initial conditions*. These are the points of the initial system that, if the transient force is activated, give origin to trajectories ending up in the safe region of the final system. It is particularly important to find if there exists an intersection between the safe zone of the initial system, that surrounds the acceptable functioning mode of the initial system, and the image at $t = 0$ of the safe zone of the final system (obviously contained in the transformed boundary), as shown in fig. 3. If such an intersection is empty then it is not possible to transfer the system from the initial safe zone

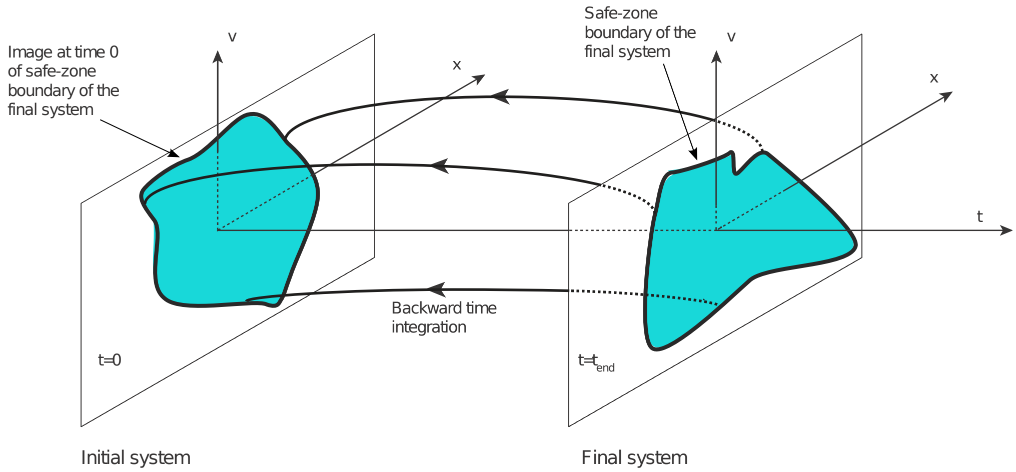

Figure 2: The boundary of the safe-zone at t_{end} is integrated backward in time to find its image at $t = 0$ [4].

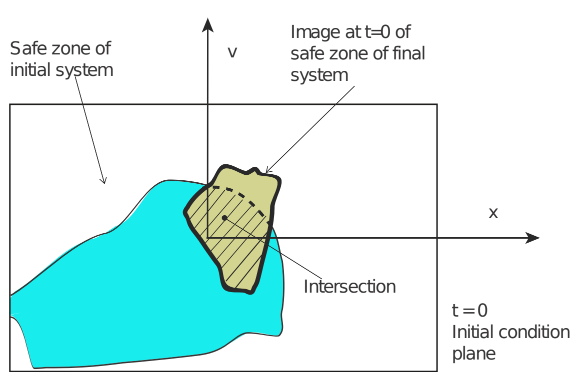

Figure 3: The image at $t = 0$ of the safe-zone at t_{end} may overlap the safe zone of the initial system only in part. If the intersection is empty, then it is not possible to bring the system from the initial safe zone to the final one under the action of the considered transient force [4].

to the final one under the action of the transient force.

3 A NONLINEAR SYSTEM AS A PROTOTYPE

In the present section we introduce a simple model as a prototype which satisfies all assumptions made in sec. 2. The main idea presented in sec. 2 will be applied to the prototype in sec. 4. This system is a nonlinear oscillator, chosen for its simplicity, and it is characterised by (at least) two coexisting attractors, one of which is representing an acceptable solution whereas the other is a solution which is assumed to be not acceptable.

The system is governed by the following equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} - k_1x + k_2x^2 + k_3x^3 = a \cdot p(t), \tag{3}$$

where $a \cdot p(t)$ is a function of time and represents the forcing term which may contain the transient force. The system parameters have the following values: $m = 1$, $c = 0.25$, $k_1 = k_2 = k_3 = 1$. The unforced nonlinear oscillator has three fixed points whose coordinates are $x_1 = 0$, $x_2 = (-1 + \sqrt{5})/2 \approx 0.618$ and $x_3 = (-1 - \sqrt{5})/2 \approx -1.618$. The potential $V$of the restoring force is as follows:

$$V(x) = -k_1\frac{x^2}{2} + k_2\frac{x^3}{3} + k_3\frac{x^4}{4} + \text{constant}, \tag{4}$$

which is shown in fig. 4. The constant in eq. (4) is assumed to be nought. The fixed point at the origin is a saddle point and $x_{2,3}$ are two point attractors (local minima of $V(x)$). The stable manifolds of the saddle point divide the phase plane of the unforced system in the two basins of attraction of the fixed points $x_{2,3}$, as shown in fig. 5. If a small-amplitude harmonic external force is added

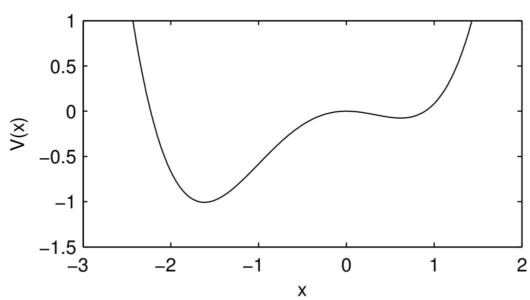

Figure 4: Potential of the restoring force.

to the system the fixed points are transformed in limit cycles of the same nature: the saddle point becomes a saddle limit cycle and the two point attractors become two attracting limit cycles. The stable manifolds of the saddle cycle divide the phase space of the forced system in the two basins of attraction of the attracting limit cycles. A Poincaré section of the phase space with angle phase of zero radians would provide a figure similar to that of the basins of attraction of the unforced system, so that fig. 5 is a good approximation of the basins of attractions of the forced system.

It is assumed that the fixed point $x_2 = (-1 + \sqrt{5})/2$ and the corresponding stable limit cycle represent the acceptable functioning modes of the system, whereas the fixed point $x_3 = -(1+\sqrt{5})/2$ and the corresponding stable limit cycle represent the unacceptable functioning modes. The system

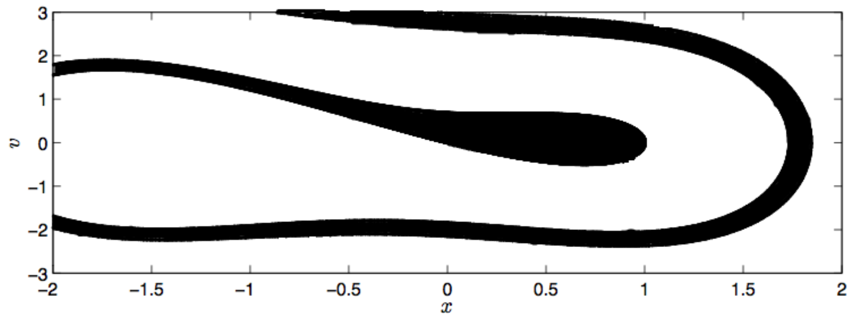

Figure 5: Basins of attraction of the attracting fixed points: the black region is the basin of x_2 and the white region is the basin of x_3[4].

is *off* when the external force $a \cdot p(t)$ is equal to zero for a certain interval of time. The system is *on* if $a \cdot p(t)$ is an harmonic force with constant amplitude, $a \cdot p(t) = a_0 \cos(\omega t)$.

The transition between the states *off* and *on* of the system takes place with a continuous variation in time of the amplitude $a(t)$, which is a function of time only during the transient and is constant before and after it.

The *switching-on* problem (fig. 1) is defined as follows:

1. the initial system is unforced, $a(t) = 0$ when $t < 0$;
2. $a(t) = a_0 \left(\frac{t}{t_{end}}\right)$ when $0 \le t \le t_{end}$;
3. the final system is forced with constant amplitude, $a(t) = a_0$ when $t > t_{end}$.

This problem is the subject of the present study. We chose the force amplitude to increase linearly, but any other continuous variation could be assumed. In fact, other force variation laws have been considered but no significant difference in the results from the linear case has been observed.

For completeness sake, we report also the definitions of swithicing-off and transient forces, as describe in [4]. The *switching-off* problem is defined as follows:

1. the initial system is forced with constant amplitude, $a(t) = a_0$, when $t < 0$;
2. $a(t) = a_0 \left(1 - \frac{t}{t_{end}}\right)$ when $0 \le t \le t_{end}$;
3. the final system is unforced, $a(t) = 0$ when $t > t_{end}$;

where t_{end} is the end of the transient, and $0 \le t \le t_{end}$ represents the duration of the transient.

Lastly, the *full-transient* problem is defined as follows:

1. the initial system is unforced, $a(t) = 0$, when $t < 0$;
2. the force varies continuously when $0 \le t \le t_{end}$;
3. the final system is unforced when $t > t_{end}$.

4 NUMERICAL RESULTS

The prototype model presented in sec. 3 is subjected to the switching-on force described in sec. 2. However, the method we propose can be applied conceptually in the same manner to other types of transient systems, once one knows i) the duration of the transient; ii) the mathematical law which describes the transient state; iii) the initial and final steady systems.

The transient-force parameters are $\omega = 3\ rad/s$ and $a_0 = 0.2$. In all examples of the present section the safe region of the initial system is chosen to coincide with the basin of attraction of the relevant acceptable attractor. At the beginning of the transient the forcing amplitude is zero and it increases linearly in time reaching the value a_0 after t_{end} seconds. We want to know the set of conditions in the initial phase plane (at $t = 0$) from which it is safe to switch on the excitation. The black line of fig. 6a represents the boundary of the safe zone of the forced system. This is the basin boundary (computed with a simple brute force technique) limited laterally by the line $x = -0.5$. The points of the boundary are integrated backward in time under the action of the excitation which, backward in time, is decreasing (fig. 1). More precisely, the integration backward in time of the boundary of the safe region of the forced system is performed choosing as initial conditions a set of closely spaced points along the black curve in fig. 6a and integrating the following system:

$$\dot{x} = -v, \tag{5}$$

$$\dot{v} = -\left(a(t)\cos(\omega t) - cv + x - x^2 - x^3)\right), \tag{6}$$

$$a(t) = a_0\left(1 - \tfrac{t}{t_{end}}\right), \tag{7}$$

$$0 \leq t \leq t_{end},\ t_{end} = 2\pi/\omega. \tag{8}$$

The integration of the above equations provides the transformed boundary shown as a black curve in fig. 6b. The same fig. 6b shows in grey the basin of attraction of the unforced system (initial system): there are points of the basin of attraction of the unforced system that are not safe in the transient case and, vice versa, points out of the basin of the unforced system which are safe with the transient switching-on force, as shown in fig. 6b-c and 7 . Figure 6c shows a magnified view of of fig. 6b. Two relevant points are chosen in order to show the effect of the transient force during the switching-on procedure. Point A (black circle) is inside the region of plane contained within the transformed boundary but outside the basin of attraction of the unforced system. Conversely, point B (black square) lies outside the region limited by the black curve but inside the basin of attraction of the unforced system. Therefore, if the periodic forcing were not switched on (no transient procedure), point A would be attracted by the unacceptable fixed point because it lies outside the basin of attraction of the unforced system. Contrariwise, point B would be attracted by the acceptable fixed point because B lies within the basin of attraction of the unforced system. This is not the case if the effect of the transient force with time-varying amplitude is considered. Time integration confirms this as shown in fig. 7. Results from time integration are shown both in the phase plane (fig. 7a) and as time histories (fig. 7b). The transient force representing the switching-on procedure, acts up to the vertical dotted lines in fig. 7b. The time integration shows that point A converges to the acceptable periodic attractor, whereas point B to the unacceptable periodic attractor of the final system. This is because point A belongs to the set of safe transient initial conditions, whereas point B does not. The engineering significance is that not all points which are safe when the system is off are also safe as conditions from which switching on the system.

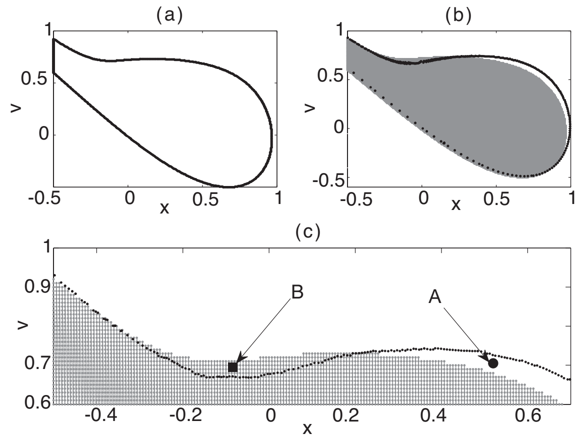

Figure 6: Switching-on problem: (a) safe-zone in the final system; (b) superposition of the transformed boundary of the safe-zone of the final system (dotted line) with the basin of attraction of the initial system (grey zone); (c) magnified view of a portion of figure (b).

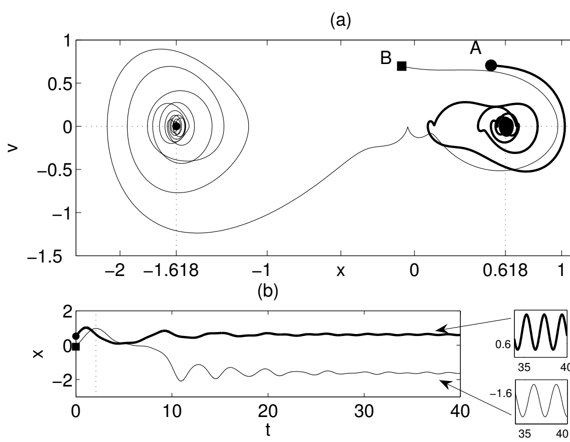

Figure 7: Switching-on problem: (a) phase portrait and (b) time histories of the two trajectories having points A and B of figure 6c as initial conditions.

4.1 Remarks on computations

The basin boundaries shown in this paper in some cases are those of a planar unforced system and in other cases belong to a three dimensional forced system. In the first case the basin boundaries are accurately located by selecting a large number of closely spaced initial conditions along the direction of the eigenvector corresponding to the stable manifold and integrating them backward in time [7, 8]. In the second case a brute force technique is used to find the basins of attraction and therefore the boundaries are approximately located with a bisection method applied to pairs of adjacent points of a grid in the plane of initial conditions. The basin boundaries of the 3-D systems could be computed in a more accurate way [7, 8, 9], but that would not contribute in any significant manner to the present work.

The reverse time integration has been carried out by changing the signs of the differential equations rather than by using a negative time step increment. This has been done to maintain in the reverse time integration the usual accuracy and stability properties of the time integration algorithms, which are usually implemented for forward integration.

5 CONCLUSIONS

This paper presents an approach to tackle transient problems by extending the tools and concepts of the theory of dynamical systems. In [4] and the present work we consider systems that are transient because subjected to transient forces. A transient force drives a system from a known initial steady state to a target final steady state in a finite time. The type of force considered in this work is a switching-on force which represents a generic switching-on procedure of a dynamical system. In other words, this type of force drives an unforced system to a forced condition.

The crucial point of this approach is to interpret the action of a transient force as that of a map acting between the two steady systems. The safe zone, e.g. part of the basin of attraction of a desired attractor, of the target system is mapped backward in time by integration. Doing so, we determine the set of initial conditions which, under the action of the transient force, give origin to a trajectory ending up in the safe zone of the target final steady system.

This conceptual method has been tested out on a simple mathematical model: a one-degree-of-freedom nonlinear oscillator.

The idea presented in the paper has been explained by assuming the system to be three-dimensional with states x, v and t. If the dimension of the system is larger, we lose the possibility to visualise the entire safe zones. The detection of the intersection would become a trickier problem, as the number of degrees of freedom increases. Moreover, the boundary of the safe zones would be of larger dimension as well. Therefore, the integration backward in time should be applied to a larger number of points, making the procedure more time-consuming.

The method presented, however, can be applied to any transient system, as defined by the authors, once one knows: i) the duration of the transient; ii) the mathematical law which describes the transient system; iii) the initial and final steady systems.

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