CONSERVATION LAWS FOR A NONLINEAR SYSTEM DESCRIBING THE PROPAGATION OF TRIADIC WAVES

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1 INTRODUCTION

When two waves, be they sound waves, waves in a liquid or electromagnetic waves, of different wavenumbers travel at the same speed, they may interact nonlinearly and resonance can arise. In this paper we shall be especially concerned with capillary-gravity waves (that is, waves which are subject to the twin restoring forces of gravity and surface tension) on the surface of deep water. However, our theory could just as well be applied to any physical wave system where the evolution of the waves is governed by an asymptotic balance between the effects of nonlinearity and dispersion. One obvious generalization is to interfacial waves between two fluids, but other examples are light pulses propagating along an optical fibre[2]; waves in a hot electron plasma[9,12] or 'beat' waves. In this report we shall be studying the case of 'one-three' (i.e. 'triadic') resonance when the interaction occurs between the fundamental mode and its third harmonic (i.e. a wave with a third of the wavelength of the fundamental). The particular system $(\mathcal{S})$ which shall be dealing with is given by the coupled equations :

$$iA_T + \frac{3}{32}A_{XX} - \frac{69}{8}|B|^2 A - \frac{77}{16}|A|^2 A + \frac{369}{16}BA^{*2} = 0, \qquad (\mathcal{S}a)$$

$$iB_T + \frac{11}{96}B_{XX} + \frac{783}{80}|B|^2 B - \frac{69}{8}|A|^2 B + \frac{123}{16}A^3 = 0. \qquad (\mathcal{S}b)$$

Here, A and B are the coefficients of the first and third harmonics respectively, X is a position variable, T is a time variable, the suffices represent differentiation with respect to the specified variable, $i^2 = -1$ and the star * denotes a complex conjugate. Further details and an outline derivation of $(\mathcal{S})$ will be provided in section 2. The third harmonic resonance under consideration here is of particular importance because this and the associated second harmonic resonance are those most likely to occur and be observable in nature, and these are also the easiest to reproduce in the laboratory. Indeed, by means of extremely accurate ripple-tank experiments, Perlin and Hammack[13] succeeded in creating third harmonic resonant water waves. These waves were excited by a wave-maker of frequency 8.4 Hz and they were measured to evolve with a wavelength of 2.99 cm. Others who have studied this phenomenon are Veheest[15], Nayfeh[11] and Jones[3,5,6] Despite these reports, however, the case of third harmonic resonance has, perhaps inexplicably when one considers its importance, attracted the attention of relatively few researchers.

The system $(\mathcal{S})$ bears a similarity to the better known pair of coupled nonlinear Schrödinger (NLS) equations which are obtained when the final term of each of the equations of $(\mathcal{S})$ is absent. Unlike the system under consideration here, these coupled NLS equations have received a good measure of attention in the literature, see for example Forest et al[1], Jones[4] and Wadati et al[16].

However the addition of these complicating extra terms has the effect of rendering the system (S) much more difficult to study, and its properties are totally different. For one thing, it might be noted that, unlike the standard coupled NLS case, there is no solution in which B is zero and A is non-zero. Recently, Jones[7] found explicit formulae for a whole family of travelling wave solutions to the standard coupled NLS system which exist for all values of the coefficients. Unfortunately, it does not appear to be possible to extend his methods to (S). Some explicit solutions of (S) are known, for instance some sinusoidal Stokes-type solutions were presented in Jones[3] and later a class of soliton type solutions was given in Jones[6]. However these solutions are of a rather restricted nature and comprise a much narrower set of solutions than those known to exist in the case of the standard coupled nonlinear Schrödinger equations.

It is the lack of explicit solutions obtained by purely analytical methods which leads us to employ alternative techniques in order to extract information concerning the properties of the equations. One such technique is the quest for *conservation laws*.

A conservation law may be defined as a statement that a particular integral quantity $\int_{-\infty}^{\infty} D(X,T)\,dX$ is a constant, i.e. "is conserved". In this investigation we shall derive three previously unknown such laws for the system (S), all of which have physical interpretations. We also make some suggestions for further avenues of research and speculate on the form of any subsequent conservation laws which may exist.

The coupled nonlinear Schrödinger equations are, of course, a generalization of the celebrated nonlinear Schrödinger equation (NLS) which, in canonical form, is $iU_T + U_{XX} + \alpha |U|^2 U = 0.$ The importance of this equation in acoustics, applied mathematics and physics can hardly be overestimated. Its relevance to the theory of water waves was first realized by Zakharov[17] who demonstrated that it may be used to describe the evolution of a nearly monochromatic wavetrain. Later Zakharov & Shabat[18] then showed that there are infinitely many conservation laws for the single NLS equation, and an obvious question is whether such laws exist for the coupled system.

2 THE EVOLUTION EQUATIONS

In this section we indicate the main steps in deriving of the system (S) of nonlinear differential equations which describe the physical scenario. Further details may be found in Jones[3], see also Nayfeh[11]. Our method is that of multiple scales and for other instances of the use of this method consult[4,8,12]. The scenario of interest is the two dimensional motion of an ideal fluid in a deep channel of infinite horizontal extent. We shall impose a two-dimensional (x, z) Cartesian co-ordinate system, so that in the undisturbed state the equation for the flat surface is given by $z = 0$. The fluid is constrained by gravity g, which acts in the negative z direction and surface tension which acts at the free surface. Of interest are the small amplitude waves which are formed by the interaction between the fundamental mode and its third harmonic. We shall normalize and assume the wave number of the fundamental mode is unity so that its wavelength is 2π while we denote its angular frequency by ω. We also introduce the small positive parameter ε which acts as a measure of the wave steepness. The motion is regarded as irrotational and hence there exists a velocity potential $\varphi(x,z,t)$ such that the velocity is given by $(u,v) = (\varphi_x, \varphi_y)$.

We also introduce a function $H(x, t)$ so that the wave profile is given by $\mathbf{z} = H$.

Classically, see e.g. Milne-Thompson[10], the velocity potential is a harmonic function and hence satisfies

$$\nabla^2 \varphi = 0, \quad z \le H. \tag{2.1}$$

The associated boundary conditions which apply on the free surface in this particular resonance are are (see Jones[3,5]) :

$$H_t - \varphi_z + \varphi_x H_x = 0, \quad z = H, \tag{2.2}$$

and

$$\varphi_t + \frac{3\omega^2}{4} gH + \frac{1}{2}(\varphi_x^2 + \varphi_z^2) - \frac{\omega^2 H_{xx}}{4(1+H_x^2)^{3/2}} = 0, \; z = H. \tag{2.3}$$

The next step is to expand the expressions for the and wave profile and velocity potential as power series in powers of the "small parameter" ε . The expression for the wave profile is :

$$H = \varepsilon A \cos(x - \omega t) + \varepsilon B \cos 3(x - \omega t) + O(\varepsilon^2) \tag{2.4}$$

and that for the velocity potential is similar. The crucial point to note is that the coefficients A and B are not constants but are functions of the "slow variables" $\varepsilon x, \varepsilon^2 x, ..., \varepsilon t, \varepsilon^2 t, ...$ only. We introduce the notation $X = \varepsilon x, T_0 = \varepsilon t$ for the "slow variables" and $T = \varepsilon^2 t$ for the "very slow variable". This notation may look a little peculiar, but it turns out that it is the variables X and T which assume the lions share in the calculations. The next stage in the derivation is to substitute the expansions of H and φ into the boundary conditions and match like terms. The terms of order ε are matched already by the choice of coefficients in the boundary conditions. When we match terms of order ε^2 we obtain an expression for the further coefficients of the expansion of H leading to the expression

$$\begin{aligned} H = \varepsilon A \cos(x - \omega t) + \varepsilon B \cos 3(x - \omega t) + 4\varepsilon^2 (4BA^* - A^2) \cos 2(x - \omega t) + \\ 8\varepsilon^2 AB \cos 4(x - \omega t) + \frac{12}{5} \varepsilon^2 B^2 \cos 6(x - \omega t) + O(\varepsilon^3). \end{aligned} \tag{2.5}$$

At this order, we also obtain the relations $A_{T_0} + \frac{3}{4} A_X = 0$ and $B_{T_0} + \frac{5}{4} B_X = 0,$ which explain why we never need to include derivatives with respect to T_0 in our equations. At order ε^3, when we match terms of the form $\cos(x - \omega t)$ and $\cos 3(x - \omega t)$, we finally obtain following system (*S)* of evolution equations which will be our main topic of interest in the remainder of this paper :

$$iA_T + \frac{3}{32} A_{XX} - \frac{69}{8} |B|^2 A - \frac{77}{16} |A|^2 A + \frac{369}{16} BA^{*2} = 0, \tag{Sa}$$

$$iB_T + \frac{11}{96} B_{XX} + \frac{783}{80} |B|^2 B - \frac{69}{8} |A|^2 B + \frac{123}{16} A^3 = 0. \tag{Sb}$$

The angular frequency ω has been eliminated from (S) by standard scalings. It now should be clear why the case of the 1-3 resonance is different from the others. For, if we consider the typical M, N resonance where $1 \le N < M$, then in general the quantities $N, M, 2N, 2M, M \pm N$ are distinct. However, this is not the case if $N = 1$ and $M = 3$ and accounts for the additional terms in (S). (Note that, for the same reasons, the case $N = 1, M = 2$ must also be considered separately. Here the interaction occurs at the quadratic level and the equations are completely different, see McGoldrick[8].)

It is possible to generalize the system (S) in a number of directions. First, observe that it is easy to generalize it to model three dimensional waves, we just need to include terms of the form A_{YY} and B_{YY} , where Y is the "slow variable" orthogonal to X. The system may also be extended to include interfacial waves in fluids. Examples can include that of wind blowing over the ocean, or internal flows such as those along a pipe transporting both oil and water, or the stratified waves which arise at an interface between fresh and salty water. In these cases, the coefficients are more complicated and consist of functions of V (the *velocity ratio* of the fluids) and ρ (the *density ratio* of the fluids). A further possible generalization is to flows in a channel of finite depth of one or both of the fluids. In this case a further function, the *mean flow term* must be included, and three evolution equations are required.

Throughout this paper we ignore the effects of dissipation and also assume that the resonance is exact. However, both these effects are important in reality, because all physical systems involve dissipation to a greater or lesser extent. Also, it is much more likely that the ratio of the wavelengths of the interacting waves is very close to the value three rather than being precisely equal three, or that the frequency of the wave generator is extremely near to the critical value rather than being exactly equal to it. Both these imperfections (and any other physical 'detuning') in the equations may be accounted for by adjoining terms of the form $i\delta A$ and $i\delta B$ to the evolution equations, where δ is a small parameter. For an account of dissipation in the case of the single NLS equation see Segur et al[14] , and for the standard coupled NLS system see Jones[7].

3. LAGRANGIAN FORMULATION

Our first main result is to show that the system (S) is Lagrangian. The Lagrangian density for (S) is given by :

$$\mathscr{L} = \frac{i}{2}\left(A^* A_T - A^*_T A + B^* B_T - B^*_T B \right) - \frac{3}{32} A^*_X A - \frac{11}{96} B^*_X B - \frac{77}{32}|A|^4 + \frac{783}{160}|B|^4 - \frac{69}{8}|A|^2|B|^2 + \frac{123}{16}A^3 B^* + \frac{123}{16}A^{*3}B. \tag{3.1}$$

The Euler-Lagrange equations are then :

$$\frac{\partial}{\partial T}\left(\frac{\partial \mathscr{L}}{\partial A^*_T}\right) + \frac{\partial}{\partial X}\left(\frac{\partial \mathscr{L}}{\partial A^*_X}\right) - \frac{\partial \mathscr{L}}{\partial A^*} = 0, \tag{3.2}$$

and

$$\frac{\partial}{\partial T}\left(\frac{\partial \mathscr{L}}{\partial B^*_T}\right) + \frac{\partial}{\partial X}\left(\frac{\partial \mathscr{L}}{\partial B^*_X}\right) - \frac{\partial \mathscr{L}}{\partial B^*} = 0 \tag{3.3}$$

which are easily seen to yield (S) and its complex conjugate.

4. CONSERVATION LAWS

A *conservation law* for the system (S) is a relation of the form $\frac{\partial}{\partial T}(D)+\frac{\partial}{\partial X}(F)=0,$ where D and F depend on $A,\ A^*,\ B,\ B^*$ and their derivatives. We call D the *density* and F the *flux*. We shall derive the existence of three such conservation laws for (S).

4.1 THE "MASS" CONSERVATION LAW

In order to derive this first law we multiply (S_a) by A^*, its conjugate by A and subtract. This, together with the analogous operations applied to (S_b), then yields the two equations :

$$i(A_T A^* + A_T^* A) + \frac{3}{32}(A_{XX} A^* - A_{XX}^* A) + \frac{369}{16}(BA^{*3} - B^* A^3) = 0 \tag{4.1}$$

and

$$i(B_T B^* + B_T^* B) + \frac{11}{32}(B_{XX} B^* - B_{XX}^* B) + \frac{123}{16}(A^3 B^* - A^{*3} B) = 0. \tag{4.2}$$

Observe first that in the absence of the final terms each of these equations would separately render a conservation law. However we must proceed by multiplying the second equation by 3 and adding upon which we obtain

$$i\{A_T A^* + A_T^* A + 3(B_T B^* + B_T^* B)\} + \frac{3}{32}(A_{XX} A^* - A_{XX}^* A) + \frac{33}{32}(B_{XX} B^* - B_{XX}^* B) = 0. \tag{4.3}$$

It is easy to verify that this may be re-expressed as

$$\frac{\partial}{\partial T}(|A|^2 + 3|B|^2) + \frac{3}{32}\frac{\partial}{\partial X}(A_X A^* - A_X^* A) + \frac{33}{32}\frac{\partial}{\partial X}(B_X B^* - B_X^* B) = 0, \tag{4.4}$$

which is our first conservation law.

Integrating this with respect to X and assuming sufficiently rapid decay as $X \to \pm\infty$ yields

$$\int_{-\infty}^{\infty} (|A|^2 + 3|B|^2)\, dX \quad = \quad C \tag{4.5}$$

where C is a constant, which indicates that "mass" is conserved.

This conservation law differs from the analogous one for the more general M, N resonance in that for the "triadic" situation considered here it is only the ***total*** mass $|A|^2 + 3|B|^2$ which is conserved, whereas in the general case the individual masses corresponding to each individual harmonic are themselves conserved. So even at this early stage, we may observe a fundamental difference between the "triadic" and other resonances.

4.2 THE MOMENTUM CONSERVATION LAW

In order to derive this second law, we first multiply (S_a) by A_X^*, its conjugate by A_X and add, which yields :

$$i(A_T A_X^* - A_T^* A_X) + \frac{3}{32}(A_{XX} A_X^* + A_{XX}^* A_X) -$$
$$\frac{77}{16}(|A|^2 AA_X^* + |A|^2 A^* A_X) - \frac{69}{8}(|B|^2 AA_X^* + |B|^2 A^* A_X) +$$
$$\frac{369}{16}(BA^{*2} A_X^* + BA^2 A_X) = 0. \quad (4.6)$$

Next, we differentiate (*Sa*) with respect to X and multiply by A^*, and analogously with the conjugate. Adding then results in :

$$i(A_{TX} A^* - A_{TX}^* A) + \frac{3}{32}(A_{XXX} A^* + A_{XXX}^* A) - \frac{77}{16}((|A|^2 A)_X A^* + (|A|^2 A^*)_X A) -$$
$$\frac{69}{8}((|B|^2 A)_X A_X^* + (|B|^2 A^*)_X A) + \frac{369}{16}((BA^{*2})_X A^* + (B^* A^2)_X A) = 0. \quad (4.7)$$

Clearly, there are two analogous equations corresponding to (*Sb*). Then subtracting (4.7) from (4.6), and proceeding similarly with the other equations, presents us with

$$i(A_T A_X^* - A_T^* A_X - A_{TX} A^* + A_{TX}^* A) + \frac{3}{32}(A_{XX} A_X^* + A_{XX}^* A_X - A_{XXX} A^* - A_{XXX}^* A) +$$
$$i(B_T B_X^* - B_T^* B_X - B_{TX} B^* + B_{TX}^* B) + \frac{11}{96}(B_{XX} B_X^* + B_{XX}^* B_X - B_{XXX} B^* - B_{XXX}^* B) -$$
$$\frac{77}{16}[|A|^2 AA_X^* + |A|^2 A^* A_X - (|A|^2 A)_X A^* - (|A|^2 A^*)_X A] +$$
$$\frac{783}{80}[|B|^2 BB_X^* + |B|^2 B^* B_X - (|B|^2 B)_X B^* - (|B|^2 B^*)_X B] -$$
$$\frac{69}{8}[|B|^2 AA_X^* + |B|^2 A^* A_X + |A|^2 BB_X^* + |A|^2 B^* B_X -$$
$$(|B|^2 A)_X A^* - (|B|^2 A^*)_X A - (|A|^2 B)_X B^* - (|A|^2 B^*)_X B] +$$
$$\frac{123}{16}[3(BA^{*2} A_X^* + B^* A^2 A_X) + A^3 B_X^* + A^{*3} B_X -$$
$$3((BA^{*2})_X A^* + (B^* A^2)_X A) - (A^3)_X B^* - (A^{*3})_X B] = 0. \quad (4.8)$$

After some simplification, this can be re-written as :

$$i\frac{\partial}{\partial T}(AA_X^* - A^* A_X + BB_X^* - B^* B_X) +$$
$$\frac{3}{32}\frac{\partial}{\partial X}(A_X A_X^* - A_{XX} A^* - A_{XX}^* A) + \frac{11}{96}\frac{\partial}{\partial X}(B_X B_X^* - B_{XX} B^* - B_{XX}^* B) +$$
$$\frac{77}{16}\frac{\partial}{\partial X}|A|^4 - \frac{783}{80}\frac{\partial}{\partial X}|B|^4 + \frac{69}{4}\frac{\partial}{\partial X}(|A|^2|B|^2) - \frac{123}{8}\frac{\partial}{\partial X}(A^{*3}B + A^3 B_X^*) = 0\,. \quad (4.9)$$

This is the second conservation law, from which we may deduce that

$$\int_{-\infty}^{\infty} (AA_X^* - A^* A_X + BB_X^* - B^* B_X) dX = C \tag{4.10}$$

where C is some constant.

This may be regarded as the conservation of "momentum".

4.3 THE "ENERGY" CONSERVATION LAW

In order to derive this law, we first multiply (*Sa)* by A_T^*, its conjugate by A_T and add. Doing the same to (*Sa)* and putting everything together gives us

$$\frac{3}{32}(A_{XX} A_T^* + A_{XX}^* A_T) + \frac{11}{16}(B_{XX} B_T^* + B_{XX}^* B_T) -$$
$$\frac{77}{16}(|A|^2 AA_T^* + |A|^2 A^* A_A) + \frac{783}{80}(|B|^2 BB_T^* + |B|^2 B^* B) -$$
$$\frac{69}{8}(|B|^2 AA_T^* + |B|^2 A^* A_T + |A|^2 BB_T^* + |A|^2 B^* B_T) -$$
$$\frac{123}{16}(3(BA^{*2} A_T^* + B^* A^2 A_T) + A^3 B_T^* + A^{*3} B_T) = 0.$$

(4.11)

It is not hard to see that this may be re-written as

$$\frac{\partial}{\partial X}[\frac{3}{32}(A_T^* A_X + A_T A_X^*) + \frac{11}{96}(B_T^* B_X + B_T B_X^*)] -$$
$$\frac{\partial}{\partial T}[\frac{3}{32} A_X A_X^* + \frac{11}{96} B_X B_X^* + \frac{77}{32}|A|^4 -$$
$$\frac{783}{160}|B|^4 + \frac{69}{8}|A|^2 |B|^2 + \frac{123}{16}(A^3 B^* + A^{*3} B)] = 0.$$

(4.12)

This is the third conservation law from which we may deduce

$$\int_{-\infty}^{\infty} \frac{3}{32} A_X A_X^* + \frac{11}{96} B_X B_X^* + \frac{77}{32}|A|^4 - \frac{783}{160}|B|^4 + \frac{69}{8}|A|^2 |B|^2 + \frac{123}{16}(A^3 B^* + A^{*3} B)\, dX = C$$

(4.13)

where C is some constant. This may be interpreted as the conservation of "energy". Observe that this is the first of our conservation laws to include the nonlinear terms of (*S).*

5 CONCLUSIONS

We have studied the properties of a system of coupled nonlinear partial differential equations which model a resonant interaction between the fundamental mode and its third harmonic. These equations are applicable to a wide variety of physical situations such as waves propagating through water, air, plasmas or optical fibres. Our first main result was to demonstrate that our system admits a Lagrangian formulation. Then we proceeded to derive three conservation laws representing the conservation of mass, momentum and energy. As first shown in[18], the single NLS equation possesses an infinity of conservation laws. However, even for the more standard (and much more studied) coupled NLS equations much less is known. Indeed Wadati et al[16] who found analogous laws to ours for that system stated that 'we could not find further conservation laws'. As an example of the type of results that might (or might not) exist we recall the single NLS equation which is $iU_T + U_{XX} + \alpha |U|^2 U = 0.$ The fourth conservation law for this single NLS equation is

$$\int_{-\infty}^{\infty} (UU^*_{XXX} + \frac{3}{2}\alpha |U|^2 UU^*_X) dX = C \ . \qquad (5.1)$$

where C is some constant. Finding an analogous result to this, even for the ordinary coupled NLS system as opposed to the one considered here, would be very interesting and it is hoped that someone may take up this challenging problem.

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