LOW ENERGY $\bar{K}$-MESON-NUCLEON SCATTERING AND THE ELASTIC SCATTERING OF PIONS BY ALPHA-PARTICLES.

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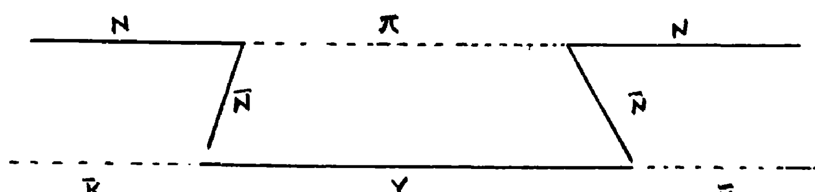
Summary.

The first part of the Thesis (Parts 1 and 2) describes a theoretical investigation on the S-wave $\bar{K}$-nucleon scattering. Although the $\bar{K}$-meson-nucleon interaction has received considerable theoretical study, the details of the mechanism remain obscure. In an interaction like this where absorptive processes are known to be quite strong, the requirements of unitarity imply a close relationship between the various channels in the reaction. It is therefore important in any consideration of $\bar{K}$-meson-nucleon scattering to take into account virtual processes involving pions and hyperons.

This investigation studies the possible importance of one particular virtual process namely scattering via the elementary virtual production of pairs. This is achieved by using a reduced Hamiltonian for which the processes

$$\pi \longleftrightarrow N + \bar{N} \qquad \bar{K} \longleftrightarrow Y + \bar{N}$$

are allowed and the only Feynman diagram for $\bar{K}$ + N elastic scattering is that shown in the figure below.



The state vector contains terms in the configurations $(\bar{K} + N)$, $(N + \bar{N} + Y)$ and $(\pi + Y)$ and the problem can be solved exactly to obtain the S-matrix for the reactions

$$\begin{aligned} \bar{K} + N \longrightarrow\ & \bar{K} + N \\ & \pi + \Lambda \\ & \pi + \Sigma \end{aligned}$$

Another version of the model in which $\pi \longleftrightarrow B + \bar{B}$ (where B is a baryon) is also allowed is investigated but clearly higher configurations can now occur and the model is not exactly soluble. Apart from its application to $\bar{K}$ + N scattering, the model is of interest in itself as it contains three open channels and an application of Tamm-Dancoff approximation leads to a system of coupled singular integral equations which have to be solved numerically. This was done on the electronic computer of the Glasgow University and both the models were studied for various combinations of coupling constants in the theory.

Because a reduced model Hamiltonian has been used, the coupling constants are not directly comparable with those employed in calculations using the full

Hamiltonian. However it has been found that for reasonable values of the coupling constants, a substantial fraction of the observed cross-sections can be obtained with this process. The model also correctly predicts the sign of the real part of the scattering amplitude for certain values of the coupling constants. The conclusion from this investigation is therefore that in S-states, pair production by $\bar{K}$-mesons and pions must be taken into account in a future relativistic theory.

The second part (Part 3) of the thesis describes a calculation on the elastic scattering of pions by alpha-particles. The interaction of pions with alpha particles has not been investigated theoretically so far and in the present study, a variational method which has been found quite successful in the pion-deuteron scattering has been applied to this problem. This method takes into account effects of multiple scattering quite simply and is an improvement over pure impulse approximation. The results of the calculations show that multiple scattering corrections are small and agreement with experimental results without such corrections is reasonably good.

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1.1 Introduction.

The problem of interaction of negative heavy mesons with nucleons is one of considerable theoretical interest. The experimental results available at present on the scattering and absorption of negative K-mesons on protons are limited. Theoretical investigations have therefore been principally concerned with fixing such parameters of the theory as the coupling constants and low energy scattering lengths. As is however well known, it has not been possible to construct as yet a satisfactory theory of strong interactions of elementary particles. Quantum field theory which has had such notable successes in electrodynamics can offer only a semi-quantitative description of low energy pion-nucleon systems. The success of the celebrated Chew-Low (1) theory appears now to be largely accidental and the point of view has been expressed that field theory is inconsistent and will eventually die. On the other hand, as Chew (2) has pointed out, many apparently valid general principles (such as the substitution law) have been discovered by studying the dubious concept of the local field. In any case, quantum field theory is the only apparatus we have at present for a detailed description of elementary particle interactions phenomena.

In pion physics, significant progress was made before the era of dispersion relations by considering the pion-nucleon interaction within the framework of various approximation schemes. The most straightforward approximation, of course, involves power series expansions of the scattering matrix in the coupling constant. In the case of meson theories, such an expansion is meaningless on account of the large coupling constant.

On the other hand, the usefulness and validity of the methods of Bethe and Saltpeter (3) and of Tamm (4) and Dancoff (5) have been extensively investigated. Chew (6) has applied the Tamm-Dancoff method to obtain the meson-nucleon potential. His calculation of the scattering with pseudovector coupling in the pseudoscalar meson theory was in reasonable agreement with experimental results.

Indications have been obtained that the K-meson-

Baryon coupling constants are very much weaker than the pion-nucleon coupling. It seems therefore reasonable to attempt to investigate K-meson-Baryon interaction phenomena by means of an approximate method like that of Tamm and Dancoff.

The difficulty in the study of strong interactions is not only the lack of a framework within which to make plausible approximations but in this particular case there is the added complication of eight coupling constants of which only one is known with any amount of certainty. Also since the relative parities of the K-mesons with the hyperons are not yet known, one is forced to make a choice between various possibilities. One may however proceed in either of two ways. One may use the possibility of determining the parity from K-meson-nucleon scattering data by using a zero angle dispersion relation. In this case one, of course, ignores the question of the validity of the dispersion relations which have not yet been rigorously proved and one also makes the assumption that the contribution from the unphysical region which is present even for forward angle due to the absorptive processes in the K^--p reaction is negligible. It has however been recognised that the structure of the unphysical region may be quite complicated, with the possibility of a cusp in the scattering amplitude where a new channel opens, although indications from perturbation calculations seem to favour a smooth extrapolation.

Alternatively, one can attempt to construct a complete dynamical theory in which from certain specified assumptions about the interaction Hamiltonian, one calculates cross-sections which are then compared with experiment. Various symmetry principles have been proposed from time to time to reduce the number of coupling constants and one of the most interesting in this connection is that of Gell-Mann (7). He adopted the point of view that, as a first approximation, one may neglect all the baryon mass differences thereby obtaining the highest degree of symmetry, which is later reduced by secondary perturbations. He replaced the isobaric spin singlet Λ and the triplet Σ by two doublets. The four baryon doublets thus obtained (including the N and Ξ doublets) were assumed to have exactly the same strong interaction with the π-mesons. The K-meson couplings were assumed to be moderately strong and responsible for the mass differences and different behaviour of the baryons. This is the well-known global symmetry of Gell-Mann.

It has however been shown by Salam (8) that if the elements of the T^{-1} matrix for the K^{-}-p interaction relating to pion-hyperon processes are taken from the pion-nucleon scattering, the results obtained for the Σ^{-}/Σ^{+} ratio do not agree with experiment. If however one uses the principle of restricted symmetry (9) that is the equality of π-Λ and π-Σ coupling constants without any reference to the pion-nucleon coupling, then it is found (10) that the results are quite consistent with the experimental data on hyperon production ratios.

It is therefore to be emphasised that in any attempt at calculation of the properties of the strong interaction, one is forced to make simplifying assumptions so that one eventually ends up with a rather simplified model. On the other hand, recently there has been a great promise of a very comprehensive dynamical theory in the double dispersion relations of Chew and Mandelstam (11). It has been recognised that in the theory of strong interactions pion-pion interactions play a central role and double dispersion relations have been used together with unitarity to provide a complete dynamical theory where although the underlying short range forces are not properly understood, long range interactions due to exchange of one or two-particle systems can be handled in a consistent way. This type of approach (2) has also opened up the possibility of deciding whether the Λ and Σ are 'elementary' or bound states.

In the present far less ambitious model, s-wave $\bar{K}$-N interaction is treated systematically on the assumption of pair creation in the intermediate states. From the comparison between the predictions and experimental cross-sections, we find that the model is quite promising for certain choice of coupling constants. The qualitative result of this investigation is therefore that in any realistic theory of low energy $\bar{K}$-N scattering, virtual pair creation should be explicitly taken into account since it accounts for a large part of the experimental situation.

1.2 Gell-Mann-Nakano-Nishijima Scheme.

It is customary to classify the elementary particles in the following way:

(a) The photon. Its rest mass is zero, spin 1 and it interacts with all charged particles through a universal constant e where $e^2/4\pi = 1/137$.

(b) Leptons. These are neutrino, anti-neutrino, electron, positron, negative and positive μ-mesons, light particles, spin $\frac{1}{2}$ and possessing no strong couplings.

(c) Mesons. These are Bosons of intermediate mass possessing strong couplings. There are two sub-groups; pions and K-mesons. Both occur with charges $\pm$ and 0.

(d) Baryons. These are fermions possessing strong couplings and satisfying the law of conservation of baryons. This law states that baryons can not be created or destroyed except in the baryon-anti-baryon pair production and annihilation. The baryons are divided into two subgroups (i) Nucleons comprising neutron and proton and (ii) Hyperons consisting of Λ, Σ and Ξ particles. The latter have masses greater than nucleons. All the baryons are expected to have anti-particles.

All the new particles i.e. the K-mesons and the hyperons have one rather surprising property in common. They are produced with a remarkably high abundance even in medium-energy collisions but have a relatively long life-time. In order to account for the large production cross-section, the interaction of K-particles and hyperons with pions and nucleons must be strong of the same order of magnitude as the pion-nucleon interaction. With such a large value of the interaction coupling constant, one would obtain for a hyperon a life-time of about 10^{-22} sec. On the other hand, the decay of a hyperon to a system of pions and nucleons is known to have a life-time of order 10^{-10} sec. or longer.

To account for this paradox of copious production and long life-time, Pais (12) was led to the hypothesis of associated production of strange particles. He pointed out that experimentally one never encounters the single production of a hyperon or a K-meson; at least two strange particles are always involved in the production process. Interactions in which only one strange particle is involved, as in the decay processes are weak. In other words, one has in general three types of elementary particle interactions: (i) strong interactions such as the pion-nucleon and nucleon-nucleon interactions, (ii) electromagnetic interactions and (iii) weak interactions as in beta decay.

The outstanding property of strong interactions is their charge independence or conservation of isotopic spin. A group of particles of nearly the same mass and other properties constitute an isotopic-spin multiplet. If there are $(2T + 1)$ particles in the group, then T is the isotopic-spin quantum number of the multiplet and each member of the multiplet is characterised by a value T_3 which takes on values $-T, -T+1, \ldots, T-1, T$. As T_3 varies, there is a variation in the electric charge which increases in steps of e as T_3 increases by one. For pions and nucleons, one can write quite generally

$$Q = T_3 + \tfrac{1}{2}N$$

where N is the number of nucleons. Nakano and Nishijima (13), Nishijima (14) and Sachs (15) have generalized this relation to include the hyperons as follows:

$$Q = T_3 + \tfrac{1}{2}(B + S)$$

where B is the baryon quantum number and S is the 'strangeness'. Nucleons and hyperons have B = 1, whereas the K-particles have B = 0. The assignment of strangeness is not unique. It depends on the assignment of isotopic spin. In the scheme of Gell-Mann (16) and Nishijima, the parameter S has the value 0 for pions and nucleons, +1 for K-mesons, -1 for $\overline{K}$-mesons, Λ and Σ hyperons and -2 for the cascade particle. The quantum number S is conserved by both strong and electromagnetic interactions; only the weak interactions can violate conservation of strangeness.

The assignment of isotopic spin and strangeness is summarized below.

Table

T	T_3	B	S	Q	Particle	Spin
1	1,0,-1	0	0	1,0,-1	π^+,π^0,π^-	0
$\frac{1}{2}$	$\frac{1}{2},-\frac{1}{2}$	1	0	1,0	p,n	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2},-\frac{1}{2}$	0	1	1,0	K^+,K^0	0
1	1,0,-1	1	-1	1,0,-1	$\Sigma^+,\Sigma^0,\Sigma^-$	$\frac{1}{2}$
0	0	1	-1	0	Λ	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2},-\frac{1}{2}$	1	-2	0,-1	Ξ^0,Ξ^-	$\frac{1}{2}$?

We have not included the anti-particles here. The general rule is that the strangeness of the anti-particles is negative of the strangeness of the particles. This leads to the interesting conclusion that there should exist two neutral K-mesons, one with strangeness +1 and the other with strangeness -1. This has been experimentally verified.

There are small mass differences among the members of the same family as, for example, the reported mass difference between $\overline{K}^{0}$ and K^{-} of 3.7 ± 0.7 MeV and that between Σ^{-} and Σ^{+} of 7.1 ± 0.4 MeV. We will disregard the small mass differences and utilize the isotopic spin formalism. Recently the property of charge independence has been experimentally verified (37) in the reactions $\pi^{+} + p \rightarrow \Sigma^{+} + K^{+}$, $\pi^{-} + p \rightarrow \Sigma^{0} + K^{0}$ and $\pi^{-} + p \rightarrow \Sigma^{-} + K^{+}$. Under charge independence, the cross-sections of these reactions satisfy the triangular inequality $\sqrt{\sigma(\Sigma^{0})} \leq \sqrt{\sigma(\Sigma^{+})} + \sqrt{\sigma(\Sigma^{-})}$. Previous experiments seemed to show a contradiction for backward produced Σ's but the new experiments have found no such contradiction.

1.3 Summary of Experimental Results.

A large body of experimental information on K^--meson decays and interactions has recently been published. Most of this has come from experiments with plates or with bubble chambers at the Berkeley bevatron. Freden, Gilbert and White (17) have collected all the presently available data, summarized and combined them with their new measurements at high energy. Earlier summaries have been given by Kaplon (18) and Ascoli et al. (19).

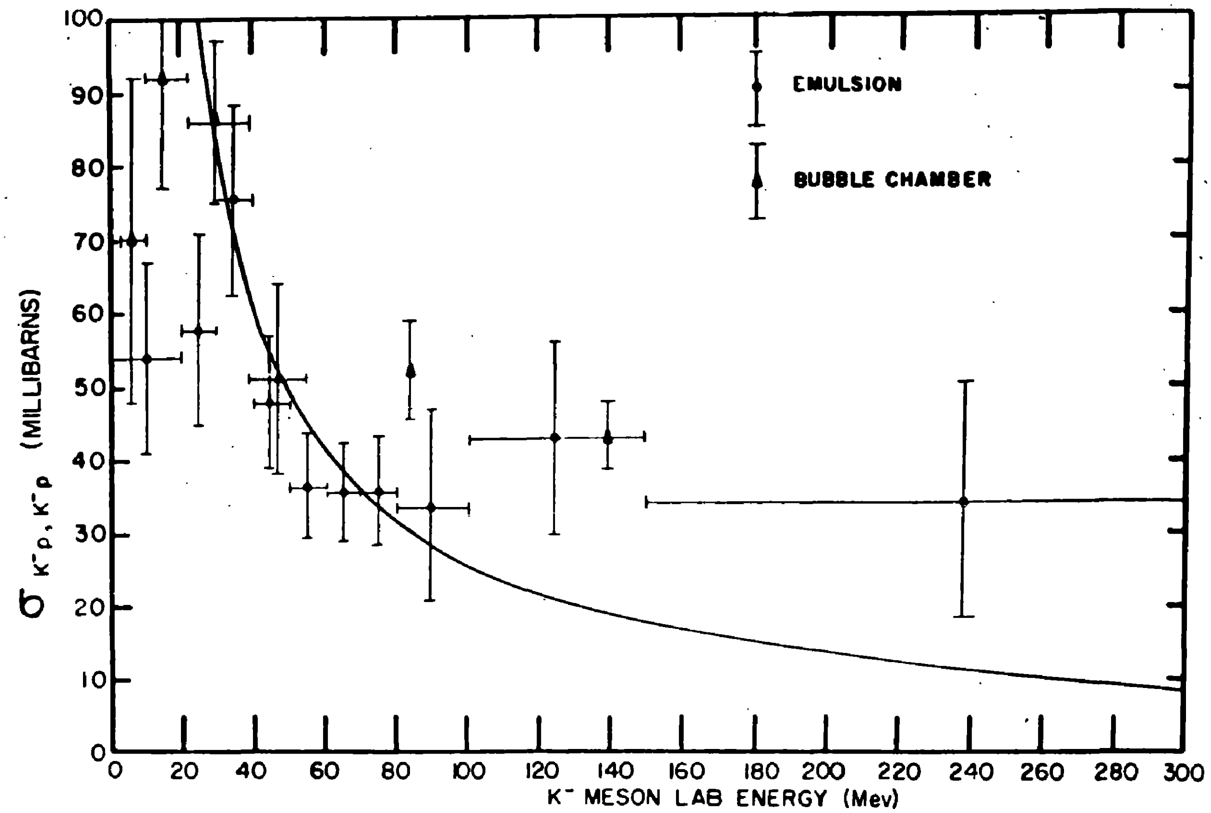
The following reactions of K^--mesons on free protons are allowed by conservation of baryons, charge and strangeness:

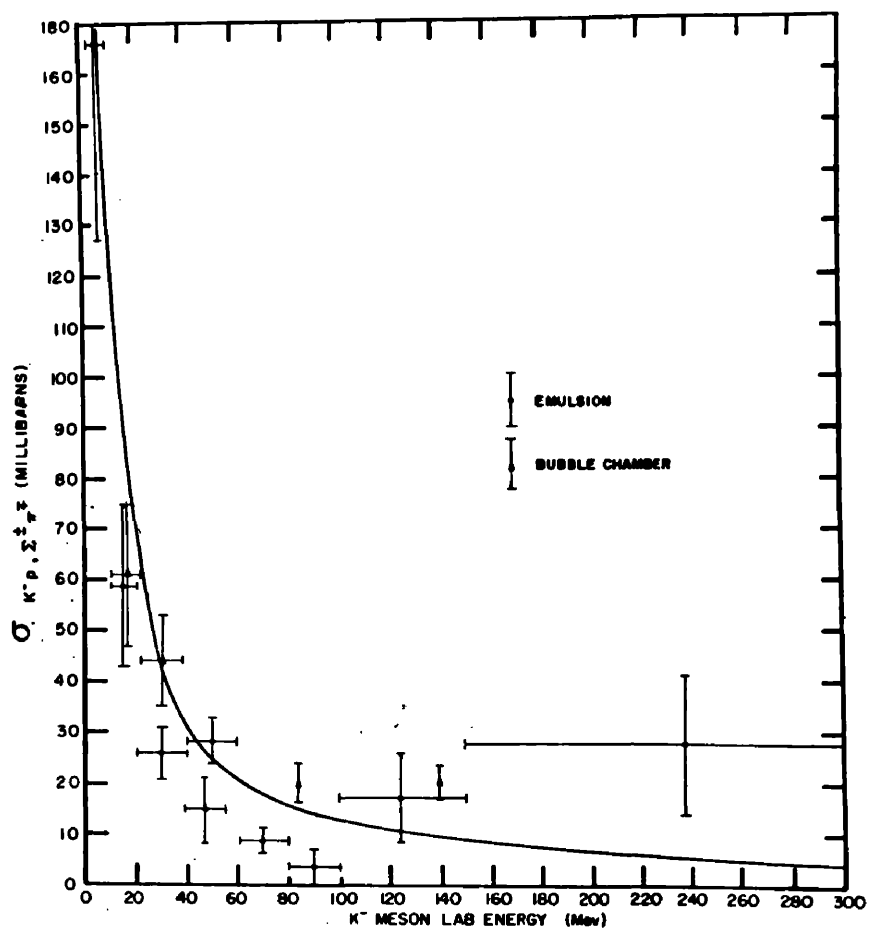
$$\begin{aligned} K^- + p \longrightarrow\ & K^- + p && (1)\\ & \bar{K}^0 + n && (2)\\ & \Sigma^+ + \pi^- && (3)\\ & \Sigma^- + \pi^+ && (4)\\ & \Sigma^0 + \pi^0 && (5)\\ & \Lambda + \pi^0 && (6) \end{aligned}$$

and those on neutrons are

$$\begin{aligned} K^- + n \longrightarrow\ & K^- + n && (7)\\ & \Sigma^- + \pi^0 && (8)\\ & \Sigma^0 + \pi^- && (9)\\ & \Lambda + \pi^- && (10) \end{aligned}$$

The total K^- + p elastic scattering cross-section


EMULSION
BUBBLE CHAMBER
σ K⁻p, K⁻p (MILLIBARNS)
K⁻ MESON LAB ENERGY (Mev)

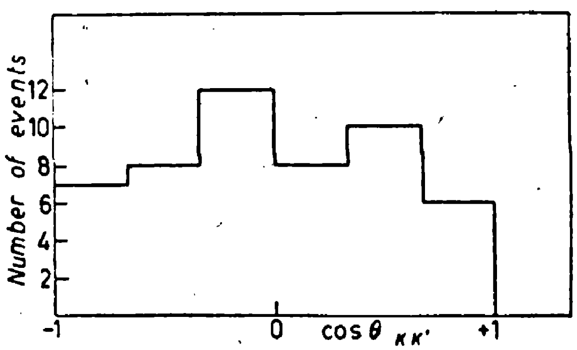

EMULSION
BUBBLE CHAMBER
σ K⁻p, Σ±π∓ (MILLIBARNS)
K⁻ MESON LAB ENERGY (Mev)

as a function of K^--meson incident energy in the laboratory system appears in Fig. 1. The curve for $\pi ƛ^2$ is also plotted in the figure where $ƛ$ is the K^--meson wavelength in the centre of mass system divided by 2π.

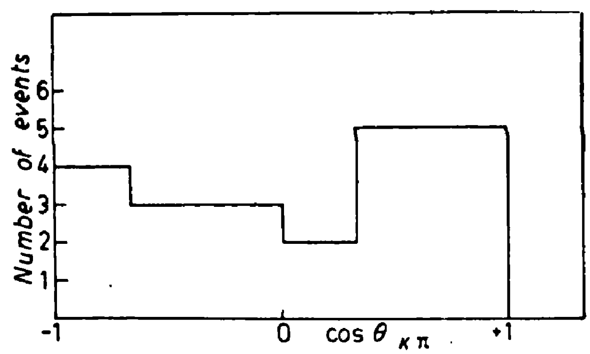
The main features of the (K^-p, K^-p) data are the possible peak in the cross-section at about 30 MeV, a sharp decrease to about 40 mb and then a fairly flat curve up to 300 MeV. However the suggestion of a peak can not be statistically substantiated specially because the emulsion data seem to show a levelling-off in the cross-section at about 30 MeV. It is important that more data be obtained in this region to settle this important point which has given rise to a resonance hypothesis in the K^--p scattering.

The K^--p charge exchange scattering has also been investigated (20). The cross-section rises from threshold, reaches a maximum of about 15 mb at 150 MeV/C and then decreases to 4 mb at 418 MeV/C (Fig. 2).

The total cross-section for capture to give $\Sigma^{\mp}\pi^{\pm}$ appears in Fig. 3 together with the curve $\pi ƛ^2/2$ as given in reference 17. The evidence for a decrease in



Angular distribution of K^--p elastic scattering. Angle $\theta_{KK'}$ is between the incoming K- and outgoing K-particles in the center of mass system. A distribution function: $1+2A \cos\theta+B(3\cos^2\theta-1)$ fitted to the histogram yields: $A = -0.06\pm0.4$ and $B=-0.05\pm0.2$.



Angular distribution of K^--p inelastic scattering. Angle $\theta_{K\pi}$ is between the incoming K-particle and the outgoing π-meson in the center of mass system. A distribution function $1+2A\cos\theta+B(3\cos^2\theta-1)$ fitted to the histogram yields: $A = 0.1\pm0.2$ and $B = 0.3\pm0.3$.

the inelastic K^--p scattering cross-section with increasing energy appears to be sound. The collected data on angular distributions of elastic and reaction cross-sections as given in reference 19 are shown in Figs. 3 and 4. These data are clearly consistent with isotropy.

If we believe this evidence of the angular distribution to be isotropic, this suggests that the interaction of K^--mesons with protons is predominantly S-wave. In discussing low-energy K^--p interaction therefore, we shall assume that S-wave processes are predominant and neglect all higher angular momentum waves.

1.4 Phenomenological Theory. Effective Range Approach.

The S-wave $\bar{K}$-N scattering amplitudes have been the subject of numerous discussions (21) from a phenomenological effective range point of view. From isotopic-spin consideration alone, one can write the cross-sections for elastic and charge-exchange scattering in the form

$$\sigma_{el.} = \frac{\pi}{4k^2} \left| \eta_0 e^{2i\alpha_0} + \eta_1 e^{2i\alpha_1} - 2 \right|^2 \tag{1}$$

$$\sigma_{c.e.} = \frac{\pi}{4k^2} \left| \eta_0 e^{2i\alpha_0} - \eta_1 e^{2i\alpha_1} \right|^2 \tag{2}$$

where $\delta_T = \alpha_T + i\beta_T$ is the complex phase-shift for isotopic spin T = 0,1 and $\eta_T = \exp(-2\beta_T)$. These phase-shifts are related to the scattering amplitudes by the relation

$$k \cot \delta_T = 1/A_T$$

where k is the centre of momentum and A_T is the scattering amplitude. The absorption cross-sections in T = 0, and T = 1 are given by

$$\sigma_T = \frac{\pi}{k^2} \left(1 - \eta_T^2 \right) \tag{3}$$

and are related to the cross-sections for hyperon production by

$$\sigma_0 = 6\,\sigma\,(\Sigma^0) \tag{4}$$

$$\sigma_1 = 2\,\sigma\,(\Sigma^+ + \Sigma^- + \Lambda) - 4\,\sigma\,(\Sigma^0) \tag{5}$$

One first determines the quantities η_0 and η_1 , from hyperon production cross-sections using eqns. (3) to (5) and one can then apply a simple graphical method (22) to obtain a_o and a_1.

Lacking detailed data on neutral hyperon production, Dalitz and Tuan (23) made the assumptions:

(a) $\sigma(\Lambda) = \epsilon\,\sigma_1 \qquad \epsilon = 0.2$

(b) $\dfrac{\beta_o}{\beta_1} = 2$

Using the experimental data at a laboratory momentum of 175 MeV/C, where $\sigma_{el.} = 86$ mb, $\sigma_{c.e.} = 14$ mb, $\sigma(\Sigma^+ + \Sigma^-) = 45$ mb, they found the four solutions:

1) $A_o = 0.2 + i\ 0.76 \qquad A_1 = 1.62 + i\ 0.38 \qquad$ (a+)

2) $A_o = 1.88 + i\ 0.82 \qquad A_1 = 0.4 + i\ 0.41 \qquad$ (b+)

and the solutions (a-), (b-) obtained by reversing the sign of the real parts of both A_o and A_1.

In an attempt to distinguish between these four solutions, Dalitz and Tuan have calculated the K^--p elastic scattering cross-sections neglecting Coulomb effects (Figs. 1 and 2).

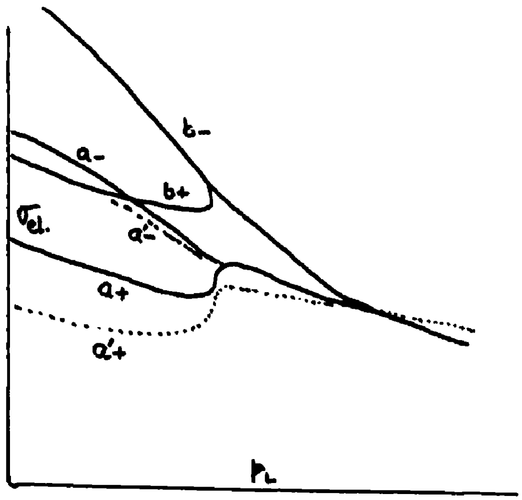


Fig. 1

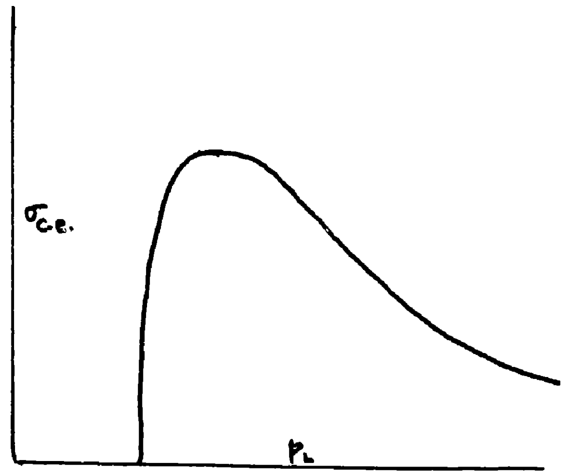


Fig. 2

Below the threshold for $\overline{K}^0$ production (90 MeV/C), the four sets of solutions show their greatest differences but in this momentum range it is very difficult to get accurate data so that it has not been possible to distinguish between the four solutions.

Jackson and Wyld (24) have calculated the K^--p elastic scattering cross-section including the Coulomb effects and found that solutions (a-) and (b-) more nearly follow the emulsion data, owing to the destructive interference with the Coulomb scattering. From this, it is concluded that K^--p interaction potential is repulsive which in turn leads to the conclusion from dispersion relations that K^--meson is scalar.

However, the re-analysis of the Berkely K^--p data presented by Alvarez (25) at the Kiev Conference shows that the Dalitz solutions are not at all precisely determined as was previously thought. The errors in the data are such that the above solutions can only be considered as tentative. It also appears that an angular distribution for elastic K^--p scattering at 172 MeV/C has been obtained at Berkeley which clearly shows a constructive interference between the nuclear and Coulomb scattering. This evidence will confirm the earlier indication that K^--p interaction is attractive and therefore that K^--meson is pseudo-scalar.

1.5 <u>Field Theory. Soluble Model.</u>

A simplified model which contains all the essential features of strong interactions and at the same time can be exactly solved has been studied by Amati and Vitale (26). It is essentially the Lee model (27), where the virtual K and π mesons in the intermediate states are disregarded and the rather unreliable approximation of no recoil is made to obtain a soluble theory.

Assuming a scalar K-meson, the model gives rise to the following integral equation

$$(\omega-\omega_0)\,\chi_T(k) = \int K_T(k,k')\,\chi_T(k')\,dk'$$

where the kernel is given by

$$K_T(k,k') = \frac{2G_Y^2 v(k)v(k')}{\sqrt{4\,\omega_k\,\omega_{k'}}}\;\frac{M_Y - M_N - \omega_0}{2\pi^2\,(M_Y-M_N-\omega_k)(M_Y-M_N-\omega_{k'})}$$

G_Y being the re-normalized coupling constant and $Y = \Lambda$ or Σ particle. Owing to the separability of the kernels, this integral equation can be solved by the Schmidt method and the phase-shift calculated from the equation

$$f_T(k_0) = \frac{1}{\pi}\tan\delta_T$$

The scattering cross-section then follows from the expression

$$\sigma(\omega) = \frac{4\pi}{k^2} \sin^2 \delta_T$$

With a choice of coupling constants $G_\Lambda^2/4\pi = G_\Sigma^2/4\pi = 0.3$ a cross-section of the order of 10 mb. is predicted for both isotopic spin states. Moreover the tangent of the phase-shift turns out to be negative indicating a repulsive K^--p potential for both values of the isotopic spin.

The reasons for the failure of the Tamm-Dancoff calculation of Amati and Vitale have been pointed out by Ceolin, Dallaporta and Taffara (28). Amati and Vitale have considered the lowest order diagram (Fig. 1) but on

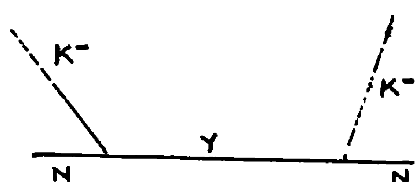


Fig. 1

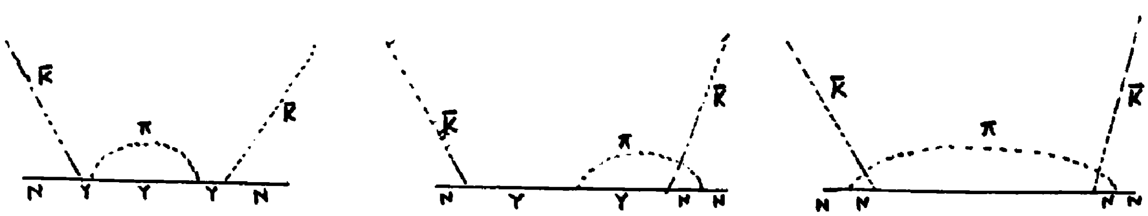
account of the possible capture reactions of negative K-mesons, it is not justified to treat scattering independent of capture. The Tamm-Dancoff calculation should take into account the virtual capture diagrams which can be included quite naturally in two-meson approximation.

The complete set of Tamm-Dancoff equations in this approximation is then

$$(E - M_N - \omega_k)\,\chi_{N\bar{K}}(k) = \int dk'\,\chi_{N\bar{K}}(k')\,\bar{K}_{N\bar{K}}(k,k') + \int dk'\,\chi_{\Lambda\pi}(k')\,K_{\Lambda\pi}(k,k') + \int dk'\,\chi_{\Sigma\pi}(k')\,K_{\Sigma\pi}(k,k') \quad .$$

and similar equations for $\chi_{\Lambda\pi}$ and $\chi_{\Sigma\pi}$. The different kernels are obtained in the usual manner.

This procedure essentially leads to the calculation of the following diagrams:



in addition to the diagram considered by Amati and Vitale. The results relating to the new terms are of opposite sign from the old term and of about the same magnitude and therefore contributes to correct the previous result in the right direction. No quantitative calculations were however made, since approximations required were rather drastic. Ferreira (29) has made a perturbation calculation of the third diagram above and finds that the contribution is always large enough to compensate the repulsive second order diagram of Amati and Vitale and to change it to an attraction. This is true for either scalar or pseudo-scalar case and a coupling constant $G_{\pi}^{2} = 1.5$.

We may thus conclude that in the scattering process of $\bar{K}$ with nucleons, pions and K-mesons should be considered in the intermediate states and will probably give an important contribution to the process. The neglect of all these could be justified if capture turned out to be a small fraction of scattering but experimentally it is known that at least at low energy this is not the case.

1.6. Dispersion Relation Theory.

Although the dispersion relations for K-meson scattering from nucleons have not yet been rigorously proved because of the difficulty of satisfying certain mass inequalities, we can formally write them down in complete analogy with the pion-nucleon case. For the forward elastic scattering, these relations take the form

$$D_+(\omega) = \frac{P}{\pi}\int_0^\infty \frac{A_+(\omega')}{\omega'-\omega}\,d\omega' + \frac{1}{\pi}\int_0^\infty \frac{A_-(\omega')}{\omega'+\omega}\,d\omega' \qquad (1)$$

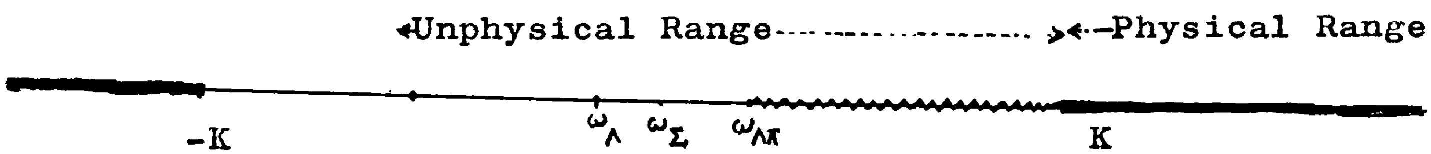
$$D_-(\omega) = \frac{P}{\pi}\int_0^\infty \frac{A_-(\omega')}{\omega'-\omega}\,d\omega' + \frac{1}{\pi}\int_0^\infty \frac{A_+(\omega')}{\omega'+\omega}\,d\omega' \qquad (2)$$

where ω is the incident K-meson energy in the laboratory system. For $\omega > K$ (K is the mass of the K-meson in units of $\hbar$ = c = pion mass μ = 1), we can use the optical theorem

$$A(\omega) = \frac{k}{4\pi}\,\sigma_{tot}(\omega) \qquad (3)$$

In order to make practical use of Eqns. (1) and (2), one must have information about $A_\pm(\omega)$ below the physical threshold $\omega = K$.

In the pion-nucleon forward scattering case, there is only a discrete bound state below the physical threshold. In the K-meson-nucleon case however $A_-(\omega)$ has contributions below the physical threshold from the continuum of πY and $2\pi Y$ states. These are graphically shown below.



Following the standard procedure, the contribution from the pole terms can be easily calculated and we get

$$A_-(\omega) = \frac{\pi}{2}\, \delta(\omega - \omega_Y)\, g^2_{NKY}\, (\omega + M \pm Y)$$

where

$$\omega_Y = \frac{Y^2 - M^2 - K^2}{2M}$$

and the positive sign is to be taken for the scalar case and the negative for the pseudo-scalar. It is this fortunate change of sign which makes the K-meson-nucleon dispersion relations so sensitive to the K-meson parity. We can then write down the dispersion relations in the final form

$$D_{\pm}(\omega) = \frac{X^{Y}}{\omega_{Y} \pm \omega} + \frac{1}{4\pi^2}\int_{K}^{\infty} k' d\omega' \left[\frac{\sigma_{+}(\omega')}{\omega' \mp \omega} + \frac{\sigma_{-}(\omega')}{\omega' \pm \omega} \right]$$

$$+ \frac{1}{\pi}\int_{\omega_{\Lambda\pi}}^{K} \frac{A_{-}(\omega')}{\omega' \pm \omega} d\omega'$$

where

$$X^{Y} = \frac{g_{NKY}^2}{4M}\left[(Y \pm M)^2 - K^2\right] \qquad \omega_{\Lambda\pi} = \frac{(\Lambda + \pi)^2 - M^2 - K^2}{2M}$$

Of course, the dispersion relations as written down here do not converge unless $\sigma(\omega)$ falls faster than $1/\omega$. To secure better convergence, various forms of subtracted dispersion relations have been proposed.

Matthews and Salam (30) have studied the dispersion relations in the form

$$D_{-}(K) - D_{+}(K) = \frac{1}{4\pi^2}\int_{K}^{\infty}\left[\sigma_{sc}^{-}(\omega') - \sigma_{sc}^{+}(\omega')\right]\left[\frac{1}{\omega' - K} - \frac{1}{\omega' + K}\right] k' d\omega'$$

$$+ \frac{1}{4\pi^2}\int_{\omega_{\Lambda\pi}}^{K} |k'| \, \sigma_{ab}^{-}(\omega')\left[\frac{1}{\omega' - K} - \frac{1}{\omega' + K}\right] d\omega' + B.S.$$

where B.S. means bound-state contributions. With the preliminary experimental results, they could only make a rough evaluation of the first integral in the energy range K-2K and gave arguments to show that the contribution from the second integral was negligible.

This is, however, by no means certain. Using their effective range solutions ($a^{\pm}$, $b^{\pm}$), Dalitz and Tuan have attempted an analytic continuation of $A-(\omega)$ into the unphysical region and found that the contribution may quite well be large. However, as has been pointed out, the solutions of Dalitz and Tuan are based on preliminary data and may well undergo significant changes with improved results from experiments.

Other forms of strongly convergent dispersion relations have been studied by Igi (31), Kerth et al. (32) and Amati-Vitale (33). Using a form of effective range relation proposed by Amati (34), Selleri (35) has discussed the K^{+}-p scattering data writing the cross-section in the form

$$\sigma^{+}(\omega) = \sigma^{+}(K) + b(\omega - K) \qquad K < \omega < 1.4K$$

A future determination of b is likely to decide whether K-meson is scalar or pseudo-scalar from the dispersion relation of Amati. If present indications of a weak dependence of $\sigma^{+}(\omega)$ on ω in the low-energy region is accepted, then K-meson is likely to be pseudo-scalar with respect to both hyperons.

Very recently Kycia, Kerth and Baender (36) have used a form of subtracted dispersion relation which has several advantages over all other previous forms. In their form, the cross-section integrals converge rapidly and depend more on $\sigma_+(\omega)$ than on $\sigma_-(\omega)$. These integrals converge even if $\sigma_\pm(\omega)$ go to a constant as ω goes to infinity. An additional advantage is that the real parts of the forward scattering amplitudes are used at energies at which they are known from experiment. Furthermore, the importance of the unphysical region is decreased in the form of dispersion relation used by these authors. In spite of all these advantages, the results of these authors indicate that even with the most recently available data it is difficult from subtracted dispersion relations to arrive at unambiguous conclusions as to the nature of the K-meson hyperon coupling.

1.7 Conclusions.

In the fore-going chapters, we have summarized the various investigations on $\bar{K}$-N scattering and have noticed that although considerable progress has been achieved in correlating the preliminary experimental data by means of zero-range analysis, not much success has been attained in fixing the parities or coupling constants of the K-mesons. Dispersion theoretic approach which has been so successful in the corresponding pion-nucleon phenomena is here plagued by a large unphysical region contribution which it is difficult to estimate properly. On the other hand, the zero-range analysis of the K^--p scattering data at 175 MeV/C has given four possible sets $(a\pm)$, $(b\pm)$ of the complex scattering amplitudes of which the constructive Coulomb-nuclear interference in K^--p scattering seem to favour the solutions (a+) and (b+). The elastic cross-sections from emulsion data show a maximum at about 20 MeV and this has been interpreted as due to destructive Coulomb-nuclear interference, as required by (a-) and (b-) amplitudes. Neither of these conclusions is at all convincing and clearly more experimental data will be required before the sign of the Coulomb-nuclear interference term can be settled.

As regards field theoretic calculations, a model for K-meson-nucleon scattering has been studied by Amati and Vitale but the results of such calculations are in complete disagreement with experiment. The failure of this model to explain the K-meson-nucleon cross-section is hardly surprising since it does not take recoil into account and neglects the effect of π-mesons in the intermediate states. The outstanding feature of the cross-section data available for the K^--p scattering and reaction processes is the strong absorption leading to pion hyperon states of all possible change combinations

$$K^- + p \rightarrow Y + \pi$$

where Y stands for Λ and Σ hyperons. For example, at a laboratory energy of 30 MeV for K^--meson, the cross-section for Σ^+ and Σ^- production amounts to 44 ± 8 mb. All of the differential cross-sections observed for the elastic scattering, charge exchange and reaction processes in K^--p collisions are found to be essentially isotropic at this energy. The available evidence therefore indicates that it is the s-state interaction which plays the dominant role in the K^--nucleon processes at low energy. We can therefore compare the observed absorption cross-section with $\pi \lambda\!\!\!^{-2}$, the geometrical maximum cross-section possible for s-wave interaction which is 103 mb. at this energy. Since the absorption cross-section is almost half the geometrical cross-section, the competition of these absorptive processes will indeed have a marked influence on the scattering processes. Quite generally, in any situation where the cross-section for the reaction processes reaches a considerable fraction of the geometrical limit, the requirements imposed by unitarity imply significant relationships between the reaction processes and the scattering processes in the various channels. Each strong scattering in one channel has an appreciable reactive effect on all other channels of the same quantum numbers.

The existence of such reactive effects casts doubt on the validity of perturbation methods for K-meson-nucleon interaction phenomena. This is clear from the fact that reactive effects do not appear in the lowest order term but are manifestly important in the higher-order terms of perturbation theory. Since perturbation theory is useful only when higher order terms are small, it follows that success of such calculation requires small reactive effects. Therefore the data on low-energy $\bar{K}$-N processes lead directly to the conclusion that perturbation expansions are of very doubtful validity in $\bar{K}$-N scattering. We have however constructed a model which explicitly takes pion-effects into account in the K^--p scattering using a state vector which comprises baryon-anti-baryon pairs and a variational method for obtaining the integral equations for the three coupled channels. This represents a generalization of the model of Bosco and Stroffolini (38) on s-wave pion-nucleon scattering. In the following chapters, we give details of calculation on the model.

Part Two.

2.8 The Lee Model and its Difficulties.

The exactly soluble model of a field theory constructed in 1954 by T.D. Lee (39) has been the subject of numerous careful investigations. Although various other examples (40) have been discussed since then, it is Lee's model which clearly brings out the meaning of renormalization and also reveals a number of difficulties of a fundamental nature in all these theories.

The model contains three types of particles - the V-particle, the N-particle and the θ-particle which transform into each other thus

$$V \longleftrightarrow N + \theta \tag{1}$$

and it is defined by the Hamiltonian

$$H = H_o + H_I \tag{2}$$

$$H_o = \sum_p m_v \psi_v^*(p)\psi_v(p) + \sum_q m_N \psi_N^*(q)\psi_N(q) + \sum_k \omega_k a_k^* a_k \tag{3}$$

$$H_I = \sum_{k,q} \frac{g_0}{\sqrt{2\omega\Omega}} \left[\psi_v^*(p)\psi_N(q)a_k + c.c.\right]\delta(p-k-q) \tag{4}$$

where $\omega_k^2 = \mu^2 + K^2$ and it has been assumed that the energy of the V and N particles does not depend on their momentum. In this respect, the Lee model is rather far from physical reality.

The commutation relations of the field operators are the conventional ones and there are two constants of motion which are

$$n_V + n_N = \text{const.} \quad \text{and} \quad n_V + n_\theta = \text{const.}$$

where n represents number operator for the respective field. Because of this situation, the problem is exactly soluble in a simple form. We can in fact express the physical V-state as a superposition of two states, one bare V-state and another state of N plus θ:

$$|\mathbf{V}\rangle = Z_2^{1/2}\left[\,|V\rangle + \sum_k f(k)\, a_k^* \,|N\rangle\right] \tag{5}$$

where $Z_2^{1/2}$ is a normalization constant and f(k) is proportional to the probability amplitude for finding a θ-particle in a physical V-state.

Application of the Schrodinger equation

$$H\,|\mathbf{V}\rangle = E_v\,|\mathbf{V}\rangle$$

together with the orthonormality of the bare eigenstates and a normalization condition for the physical

V-particle then yields:

$$f(k) = \frac{g_0}{\sqrt{2\omega\Omega}} \frac{1}{E_V - m_N - \omega} \tag{6}$$

$$m_V - E_V = \frac{g_0^2}{4\pi^2}\int \frac{k\,d\omega}{\omega + m_N - E_V} = \delta m_V \tag{7}$$

$$Z_2^{-1} = 1 + \frac{g_0^2}{4\pi^2}\int \frac{k\,d\omega}{(\omega + m_N - E_V)^2} \tag{8}$$

The divergent quantities δm_V and Z_2^{-1} serve to renormalize mass and coupling constant.

The next simple state we can solve for is $|N + \theta\rangle$ which can be written as

$$|N+\theta\rangle = c|V\rangle + \sum_k \chi(k)\, a_k^* |N\rangle \tag{9}$$

and on application of the Schrodinger equation, one obtains the integral equation satisfied by $\chi(k)$:

$$(\omega - \omega_0)\chi(k) = \frac{g_c^2}{16\pi^3}\int \frac{d^3k'}{\sqrt{\omega\omega'}} \frac{m_V - m_N - \omega_0}{(m_V - m_N - \omega)(m_V - m_N - \omega')}\chi(k') \tag{10}$$

where $g_c^2 = g_0^2 Z_2$. Since the kernel of the integral equation is separable, one can solve it easily and the s-wave phase-shift δ is given by

$$\frac{1}{\pi}\tan\delta = \frac{g_c^2}{4\pi}\frac{k}{(m_V - m_N - \omega)(1 + D(k))}$$

$$D(k) = \frac{g_c^2}{16\pi^3}\int \frac{m_V - m_N - \omega}{\omega'(\omega - \omega')(m_V - m_N - \omega')^2}\, d^3k' \tag{11}$$

Everything is finite here and this is the most significant fact that after performing the mass and coupling constant renormalization, the phase-shift for scattering turns out to be finite. In this model, only the V-particle operator requires normalization. In general, all fields have to be renormalized. The relation between g_o^2 and g_c^2 is

$$\frac{g_o^2}{4\pi} = \frac{\frac{g_c^2}{4\pi}}{1 - \frac{g_c^2}{4\pi} I} \tag{12}$$

where

$$I = \frac{1}{\pi} \int \frac{k\, d\omega}{(\omega + m_V - E_V)^2}$$

The integral I is divergent and if we use a cut-off in the integral, there will be a value of the cut-off for which $\left(\frac{g_c^2}{4\pi}\right) I > 1$ so that $g_o^2/4\pi < 0$ provided of course $(g_c^2/4\pi) > 0$. In this case g_o is imaginary and the Hamiltonian is no longer hermitian. Also Z_2^{-1} becomes $-\infty$ in the point-source limit and this contradicts the probability interpretation given for this quantity. Källen and Pauli (41) have shown that this is an essential difficulty of the model and introduction of the indefinite metric does not save the situation as the S-matrix is non-unitary. Heisenberg (42) has therefore favoured the idea of dividing all space into Hilbert space I containing normal states of the system and having a positive definite metric while the other part called Hilbert space II contains states of a different category. These latter states are composed of one normal state and one 'ghost state' of the same mass.

This kind of difficulty has given rise to the question whether such features are common to all field theories. The defect perhaps arises from the failure to build a theory with consistent transformation properties, e.g. in a fully relativistic theory after renormalization such divergences do not persist. Therefore a self-consistent field theory can perhaps be realised if every field operator and vertex function is renormalized as in electrodynamics.

2.9 A Model for S-wave Pion-Nucleon Scattering.

Of the several exactly soluble field theories discussed so far in the literature, Bosco and Stroffolini's (43) model for s-wave pion-nucleon scattering is closest to the model we are discussing. A variation of Lee's model was discussed by Machida (44) and more recently Goldstein (45). In each of these theories, the renormalization constants are cut-off dependent, implying an imaginary value for the unrenormalized coupling constant as the cut-off exceeds a certain critical value. For this situation, Kallen and Pauli have shown that a 'ghost state' is to be expected. Fried (46) has constructed a model in which the coupling constant renormalization is finite if the fermions of the theory are assumed non-relativistic. For the unrenormalized coupling constant to be real, the renormalized coupling constant must satisfy an inequality involving mass ratios; if this inequality is violated, a single boson 'ghost-state' appears.

In the model of Bosco and Stroffolini, the interaction Hamiltonian is that part of the fully relativistic pseudo-scalar Hamiltonian which corresponds to the process

$$\pi \longleftrightarrow N + \bar{N} \tag{1}$$

The physical meson state is defined by

$$|\Pi_s(k)\rangle = N^{1/2}(k)\Big[\,|\pi_s(k)\rangle + \sum_{p_1,p_2} f_s(p_1,p_2)\,|a^\dagger_{p_1} b^\dagger_{p_2}\rangle\Big] \tag{2}$$

where $a^\dagger_p$, a_p and $b^\dagger_p$, b_p are the creation and destruction operators for the nucleons and the anti-nucleons. Substituting this in the Schrodinger equation

$$H\,|\Pi_s(k)\rangle = w_k\,|\Pi_s(k)\rangle \tag{3}$$

and from the normalization condition one obtains as usual

$$(w_k - \omega_k - \delta\omega_k) = \frac{g^2}{\pi^2}\,\frac{1}{\omega_k(w_k - 2M)}\,\frac{(\xi M)^3}{3} \tag{4}$$

$$N^{-1}(k) = 1 + \frac{g^2}{\pi^2}\,\frac{1}{\omega_k(w_k - 2M)^2}\,\frac{(\xi M)^3}{3} \tag{5}$$

where nucleon recoil has been neglected and the cut-off ξ

has been introduced on the momenta. The renormalized coupling constant can be defined by the relation

$$g_c^2 = \lim_{k \to 0} N(k)\, g^2 \tag{6}$$

so that

$$g_c^2 = \frac{g^2}{1 + g^2 A(o)} \tag{7}$$

where

$$A(k) = \frac{1}{\pi^2} \frac{1}{\omega_k (w_k - 2M)^2} \frac{(\xi M)^3}{3} \tag{8}$$

The pion-nucleon scattering state is defined by

$$|N+\Theta\rangle = \sum_{s,p,k} \chi_s(p,k)\, a_p^\dagger |\Pi_s(k)\rangle + \sum_{p_1 p_2 p_3} \varphi(p_1 p_2 p_3) |\, a_{p_1}^\dagger a_{p_2}^\dagger b_{p_3}^\dagger \rangle \tag{9}$$

The integral equation satisfied by the amplitude $\chi_s(p)$ in the barycentric system is then obtained in the usual manner in the form

$$(M + \omega_p - E)\, h(p) \chi_s(p) = - \frac{g^2 N(p)}{4\pi^2} \tau^r \tau^s \int \frac{q^2 \chi_r(q)\, dq}{\sqrt{\omega_p \omega_q}\, (3M - E)} \tag{10}$$

where

$$h(p) = 1 - \frac{g^2 N(p)}{\pi^2} \frac{M + \omega_p - E}{\omega_p (\omega_p - 2M)^2 (3M - E)} \frac{(\xi M)^3}{3} \tag{11}$$

Separating into isotopic spin states and using

(6) and (7),

$$(M+\omega_p-E)h(p)\chi_{\frac{3}{2},\frac{1}{2}}(p)=\begin{pmatrix}-2\\1\end{pmatrix}\frac{g_c^2}{4\pi^2}\int\frac{q^2dq\ \ \chi_{3/2,1/2}(q)}{\sqrt{\omega_p\omega_q}\ (3M-E)\left[1+g_c^2\{A(p)-A(q)\}\right]} \tag{12}$$

writing

$$\chi(p) = \delta(M+w_p-E) + P\frac{1}{M+w_p-E}f(p)$$

and then solving by the Schmidth method for separable kernels, one obtains the phase-shifts for S-wave scattering.

It is found that the signs of the phase-shifts are given correctly and they are independent of the values of $g_c^2/4\pi$ provided the cut-off is chosen in a reasonable manner.

Using a set of s-wave pion-nucleon scattering phase-shifts given by Orear, Bosco and Stroffolini determined $g_c^2/4\pi$ to be equal to 1 and pointed out that this value is of the same order of magnitude as the renormalised constant obtained by Goldberger, Deser and Thirring. The conclusion then was drawn that the nucleon-anti-nucleon cloud of the pion is coupled with the pion much more weakly than the pion cloud is coupled to the nucleon. Moreover, since agreement with experimental results was quite reasonable, Bosco and Stroffolini suggested that s-wave pion-nucleon scattering at very low energy is mainly due to the production of a virtual pair in the cloud of the meson.

2.10 Fundamental Interaction Lagrangian and the Present Model.

In the Gell-Mann-Nishijima scheme, there is one iso-scalar Λ -hyperon, three isospinors

$$N = \binom{p}{n}, \quad \Xi = \begin{pmatrix} \Xi^{0} \\ \Xi^{-} \end{pmatrix} \qquad K = \binom{K^{+}}{K^{0}}$$

and two iso-vectors

$$\underline{\pi} = \begin{Bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{Bmatrix} \quad \underline{\Sigma} = \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix}$$

where as usual

$$\pi^{\pm} = (\pi_1 \mp i\, \pi_2)/\, 2 \qquad \pi_o = \pi_3$$

and similarly for $\Sigma^{\pm,0}$.

Assuming now

1) charge independence
2) charge and nucleon number conservation

and 3) three-field interactions (Yukaka coupling)

we can write down the strong interaction Lagrangian in the following form:

$$\begin{aligned} L_{strong} = {} & G_{NN\pi}\, \bar{N}\, (i\gamma_5)\tau\cdot\pi\, N + G_{\Xi\Xi\pi}\, \bar{\Xi}\, \Xi (i\gamma_5)\tau\cdot\pi \\ & + G_{\Lambda\Sigma\pi}\, \bar{\Lambda}\, (i\gamma_5)\, \pi\cdot\Sigma + h.c. + G_{\Sigma\Sigma\pi}\, i\, \underline{\Sigma}(i\gamma_5) \times \underline{\Sigma}\cdot\underline{\pi} \\ & + G_{NK\Lambda}\, \bar{N}\, \Gamma K \Lambda + h.c. + G_{NK\Sigma}\, \bar{N}\, \Gamma\, \tau\cdot\Sigma K + h.c. \\ & + G_{\Xi K\Lambda}\, \bar{\Xi}\, \Gamma' \dot{K} \Lambda + h.c. + G_{\Xi K\Sigma}\, \bar{\Xi}\, \tau\cdot\Sigma\, \dot{K} + h.c. \end{aligned}$$

where

$$\dot{K} = - i\tau_2 K^* = \begin{pmatrix} -\bar{K}^0 \\ K^- \end{pmatrix}$$

The symbols here have their usual meanings i.e. each represents the annihilation operator of the corresponding particle; the operator Γ is 1 or $i\gamma_5$ for even or odd K-meson-hyperon parity and Γ' is 1 or $i\gamma_5$ if the parity of Ξ is equal or opposite to the K-hyperon parity.

From isotopic spin considerations alone, a term

$$g_{KK\pi} K^* \tau \cdot \pi K' + h.c.$$

should also be included. As however π is pseudo-scalar such a KKπ-interaction can only be present if the two K-fields K, K' exist with opposite parity (Schwinger (47); Pais (48)). We do not consider this possibility in the present model. We will assume a pseudo-scalar K-meson and for simplicity neglect all effects due to the Ξ-particle so that the Lagrangian can be written as

$$L = G_{NN\pi} P_{NN\pi} + G_{\Lambda\Sigma\pi} P_{\Lambda\Sigma\pi} + G_{\Sigma\Sigma\pi} P_{\Sigma\Sigma\pi} + G_{\Lambda NK} P_{\Lambda NK} + G_{\Sigma NK} P_{NK\Sigma}$$

where

$$P_{NN\pi} = i\left[\left(\bar{p}\gamma_5 p - \bar{n}\gamma_5 n\right)\pi^0 + \sqrt{2}\left(\bar{p}\gamma_5 n\pi^+ + \bar{n}\gamma_5 p\pi^-\right)\right]$$

$$P_{\Lambda\Sigma\pi} = i\left[\bar{\Sigma}^0\gamma_5\Lambda\pi + \bar{\Sigma}^+\gamma_5\Lambda\pi^+ + \bar{\Sigma}^-\gamma_5\Lambda\pi^-\right] + h.c.$$

$$P_{\Sigma\Sigma\pi} = i\Big[\left(\bar{\Sigma}^+\gamma_5\Sigma^+ - \bar{\Sigma}^-\gamma_5\Sigma^-\right)\pi^0 + \left(\bar{\Sigma}^0\gamma_5\Sigma^- - \bar{\Sigma}^+\gamma_5\Sigma^0\right)\pi^+ + \left(\bar{\Sigma}^-\gamma_5\Sigma^0 - \bar{\Sigma}^0\gamma_5\Sigma^+\right)\pi^-\Big]$$

$$P_{\Lambda NK} = i\left[\bar{p}\gamma_5\Lambda K^+ + \bar{n}\gamma_5\Lambda K^0\right] + h.c.$$

$$P_{\Sigma NK} = i\left[\bar{p}\gamma_5\Sigma^0 K^+ - \bar{n}\gamma_5\Sigma^0 K^0 + \sqrt{2}\left(\bar{n}\gamma_5\Sigma^- K^+ + \bar{p}\gamma_5\Sigma^+ K^0\right)\right] + h.c.$$

Although it seems likely that the baryon masses, especially the mass splittings are consequences of strong interactions (Bransden and Moorhouse (49)), no attempt will be made here to discuss the masses in this sense. We shall consider the masses to be given quantities, even though future developments may show how these quantities originate.

The model we are considering is a generalization of the model of Bosco and Stroffolini and there are two versions of it. In the first version designated model A, we retain that part of the interaction Lagrangian which corresponds to the process

$$\pi \longleftrightarrow N + \bar{N}$$

and

$$\bar{K} \longleftrightarrow Y + \bar{N} \qquad (1)$$

The second version of the model is called model B, the additional interactions

$$\pi \longleftrightarrow \Sigma + \bar{\Sigma}\ ,\ \Sigma + \bar{\Lambda}\ ,\ \Lambda + \bar{\Sigma} \qquad (2)$$

are allowed.

In order to avoid difficulties connected with vacuum diagrams, we shall neglect the pair effect

$$\text{vacuum} \longleftrightarrow \pi + B + \bar{B}$$

where B stands for N, Σ, Λ.

It must be pointed out here that the neglect of the effects of Ξ -particle are not serious since in the present model they only contribute to the self-energy terms in which however we are forced to use a cut-off.

It will thus be seen that the principal objective of the present investigation is the possible importance of one particular virtual process, namely scattering via the virtual production of pairs (1) and (2) and hence the only Feynman diagram for $\bar{K}$-N elastic scattering considered is the following Figs. 1 and 2.

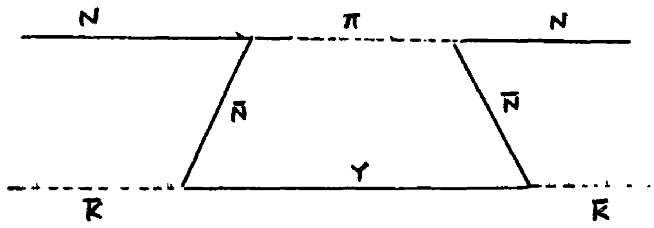


Fig. 1 (Model A)

No other graph is possible in this order. In the higher order, we may have Fig. 2

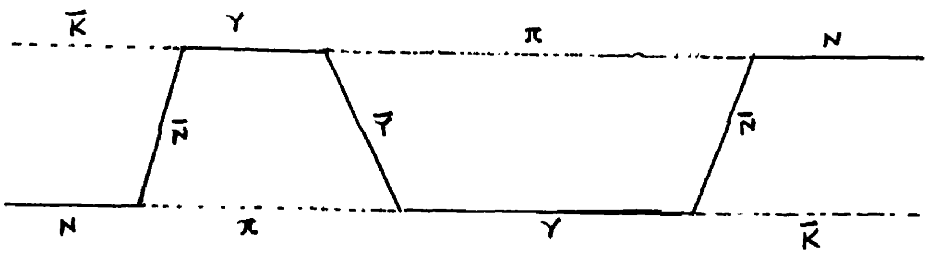


Fig. 2 (Model B)

Clearly higher configurations can occur in model B and the problem is no longer exactly soluble.

2.11 The Truncated Hamiltonian.

The total Hamiltonian of strong interaction can be written as

$$H = H_o + H_I$$

where the free-field Hamiltonian H_o is given by

$$\begin{aligned} H_o = {} & \sum_{p} E_N(p)\psi_N^\dagger(p)\psi_N(p) + \sum_{p} E_Y(p)\psi_Y^\dagger(p)\psi_Y(p) \\ & + \sum_{k} \omega_k \phi_\pi^*(k)\phi_\pi(k) + \sum_{k} w_k \phi_K^*(k)\phi_K(k) \end{aligned} \tag{1}$$

+ renormalization counter terms,

and the interaction Hamiltonian is given by

$$\begin{aligned} H_I = \sum_{p,q,k} \Big[& G_{NN\pi}\bar{\psi}_N(p)i\gamma_5 \tau^N\cdot\phi_\pi(k)\psi_N(q) + G_{\Lambda\Sigma\pi}\bar{\psi}_\Lambda(p)i\gamma_5 \pi(k)\psi_\Sigma(q) + h.c. \\ & + G_{\Sigma\Sigma\pi}\bar{\psi}_\Sigma(p)(i\gamma_5)\times\psi_\Sigma(q)\cdot\phi_\pi(k) + G_{NK\Lambda}\bar{\psi}_N(p)i\gamma_5\psi_\Lambda(q)\phi_K(q) + h.c. \\ & + G_{NK\Sigma}\bar{\psi}_N(p)i\gamma_5 \tau^\Sigma_\rho \psi_\Sigma^\rho(q)\phi_K(q) + h.c.\Big] \end{aligned} \tag{2}$$

The nucleon field operator can be expanded in the form

$$\psi_N(p) = \frac{1}{(2\pi)^{3/2}}\int\sqrt{\frac{M}{E_N(p)}}\sum_{r=1}^{3}\left[a_r^N(p)\,\omega_r(p)e^{ip\cdot x} + b_r^{\bar{N}\dagger}(p)\,v_r(p)e^{-ip\cdot x}\right]d^3p \tag{3}$$

where the a_N and $b^\dagger_{\bar{N}}$ are the destruction and creation operators of nucleons and anti-nucleons and the Dirac spinors $\omega(p)$ and $\upsilon(p)$ for positive and negative energy states respectively and are normalized in such a way that

$$\sum_{r} \bar{\omega}_r(p)\omega_r(p) = 1 \qquad\qquad \sum_{r} \bar{\upsilon}_r(p)\upsilon_r(p) = -1$$

The ω and υ satisfy the Dirac equation

$$(\gamma.p - iM)\,\omega(p) = 0$$

$$(\gamma.p + iM)\,\upsilon(p) = 0$$

We can similarly expand the hyperon fields.

The K-meson and the π-meson field operators can be decomposed in the form

$$\phi^{\alpha}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{1}{\sqrt{2\omega_k}} \left[c^{\alpha}(k)\, e^{ik.x} + c^{\alpha *}(k)\, e^{-ik.x} \right] d^3k \tag{4}$$

The conjugate momenta to ϕ^{α} is

$$\Pi^{\alpha}(x) = \frac{i}{(2\pi)^{3/2}} \int \sqrt{\frac{\omega_k}{2}} \left[c^{\alpha}(k) - c^{\alpha *}(k) \right] e^{-ik.x}\, d^3k$$

The energies of the baryons and the mesons are given by

$$E_B(p) = \sqrt{p^2 + B^2} \qquad w_k = \sqrt{k^2 + K^2} \qquad \omega_{\pi} = \sqrt{k^2 + \mu^2} \tag{5}$$

where B stands for the mass of the baryon, K is the mass of the hyperon and the mass of the pion is μ.

The commutation rules satisfied by the operators are as follows:

$$\left\{ a_r^B(p) \,,\, a_s^{B\dagger}(q) \right\} = \delta_{rs}\, \delta(p-q)$$

$$\left\{ b_r^{\bar{B}}(p) \,,\, b_s^{\bar{B}\dagger}(q) \right\} = \delta_{rs}\, \delta(p-q)$$

$$\left[c^{\alpha}(k) \,,\, c^{\alpha' *}(k') \right] = \delta_{\alpha\alpha'}\, \delta(k-k')$$

All other commutators or anti-commutators vanish.

Retaining only the terms corresponding to the processes

$$\pi \longleftrightarrow N + \bar{N},\ \Sigma + \bar{\Sigma},\ \Sigma + \bar{\Lambda}, \qquad \text{and} \quad \Lambda + \bar{\Sigma}$$

$$\bar{K} \longleftrightarrow N + \Sigma \quad \text{and} \quad \bar{N} + \Lambda$$

we obtain the interaction Hamiltonian of the model B.

$$\begin{aligned} H_{int} = \sum_{p,q,k} \Big[& G_{NN\pi} \left(a_N^{\dagger}(p)\, b_{\bar{N}}^{\dagger}(q)\, c_{\pi}(k)\, \Gamma_1^{\lambda}(p,q,k) + c.c. \right) \\ & + G_{\Lambda\Sigma\pi} \left(a_{\Lambda}^{\dagger}(p)\, b_{\bar{\Sigma}}^{\dagger}(q)\, c_{\pi}(k)\, \Gamma_2(p,q,k) + c.c. \right) \\ & + G_{\Sigma\Lambda\pi} \left(a_{\Sigma}^{\dagger}(p)\, b_{\bar{\Lambda}}^{\dagger}(q)\, c_{\pi}(k)\, \Gamma_3(p,q,k) + c.c. \right) \\ & + G_{\Sigma\Sigma\pi} \left(a_{\Sigma}^{\dagger}(p)\, b_{\bar{\Sigma}}^{\dagger}(q)\, c_{\pi}(k)\, \Gamma_4^{\rho}(p,q,k) + c.c. \right) \\ & + G_{NK\Lambda} \left(a_{\Lambda}^{\dagger}(p)\, b_{\bar{N}}^{\dagger}(q)\, c_{\bar{K}}(k)\, \Gamma_5(p,q,k) + c.c. \right) \\ & + G_{NK\Sigma} \left(a_{\Sigma}^{\dagger}(p)\, b_{\bar{N}}^{\dagger}(q)\, c_{\bar{K}}(k)\, \Gamma_6(p,q,k) + c.c. \right) \Big] \end{aligned} \tag{6}$$

where
$$\Gamma_1^{\lambda} = \sqrt{\frac{M^2}{2E_N(p)E_N(q)\omega(k)}}\;\bar{\omega}_N(p)\, i\gamma_5\, \tau_\lambda\, v_{\bar{N}}(q)\;\delta(p+q-k)$$
$$\Gamma_2 = \sqrt{\frac{\Lambda\Sigma}{2E_\Lambda(p)E_\Sigma(q)\omega(k)}}\;\bar{\omega}_\Lambda(p)\, i\gamma_5\, v_{\bar{\Sigma}}(q)\;\delta(p+q-k)$$
$$\Gamma_3 = \sqrt{\frac{\Sigma\Lambda}{2E_\Sigma(p)E_\Lambda(q)\omega(k)}}\;\bar{\omega}_\Sigma(p)\, i\gamma_5\, v_{\bar{\Lambda}}(q)\;\delta(p+q-k)$$
$$\Gamma_4^{\rho} = \sqrt{\frac{\Sigma^2}{2E_\Sigma(p)E_\Sigma(q)\omega(k)}}\;\bar{\omega}_\Sigma(p)\, i\gamma_5\, \tau_\Sigma^{\rho}\, v_{\bar{\Sigma}}(q)\;\delta(p+q-k)$$
$$\Gamma_5 = \sqrt{\frac{M\Lambda}{2E_\Lambda(p)E_N(q)w(k)}}\;\bar{\omega}_\Lambda(p)\, i\gamma_5\, v_{\bar{N}}(q)\;\delta(p+q-k)$$
$$\Gamma_6^{\rho} = \sqrt{\frac{M\Sigma}{2E_\Sigma(p)E_N(q)w(k)}}\;\bar{\omega}_\Sigma(p)\, i\gamma_5\, \tau_\Sigma^{\rho}\, v_{\bar{N}}(q)\;\delta(p+q-k)$$

In the following we will use the following projection operator

$$\Lambda_+^i(p) = \sum_r \omega_r^i(p)\,\bar{\omega}_r^i(p)$$
$$= \frac{\gamma\cdot p + M_i}{2M_i} = \frac{\beta E_i(p) + M_i - i\beta\gamma_5\,\sigma\cdot p}{2M_i}$$

and the large and small components will be factored out by the use of

$$\left(1 + \frac{i\gamma_5\,\sigma\cdot p}{E(p)+M}\right)$$

2.12 One Particle States.

The eigenstates of the total Hamiltonian $H = H_o + H_{int}$ are most conveniently expressed in terms of the free-particle states. It will be seen at once that the vacuum of bare particles is also an eigenstate of H with eigenvalue zero and that the bare baryon and anti-baryons are also eigenstates of H with eigenvalues $E_B(\mathbf{p})$:

$$\begin{aligned} H\,|0\rangle &= 0 \\ H\,\psi_B^\dagger(\mathbf{p})|0\rangle &= E_B(\mathbf{p})\,\psi_B^\dagger(\mathbf{p})|0\rangle \end{aligned} \tag{1}$$

Now because of the possibility

$$\pi \longleftrightarrow N + \overline{N}$$

the state of one clothed π-meson is no longer equal to the state of one bare meson. We can however express the physical one-meson state as the superposition of a bare meson and a nucleon-anti-nucleon pair state:

$$|\Pi_\lambda(-\mathbf{p})\rangle = N_3(\mathbf{p})\Big[\,|\pi_\lambda(-\mathbf{p})\rangle + \sum_{\mathbf{p},\mathbf{q}} F_1^\lambda(\mathbf{q},-\mathbf{q}-\mathbf{p})\,a_r^{N\dagger}(\mathbf{q})\,b_s^{\bar{N}\dagger}(-\mathbf{q}-\mathbf{p})|0\rangle\Big] \tag{2}$$

in the centre of mass system. Here $N_3(\mathbf{p})$ is the normalization constant and $F(\mathbf{q},-\mathbf{q}-\mathbf{p})$ represents the probability amplitude of finding a nucleon-anti-nucleon pair in the physical π-meson state.

It is now required that the above physical pion-state be an eigenstate of H belonging to the eigenvalue $E_\pi(\mathbf{p})$ so that we must solve the Schrodinger equation

$$(H_o + H_{int}) \,|\, \Pi_\lambda(-\mathbf{p})\rangle = E_\pi(\mathbf{p}) \,|\, \Pi_\lambda(-\mathbf{p})\rangle \tag{3}$$

This leads to the result

$$F_1^\lambda(q,-q-p) = -\frac{\Gamma_1^\lambda(q,-q-p)}{E_N(q) + E_N(p+q) - E_\pi(p)} \tag{4}$$

where

$$\Gamma_1^\lambda(q,-p-q) = \frac{G_{NN\pi}}{(2\pi)^{3/2}} \sqrt{\frac{M^2}{2E_N(q)E_N(p+q)\omega(p)}}\; \bar{\omega}_N(q)\, i\gamma_5 \tau^\lambda v_N(p+q)$$

Also since $\omega_\pi(\mathbf{p})$ is the energy of the bare-meson

$$\omega_\pi(p) - E_\pi(p) = -\int \Gamma_1^{\lambda *}(q,-p-q)\, F_1^\lambda(q,-p-q)\, d^3q$$

$$= \frac{G^2_{NN\pi}}{16\pi^3} \int \frac{M^2}{E_N(q)E_N(p+q)\omega(p)} \frac{|Q(p,q)|^2}{E_N(q) + E_N(p+q) - E_\pi(q)}\, d^3q$$

where

$$|Q(p,q)|^2 = \sum_{r,s} \left[v_s^{\bar{N}}(p+q)\gamma_5 \tau^\lambda \omega_r^N(q)\right]\left[\bar{\omega}_r^N(q)\gamma_5\tau^\lambda v_s^{\bar{N}}(p+q)\right]$$

$$= \eta\, \frac{q\cdot(p+q) + M^2}{M^2}$$

where $\eta = 2$ is the isotopic spin factor. Thus the self-energy of the π-meson is given by

$$\omega_\pi(p) - E_\pi(p) = -\frac{G^2_{NN\pi}}{16\pi^3}\eta \int \frac{d^3q}{E_N(q)E_N(p+q)\omega(p)} \frac{M^2 + E_N(q)E_N(p+q) - q\cdot(p+q)}{E_N(q) + E_N(p+q) - E_\pi(p)} \tag{5}$$

It will thus be seen that the mass renormalization for the π-meson here is more singular, in the limit of no cut-off, than the corresponding V-particle mass renormalization found by Lee. The reason is that we have not neglected the momentum dependence of the nucleon energy as was done by Lee.

Now from the normalization condition

$$\langle \Pi(-p) | \Pi(-p) \rangle = 1$$

we have

$$N_3^{-2}(p) = 1 + \frac{G^2_{NN\pi}}{16\pi^3}\eta \int \frac{d^3q}{E_N(q)E_N(p+q)\omega(p)} \frac{M^2 + E_N(q)E_N(p+q) - q\cdot(p+q)}{\left[E_N(q) + E_N(p+q) - E_\pi(p)\right]^2} \tag{6}$$

We now define the constant of charge renormalization by

$$Z_3 = \lim_{p\to 0} N_3^2(p)$$

and the renormalized coupling constant by

$$\bar{G}^2_{NN\pi} = Z_3 G_{NN\pi} \tag{7}$$

Hence in the present model, the renormalized coupling constant is given by

$$\bar{G}^2_{NN\pi} = \frac{G^2_{NN\pi}}{1 + G^2_{NN\pi} A(0)}$$

where

$$A(0) = \eta \int \frac{d^3q}{16\pi^3} \frac{2M^2}{\omega_\pi(0)\, E_N^2(q) \left(2E_N(q) - \mu\right)^2} \tag{8}$$

If we neglect recoil and introduce a cut-off, we of course get the expression given by Bosco and Stroffolini.

In an exactly similar fashion, the state of one clothed $\bar{K}$-meson can be found and the renormalized K-meson coupling constants defined. The physical $\bar{K}$-meson state is given by

$$|\bar{K}(-p)\rangle\rangle = N_4(p)\Big[\, |\bar{K}(-p)\rangle + \sum_{p,q,Y} f_Y(q, p-q)\, a_Y^{Y\dagger}(q)\, b_s^{\bar{N}\dagger}(-q-p)|0\rangle \Big] \tag{9}$$

Hence as before we obtain from the Schrodinger equation

$$H\, |\bar{K}(-p)\rangle\rangle = E_K(p)\, |\bar{K}(-p)\rangle\rangle$$

the self-energy of the $\bar{K}$-meson in the form

$$w_K(p) - E_K(p) = \sum_Y \eta_Y \frac{G^2_{NKY}}{16\pi^3} \int \frac{d^3q}{E_Y(q) E_N(p+q)\, w(p)} \frac{MY + E_N(q) E_N(p+q) - q\cdot(q+p)}{E_Y(q) + E_N(p+q) - E_K(p)}$$

$$\eta_\Lambda = 1 \qquad \eta_\Sigma = 3 \tag{10}$$

and from the normalization condition we get

$$\bar{N}_4^2(p) = 1 + \sum_Y \frac{G^2_{NKY}}{16\pi^3} \int \frac{d^3q}{E_Y(q)E_N(p+q)w(p)} \frac{MY + E_Y(q)E_N(p+q) - q\cdot(p+q)}{[E_Y(q) + E_N(p+q) - E_K(p)]^2} \tag{11}$$

As before, the renormalized K-coupling constants are defined by

$$\bar{G}^2_{NKY} = \frac{G^2_{NKY}}{1 + \sum_Y G^2_{NKY} B_Y(0)} \tag{12}$$

where

$$B_Y(0) = \frac{\eta_Y}{16\pi^3} \int \frac{MY + E_N(q)E_Y(q) + q^2}{E_N(q)E_Y(q)w_K(0)[E_N(q)+E_Y(q)-K]^2} d^3q$$

In the more elaborate version of the model, we allow the processes

$$\pi \longleftrightarrow N + \bar{N},\ \Sigma + \bar{\Sigma},\ \Sigma + \bar{\Lambda},\ \Lambda + \bar{\Sigma}$$

and in this scheme, the self-energy of the π-meson becomes

$$\omega_\pi(p) - E_\pi(p) = \sum_i \frac{G_i^2}{16\pi^3} \eta_i \int \frac{d^3q}{E_i(q)E_{i'}(p+q)\omega_i(p)} \frac{M_iM_{i'} + E_i(q)E_{i'}(p+q) - q\cdot(p+q)}{E_i(q) + E_{i'}(p+q) - E_\pi(p)} \tag{13}$$

where the following notation has been used

$$M_1 = M_{1'} = M \qquad G_1 = G_{NN\pi}$$
$$M_2 = \Lambda \; M_{2'} = \Sigma \qquad G_2 = G_{\Lambda\Sigma\pi}$$
$$M_3 = M_{3'} = \Sigma \qquad G_3 = G_{\Sigma\Sigma\pi}$$

The renormalized π-meson coupling constants are now given by

$$\bar{G}_i^2 = \frac{G_i^2}{1 + \sum_i G_i^2 A_i(0)}$$

where

$$A_i(0) = \eta_i \int \frac{d^3q}{16\pi^3 E_i(q) E_{i'}(q)\, \omega_\pi(0)} \; \frac{M_i M_{i'} + E_i(q) E_{i'}(q) - q^2}{[E_i(q) + E_{i'}(q) - \mu]^2} \qquad (14)$$

We notice that in the expressions (8), (12) and (14), the energies of the bare-mesons occur in the denominator and we can easily see from equations (5), (10) and (13) that the bare particle masses tend to the real particle ones as the coupling constants tend to zero. For small coupling constants, we can therefore replace $\omega_\pi(o)$ and $\omega_K(o)$ by pion-mass and K-meson mass respectively.

In this connection, we may point out that with a reasonable choice of cut-off (0.7 nucleon mass) and renormalized coupling constant of the order unity, Bosco and Stroffolini have found that the correction to the real particle mass by mass renormalization is very small.

2.13 Two Particle States.

In order to describe the scattering of a physical $\bar{K}$-meson from a physical nucleon, we will use a Tamm-Dancoff expansion for the two particle wave-functional and solve the Schrodinger equation satisfied by the wave-functional to obtain the scattering amplitudes.

The Schrodinger equation for the wave functional is

$$(H_o + H_{int}) \,|\Psi\rangle = E\,|\Psi\rangle \tag{1}$$

Expanding $|\Psi\rangle$ in a series of eigenfunctions of H_o containing states with one baryon, m mesons and n baryon-antibaryon pairs we have

$$|\Psi\rangle = \sum_{\lambda,m,n} \chi_\lambda^{m,n} \,|\, \lambda, m, n\rangle \tag{2}$$

where λ specifies momenta, spins etc. of the system. In the present model, we limit ourselves to states with no more than one baryon-antibaryon pair. Thus we write the wave-functional in the barycentric system in the form

$$\begin{aligned}
|\Psi\rangle = \sum_{p} \Big[& \chi_1(p)\, a_r^{N\dagger}(p)\,|\bar{K}(-p)\rangle + \chi_2(p)\, a_r^{\Lambda\dagger}(p)\,|\pi(-p)\rangle + \chi_3(p)\, a_r^{\Sigma\dagger}(p)\,|\pi(-p)\rangle \Big] \\
+ \sum_{p,q} \Big[& \chi_4(p,q)\, a_t^{N\dagger}(p)\, b_r^{\bar{N}\dagger}(-q-p)\, a_s^{\Lambda\dagger}(q) + \chi_5(q,p)\, a_t^{N\dagger}(p)\, b_r^{\bar{N}\dagger}(-q-p)\, a_s^{\Sigma\dagger}(q) \\
& + \chi_6(q,p)\, a_t^{\Lambda\dagger}(p)\, b_r^{\bar{\Sigma}\dagger}(-q-p)\, a_s^{\Lambda\dagger}(q) + \chi_7(q,p)\, a_t^{\Sigma\dagger}(p)\, b_r^{\bar{\Sigma}\dagger}(-q-p)\, a_s^{\Lambda\dagger}(q) \\
& + \chi_8(q,p)\, a_t^{\Sigma\dagger}(p)\, b_r^{\bar{\Sigma}\dagger}(-q-p)\, a_s^{\Sigma\dagger}(q) + \chi_9(q,p)\, a_t^{\Sigma\dagger}(p)\, b_r^{\bar{\Lambda}\dagger}(-q-p)\, a_s^{\Sigma\dagger}(q) \\
& + \chi_{10}(q,p)\, a_t^{\Lambda\dagger}(p)\, b_r^{\bar{\Lambda}\dagger}(-q-p)\, a_s^{\Sigma\dagger}(q) \Big] |0\rangle
\end{aligned}$$

.....(3)

It is then necessary to solve the Schrodinger equation to obtain the three amplitudes χ_1, χ_2 and χ_3 from which one wishes to obtain the amplitude χ_1.

It may be noted here that in model A i.e. the exactly soluble version of the model, only the first four terms occur.

To obtain the equations for the amplitudes for the open channels $\bar{K} + N$, $\Lambda + \pi$, and $\Sigma + \pi$ respectively (we do not consider energies above which the processes $B + \bar{B} + Y$ are possible), $|\Psi\rangle$ is inserted into the expression

$$I = \langle \Psi | \, E - H \, | \Psi \rangle$$

and the condition $\delta I = 0$ for the variation in the χ_i is imposed. This gives the following equations for $\chi_4 \cdots$ $\cdots \cdots \chi_9, \chi_{10}$. We have

$$\left[E - E_N(p) - E_\Lambda(s) - E_N(p+s)\right] \chi_4(s)$$
$$= -\Big[\langle 0 | \, a_i^N(p) \, b_1^{\bar{N}}(-p-q) a_S^\Lambda(q) (E-H) \, a_{i'}^{N\dagger}(s) \, | \bar{K}(-s) \rangle \, \chi_1(s)$$
$$+ \langle 0 | \, a_i^N(p) \, b_1^{\bar{N}}(-p-q) \, a_S^\Lambda(q) \, (E-H) \, a_{i'}^{N\dagger}(s) \, | \, \Pi(-s) \rangle \, \chi_2(s) \qquad (4)$$

and a similar equation for χ_5 with Λ replaced by Σ

Also

$$\left[E - E_\Lambda(p) - E_\Lambda(s) - E_\Sigma(p+s)\right] \chi_6(s)$$
$$= - \langle 0 | \, a_i^\Lambda(p) \, b_1^{\bar{\Sigma}}(-p-q) a_S^\Lambda(q) \, (E-H) \, a_{i'}^{\Lambda\dagger}(s) \, | \, \Pi(-s) \rangle \, \chi_2(s) \qquad (5)$$

and similar equations can be written down for $\chi_7, \ldots \chi_{10}$.

Now from the condition

$$\langle \bar{K}(-p) | a_1^N(p)(E-H) | \Psi \rangle = 0$$

we have

$$\langle \bar{K}(-p) | a_1^N(p)(E-H) a_{r'}^{N\dagger}(s) | \bar{K}(-s) \rangle \chi_1(s)$$

$$+ \langle \bar{K}(-p) | a_1^N(p)(E-H) a_{r'}^{\Lambda\dagger}(s) | \Pi(-s) \rangle \chi_2(s)$$

$$+ \langle \bar{K}(-p) | a_1^N(p)(E-H) a_{r'}^{\Sigma\dagger}(s) | \Pi(-s) \rangle \chi_3(s)$$

$$+ \langle \bar{K}(-p) | a_1^N(p)(E-H) a_{r'}^{N\dagger}(q) b_{t'}^{\bar{N}\dagger}(-q-s) a_{s'}^{\Lambda\dagger}(s) | 0 \rangle \chi_4(s)$$

$$+ \langle \bar{K}(-p) | a_1^N(p)(E-H) a_{r'}^{N\dagger}(q) b_{t'}^{\bar{N}\dagger}(-q-s) a_{s'}^{\Sigma\dagger}(s) | 0 \rangle \chi_5(s) = 0 \qquad (6)$$

Substituting (4) and a similar expression for χ_5 into (6), one obtains

$$\Big[\langle \bar{K}(-p) | a_r^N(p)(E-H) \{ a_{r'}^{N\dagger}(s) | \bar{K}(-s) \rangle \chi_1(s) + a_{r'}^{\Lambda\dagger}(s) | \Pi(-s) \rangle \chi_2(s)$$

$$+ a_{r'}^{\Sigma\dagger}(s) | \Pi(-s) \rangle \chi_3(s) \}$$

$$- \bar{D}^{1}(q,s) \langle \bar{K}(-p) | a_1^N(p)(E-H) a_1^{N\dagger}(q) b_{t'}^{\bar{N}\dagger}(-q-s) a_{s'}^{Y\dagger}(s) | 0 \rangle \langle 0 |$$

$$a_{r''}^N(q') b_{t''}^{\bar{N}}(-q'-s') a_{s''}^Y(s')(E-H) a_{r'''}^{N\dagger}(s) \{ | \bar{K}(-s) \rangle \chi_1(s)$$

$$+ | \Pi(-s) \rangle \chi_2(s) \} \Big] = 0 \qquad (7)$$

where

$$D_{q,s} = E - E_N(q) - E_Y(s) - E_N(q+s)$$

Two similar equations obtained from the conditions

$$\langle \Pi(-p) | a_r^Y(p)(E-H) | \Psi \rangle = 0$$

are relegated to the appendix.

2.14 The Integral Equations.

We will reduce the equation (7) of the previous chapter separately and establish the integral equation satisfied by χ_i .

The diagonal terms are

$$\langle \bar{K}(-p) |\, a^{N}_{\gamma}(p)\,(E-H)\, a^{N\dagger}_{\gamma'}(s)\, | \bar{K}(-s)\rangle$$

$$- D^{-1}_{q,s} \langle \bar{K}(-p) | a^{N}_{\gamma}(p)\,(E-H)\, a^{N\dagger}_{\gamma'}(q)\, b^{\bar{N}\dagger}_{t'}(-q-s)\, a^{\gamma\dagger}_{\zeta'}(s) | 0 \rangle\langle 0 | a^{N}_{\gamma''}(q')\, b^{\bar{N}}_{t''}(-q'-s')\, a^{\gamma}_{\zeta''}(s')(E-H)\, a^{N\dagger}_{\gamma'''}(s) | \bar{K}(-s)\rangle$$

The first term

$$= \left[E - E_N(p) - w(p) \right] N_4(p) + \sum_{\gamma} \left[E - E_N(p) - E_\gamma(s) - E_N(s+q) \right] | f_\gamma(p,s) |^2 N_4(p)$$

$$+ \sum_{\gamma} f_\gamma(p,s)\, \Gamma^*(p,s)\, N_4(p)$$

$$= N_4(p) \left\{ \left[E - E_N(p) - w(p) \right] + \sum_{\gamma} \left[E - E_N(p) - E_N(s+q) - E_\gamma(s) + E_N(s+q) + E_\gamma(s) - w(p) \right] | f_\gamma(p,s) |^2 \right\}$$

$$= N_4(p) \left[E - E_N(p) - w(p) \right] \left[1 + \sum_{\gamma} | f_\gamma(s,p) |^2 \right]$$

$$= N_4^{-1}(p) \left[E - E_N(p) - w(p) \right]$$

The second term

$$= \sum_{\gamma} \Big\{ \left[E - E_N(q) - E_N(q+s) - E_\gamma(s) \right] | f_\gamma(p,s) |^2 + \Gamma^*_\gamma(p,s)\, f_\gamma(s,p) + \text{c.c.}$$

$$+ D^{-1}_{q,s} | \Gamma_\gamma(p,s) |^2 \Big\}\, N_4(p)$$

$$= \sum_{\gamma} N_4(p)\, | f_\gamma(p,s) |^2 \left[D + 2\left(E_N(p+s) + E_\gamma(s) - w(q) \right) + \frac{\left[E_N(p+s) + E_\gamma(s) - w(p) \right]^2}{D} \right]$$

$$= \sum_{\gamma} N_4(p)\, | f_\gamma(p,s) |^2 \left[E - E_N(p) - w(p) \right]^2 D^{-1}_{q,s}$$

Thus the total diagonal term is

$$\mathcal{N}_4^{-1}(p)\left[E - E_N(p) - w(p)\right] h_1(p)\chi_1(p)$$

where

$$h_1(p) = 1 - \mathcal{N}_4^2(p)\left[E - E_N(p) - w(p)\right] \sum_Y \frac{\eta_Y}{16\pi^3} \int \frac{d^3s}{E_N(p+s)E_Y(s)w(p)} \frac{MY + E_N(p+s)E_Y(s) - s\cdot(s+p)}{\left[E_N(p+s)+E_Y(s)-w(p)\right]\left[E_N(p+s)+B\right]}$$

$$B = E_N(p) + E_Y(s) - E$$

and the isotopic spin factor η_Y is given by

$$\eta_\Lambda = G^2_{NK\Lambda} \qquad \eta_\Sigma = 3\,G^2_{NK\Sigma}$$

We now consider the non-diagonal terms. These are

$$\langle \bar{K}(-p) |\, a^N_r(p)\,(E-H)\, a^{Y\dagger}_{r'}(s)\, |\Pi(-s)\rangle$$

$$- D^{-1}_{q,s} \langle \bar{K}(-p) |\, a^N_r(p)(E-H)\, a^{N\dagger}_{r'}(q)\, b^{\bar{N}\dagger}_{s'}(-q-s)\, a^{Y\dagger}_{t}(s)|0\rangle\langle 0|\, a^{N}_{r''}(q')\, b^{\bar{N}}_{s''}(-q'-s')\, a^{Y}_{t''}(s')(E-H)\, a^{Y\dagger}_{r'''}(s)\,|\Pi(-s)\rangle$$

The first term

$$= \mathcal{N}_3(s)\Big[\langle 0|\Big\{\bar{K}(-p) + \sum_Y a^{Y}_{r}(k)\, b^{N}_{s}(-k-p)\, f^*_Y(k,p)\Big\} a^N_r(p)\,(E-H)\, a^{Y\dagger}_{r'}(s)\Big\{\pi^\dagger(-s) + \sum_i F_i(q,s)\, a^{B\dagger}_{r'}(q)\, b^{\bar{B}\dagger}_{s'}(-q-s)\Big\}|0\rangle\Big]$$

$$= \mathcal{N}_3(s)\Big\{\left[E - E_Y(s) - E_N(p) - E_N(p+s)\right] f^*_Y(p,s)\, F_i(p,s)$$

$$+ \Gamma_1^*(p,s)\, F_1(p,s) + \Gamma_Y(p,s)\, f^*_Y(p,s)\Big\}$$

The second term

$$= \mathcal{N}_3(s)\Big[\Gamma_1^*(p,s)\,\Gamma_Y'(p,s)\, D^{-1}(p,s) + f^*_Y(p,s)\,\Gamma_Y(p,s) + \Gamma_1^*(s,p)\, F_1(p,s)$$

$$+ D(p,s)\, F_1(p,s)\, f^*_Y(p,s)\Big]$$

$$D(p,s) = E - E_N(p) - E_Y(s) - E_N(p+s).$$

Hence the total non-diagonal term

$$= - N_3(s) \frac{\Gamma_1^*(p,s)\ \Gamma_Y(p,s)}{E - E_Y(s) - E_N(p) - E_N(p+s)}$$

$$= - N_3(s) \sqrt{\frac{M^2 MY}{4\, E_N(p) E_Y(s) E_N^2(p+s) w(p) \omega(s)}} \frac{G_{NN\pi} G_{NK\Lambda} \left[\bar{\omega}_r^N(p) \gamma_5 \tau^\lambda v_s^{\bar{N}}(p+s)\right]\left[\bar{v}_s^{\bar{N}}(p+s)\gamma_5 \xi_Y \omega_{r'}^N(s)\right]}{E - E_Y(s) - E_N(p) - E_N(p+s)}$$

where $\xi_\Lambda = 1 \qquad \xi_\Sigma = \tau^\Sigma$

In this way Eq. (7) of the previous page is reduced to

$$\left[E - E_N(p) - w(p)\right] h_1(p) \chi_{1\omega}^\alpha(p) = \sum_{j=1}^{3} \int K_{1j}(p,s) \chi_{j\omega}^\delta(s)\, d^3s \qquad \ldots.(1)$$

where

$$K_{11}(p,s) = 0$$

$$K_{12}(p,s) = N_4(p) N_3(s) \frac{G_{12}(T)}{16\pi^3} \sqrt{\frac{M^2 M\Lambda}{E_N^2(p+s) E_\Lambda(s) E_N(p) w(p) \omega(s)}}$$

$$\frac{\left[\bar{\omega}_r^N(p)\gamma_5 v_s^{\bar{N}}(p+s)\right]\left[\bar{v}_s^{\bar{N}}(p+s)\gamma_5\, \omega_{r'}^Y(s)\right]}{E - E_N(p) - E_Y(s) - E_N(p+s)} \qquad \ldots.(2)$$

and $K_{13}(p,s)$ is the same with Λ replaced by Σ

Eq. (1) above is the integral equation for χ_1 and we can obtain similar equations for χ_2 and χ_3 in exactly the same manner. These are

$$\left[E - E_\Lambda(p) - \omega(p)\right] h_2(p) \chi_{2\omega}^\alpha(p) = \sum_{j=1}^{3} \int K_{2j}(p,s) \chi_{j\omega}^\delta(s)\, d^3s \qquad (3)$$

and

$$\left[E - E_\Sigma(p) - \omega(p)\right] h_3(p) \chi^{\delta}_{3\omega}(p) = \sum_{j=1}^{3} \int K_{3j}(p, s) \chi^{\delta}_{j\omega}(s)\, d^3 s \tag{4}$$

The various kernels can be found in the appendix.

An alternative but much simpler method of obtaining the integral equations may be mentioned here. We start as before from the Schrödinger equation for the wave-functional

$$(H_o + H_{int})|\Psi\rangle = E|\Psi\rangle \tag{5}$$

and then expand $|\Psi\rangle$ in a series of eigenfunctions of H_o containing states with one baryon, m mesons and n baryon-antibaryon pairs.

$$|\Psi\rangle = \sum_{\lambda, m, n} a^{m,n} |\lambda, m, n\rangle \tag{6}$$

where λ specifies momenta, spins etc. of the system. Substituting (6) into (5)

$$\left[E - E_\lambda^{m,n}\right] a_\lambda^{m,n} = \sum_q \sum_p \sum_\mu \langle \lambda, m, n | H_{int} | \mu, p, q \rangle a_\mu^{p,q} \tag{7}$$

In the present model we limit ourselves to states with no more than one baryon-antibaryon pair. This results in the following equations

$$\left(E - E_\lambda^{1,0}\right) a_\lambda^{1,0} = \sum_\mu \langle \lambda, 1, 0 | H_{int} | \mu, 0, 1 \rangle a_\mu^{0,1} \tag{8}$$

and

$$(E - E_{\mu}^{0,1})\, a_{\mu}^{0,1} = \sum_{\rho} \langle \mu, 0, 1 |\, H_{int}\, | \rho, 1, 0 \rangle\, a_{\rho}^{1,0} \tag{9}$$

Hence substituting (9) into (8),

$$(E - E_{\lambda}^{1,0})\, a_{\lambda}^{1,0} = \sum_{\mu,\rho} \Big[\langle \lambda, 1, 0 |\, H_{int}\, | \mu, 0, 1 \rangle \langle \mu, 0, 1 |\, H_{int}\, | \rho, 1, 0 \rangle \times \frac{a_{\rho}^{1,0}}{E - E_{\mu}^{0,1}} \tag{10}$$

There are two types of mesons in the theory. The $\bar{K}$-N amplitude for example satisfies the integral equation

$$\Big[E - E_N(p) - w(p)\Big] a_1(p) = N_4(p) N_3(s) \Big[G_{NN\pi} G_{KK\Lambda} \frac{\Gamma_1(p,s)\, \Gamma_5^*(p,s)}{E - E_N(p) - E_{\Lambda}(s) - E_N(p+s)}\, a_2(s) + G_{NN\pi} G_{NK\Sigma} \frac{\Gamma_1(p,s)\, \Gamma_6^*(p,s)}{E - E_{\Sigma}(s) - E_N(p) - E_N(p+s)}\, a_3(s) \Big] \tag{11}$$

where $N_3(p)$ and $N_4(s)$ are normalization constants of the π-meson and $\bar{K}$-meson 'physical' states since $| \rho, 1, 0 \rangle$ involves a 'physical' meson in Eqn. (10). It will be noticed that there is no term involving $a_1(s)$ on the right of Eqn. (11). The reason is of course that $\bar{K} + N \longrightarrow \bar{N} + Y + N$ and this intermediate state does not lead back to $\bar{K} + N$ but gives rise to $\pi + Y$ state.

In Eqn. (10) we have assumed that $\lambda \neq \rho$. When however $\lambda = \rho$, we get the self-energy contribution which for the $\overline{K}N$ amplitude is the following

$$[E - E_N(p) - w(p)]\, S_1(p) = \sum_Y \lambda_Y \int \frac{d^3s}{16\pi^3} \frac{G^2_{NK\Lambda} N_4^2(p) MY}{E_N(p+s)E_Y(s)w(p)} \frac{[\bar{\omega}(s)\gamma_5 v(p+s)][\bar{v}(p+s)\gamma_5\omega(s)]}{E - E_N(p) - E_Y(s) - E_N(p+s)}$$

where λ_Y is the isotopic spin factor.

The simplest procedure for renormalization of the Tamm-Dancoff equation has been described by Bethe-de Hoffmann (50). The self-energy terms are expanded in a series in powers of the difference between the energy of free-particles of momentum p and the actual energy of the system. The first two terms in the series are dropped and the rest is considered the renormalized self-energy term.

Thus expanding D^{-1} in powers of $[E_N(p) + w(p) - E]$ where D is the denominator $E - E_N(p) - E_Y(s) - E_N(p+s)$ we have

$$\frac{1}{E - E_N(p) - E_Y(s) - E_N(p+s)} = -\frac{1}{E_N(p+s) + E_Y(s) - w(p)} - \frac{E - E_N(p) - w(p)}{[E_N(p+s) + E_Y(s) - w(p)]^2} + \cdots$$

Thus from the above prescription, the renormalized self-energy term is

$$[E - E_N(\mathbf{p}) - w(\mathbf{p})]\, \bar{S}_1(\mathbf{p}) = -\sum_Y \lambda_Y \frac{N_4^2(\mathbf{p})\, G^2_{NKY}}{16\pi^3} \int \frac{d^3 s}{E_N(\mathbf{p}+\mathbf{s})\, E_Y(\mathbf{s})\, w(\mathbf{p})} \left[MY + \mathbf{s}\cdot(\mathbf{s}+\mathbf{p}) \right]$$

$$\left[\frac{1}{E - E_N(\mathbf{p}+\mathbf{s}) - E_Y(\mathbf{s}) - E_N(\mathbf{p})} + \frac{1}{E_N(\mathbf{p}+\mathbf{s}) + E_Y(\mathbf{s}) - w(\mathbf{p})} + \frac{E - E_N(\mathbf{p}) - w(\mathbf{p})}{[E_N(\mathbf{p}+\mathbf{s}) + E_Y(\mathbf{s}) - w(\mathbf{p})]^2} \right]$$

$$= -\sum_Y \lambda_Y \frac{N_4^2(\mathbf{p})\, G^2_{NKY}}{16\pi^3} \int \frac{d^3 s}{E_N(\mathbf{p}+\mathbf{s})\, E_Y(\mathbf{s})\, w(\mathbf{p})} \times$$

$$\frac{[MY + E_N(\mathbf{p}+\mathbf{s})E_Y(\mathbf{s}) - \mathbf{s}\cdot(\mathbf{s}+\mathbf{p})][E - E_N(\mathbf{p}) - w(\mathbf{p})]^2}{[E_N(\mathbf{p}+\mathbf{s}) + E_Y(\mathbf{s}) - w(\mathbf{p})]^2 [E - E_N(\mathbf{p}+\mathbf{s}) - E_N(\mathbf{p}) - E_Y(\mathbf{s})]}$$

Thus Eqn. (11) is replaced by an equation where the wave-function on the left-hand side $a_1(\mathbf{p})$ is multiplied by the factor $h_1(\mathbf{p})$ where

$$h_1(\mathbf{p}) = 1 - \sum_Y \lambda_Y \frac{N_4^2(\mathbf{p})\, G^2_{NKY}}{16\pi^3} \int \frac{d^3 s}{E_N(\mathbf{p}+\mathbf{s})\, E_Y(\mathbf{s})\, w(\mathbf{p})} \frac{E - E_N(\mathbf{p}) - w(\mathbf{p})}{[E_N(\mathbf{p}+\mathbf{s}) + E_Y(\mathbf{s}) - w(\mathbf{p})]^2} \times$$

$$\frac{MY + E_N(\mathbf{p}+\mathbf{s})E_Y(\mathbf{s}) - \mathbf{s}\cdot(\mathbf{s}+\mathbf{p})}{E_N(\mathbf{p}+\mathbf{s}) + E_N(\mathbf{p}) + E_Y(\mathbf{s}) - E}$$

where

$$\lambda_Y = 1 \qquad Y = \Lambda$$
$$\quad\;\; 3 \qquad \;\; = \Sigma$$

2.15 Reduction to Large Components and Elimination of Angular and Spin Dependence.

In the last chapter, we have obtained the integral equations in the form

$$[E - E_i(p) - \omega_i(p)]\, h_i(p)\, \chi^{\alpha}_{i\omega}(p) = \sum_{j=1}^{3} \int K_{ij}(p,s)\, \chi^{\delta}_{j\omega}(s)\, d^3s$$

where the kernels are of the form

$$K_{ij}(p,s) = N_i(p) N_j(s) \frac{C_{ij}(T)}{16\pi^3} \sqrt{\frac{M_{ij}^2 M_i M_j}{E_i(p) E_j(s) \omega_i(p) \omega_j(s)}} \cdot \frac{1}{E_{ij}(p+s)} \frac{[\bar{\omega}^i_r(p)\, \gamma_5\, v^{ij}_s(p+s)][v^{ij}_s(p+s)\gamma_5\, \omega^j_r(s)]}{E - E_i(p) - E_j(s) - E_{ij}(p+s)} \qquad (1)$$

We next use the abbreviation

$$\sum_{\omega} \omega(p)\, \chi^{\alpha}_{i\omega}(p) = \chi^{\alpha}_i(p)$$

so that $\chi^{\alpha}_i(p)$ is a spinor. This yields

$$[E - E_i(p) - \omega_i(p)]\, h_i(p)\, \chi^{\alpha}_i(p) = \sum_{j=1}^{3} \int K_{ij}(p,s)\, \chi^{\delta}_j(s)\, d^3s$$

where now

$$K_{ij}(p,s) = N_i(p) N_j(s) \frac{C_{ij}(T)}{16\pi^3} \sqrt{\frac{M_{ij}^2 M_i M_j}{E_i(p) E_j(s) \omega_i(p) \omega_j(s)}} \cdot \frac{1}{E_{ij}(p+s)} \frac{\Lambda^i_+(p)\, \gamma_5\, \Lambda^{ij}_-(p+s)\, \gamma_5}{E - E_i(p) - E_j(s) - E_{ij}(p+s)}$$

$$= N_i(p) N_j(s) \frac{C_{ij}(T)}{16\pi^3} \sqrt{\frac{M_{ij}^2 M_i M_j}{E_i(p) E_j(s) \omega_i(p) \omega_j(s)}} \frac{1}{E_{ij}(p+s)} \frac{\Lambda^i_+(p)\, \Lambda^{ij}_+(p+s)}{E_{ij}(p+s) + E_i(p) + E_j(s) - E} \qquad (2)$$

It is well-known that by reduction to large components we can deal with states of given parity. For this purpose we write the four component wave-function $\chi(p)$ in terms of the two-component Pauli spinors $\chi_+(p)$ and $\chi_-(p)$:

$$\chi(p) = \begin{pmatrix} \chi_+(p) \\ \chi_-(p) \end{pmatrix}$$

and by eliminating the small components in favour of the large, we can write $\chi(p)$ in the form

$$\chi_i(p) = \left(1 + \frac{i\gamma_5\, \sigma\cdot p}{E_i(p)+M_i} \right) \chi_{i+}(p)$$

We insert this in Eqn. (2) and write

$$\Lambda_+^i(p) = \frac{\beta E_i(p) + M_i - \beta\gamma_5 \sigma\cdot p}{2M_i}$$

Picking out large components by neglecting terms involving γ_5 as well as putting $\beta = 1$ and also remembering that $\gamma_5^2 = 1$, we have

$$\left[\beta E_i(p)+M_i - \beta\gamma_5\, \sigma\cdot p\right]\left[\beta E_{ij}(p+s)+M_{ij}+\beta\gamma_5\sigma\cdot(p+s)\right]\left[1+\frac{i\gamma_5\sigma\cdot p}{E_j(s)+M_j}\right]$$
$$= E_{ij}(p+s)\left[\frac{\sigma\cdot p\,\sigma\cdot s}{E_j(s)+M_j} + E_i(p)+M_i\right] + \frac{\sigma\cdot p\,\sigma\cdot s}{E_j(s)+M_j}\left[E_i(p)+E_j(s)+M_i+M_j-M_{ij}\right]$$
$$+ \left[E_i(p)+M_i\right]\left[E_i(p)+E_j(s)+M_{ij}-M_i-M_j\right]$$

Thus the integral equations can be written as

$$\left[E - E_i(p) - \omega_i(p)\right] h_i(p)\, \chi_i^\alpha(p) = \sum_j \int K_{ij}(p,s)\, \chi_j^\delta(s)\, d^3s$$

where

$$K_{ij}(p,s) = \frac{N_i(p)\,N_j(s)\,C_{ij}(T)}{64\pi^3}\sqrt{\frac{M_j}{M_i\,E_i(p)\,E_j(s)\,\omega_i(p)\,\omega_j(s)}}\ \times$$

$$\left[\ \frac{1}{E_{ij}(x)+B_{ij}}\left\{\frac{\sigma\cdot p\ \sigma\cdot s}{E_j(s)+M_j} + E_i(p)+M_i\right\}\right.$$

$$+\frac{1}{E_{ij}(x)\left[E_{ij}(x)+B_{ij}\right]}\left\{\left[E_i(p)+M_i\right]\left[E_i(p)+E_j(s)+M_{ij}-M_i-M_j\right]\right.$$

$$\left.\left.+\frac{\sigma\cdot p\ \sigma\cdot s}{E_j(s)+M_j}\Big(E_i(p)+E_j(s)+M_i+M_j-M_{ij}\Big)\right\}\right] \tag{3}$$

where

$$B_{ij} = E_i(p) + E_j(s) - E$$

$$E_{ij}(x) = \left(p^2+s^2+2psx+M_{ij}^2\right)^{\frac{1}{2}}$$

and

$x = \cos\theta$, θ being the angle between $\mathbf{p}$ and $\mathbf{s}$.

We will follow the method of Dyson et al. (51) to perform the angular integration. For this purpose we write

$$\frac{1}{B+E(x)} = \sum_{n=0}^{\infty} \mathcal{Y}_n P_n(x) \tag{4}$$

and

$$\frac{1}{E(x)\left[B+E(x)\right]} = \sum_{n=0}^{\infty} \mathcal{Z}_n P_n(x) \tag{5}$$

and define the operators S_n and R_n by

$$S_n \chi(s) = \frac{1}{4\pi} \int d\Omega_s \, P_n(\theta_s - \theta_p) \chi(s)$$

$$R_n \chi(s) = \frac{1}{4\pi} \int d\Omega_s \, P_n(\theta_s - \theta_p) \frac{\sigma \cdot p \, \sigma \cdot s}{ps} \chi(s)$$

then the kernels become

$$K_{ij}(p,s) = N_i(p) N_j(s) \frac{C_{ij}(T)}{16\pi^2} \sqrt{\frac{M_j}{M_i \quad E_i(p) E_j(s) \omega_i(p) \omega_j(s)}} \quad L_{ij}(p,s)$$

where

$$L_{ij}(p,s) = \sum_n \Big[\Big(\frac{ps}{E_j(s)+M_j} R_n + [E_i(p)+M_i] S_n \Big) \mathcal{Y}_n$$

$$+ \Big\{ [E_i(p)+M_i][E_i(p)+E_j(s)-M_i-M_j+M_{ij}] S_n + \frac{ps}{E_j(s)+M_j} [E_i(p)+E_j(s)+M_i+M_j-M_{ij}] R_n \Big\} \mathcal{Z}_n \Big]$$

and the integral equations are

$$[E - E_i(p) - \omega_i(p)] h_i(p) \chi_i(p) = \sum_{j=1}^{3} \int K_{ij}(p,s) \chi_j(s) \, s^2 ds \qquad (6)$$

We now make a slight change in notation in (4) and (5) and write

$$\mathcal{Y}_n = \frac{2n+1}{2} \int_{-1}^{1} \frac{dx}{B + E(x)} P_n(x) = \frac{2n+1}{2} X_n$$

$$\mathcal{Z}_n = \frac{2n+1}{2} \int_{-1}^{1} \frac{dx}{E(x)[B + E(x)]} P_n(x) = \frac{2n+1}{2} Y_n$$

where for convenience we have defined the integrals

$$X_n = \int_{-1}^{1} \frac{P_n(x)\, dx}{B + E(x)}$$
$$Y_n = \int_{-1}^{1} \frac{P_n(x)\, dx}{E(x)[B + E(x)]} \tag{7}$$

Considering states with definite angular momentum l and total spin j, it has been shown by Dyson et al. that the sum over n reduces to a single term and that S_n and R_n have the eigenvalues $\delta_{n,l}/(2n+1)$ and $\delta_{n,l\pm1}/2n+1$ for j = $l \pm 1/2$ respectively. Hence

$$L_{ij}(p,s) = \frac{1}{2}\Big[\frac{ps}{E_j(s)+M_j}\delta_{n,l\pm1}X_n + \big(E_i(p)+M_i\big)\delta_{n,l}X_n$$
$$+ \Big\{\big(E_i(p)+M_i\big)\big(E_i(p)+E_j(s)-M_i-M_j+M_{ij}\big)\Big\}\delta_{n,l}Y_n$$
$$+ \frac{ps}{E_j(s)+M_j}\big(E_i(p)+E_j(s)+M_i+M_j-M_{ij}\big)\delta_{n,l\pm1}Y_n\Big] \tag{8}$$

Thus the final integral equations are

$$\big[E - E_i(p) - \omega_i(p)\big]h_i(p)\chi_i(p) = \sum_{j=1}^{3}\int K_{ij}(p,s)\,\chi_j(s)\,s^2 ds$$

where

$$K_{ij}(p,s) = \frac{N_i(p)N_j(s)C_{ij}(T)}{16\pi^2}\sqrt{\frac{M_j}{M_i E_i(p) E_j(s)\,\omega_i(p)\omega_j(s)}}\; L_{ij}(p,s) \tag{9}$$

If we neglect recoil for the s-wave kernel, we immediately get

$$\left[E - E_i(p) - \omega_i(p)\right] h_i(p) \chi_i(p) = \frac{G_{ij}^2}{4\pi^2} \int_0^\infty \frac{s^2 ds}{\sqrt{\omega(p)\omega(s)}} C_{ij}(T) \frac{\chi_j(s)}{3M - E}$$

which agrees with the equation of Bosco and Stroffolini. In our actual calculation, we however do not make this approximation.

2.16. Non-Singular Amplitude and the Born Approximation.

We have obtained the coupled integral equations

$$[E - E_i(p) - \omega_i(p)]\, h_i(p)\, \chi_i(p) = \sum_{j=1}^{3} \int_0^\infty K_{ij}(p,s)\, \chi_j(s)\, s^2 ds \tag{1}$$

and it is at once apparent that $\chi_i(p)$ is singular on the energy shell. If $\chi_i(p)$ were a regular function, the left sides of the above equations would vanish for $p = k_i$ where k_i are the roots of the equations

$$E_i(p) + \omega_i(p) - E = 0$$

Since the right sides do not in general vanish identically for $p = k_i$, the amplitude must be singular at $p = k_i$. The amplitude can be written generally

$$\chi_i(p) = \lambda_i\, \delta\left(E - E_i(p) - \omega_i(p)\right) + P\, \frac{1}{E - E_i(p) - \omega_i(p)}\, f_i(p) \tag{2}$$

where P denotes the principal value and $f_i(p)$ is a non-singular function. Eq. (1) then becomes

$$f_i(p) = \sum_j \int K_{ij}(p,s)\left[\lambda_j\, \delta\left(E - E_j(s) - \omega_j(s)\right) + P\, \frac{1}{E - E_j(s) - \omega_j(s)}\, f_j(s)\right] s^2 ds \tag{3}$$

Here the principal value is, of course, taken of the whole part of the integral.

In a single channel reaction like that considered by Dyson et al., the identification may be made that

$$f(k_o) = -\frac{1}{\pi} \tan\delta$$

where k_o is the incident momentum and δ is the scattering phase-shift. In the present case, there are three open channels and it appears more suitable to deal with the S-matrix directly as will be shown in Chapter 18. Rewriting Eq. (3), we have

$$f_i(p) = \sum_j \left[\lambda_j K_{ij}(p,k_j) \left(\frac{\partial s}{\partial \mathcal{E}_j(s)}\right)_{s=k_j} k_j^2 + P\int_0^\infty \frac{K_{ij}(p,s)\, s^2}{E - E_j(s) - \omega_j(s)} f_j(s)\, ds \right. \tag{4}$$

where

$$\mathcal{E}_j(s) = E_j(s) + \omega_j(s)$$

Since

$$\left(\frac{\partial s}{\partial \mathcal{E}_j(s)}\right)_{s=k_j} = \frac{E_j(k_j)\,\omega_j(k_j)}{k_j E}$$

where E is the total centre of mass energy of the system, we can simplify the first term of Eq. (4) into

$$\sum_{j=1}^{3} \lambda_j \frac{k_j}{E} \sqrt{\frac{M_j\, E_j(k_j)\,\omega_j(k_j)}{M_i\, E_i(p)\,\omega_i(p)}}\; L_{ij}(p,k_j) \tag{5}$$

where the factor $L_{ij}(p, k_j)$ is given in Eq. (8) of the last chapter with s replaced by k_j. Since we are considering only S-waves, we have

$$L_{ij}(p,s) = \frac{1}{2}\left[\frac{ps}{E_j(s)+M_j}\left\{X_1^{ij} + \left(E_i(p)+E_j(s)+M_i+M_j-M_{ij}\right)Y_1^{ij}\right\}\right.$$
$$\left. + \left(E_i(p)+M_i\right)\left\{X_0^{ij} + \left(E_i(p)+E_j(s)-M_i-M_j+M_{ij}\right)Y_0^{ij}\right\}\right]$$

where

$$X_0^{ij}(p,s) = \int_{-1}^{1}\frac{dx}{B_{ij}+E_{ij}(x)} \qquad X_1^{ij}(p,s) = \int_{-1}^{1}\frac{x\,dx}{B_{ij}+E_{ij}(x)}$$

and

$$Y_0^{ij}(p,s) = \int_{-1}^{1}\frac{dx}{E_{ij}(x)\left[B_{ij}+E_{ij}(x)\right]} \qquad Y_1^{ij}(p,s) = \int_{-1}^{1}\frac{x\,dx}{E_{ij}(x)\left[B_{ij}+E_{ij}(x)\right]}$$

These integrations can be performed by elementary means and we obtain

$$X_0^{ij}(p,s) = \frac{1}{2ps}\left[E_{ij}(p+s)-E_{ij}(p-s) - B_{ij}(p,s)\ln\frac{B_{ij}+E_{ij}(p+s)}{B_{ij}+E_{ij}(p-s)}\right]$$

$$X_1^{ij}(p,s) = \frac{1}{2p^2s^2}\left[\frac{1}{3}\left\{E_{ij}^3(p+s)-E_{ij}^3(p-s)\right\} - \frac{1}{2}B_{ij}\left\{E_N^2(p+s)-E_N^2(p-s)\right\}\right.$$
$$\left. -\left(A_{ij}-B_{ij}^2\right)\left\{E_{ij}(p+s)-E_{ij}(p-s)\right\} + B_{ij}\left(A_{ij}-B_{ij}^2\right)\ln\frac{B_{ij}+E_{ij}(p+s)}{B_{ij}+E_{ij}(p-s)}\right]$$

$$Y_0^{ij}(p,s) = \frac{1}{ps}\ln\frac{B_{ij}+E_{ij}(p+s)}{B_{ij}+E_{ij}(p-s)}$$

$$Y_1^{ij}(p,s) = \frac{1}{2p^2s^2}\left[\frac{1}{2}\left\{E_{ij}^2(p+s) - E_{ij}^2(p-s)\right\} - B_{ij}\left\{E_{ij}(p+s) - E_{ij}(p-s)\right\} - (A_{ij} - B_{ij}^2)\ln\frac{B_{ij} + E_{ij}(p+s)}{B_{ij} + E_{ij}(p-s)}\right]$$

where $B_{ij} = E_i(p) + E_j(s) - E$, $A_{ij} = M_{ij}^2 + p^2 + s^2$ and $E_{ij}^2(p+s) = M_{ij}^2 + (p+s)^2$

For small p and s we can write

$$X_0 = \frac{2}{\bar{E}}\,\frac{1}{1+\alpha} \qquad\qquad Y_0 = \frac{2}{\bar{E}^2}\,\frac{1}{1+\alpha}$$

$$X_1 = -\frac{2}{3}\,\frac{r}{\bar{E}}\,\frac{1}{(1+\alpha)^2} \qquad\qquad Y_1 = -\frac{2}{3}\,\frac{r}{\bar{E}^2}\,\frac{\alpha+2}{(\alpha+1)^2}$$

where $\alpha = B/\bar{E}$, $r = ps/\bar{E}^2$ and $2\bar{E} = E_N(p+s) + E_N(p-s)$. It will be seen that X and Y decrease in magnitude as we go from X_0, Y_0 to X_1 and Y_1.

Thus for small p and k, we can neglect $X_1^{ij}(p,k_j)$ and $Y_1^{ij}(p,k_j)$ and $L_{ij}(p,k_j)$ is positive and hence the Born approximation to the amplitude is positive resulting in a negative phase-shift δ since $f(k_0) = -\frac{1}{\pi}\tan\delta$. This is the same result as that of Amati and Vitale.

In the next chapter we give details of the numerical method of solution of Eq. (4).

2.17 Method of Numerical Solution.

We wish to solve the coupled integral equations

$$f_i(p) = f_i^B(p) + \sum_j \int_0^\infty G_{ij}(p,s)\, f_j(s)\, ds \tag{1}$$

where

$$f_i^B(p) = \sum_j \lambda_j \int_0^\infty K_{ij}(p,s)\, \delta\left(E - E_j(s) - \omega_j(s)\right) \tag{2}$$

and

$$G_{ij}(p,s) = \frac{K_{ij}(p,s)\, s^2}{E - E_j(s) - \omega_j(s)} \tag{3}$$

Let the solutions of the equations

$$E - E_i(p) - \omega_i(p) = 0$$

be $p = k_i$, then $G_{ij}(p,s)$ have singular points at $s = k_i$ of the type $(k_j - s)^{-1}$. We now introduce

$$x_i = p/k_i \qquad x_j = s/k_j$$

so that

$$f_i(k_i x_i) = f_i^B(k_i x_i) + \sum_j \int_0^\infty G_{ij}(k_i x_i, k_j x_j)\, f_j(x_j k_j)\, k_j\, dx_j \tag{4}$$

and hence the singularity in (4) occurs at $x_j = 1$.

Writing

$$F_i(x) = f_i(k_i x_i)$$

$$k_i G_{ij}(k_i x_i k_j x_j) = H_{ij}(x_i x')$$

we have

$$F_i(x) = F_i^B(x) + \sum_j \int_0^\infty H_{ij}\ (x, x')\ F_j(x')\, dx' \qquad (5)$$

We now divide the range of integration into three parts: (0,1), (1,2) and (2, ∞) so that Eq. (5) takes the form

$$F_i(x) = F_i^B(x) + \sum_j \Big[\int_0^1 \Big\{ H_{ij}(x,x') F_j(x') + H_{ij}(x, 2-x')\ F_j(2-x') \Big\} dx' + \int_2^\infty H_{ij}(x,x') F_j(x')\, dx' \Big] \qquad (6)$$

We have seen above that the kernels $H_{ij}(x,x')$ have singularities at $x_j' = 1$. To avoid this singularity in the integration, we use the method described by Gammel (51). We write

$$M_{ij}(x,x') = \frac{1}{2} \big[H_{ij}(x,x') + H_{ij}(x, 2-x') \big]$$

$$N_{ij}(x,x') = \frac{1}{2} \big[H_{ij}(x,x') - H_{ij}(x, 2-x') \big]$$

and

$$F_i(x) = \frac{1}{2} \big[Z_i(x) + (x-1)\, Y_i(x) \big]$$

$$F_i(2-x) = \frac{1}{2} \big[Z_i(x) - (x-1)\, Y_i(x) \big]$$

Then equations (6) become

$$\frac{1}{2}\big[Z_i(x) + (x-1)Y_i(x)\big] = F_i^B(x) + \sum_j \Big[\int_0^1 \Big\{ M_{ij}(x,x') Z_j(x') + N_{ij}(x,x') Y_j(x') \Big\} dx' + \int_2^\infty H_{ij}(x,x')\, F_j(x')\, dx' \Big] \qquad x \le 1 \qquad (7)$$

$$\tfrac{1}{2}\left[Z_i(2-x)+(x-1)Y_i(2-x)\right] = \text{same thing} \qquad 1 \leq x \leq 2 \tag{8}$$

$$F_i(x) = \text{same thing} \qquad x \geq 2 \tag{9}$$

The kernels $M_{ij}(x,x')$ and $N_{ij}(x,x')$ are non-singular.

Replacing Eqns. (7) to (9) by a system of linear equations and using an n-point Gaussian formula for the integration, we have

$$\begin{aligned}\tfrac{1}{2}\left[Z_i(x_\mu)+(x_\mu-1)Y_i(x_\mu)\right] = F_i^B(x_\mu) + \sum_j \Big[\sum_{\lambda=1}^{n} \omega_\lambda \big\{ M_{ij}(x_\mu,x_\lambda) Z_j(x_\lambda) \\ + N_{ij}(x_\mu,x_\lambda) Y_j(x_\lambda) \big\} \\ + \sum_{\lambda=n}^{2n+1} \omega_\lambda H_{ij}(x_\mu,x_\lambda) F_j(x_\lambda) \Big] \qquad 1 \leq \mu \leq n\end{aligned} \tag{10}$$

$$\tfrac{1}{2}\left[Z_i(2-x_\mu)+(x_\mu-1)Y_i(2-x_\mu)\right] = \text{same thing} \qquad 1 \leq \mu \leq n \tag{11}$$

$$F_i(x\mu) = \text{same thing} \qquad n \leq \mu \leq 2n+1 \tag{12}$$

where ω_λ are the Gaussian weights.

Solving for Z, Y, F, we have

$$\begin{aligned}Z_i^B(x_\mu) = \sum_{\lambda=1}^{n} \Big[\delta_{ij}\delta_{\lambda\mu} - \omega_\lambda \big\{ M_{ij}(x_\mu,x_\lambda) + M_{ij}(2-x_\mu,x_\lambda) \big\} \Big] \\ - \sum_{\lambda=1}^{n} \omega_\lambda \big\{ N_{ij}(x_\mu,x_\lambda) + N_{ij}(2-x_\mu,x_\lambda) \big\} \\ - \sum_{\lambda=n}^{2n+1} \omega_\lambda \big\{ H_{ij}(x_\mu,x_\lambda) + H_{ij}(2-x_\mu,2-x_\lambda) \big\} \qquad 1 \leq \mu \leq n\end{aligned} \tag{13}$$

$$
\begin{aligned}
Y_i^B(x_\mu) = {} & \sum_{\lambda=1}^{n}\Big[-\omega_\lambda\big\{M_{ij}(x_\mu,x_\lambda) - M_{ij}(2-x_\mu,x_\lambda)\big\}\Big] \\
& + \sum_{\lambda=1}^{n}\Big[\delta_{ij}\delta_{\lambda\mu}(x_\mu-1) - \omega_\lambda\big\{N_{ij}(x_\mu,x_\lambda) - N_{ij}(2-x_\mu,2-x_\lambda)\big\} \\
& - \sum_{\lambda=n}^{2n+1}\omega_\lambda\big\{H_{ij}(x_\mu,x_\lambda) - H_{ij}(2-x_\mu,2-x_\lambda)\big\}
\end{aligned}
$$

$$1 \leq \mu \leq n \tag{14}$$

$$
\begin{aligned}
F_i^B(x_\mu) = {} & \sum_{\lambda}\Big[-\omega_\lambda\big\{M_{ij}(x_\mu,x_\lambda) + N_{ij}(x_\mu,x_\lambda)\big\} \\
& + \big\{\delta_{ij}\delta_{\lambda\mu} - \omega_\lambda H_{ij}(x_\mu,x_\lambda)\big\}\Big] \qquad n \leq \mu \leq 2n+1
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
Z_i^B(x_\mu) &= F_i^B(x_\mu) + F_i^B(2-x_\mu) \\
Y_i^B(x_\mu) &= F_i^B(x_\mu) - F_i^B(2-x_\mu)
\end{aligned}
\tag{16}
$$

On account of the fact that we have used Gaussian integration, we have the same number of equations as there are unknowns and therefore no additional condition is necessary on the functions Z, Y and F.

Finally then if we write the kernels $K_{ij}(p,s)$

in the form

$$K_{ij}(p,s) = \begin{pmatrix} P & Q & R \\ S & C & T \\ U & V & W \end{pmatrix} \quad \begin{matrix} s \; p \rightarrow \\ \downarrow \end{matrix}$$

the re-arranged kernels take the form

$$\mathcal{K}_{ij}(p,s) = \begin{pmatrix} Z_iZ_j & Z_iY_j & Z_iF_j \\ Y_iZ_j & Y_iY_j & Y_iF_j \\ F_iZ_j & F_iY_j & F_iF_j \end{pmatrix}$$

where

$$\begin{aligned} Z_iZ_j &= \delta_{ij}\delta_{\lambda\mu} - \omega_\lambda \left\{ M_{ij}(x_\mu, x_\lambda) + M_{ij}(2-x_\mu, x_\lambda) \right\} \\ &= \delta_{ij}\delta_{\lambda\mu} + \tfrac{1}{2}\omega_\lambda \left[\frac{K_{ij}(x_\mu, x_\lambda)}{E_j(x_\lambda)+\omega_j(x_\lambda)-E} + \frac{K_{ij}(x_\mu, 2-x_\lambda)}{E_j(2-x_\lambda)+\omega_j(2-x_\lambda)-E} \right. \\ &\qquad \left. + \frac{K_{ij}(2-x_\mu, x_\lambda)}{E_j(x_\lambda)+\omega_j(x_\lambda)-E} + \frac{K_{ij}(2-x_\mu, 2-x_\lambda)}{E_j(2-x_\lambda)+\omega_j(2-x_\lambda)-E} \right] \\ &= \delta_{ij}\delta_{\lambda\mu} + \frac{1}{2}\frac{\omega_\lambda}{\mathcal{E}_j(x_\lambda)} \left[(P+S) + f(x_\lambda)(Q+C) \right] \end{aligned}$$

with

$$f(x_\lambda) = \frac{\mathcal{E}_j(x_\lambda)}{\mathcal{E}_j(2-x_\lambda)} \quad \text{and} \quad \mathcal{E}_j(x_\lambda) = E_j(x_\lambda) + \omega_j(x_\lambda) - E$$

$$Z_iY_j = \frac{\omega_\lambda(x_\lambda - 1)}{2\,\mathcal{E}_j(x_\lambda)} \left[(P+S) - f(x_\lambda)(Q+C) \right]$$

$$Z_iF_j = \frac{\omega_\lambda}{\mathcal{E}_j(x_\lambda)} (R+T)$$

$$Y_iZ_j = \frac{\omega_\lambda}{2\,\mathcal{E}_j(x_\lambda)} \left[(P-S) + f(x_\lambda)(Q-C) \right]$$

$$Y_iY_j = \delta_{ij}\delta_{\lambda\mu}(x_\lambda - 1) + \frac{\omega_\lambda(x_\lambda-1)}{2\,\mathcal{E}_j(x_\lambda)} \left[(P-S) - f(x_\lambda)(Q-C) \right]$$

$$Y_iF_j = \frac{\omega_\lambda}{\mathcal{E}_j(x_\lambda)}(R-T)$$

$$F_iZ_j = \frac{\omega_\lambda}{2\mathcal{E}_j(x_\lambda)}\left[U + f(x_\lambda)V\right]$$

$$F_iY_j = \frac{\omega_\lambda(x_\lambda - 1)}{2\mathcal{E}_j(x_\lambda)}\left[U - f(x_\lambda)V\right]$$

$$F_iF_j = \delta_{ij}\delta_{\lambda\mu} + \frac{\omega_\lambda}{\mathcal{E}_j(x_\lambda)}W$$

All the operations have thus been put into matrix form and are now particularly suitable for high speed computation on an electronic computer. Gaussian integration formula was used for the integration of both the angular integrals X and Y as well as for the integral in Eqn. (1). The angular integrations were written as

$$X_o = \frac{1}{B+\sqrt{A+2ps}}\left[2 + \sum_i \omega_i F(x_i)\right]$$

$$X_1 = \frac{1}{B+\sqrt{A+2ps}}\sum_i \omega_i x_i F(x_i)$$

$$Y_o = \frac{1}{B+\sqrt{A+2ps}}\,\frac{1}{\sqrt{A+2ps}}\left[2 + \sum_i \omega_i G(x_i)\right]$$

$$Y_1 = \frac{1}{B+\sqrt{A+2ps}}\,\frac{1}{\sqrt{A+2ps}}\sum_i \omega_i x_i G(x_i)$$

where $$F(x) = \frac{\sqrt{A+2ps} - \sqrt{A+2psx}}{B+\sqrt{A+2psx}}$$

$$G(x) = F(x)\left[\frac{B+\sqrt{A+2ps}+\sqrt{A+2psx}}{\sqrt{A+2psx}}\right]$$

and w_i are the Gaussian weights with x_i the corresponding pivotal points. A four-point Gaussian formula was found quite adequate and checked accurately with the analytic formulae given in the previous page. Programmes were written under the General Interpretative Scheme of Deuce to produce the re-arranged kernels and the linear simultaneous equations (13) - (15) were then solved by a Basic Programme to obtain the scattering amplitudes $f_i(p)$.

2.18 The S-matrix.

In the previous chapter, we have described the method of numerical solution to obtain the scattering amplitude $f_i(p)$ from the coupled integral equations

$$f_i(p) = f_i^B(p) + \sum_j \int_0^\infty K_{ij}(p,s) f_j(s) s^2 ds \tag{1}$$

Now various boundary conditions may be used for $f_i(p)$, depending on the values of λ_i and on the way the integral over s is taken. In numerical work, it is easiest to deal with real functions i.e. we use standing waves in all channels instead of travelling waves so that we take the principal value of the integral over s. The most convenient choices are standing waves of the form cos $k_i(r_i-b_i)$ and sin $k_i(r_i-b_i)$ where b_i are the channel radii. The first of these has zero derivative at $r_i = b_i$ and the second has zero value at $r_i = b_i$. We thus define the basic set Φ_i by

$$\Phi_i = \phi_{ij}(r_j)\chi_j \tag{2}$$

where χ_j is the spin-dependent part and

$$\phi_{ij}(r_j) = \left[\frac{A_j}{k_j} \sin k_j r_j + B_j \cos k_j r_j \right] r_j^{-1}$$

where we have put $b_i = 0$ for convenience.

We must now compare this with the Fourier transform of our Tamm-Dancoff scattering wave function $\chi_i(p)$. The configuration space wave function for s-waves corresponding to $\chi_i(p)$ is given by

$$\psi_i(r) = \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty \chi_i(p)\, j_0(pr)\, p^2 dp$$

where asymptotically

$$j_0(pr) \rightarrow \frac{\sin pr}{pr}$$

Let the asymptotic form of $\psi_i(r)$ be written as $\phi_i(r)$

Then

$$r\phi_i(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin pr\; \chi_i(p)\, p\,dp$$

Substituting

$$\chi_i(p) = \lambda_i\, \delta(E-E_i(p)-\omega_i(p)) + P\frac{1}{E-E_i(p)-\omega_i(p)} f_i(p)$$

we have

$$r\phi_i(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty p\,dp \sin pr \left[\lambda_i\, \delta(E-E_i(p)-\omega_i(p)) + P\frac{1}{E-E_i(p)-\omega_i(p)} f_i(p)\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty p\,dp \sin pr \left[\lambda_i\, \delta(E-E_i(p)-\omega_i(p)) + P\frac{1}{E-E_i(p)-\omega_i(p)} f_i(k_i) + g_i(p)\right] \quad \ldots\ldots (3)$$

where $g_i(p)$ does not contain singularities at $p = k_i$. The contribution of small $r < r_0$ to this integral is given by this term but since the behaviour of $\psi(r)$ is not known there, this finite term will remain undetermined. But the terms which become singular near $p = k_i$ are the leading terms in Eq. (3) and these are given by the asymptotic behaviour of $\phi_i(r)$. Thus asymptotically

$$r\phi_i(r) \sim \sqrt{\frac{2}{\pi}} \int \sin pr \left[\lambda_i \ \delta(E - E_i(p) - \omega_i(p)) + P \frac{1}{E - E_i(p) - \omega_i(p)} f_i(k_i) \right] p\,dp$$

Since however

$$\frac{1}{E - E_i(p) - \omega_i(p) + i\epsilon} = P \frac{1}{E - E_i(p) - \omega_i(p)} - i\pi\,\delta\left(E - E_i(p) - \omega_i(p)\right)$$

and

$$\int \frac{p\,dp \ \sin pr}{E - E_i(p) - \omega_i(p) + i\epsilon} = -\pi k_i \left(\frac{\partial p}{\partial \mathcal{E}_i(p)} \right)_{p=k_i} e^{ik_i r} \qquad \mathcal{E}_i(p) = E_i(p) + \omega_i(p)$$

we have

$$r\phi_i(r) \sim \sqrt{\frac{2}{\pi}} \left[\lambda_i \sin k_i r - \pi f(k_i) \cos k_i r \right] k_i \left(\frac{\partial p}{\partial \mathcal{E}_i(p)} \right)_{p=k_i} \qquad (4)$$

Comparing (2) and (4) we have

$$A_i = \sqrt{\frac{2}{\pi}}\ \lambda_i\ k_i^2 \left(\frac{\partial p}{\partial \mathcal{E}_i(p)} \right)_{p=k_i}$$

$$B_i = -\sqrt{\frac{2}{\pi}}\ \pi f(k_i)\ \ k_i \left(\frac{\partial p}{\partial \mathcal{E}_i(p)} \right)_{p=k_i}$$

Since there are three coupled equations, three independent solutions of $f_i(k_i)$ may be found corresponding to three different values of λ_i and we will have the amplitudes A_i^{μ} and B_i^{μ} ($\mu = 1,2,3$) from which the S-matrix can be determined.

Now the S-matrix is defined by means of a basic set of travelling wave functions. The wave function describing a reaction initiated through channel i is denoted by Ψ_i . The behaviour of Ψ_i in the region of configuration space corresponding to some channel j is

$$\Psi_i = \psi_{ij}(r_j)\,\chi_\beta$$

where if $i \neq j$,

$$\psi_{ij}(r_j) = -\frac{1}{r_j}\, S_{ij}\, e^{ik_j r_j}\, \frac{1}{\sqrt{I_j}}$$

and in channel i,

$$\psi_{ii}(r_i) = \frac{1}{r_i}\left[e^{-ik_i r_i} - S_{ii}\, e^{ik_i r_i} \right] \frac{1}{\sqrt{I_i}}$$

where I_i is the 'current' in the channel i.

Expressing the complete set of functions Φ_i as linear superpositions of the complete set Ψ_i ,

$$\Phi_i = \sum_j C_{ij}\, \Psi_j$$

we determine the coefficients C_{ij} by comparing the behaviour of the right and left-sides of this equation in an arbitrary channel

Equating the coefficients of $\exp(ik_l r_l)$ we have

$$\frac{1}{2}\left[\frac{A_i^\mu}{k_i} \delta_{il} + iB_l^\mu \right] = -i \sum_j C_{ij} S_{jl} \frac{1}{\sqrt{I_l}} \tag{6}$$

The coefficients of $\exp(-ik_l r_l)$ lead to the equation

$$\frac{1}{2}\left[\frac{A_i^\mu}{k_i} \delta_{il} - iB_i^\mu \right] = -i C_{il} \frac{1}{\sqrt{I_l}} \tag{7}$$

Substituting (7) into (6)

$$\frac{A_i^\mu}{k_i} \delta_{il} + i B_l^\mu = \sum_j \left[\frac{A_j^\mu}{k_j} \delta_{ij} - iB_j^\mu \right] S_{jl} \sqrt{\frac{I_j}{I_l}}$$

In the centre of mass system,

$$I_i = k_i \left[\frac{1}{E_i(k_i)} + \frac{1}{\omega_i(k_i)} \right] = \frac{k_i E}{E_i(k_i)\omega_i(k_i)}$$

Using (5) then we obtain

$$\sqrt{k_l E_l \omega_l} \left[\lambda_l^\mu - i\pi f_l^\mu(k_l) \right] = \sum_j \left[\lambda_j^\mu + i\pi f_j^\mu(k_j) \right] S_{jl} \sqrt{k_j E_j \omega_j} \tag{8}$$

If the S-matrix is complex, we can write

$$S = s + ir$$

and equate the real and imaginary parts of (8)

$$\sqrt{k_i E_i \omega_i}\, \lambda_i^\mu = \sum_j \left[\lambda_j^\mu s_{ji} - \pi f_j^\mu(k_j) r_{ji} \right] \sqrt{k_j E_j \omega_j} \tag{9}$$

$$-\pi\sqrt{k_i E_i \omega_i}\; f_i^{\mu}(k_i) = \sum_j \left[\lambda_j^{\mu} s_{ji} + \pi f_j^{\mu}(k_j)\, r_{ji}\right] \sqrt{k_j E_j \omega_j} \qquad (10)$$

Treating s_{ij} and r_{ij} as eighteen unknown independent quantities, we can determine them by arbitrarily choosing λ_i for each solution. For example one can write

1) $\lambda_1^1 = \lambda_2^1 = \lambda_3^1 = 1$

2) $\lambda_1^2 = \lambda_2^2 = 1 \qquad \lambda_3^2 = -1$

3) $\lambda_1^3 = 1 \qquad \lambda_2^3 = \lambda_3^3 = -1$

The choice is quite arbitrary and must be revised in the light of numerical solutions which must be as different as possible.

Finally a good check on the S-matrix can be obtained from its symmetry property $S_{ij} = S_{ji}$ and the unitarity relation $S^{\dagger}S = 1$. These properties were checked in the present calculation up to the 5th place of decimals.

2.19 The Cross-sections.

The cross-sections leading from an entrance channel i to an exit channel j is given by

$$\sigma_{ij} = \frac{4\pi k_j}{k_i} \left| T_{ij} \right|^2 \tag{1}$$

where k_i is the centre of mass momentum in the channel i. The S-matrix is given by

$$S_{ij} = \delta_{ij} + 2i \sqrt{k_i}\, T_{ij} \sqrt{k_j} \tag{2}$$

From isotopic spin considerations alone, one can show that the T-matrix elements for the various reactions are given by the following relations.

$$\begin{array}{lll}
K^- + p \rightarrow & K^- + p & T_{el.} = \frac{1}{2}(T^0_{11} + T^1_{11}) \\
& \bar{K}^\circ + n & T_{c.e.} = \frac{1}{2}(T^0_{11} - T^1_{11}) \\
& \Sigma^+ + \pi^- & T_{\Sigma^+\pi^-} = \frac{1}{\sqrt{6}} T^0_{13} - \frac{1}{2} T^1_{13} \\
& \Sigma^- + \pi^+ & T_{\Sigma^-\pi^+} = \frac{1}{\sqrt{6}} T^0_{13} + \frac{1}{2} T^1_{13} \\
& \Sigma^\circ + \pi^\circ & T_{\Sigma^\circ\pi^\circ} = \frac{1}{\sqrt{6}} T^0_{13} \\
& \Lambda + \pi^\circ & T_{\Lambda\pi} = \frac{1}{\sqrt{2}} T^1_{12}
\end{array} \tag{3}$$

Using (1), (2) and (3) one can obtain the cross-sections for the various scattering and production processes.

We may also note here that for the low energies we are considering, the Coulomb and mass-difference (between $\overline{K}^{o}$ and K^{-}) corrections can not be neglected. A number of authors have shown how to modify the Wigner R-matrix formalism to include these corrections. If δ is the complex scattering phase-shift, we have

$$S = e^{2i\delta}$$

and defining the amplitude A by the relation

$$kA = \tan\delta$$

it has been shown by Jackson & Wyld that the mass-difference can be phenomenologically taken into account by writing the elastic differential cross-section in the form

$$\frac{d\sigma_{el.}}{d\Omega} = \left| \frac{A_o + A_1 - 2ik'A_oA_1}{\Delta} \right|^2$$

where $\Delta = 1 - \frac{i}{2}(k+k')(A_o+A_1) - kk'A_oA_1$; k_o, k' represent the centre of mass momenta for the systems $K^{-}+p$ and $\overline{K}^{o}+n$ respectively. Introducing Coulomb correction, this relation is transformed into

$$\frac{d\sigma_{el.}}{d\Omega} = \left| \frac{\eta\, e^{2i\eta \ln \sin\theta/2}}{2k \sin^2\theta/2} + C^2\, \frac{A_0 + A_1 - 2ik' A_0 A_1}{\Delta} \right|^2$$

where the penetration factor C is given by

$$C^2 = 2\pi\eta \Big/ \left(1 - e^{-2\pi\eta}\right) \qquad \eta = \frac{1}{kB} \qquad B = \frac{\hbar^2}{\mu e^2}$$

and the Δ now becomes

$$\Delta = 1 - \frac{i}{2}(A_0 + A_1)\left[k' + C^2 k(1 - i\tan\alpha)\right) - A_0 A_1 k' C^2 k (1 - i\tan\alpha)$$

and

$$\tan\alpha = -\frac{2\eta}{C^2}\left[\ln 2kR + 2\gamma + \mathrm{Re}\,\psi(i\eta)\right] \qquad \gamma = \text{Euler constant.}$$

Dalitz and Tuan have recently given a derivation of these relations within the 'effective range' formalism. Since the primary interest in the present work is a study of the behaviour of the model, it was not felt worthwhile to include the above corrections.

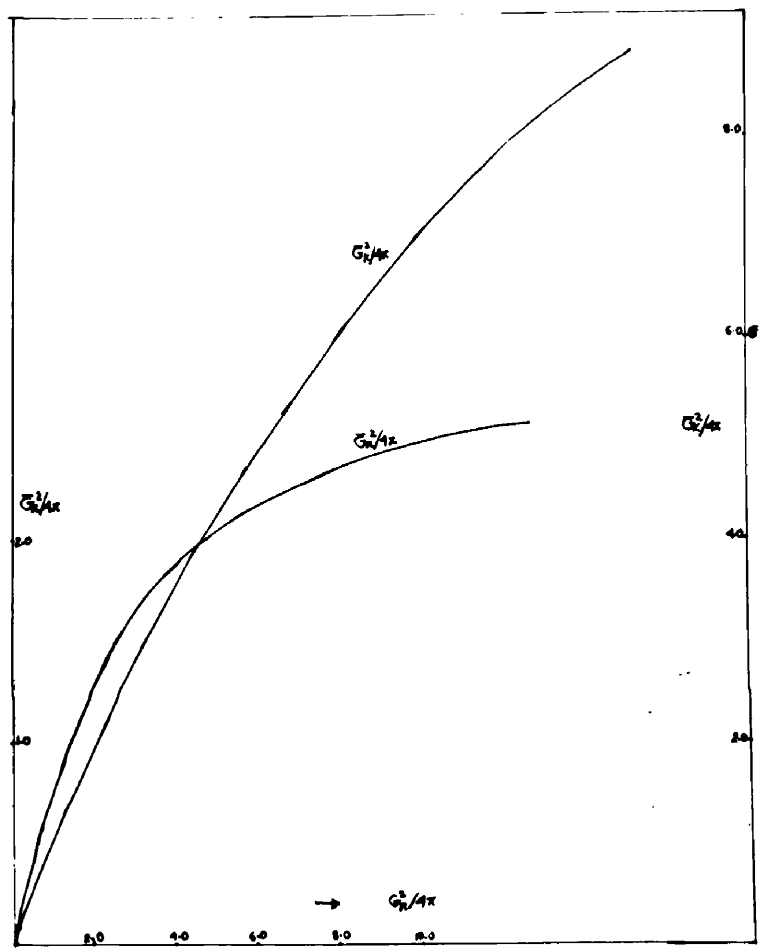


Fig.1

THE RENORMALISED COUPLING CONSTANT AS A FUNCTION OF

THE UNRENORMALISED CONSTANT.

2.20 The Results.

In order to reduce the number of coupling constants, we have put

$$3G^2_{KN\Sigma}/4\pi = G^2_{NK\Lambda}/4\pi = G^2_K/4\pi$$

in model A and in the second version of the model, the pion coupling constants were made equal

$$G^2_{NN\pi}/4\pi = G^2_{\Lambda\Sigma\pi}/4\pi = G^2_{\Sigma\Sigma\pi}/4\pi = G^2_\pi/4\pi$$

It should be noted that in common with the Lee model and several other models, while the unrenormalized coupling constants may take any real value, there is only a limited range in which the renormalized coupling constants must lie if the theory is to remain hermitian and ghost states are to be avoided.

This follows from the definition in model B,

$$G_i^2 = \frac{\mathcal{G}_i^2}{1 - \sum_i \mathcal{G}_i^2 I}$$

where we must have $\sum_i \mathcal{G}_i^2 I < 1$ in order that $G_i^2 > 0$. The relation between the renormalized and the unrenormalized coupling constants depends on the value of the cut-off chosen when evaluating the integral I. In the present work, two cut-offs $k_{max} = M_\Lambda$ and $0.5\ M_\Lambda$ were initially tried and in Fig. 1, one example of the

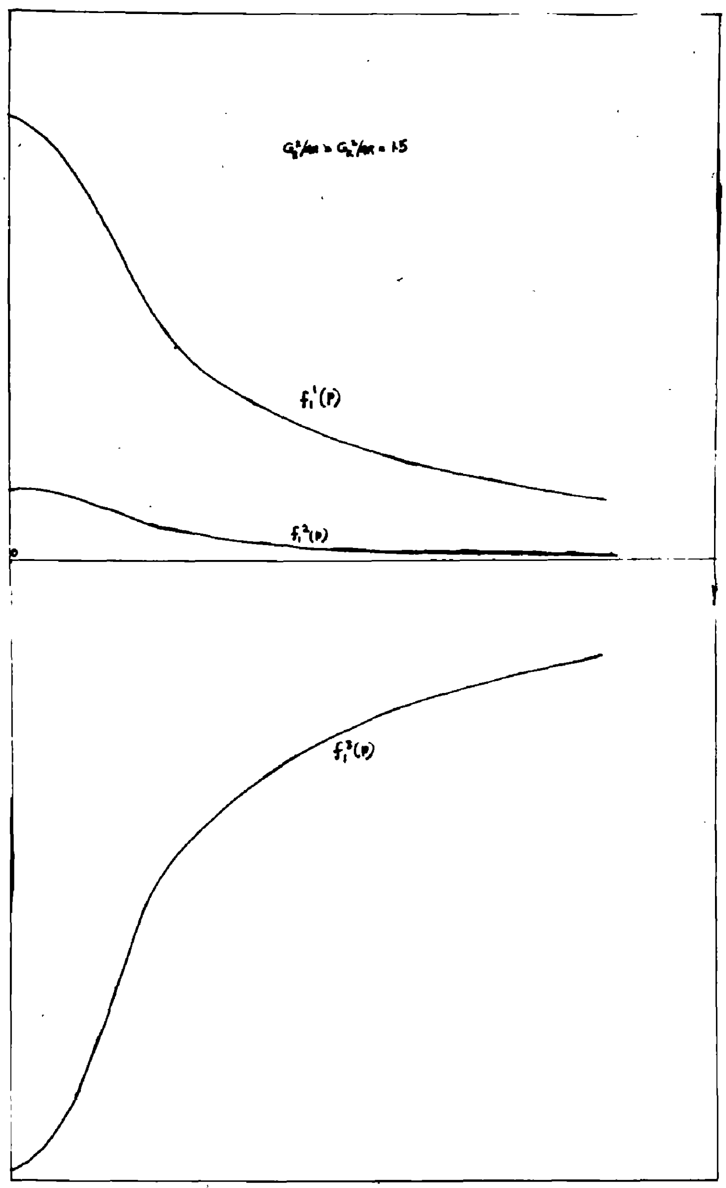


Fig. 2

THREE INDEPENDENT SCATTERING AMPLITUDES FOR THE FIRST CHANNEL ON MODEL A .

relationship between $\mathcal{G}$ and G is shown for a cut-off at 0.5 M . In the remainder of the figures, for convenience, the cross-sections are displayed as a function of the unrenormalized coupling constants G for this cut-off.

In Fig. 2, we have displayed typical scattering amplitudes for the channel $\bar{K}$+N obtained from solution of the integral equations in the three cases $\lambda^{\mu}(\mu = 1,2,3)$. It will be noticed that the extrapolated amplitudes $f_1^{\mu}(k_1)$ are widely different in the three cases and the solutions fall off smoothly. It may be mentioned here that as recoil effects are included in the calculations, the scattering equations do not require any additional cut-off. The integrations were however arbitrarily terminated when the scattering amplitudes have fallen off sufficiently. The limitation of storage space in the DEUCE was another factor which had to be borne in mind in this connection.

The energy variation of the elastic and inelastic cross-sections for the case

$$G_K^2/4\pi = 5.1$$

$$G_\pi^2/4\pi = 4.8$$

is shown for model B in Fig. 3 for a laboratory $\bar{K}$-meson

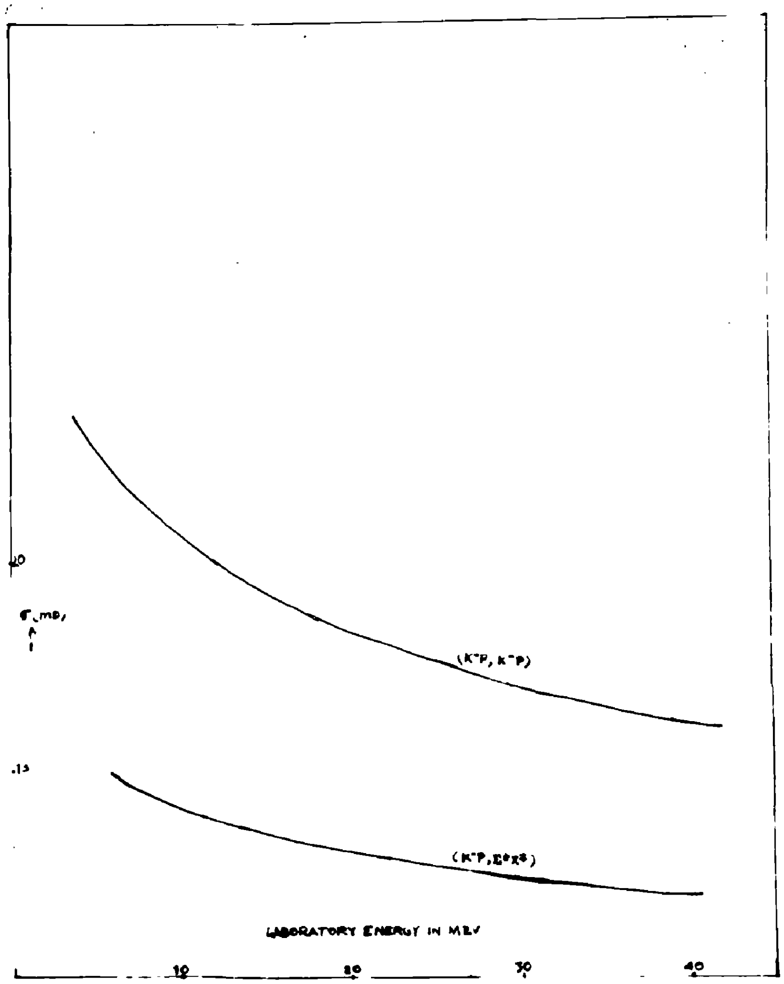


Fig. 3

THE ENERGY VARIATION OF THE CROSS-SECTIONS FOR ELASTIC SCATTERING OF K^--MESONS ON PROTONS AND THE PRODUCTION OF $\Sigma^{\pm}$ HYPERONS BY K^--MESONS ON PROTONS FOR $G_\pi^2/4\pi = G_K^2/4\pi = 5.0$ ON MODEL B.

energy of up to 40 MeV. It will be noticed that this set of coupling constants can not reproduce the experimental cross-sections. Experimentally, the elastic cross-section is about 90 mb at 20 MeV from bubble chamber studies and 55 mb from emulsion measurements. The production cross-section for charged $\Sigma^{\pm}$ hyperons is about 60 mb.

The above unrenormalized coupling constants correspond to the following renormalized coupling constants

$$g_K^2/4\pi = 4.2$$

$$g_\pi^2/4\pi = 2.0$$

The Fig. 4 displays the variation on model B of the elastic cross-section with unrenormalized coupling constant $G_K^2/4\pi$ for various values of $G_\pi^2/4\pi$. The production cross-section for $\Sigma^{\pm}$ hyperons is also shown for one value of $G_\pi^2/4\pi$.

The Fig. 5 shows the variation on model A of different elastic and production cross-sections for the coupling constant $G_K^2/4\pi = 1.5$. The pronounced maxima in the elastic cross-sections $\Sigma\pi \rightarrow \Sigma\pi$ for isotopic spin T = 1 also occurs in the isotopic spin state T = 0 which is not shown here.

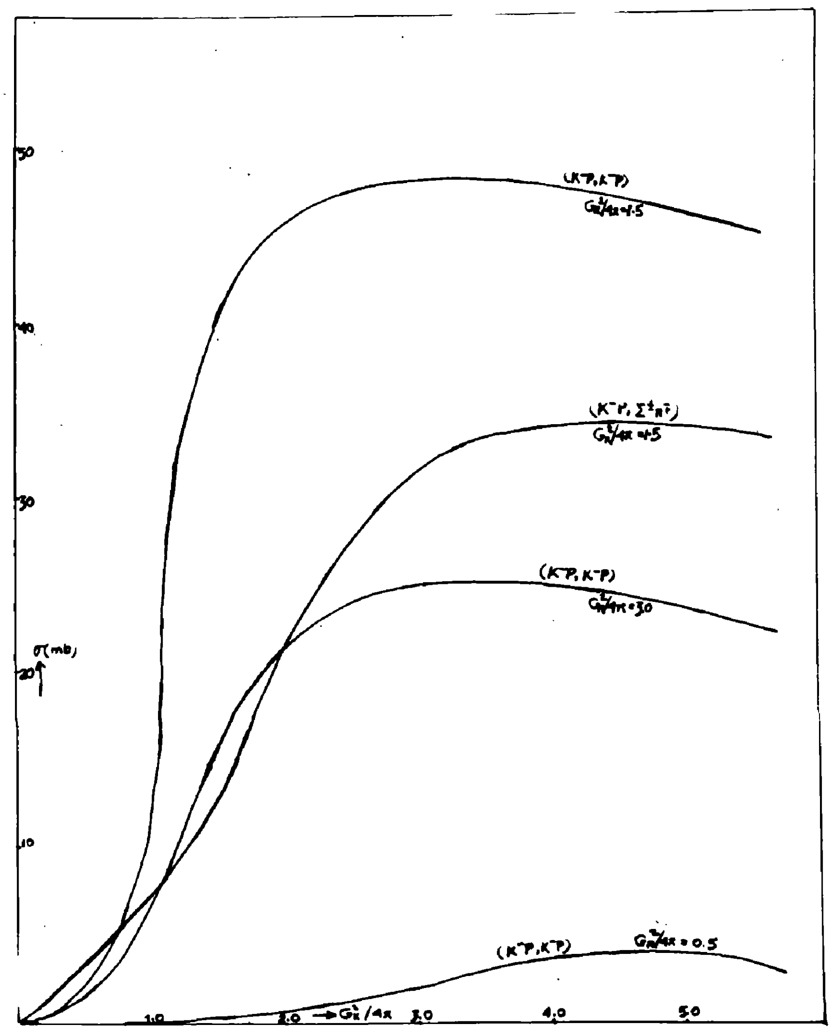


Fig. 4.4

THE VARIATION ON MODEL B OF THE ELASTIC CROSS-SECTION WITH UNRENORMALISED COUPLING CONSTANT $G_K^2/4\pi$ FOR VARIOUS $G_\pi^2/4\pi$. THE PRODUCTION CROSS-SECTION FOR $\Sigma^\pm$-HYPERON IS ALSO SHOWN FOR ONE VALUE OF $G_\pi^2/4\pi$.

The Fig. 6 shows the variation on model B of elastic cross-sections with pion coupling constants $G_{\pi}^2/4\pi$ for various values of $G_K^2/4\pi$.

The figures (4) to (6) display the variation of the cross-sections with coupling constants at a single energy of 20 MeV in the laboratory system. In all these displayed results G_{π} and G_K were taken to be of the same sign. The distinct case where G_{π} and G_K are of opposite sign has also been investigated. The Fig. 7 shows the variation of cross-sections for elastic and production processes against $G_K^2/4\pi$ in this case.

The S-matrices obtained in all the above cases are given in the appendix. The amount of electronic machine time required for each point on the graphs turned out to be roughly eight hours and no detailed energy variation for the best coupling constants obtained for the model could be attempted.

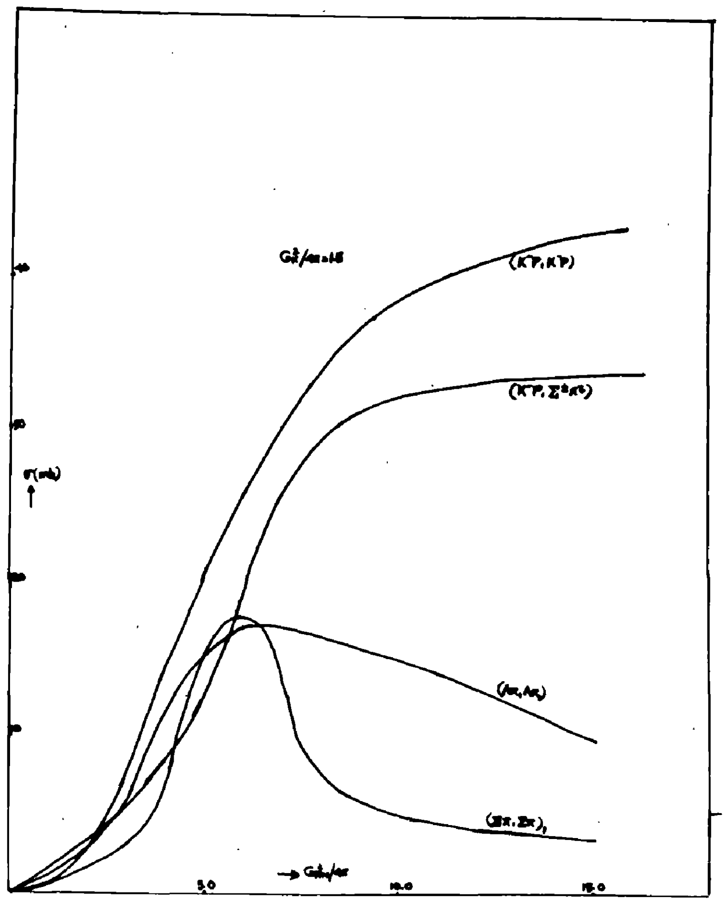


Fig. 5

THE VARIATION ON MODEL A OF DIFFERENT ELSTIC AND PRODUCTION CROSS-SECTIONS FOR THE COUPLING CONSTANT $G_K^2/4\pi = 1.5$.

2.21 <u>Discussion of the Results and Conclusions</u>.

The model we have discussed here is based on the assumption of the pseudo-scalar character of the $\bar{K}$-meson and even Λ and Σ hyperon relative parity. The possibility of the scalar character of the coupling is not in accord with the photo-production data as shown in a preliminary analysis of Moravcsik (58) and also with the Tamm-Dancoff calculations mentioned by Bialkowski and Jurewickz (53) on $K^{\pm}$ meson-nucleon scattering. Experimentally, the existence of the reaction

$$K^- + He^4 \longrightarrow H^4 + \pi^-$$

would prove the pseudo-scalar nature of the $KN\Lambda$-coupling provided the hyperfragment is in the ground state and of spin zero (Dalitz 54, Block et al. 65 and Day & Snow 56). Recent investigations seem to indicate that the K-meson will turn out to be pseudoscalar at least with respect to the Λ-hyperon. The question of the relative Σ and Λ parity has not been definitely answered as yet. The possibility that it might be odd has been mentioned by Barshray (57) and Gursey (58).

The hypothesis of a three or four boson-field interactions of the type $KK\pi$ or $K^{\dagger}K\pi\pi$ has also been

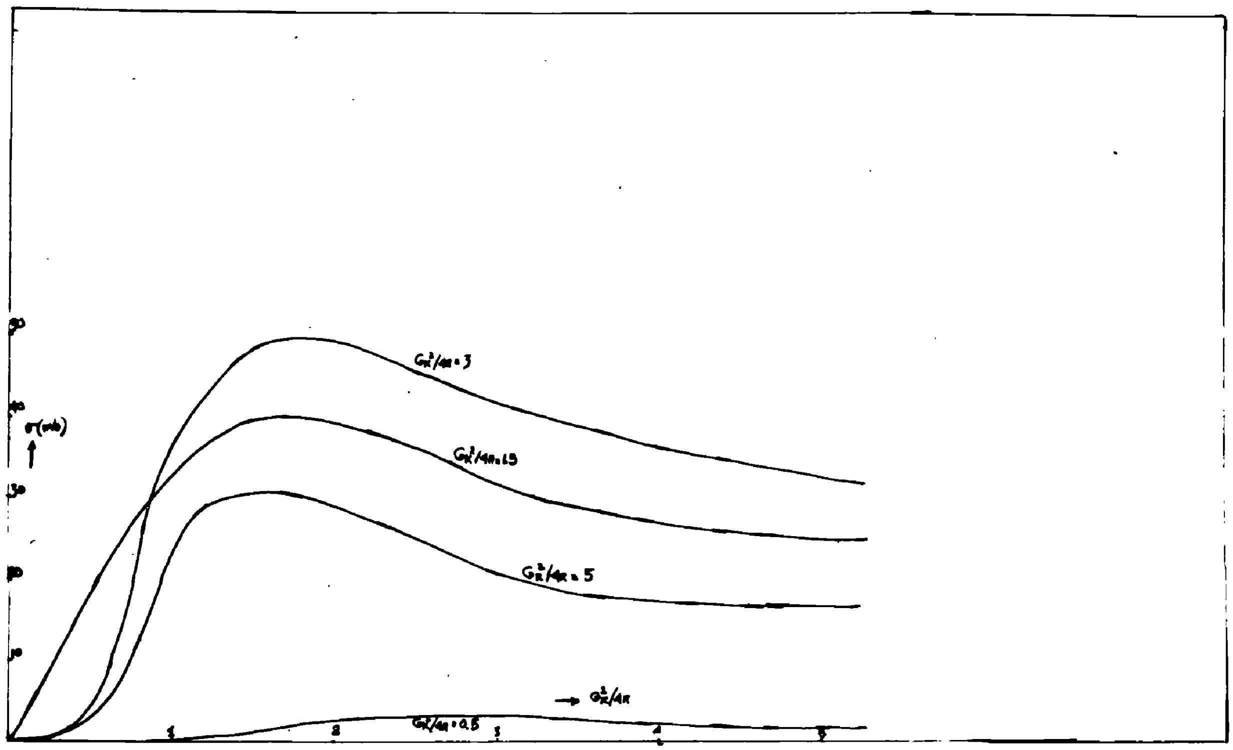


Fig. 6

THE VARIATION ON MODEL B OF ELASTIC CROSS-SECTIONS AGAINST PION COUPLING CONSTANT $G_\pi^2/4\pi$ FOR VARIOUS VALUES OF $G_K^2/4\pi$:

discussed and Pais (59) has advanced arguments in favour of an odd relative parity between K^+ and K^0.

From all these considerations it is apparent that the fundamental interaction Lagrangian is by no means completely certain and the reduced Hamiltonian we have discussed can only be considered as a rather crude model likely to give qualitative indications of a certain aspect of the interaction mechanism. Because a reduced model Hamiltonian has been used, the coupling constants G_π and G_K are not directly comparable with those employed in calculations using the full Hamiltonian. However, from a study of the graphs it will be seen that for reasonable values of G_π and G_K, a substantial fraction of the observed cross-sections can be obtained with the process considered namely pair creation in the intermediate states. In fact with coupling constants

$$G_\pi^2/4\pi = 1.5 \qquad G_K^2/4\pi = 3$$

which corresponds to the renormalized coupling constants

$$g_\pi^2/4\pi = 1.0 \qquad g_K^2/4\pi = 2.0$$

the cross-section $\sigma_{el.}$ turns out to be about 50 mb at 20 MeV which compares well with the emulsion data but

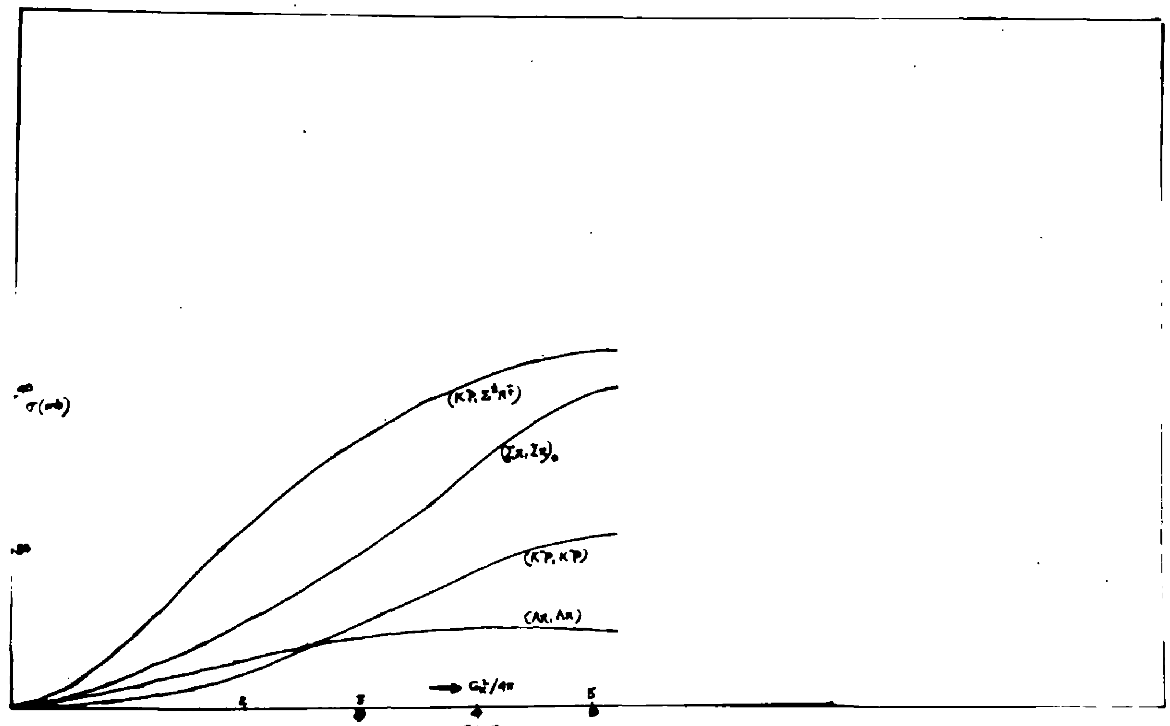


Fig. 7

THE VARIATION OF CROSS-SECTIONS FOR ELASTIC AND PRODUCTION PROCESSES AGAINST $G_K^2/4\pi$ IN THE CASE WHEN $G_K/\sqrt{4\pi}$ IS NEGATIVE, RELATIVE TO $G_\pi/\sqrt{4\pi}$.

not with the bubble chamber result. However a quantitative agreement with experimental data is not of particular importance in such model calculations. The coupling constants predicted by the model do however merit a little discussion. According to the results obtained by Matthews and Salam and by Igi, it appears that the value of the coupling constant $g_K^2/4\pi$ is of the order of 4. This value may of course undergo considerable change with increasing accuracy of experimental data. Recently Kycia, Kerth and Baender (60) have concluded from dispersion relation analysis of the most recent data that if K-mesons were scalar, $g_K^2/4\pi$ would be less than about 0.6 and if pseudo-scalar less than about 10.

The question of the π-coupling constants is still more difficult as the only one known with any certainty is $g_{NN\pi}^2/4\pi$. In the present calculations we were forced to use

$$G_{NN\pi}^2/4\pi = G_{\Lambda\Sigma\pi}^2/4\pi = G_{\Sigma\Sigma\pi}^2/4\pi$$

in order principally to cut down computation time. This global symmetry is known to lead to wrong branching ratios of the Σ and Λ productions. The branching ratios of the reactions $K^- + p \to \Sigma^\pm + \pi^\mp$ with K^--mesons nearly at rest indicate that $|\delta_1 - \delta_0| \sim 62^\circ$ where δ_1 and δ_0 are phase-shifts of the π-Σ scattering in the final state of isotopic spin 1 and 0 respectively. Kawarabayashi (61) and Capps (62) have pointed out that $g_{\Lambda\Sigma K}/\sqrt{4\pi}$ and $g_{\Sigma\Sigma K}/\sqrt{4\pi}$ should be much different from each other in order to give the large phase-difference.

The best value obtained in the present model (model B) is

$$g_{\pi}^{2}/4\pi \sim 1$$

which should be compared with the result of Bosco and Stroffolini who also noticed that a coupling constant $g^{2}_{NN\pi}/4\pi$ of the order of unity reproduced the S-wave pion-nucleon phase-shifts reasonably well.

As regards the sign of the K^--p interaction, experiments seem to indicate that in the $K^{\pm}$-p scattering at low energies, the S-wave K^+-p interaction is repulsive and K^--p interaction is attractive. In the Born approximation the K^+-p interaction is attractive or repulsive according as NKY-coupling is scalar or pseudo-scalar whereas the K^--p interaction is repulsive irrespective of the coupling.

Recently Ferreira (63) has made a 4th order perturbation calculation for the K^--p scattering at the threshold and pointed out that for any combination of the NYK-coupling, the 4th order term is of opposite sign to the Born term and large enough to compensate for or exceed the Born term. The conclusions of Ceolin, Dallaporta and Taffara (64) agree with this.

In the present model, we have observed a similar behaviour. It is well-known that the sign of the interaction 'potential' depends on the sign of the real part of the K^--p s-state scattering length. If we define (Hamilton 65) the elastic scattering amplitude by

$$f(\theta) = -(2\pi)^2 k \left(\frac{\partial k}{\partial E}\right) \langle k | T | k \rangle$$

then according to Dalitz and Tuan (66), a negative sign of the real part of the forward scattering amplitude corresponds to an attractive K^--p interaction. This sign convention agrees with the one used by Nogami (67) and Jackson and Wyld (68) according to whom a positive sign of the real part of the tangent of the phase-shift corresponds to an attractive potential since by definition

$$T = e^{i\delta} \sin\delta / k$$

In the effective range analysis of Dalitz and Tuan, one writes $kA_T = \tan \delta_T$ and $A_T = a_T + ib_T$ and it can be shown that ReT has the same sign as the pair (a_0, a_1) when they have a common sign. The positive solutions $(a+, b+)$ of Dalitz and Tuan therefore correspond to an attractive potential. Since the K^--p Coulomb amplitude is essentially real and positive, the positive sign for ReT corresponds to a constructive interference between Coulomb and nuclear scattering.

In the present calculations in model A, ReT_{11} which corresponds to elastic $\bar{K}N$-scattering is positive for the coupling constant $G^2_{NN\pi}/4\pi$ varying between 1 and 5 and for both isotopic spin states.

In the model B, ReT_{11} is positive for the coupling constant $G_\pi^2/4\pi$ up to 1.5 and $G_K^2/4\pi$ up to 1.5 and for both isotopic spins.

The same sign in both isotopic spin states agrees with the solutions of Dalitz and Tuan.

Lastly, we may mention an interesting behaviour of the cross-sections observed in the present calculations. From figs. (4), (5), (6) it can be seen that the cross-section increases in the way demanded by low order perturbation theory, for small G_π or G_K. However for large values of coupling, the coupling between the three channels becomes important and this results in a definite maximum in the cross-section as a function of G. This phenomena is known in other connections, for example the coupled equations of the type occurring in electron scattering by atoms have been investigated by Massey and Mohr with somewhat similar results.

Chapter Three.

On the Elastic Scattering of Pions by Alpha-Particles.

3.1 Introduction.

Although the elastic scattering of pions by deuterons has been studied by a number of authors, no attempt seems to have been made so far to study theoretically the interaction of pions with alpha-particles. In the impulse approximation used by Fernbach, Green and Watson (1) for pion-deuteron scattering, the scattering amplitude is expressed as a superposition of free nucleon scattering amplitudes and the effects of multiple scattering, nuclear binding and off the energy shell nature of pion-free nucleon scattering matrix neglected. Recently, Rockmore (2) has evaluated some of the corrections to the impulse approximation for pion-deuteron scattering. In particular he has extended Brueckner's (3) method to take into account the spin dependence of the pion-nucleon scattering matrix. From a perturbation calculation he finds the correction due to nuclear binding to be of the order of ten per cent and he also shows that Brueckner's neglect of the off-the-energy shell matrix elements in multiple scattering can not be justified.

The Brueckner method consists essentially in assuming that the pion propagator between two nucleons at r_1 and r_2 is of the form $e^{ik|r-r_i|}/|r-r_i|$. As has been pointed out by De Alfaro and Stroffolini (4) this assumption is correct only beyond the range of the pion interaction with both nucleons. Using a propagator which does not have singularities at small distances, they find that multiple scattering corrections are larger than the double scattering corrections and accordingly regard the agreement between experiments and Rockmore calculations as fortuitous.

Recently Bransden and Moorhouse (5) have considered a variational method for the problem which includes the effects of multiple scattering quite simply and found them to be quite small. It was therefore suggested that the method could be applied to processes such as pion-alpha particle scattering for which this correction was expected to be large.

Kozodaev (6) has experimentally investigated the interaction of pions with helium nuclei and found considerable decrease in elastic scattering cross-section at small angles. This is possibly due to an interference between Coulomb and nuclear scattering and it was concluded that interaction between π-mesons and alpha-particles is repulsive.

We have carried out a calculation of the pion-alpha particle scattering problem following the method of Bransden and Moorhouse and compared the results with the two differential cross-section curves presented by Kozodaev et al. In Chapter 2 we formulate the method and in Chapter 3 give the results. In Chapter 4 we discuss the results and in the appendix give a formula for multiple scattering.

3.2 Formulation of the Method.

The method has been described in detail by Bransden and Moorhouse and we shall show here the necessary modifications for pion-alpha particle scattering. Let the Hamiltonian of the target nucleus be H_n and the interaction between the meson field and the nucleon field be H_{int}. The unperturbed Hamiltonian is

$$H_o = H_n + \omega_o \qquad \omega_o^2 = \mu^2 + k^2 \tag{1}$$

where ω_o is the energy of the incident meson. The eigen-functions of H_o are $\Phi_{n,k}(r_1, r_2, \cdots)$ where $r_1, r_2, \cdots$ are the co-ordinates of the target nucleons:

$$\Phi_{n,k} = \phi_n(r_1 \cdots r_4)\, e^{ikz} \tag{2}$$

The ϕ_n are the normalized states of target helium nucleus with ϕ_o the ground state.

The eigenstates of the perturbed Hamiltonian H which in the absence of the interaction Hamiltonian H_{int} would be $\Phi_{o,k}$ is in its presence written as Ψ_k . Then the Schrodinger equation which describes the scattering is

$$(H_o + H_{int})\, \Psi_k = E_K \Psi_K \tag{3}$$

This equation has the boundary condition that at large distances from the scatterer

$$\Psi_k \rightarrow \phi_o(r_1, \cdots r_4)\, e^{ikz} = \Phi_{o,k} \tag{4}$$

Let us suppose now that Ψ_a in Eq. (3) describes a scattering event which originates in an eigenstate of H_o say ϕ_a. In terms of Møller (7) wave-matrix Ω, we write

$$\Psi_a = \Omega\, \phi_a \tag{5}$$

where Ω satisfies the Schwinger (8) expression

$$\Omega = 1 + \frac{1}{E - H_o + i\epsilon} H_{int}\, \Omega \tag{6}$$

operating on the state ϕ_a. The eq. (5) gives the scattering solution of eq. (3) with outgoing wave boundary condition. Here

$$E = E_a \text{ and } E_a\, \phi_a = H_o\, \phi_a \tag{7}$$

The transition operator is now defined by

$$\begin{aligned} T &= H_{int}\, \Omega \\ &= H_{int} + H_{int} \frac{1}{E - H_o + i\epsilon} T \end{aligned} \tag{8}$$

Instead of attempting to solve equation (8), Bransden and Moorhouse use a variational method to obtain the transition matrix.

Schwinger (8) has given the following variational expression for the transition matrix

$$T_{ba} = \frac{(\Psi_b^-, H_{int}\phi_a)(\phi_b, H_{int}\Psi_a^+)}{(\Psi_b^-, H_{int}\Psi_a^+) - \left(\Psi_b^-, H_{int}\frac{1}{E-H_0+i\epsilon}H_{int}\Psi_a^+\right)} \tag{9}$$

which should be stationary when $E = E_a = E_b$ for variations of Ψ_a^+ and Ψ_b^- about the correct solutions of Eq. (3). Chew (9) has suggested the simple trial functions $\Psi_a^+ = \phi_a$ and $\Psi_b^- = \phi_b$ namely the unperturbed eigenfunctions for use in the variational expression for the K-matrix.

Then the elastic scattering of pions by alpha particles is given by

$$T_{k'k} = \frac{L^2_{k'k}}{L_{k'k} - N_{k'k}} \tag{10}$$

where

$$L_{k'k} = (\phi_b, H_{int}\phi_a) \tag{11}$$

and

$$N_{k'k} = \left(\phi_b, H_{int}\frac{1}{E-H_0+i\epsilon}H_{int}\phi_a\right) \tag{12}$$

Here $\underline{k}$ and $\underline{k}'$ are the incident and scattered momenta of the pions. These equations are of course indefinite until an explicit form of H_{int} has been fixed. This is obtained from the fixed source meson theory with the interaction Hamiltonian

$$\mathcal{H}(\mathbf{r}) = \frac{\sqrt{4\pi}f}{\mu} \tau_k \frac{i\sigma\cdot k}{\sqrt{2\omega_k}} v(k) \; e^{-ik\cdot r} \tag{13}$$

where $v(k)$ is a cut-off function. Now applying the method of Tamm and Dancoff, we obtain the following equation for the scattering of a pion from a nucleon situated at $\mathbf{r}$:

$$(E-\omega_k)\psi(\mathbf{r},\mathbf{k}) = 4\pi f^2 \int_0^{k_{max}} \frac{d^3k'}{(2\pi)^3 2\sqrt{\omega_k \omega_{k'}}} \left[\frac{\sigma\cdot k' \, \sigma\cdot k \;\; \tau_{k'} \tau_k \;\; e^{i(k'-k)\cdot r}}{E - \omega_k - \omega_{k'}} + \frac{\sigma\cdot k \;\; \sigma\cdot k' \, \tau_k \tau_{k'} \;\; e^{i(k-k')\cdot r}}{E} \right] \Psi(\mathbf{r},\mathbf{k}') \tag{14}$$

where we have used a square cut-off in momentum space at momentum k_{max} and the nucleon has been assumed to be at rest.

It is well-known that the scattering of pions on nucleons in the energy range 80 to 400 MeV incident pion energy is predominantly in the isotopic spin state 3/2 and spin state 3/2. It is therefore a good approximation to take the pion-nucleon interaction as being due to a potential operating on the (3-3) state only. The only part of the kernel of eq. (14) which contributes is the first or cross-over term in the square bracket and we may write the potential in the form

$$\langle k' | V_{33} | k \rangle = \frac{8\pi f^2}{3\mu^2} \frac{v(k)\, v(k')\, kk'}{\sqrt{\omega_k \omega_{k'}}} \frac{P_{33}(k',k)}{\omega_0 - \omega_k - \omega_{k'}} e^{i(k'-k).r} \tag{15}$$

where P_{33} is the projection operator for the 3-3 state

$$P_{33} = \left(\delta_{k'k} - \tfrac{1}{3}\tau_{k'k}\right)\left(3\,k'.k - \sigma.k'\sigma.k\right) \tag{16}$$

With the interaction potential so defined, one might solve the pion-nucleon scattering from the equation

$$\left(\omega_0 - \omega_k + \frac{1}{2M^2}\nabla^2\right)\Psi(r,k) = \int \frac{d^3k'}{(2\pi)^3} \langle k' | V_{33} | k \rangle \Psi(r,k') \tag{17}$$

by the usual method of resolving Ψ into partial waves. All phase-shifts are zero except the (3-3) phase-shift and this can be found by employing Schwinger's variational expression for the K-matrix in pion-nucleon scattering. This is completely equivalent to the variational method of Cini and Fubini (10) as applied by Sartori and Wataghin (11). Instead of doing this, Bransden and Moorhouse obtained the pion-nucleon cross-section from the T-matrix using (9) and (17) and found the correct position and height of the resonance for a coupling constant

$$f^2 = 0.09$$

and cut-off $\omega_m = \sqrt{1 + k_m^2} = 6.8$. With these values, the interaction potential given by Eq. (15) is substituted in the pion-alpha particle scattering equation. The transition matrix for elastic pion alpha particle scattering can then be found from eqs. (10) - (12). The results are given in the following section.

3.3 Evaluation and Results.

We consider the elastic scattering of π-mesons by alpha particles. We write the wave-function of the alpha particle in the form

$$\phi_\alpha = \sqrt{N}\; e^{-\sigma^2 \sum r_{ij}^2} \chi_\alpha$$

where

$$\chi_\alpha = \frac{1}{\sqrt{6}}\left\{ \sigma_{12;34}\, \chi_{12}\chi_{34} - \sigma_{13;24}\, \chi_{13}\chi_{24} + \sigma_{14;23}\, \chi_{14}\chi_{23} \right\}$$

$$\sigma_{12;34} = \alpha_1\alpha_2\beta_3\beta_4 + \beta_1\beta_2\alpha_3\alpha_4$$

$$\chi_{12} = \frac{1}{\sqrt{2}}\left\{ p_1 n_2 - n_1 p_2 \right\}$$

Here $\sqrt{N}$ is the normalization constant and σ is a parameter to be chosen to fit the experimental binding energy of the alpha particle or its rms radius. Hofstadter (12) et al. have estimated the rms radius of the alpha particle to be 1.41 x 10^{-13} cm.

The principal defect of the Gaussian wave-function is its bad asymptotic behaviour as it falls off too rapidly with increasing separation of the nucleons. In order to improve the asymptotic behaviour of the function, one may employ wave-functions of the type (13):

$$\sqrt{N}\; e^{-\sigma_I \sqrt{r_{ij}^2}} \Big/ \left(r_{ij}^2\right)^n$$

where n = 1/2 leads to the simplest mathematical analysis.

In the following calculations only a Gaussian wave-function was used.

We take the incoming pions along the Z-axis and consider the scattering into an angle (θ, ϕ). Thus

$\underline{k} = (0,0,k) \quad \underline{k}' = (k \sin \Theta \cos \phi,\ k \sin \Theta \sin \phi,\ k \cos \Theta)$

and eq. (10) can be written as

$$T_{k'k}(\theta) = \frac{L^2_{k'k}(\theta)}{L_{k'k}(\theta) - N_{k'k}(\theta)}$$

with

$$L_{k'k}(\theta) = -\frac{128}{9}\frac{f^4 k^2}{\mu^2\omega^2}\cos\theta \; e^{-3q^2/32\sigma^2}$$

$$N_{k'k}(\theta) = -\frac{8\pi}{9}\frac{f^2\omega}{\mu^2} L_{k'k}(\theta)\, I(k) + \left(\frac{8\pi f^2}{3\mu^2}\right)^2 \frac{k^2}{\omega} \cos\theta \; e^{-(q^2+2P^2)/32\sigma^2} J(k)$$

where

$$I(k) = \int_0^{k_{max}} \frac{p^4\,dp}{\omega_p^3(\omega_p - \omega_0)}$$

$$J(k) = \int_0^{k_{max}} \frac{p^4\,dp}{\omega_p^3(\omega_p - \omega_0)} e^{-p(p-2P)/32\sigma^2}$$

$\underline{q} = \frac{1}{2}(\underline{k}' - \underline{k})$

and $\underline{P} = \frac{1}{2}(\underline{k}' + \underline{k})$

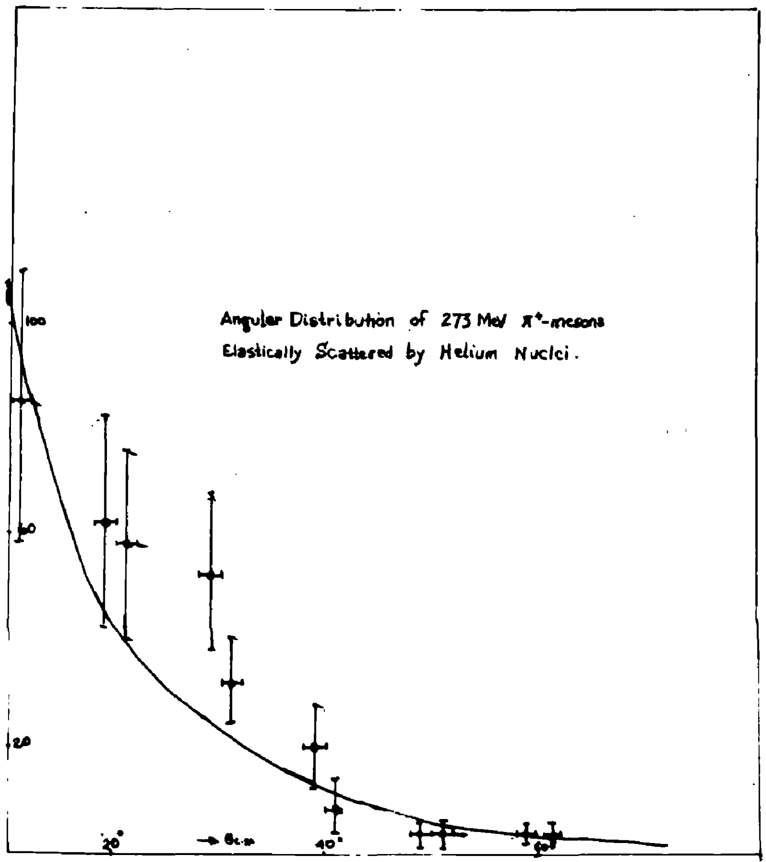


Fig. 1

ANGULAR DISTRIBUTION OF 273-MeV π^+- MESONS ELASTICALLY SCATTERED BY HELIUM NUCLEI.

In these equations $L_{k'k}(\theta)$ originates from a single scattering from the interaction potential $\langle k' | V_{33} | k \rangle$ and $N_{k'k}(\theta)$ comes from repeated scattering. The first term in $N_{k'k}$ corresponds to double scattering on the same nucleon whereas the second term corresponds to successive scattering from two nucleons. This term, which represents an effect of multiple scattering, was evaluated numerically and was found to be about 16% of the first term at forward scattering. Neglecting this term, we can write

$$T_{k'k}(\theta) = \frac{L_{k'k}(\theta)}{1 + \frac{8\pi f^2 \omega}{9\mu^2} I(k)}$$

The differential cross-section is then

$$\frac{d\sigma}{d\Omega} = \left| T_{k'k}(\theta) \right|^2 \frac{\omega^2}{4\pi^2 \left(1 + \frac{\omega}{4M}\right)^2}$$

In Figs. 1 and 2 we compare the calculated differential cross-sections in millibarns per steradian with the experimental results given by Kozodaev et al. We have also made Coulomb corrections at small angles; for higher angles it is negligible.

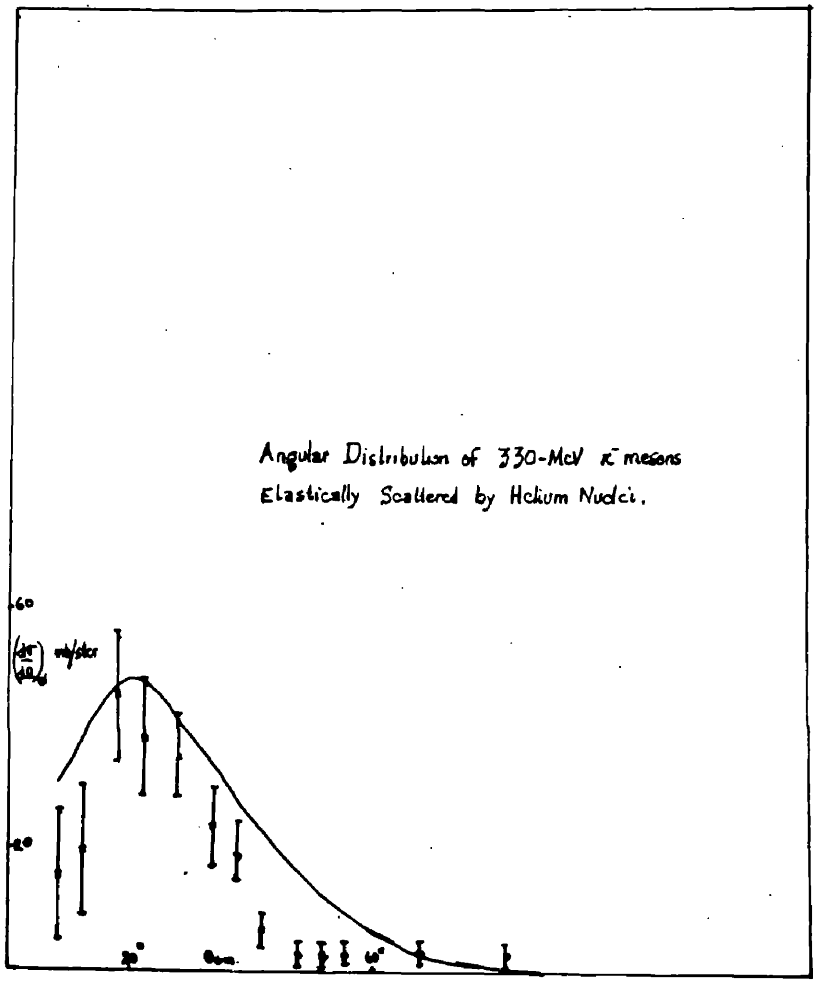


Fig. 2

ANGULAR DISTRIBUTION OF 330-MeV π^--MESONS ELASTICALLY SCATTERED BY HELIUM NUCLEI.

In these equations $L_{k'k}(\theta)$ originates from a single scattering from the interaction potential $\langle k' | V_{33} | k \rangle$ and $N_{k'k}(\theta)$ comes from repeated scattering. The first term in $N_{k'k}$ corresponds to double scattering on the same nucleon whereas the second term corresponds to successive scattering from two nucleons. This term, which represents an effect of multiple scattering, was evaluated numerically and was found to be about 16% of the first term at forward scattering. Neglecting this term, we can write

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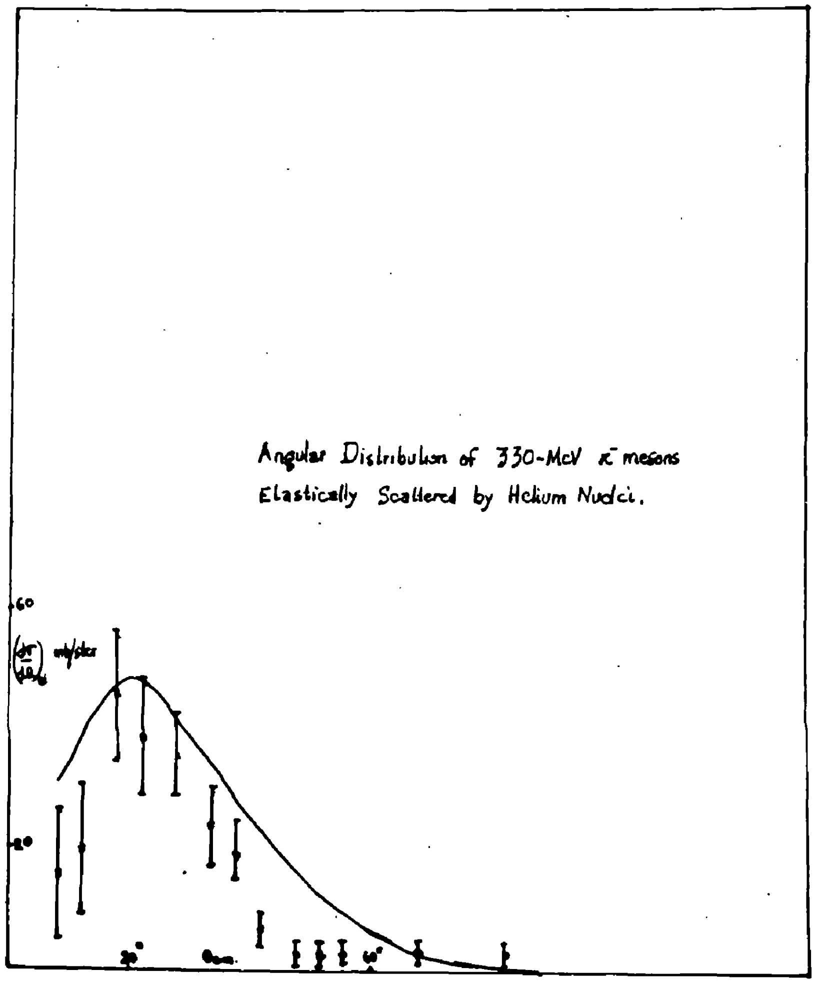

Fig. 2

ANGULAR DISTRIBUTION OF 330-MeV π^--MESONS ELASTICALLY SCATTERED BY HELIUM NUCLEI.

3.4 Discussion of the Result.

We see from the two figures that there is fairly good agreement with the experimental results. This seems to confirm the results of Bransden and Moorhouse that multiple scattering corrections predicted in this formalism are substantially less than the double scattering corrections. For this calculation and the calculation on pion-deuteron scattering, the coupling constant and the cut-off for the 3-3 pion-nucleon interaction are obtained from a variational method of solving the scattering equation. With this pion-nucleon interaction used in the pion alpha particle scattering equation, the same variational method gives reasonable agreement with experimental results.

Acknowledgment.

I would like to thank Professor J.C. Gunn for very kindly granting me the facilities of work in his department and for his continual interest and encouragement.

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Finally I would like to acknowledge a fellowship from the Colombo Plan Technical Assistance Programme and a study-leave from the University of Dacca, East Pakistan.

Appendix 1.

Using the variational method, the following equations are satisfied

$$\langle a_1^N(p)\bar{K}(-p)|E-H|\Psi\rangle = 0 \tag{1}$$

$$\langle a_1^\Lambda(p)\Pi(-p)|E-H|\Psi\rangle = 0 \tag{2}$$

etc., where the $|\Psi\rangle$ is given in Chapter 13.

Then

$$\langle a_1^N(p)\bar{K}(-p)|E-H|\chi_1(p)a_1^{N\dagger}(p)\bar{K}^\dagger(-p) + \chi_2(p)a_1^{\Lambda\dagger}(p)\Pi^\dagger(-p) + \chi_3(p)a_1^{\Sigma\dagger}(p)\Pi^\dagger(-p)$$
$$+\chi_4(q,p)a_r^{N\dagger}(q)a_s^{\Lambda\dagger}(p)b_t^{\bar{N}\dagger}(-p-q) + \chi_5(q,p)a_r^{N\dagger}(q)a_s^{\Sigma\dagger}(p)b_t^{\bar{N}\dagger}(-q-p)\rangle = 0 \quad \ldots\ldots (3)$$

$$\langle a_1^\Lambda(p)\Pi(-p)|E-H|\chi_1(p)a_1^{N\dagger}(p)\bar{K}^\dagger(-p) + \chi_2(p)a_1^{\Lambda\dagger}(p)\Pi^\dagger(-p) + \chi_4(q,p)a_t^{N\dagger}(q)b_r^{\bar{N}\dagger}(-q-p)a_s^{\Lambda\dagger}(p)$$
$$+\chi_6(q,p)a_t^{N\dagger}(q)b_r^{\Sigma\dagger}(-q-p)a_s^{\Lambda\dagger}(p)\rangle = 0 \quad \ldots\ldots (4)$$

$$\langle a_1^\Sigma(p)\Pi(-p)|E-H|\chi_1(p)a_1^{N\dagger}(p)\bar{K}^\dagger(-p) + \chi_3(p)a_1^{\Sigma\dagger}(p)\Pi^\dagger(-p) + \chi_5(q,p)a_t^{N\dagger}(q)b_1^{\bar{N}\dagger}(-q-p)a_s^{\Sigma\dagger}(p)$$
$$+\chi_7(q,p)a_t^{\Lambda\dagger}(q)b_r^{\Xi\dagger}(-q-p)a_s^{\Sigma\dagger}(p) + \chi_8(q,p)a_t^{\Sigma\dagger}(q)b_r^{\Xi\dagger}(-q-p)a_s^{\Sigma\dagger}(p)\rangle = 0 \quad \ldots\ldots (5)$$

$$\langle a_t^N(q)b_r^{\bar{N}}(-p-q)a_s^\Lambda(p)|E-H|\chi_1(p)a_1^{N\dagger}(p)\bar{K}^\dagger(-p) + \chi_2(p)a_1^{\Lambda\dagger}(p)\Pi^\dagger(-p)$$
$$+\chi_4(p,q)a_t^{N\dagger}(q)b_1^{\bar{N}\dagger}(-p-q)a_s^{\Lambda\dagger}(p)\rangle = 0 \quad \ldots\ldots (6)$$

$$\langle a_t^{N}(q)\, b_r^{\bar N}(-q-p)\, a_s^{\Sigma}(p)\, |E-H|\, \chi_1(p)\, a_r^{N\dagger}(p)\, \bar K^{\dagger}(-p) + \chi_3(p)\, a_r^{\Sigma\dagger}(p)\, \Pi^{\dagger}(-p) + \chi_5(q,p)\, a_t^{N\dagger}(q)\, b_r^{\bar N\dagger}(-p-q)\, a_s^{\Sigma\dagger}(p)\rangle = 0 \qquad \ldots.(7)$$

$$\langle a_t^{\Lambda}(q)\, b_r^{\bar\Sigma}(-p-q)\, a_r^{\Lambda}(p)\, |E-H|\, \chi_2(p)\, a_r^{\Lambda\dagger}(p)\, \Pi^{\dagger}(-p) + \chi_6(q,p)\, a_t^{\Lambda\dagger}(q)\, b_s^{\bar\Sigma\dagger}(-q-p)\, a_r^{\Lambda\dagger}(p)\rangle = 0 \qquad \ldots.(8)$$

$$\langle a_t^{\Sigma}(q)\, b_r^{\bar\Sigma}(-q-p)\, a_r^{\Lambda}(p)\, |E-H|\, \chi_3(p)\, a_r^{\Sigma\dagger}(p)\, \Pi^{\dagger}(-p) + \chi_7(q,p)\, a_t^{\Sigma\dagger}(q)\, b_s^{\bar\Sigma\dagger}(-q-p)\, a_r^{\Lambda\dagger}(p)\rangle = 0 \qquad \ldots.(9)$$

$$\langle a_t^{\Sigma}(q)\, b_r^{\bar\Sigma}(-q-p)\, a_r^{\Sigma}(p)\, |E-H|\, \chi_3(p)\, a_r^{\Sigma\dagger}(p)\, \Pi^{\dagger}(-p) + \chi_8(q,p)\, a_t^{\Sigma\dagger}(q)\, b_s^{\bar\Sigma\dagger}(-q-p)\, a_r^{\Sigma\dagger}(p)\rangle = 0 \qquad \ldots.(10)$$

From (6) - (10),

$$\chi_4(q,p) = -\langle a_t^{N}(q)\, b_r^{\bar N}(-q-p)\, a_s^{\Lambda}(p)|E-H|\, \chi_1(p)\, a_r^{N\dagger}(p)\, \bar K^{\dagger}(-p) + \chi_2(p)\, a_r^{\Lambda\dagger}(p)\, \Pi^{\dagger}(-p)\rangle \left[E - E_N(q) - E_\Lambda(p) - E_N(p+q)\right]^{-1}$$

$$\chi_5(q,p) = -\langle a_t^{N}(q)\, b_r^{\bar N}(-q-p)\, a_s^{\Lambda}(p)|E-H|\, \chi_1(p)\, a_r^{N\dagger}(p)\, \bar K^{\dagger}(-p) + \chi_3(p)\, a_r^{\Sigma\dagger}(p)\, \Pi^{\dagger}(-p)\rangle \left[E - E_N(q) - E_\Sigma(p) - E_N(p+q)\right]^{-1}$$

$$\chi_6(q,p) = -\langle a_t^{\Lambda}(q)\, b_s^{\bar\Sigma}(-p-q)\, a_r^{\Lambda}(p)|E-H|\, \chi_3(p)\, a_r^{\Lambda\dagger}(p)\, \Pi^{\dagger}(-p)\rangle \left[E - E_\Lambda(q) - E_\Sigma(p) - E_\Lambda(p+q)\right]^{-1}$$

$$\chi_7(q,p) = -\langle a_t^{\Sigma}(q)\, b_s^{\bar\Sigma}(-p-q)\, a_r^{\Lambda}(p)|E-H|\, \chi_3(p)\, a_r^{\Sigma\dagger}(p)\, \Pi^{\dagger}(-p)\rangle \left[E - E_\Lambda(p) - E_\Sigma(q) - E_\Sigma(p+q)\right]^{-1}$$

$$\chi_8(q,p) = -\langle a_t^{\Sigma}(q)\, b_s^{\bar\Sigma}(-p-q)\, a_s^{\Sigma}(p)|E-H|\, \chi_3(p)\, a_r^{\Sigma\dagger}(p)\, \Pi^{\dagger}(-p)\rangle \left[E - E_\Sigma(q) - E_\Sigma(q) - E_\Sigma(p+q)\right]^{-1}$$

Substituting these equations into Eqs. (1) - (3), we get three equations as follows:

$$\begin{aligned}\langle a_{r}^{N}(p)\bar{K}(-p)\,|\,E-H\,|\,\big[\, & a_{r}^{N\dagger}(p)\bar{K}^{\dagger}(-p)\,\chi_1(p) + a_{r}^{\Lambda\dagger}(p)\Pi^{\dagger}(-p)\,\chi_2(p)\\ & + \chi_3(p)\,a_{r}^{\Sigma\dagger}(p)\,\Pi^{\dagger}(-p)\\ & + \chi_4(q,p)\;a_{t}^{N\dagger}(q)\,b_{r}^{\bar{N}\dagger}(-q-p)\,\hat{a}_{s}^{\Lambda\dagger}(p)\\ & + \chi_5(q,p)\;a_{t}^{N\dagger}(q)\,b_{r}^{\bar{N}\dagger}(-q-p)\,a_{s}^{\Sigma\dagger}(p)\big]\rangle\\ & = 0\end{aligned}$$

..... (A)

where the fourth and fifth terms are

$$-\big[\langle a_{r}(p)\bar{K}(-p)\,|E-H|\,a_{t}^{N\dagger}(q)\,b_{r}^{\bar{N}\dagger}(-p-q)\,a_{s}^{\Lambda\dagger}(p)\rangle\langle a_{t}^{\Lambda\dagger}(q)\,b_{r}^{\bar{N}\dagger}(-p-q)\,a_{s}^{\Lambda}(p)\,|E-H|$$
$$\{a_{r}^{N\dagger}(p)\bar{K}^{\dagger}(-p)\chi_1(p) + a_{r}^{\Lambda\dagger}(p)\Pi^{\dagger}(-p)\chi_2(p)\}\rangle\,[E-E_N(q)-E_\Lambda(p)-E_N(p+q)]^{-1}$$

and the same expression with Λ replaced by Σ and χ_2 by χ_3 .

Similarly

$$\langle a_{r}^{\Lambda}(p)\Pi(-p)\,|\,E-H\,|\,[a_{r}^{N\dagger}(p)\bar{K}^{\dagger}(-p)\chi_1(p) + a_{r}^{\Lambda\dagger}(p)\Pi^{\dagger}(-p)\,\chi_2(p)]\rangle$$

$$-\langle a_{r}^{\Lambda}(p)\Pi(-p)\,|\,E-H\,|\,a_{t}^{\Lambda\dagger}(q)\,b_{r}^{\bar{\Sigma}\dagger}(-p-q)\,a_{s}^{N\dagger}(p)\rangle\langle a_{t}^{\Lambda}(q)\,b_{r}^{\Sigma}(-p-q)\,\hat{a}_{s}(p)\,|E-H|\,a_{r}^{\Lambda\dagger}(p)\Pi^{\dagger}(-p)\rangle\chi_2(p)[E-E_\Lambda(p)-E_{\Lambda}(q)-E_{\Sigma}(p+q)]^{-1}$$

$$-\langle a_{r}^{\Lambda}(p)\Pi(-p)\,|E-H|\,a_{t}^{N\dagger}(q)\,b_{r}^{\bar{\Sigma}\dagger}(-p-q)\,a_{s}^{\Lambda\dagger}(p)\langle a_{r}^{N}(q)\,b_{r}^{\bar{N}}(-p-q)\,\hat{a}_{s}(p)\,|E-H\rangle|\,a_{r}^{N\dagger}(p)\bar{K}^{\dagger}(-p)\chi_1(p)$$
$$+ a_{r}^{\Lambda\dagger}(p)\Pi^{\dagger}(-p)\chi_2(p)\rangle[E-E_\Lambda(p)-E_N(q)-E_N(p+q)]^{-1} = 0 \quad .....\quad (B)$$

and

$$\langle a_1^{\Sigma}(p)\Pi(-p)|E-H|\left[a_1^{N\dagger}(p)\bar{K}^{\dagger}(-p)\chi_1(p)+a_1^{\Sigma\dagger}(p)\Pi^{\dagger}(-p)\chi_3(p)\right]\rangle$$

$$-\langle a_1^{\Sigma}(p)\Pi(-p)|E-H|a_t^{\Sigma\dagger}(q)b_1^{\bar{\Xi}\dagger}(-p-q)a_s^{N\dagger}(p)\rangle\langle a_t^{\Sigma}(q)b_1^{\bar{\Xi}}(-p-q)a_1^{N}(p)|E-H|a_1^{\Sigma\dagger}(p)\Pi^{\dagger}(-p)\rangle\chi_3(p)\left[E-E_{\Sigma}(q)-E_{N}(p)-E_{\Xi}(p+q)\right]^{-1}$$

$$-\langle a_1^{\Sigma}(p)\Pi(-p)|E-H|a_t^{\Sigma\dagger}(q)b_1^{\bar{\Xi}\dagger}(-p-q)a_s^{\Sigma\dagger}(p)\rangle\langle a_t^{\Sigma}(q)b_1^{\bar{\Xi}}(-p-q)a_s^{\Sigma}(p)|E-H|a_1^{\Sigma\dagger}(p)\Pi^{\dagger}(-p)\rangle\chi_3(p)\left[E-E_{\Sigma}(q)-E_{\Sigma}(p)-E_{\Xi}(p+q)\right]^{-1}$$

$$-\langle a_1^{\Sigma}(p)\Pi(-p)|E-H|a_t^{N\dagger}(q)b_1^{\bar{N}\dagger}(-p-q)a_s^{\Sigma\dagger}(p)\rangle\langle a_t^{N}(q)b_1^{\bar{N}}(-p-q)a_s^{\Sigma}(p)|E-H|a_1^{N\dagger}(p)\bar{K}^{\dagger}(-p)\chi_1(p)$$

$$+a_1^{\Sigma\dagger}(p)\Pi^{\dagger}(-p)\chi_3(p)\rangle\left[E-E_{\Sigma}(p)-E_{N}(q)-E_{N}(p+q)\right]^{-1}$$

$$=0$$

..... (c)

Appendix 2.

The various kernels in Chapter 14 are given below

$$K_{12}(p,s) = \frac{-N_4(p)N_3(s)}{16\pi^3} \sqrt{\frac{M^2 M\Lambda}{4E_N(p)E_\Lambda(s)\omega(p)\omega(s)}} \cdot \frac{C_{12}(T)}{E_N(p+s)} \cdot \frac{\varpi_+^N(p)\,\Lambda_+^N(p+s)\,\omega_s^\Lambda(s)}{E-E_\Lambda(s)-E_N(p)-E_N(p+s)}$$

$K_{13}(p,s) = K_{12}(p,s)$ with Λ replaced by Σ and $G_{12}(T)$ replaced by $G_{13}(T)$

$K_{21}(p,s) = K_{12}(s,p)$ and $M \longleftrightarrow \Lambda$.

$$K_{22}(p,s) = -\frac{N_3(p)N_3(s)}{16\pi^3} \sqrt{\frac{\Lambda^2\Sigma^2}{4E_\Lambda(p)E_\Lambda(s)\omega(p)\omega(s)}} \frac{G_{22}(T)}{E_\Sigma(p+s)} \frac{\varpi_+^N(p)\,\Lambda_+^\Sigma(p+s)\,\omega_s^\Lambda(s)}{E-E_\Lambda(p)-E_\Lambda(s)-E_\Sigma(p+s)}$$

$$K_{23}(p,s) = \frac{-N_3(p)N_3(s)}{16\pi^3} \sqrt{\frac{\Sigma^2\Sigma\Lambda}{4E_\Lambda(p)E_\Sigma(s)\omega(p)\omega(s)}} \frac{G_{23}(T)}{E_\Sigma(p+s)} \frac{\varpi_+^N(p)\,\Lambda_+^\Sigma(p+s)\,\omega_s^\Sigma(s)}{E-E_\Lambda(p)-E_\Sigma(s)-E_\Sigma(p+s)}$$

$$K_{31}(p,s) = \frac{-N_3(p)N_3(s)}{16\pi^3} \sqrt{\frac{M^2 M\Sigma}{4E_\Sigma(p)E_N(s)\omega(s)\omega(p)}} \frac{G_{31}(T)}{E_N(p+s)} \frac{\varpi_+^N(p)\,\Lambda_+^N(p+s)\,\omega_s^N(s)}{E-E_\Sigma(p)-E_N(s)-E_N(p+s)}$$

$K_{32}(p,s) = K_{32}(s,p)$

$$K_{33}(p,s) = -N_3(p)N_3(s)\Bigg[\frac{G_{33}^{(1)}(T)}{16\pi^3} \sqrt{\frac{\Lambda^2\Sigma^2}{4E_\Sigma(p)E_\Sigma(s)\omega(p)\omega(s)}} \frac{1}{E_\Lambda(p+s)} \frac{\varpi_+^\Sigma(p)\,\Lambda_+^\Lambda(p+s)\,\omega_s^\Sigma(s)}{E-E_\Sigma(p)-E_\Sigma(s)-E_\Lambda(p+s)}$$

$$\frac{G_{33}^{(2)}(T)}{16\pi^3} \sqrt{\frac{\Sigma^2\Sigma^2}{4E_\Sigma(p)E_\Sigma(s)\omega(p)\omega(s)}} \frac{1}{E_\Sigma(p+s)} \frac{\varpi_+^\Sigma(p)\,\Lambda_+^\Sigma(p+s)\,\omega_s^\Sigma(s)}{E-E_\Sigma(p)-E_\Sigma(s)-E_\Sigma(p+s)}\Bigg]$$

Here $C_{ij}(T)$ is the isotopic-spin factor

$$C(1) = \begin{pmatrix} 0 & \sqrt{2}\, G_{\Lambda K} G_{N\pi} & 2\, G_{\Sigma K} G_{N\pi} \\ \sqrt{2}\, G_{\Lambda K} G_{N\pi} & G_{\Lambda\pi}^2 & -\sqrt{2}\, G_{\Lambda\pi} G_{\Sigma\pi} \\ 2 G_{\Sigma K} G_{N\pi} & -\sqrt{2} G_{\Lambda\pi} G_{\Sigma\pi} & -G_{\Lambda\pi}^2 + G_{\Sigma\pi}^2 \end{pmatrix}$$

$$C(0) = \begin{pmatrix} 0 & \sqrt{6}\, G_{N\pi} G_{\Sigma K} \\ \sqrt{6}\, G_{N\pi} G_{\Sigma K} & G_{\Lambda\pi}^2, -2 G_{\Sigma\pi}^2 \end{pmatrix}$$

Introducing the abbreviations

$$E_{12} = E_{13} = E_N \qquad N_1(P) = N_4(P) \qquad N_2(s) = N_3(s)$$

$$E_{22} = E_{23} = E_\Sigma$$

$$E_{33}^{(1)} = E_\Lambda \quad E_{33}^{(2)} = E_\Sigma$$

we can write the kernels as

$$K_{ij}(p,s) = -\frac{N_i(p)\, N_j(s)}{16\pi^3} \sqrt{\frac{M_i M_j\; M_{ij}^{*}}{4 E_i(p) E_j(s) \omega_i(p) \omega_j(s)}}\; \frac{C_{ij}(T)}{E_{ij}(p+s)}$$

$$\frac{\bar{\omega}_r^i(p) \quad \Lambda_+^{ij}(p+s)\, \omega_s^j(s)}{E - E_i(p) - E_j(s) - E_{ij}(p+s)}$$

Appendix 3.

Programme for the Self-Energy and Normalization Functions

This programme was made under the Tabular Interpretative Programme (T.I.P.) scheme of the Deuce. In T.I.P., the data storage space consists of 128 columns each having 30 rows and 128 additional spaces for constants. The codeword is of four parts in the form a, b, c, r or N_a, N_b, N_c, r where a or Na are in general column or constant numbers respectively and r is the codenumber of the operation.

The programme is given below.

Programme T(1).

No.	a	b	c	r	Notes.
0	16	0	N31	4	Read $k_i, M_i, M_j, M_{ij}, \mu_i, \Delta, N, E, x_\mu, a_\mu$ $(0 \leq \mu \leq 3)$
1	N98	N98	N98	3	
2	N108	N108	N108	3	
3	24	0	N51	4	Read x the pivotal points
4	10	0	0	4	Read Δ
5	0	0	0	15	$s = n\Delta$
6	10	0	1	4	Read ω_{s_i}
7	N32*	N32*	N78*	0	M_i^2, M_j^2, M_{ij}^2, μ_i^2.
8	0	4	7	16	
9	0	0	0	18	

No.	a	b	c	r	Notes.
10	0	0	2	0	s^2
11	N79	2	3	2	$M_j^2 + s^2$
12	3	0	3	6	$E_j(s)$
13	N80	2	4	2	$s^2 + M_{ij}^2$
14	4	0	4	6	$E_{ij}(s)$
15	N33	N34	N82	0	$M_i M_{ij}$
16	N82	2	5	3	$M_i M_{ij} - s^2$
17	4	3	6	0	$E_j(s) E_{ij}(s)$
18	5	6	7	2	$M_i M_{ij} + E_i(s) E_{ij}(s) - s^2$
19	3	4	9	2	$E_{ij}(s) + E_{ij}(s)$
20	9	N35	8	3	$E_{ij}(s) + E_i(s) - \mu_i$
21	8	8	8	0	
22	8	6	8	0	
23	2	8	8	1	
24	8	7	8	0	Integrand for $B_i^{(j)}(0)$
25	8	1	8	0	Multiplied by Simpson weights
26	8	0	8	15	Add
27	8	9	N90	13	Integral to N90
28	N90	N4	N90	1	Divide by π
29	N90	N35	N90	1	$B_i^{(j)}(0)$
30	N31	N51*	N51*	0	$k_i x = p$
31	0	N51*	30	0	ps

No.	a	b	c	r	Notes.
32	N51*	N51*	N83	0	p^2
33	N83	N78	N84	2	$M_i^2 + p^2$
34	N84	0	N85	6	$E_i(p)$
35	N83	N81	N86	2	$p^2 + \mu_i^2$
36	N86	0	N86	6	$\omega_i(p)$
37	N87	N85	N87	2	$E_i(p) + \omega_i(p)$
38	N87	N38	N89	3	$E_i(p) + \omega_i(p) - E$
39	3	N86	11	3	$E_j(s) - \omega_i(p)$
40	N85	N38	N88	3	$E_i(p) - E$
41	N88	3	14	2	$E_i(p) + E_j(s) - E$
42	N39*	30	10	0	$ps\ x\mu$
43	10	10	12	2	$2ps\ x\mu$
44	2	12	12	2	$s^2 + 2ps\ x\mu$
45	12	N83	12	2	$p^2 + s^2 + 2psx\mu$
46	12	N80	12	2	$M_{ij}^2 + p^2 + s^2 + 2ps\ x\mu$
47	12	0	12	6	$E_{ij}(x\mu)$
48	12	11	13	2	$E_{ij}(x\mu) + E_j(s) - \omega_i(p)$
49	13	13	13	0	square
50	12	14	34	2	$E_{ij}(x\mu) + E_i(p) + E_j(s) - E$
51	12	3	15	0	$E_{ij}(x\mu)E_j(s)$
52	15	13	18	0	Denom. in $B_i^j(p)$
53	18	34	34	0	" in $h_i^j(p)$

No.	a	b	c	r	Notes.
54	5	10	16	3	$M_i M_{ij} - s^2 - ps\ x\mu$
55	15	16	16	2	$M_i M_{ij} + E_{ij}(x\mu) E_j(s) - s^2 - psx\mu$
56	16	2	16	0	$s^2 x$ "
57	16	18	20	1	Integrand of $B_i^j(p)$
58	16	34	21	1	" " $h_i^j(p)$
59	20	1	20	0	Multiply by ω_{s_i} (Simpson weights
60	21	1	21	0	"
61	20	0	20	15	Add
62	21	0	21	15	Add
63	20	9	N91 *	13	Integral to B_i^j to N91
64	21	9	N101*	13	" h_i^j to N101
65	N91*	N43*	N91*	0	Multiply by ω_p(Gaussian wgts.)
66	N101*	N43*	N101*	0	"
67	N91*	N98	N98	2	Accumulate in N98
68	N101*	N108	N108	2	" N108
69	0	4	42	16	
70	0	0	0	18	
71	0	0	72	19	
72	N2	N4	N99	0	2π
73	N99	N86	N99	0	$2\pi\ \omega_i(p)$
74	N98	N99	N98	1	$B_i^j(p)$
75	N108	N99	N108	1	[illegible]
76	N98	N90	N98	3	$A_i^j(p)$

No.	a	b	c	r	Notes.
77	N108	N89	N108	0	$h_i^{(j)}(p)$
78	N98	0	0	5	Punch $A_i^{(j)}(p)$
79	N98	0*	24	13	Move to col. 24
80	N98	N98	N98	3	
81	N108	0	0	5	Punch $h_i^{(j)}(p)$
82	N108	0*	25	13	$h_i^{(j)}(p)$ in 25
83	N108	N108	N108	3	
84	0	24	30	16	
85	0	0	0	18	
86	24	0	0	5	Punch $A_i^j(p)$
87	25	0	0	5	" $h_i^{(j)}(p)$
88	24	24	24	3	
89	25	25	25	3	
90	0	0	0	31	

Programme T(2)

No.	a	b	c	r	Notes.
0	9	0	N31	4	$k_i, G_1, G_2, G_3, M_i, \mu_i, M, \Lambda, \Sigma$
1	0	0	16	4	x
2	0	0	1	4	$h_i^{(1)}$
3	0	0	0	4	$h_i'^{(2)}$
4	0	0	2	4	$h_i''^{(2)}$
5	0	0	3	4	$h_i^{(3)}$

No.	a	b	c	r	Notes.
6	0	2	2	2	$h_i^{(2)}$
7	N32	1	1	0	$G_1 h_i^{(1)}$
8	N33	2	2	0	$G_2 h_i^{(2)}$
9	N34	3	3	0	$G_3 h_i^{(3)}$
10	1	2	2	2	
11	2	3	4*	2	$\Sigma G_j h_i^{(j)}$
12	0	2	2	16	$\Sigma G_j A_i^{(j)}$
13	N1	5	5	2	$A_i(p)$
14	5	0	0	5	Punch $A_i(p)$
15	4	5	7	1	$A_i^{-1}(p) \sum_j G_j h_i^{(j)}(p)$
16	N1	7	7	2	$h_i(p)$
17	N31	16	16	0	$p = k_i x$
18	16	16	17	0	p^2
19	N35*	N35*	N40*	0	M_i^2
20	N40*	17	18*	2	$M_i^2 + p^2$
21	18*	0	18*	6	$E_i(p)$
22	0	2	19	16	$\omega_i(p)$
23	18	19	20	0	$E_i(p) \ \omega_i(p)$
24	5	0	0	30	Punch $A_i(p)$
25	5	20	20	0	$A_i(p) E_i(p) \ \omega_i(p)$
26	N35	20	20	0	$M_i A_i(p) E_i(p) \ \omega_i(p)$
27	N37*	20	22	1	$M_j / M_i A_i(p) E_i(p) \ \omega_i(p)$

No.	a	b	c	r	Notes.
28	22	0	21	6	
29	21	7	21	1	$C_{ij}(p) = \frac{1}{h_i(p)}\sqrt{\frac{M_j}{M_i\, E_i(p)\omega_i(p)\omega_i(p)\, A_i(p)}}$
30	21	0	0	30	Punch in binary
31	21	0	0	5	" decimals.
32	0	3	27	16	
33	7	0	0	5	Punch $h_i(p)$
34	0	0	0	31	

Here

$$h_i(p) = 1 + \sum_j \lambda_j \frac{\sigma_i^{j\,2}}{4\pi} h_i^j(p) \Big/ A_i(p)$$

$$A_i(p) = 1 + \sum \lambda_j \frac{\sigma_i^{j\,2}}{4\pi} A_i^j(p)$$

Appendix 4.

Programme for the Kernels.

This programme was made under the General Interpretative Scheme of the Deuce. The G.I.P. is a programme for controlling standard programmes called 'Bricks' and is particularly suitable for matrix operations. The function of G.I.P. is to read and store on the drum a number of standard bricks and then to obey them in a manner specified by a series of codewords. Each codeword again is written in the form a,b,c,r which instructs G.I.P. to obey the rth of the stored bricks and provide that brick with parameters a b and c which usually specify what data is to be used and what is to be done with the results.

The programme is given below.

Programme G(1)

Bricks required

1. Read Binary Matrix
2. Punch

3-4. Term by Term Matrix Arithmetic

5. Term by Term Matrix Square Root
6. Select Element
7. Expand Scalar
8. Scalar Mult
9. Form a zero-matrix.

Codewords:

No.	a	b	c	r	Notes
0	0	1	1	47	
1	0	2	2	47	
2	0	0	0	1	Read $k_i, k_j, 2, M_i, M_j, M_{ij}, E, x_0, x_0, \omega_0, x_1, x_1, \omega_1, x_2, x_2, \omega_2, x_3, x_3, \omega_3$
3	0	0	197	1	Read x 1 x 24 matrix
4	0	0	184	1	Read x 12 x 24 matrix
5	93	95	6	42	Select k_i
6	197	0	197	8	$k_i x = p$
7	93	95	8	42	Select k_j
8	184	0	184	8	$k_j x = s$
9	3	0	0	48	
10	197	197	1	3	p^2
11	3	0	0	48	
12	184	184	2	3	s^2
13	3	0	0	48	
14	197	184	14	3	ps
15	93	95	16	42	Select 2
16	14	0	160	8	2ps
17	93	95	18	42	Select M_i
18	1	24	26	7	Expand
19	3	0	0	48	
20	26	26	197	3	M_i^2
21	1	0	0	48	

No.	a	b	c	r	Notes.
22	197	1	39	3	$M_i{}^2 + p^2$
23	17	21	24	46	
24	2	197	40	3	$M_j^2 + s^2$
25	17	21	26	46	
26	1	197	172	3	$M_{ij}^2 + p^2$
27	39	0	1	5	$E_i(p)$
28	40	0	40	5	$E_j(s)$
29	1	0	0	48	
30	172	2	52	3	$M_{ij}^2 + p^2 + s^2$
31	52	0	2	5	$E_{ij}(0)$
32	1	0	0	48	
33	1	40	64	3	$E_i(p) + E_j(s)$
34	17	18	35	46	Select E
35	2	0	0	48	
36	64	26	64	3	$B_{ij}(p,s)$
37	1	0	0	48	
38	64	2	76	3	$B_{ij} + E_{ij}(0)$
39	12	24	100	9	zero-matrix
40	12	24	112	9	"
41	12	24	124	9	"
42	12	24	136	9	"
43	93	95	44	42	Select $x\mu$
44	160	0	14	8	2ps $x\mu$

No.	a	b	c	r	Notes.
45	1	0	0	48	
46	52	14	14	3	$M_{ij}^2 + p^2 + s^2 + 2ps\ x\mu$
47	14	0	14	5	$E_{ij}(x)$
48	2	0	0	48	
49	2	14	88	3	$E_{ij}(0) - E_{ij}(x)$
50	1	0	0	48	
51	14	64	184	3	$E_{ij}(x) + B_{ij}$
52	4	0	0	48	
53	88	184	184	3	$F_{ij}(x)$
54	93	95	55	42	Select
55	184	0	27	8	$x\ F_{ij}(x)$
56	93	95	57	42	Select
57	184	0	184	8	$F_{ij}(x_\mu)$
58	27	0	27	8	$x\ F(x)$
59	1	0	0	48	
60	184	100	100	3	
61	1	0	0	48	
62	27	112	112	3	
63	1	0	0	48	
64	76	14	148	3	$B_{ij} + E_{ij}(0) + E_{ij}(x)$
65	4	0	0	48	
66	148	14	14	3	
67	3	0	0	48	

No.	a	b	c	r	Notes.
68	184	14	88	3	$\omega G_{ij}(x)$
69	3	0	0	48	
70	27	14	14	3	$\omega x G_{ij}(x)$
71	1	0	0	48	
72	88	124	124	3	$\sum_\mu \omega_\mu G(x_\mu)$
73	1	0	0	48	
74	14	136	136	3	$\sum_\mu \omega_\mu x_\mu G(x_\mu)$
75	94	95	43	37	
76	95	95	95	39	
77	0	2	0	6	Select 2
78	1	24	26	7	Expand
79	1	0	0	48	
80	100	26	100	3	$2 + \sum_\mu \omega_\mu F(x_\mu)$
81	1	0	0	48	
82	124	26	124	3	$2 + \sum_\mu \omega_\mu G(x_\mu)$
83	4	0	0	48	
84	124	2	88	3	$[2 + \sum \omega_\mu G(x_\mu)] E^{-1}_{ij}(0)$
85	4	0	0	48	
86	136	2	52	3	
87	4	0	0	48	
88	100	76	14	3	$X_0^{ij}(p,s)$
89	4	0	0	48	
90	112	76	27	3	$X_1^{ij}(p,s)$

No.	a	b	c	r	Notes.
91	0	92	0	47	
92	0	0	1	47	
93	0	0	0	6	
94	0	19	0	0	
95	0	0	0	0	
0	4	0	0	48	
1	88	76	64	3	$Y_o^{ij}(p,s)$
2	4	0	0	48	
3	52	76	76	3	$Y_1^{ij}(p,s)$
4	62	63	5	42	Select M_i
5	1	24	26	7	Expand
6	1	0	0	48	
7	26	1	39	3	$E_i(p) + M_i$
8	2	0	0	48	
9	1	26	38	3	$E_i(p) - M_i$
10	4	6	11	46	Select M_j
11	40	26	52	3	$E_j(s) + M_j$
12	2	0	0	48	
13	40	26	88	3	$E_j(s) - M_j$
14	1	0	0	48	
15	39	52	124	3	$E_i(p) + E_j(s) + M_i + M_j$
16	1	0	0	48	
17	38	88	88	3	$E_i(p) + E_j(s) - M_i - M_j$
18	4	6	19	46	Select M_{ij}

No.	a	b	c	r	Notes.
19	26	88	88	3	$E_i(p) + E_j(s) - M_i - M_j + M_{ij}$
20	3	0	0	48	
21	64	88	88	3	$Y_o^{ij}[E_i(p) + E_j(s) - M_i - M_j + M_{ij}]$
22	1	0	0	48	
23	88	14	14	3	$X_o^{ij} + (E_i(p)+E_j(s)-M_i-M_j+M_{ij})Y_o^{ij}$
24	3	0	0	48	
25	39	14	14	3	
26	2	0	0	48	
27	124	26	88	3	
28	3	0	0	48	
29	88	76	76	3	
30	1	0	0	48	
31	76	27	27	3	$X_1^{ij} + (E_i(p)+E_j(s)+M_i+M_j-M_{ij})Y_o^{ij}$
32	0	2	0	6	Select 2
33	52	0	52	8	$2[E_j(s)+M_j]$
34	4	0	0	48	
35	27	52	52	3	
36	3	0	0	48	
37	160	52	52	3	
38	1	0	0	48	

No.	a	b	c	r	Notes.
39	14	52	52	3	$L_{ij}(p,s)$
40	52	0	0	2	Punch
41	0	0	0	47	
42	0	3	0	6	
43	0	0	0	0	

Programme - G(2)

Bricks Required:

1. Read Binary Matrix
2. Punch

3-4. Term by Term Matrix Arithmetic

5-6. Diagonal Post-Mult.

7. Select Element
8. Expand Scalar
9. Compound Rows
10. Extract Submatrix
11. Term by Term Matrix Sq. Root
12. Transpose

Codewords:

No.	a	b	c	r	Notes
0	0	1	1	47	
1	0	2	2	47	
2	0	0	0	1	Weights
3	0	0	1	1	pivotal points
4	0	0	2	1	k_j, M_j, μ_j, $E, C_{ij}(T)$
5	0	0	3	1	$C_{ij}(p)$ 1 x 24 matrix
6	0	0	23	1	$A_j(k_j x)$ "
7	0	0	24	1	$L_{ij}(k_j x_1 k_i x)$ first half
8	0	0	48	1	" second half
9	0	0	22	1	ξ 1 x 24 matrix

No.	a	b	c	r	Notes.
10	16	18	11	46	Select and expand k_j
11	4	1	5	3	$s = k_j x$
12	3	0	0	48	
13	5	5	9	3	s^2
14	24	48	60	9	$L_{ij}(s,p)$
15	60	0	60	12	$L_{ij}(p,s)$
16	94	95	17	42	Select M_j
17	1	24	4	8	Expand
18	3	0	0	48	
19	4	4	5	3	M_j^2
20	1	0	0	48	
21	9	5	6	3	$M_j^2 + s^2$
22	6	0	7	11	$E_j(s)$
23	16	21	24	46	
24	6	0	8	11	$\omega_j(s)$
25	1	0	0	48	
26	7	8	10	3	$E_j(s) + \omega_j(s)$
27	3	0	0	48	
28	7	8	11	3	$E_j(s)\,\omega_j(s)$
29	16	17	30	46	
30	2	0	0	48	
31	10	4	12	3	$\mathcal{E}_j(s) = E_j(s) + \omega_j(s) - E$
32	3	0	0	48	

No.	a	b	c	r	Notes.
33	11	23	8	3	$E_j(s)\ \ \omega_j(s)\ A_j(s)$
34	8	0	8	11	$[E_j(s)\ \ \omega_j(s)\ A_j(s)]^{1/2}$
35	4	0	0	48	
36	9	8	8	3	
37	60	8	24	5	
38	12	0	1	32	"Diagonal Pre-mult" Brick
39	3	24	60	12	$K_{ij}(p,s)$
40	0	0	42	33	
41	0	0	0	0	
42	60	0	4	48	
43	0	4	84	10	Extract P
44	60	0	4	48	
45	4	8	88	10	Extract Q
46	60	4	8	48	
47	0	4	116	10	Extract S
48	60	4	8	48	
49	4	8	120	10	Extract C
50	1	0	0	48	
51	84	116	112	3	P + S
52	1	0	0	48	
53	88	120	124	3	Q + C
54	12	0	1	48	

No.	a	b	c	r	Notes.
55	0	4	10	10	Extract $\varepsilon_j(x_\lambda)$ $0 \leq \lambda \leq 3$
56	12	0	1	48	
57	4	8	11	10	Extract $\varepsilon_j(2-x_\lambda)$
58	4	0	0	48	
59	10	11	6	3	f(x)
60	124	6	128	5	f(x) (Q + C)
61	1	0	0	48	
62	112	128	142	3	(p+s) + f(x) (Q + C)
63	22	0	1	48	
64	4	8	14	10	Extract $(x_\lambda - 1)$ $0 \leq \lambda \leq 3$
65	0	0	1	48	
66	0	4	15	10	Extract $\frac{1}{2}\omega_\lambda$
67	4	0	0	48	
68	15	10	16	3	
69	142	16	146	5	$Z_i Z_j$
70	2	0	0	48	
71	112	128	150	3	(P + S) - f(x) (Q + C)
72	3	0	0	48	
73	14	16	7	3	
74	150	7	154	5	$Z_i Y_j$
75	2	0	0	48	

No.	a	b	c	r	Notes.
76	84	116	116	3	P - S
77	2	0	0	48	
78	88	120	120	3	Q - C
79	120	6	84	5	f(x) (Q - C)
80	1	0	0	48	
81	116	84	88	3	(P-S) + f(x) (Q - C)
82	88	16	138	5	$Y_i Z_j$
83	2	0	0	48	
84	116	84	162	3	P-S - f(x) (Q - C)
85	162	7	166	5	$Y_i Y_j$
86	60	0	4	48	
87	8	24	24	10	Extract R
88	60	4	8	48	
89	8	24	40	10	Extract T
90	1	0	0	48	
91	24	40	104	3	R + T
92	0	93	0	47	
93	0	0	1	47	
94	2	0	0	7	
95	0	0	0	0	
0	2	0	0	48	
1	24	40	120	3	R - T
2	0	0	1	48	

No.	a	b	c	r	Notes.
3	8	24	5	10	Extract $\omega_\lambda \quad 8 \leq \lambda \leq 23$
4	12	0	1	48	
5	8	24	3	10	Extract $\varepsilon_j(x_\lambda)$ "
6	4	0	0	48	
7	5	3	1	3	
8	104	1	24	5	Z_iF_j
9	120	1	40	5	Y_iF_j
10	60	8	24	48	
11	0	4	84	10	Extract U
12	60	8	24	48	
13	4	8	92	10	Extract V
14	92	6	88	5	$f(x)V$
15	1	0	0	48	
16	84	88	104	3	$U + f(x)V$
17	2	0	0	48	
18	84	88	88	3	$U - f(x)V$
19	104	16	120	5	F_iZ_j
20	88	7	104	5	F_iY_j
21	60	8	24	48	
22	8	24	88	10	Extract W
23	88	1	170	5	F_iF_j
24	10	0	1	32	'Comp. Col.' Brick
25	146	154	124	10	Z_iZ_j, Z_iY_j

No.	a	b	c	r	Notes.
26	124	24	146	10	$(Z_iZ_j,\ Z_iY_j,\ Z_iF_j)$
27	138	166	28	10	$Y_iZ_j\ Y_iY_j$
28	28	40	150	10	$(Y_iZ_j,\ Y_iY_j,\ Y_iF_j)$
29	120	104	88	10	$F_iZ_j,\ F_iY_j$
30	88	170	24	10	$(F_iZ_j,\ F_iY_j,\ F_iF_j)$
31	146	150	154	9	
32	154	24	60	9	
33	10	0	1	32	'Scalar Mult' Brick
34	2	0	0	7	Select k_j
35	60	0	60	10	
36	2	4	0	7	Select $C_{ij}(T)$
37	60	0	60	10	$K_{ij}(p,s)$
38	0	0	43	33	
39					
40					
41					
42					$a_{ij}(\nu,s)$
43	60	0	0	2	Punch
44	0	0	0	47	

Note:

(i) $\xi = [1, 1, 1, 1, (x_0-1), (x_1-1), (x_2-1), (x_3-1), 1, \cdots \cdots \cdots 1]$ 1 x 24 matrix

(ii) For a_{33} punch out $K_{33}^{(1)}$ and $K_{33}^{(2)}$. Then use the following programme - G(3).

(iii) For a_{22}, use a zero-matrix for instruction (1) in the next programme - G(3).

Programme - G(3)

Bricks Required:

1. Read Binary Matrix
2. Punch

3-4. Term by Term Matrix Arithmetic

5. Expand Diagonal

Codewords:

No.	a	b	c	r	Notes
0	0	0	0	1	$K_{33}^{(1)}$
1	0	0	24	1	$K_{33}^{(2)}$
2	1	0	0	48	
3	0	24	24	3	K_{33}
4	0	0	0	1	
5	0	0	1	5	Expand
6	1	0	0	48	
7	1	24	24	3	a_{33}
8	24	0	0	2	Punch
9	0	0	0	33	

Appendix 5.

The Born Terms.

We have defined

$$f_i^B(p_r) = \begin{bmatrix} Z_i^B(p_r) \\ Y_i^B(p_r) \\ F_i^B(p_r) \end{bmatrix} \quad \begin{matrix} 0 \le r \le 3 \\ 0 \le r \le 3 \\ 8 \le r \le 23 \end{matrix}$$

where

$$Z_i^B(p_r) = \sum_j \lambda_j \left[f_{ij}^B(p_r) + f_{ij}^B(2-p_r) \right] \qquad 0 \le r \le 3$$

$$Y_i^B(p_r) = \sum_j \lambda_j \left[f_{ij}^B(p_r) - f_{ij}^B(2-p_r) \right] \qquad 0 \le r \le 3$$

$$F_i^B(p_r) = f_i^B(p_r) = \sum_j \lambda_j \, f_{ij}^B(p_r) \qquad 8 \le r \le 23$$

$$f_{ij}^B(p_r) = \frac{C_{ij}(T)}{32\pi^2} \sqrt{\frac{M_j E_j(k_j)\omega_j(k_j)}{M_i E_i(p_r)\omega_i(p_r)}} \, \frac{1}{h_i(p_r)} \, \frac{1}{\sqrt{A_i(p_r)A_j(k_j)}} \, \frac{k_j}{E} \, L_{ij}(p_r k_j)$$

For each solution, one can fix the λ_i which are arbitrary: e.g.

i) $\lambda_1 = \lambda_1 = \lambda_3 = 1$

ii) $\lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = -1$

iii) $\lambda_1 = 1 \qquad \lambda_2 = \lambda_3 = -1$

The choice must be revised in the light of numerical solutions which must be as different as possible.

Programme - T(3)

No.	a	b	c	r	Notes.
0	4	0	N32	4	M_i, M_j, M_{ij}, μ_j
1	N32*	N32*	N36*	0	
2	0	4	1	16	
3	2	0	N40	4	k_i, k_j
4	N40	N40	N46	0	k_i^2
5	2	0	N14	4	E, $A_j(k_j)$
6	4	0	N70	4	
7	4	0	N80	4	
8	0	0	0	4	x
9	0	N40	28	0	$k_i x = p$
10	28	28	2	0	p^2
11	N41	N41	N4	0	k_j^2
12	28	N41	35	0	pk_j
13	35	35	3	2	$2pk_j$
14	N36	2	5	2	$M_i^2 + p^2$
15	N38	2	4	2	$M_{ij}^2 + p^2$
16	N37	N4	N5	2	$M_j^2 + k_j^2$
17	N39	N4	N7	2	$\mu_j^2 + k_j^2$
18	5	0	5	6	$E_i(p)$
19	N5	0	N5	6	$E_j(k_j)$
20	N7	0	N7	6	$\omega_j(k_j)$
21	N5	N7	N8	0	$E_j(k_j)\ \omega_j(k_j)$

No.	a	b	c	r	Notes.
22	N8	0	N8	6	
23	N41	N8	N8	0	$k_j\left(E_j(k_j)\omega_j(k_j)\right)^{1/2}$
24	N8	N14	N8	1	
25	5	N5	7	2	$E_i(p) + E_j(k_j)$
26	7	N14	7	3	$B_{ij}(p, k_j)$
27	N4	4	4	2	$M_{ij}^2 + p^2 + k_j^2$
28	4	0	13	6	$E_{ij}(0)$
29	7	13	14	2	$B_{ij} + E_{ij}(0)$
30	13	14	15	0	
31	N0	24	20*	13	
32	0	4	31	16	
33	0	0	0	18	
34	3	N80*	16	0	$2pk_jxg_i$
35	4	16	16	2	$M_{ij}^2 + p^2 + k_j^2 + 2pk_j\,xg_i$
36	16	0	16	6	$E_{ij}(x)$
37	14	16	17	2	$B_{ij} + E_{ij}(x) + E_{ij}(0)$
38	13	16	18	3	$E_{ij}(0) - E_{ij}(x)$
39	7	16	19	2	$B_{ij} + E_{ij}(x)$
40	18	19	18	1	$F_{ij}(x)$
41	17	16	17	1	
42	17	18	19	0	$G_{ij}(x)$
43	N80*	18	16	0	$x\,F(x)$

No.	a	b	c	r	Notes
44	N80*	19	17	0	$x\ G(x)$
45	N70*	18	18	0	$\omega F(x)$
46	18	20	20	2	$\Sigma\omega_\mu F(x_\mu)$
47	N70*	16	16	0	$\omega x F(x)$
48	16	21	21	2	$\Sigma\omega_\mu x_\mu F_{ij}(x_\mu)$
49	N70*	19	19	0	$\omega G(x)$
50	19	22	22	2	$\Sigma\omega_\mu G(x_\mu)$
51	N70*	17	17	0	$\omega x G(x)$
52	17	23	23	2	$\Sigma\omega_\mu x_\mu G(x_\mu)$
53	0	4	36	16	
54	0	0	0	18	
55	N2	20	20	2	$2 + \Sigma\omega_\mu F(x_\mu)$
56	N2	22	22	2	$2 + \Sigma\omega_\mu G(x_\mu)$
57	22	13	22	1	
58	23	13	23	1	
59	20*	14	20*	1	$X_o^{ij},\ X_1^{ij},\ Y_o^{ij},\ Y_1^{ij}$
60	0	4	59	16	
61	0	0	0	18	
62	N32	5	8	2	$E_i(p) + M_i$
63	N33	N5	N6	2	$E_j(k_j) + M_j$
64	5	N32	10	3	$E_i(p) - M_i$
65	N5	N33	N11	3	$E_j(k_j) - M_j$

No.	a	b	c	r	Notes
66	8	N6	12	2	$E_i(p) + E_j(k_j) + M_i + M_j$
67	12	N34	12	3	$E_i(p) + E_j(k_j) + M_i + M_j - M_{ij}$
68	10	N11	10	2	$E_i(p) + E_j(k_j) - M_i - M_j$
69	10	N34	10	2	$E_i(p) + E_j(k_j) - M_i - M_j + M_{ij}$
70	10	22	10	0	
71	12	23	11	0	
72	20	10	10	2	
73	21	11	11	2	
74	35	N6	35	1	$pk_j/[E_j(k_j) + M_j]$
75	8	10	10	0	
76	35	11	11	0	
77	10	11	12	2	$L_{ij}(p,k_j)$
78	N15	0	N15	6	$\sqrt{\Lambda_j}(k_j)$
79	12	N15	12	1	$L_{ij}/\sqrt{\Lambda_j}$
80	12	N8	12	0	
81	1	0	N96	4	Read $C_{ij}(T)$
82	0	0	6	4	Read $C_{ij}(p)$
83	N96	6	6	0	
84	12	6	12	0	$f^B_{ij}(p)$
85	12	0*	N50	13	

No.	a	b	c	r	Notes.
86	12	4*	N51	13	
87	N50	N51	N52	2	Z^B_{ij}
88	N50	N51	N53	3	Y^B_{ij}
89	N52	0*	12	13	
90	N53	4*	12	13	
91	0	4	85	16	
92	12	0	0	5	Punch f^B_{ij}
93	0	0	0	31	

Appendix 6.

Typical S-Matrices.

1. The S-matrix in the model B for

$$G_\pi^2/4\pi = 0.5 \qquad G_k^2/4\pi = 0.5$$

$$T = 1$$

$S_{11} = 0.99222382 + i\ 0.01152938$

$S_{12} = 0.00296197 - i\ 0.09787091$ $S_{21} = 0.00296598 - i\ 0.09797835$

$S_{13} = 0.00762178 - i\ 0.07549031$ $S_{31} = 0.00762624 - i\ 0.07554422$

$S_{22} = 0.98628904 - i\ 0.04746687$

$S_{23} = -0.00250802 + i\ 0.12403592$ $S_{32} = -0.00250726 + i\ 0.12396850$

$S_{33} = 0.98898675 + i\ 0.02750368$

$$T = 0$$

$S_{11} = 0.99614681 + i\ 0.00715839$

$S_{13} = 0.00432825 - i\ 0.08727423$ $S_{31} = 0.00433096 - i\ 0.08732889$

$S_{33} = 0.99196700 + \quad 0.09143951$

2. $$G_\pi^2/4\pi = 0.5 \qquad G_k^2/4\pi = 1.5$$

$$T = 1$$

$S_{11} = 0.97415213 + i\ 0.03421486$

$S_{12} = 0.01486057 - i\ 0.17570214$ $S_{21} = 0.01488329 - i\ 0.17592195$

$S_{13} = 0.02114059 - i\ 0.13514694$ $S_{31} = 0.02115153 - i\ 0.13525134$

$S_{22} = 0.97060416 + i\ 0.00305161$

$S_{23} = -0.01741694 + i\ 0.16280808$ $S_{32} = -0.01740667 + i\ 0.16267036$

$S_{33} = 0.97530525 + i\ 0.05715323$

T = 0

$S_{11} = 0.98782784 + i\ 0.02120230$

$S_{13} = 0.01188320 - i\ 0.15358907$ $S_{31} = 0.01189114 - i\ 0.15369159$

$S_{33} = 0.97933301 + i\ 0.13099699$

3. $G_{\pi}^{2}/4\pi = 0.5$ $G_{k}^{2}/4\pi = 3$

T = 1

$S_{11} = 0.94939858 + i\ 0.06617713$

$S_{12} = 0.04510890 - i\ 0.25954912$ $S_{21} = 0.04518797 - i\ 0.25993420$

$S_{13} = 0.04934789 - i\ 0.19865847$ $S_{31} = 0.04937526 - 0.19883249$

$S_{22} = 0.93402696 + i\ 0.08057161$

$S_{23} = -0.05010961 + i\ 0.22168522$ $S_{32} = -0.05006733 + i\ 0.22143529$

$S_{33} = 0.94665458 + i\ 0.10154319$

T = 0

$S_{11} = 0.97384497 + i\ 0.04148895$

$S_{13} = 0.02659768 - i\ 0.22180973$ $S_{31} = 0.02667718 - i\ 0.22197248$

$S_{33} = 0.95603586 + i\ 0.18992355$

Appendix 7.

Multiple Scattering Integral in Pion-Alpha Scattering.

Consider

$$N_{k'k} = \left(\Psi_f \,,\ V_{33} \frac{1}{E - H_0 + i\epsilon} V_{33} \Psi_i \right)$$

where

$$\Psi_f = e^{-iD\cdot R} \Psi_\alpha \quad \text{and} \quad \Psi_i = \Psi_\alpha$$

D being recoil momentum of the alpha-particle and R = $\frac{1}{4}(r_1 + r_2 + r_3 + r_4)$.

This gives rise to the following integral for two successive scatterings on nucleons 1 and 2, say:

$$N \int dr_1\, dr_2\, dr_3\, dr_4 \; e^{-2\sigma^2 r_{ij}^2 + i(k\cdot r_1 - k'\cdot r_2) + i p\cdot(r_2 - r_1) + i D\cdot R}$$

Introducing the co-ordinate system

$$S_1 = \tfrac{1}{2}(r_1 + r_2 - r_3 - r_4)$$
$$S_2 = \tfrac{1}{2}(r_1 - r_2 - r_3 - r_4)$$
$$S_3 = \tfrac{1}{2}(r_1 - r_2 - r_3 + r_4)$$

we find the integral reduces to

$$N \int ds_1\, ds_2\, ds_3 \; e^{-8\sigma^2(s_1^2 + s_2^2 + s_3^2) + i s_1\cdot q + i (s_2 + s_3)\cdot(P - p)}$$

where we have omitted the momentum conserving δ- function.

Using Laplace transform, this integration reduces to

$$e^{-[q^2 + 2(P-p)]/32\sigma^2}$$

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